

MIXED GAUSSIAN PROCESSES: A FILTERING APPROACH

BY CHUNHAO CAI, PAVEL CHIGANSKY¹ AND MARINA KLEPTSZYNA²

Nankai University, The Hebrew University of Jerusalem and Université du Maine

This paper presents a new approach to the analysis of mixed processes

$$X_t = B_t + G_t, \quad t \in [0, T],$$

where B_t is a Brownian motion and G_t is an independent centered Gaussian process. We obtain a new canonical innovation representation of X , using linear filtering theory. When the kernel

$$K(s, t) = \frac{\partial^2}{\partial s \partial t} \mathbb{E}G_t G_s, \quad s \neq t$$

has a weak singularity on the diagonal, our results generalize the classical innovation formulas beyond the square integrable setting. For kernels with stronger singularity, our approach is applicable to processes with additional “fractional” structure, including the mixed fractional Brownian motion from mathematical finance. We show how previously-known measure equivalence relations and semimartingale properties follow from our canonical representation in a unified way, and complement them with new formulas for Radon–Nikodym densities.

1. Introduction. In this paper, we present a new perspective on *mixed* processes of the form

$$(1.1) \quad X_t = B_t + G_t, \quad t \in [0, T], T > 0,$$

where $B = (B_t)$ is a Brownian motion and $G = (G_t)$ is an independent Gaussian process. Such mixtures have been the subject of much research in the past, due to their importance in engineering applications (see, e.g., the survey [18]), and, more recently, have reemerged in mathematical finance in the context of option pricing.

The renewed interest was triggered by Cheridito’s paper [5], in which the author considered mixed fractional Brownian motion (fBm)

$$(1.2) \quad X_t = B_t + B_t^H, \quad t \in [0, T],$$

where $B^H = (B_t^H)$ is fBm with the Hurst exponent $H \in (0, 1]$, that is, the centered Gaussian process with covariance function

$$(1.3) \quad R(s, t) := \mathbb{E}B_t^H B_s^H = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}), \quad s, t \in [0, T].$$

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A curious change in properties of X was shown to occur at $H = \frac{3}{4}$, where it became apparent that X is a semimartingale in its own filtration if, and only if, either $H = \frac{1}{2}$ or $H \in (\frac{3}{4}, 1]$. Moreover, in the latter case, the probability measure μ^X , induced by X on its paths space, is equivalent to the Wiener measure μ^B .

Since B^H is not a semimartingale on its own, unless $H = \frac{1}{2}$ or $H = 1$, this assertion means that B^H can be “regularized” up to a semimartingale by adding to it an independent Brownian perturbation. In [5], this fact is discussed in the context of arbitrage opportunities on nonsemimartingale markets (see also [6]). A comprehensive survey of further related developments in finance can be found in [3].

Besides being of interest to the finance community, the result in [5] also led to a number of elegant generalizations and alternative proofs [2, 31, 32]. In addition, as pointed out in [7], the equivalence $\mu^X \sim \mu^B$ follows from the general theory of Shepp [28], Hitsuda [14] and Kailath [17], which, moreover, gives a formula for the density $d\mu^X/d\mu^B$.

A complementary result, obtained in [1] (see Proposition 6.5) and [31], asserts that $\mu^X \sim \mu^{B^H}$ if, and only if, $H < \frac{1}{4}$. Both proofs are based on the spectral theory of processes with stationary increments and the corresponding density is given in [31] in terms of certain reproducing kernels. However, as the author points out, a more explicit expression might be hard to obtain using this method.

The main contribution of this paper is a novel approach to the analysis of mixtures such as (1.1), based on the filtering theory of Gaussian processes. The core of our method is a new canonical innovation representation of X . Our construction reveals a new interesting connection between the probabilistic properties of X and the structure of solutions of integral equations with weakly singular kernels.

In the context of mixed fBm (1.2), all the aforementioned properties can be deduced from this representation in a unified manner for all values of H , due to an apposite choice of the fundamental martingale. Moreover, it yields the missing density $d\mu^X/d\mu^{B^H}$ for $H < \frac{1}{4}$, as well as Girsanov-type formulas for the density of μ^X with respect to measures induced by stochastic shifts of X .

The precise formulation of our results is given in the next section. Section 3 contains auxiliary results, including a relevant theory of integral equations and frequently used formulas from stochastic calculus with respect to fBm. The proofs of the main theorems appear in Sections 5–7, and in Section 8 we show how our method applies to the Riemann–Liouville process.

2. The main results.

2.1. *A background.* Let us briefly recall the essential elements of the linear innovation theory [13, 27]. A Gaussian process admits an *innovation* representation if it can be generated by a linear causal transformation of N orthogonal processes with independent increments. Such a representation is called *canonical*,

if the transformation is also causally invertible. A well-known result of Hida [12] and Cramer [8] asserts that under mild regularity conditions any Gaussian process admits a canonical innovation representation.

Certain properties of canonical representations, collectively referred to as *type*, are the unique attributes of the process. This includes the number N of innovation components, called *multiplicity*, which can be finite or infinite. For example, stationary processes have unit multiplicity, that is, $N = 1$, and the corresponding innovation representation can be found by solving the spectral factorization problem. Processes which induce equivalent measures on their paths space, have the same innovation type (see [19]).

Let us now review in greater detail the results directly relevant to the mixed processes of the form (1.1). A general criteria for equivalence, obtained by Shepp in [28], implies that $\mu^X \sim \mu^B$ if and only if

$$(2.1) \quad \mathbb{E}G_t G_s = \int_0^s \int_0^t K(u, v) du dv,$$

with a kernel $K \in L^2([0, T]^2)$. The corresponding formula for the density $d\mu^X/d\mu^B$, given in [28], involves the Carleman–Fredholm determinant and resolvent kernel of the covariance operator, associated with K .

While Shepp’s result gives a complete answer to the question of equivalence, it does not immediately reveal the innovation structure of the process X . The missing link was found by Kailath in [17], who noticed the relevance of factorization theory of Fredholm operators in Hilbert spaces, developed by Gohberg and Krein at around the same time. Using the resolvent identity (7.10) from [10], Shepp’s density formula is rewritten in [17] in the form:

$$\frac{d\mu^X}{d\mu^B}(X) = \exp\left(-\int_0^T \varphi_t(X) dX_t - \frac{1}{2} \int_0^T \varphi_t^2(X) dt\right),$$

where $\varphi_t(X) = \int_0^t L(s, t) dX_s$ with $L \in L^2([0, T]^2)$ being the unique solution of the Wiener–Hopf integral equation

$$(2.2) \quad L(s, t) + \int_0^t L(r, t)K(r, s) dr = -K(s, t), \quad 0 \leq s \leq t \leq T.$$

It follows by Girsanov’s theorem that the process

$$(2.3) \quad \bar{B}_t = X_t + \int_0^t \int_0^s L(r, s) dX_r ds$$

is a Brownian motion. Moreover, it is shown in [17] that $\mathcal{F}_t^X = \mathcal{F}_t^{\bar{B}}$ and

$$(2.4) \quad X_t = \bar{B}_t - \int_0^t \int_0^s \ell(r, s) d\bar{B}_r ds,$$

where $\ell \in L^2([0, T]^2)$ solves the Volterra equation

$$(2.5) \quad \ell(s, t) + \int_s^t \ell(r, t)L(s, r) dr = L(s, t), \quad 0 \leq s \leq t \leq T.$$

In particular, it follows that X has unit multiplicity.

A different construction of canonical representation was given by Hitsuda in [14], where $\mu^X \sim \mu^B$ is shown to hold if and only if X can be represented in the form (2.4) with *some* Brownian motion \bar{B} and *some* Volterra kernel $\ell \in L^2([0, T]^2)$. The representation is proved to be canonical and the formula (2.3) is obtained. Though kernels ℓ and L are characterized in [14] in a different way, it can be shown that, in fact, both representations coincide. A detailed discussion about the links between all the aforementioned results can be found in [7].

2.2. *A new canonical representation.* The canonical representation (2.3) and (2.4) requires that the kernel K belongs to $L^2([0, T]^2)$. On the other hand, it is well known that X may have multiplicity greater than one if, for example, K is only integrable on $[0, T]^2$. This can be seen in a simple example.

EXAMPLE 2.1. Consider the process

$$X_t = B_t + \xi \int_0^t \frac{1}{\sqrt{|1-s|}} ds,$$

where $\xi \sim N(0, 1)$ is independent of B . It is easy to see that ξ can be recovered precisely from \mathcal{F}_t^X for all $t \geq 1$. Therefore, the filtration \mathcal{F}_t^X is discontinuous at $t = 1$, with $\mathcal{F}_{t-}^X \subsetneq \mathcal{F}_t^X = \mathcal{F}_t^B \vee \sigma\{\xi\}$ for all $t \geq 1$. By uniqueness of multiplicity, X cannot be innovated by a single Brownian motion on any interval $[0, T]$ with $T > 1$. In this case, equation (2.2) has no solution on $[0, T]$, even though $K \in L^1([0, T]^2)$. In fact, discontinuity of filtration is not essential and it is possible to construct X with arbitrary multiplicity and continuous natural filtration (Example 2 on page 266 in [22] and Example D on page 72 in [13]).

The following theorem shows that X has unit multiplicity under fairly general conditions, beyond the $L^2([0, T]^2)$ case, and gives the corresponding canonical representation.

THEOREM 2.2. Let X be given by (1.1), where G satisfies (2.1) with

$$(2.6) \quad |K(s, t)| \leq C(1 + |s - t|^{-\alpha}), \quad 0 \leq \alpha < 1,$$

for some constant C . Define $\phi_s = 1 - \int_0^s L(r, s) dr$, where $L(s, t)$ is the solution of equation (2.2). Then the process

$$(2.7) \quad \bar{B}_t = \mathbb{E} \left(\int_0^t \phi_s dB_s \mid \mathcal{F}_t^X \right),$$

is a Brownian motion, satisfying

$$(2.8) \quad \bar{B}_t = \int_0^t q(s, t) dX_s,$$

with $q(s, t)$ being the unique solution of the Wiener–Hopf equation:

$$(2.9) \quad q(s, t) + \int_0^t q(r, t)K(r, s) dr = \phi(s), \quad 0 \leq s, t \leq T.$$

The representation

$$(2.10) \quad X_t = \int_0^t \hat{q}(s, t) d\bar{B}_s$$

with $\hat{q}(s, t) = -\frac{\partial}{\partial s} \int_s^t q(r, s) dr$, is canonical, that is, $\mathcal{F}_t^X = \mathcal{F}_t^{\bar{B}}$.

REMARK 2.3. 1. As can be seen from the proof in Section 4, the assertion of this theorem remains true when $K \in L^2([0, T]^2)$, without assuming the particular structure of (2.6). Moreover, $\frac{\partial}{\partial t} q(s, t) = L(s, t)$ and, therefore,

$$\bar{B}_t = \int_0^t q(s, t) dX_s = X_t + \int_0^t \int_s^t L(s, r) dr dX_s.$$

When K is square integrable, so is the solution L of (2.2) and the order of integration in the right-hand side can be interchanged, recovering the formula (2.3). Moreover, in this case $\hat{q}(t, t) = q(t, t) = 1$ and

$$\hat{q}(s, t) = 1 - \int_s^t L(r, s) dr$$

and hence

$$(2.11) \quad \begin{aligned} X_t &= \int_0^t \hat{q}(s, t) d\bar{B}_s = \bar{B}_t - \int_0^t \int_s^t L(r, s) dr d\bar{B}_s \\ &= \bar{B}_t - \int_0^t \int_0^r L(r, s) d\bar{B}_s dr. \end{aligned}$$

Comparing this with (2.4) reveals a curious relation between the solutions of the Volterra equation (2.5) and the Wiener–Hopf equation (2.2): the solution of the former on the sub-diagonal coincides with the solution of the latter on the *super-diagonal*, that is, $\ell(s, t) = L(t, s)$ for $s \leq t$. In other words, both the direct and the inverse transformation between X and \bar{B} can be expressed in terms of the single Wiener–Hopf equation (2.2), whose solution is extended to the whole rectangle $[0, T]^2$.

2. A two stage procedure can be used to construct a canonical representation for the process

$$X_t = B_t + G_t + G_t^\dagger,$$

where G^\dagger is an independent centered Gaussian process, satisfying (2.1) with $K^\dagger \in L^2([0, T]^2)$. We can first generate an intermediate process \bar{X} by applying (2.8):

$$\bar{X}_t = \int_0^t q(s, t) dX_s = \bar{B}_t + \int_0^t q(s, t) dG_s^\dagger.$$

It can readily be seen that the process, defined by the last term, satisfies (2.1) with a square integrable kernel and, therefore, can be represented canonically in the standard way.

3. When $K \in L^2([0, T]^2)$, it follows from the results of Shepp and Hitsuda, that the kernel $\ell(s, t)$ also solves the Riccati–Volterra equation

$$(2.12) \quad \ell(s, t) = K(s, t) - \int_0^{t \wedge s} \ell(s, r)\ell(t, r) dr.$$

If $G_t = \int_0^t \gamma_s ds$ with a Gaussian process γ , then the innovating Brownian motion reduces to

$$(2.13) \quad \bar{B}_t = X_t - \int_0^t \pi_s(\gamma) ds,$$

with $\pi_t(\gamma) = \mathbb{E}(\gamma_t | \mathcal{F}_t^X) = \int_0^t \ell(s, t) d\bar{B}_s$. In this form, due to Kailath [16], the canonical representation plays an important role in control and filtering theory, which goes beyond the linear setting (see [11]). If furthermore, γ is the Gauss–Markov process, (2.12) reduces to the familiar Riccati equation from the Kalman–Bucy filter.

Note that in our approach conditioning on \mathcal{F}_t^X is used in an essentially different way than in (2.13): in the lack of derivative of G_t , the innovation Brownian motion is produced by projecting a specially designed martingale (2.7). Somewhat unexpectedly, equality of filtrations can be established in this case, using only the basic theory of integral equations, which needs nothing more than weak singularity of K .

