LOCAL LIMIT THEOREM AND EQUIVALENCE OF DYNAMIC AND STATIC POINTS OF VIEW FOR CERTAIN BALLISTIC RANDOM WALKS IN I.I.D. ENVIRONMENTS

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In this work, we discuss certain ballistic random walks in random environments on $\mathbb{Z}^d$, and prove the equivalence between the static and dynamic points of view in dimension $d \geq 4$. Using this equivalence, we also prove a version of a local limit theorem which relates the local behavior of the quenched and annealed measures of the random walk by a prefactor.

1. Introduction.

1.1. *Background.* Let $d \geq 1$. A random walk in a random environment (RWRE) on $\mathbb{Z}^d$ is defined by the following procedure: let $\mathcal{M}_d$ denote the space of all probability measures on $\mathcal{E}_d = \{\pm e_i\}_{i=1