4. Condition (2.6) is borrowed from the classical theory of integral equations (see Section 3.1 below). It is satisfied by some interesting processes, related to the fBm. One example is *bifractional* Brownian motion, introduced in [15], which is a centered Gaussian process G with covariance function

$$\mathbb{E}G_t G_s = \frac{1}{2^K} ((t^{2H} + s^{2H})^K - |t - s|^{2HK}),$$

where $H \in (0, 1)$ and $K \in (0, 1]$. The representation (2.1) holds for $HK > \frac{1}{2}$ with

$$(2.14) \quad \begin{aligned} K(s, t) &= C_1(H, K)(t^{2H} + s^{2H})^{K-2} s^{2H-1} t^{2H-1} \\ &+ C_2(H, K)|t - s|^{2HK-2}, \quad t \neq s, \end{aligned}$$

where $C_1(H, K) = \frac{K(K-1)(2H)^2}{2^K} < 0$ and $C_2(H, K) = \frac{2HK(2HK-1)}{2^K} > 0$. Since $HK > \frac{1}{2}$ implies $H > \frac{1}{2}$ and $2HK - 2 \in (-1, 0)$,

$$\begin{aligned} (t^{2H} + s^{2H})^{K-2} s^{2H-1} t^{2H-1} &\leq (s \wedge t)^{2H-1} (s \vee t)^{2H(K-1)-1} \\ &= \left(\frac{s \wedge t}{s \vee t} \right)^{2H-1} (s \vee t)^{2HK-2} \leq |s - t|^{2HK-2} \end{aligned}$$

and thus the kernel (2.14) satisfies (2.6) with $\alpha := 2HK - 2$.

Other examples are the *sub-fractional* Brownian motion from [4] and the Riemann–Liouville process, which can be fitted in by a similar calculation.

Let us now return to the mixed fBm (1.2). For $H > \frac{1}{2}$, B^H satisfies (2.1) with

$$(2.15) \quad K_H(s, t) = \frac{\partial^2}{\partial s \partial t} \mathbb{E} B_t^H B_s^H = H(2H - 1)|s - t|^{2H-2}.$$

For $H > \frac{3}{4}$, this kernel is square integrable and, therefore, the mixed fBm $X = B + B^H$ can be represented canonically by (2.3)–(2.4). For $H \in (\frac{1}{2}, \frac{3}{4}]$, K_H does not belong to $L^2([0, T]^2)$, but still satisfies the assumptions of Theorem 2.2. Therefore, its canonical representation is given by (2.8) and (2.10) for all values of H in $(\frac{1}{2}, 1]$.

For $H < \frac{1}{2}$, the kernel K_H has stronger singularity than admitted by (2.6) and the covariance of B^H fails to satisfy (2.1). Nevertheless, quite remarkably (see Example 2.6), the mixed fBm can still be innovated canonically by a different martingale [cf. (2.7)].

THEOREM 2.4. (i) *Let X be defined by (1.2). The martingale $M_t = \mathbb{E}(B_t | \mathcal{F}_t^X)$ admits the representation*

$$(2.16) \quad M_t = \int_0^t g(s, t) dX_s, \quad \langle M \rangle_t = \int_0^t g(s, t) ds,$$

where $g(s, t)$ is the unique solution of the integro-differential equation

$$(2.17) \quad g(s, t) + \frac{\partial}{\partial s} \int_0^t g(r, t) \frac{\partial}{\partial r} R(r, s) dr = 1, \quad 0 < s, t \leq T,$$

with $R(s, t)$ defined in (1.3).

(ii) *The quadratic variation of M is given by*

$$(2.18) \quad \frac{d}{dt} \langle M \rangle_t = g^2(t, t) + \frac{2 - 2H}{\lambda_H} (t^{1/2-H} (\Psi g)(t, t))^2 > 0, \quad t \in [0, T],$$

where λ_H is the constant defined in (3.13) and Ψ is the operator, defined in (3.14) below. The innovation representation

$$(2.19) \quad X_t = \int_0^t \hat{g}(s, t) dM_s,$$

with

$$(2.20) \quad \hat{g}(s, t) = 1 - \frac{d}{d\langle M \rangle_s} \int_0^t g(r, s) dr,$$

is canonical, that is, $\mathcal{F}_t^X = \mathcal{F}_t^M$ for all $t \in [0, T]$.

REMARK 2.5. 1. For $H > \frac{1}{2}$ the kernel K_H in (2.15) is integrable and, therefore, the derivative in (2.17) can be interchanged with integration, so that it takes the form of the Wiener–Hopf integral equation [cf. (2.9)]

$$(2.21) \quad g(s, t) + \int_0^t g(r, t) K_H(r, s) dr = 1, \quad 0 \leq s, t \leq T.$$

In this case, the second term in (2.18) vanishes [see (3.14) below] and (2.20) becomes

$$(2.22) \quad \hat{g}(s, t) = 1 - \frac{1}{g(s, s)} \int_0^t \frac{\partial/\partial s g(r, s)}{g(s, s)} dr = 1 - \frac{1}{g(s, s)} \int_0^t L(r, s) dr.$$

The last equality holds, since $\frac{\partial/\partial s g(r, s)}{g(s, s)}$ turns to be the solution of equation (2.2). Differentiating (2.22) yields $\frac{\partial}{\partial t} \hat{g}(s, t) = -\frac{L(t, s)}{g(s, s)}$, and we obtain

$$\begin{aligned} X_t &= \int_0^t \hat{g}(s, t) dM_s = \int_0^t \hat{g}(s, s) dM_s + \int_0^t \int_s^t \frac{\partial}{\partial \tau} \hat{g}(s, \tau) d\tau dM_s \\ &= \int_0^t \hat{g}(s, s) dM_s - \int_0^t \int_s^t \frac{L(\tau, s)}{g(s, s)} d\tau dM_s. \end{aligned}$$

A calculation shows that $\hat{g}(s, s) = 1/g(s, s)$ and, therefore,

$$X_t = \bar{B}_t - \int_0^t \int_s^t L(\tau, s) d\tau d\bar{B}_s,$$

where $\bar{B}_t = \int_0^t \frac{1}{g(s, s)} dM_s$, is a Brownian motion. Similar calculations give [cf. (2.11)]

$$\bar{B}_t = X_t + \int_0^t \int_s^t L(s, r) dr dX_s,$$

and hence we are back to the innovation representation from Theorem 2.2.

2. Natural, as it may seem, the choice of the martingale $M_t = \mathbb{E}(B_t | \mathcal{F}_t^X)$ is not at all obvious and, in fact, it fails to innovate X in general, as demonstrated in the following example.

EXAMPLE 2.6. Let $m(t) = 6^{1/3} \wedge t$ and $\xi(t) = \eta m(t)$, where the random variable $\eta \sim N(0, 1)$ is independent of B . The martingale $M_t = \mathbb{E}(B_t | \mathcal{F}_t^X)$ still satisfies (2.16) where $g(s, t)$ solves the Wiener–Hopf equation (2.21), with $K_H(s, t)$ replaced by $K(s, t) = m(s)m(t)$. Its quadratic variation is $\langle M \rangle_t = \int_0^t g^2(s, s) ds$, as in Theorem 2.4 for $H > \frac{1}{2}$.

For the degenerate kernel $K(s, t) = m(s)m(t)$, equation (2.21) can be solved explicitly:

$$g(s, t) = 1 - m(s) \frac{\int_0^t m(r) dr}{1 + \int_0^t m^2(r) dr}, \quad 0 \leq s \leq t \leq T,$$

and an easy calculation shows that $g(t, t) = 0$ for all $t \geq 6^{1/3}$. On the other hand, since $K \in L^2([0, T]^2)$ the representation (2.3)–(2.4) is canonical and therefore X cannot be innovated by M , that is, $\mathcal{F}_t^M \subsetneq \mathcal{F}_t^X$ for $t \geq 6^{1/3}$. Incidentally, $\{M_t, t \in [0, T]\}$ is a sufficient statistic in the problem of estimating $\theta \in \mathbb{R}$ from the observations of $\{\theta t + X_t, t \in [0, T]\}$.

The equality $\mathcal{F}_t^X = \mathcal{F}_t^M$ in Theorem 2.4 is closely related to Krein’s method of solving integral equations with difference kernels (see Theorem 3.3 below). Krein showed that the solution of the Wiener–Hopf equation with a unit forcing function such as (2.21) does not vanish on the diagonal and that it can be used to express solutions to this equation with an arbitrary right-hand side. Remarkably, this property remains true for kernels with a somewhat more general structure (Lemma 3.6), arising in the case of mixed fBm. Thus, Theorem 2.4 gives a probabilistic interpretation of the nondegeneracy of Krein’s solutions in terms of the equality of filtrations.

3. For $H < \frac{1}{2}$, the kernel K_H in (2.15) has a stronger singularity and, consequently, the derivative and integration in equation (2.17) are no longer interchangeable and the integral equation (2.21) makes no sense. Nevertheless, (2.17) can still be solved by reduction to a different weakly singular integral equation, using tools from fractional calculus. Moreover, it turns out that, while the first term in (2.18) vanishes in this case, the second term remains strictly positive for all $t \in [0, T]$ and, consequently, the martingale M generates the same filtration as X . Martingales with such property are sometimes referred to as *fundamental* in fBm literature (see, e.g., [24]), playing the central role in related statistical problems.

Construction of a canonical representation for more general mixed Gaussian processes of the form (1.1), beyond the condition (2.6), seems to be quite a delicate problem, especially in view of Example 2.1. Our approach remains applicable to other processes of “fractional” type. One example is the Riemann–Liouville process (see [23]):

$$(2.23) \quad V_t^H = 2H \int_0^t (t - s)^{H-1/2} dV_s,$$

where V is a Brownian motion. While many properties of V^H are similar to those of B^H , there are some essential differences, at least from the standpoint of the problems under consideration.

First, the increments of V^H are not stationary, and hence the equivalence of $X = B + V^H$ and V^H for $H < \frac{1}{4}$, cannot be deduced by the spectral technique, used in [1] and [31]. Second, for $H < \frac{1}{2}$ the first partial derivative $\partial/\partial s \mathbb{E}V_t^H V_s^H$ already has a nonintegrable singularity of the diagonal and consequently, equation (2.17) makes sense only if the inner derivative is moved to the solution itself. Further details are referred to Section 8.

2.3. *Equivalence relations and density formulas.*

2.3.1. *The mixed fBm.* As discussed in the [Introduction](#), our interest in mixed fBm was motivated by the equivalence relations, discovered in [5] and [31]. We will show how these results can be derived from the canonical representation of Theorem 2.4 and, in addition, complement them with a new formula for the Radon–Nikodym density in the case $H < \frac{1}{4}$.

THEOREM 2.7. (i) *The process X defined in (1.2) is a semimartingale in its own filtration if and only if $H \in \{\frac{1}{2}\} \cup (\frac{3}{4}, 1]$. For $H \in (\frac{3}{4}, 1]$, X is a diffusion type process*

$$X_t = \bar{B}_t - \int_0^t \varphi_s(X) ds, \quad t \in [0, T],$$

where \bar{B} is a Brownian motion with $\mathcal{F}_t^{\bar{B}} = \mathcal{F}_t^X$, $\varphi_t(X) = \int_0^t L(s, t) dX_s$ and

$$L(s, t) := \frac{\partial}{\partial t} g(s, t) / \sqrt{\frac{d}{dt} \langle M \rangle_t}.$$

The measures μ^X and μ^B are equivalent, if and only if $H \in (\frac{3}{4}, 1]$, and

$$\frac{d\mu^X}{d\mu^B}(X) = \exp \left\{ - \int_0^T \varphi_t(X) dX_t - \frac{1}{2} \int_0^T \varphi_t^2(X) dt \right\}.$$

(ii) For $H \in (0, \frac{1}{4})$, X is a fractional diffusion type process

$$(2.24) \quad X_t = \bar{B}_t^H - \int_0^t \rho(s, t) \varphi_s(X) ds,$$

where \bar{B}^H is fBm with $\mathcal{F}_t^{\bar{B}^H} = \mathcal{F}_t^X$, $\varphi_t(X) = \int_0^t L(s, t) dX_s$ and

$$L(s, t) := \frac{\partial}{\partial t} g(s, t) / \sqrt{\frac{d}{dt} \langle M \rangle_t} - \frac{\partial}{\partial t} \tilde{\rho}(s, t),$$

with the kernels $\rho(s, t)$ and $\tilde{\rho}(s, t)$ are defined in (3.20) below. The measures μ^X and μ^{B^H} are equivalent if and only if $H \in (0, \frac{1}{4})$ and

$$(2.25) \quad \frac{d\mu^X}{d\mu^{B^H}}(X) = \exp \left\{ - \int_0^T \varphi_t(X) d\tilde{X}_t - \frac{1}{2} \int_0^T \varphi_t^2(X) dt \right\},$$

where $\tilde{X}_t = \int_0^t \tilde{\rho}(s, t) dX_s$.

REMARK 2.8. 1. The density formulas in both cases are given in terms of solutions of certain integral equations, rather than reproducing kernels as in [31].

Work in progress indicates that in some statistical applications, such as estimating $H > \frac{3}{4}$ from the sample $X^T = \{X_t, t \in [0, T]\}$, integral equations are a more manageable alternative.

2. A similar result holds for the mixed Riemann–Liouville process $X = B + V^H$ with V^H defined in (2.23). The precise details appear in Section 8 below.

2.3.2. *Mixed processes with drift.* The canonical representation also yields Girsanov’s-type formulas, useful in the likelihood based statistical inference. Consider the process

$$(2.26) \quad Y_t = \int_0^t \xi_s ds + X_t, \quad t \in [0, T],$$

where X is defined in (1.2) and $\xi = (\xi_t)$ is a process with continuous paths, satisfying $\mathbb{E} \int_0^T |\xi_t| dt < \infty$. Assume that ξ is adapted to a filtration $\mathcal{G} = (\mathcal{G}_t)$, with respect to which M , introduced in Theorem 2.4, is a martingale.

The choice of the filtration \mathcal{G} can vary in different applications. For example, in filtering problems ξ plays the role of unobserved state process and X is interpreted as the observation noise. If the state process and the noise are independent, then the assumption holds with $\mathcal{G}_t := \mathcal{F}_t^\xi \vee \mathcal{F}_t^X$.

If ξ_t is a function of Y_t , then (2.26) becomes a stochastic differential equation with respect to the mixed fBm X . In this case, ξ is adapted to \mathcal{F}_t^X itself, and hence the natural choice is $\mathcal{G}_t := \mathcal{F}_t^X$. For example, $\xi_t := aY_t$ with $a \in \mathbb{R}$ corresponds to the mixed fractional Ornstein–Uhlenbeck process

$$Y_t = a \int_0^t Y_s ds + X_t, \quad t \in [0, T],$$

with the drift parameter a .

Theorem 2.4 yields a formula for the density of μ^Y with respect to μ^X :

COROLLARY 2.9. *The process Y admits the representation*

$$(2.27) \quad Y_t = \int_0^t \hat{g}(s, t) dZ_s$$

with $\hat{g}(s, t)$ defined in (2.20), where

$$Z_t = \int_0^t g(s, t) dY_s, \quad t \in [0, T]$$

is a \mathcal{G} -semimartingale with decomposition

$$(2.28) \quad Z_t = M_t + \int_0^t \Xi(s) d\langle M \rangle_s$$

and

$$(2.29) \quad \Xi(t) = \frac{d}{d\langle M \rangle_t} \int_0^t g(s, t) \xi_s ds.$$

In particular, $\mathcal{F}_t^Y = \mathcal{F}_t^Z$, for all $t \in [0, T]$ and, if

$$\mathbb{E} \exp \left\{ - \int_0^T \Xi(t) dM_t - \frac{1}{2} \int_0^T \Xi^2(t) d\langle M \rangle_t \right\} = 1,$$

then $\mu^X \sim \mu^Y$ and

$$(2.30) \quad \frac{d\mu^Y}{d\mu^X}(Y) = \exp \left\{ \int_0^T \widehat{\Xi}(t) dZ_t - \frac{1}{2} \int_0^T \widehat{\Xi}^2(t) d\langle M \rangle_t \right\},$$

where $\widehat{\Xi}(t) = \mathbb{E}(\Xi(t) | \mathcal{F}_t^Y)$.

In the setting of Theorem 2.2, we have the following analog.

COROLLARY 2.10. *Let Y be the process in (2.26), where X is defined in (1.1) and satisfies the assumptions of Theorem 2.2. Then Y admits the representation*

$$Y_t = \int_0^t \hat{q}(s, t) dZ_s,$$

where the process $Z_t = \int_0^t q(s, t) dY_s$ satisfies

$$Z_t = \bar{B}_t + \int_0^t \Xi(s) ds,$$

with $\Xi(s) = \xi_s + \int_0^s L(u, s) \xi_u du$. In particular, $\mathcal{F}_t^Y = \mathcal{F}_t^Z$, for all $t \in [0, T]$ and, if

$$\mathbb{E} \exp \left\{ - \int_0^T \Xi(t) d\bar{B}_t - \frac{1}{2} \int_0^T \Xi^2(t) dt \right\} = 1,$$

then $\mu^X \sim \mu^Y$ and

$$\frac{d\mu^Y}{d\mu^X}(Y) = \exp \left\{ \int_0^T \widehat{\Xi}(t) dZ_t - \frac{1}{2} \int_0^T \widehat{\Xi}^2(t) dt \right\},$$

where $\widehat{\Xi}(t) = \mathbb{E}(\Xi(t) | \mathcal{F}_t^Y)$.

3. Notation and auxiliary results.

3.1. *Weakly singular integral equations.* In this section, we review terminology and basic theory of integral equations, relevant to our problem. We will be concerned with the Wiener–Hopf equations on the finite interval $[0, T]$, $T < \infty$

$$(3.1) \quad u(s, t) + \int_0^t u(r, t) K(r, s) dr = f(s, t), \quad 0 < s, t \leq T,$$

where the kernel $K : [0, T]^2 \mapsto \mathbb{R}$ and the forcing function $f : [0, T]^2 \mapsto \mathbb{R}$ are given.

Note that the values of $u(s, t)$ on the sub-diagonal $\{0 < s < t \leq T\}$ determine $u(s, t)$ on the super-diagonal. Hence, the problem of solving (3.1) reduces to solving it on the sub-diagonal $\{0 < s < t\}$ for all $t \in [0, T]$. In this regard, (3.1) can be interpreted as an evolution equation in the second (forward) variable. Let us stress, however, that we will consider the solution u as a function on $[0, T]^2$.

For a fixed $t \in (0, T]$, the restriction of (3.1) to the sub-diagonal:

$$(3.2) \quad u(s, t) + \int_0^t u(r, t)K(r, s) dr = f(s, t), \quad 0 < s < t,$$

is the Fredholm equation of the second kind, whose solvability is very well-known under various conditions (see, e.g., [20]).

In this paper, we will consider weakly singular symmetric nonnegative definite kernels satisfying (2.6). Iterates of K are denoted by $K^{(m)}$:

$$K^{(1)}(s, t) = K(s, t),$$

$$K^{(m)}(s, t) = \int_0^T K^{(m-1)}(s, r)K(r, t) dr, \quad m = 2, 3, \dots$$

Recall that for $0 < \alpha, \beta < 1$

$$\int_0^T |s - r|^{-\alpha} |r - t|^{-\beta} dr \leq \begin{cases} C_1 |s - t|^{1-\alpha-\beta}, & \alpha + \beta > 1, \\ C_2 \log \frac{1}{|s - t|} + C_3, & \alpha + \beta = 1, \\ C_4, & \alpha + \beta < 1, \end{cases}$$

where C_i 's are constants. Therefore, singularity improves with iterations and eventually disappears.

For weakly singular kernels, equation (3.2) is uniquely solvable in $L^1([0, t])$, provided $f(\cdot, t) \in L^1([0, t])$. If $f(\cdot, t)$ is bounded on $[0, t]$, so is the solution $u(\cdot, t)$. If the kernel K is continuous outside the diagonal and $f(\cdot, t) \in C([0, t])$, the solution $u(\cdot, t)$ is continuous on $[0, t]$, but typically its derivative has discontinuities at the end points of the interval. The comprehensive accounts of these results can be found in [20, 30] and [29].

The following lemma shows that the solution of (3.1) has at most the same type of singularity as the forcing function.

LEMMA 3.1. *Assume that $|f(s, t)| \leq c|s - t|^{-\beta}$ with constants c and $\beta \in [0, 1)$, then the solution of (3.2) satisfies*

$$|u(s, t)| \leq C|s - t|^{-\beta}, \quad s, t \in [0, T]$$

for some constant C .

PROOF. Let m_0 be an integer, such that $\tilde{f}(s, t) = \int_0^t K^{(m_0)}(s, r) f(r, t) dr$ is bounded. The function

$$\tilde{u}(s, t) = u(s, t) - f(s, t) - \sum_{m=1}^{m_0-1} \int_0^t (-1)^m K^{(m)}(s, r) f(r, t) dr,$$

solves the equation

$$(3.3) \quad \tilde{u}(s, t) + \int_0^t \tilde{u}(r, t)K(r, s) dr = (-1)^{m_0} \tilde{f}(s, t), \quad 0 \leq s \leq t.$$

Multiplying this equation by $\tilde{u}(s, t)$, integrating and using positive definiteness of the kernel K , we get

$$\int_0^t \tilde{u}^2(s, t) ds \leq \int_0^t \tilde{u}(s, t) \tilde{f}(s, t) ds,$$

and, by the Cauchy–Schwarz inequality,

$$(3.4) \quad \int_0^t \tilde{u}^2(s, t) ds \leq \int_0^t \tilde{f}^2(s, t) ds \leq \|\tilde{f}\|_\infty^2.$$

Let n_0 be the integer such that $K^{(n_0)}$ is bounded, then iterating (3.3) n_0 times gives

$$\begin{aligned} \tilde{u}(s, t) &= \tilde{f}(s, t) + \sum_{m=1}^{n_0-1} (-1)^m \int_0^t K^{(m)}(r, s) \tilde{f}(r, t) dr \\ &\quad + (-1)^{n_0} \int_0^t K^{(n_0)}(r, s) \tilde{u}(r, t) dr. \end{aligned}$$

The first two terms in the right-hand side are bounded, since \tilde{f} is bounded and

$$\sup_{s \leq T} \int_0^T K^{(m)}(s, t) dt < \infty \quad \forall m \geq 1.$$

The last term is bounded due to (3.4). It follows that $\tilde{u}(s, t)$ is bounded and therefore $|u(s, t)| \leq C_1|s - t|^{-\beta}$ for all $s < t \leq T$ with a constant C_1 . As discussed above, the solution of (3.1) on the super-diagonal is determined by the solution on the sub-diagonal and, therefore, the same bound holds for $t < s \leq T$ possibly with a different constant. \square

The following lemma shows that certain integrals of the solution are determined by its values on the diagonal.

LEMMA 3.2. *Assume $f \in C([0, T])$ does not depend on t and the partial derivative $\dot{u}(s, t) := \frac{\partial}{\partial t}u(s, t)$ exists and $\dot{u}(\cdot, t) \in L^1([0, t])$, then*

$$\int_0^t u(s, t) f(s) ds = \int_0^t u^2(s, s) ds.$$

PROOF. Multiplying (3.2) by $u(s, t)$ and integrating, we get

$$(3.5) \quad \int_0^t u^2(s, t) ds + \int_0^t \int_0^t u(r, t)u(s, t)K(r, s) dr ds = \int_0^t u(s, t) f(s) ds$$

and, consequently,

$$\begin{aligned} \frac{d}{dt} \int_0^t u(s, t) f(s) ds &= u^2(t, t) + 2u(t, t) \int_0^t u(r, t) K(r, t) dr \\ &\quad + 2 \int_0^t \dot{u}(r, t) \left(u(r, t) + \int_0^t u(s, t) K(r, s) ds \right) dr \\ &= u^2(t, t) + 2u(t, t)(f(t) - u(t, t)) + 2 \int_0^t \dot{u}(r, t) f(r) dr \\ &= -u^2(t, t) + 2 \frac{d}{dt} \int_0^t u(r, t) f(r) dr, \end{aligned}$$

which gives the claimed identity. \square

3.1.1. *Krein’s method.* For kernels with certain special structure, the solution of (3.2) with an arbitrary forcing can be expressed in terms of its solution with the unit forcing. The following theorem is an adaptation of Theorem 8.1, Section 8, Chapter IV in [10].

THEOREM 3.3. *Assume that the equation*

$$(3.6) \quad g(s, t) + \int_0^t g(r, t) K(r, s) dr = 1, \quad 0 \leq s \leq t \leq T,$$

has a unique continuous solution and $g(t, t) \neq 0, t \in [0, T]$. Then the solution of (3.2) with an arbitrary $f(\cdot, t) \in L^1([0, T])$ is given by

$$(3.7) \quad u(s, t) = g(s, t)F(t, t) - \int_s^t g(s, u) \frac{\partial}{\partial u} F(u, t) du,$$

where

$$(3.8) \quad F(\tau, t) = \frac{1}{g^2(\tau, \tau)} \frac{\partial}{\partial \tau} \int_0^\tau g(s, \tau) f(s, t) ds.$$

REMARK 3.4. The result in [10] requires that the forcing function f does not depend on t and is continuous. The extension to integrable f can be carried out through approximation of f by continuous functions in $L^1([0, t])$ in the usual way. The formulas (3.7) and (3.8), where f is allowed to depend on t , are obtained by treating t in the right-hand side of (3.2) as a fixed parameter, applying the original formula (8.7) in [10] and then equating t to the integration limit. We omit lengthy, but otherwise routine details.

The following class of kernels will be particularly useful for our purposes.

LEMMA 3.5. *Assume that f does not depend on $t, f \in C_1((0, T)) \cap C([0, T])$ and the kernel K has the form*

$$(3.9) \quad K(s, t) = \chi(s/t)|s - t|^{-\alpha}, \quad 0 \leq \alpha < 1$$

with $\chi \in C([0, \infty))$. Then the solution $u(s, t)$ of (3.1) satisfies the following properties:

- (i) $u(s, t)$ is continuously differentiable in $t \in (0, T]$ for any $s > 0, s \neq t$,
- (ii) the derivative $\dot{u}(s, t) := \frac{\partial}{\partial t}u(s, t)$ solves the equation

$$(3.10) \quad \dot{u}(s, t) + \int_0^t \dot{u}(r, t)K(r, s) dr = -u(t, t)K(s, t), \quad 0 < s < t \leq T$$

and satisfies the bound

$$|\dot{u}(s, t)| \leq C|s - t|^{-\alpha}, \quad 0 \leq s, t \leq T.$$

- (iii) $\dot{u}(\cdot, t) \in L^2([0, t])$ for $\alpha < \frac{1}{2}$.

PROOF. (i) The function $u_t(x) := u(xt, t), x \in [0, 1], t > 0$ satisfies the integral equation

$$u_t(x) + t^{1-\alpha} \int_0^1 u_t(y)K(x, y) dv = f(xt), \quad u \in [0, 1].$$

As mentioned above, the unique continuous solution exists and in the terminology of [26], any point $\lambda := t^{1-\alpha}$ is regular. The operator associated with the weakly singular kernel K maps $L^2([0, 1])$ into itself (see, e.g., Theorem 9.5.1 in [9]). It follows from, for example, the theorem on page 154 in [26], that the resolvent is analytic in λ , and hence $u_t(x)$ is continuously differentiable at $t \in (0, T]$ for all $x \in [0, 1]$. Differentiability of $u_t(x)$ with respect to $x \in (0, 1)$ for continuous $\chi(\cdot)$ can be shown by the method from [30], using the particular form of the kernel K . Therefore, the function $u(s, t) = u_t(s/t)$ is continuously differentiable at $t > 0$ for any $s \in (0, t)$ and, therefore, for $s \in (t, T]$ as well.

(ii) Equation (3.10) is obtained by taking the derivative of both sides of (3.1) and the bound follows from Lemma 3.1.

(iii) Obvious in view of (ii). \square

The crucial assumption in Theorem 3.3, inherited from Theorem 8.1 in [10], is that the solution of (3.6) does not vanish in the diagonal. This property is guaranteed for symmetric difference kernels of the form $K(s, t) = \kappa(s - t)$ with $\kappa \in L^1([-T, T])$ (Theorem 8.2 in [10]). Obviously such nondegeneracy cannot be expected to hold in general (see Example 2.6). The following lemma extends applicability of Krein’s method to kernels, introduced in Lemma 3.5:

LEMMA 3.6. *The assertion of Theorem 3.3 is true for kernels of the form (3.9).*

PROOF. We will argue that $g(t, t) \neq 0$ by contradiction. Suppose $g(t, t) = 0$ for some $t > 0$. Changing the integration variable, equation (3.6) can be rewritten as

$$g(s, t) + s^{1-\alpha} \int_0^{t/s} g(xs, t)|1 - x|^{-\alpha} \chi(x) dx = 1.$$

Taking the derivative of both sides with respect to s and multiplying by s , we get

$$\begin{aligned} s g'(s, t) + s^{2-\alpha} \int_0^{t/s} x g'(xs, t) |1-x|^{-\alpha} \chi(x) dx \\ = -(1-\alpha) s^{1-\alpha} \int_0^{t/s} g(xs, t) |1-x|^{-\alpha} \chi(x) dx, \end{aligned}$$

where $g'(s, t) = \frac{\partial}{\partial s} g(s, t)$ and we used $g(t, t) = 0$. Now change the variables back to get

$$s g'(s, t) + \int_0^t r g'(r, t) K(r, s) dr = -(1-\alpha)(1-g(s, t)).$$

Multiplying by $g(s, t)$ and integrating gives

$$\begin{aligned} -(1-\alpha) \int_0^t g(s, t) ds + (1-\alpha) \int_0^t g^2(s, t) ds \\ = \int_0^t s g'(s, t) g(s, t) ds + \int_0^t \int_0^t r g'(r, t) g(s, t) K(r, s) dr ds \\ = \int_0^t r g'(r, t) dr = - \int_0^t g(r, t) dr \end{aligned}$$

and, after a rearrangement,

$$(1-\alpha) \int_0^t g^2(s, t) ds + \alpha \int_0^t g(s, t) ds = 0.$$

By Lemma 3.2, it follows that

$$(1-\alpha) \int_0^t g^2(s, t) ds + \alpha \int_0^t g^2(s, s) ds = 0.$$

This implies that $g(s, t) = 0$ for a.e. $s \in [0, t]$, which contradicts (3.6). \square

COROLLARY 3.7. *For the kernel (3.9) and $f(s) = s^\beta$ with $\beta \geq 0$, the solution of (3.1) does not vanish on the diagonal, that is, $u(t, t) \neq 0$ for all $t \in (0, T]$.*

PROOF. The function $\tilde{u}(s, t) := s^{-\beta} u(s, t)$ solves the equation

$$\tilde{u}(s, t) + \int_0^t \tilde{u}(r, t) (r/s)^\beta K(r, s) dr = 1.$$

The claim follows, since the kernel $(r/s)^\beta K(r, s)$ satisfies the assumption of Lemma 3.6. \square

Krein’s method reveals yet another useful formula.

COROLLARY 3.8. *The function $L(s, t) = \frac{\dot{g}(s,t)}{g(t,t)}$ satisfies the equation (2.2) and*

$$L(s, t) - L(t, s) = \int_s^t L(s, \tau)L(t, \tau) d\tau, \quad s < t.$$

PROOF. Equation (2.2) readily follows from Lemma 3.5. By (3.8),

$$\begin{aligned} F(\tau, t) &= -\frac{1}{g^2(\tau, \tau)} \frac{\partial}{\partial \tau} \int_0^\tau g(r, \tau)K(r, t) dr \\ &= -\frac{1}{g^2(\tau, \tau)} \frac{\partial}{\partial \tau} (1 - g(t, \tau)) = \frac{\dot{g}(t, \tau)}{g^2(\tau, \tau)}, \end{aligned}$$

and, integrating by parts in (3.7), we get

$$\begin{aligned} L(s, t) &= \frac{\dot{g}(t, s)}{g(s, s)} + \int_s^t \frac{\dot{g}(t, \tau)}{g^2(\tau, \tau)} \dot{g}(s, \tau) d\tau \\ &= L(t, s) + \int_s^t L(s, \tau)L(t, \tau) d\tau. \end{aligned} \quad \square$$

3.2. *Stochastic integral representation.* Consider the process X defined in (1.1) and let η be a random variable, such that the pair (η, X_t) forms a Gaussian process. Then $\mathbb{E}(\eta|\mathcal{F}_t^X)$ belongs to the closure \mathcal{H}_t^X of the linear combinations of the increments of X in $L^2(\Omega, \mathbb{P})$. What is less apparent is that this conditional expectation can be expressed as a stochastic integral with respect to X . For example, it is not hard to see that such a representation is impossible in Example 2.1.

We will assume that the stochastic integral with respect to the process G is defined on a scalar product space Λ_t of deterministic functions $f : [0, t] \mapsto \mathbb{R}$, in which simple (piecewise constant) functions are dense. For $f \in \Lambda_t$

$$\int_0^t f(s) dG_s = \lim_{n \rightarrow \infty} \int_0^t f_n(s) dG_s \quad \text{in } L^2(\Omega, \mathbb{P}),$$

whenever $f_n \rightarrow f$ in Λ_t . Also we will assume that

$$(3.11) \quad \mathbb{E} \int_0^t f(s) dG_s \int_0^t h(s) dG_s = \langle f, g \rangle_{\Lambda_t},$$

and either $L^2([0, t]) \subseteq \Lambda_t$ or Λ_t is complete.

All these assumptions hold for a variety of familiar processes, including fBm. Let us stress, however, that they do not exclude the possibility of \mathcal{H}_t^G being strictly larger than the image of Λ_t in $L^2(\Omega, \mathbb{P})$ under the stochastic integral. For example, not all random variables in $\mathcal{H}_t^{B^H}$ can be expressed as stochastic integrals with respect to B^H (see [25]).

LEMMA 3.9. *Under the above assumptions,*

$$\mathbb{E}(\eta|\mathcal{F}_t^X) = \mathbb{E}\eta + \int_0^t h(s, t) dX_s,$$

with a unique function $h(\cdot, t) \in L^2([0, t]) \cap \Lambda_t$.

PROOF. Following the arguments of the proof of Lemma 10.1 in [21], let $t_i = ti/2^n, i = 0, \dots, 2^n$ and $\mathcal{F}_{t,n}^X = \sigma\{X_{t_i} - X_{t_{i-1}}, i = 1, \dots, 2^n\}$. Then $\mathcal{F}_{t,n}^X \nearrow \mathcal{F}_t^X$ and by the martingale convergence

$$(3.12) \quad \lim_n \mathbb{E}(\eta|\mathcal{F}_{t,n}^X) = \mathbb{E}(\eta|\mathcal{F}_t^X) \quad \text{in } L^2(\Omega, \mathbb{P}).$$

By the normal correlation theorem,

$$\mathbb{E}(\eta|\mathcal{F}_{t,n}^X) = \mathbb{E}\eta + \sum_{i=1}^{2^n} h_{i-1}^n (X_{t_i} - X_{t_{i-1}}),$$

with constants $h_{i-1}^n, i = 1, \dots, 2^n$. Define

$$h_n(s, t) := \sum_{i=1}^{2^n} h_{i-1}^n \mathbf{1}_{\{s \in [t_{i-1}, t_i)\}},$$

then

$$\mathbb{E}(\eta|\mathcal{F}_{t,n}^X) = \mathbb{E}\eta + \int_0^t h_n(s, t) dB_s + \int_0^t h_n(s, t) dG_s,$$

and by independence of B and G

$$\mathbb{E}(\mathbb{E}(\eta|\mathcal{F}_{t,n}^X) - \mathbb{E}(\eta|\mathcal{F}_{t,m}^X))^2 = \|h_n - h_m\|_2^2 + \|h_n - h_m\|_{\Lambda_t}^2.$$

Therefore, by (3.12)

$$\begin{aligned} & \limsup_n \sup_{m \geq n} (\|h_n - h_m\|_2^2 + \|h_n - h_m\|_{\Lambda_t}^2) \\ &= \limsup_n \sup_{m \geq n} \mathbb{E}(\mathbb{E}(\eta|\mathcal{F}_{t,n}^X) - \mathbb{E}(\eta|\mathcal{F}_{t,m}^X))^2 = 0, \end{aligned}$$

and hence $h_n \rightarrow h$ in $L^2([0, t])$ by its completeness. Since we assumed that either $L^2([0, t]) \subseteq \Lambda_t$ or Λ_t is complete, $h \in \Lambda_t$ as well. The claimed representation now follows, since

$$\begin{aligned} & \mathbb{E}\left(\mathbb{E}(\eta|\mathcal{F}_t^X) - \mathbb{E}\eta - \int_0^t h(s, t) dB_s - \int_0^t h(s, t) dB_s^H\right)^2 \\ & \leq 3\mathbb{E}(\mathbb{E}(\eta|\mathcal{F}_t^X) - \mathbb{E}(\eta|\mathcal{F}_{t,n}^X))^2 + 3\|h - h_n\|_2^2 + 3\|h - h_n\|_{\Lambda_t}^2 \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

The uniqueness of h is obvious. \square

REMARK 3.10. In particular, the conclusion of Lemma 3.9 holds, when G satisfies (2.1) with K , such that

$$\sup_{s \leq T} \int_0^T |K(s, t)| dt < \infty.$$

In this case, (3.11) holds with

$$\langle f, g \rangle_{\Lambda_t} = \int_0^t \int_0^t f(s)g(s)K(s, r) ds dr,$$

and $L^2([0, t]) \subseteq \Lambda_t$. Note that weakly singular kernels as in (2.6) fit this framework.

3.3. *Fractional Brownian motion.* In the proofs below, we will frequently use a number of well-known formulas, related to fBm. Our main references are [24] and [25].

3.3.1. *Constants.*

$$\begin{aligned} c_H &= \frac{1}{2H\Gamma(3/2 - H)\Gamma(H + 1/2)}, \\ \lambda_H &= \frac{2H\Gamma(H + 1/2)\Gamma(3 - 2H)}{\Gamma(3/2 - H)}, \\ \beta_H &= c_H^2 \left(\frac{1}{2} - H\right)^2 \frac{\lambda_H}{2 - 2H}. \end{aligned} \tag{3.13}$$

3.3.2. *Spaces and operators.* For $f : [0, t] \mapsto \mathbb{R}$, define the operators

$$(\Psi f)(s, t) = -2H \frac{d}{ds} \int_s^t f(r)r^{H-1/2}(r - s)^{H-1/2} dr, \quad 0 \leq s \leq t, \tag{3.14}$$

and

$$(\Phi f)(s) = \frac{d}{ds} \int_0^s f(r)r^{1/2-H}(s - r)^{1/2-H} dr, \quad 0 \leq s \leq t.$$

These formulas can be readily expressed in terms of the Riemann–Liouville fractional integrals and derivatives; see [25]. The respective inverse operators are given by

$$(\Psi^{-1}g)(s, t) = -c_H s^{1/2-H} \frac{d}{ds} \int_s^t g(r, t)(r - s)^{1/2-H} dr \tag{3.15}$$

and

$$(\Phi^{-1}g)(s) = 2H c_H s^{H-1/2} \frac{d}{ds} \int_0^s g(r)(s - r)^{H-1/2} dr. \tag{3.16}$$

Let us define the space

$$\Lambda_t^{H-1/2} := \left\{ f : [0, t] \mapsto \mathbb{R} \text{ such that } \int_0^t (s^{1/2-H}(\Psi f)(s, t))^2 ds < \infty \right\},$$

with the scalar product

$$(3.17) \quad \langle f, g \rangle_{\Lambda_t^{H-1/2}} := \frac{2-2H}{\lambda_H} \int_0^t s^{1-2H}(\Psi f)(s, t)(\Psi g)(s, t) ds.$$

The inclusion $L^2([0, t]) \subset \Lambda_t^{H-1/2}$ holds for $H > \frac{1}{2}$ and fails for $H < \frac{1}{2}$ (see Remark 4.2 in [25]).

The expression in (3.17) can be rewritten as

$$(3.18) \quad \langle f, g \rangle_{\Lambda_t^{H-1/2}} = H \int_0^t f(r) \frac{d}{dr} \int_0^t g(u) |r-u|^{2H-1} \text{sign}(r-u) du dr,$$

which for $H > \frac{1}{2}$ becomes

$$\langle f, g \rangle_{\Lambda_t^{H-1/2}} = \int_0^t \int_0^s K_H(u, v) f(u) g(v) du dv,$$

with kernel K_H defined in (2.15).

Finally, for any $H \in (0, 1)$ and $f, g \in L^2([0, t]) \cap \Lambda_t^{H-1/2}$, the following identity holds:

$$(3.19) \quad \int_0^t f(s)g(s) ds = c_H \int_0^t (\Psi f)(s, t)(\Phi g)(s) ds.$$

3.3.3. *Representation formulas for fBm.* The stochastic integral with respect to fBm, $\int_0^t f(s) dB_s^H$, can be defined for $f \in \Lambda_t^{H-1/2}$ in the usual way (see [25]). Integrals with kernels

$$(3.20) \quad \begin{aligned} \rho(s, t) &= \sqrt{\frac{2-2H}{\lambda_H}} s^{1/2-H} (\Psi 1)(s, t), \\ \tilde{\rho}(s, t) &= \sqrt{\frac{2-2H}{\lambda_H}} (\Psi^{-1} u^{H-1/2})(s, t), \end{aligned}$$

transform fBm into standard Brownian motion and vice versa. Namely,

$$B_t^H = \int_0^t \rho(s, t) dW_s,$$

where

$$W_t = \int_0^t \tilde{\rho}(s, t) dB_s^H$$

is a standard Brownian motion. More generally,

$$\int_0^t f(s) dB_s^H = \int_0^t \sqrt{\frac{2-2H}{\lambda_H}} s^{1/2-H} (\Psi f)(s, t) dW_s$$

and

$$\int_0^t f(s) dW_s = \int_0^t \sqrt{\frac{2-2H}{\lambda_H}} (\Psi^{-1} u^{H-1/2} f(u))(s, t) dB_s^H.$$

It follows that

$$\mathbb{E} \int_0^t f(u) dB_u^H \int_0^s g(v) dB_v^H = \langle f, g \rangle_{\Lambda_t^{H-1/2}}.$$

4. Proof of Theorem 2.2. As mentioned in Section 3.1, for kernels of the form (2.6), equation (2.2) has a unique solution $L(\cdot, t) \in L^1([0, t])$. Let

$$W_t = \int_0^t \phi(s) dB_s,$$

with $\phi(s) = 1 - \int_0^s L(r, s) dr$, and define the martingale $\bar{B}_t := \mathbb{E}(W_t | \mathcal{F}_t^X)$. By Lemma 3.9 (see Remark 3.10)

$$\bar{B}_t = \int_0^t q(s, t) dX_s,$$

where q solves the integral equation (2.9). A direct substitution shows that the unique solution is given by

$$q(s, t) = 1 + \int_s^t L(s, r) dr,$$

and hence

$$\langle \bar{B} \rangle_t = \mathbb{E} \bar{B}_t^2 = \mathbb{E} \bar{B}_t W_t = \int_0^t q(s, t) \phi(s) ds \stackrel{\dagger}{=} \int_0^t q^2(s, s) ds = t,$$

where the equality \dagger holds by Lemma 3.2 and the last equality holds since $q(s, s) = 1$. By the Lévy theorem, \bar{B} is a Brownian motion in filtration \mathcal{F}_t^X .

Let us check that $\mathcal{F}_t^{\bar{B}} = \mathcal{F}_t^X$. Since $\mathcal{F}_t^{\bar{B}} \subseteq \mathcal{F}_t^X$, the process $D_t = X_t - \mathbb{E}(X_t | \mathcal{F}_t^{\bar{B}})$ is \mathcal{F}_t^X -adapted, and hence admits the representation

$$D_t = \int_0^t h(s, t) dX_s,$$

for some $h(\cdot, t) \in L^2([0, T])$. On the other hand, by the orthogonality property of conditional expectation, $\mathbb{E} D_t \bar{B}_s = 0$ for all $s \leq t$. Let us show that this condition

implies $h(s, t) = 0$ for all $s \leq t$, that is, $D_t = 0$. To this end, we have

$$\begin{aligned} \mathbb{E}D_t \bar{B}_s &= \mathbb{E} \int_0^t h(u, t) dX_u \int_0^s q(u, s) dX_u \\ &= \int_0^s h(u, t)q(u, s) du + \int_0^t h(u, t) \int_0^s q(v, s)K(v, u) dv du \\ &= \int_0^s h(u, t)q(u, s) du + \int_0^t h(u, t)(\varphi(u) - q(u, s)) du \\ &= \int_0^t h(u, t)\varphi(u) du - \int_s^t h(u, t)q(u, s) du, \end{aligned}$$

where we used (2.9). Since $\mathbb{E}D_t \bar{B}_s = 0$ for all $s \leq t$, taking the derivative with respect to s , we obtain the Volterra equation for $h(s, t)$:

$$(4.1) \quad h(s, t) - \int_s^t h(u, t)L(u, s) du = 0, \quad s \leq t.$$

Recall that L solves equation (2.2), and thus by Lemma 3.1, $|L(s, t)| \leq C|s - t|^{-\alpha}$ with a constant C . Iterating (4.1) sufficient number of times, it follows that h also solves the Volterra equation with a bounded kernel. Such equations are well known to have a unique solution, and hence $h(s, t) \equiv 0$. Consequently, $X_t = \mathbb{E}(X_t | \mathcal{F}_t^{\bar{B}})$ and $\mathcal{F}_t^{\bar{B}} = \mathcal{F}_t^X$.

Finally, the inverse transformation (2.10) follows from the normal correlation theorem:

$$\begin{aligned} \hat{q}(s, t) &= \frac{\partial}{\partial s} \mathbb{E}X_t \bar{B}_s = \frac{\partial}{\partial s} \mathbb{E}X_t \int_0^s q(r, s) dX_r \\ &= \frac{\partial}{\partial s} \left(\int_0^s q(r, s) dr + \int_0^t \int_0^s K(r, u)q(r, s) dr du \right) \\ &= \frac{\partial}{\partial s} \left(\int_0^s q(r, s) dr + \int_0^t (\varphi(u) - q(u, s)) du \right) \\ &= -\frac{\partial}{\partial s} \int_s^t q(r, s) dr. \end{aligned}$$

5. Proof of Theorem 2.4. Let us first sketch the main steps of the proof. Our candidate for the innovation process is the martingale $M_t = \mathbb{E}(B_t | \mathcal{F}_t^X)$. We argue that it can be represented as a stochastic integral with respect to X and then identify the integrand as the solution of equation (2.17). This equation is solved by reduction to a weakly singular integral equation, whose precise form depends on whether H is greater or less than $\frac{1}{2}$. The quadratic variation $\langle M \rangle_t$ is then expressed as an integral of the solution on the diagonal, which gives the formula (2.18).

The claimed assertions are obvious for the case $H = \frac{1}{2}$, which we exclude from the consideration thereafter.

5.1. Proof of part (i).

5.1.1. The equation (2.17) and its alternative forms. The following theorem proves part (i) of Theorem 2.4 and elaborates on the structure of equation (2.17).

THEOREM 5.1. The representation (2.16) holds with $g(s, t)$, $s \leq t$, being the unique continuous solution of the following equations:

- (i) for $H \in (0, 1]$, the integro-differential equation (2.17),
- (ii) for $H \in (0, 1)$, the fractional integro-differential equation

$$(5.1) \quad c_H(\Phi g)(s) + \frac{2 - 2H}{\lambda_H}(\Psi g)(s, t)s^{1-2H} = c_H(\Phi 1)(s), \quad s \in (0, t],$$

- (iii) for $H \in (\frac{1}{2}, 1]$, the weakly singular integral equation

$$(5.2) \quad g(s, t) + \int_0^t g(r, t)K_H(r, s) dr = 1, \quad s \in (0, t],$$

with kernel K_H , defined in (2.15),

- (iv) for $H \in (0, \frac{1}{2})$, the weakly singular integral equation

$$(5.3) \quad \begin{aligned} g(s, t) + \beta_H t^{-2H} \int_0^t g(r, t)K^H\left(\frac{r}{t}, \frac{s}{t}\right) dr \\ = c_H s^{1/2-H} (t - s)^{1/2-H}, \quad s \in [0, t], \end{aligned}$$

with the kernel

$$(5.4) \quad K^H(u, v) = |u - v|^{-2H} N(u, v),$$

where $N \in C([0, 1]^2)$ is given in (5.6) below.

PROOF. By Lemma 3.9, there exists a function $g(\cdot, t) \in L^2([0, t]) \cap \Lambda_t^{H-1/2}$, such that

$$M_t = \mathbb{E}(B_t | \mathcal{F}_t^X) = \int_0^t g(s, t) dX_s.$$

To verify the representation (2.16), we have to check that $g(s, t)$ uniquely solves each one of the equations in (i)–(iv). To this end, we will argue that $g(s, t)$ satisfies the equation from (ii) for almost every $s \in [0, t]$. Then we show that this equation reduces to (iii) for $H > \frac{1}{2}$ and to (iv) for $H < \frac{1}{2}$. These weakly singular integral equations are well known to have unique continuous solutions and, therefore, $g(s, t)$, in fact, satisfies (ii) for all $s \in [0, t]$. Finally, we will argue that (ii) and (i) share the same solution.

For any test function $\varphi \in L^2([0, t]) \cap \Lambda_t^{H-1/2}$, the orthogonality property of the conditional expectation implies

$$\begin{aligned}
 0 &= \mathbb{E} \left(B_t - \int_0^t g(s, t) dX_s \right) \int_0^t \varphi(s) dX_s \\
 &= \int_0^t \varphi(s) ds - \int_0^t \varphi(s) g(s, t) ds \\
 (5.5) \quad &\quad - \frac{2-2H}{\lambda_H} \int_0^t s^{1-2H} (\Psi g)(s, t) (\Psi \varphi)(s, t) ds \\
 &= \int_0^t (\Psi \varphi)(s, t) \left(c_H(\Phi 1)(s) ds - c_H(\Phi g)(s, t) \right. \\
 &\quad \left. - \frac{2-2H}{\lambda_H} s^{1-2H} (\Psi g)(s, t) \right) ds,
 \end{aligned}$$

where we used the identity (3.19). Since φ can be an arbitrary differentiable function, $g(s, t)$ satisfies (5.1) for almost all $s \in [0, t]$.

Applying the transformation (3.16) with $H > \frac{1}{2}$ to equation (5.1), a direct calculation shows that $g(s, t)$ satisfies (5.2). This weakly singular equation is well known to have a unique solution (see, e.g., [30]), continuous on $[0, t]$. Since the transformation (3.16) is invertible, $g(s, t)$ is also the unique continuous solution of (5.1) for $H > \frac{1}{2}$.

Similarly, equation (5.3) is obtained for $H < \frac{1}{2}$ by multiplying (5.1) by $\frac{\lambda_H}{2-2H} s^{2H-1}$ and applying the transformation Ψ^{-1} . A calculation shows that

$$K^H(u, v) = (uv)^{1/2-H} \int_{u \vee v}^1 r^{2H-1} (r-u)^{-1/2-H} (r-v)^{-1/2-H} dr,$$

and changing the integration variable to $x := \frac{1-v}{u-v} \frac{r-u}{1-r}$ we get (5.4), where

$$\begin{aligned}
 (5.6) \quad N(u, v) &= \left(\frac{a}{b} \right)^{1/2-H} \\
 &\quad \times \int_0^\infty x^{-1/2-H} (1+x)^{-1/2-H} \left(1 + \left(1 - \frac{a}{b} \right) x \right)^{2H-1} dx,
 \end{aligned}$$

with

$$a = \frac{u}{1-u} \wedge \frac{v}{1-v}, \quad b = \frac{u}{1-u} \vee \frac{v}{1-v}.$$

For $H < \frac{1}{2}$ the function N is continuous and thus kernel K^H is weakly singular. Since the right-hand side of (5.3) is a continuous function for $H < \frac{1}{2}$, this equation has a unique solution, continuous on $[0, t]$. This completes the proof of (iv) and, in turn, of (ii).

Further, the identity (3.18) and orthogonality property (5.5) imply

$$\begin{aligned} 0 &= \int_0^t \varphi(s) ds - \int_0^t \varphi(s)g(s, t) ds - \frac{2-2H}{\lambda_H} \int_0^t s^{1-2H} (\Psi g)(s, t)(\Psi \varphi)(s, t) ds \\ &= \int_0^t \varphi(s) \left(1 - g(s, t) - H \frac{d}{ds} \int_0^t g(r, t) |s - r|^{2H-1} \text{sign}(s - r) dr \right) ds. \end{aligned}$$

The assertion (i) follows, in view of arbitrariness of φ and unique solvability of (5.1). Finally, for $t \in [0, T]$,

$$\langle M \rangle_t = \mathbb{E}M_t^2 = \mathbb{E}B_t M_t = \mathbb{E}B_t \int_0^t g(s, t) dX_s = \int_0^t g(s, t) ds. \quad \square$$

5.2. *Proof of part (ii) for $H > \frac{1}{2}$.* Note that in this case, the derivative and integration in (3.14) can be interchanged, and hence

$$(\Psi g)(s, t) = 2H \left(H - \frac{1}{2} \right) \int_s^t g(r, t) r^{H-1/2} (r - s)^{H-3/2} dr, \quad 0 \leq s \leq t,$$

and $(\Psi g)(t, t) = 0$ for all $t \in [0, T]$. Therefore, (2.18) holds by (2.16) and Lemma 3.2, and, in fact,

$$(5.7) \quad \langle M \rangle_t = \int_0^t g^2(s, s) ds > 0.$$

Since $g(t, t) > 0$ for all $t \in [0, T]$ there exists a function $\hat{g}(\cdot, t) \in L^2([0, t])$, such that

$$\mathbb{E}(X_t | \mathcal{F}_t^M) = \int_0^t \hat{g}(s, t) dM_s, \quad t \in [0, T].$$

By the normal correlation theorem,

$$\hat{g}(s, t) = \frac{d}{d\langle M \rangle_s} \mathbb{E}X_t M_s,$$

and the formula (2.20) follows, since

$$\begin{aligned} \mathbb{E}X_t M_s &= \int_0^s g(r, s) \frac{\partial}{\partial r} \mathbb{E}X_t X_r dr \\ &= \int_0^s g(r, s) dr + H \int_0^s g(r, s) (r^{2H-1} + (t - r)^{2H-1}) dr \\ (5.8) \quad &= \langle M \rangle_s + \int_0^t H \frac{d}{d\tau} \int_0^s g(r, s) |r - \tau|^{2H-1} \text{sign}(r - \tau) dr d\tau \\ &= \langle M \rangle_s + \int_0^t (1 - g(\tau, s)) d\tau. \end{aligned}$$

It is left to check that $\mathcal{F}_t^X = \mathcal{F}_t^M$, that is, $\mathbb{E}(X_t | \mathcal{F}_t^M) = X_t$ or

$$(5.9) \quad \mathbb{E}(X_t - \mathbb{E}(X_t | \mathcal{F}_t^M))^2 = \mathbb{E}X_t^2 - \mathbb{E}(\mathbb{E}(X_t | \mathcal{F}_t^M))^2 = 0.$$

Since $X_0 = \mathbb{E}(X_0|\mathcal{F}_0^M) = 0$, (5.9) holds if

$$\frac{\partial^2}{\partial t \partial s} \int_0^{t \wedge s} \hat{g}(r, t) \hat{g}(r, s) d\langle M \rangle_r = K_H(t, s), \quad s < t,$$

or, by (5.7), if

$$(5.10) \quad \dot{\hat{g}}(s, t) \hat{g}(s, s) g^2(s, s) + \int_0^s \dot{\hat{g}}(r, t) \dot{\hat{g}}(r, s) g^2(r, r) dr = K_H(t, s).$$

Further, we have

$$\begin{aligned} \hat{g}(t, t) &= 1 - \frac{1}{g^2(t, t)} \int_0^t \dot{\hat{g}}(s, t) ds = 1 - \frac{1}{g^2(t, t)} \left(\frac{d}{dt} \int_0^t g(s, t) ds - g(t, t) \right) \\ &= 1 - \frac{1}{g^2(t, t)} (g^2(t, t) - g(t, t)) = \frac{1}{g(t, t)}, \end{aligned}$$

and, since $\dot{\hat{g}}(s, t) g(s, s) = -L(t, s)$, (5.10) becomes

$$(5.11) \quad -L(t, s) + \int_0^s L(t, r) L(s, r) dr = K_H(t, s).$$

Recall that the function L , satisfies equation (2.2). Rearranging the terms, multiplying by $L(s, u)$ and integrating gives

$$\begin{aligned} \int_0^s L(t, u) L(s, u) du + \int_0^s K_H(t, u) L(s, u) du \\ &= - \int_0^s \int_0^u L(r, u) L(s, u) K_H(r, t) dr du \\ &= - \int_0^s \left(\int_r^s L(r, u) L(s, u) du \right) K_H(r, t) dr \\ &= - \int_0^s (L(r, s) - L(s, r)) K_H(r, t) dr, \end{aligned}$$

where we used Corollary 3.8. The second term on the left-hand side and the last term on the right-hand side cancel out and we get

$$\int_0^s L(t, u) L(s, u) du = - \int_0^s L(r, s) K_H(r, t) dr = L(t, s) + K_H(s, t),$$

which verifies (5.11) and, therefore, (5.9), thus completing the proof.

5.3. *An auxiliary process \tilde{X} .* The proof of (ii) of Theorem 2.4 in the case $H < \frac{1}{2}$ relies on the auxiliary process

$$(5.12) \quad \tilde{X}_t = \int_0^t \tilde{\rho}(s, t) dX_s,$$

where the kernel $\tilde{\rho}$ is defined in (3.20). In this section, we explore some of its properties.

LEMMA 5.2. $\tilde{X} = \tilde{B} + \tilde{U}$, where \tilde{B} is a Brownian motion in its own filtration and \tilde{U} is a centered Gaussian process with the covariance function, satisfying

$$(5.13) \quad \tilde{K}_H(s, t) := \frac{\partial^2}{\partial s \partial t} \mathbb{E} \tilde{U}_s \tilde{U}_t = |t - s|^{-2H} \chi(t/s), \quad s \neq t,$$

where

$$\chi(u) = \beta_H (u \wedge u^{-1})^{1/2-H} L\left(\frac{1}{u \vee u^{-1} - 1}\right), \quad u \in \mathbb{R}_+,$$

and

$$L(v) = \int_0^v r^{-1/2-H} (1+r)^{-1/2-H} \left(1 - \frac{r}{v}\right)^{1-2H} dr.$$

Moreover, $\mathcal{F}_t^X = \mathcal{F}_t^{\tilde{X}}$ for all $t \in [0, T]$.

PROOF. It is well known (see, e.g., [24]), that the integral transformation (5.12) is invertible:

$$X_t = \int_0^t \rho(s, t) d\tilde{X}_s, \quad t \in [0, T],$$

where ρ is defined in (3.20). In particular, $\mathcal{F}_t^X = \mathcal{F}_t^{\tilde{X}}$, $t \in [0, T]$.

Further, it follows from [24] that the process $\tilde{B}_t = \int_0^t \tilde{\rho}(s, t) dB_s^H$ is the Brownian motion in its own filtration. Hence, $\tilde{X} = \tilde{B} + \tilde{U}$ with

$$\tilde{U}_t = \int_0^t \tilde{\rho}(s, t) dB_s,$$

and

$$\tilde{K}_H(s, t) = \frac{\partial^2}{\partial s \partial t} \mathbb{E} \tilde{U}_t \tilde{U}_s = \frac{\partial^2}{\partial s \partial t} \int_0^{s \wedge t} \tilde{\rho}(r, s) \tilde{\rho}(r, t) dr = \int_0^{s \wedge t} \dot{\tilde{\rho}}(r, s) \dot{\tilde{\rho}}(r, t) dr,$$

where $\dot{\tilde{\rho}}(s, t) = \frac{\partial}{\partial t} \tilde{\rho}(s, t)$ and we used the property $\tilde{\rho}(s, s) = 0$. The expression in (5.13) is obtained by a direct calculation, using the explicit expression (3.20) for $\tilde{\rho}(s, t)$. \square

5.4. *The martingale M for $H < \frac{1}{2}$.* The structure of the martingale M and its relation to the process \tilde{X} are described in detail in the following lemma.

LEMMA 5.3. For $H < \frac{1}{2}$ and $t \in [0, T]$,

$$(5.14) \quad M_t = \int_0^t p(s, t) d\tilde{X}_s, \quad \langle M \rangle_t = \int_0^t p^2(s, s) ds,$$

where

$$(5.15) \quad p(s, t) := \sqrt{\frac{2-2H}{\lambda_H}} s^{1/2-H} (\Psi g)(s, t)$$

solves the equation

$$(5.16) \quad p(s, t) + \int_0^t p(r, t) \tilde{K}_H(r, s) dr = \sqrt{\frac{2-2H}{\lambda_H}} s^{1/2-H}, \quad 0 \leq s \leq t \leq T.$$

Moreover, $p^2(t, t) > 0$ for all $t > 0$ and $\mathcal{F}_t^M = \mathcal{F}_t^{\tilde{X}}$.

PROOF. Let us set $C := \sqrt{\frac{\lambda_H}{2-2H}}$ for brevity. The equation (5.16) is obtained from equation (5.1) by replacing g in the first term with [see (3.15)]

$$\begin{aligned} g(s, t) &= -c_H s^{1/2-H} \frac{d}{ds} \int_s^t (\Psi g)(r, t) (r-s)^{1/2-H} dr \\ &= -c_H C s^{1/2-H} \frac{d}{ds} \int_s^t p(r, t) r^{H-1/2} (r-s)^{1/2-H} dr. \end{aligned}$$

Indeed,

$$\begin{aligned} c_H(\Phi g)(s) &= -c_H^2 C \frac{d}{ds} \int_0^s r^{1-2H} (s-r)^{1/2-H} \frac{d}{dr} \\ &\quad \times \int_r^t p(u, t) u^{H-1/2} (u-r)^{1/2-H} du dr \\ &= s^{1/2-H} C \int_0^t p(u, t) \beta_H(su)^{H-1/2} \\ &\quad \times \int_0^{s \wedge u} r^{1-2H} (s-r)^{-1/2-H} (u-r)^{-1/2-H} dr du \\ &= s^{1/2-H} C \int_0^t p(u, t) \tilde{K}_H(u, s) du, \end{aligned}$$

where we used the definition (5.13) of \tilde{K}_H . Equation (5.16) follows, since

$$c_H(\Phi 1)(s) = \frac{2-2H}{\lambda_H} s^{1-2H}.$$

Integrating by parts, we get the first formula in (5.14):

$$\begin{aligned} &\int_0^t p(s, t) d\tilde{X}_s \\ &= p(t, t) \tilde{X}_t - \int_0^t \tilde{X}_s p'(s, t) ds \\ &= p(t, t) \tilde{X}_t - \int_0^t \int_0^s \tilde{\rho}(r, s) dX_r p'(s, t) ds \end{aligned}$$

$$\begin{aligned} &= \int_0^t p(t, t) \tilde{\rho}(r, s) dX_r - \int_0^t \int_r^t \tilde{\rho}(r, s) p'(s, t) ds dX_r \\ &= \int_0^t \int_r^t \tilde{\rho}'(r, s) p(s, t) ds dX_r = \int_0^t g(s, t) dX_s, \end{aligned}$$

where the last equality holds by a direct calculation, using the definitions (5.12) and (5.15).

The second formula is obtained, using the identity (3.19):

$$\begin{aligned} \langle M \rangle_t &= \int_0^t g(s, t) ds = \frac{2 - 2H}{\lambda_H} \int_0^t s^{1-2H} (\Psi g)(s, t) ds \\ &= \sqrt{\frac{2 - 2H}{\lambda_H}} \int_0^t p(s, t) s^{1/2-H} ds = \int_0^t p^2(s, s) ds, \end{aligned}$$

where the last equality holds by Lemma 3.2 and $p^2(t, t) > 0$ for all $t > 0$ by Corollary 3.7.

It is left to verify the inclusion $\mathcal{F}_t^{\tilde{X}} \subseteq \mathcal{F}_t^M$, or equivalently, $\mathbb{E}(\tilde{X}_t | \mathcal{F}_t^M) = \tilde{X}_t$. Since $\tilde{X}_t - \mathbb{E}(\tilde{X}_t | \mathcal{F}_t^M)$ is measurable with respect to $\mathcal{F}_t^{\tilde{X}}$ and $\tilde{K}_H(s, t)$ is weakly singular, it admits the representation (see Remark 3.10)

$$\tilde{X}_t - \mathbb{E}(\tilde{X}_t | \mathcal{F}_t^M) = \int_0^t h(r, t) d\tilde{X}_r$$

with $h(\cdot, t) \in L^2([0, t])$. By the orthogonality property of the conditional expectation,

$$\mathbb{E}M_s \int_0^t h(r, t) d\tilde{X}_r = 0, \quad s \leq t.$$

Let us show that this condition implies that $h(s, t) = 0$ for all $s \leq t$, therefore, completing the proof. Indeed, for $s \leq t$,

$$\begin{aligned} &\mathbb{E}M_s \int_0^t h(r, t) d\tilde{X}_r \\ &= \mathbb{E} \int_0^s p(u, s) d\tilde{X}_u \int_0^t h(r, t) d\tilde{X}_r \\ &= \int_0^s p(r, s) h(r, t) dr + \int_0^t h(r, t) \int_0^s p(u, s) \tilde{K}(r, u) du dr \\ &= \int_0^s p(r, s) h(r, t) dr + \int_0^t h(r, t) (Cr^{1/2-H} - p(r, s)) dr \\ &= C \int_0^t h(r, t) r^{1/2-H} dr - \int_s^t h(r, t) p(r, s) dr. \end{aligned}$$

Taking the derivative with respect to s , we get

$$h(s, t) p(s, s) - \int_s^t h(r, t) \dot{p}(r, s) dr = 0$$

and, since $p(s, s) > 0$, it follows that $h(s, t)$ solves the Volterra equation

$$(5.17) \quad h(s, t) - \int_s^t h(r, t) \tilde{L}(r, s) dr = 0,$$

where $\tilde{L}(s, t) = \dot{p}(r, s)/p(s, s)$ solves the equation [cf. (2.2)]

$$\tilde{L}(s, t) + \int_0^t \tilde{L}(r, t) \tilde{K}_H(r, s) dr = -\tilde{K}_H(r, s), \quad 0 \leq s \leq t \leq T.$$

By Lemma 3.1, $|\tilde{L}(s, t)| \leq C_1|s - t|^{-2H}$ for some constant C_1 . By a sufficient number of iterations, (5.17) becomes the Volterra equation with a continuous kernel, which has a unique solution $h(s, t) \equiv 0$. \square

5.5. *Proof of part (ii) for $H < \frac{1}{2}$.* Equation (5.3) with $s := t$ yields $g(t, t) = 0$, $t \geq 0$, since $K^H(u, 1) \equiv 0$. Hence, the formula (2.18) holds by Lemma 5.3. The calculations in (5.8) are valid for any H , and hence

$$\mathbb{E}(X_t | \mathcal{F}_t^M) = \int_0^t \hat{g}(s, t) dM_s,$$

where $\hat{g}(s, t)$ is given by (2.20). Finally, $\mathcal{F}_t^M = \mathcal{F}_t^{\tilde{X}} = \mathcal{F}_t^X$, by Lemmas 5.3 and 5.2.

6. Proof of Theorem 2.7. As discussed in the Introduction, some of the assertions in this theorem have been previously proved by a number of authors, using different methods. Our objective is to show how all these results can be deduced from the canonical representation of Theorem 2.4. The original contribution here is the new density formula (2.25).

6.1. *Proof of (i).* The fBm B^H and hence also X have infinite quadratic variation for $H \in (0, \frac{1}{2})$. Hence, X is not a semimartingale in its own filtration and a fortiori μ^X and μ^W are singular. For $H = \frac{1}{2}$, the statement of the theorem is evident. Below we focus on the case $H \in (\frac{1}{2}, 1]$.

REMARK 6.1. The fact that X is not a semimartingale for $H \in (\frac{1}{2}, \frac{3}{4}]$ implies singularity of μ^X and μ^W , but not vice versa. For the sake of completeness, we prove both assertions directly, showing how they stem from the same property of the kernel K_H in (2.15).

6.1.1. *Equivalence for $H \in (\frac{3}{4}, 1]$.* Recall that for $H > \frac{1}{2}$ the second term in (2.18) vanishes and

$$\langle M \rangle_t = \int_0^t g^2(s, s) ds, \quad t \in [0, T].$$

By Lemma 3.6, $g(t, t) > 0$ for all $t \geq 0$ and the process

$$(6.1) \quad \bar{B}_t = \int_0^t \frac{1}{g(s, s)} dM_s$$

is a Brownian motion in filtration \mathcal{F}_t^X . On the other hand,

$$\begin{aligned} M_t &= \int_0^t g(s, t) dX_s = \int_0^t g(s, s) dX_s + \int_0^t (g(r, t) - g(r, r)) dX_r \\ &= \int_0^t g(s, s) dX_s + \int_0^t \int_r^t \dot{g}(r, s) ds dX_r \\ &= \int_0^t g(s, s) dX_s + \int_0^t \int_0^s \dot{g}(r, s) dX_r ds, \end{aligned}$$

where the last equality holds since $\dot{g}(\cdot, s) \in L^2([0, s])$ by Lemma 3.5. Hence,

$$\bar{B}_t = \int_0^t \frac{1}{g(s, s)} dM_s = X_t + \int_0^t \int_0^s \frac{\dot{g}(r, s)}{g(s, s)} dX_r ds =: X_t + \int_0^t \varphi_s(X) ds.$$

The desired claim follows by the Girsanov theorem (Theorem 7.7 in [21]), once we check

$$(6.2) \quad \int_0^T \mathbb{E} \varphi_t^2(\bar{B}) dt < \infty \quad \text{and} \quad \int_0^T \mathbb{E} \varphi_t^2(X) dt < \infty.$$

Since $\varphi_t(\cdot)$ is a linear functional of $X = B + B^H$ and B and B^H are independent, it is enough to check only the latter condition. The function $L(s, t) = \frac{\dot{g}(s, t)}{g(t, t)}$ satisfies (2.2), and hence for $H > 3/4$

$$\begin{aligned} \mathbb{E} \varphi_t^2(X) &= \mathbb{E} \left(\int_0^t L(r, t) dX_r \right)^2 \\ &= \int_0^t L^2(s, t) ds + \int_0^t \int_0^t L(s, t) L(r, t) K_H(r, s) dr ds \\ &= \int_0^t L(s, t) \left(L(s, t) + \int_0^t L(r, t) K_H(r, s) dr \right) ds \\ &= - \int_0^t L(s, t) K_H(s, t) ds \\ &\leq \left(\int_0^t L^2(s, t) ds \right)^{1/2} \left(\int_0^t K_H^2(s, t) ds \right)^{1/2} \\ &= C_1 \left(\int_0^t L^2(s, t) ds \right)^{1/2} t^{2H-3/2}. \end{aligned}$$

Since the kernel is positive definite, multiplying (2.2) by $L(s, t)$ and integrating gives

$$\int_0^t L^2(s, t) ds \leq - \int_0^t L(s, t) K_H(s, t) ds \leq C_1 \left(\int_0^t L^2(s, t) ds \right)^{1/2} t^{2H-3/2},$$

and consequently

$$\left(\int_0^t L^2(s, t) ds \right)^{1/2} \leq C_1 t^{2H-3/2}.$$

Plugging this bound back gives $\mathbb{E}\varphi_t^2(X) \leq C_1^2 t^{4H-3}$ and in turn

$$\int_0^T \mathbb{E}\varphi_t^2(X) dt \leq C_1^2 \int_0^T t^{4H-3} dt = C_2 T^{4H-2},$$

which verifies (6.2) and completes the proof.

6.1.2. *Singularity for $H \in (\frac{1}{2}, \frac{3}{4}]$.* Suppose there exists a probability measure \mathbb{Q} , equivalent to \mathbb{P} , under which X is a Brownian motion in its natural filtration. Since the semimartingale property is preserved under equivalent change of measure, the \mathbb{P} -martingale

$$M_t = \int_0^t g(s, t) dX_s, \quad t \in [0, T],$$

must be a semimartingale under \mathbb{Q} . Since X is assumed to be a Brownian motion under \mathbb{Q} , this is equivalent to saying that the process

$$L_t := \int_0^t g(s, t) d\bar{B}_s,$$

with the Brownian motion \bar{B} , defined in (6.1), must be a semimartingale under \mathbb{P} .

We will argue by contradiction that this is impossible for $H \leq \frac{3}{4}$. To this end, we define

$$\psi(s, t) = - \int_s^t g(r, r) \sum_{m=1}^{n_0-1} (-1)^m K_H^{(m)}(r, s) dr, \quad 0 < s < t \leq T,$$

where n_0 is the least integer greater than $\frac{1}{4H-2}$. Note that $\psi(\cdot, t) \in L^2([0, t])$ and define the processes

$$U_t := \int_0^t \psi(s, t) d\bar{B}_s,$$

$$V_t := \int_0^t (g(s, t) - g(s, s) + \psi(s, t)) d\bar{B}_s,$$

so that

$$L_t = V_t + \int_0^t g(s, s) d\bar{B}_s - U_t.$$

The second term is a martingale in filtration \mathcal{F}_t^X and hence, in order to argue that L is not a semimartingale, it is enough to show that:

- (a) U has zero quadratic variation, but infinite variation,
- (b) V has bounded variation.

PROOF OF (a). To verify this assertion, we will need an estimate for the variance of increments of U . To this end, for any two points $t_1, t_2 \in [0, T]$, such that $0 < t_2 - t_1 < 1$,

$$\begin{aligned}
 \mathbb{E}(U_{t_2} - U_{t_1})^2 &= \mathbb{E}\left(\int_{t_1}^{t_2} \psi(s, t_2) d\bar{B}_s + \int_0^{t_1} (\psi(s, t_2) - \psi(s, t_1)) d\bar{B}_s\right)^2 \\
 (6.3) \qquad &= \int_{t_1}^{t_2} \psi^2(s, t_2) ds + \int_0^{t_1} (\psi(s, t_2) - \psi(s, t_1))^2 ds.
 \end{aligned}$$

To bound the first term, note that

$$\begin{aligned}
 \psi^2(s, t_2) &\leq \|g\|_\infty^2 n_0 \sum_{m=1}^{n_0-1} \left(\int_s^{t_2} K_H^{(m)}(s, r) dr\right)^2 \\
 &\leq C_1 \sum_{m=1}^{n_0-1} (t_2 - s)^{(4H-2)m} \leq C_2 (t_2 - s)^{4H-2},
 \end{aligned}$$

where $\|g\|_\infty = \sup_{r \leq T} |g(r, r)| < \infty$, and consequently

$$\int_{t_1}^{t_2} \psi^2(s, t_2) ds \leq C_3 (t_2 - t_1)^{4H-1}.$$

For the second term, we have

$$\begin{aligned}
 &\int_0^{t_1} (\psi(s, t_2) - \psi(s, t_1))^2 ds \\
 &= \int_0^{t_1} \left(\sum_{m=1}^{n_0-1} \int_{t_1}^{t_2} (-1)^m g(r, r) K_H^{(m)}(s, r) dr\right)^2 ds \\
 (6.4) \qquad &= \sum_{m=1}^{n_0-1} \sum_{\ell=1}^{n_0-1} \int_0^{t_1} \int_{t_1}^{t_2} \int_{t_1}^{t_2} (-1)^{m+\ell} g(r, r) g(\tau, \tau) \\
 &\quad \times K_H^{(m)}(s, r) K_H^{(\ell)}(s, \tau) dr d\tau ds.
 \end{aligned}$$

The dominating term in this sum corresponds to $m = 1, \ell = 1$:

$$\int_0^{t_1} \left(\int_{t_1}^{t_2} g(r, r) K_H(r, s) dr\right)^2 ds.$$

We have

$$\begin{aligned}
 &\int_0^{t_1} \left(\int_{t_1}^{t_2} K_H(r, s) dr\right)^2 ds \\
 (6.5) \qquad &= H^2 \int_0^{t_1} ((t_2 - t_1 + s)^{2H-1} - s^{2H-1})^2 ds \\
 &= H^2 (t_2 - t_1)^{4H-1} \int_0^{t_1/(t_2-t_1)} ((1 + u)^{2H-1} - u^{2H-1})^2 du.
 \end{aligned}$$

The increasing function

$$\gamma(y) := H^2 \int_0^y ((1+u)^{2H-1} - u^{2H-1})^2 du, \quad y \geq 0$$

satisfies

$$\begin{aligned} \lim_{y \rightarrow \infty} \gamma(y) &= \gamma_H, & H \in \left(\frac{1}{2}, \frac{3}{4}\right), \\ \lim_{y \rightarrow \infty} \frac{\gamma(y)}{\log y} &= \gamma_{3/4}, & H = \frac{3}{4}, \end{aligned}$$

with positive constants γ_H . The function $r \mapsto g(r, r)$ is positive and continuous on $[0, T]$, and hence

$$c_4 \leq \int_0^{t_1} \left(\int_{t_1}^{t_2} g(r, r) K_H(s, r) dr \right)^2 / (t_2 - t_1)^{4H-1} \gamma\left(\frac{t_1}{t_2 - t_1}\right) \leq C_4$$

with some positive constants c_4, C_4 for all sufficiently small $t_2 - t_1$. A similar calculation shows that the rest of the terms in (6.4) converge to zero as $t_2 - t_1 \rightarrow 0$ at a faster rate and assembling all parts together, we obtain

$$(6.6) \quad c_5 \leq \mathbb{E}(U_{t_2} - U_{t_1})^2 / (t_2 - t_1)^{4H-1} \gamma\left(\frac{t_1}{t_2 - t_1}\right) \leq C_5.$$

Now let $0 = t_0 < t_1 < \dots < t_n = T$ be an arbitrary partition, then for all $H \in (\frac{1}{2}, \frac{3}{4}]$

$$\begin{aligned} \mathbb{E} \sum_{i=1}^n (U_{t_i} - U_{t_{i-1}})^2 &\leq C_5 \sum_{i=1}^n (t_i - t_{i-1})^{4H-1} \gamma\left(\frac{T}{t_i - t_{i-1}}\right) \\ &\leq C_6 \max_i (t_i - t_{i-1})^{4H-2} \log \frac{1}{t_i - t_{i-1}} \xrightarrow{n \rightarrow \infty} 0, \end{aligned}$$

that is, U has zero quadratic variation.

On the other hand, since the process U is Gaussian

$$\begin{aligned} \mathbb{E} \sum_{i=1}^n |U_{t_i} - U_{t_{i-1}}| &\geq \sqrt{\frac{2}{\pi}} c_5 \sum_{i: t_i \geq T/2} (t_i - t_{i-1})^{2H-1/2} \gamma^{1/2}\left(\frac{T/2}{t_i - t_{i-1}}\right) \\ &\geq c_6 \min_i (t_i - t_{i-1})^{2H-3/2} \gamma^{1/2}\left(\frac{T/2}{t_i - t_{i-1}}\right) \xrightarrow{n \rightarrow \infty} \infty, \end{aligned}$$

which implies that U has infinite variation (see, e.g., Theorem 4, Chapter 4, Section 9 in [22]). \square

PROOF OF (b). For $0 < s < t \leq T$

$$\dot{\psi}(s, t) := \frac{\partial}{\partial t} \psi(s, t) = -g(t, t) \sum_{m=1}^{n_0-1} (-1)^m K_H^{(m)}(s, t)$$

and hence

$$\begin{aligned} & \int_0^t \dot{\psi}(s, t) K_H(r, s) dr \\ &= - \int_0^t \left(g(t, t) \sum_{m=1}^{n_0-1} (-1)^m K_H^{(m)}(s, t) \right) K_H(r, s) dr \\ &= -g(t, t) \sum_{m=1}^{n_0-1} (-1)^m K_H^{(m+1)}(s, t) = g(t, t) \sum_{m=2}^{n_0} (-1)^m K_H^{(m)}(s, t) \\ &= g(t, t) K_H(s, t) - \dot{\psi}(s, t) + (-1)^{n_0} g(t, t) K_H^{(n_0)}(s, t). \end{aligned}$$

Adding this expression to the equation for $\dot{g}(s, t)$ [see (3.10)], we get

$$\begin{aligned} & (\dot{g}(s, t) + \dot{\psi}(s, t)) + \int_0^t (\dot{g}(r, t) + \dot{\psi}(r, t)) K_H(r, s) dr \\ &= (-1)^{n_0} g(t, t) K_H^{(n_0)}(s, t). \end{aligned}$$

By the choice of n_0 , the right-hand side is square integrable and so is the function $\dot{g}(s, t) + \dot{\psi}(s, t)$, $s \in (0, t)$. Since $\psi(s, s) = 0$,

$$\begin{aligned} V_t &= \int_0^t (g(s, t) - g(s, s) + \psi(s, t)) d\bar{B}_s \\ &= \int_0^t \int_s^t (\dot{g}(s, r) + \dot{\psi}(s, r)) dr d\bar{B}_s \\ &= \int_0^t \int_0^r (\dot{g}(s, r) + \dot{\psi}(s, r)) d\bar{B}_s dr, \end{aligned}$$

and hence V has bounded variation. \square

6.1.3. X is not a semimartingale for $H \in (\frac{1}{2}, \frac{3}{4}]$. We will use the representation (2.20), where, for $H > \frac{1}{2}$,

$$\hat{g}(s, t) = 1 - \frac{1}{g(s, s)} \int_0^t L(r, s) dr.$$

We have

$$\begin{aligned} X_t &= M_t - \int_0^t \frac{1}{g(s, s)} \int_0^t L(\tau, s) d\tau dM_s \\ &= M_t - \int_0^t \int_0^t L(\tau, s) d\tau d\bar{B}_s \\ &= M_t - \int_0^t \int_0^s L(\tau, s) d\tau d\bar{B}_s - \int_0^t \int_s^t L(\tau, s) d\tau d\bar{B}_s =: M_t - N_t - U_t, \end{aligned}$$

where \bar{B} is the Brownian motion, defined by (6.1). By Lemma 3.1, the function $\int_0^s L(\tau, s) d\tau$ is bounded, and hence $M - N$ is a martingale in filtration \mathcal{F}_t^X . To argue that X is not a semimartingale in its own filtration, we will show that U has zero quadratic variation, but infinite variation.

Let n_0 be the least integer greater than $\frac{1}{4H-2}$. It then follows from (2.2) that the function

$$Q(s, t) := \int_0^t L(r, t) K_H^{(n_0-1)}(r, s) dr$$

satisfies

$$Q(s, t) + \int_0^t Q(r, t) K_H(r, s) dr = -K_H^{(n_0)}(s, t),$$

and hence $Q(\cdot, t) \in L^2([0, t])$. Iterating the equation (2.2), we get

$$(6.7) \quad L(s, t) = \sum_{m=1}^{n_0-1} (-1)^m K_H^{(m)}(s, t) + (-1)^{(n_0-1)} Q(s, t).$$

Define $\phi(s, t) := \int_s^t L(\tau, s) d\tau$, then, similar to (6.3),

$$(6.8) \quad \mathbb{E}(U_{t_2} - U_{t_1})^2 = \int_{t_1}^{t_2} \phi^2(s, t_2) ds + \int_0^{t_1} (\phi(s, t_2) - \phi(s, t_1))^2 ds.$$

By (6.7),

$$\begin{aligned} \phi^2(s, t) &\leq C_1 \sum_{m=1}^{n_0-1} \left(\int_s^t K_H^{(m)}(\tau, s) d\tau \right)^2 + C_1 \left(\int_s^t Q(\tau, s) d\tau \right)^2 \\ &\leq C_2 |t - s|^{4H-2} \end{aligned}$$

and hence the first term in (6.8) is bounded by

$$\int_{t_1}^{t_2} \phi^2(s, t_2) ds \leq \int_{t_1}^{t_2} C_2 (t_2 - s)^{4H-2} ds \leq C_3 (t_2 - t_1)^{4H-1}.$$

Further,

$$\begin{aligned} \int_0^{t_1} (\phi(s, t_2) - \phi(s, t_1))^2 ds &= \int_0^{t_1} \left(\int_{t_1}^{t_2} L(\tau, s) d\tau \right)^2 ds \\ &= \int_0^{t_1} \int_{t_1}^{t_2} \int_{t_1}^{t_2} L(\tau, s) L(r, s) d\tau dr ds. \end{aligned}$$

After plugging in the expression (6.7), the dominating term is readily seen to be given by (6.5), and hence as in the previous section the bound (6.6) holds. Consequently, U has infinite variation and zero quadratic variation and thus X is not a semimartingale.

6.2. Proof of (ii).

6.2.1. Equivalence for $H < \frac{1}{4}$. By Lemma 5.3,

$$\begin{aligned} M_t &= \int_0^t p(s, t) d\tilde{X}_s = \int_0^t p(s, s) d\tilde{X}_s + \int_0^t \int_s^t \dot{p}(s, r) dr d\tilde{X}_s \\ &= \int_0^t p(s, s) d\tilde{X}_s + \int_0^t \int_0^r \dot{p}(s, r) d\tilde{X}_s dr. \end{aligned}$$

The last equality holds since $\dot{p}(\cdot, t) \in L^2([0, t])$ for $H < \frac{1}{4}$ by Lemma 3.5. Hence,

$$\tilde{X}_t = \bar{W}_t - \int_0^t \tilde{\varphi}_s(\tilde{X}) ds,$$

where $\bar{W}_t = \int_0^t \frac{dM_s}{p(s, s)}$ is a Brownian motion in filtration $\mathcal{F}_t^{\tilde{X}}$ and

$$\begin{aligned} \tilde{\varphi}_t(\tilde{X}) &= \int_0^t \tilde{L}(s, t) d\tilde{X}_s = \int_0^t \sqrt{\frac{2-2H}{\lambda_H}} (\Psi^{-1} u^{H-1/2} \tilde{L}(u, t))(s, t) dX_s \\ &=: \int_0^t L(s, t) dX_s =: \varphi_t(X), \end{aligned}$$

with $\tilde{L}(s, t) := \frac{\dot{p}(s, t)}{p(t, t)}$. A calculation shows that

$$(6.9) \quad L(s, t) = \frac{\dot{g}(s, t)}{p(t, t)} - \dot{\rho}(s, t).$$

Since the kernel in (5.4) is weakly singular, by Lemma 3.5, the solution $g(s, t)$ of (5.3) is differentiable with respect to the second variable. Taking the derivative of (5.1), we obtain

$$c_H(\Phi \dot{g})(s) + \frac{2-2H}{\lambda_H} (\Psi \dot{g})(s, t) s^{1-2H} = 0, \quad 0 < s < t \leq T,$$

since $g(t, t) = 0$ for $H < \frac{1}{2}$. Multiplying this equation by $\frac{\lambda_H}{2-2H} s^{2H-1}$ and applying Ψ^{-1} , it can be seen that $\dot{g}(s, t)$ satisfies [cf. (3.10)]

$$(6.10) \quad \dot{g}(s, t) + \beta_H t^{-2H} \int_0^t \dot{g}(r, t) K^H\left(\frac{r}{t}, \frac{s}{t}\right) dr = p(t, t) \dot{\rho}(s, t),$$

and plugging (6.9) into this equation yields

$$L(s, t) + \beta_H t^{-2H} \int_0^t (L(r, t) + \dot{\rho}(r, t)) K^H\left(\frac{r}{t}, \frac{s}{t}\right) dr = 0.$$

After applying Ψ and rearranging the terms, it becomes

$$L(s, t) + \frac{\partial}{\partial s} \int_0^t L(r, t) \frac{\partial}{\partial r} R(r, s) dr = -\dot{\rho}(s, t).$$

Assembling all the parts together, we get

$$\tilde{X}_t = \overline{W}_t - \int_0^t \varphi_s(X) ds,$$

and, consequently, the representation (2.24):

$$X_t = \int_0^t \rho(s, t) d\overline{W}_t - \int_0^t \rho(s, t) \varphi_s(X) ds =: \overline{B}_t^H - \int_0^t \rho(s, t) \varphi_s(X) ds.$$

The density (2.25) is obtained by Girsanov’s change of measure as in Section 6.1.1, under which \tilde{X} is a Brownian motion and, therefore, X is an fBm.

6.2.2. *Singularity for $H \geq \frac{1}{4}$.* The claim is obvious for $H = \frac{1}{2}$. For $H > \frac{1}{2}$, the process X has positive quadratic variation, and hence cannot be equivalent to fBm with $H > \frac{1}{2}$, whose quadratic variation vanishes.

To prove singularity for $H \in [\frac{1}{4}, \frac{1}{2})$, suppose there is a probability \mathbb{Q} , equivalent to \mathbb{P} , under which X is an fBm with the Hurst exponent H in its own filtration. Then $\tilde{X}_t = \int_0^t \tilde{\rho}(s, t) dX_s$, with $\tilde{\rho}(s, t)$ defined in (3.20), is a Brownian motion under \mathbb{Q} . By calculations as in Section 6.1.3, it can be seen that \tilde{X} is not a semimartingale for $H \in [\frac{1}{4}, \frac{1}{2})$, thus obtaining a contradiction.

7. Proof of Corollaries 2.9 and 2.10. The proofs of Corollaries 2.9 and 2.10 follow the same pattern and we will omit the details for the latter. The representation (2.28) is obvious in view of (2.16) and the definition (2.29). To prove the inversion formula (2.27), we have to check that

$$(7.1) \quad \int_0^t \xi_s ds = \int_0^t \hat{g}(s, t) \Xi(s) d\langle M \rangle_s, \quad t \in [0, T].$$

Since this is a pathwise statement and ξ is the only random object, no generality will be lost if ξ_t is assumed to be deterministic. For $\xi \in L^2([0, t])$, we have

$$\begin{aligned} \mathbb{E} \left(\int_0^t \xi_s dB_s \middle| \mathcal{F}_t^X \right) &= \mathbb{E} \left(\int_0^t \xi_s dB_s \middle| \mathcal{F}_t^M \right) \\ &= \int_0^t \frac{d}{d\langle M \rangle_s} \left(\mathbb{E} M_s \int_0^t \xi_r dB_r \right) dM_s \\ &= \int_0^t \frac{d}{d\langle M \rangle_s} \left(\mathbb{E} \int_0^s g(r, s) dX_r \int_0^t \xi_r dB_r \right) dM_s \\ &= \int_0^t \Xi(s) dM_s, \end{aligned}$$

and, using the representation (2.19), we obtain (7.1):

$$\int_0^t \xi_s ds = \mathbb{E} X_t \int_0^t \xi_s dB_s = \mathbb{E} X_t \int_0^t \Xi(s) dM_s = \int_0^t \hat{g}(s, t) \Xi(s) d\langle M \rangle_s.$$

The formula (2.30) follows from Theorem 7.13 in [21], once we check

$$(7.2) \quad \int_0^T \Xi^2(t) d\langle M \rangle_t < \infty, \quad \mathbb{P}\text{-a.s.}$$

and

$$(7.3) \quad \mathbb{E} \int_0^T |\Xi(t)| d\langle M \rangle_t < \infty.$$

Let us first consider the case $H > \frac{1}{2}$, for which $d\langle M \rangle_t/dt = g^2(t, t) > 0$. By definition (2.29) and continuity of ξ_t

$$\Xi(t)g(t, t) = \xi_t + \int_0^t L(s, t)\xi_s ds,$$

where $L(s, t)$ solves (2.2). By (ii) of Lemma 3.5, $|L(s, \tau)| \leq c_1|s - \tau|^{2H-2}$ with a constant c_1 and, therefore,

$$\begin{aligned} \left| \int_0^\tau L(s, \tau)\xi_s ds \right| &\leq \left(\int_0^\tau |L(s, \tau)|\xi_s^2 ds \right)^{1/2} \left(\int_0^\tau |L(s, \tau)| ds \right)^{1/2} \\ &\leq c_2 \left(\int_0^T |L(s, \tau)|\xi_s^2 ds \right)^{1/2}, \end{aligned}$$

where $c_2^2 = c_1 \sup_{\tau \in [0, T]} \int_0^T |s - \tau|^{2H-2} ds$. Consequently,

$$\begin{aligned} \int_0^T \Xi^2(t) d\langle M \rangle_t &\leq 2 \int_0^T \xi_t^2 dt + 2 \int_0^T \left(\int_0^t L(s, t)\xi_s ds \right)^2 dt \\ &\leq 2 \int_0^T \xi_s^2 ds + 2c_2^2 \int_0^T \xi_s^2 \int_0^T |L(s, t)| dt ds \\ &\leq 2(1 + c_2^4) \int_0^T \xi_t^2 dt < \infty, \end{aligned}$$

which proves (7.2). Condition (7.3) is verified similarly:

$$\begin{aligned} \mathbb{E} \int_0^T |\Xi(t)| d\langle M \rangle_t &\leq c_3 \mathbb{E} \int_0^T |\xi_t| dt + c_3 \mathbb{E} \int_0^T |\xi_s| \int_0^T |L(s, t)| ds dt \\ &\leq c_3(1 + c_2^2) \mathbb{E} \int_0^T |\xi_t| dt < \infty, \end{aligned}$$

where $c_3 := \sup_{t \in [0, T]} g(t, t)$.

For $H < \frac{1}{2}$, by Lemma 5.3, $d\langle M \rangle_t/dt = p^2(t, t) > 0$ and, therefore,

$$\Xi(t)p(t, t) = \xi_t + \int_0^t \frac{\dot{g}(s, t)}{p(t, t)} \xi_s ds.$$

Dividing both sides of equation (6.10) by $p(t, t)$, we see that $H(s, t) := \dot{g}(s, t)/p(t, t)$ solves the equation

$$H(s, t) + \beta_H t^{-2H} \int_0^t H(r, t) K^H\left(\frac{r}{t}, \frac{s}{t}\right) dr = \dot{\rho}(s, t),$$

where $|K^H(s, t)| \leq c_4 |s - t|^{-2H}$ and $|\dot{\rho}(s, t)| \leq c_5 |s - t|^{-1/2-H}$. Therefore, by Lemma 3.1, $|H(s, t)| \leq c_6 |s - t|^{-1/2-H}$ and the claim follows by the same arguments as in the case $H < \frac{1}{2}$.

8. The mixed Riemann–Liouville process. In this section, we outline the results, obtained by our method, for the mixed Riemann–Liouville process:

$$X_t = B_t + V_t^H, \quad t \in [0, T],$$

where V^H is defined in (2.23).

As mentioned in the Introduction, V^H shares many properties with B^H . In particular, the respective stochastic calculus builds on operators similar to those defined in (3.14)–(3.16). In this case, they are defined in a slightly different way:

$$(\Psi f)(s, t) = -2H \frac{d}{ds} \int_s^t f(r)(r - s)^{H-1/2} dr, \quad 0 \leq s \leq t,$$

$$(\Phi f)(s) = c_H \frac{d}{ds} \int_0^s f(r)(s - r)^{1/2-H} dr,$$

and

$$(\Psi^{-1} g)(s, t) = -c_H \frac{d}{ds} \int_s^t (r - s)^{1/2-H} g(r) dr,$$

$$(\Phi^{-1} g)(s) = \frac{2H}{c_H} \frac{d}{ds} \int_0^s g(r)(s - r)^{H-1/2} dr.$$

Stochastic integrals with respect to V^H can be defined on the space

$$\Lambda_t := \left\{ f : [0, t] \mapsto \mathbb{R} \text{ such that } \int_0^t (\Psi f)^2(s, t) ds < \infty \right\},$$

with the scalar product

$$\langle f, g \rangle_{\Lambda_t} = \int_0^t (\Psi f)(s)(\Psi g)(s) ds.$$

The formula (3.19) remains valid and the kernels ρ and $\tilde{\rho}$ become [cf. (3.20)]

$$\rho(s, t) = (\Psi 1)(s, t), \quad \tilde{\rho}(s, t) = (\Psi^{-1} 1)(s, t),$$

so that

$$V_t^H = \int_0^t \rho(s, t) dW_s,$$

where

$$W_t = \int_0^t \tilde{\rho}(s, t) dV_s^H,$$

is a Brownian motion with $\mathcal{F}_t^W = \mathcal{F}_t^{V^H}$. As before, we have

$$\mathbb{E} \int_0^t f(u) dV_u^H \int_0^t f(r) dV_r^H = \langle f, g \rangle_{\Lambda_t}.$$

For $H > \frac{1}{2}$, the covariance function of V^H

$$(8.1) \quad R(s, t) = \mathbb{E} V_t^H V_s^H = (2H)^2 \int_0^{s \wedge t} (t-r)^{H-1/2} (s-r)^{H-1/2} dr,$$

satisfies

$$(8.2) \quad K_H(s, t) := \frac{\partial^2 R(s, t)}{\partial t \partial s} = H^2 (2H - 1)^2 |t - s|^{2H-2} \chi\left(\frac{s}{t}\right), \quad s \leq t$$

with $\chi \in C([0, 1])$ given by

$$\chi(u) = \int_0^{u/(1-u)} \tau^{H-3/2} (1 + \tau)^{H-3/2} d\tau.$$

Repeating the proofs with these modifications gives the following analogs of the main results.

THEOREM 8.1. (i) *Theorem 2.4 remains valid with $g(s, t)$ solving the equation*

$$(8.3) \quad g(s, t) - \frac{\partial}{\partial s} \int_0^t R(r, s) \frac{\partial}{\partial r} g(r, t) dr + g(t, t) \frac{\partial}{\partial s} R(s, t) = 1, \quad 0 < s, t \leq T,$$

where $R(s, t)$ is defined in (8.1), and (2.18) is replaced with

$$\frac{d}{dt} \langle M \rangle_t = g^2(t, t) + (\Psi g)^2(t, t) > 0, \quad t \in [0, T].$$

(ii) *Theorem 2.7 remains valid with \overline{B}^H being replaced with the Riemann–Liouville process \overline{V}^H .*

(iii) *Corollary 2.9 remains valid.*

Note that equation (8.3) can be obtained formally from (2.17) through integration by parts. The reason for such a twist is that the first derivative $\partial R(s, t)/\partial s$ of the covariance function $R(s, t)$ is not integrable for $H < \frac{1}{2}$. Let us note that (8.3) also reduces to a weakly singular integral equation with the kernel K_H from (8.2) for $H > \frac{1}{2}$ and the kernel

$$K^H(u, v) = |u - v|^{-2H} \int_0^{(1-u)/|u-v|} \tau^{-1/2-H} (1 + \tau)^{-1/2-H} d\tau$$

for $H < \frac{1}{2}$ (cf. Theorem 5.1).

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C. CAI
SCHOOL OF MATHEMATICAL SCIENCE
NANKAI UNIVERSITY
94 WEIJIN ROAD, NANKAI
TIANJIN 300071
CHINA
E-MAIL: chunhao_cai@nankai.edu.cn

P. CHIGANSKY
DEPARTMENT OF STATISTICS
THE HEBREW UNIVERSITY OF JERUSALEM
MOUNT SCOPUS
JERUSALEM 91905
ISRAEL
E-MAIL: Pavel.Chigansky@mail.huji.ac.il

M. KLEPTSYNA
LABORATOIRE MANCEAU DE MATHÉMATIQUES
UNIVERSITÉ DU MAINE
AVENUE OLIVIER MESSIAEN
72085 LE MANS
FRANCE
E-MAIL: marina.kleptsyna@univ-lemans.fr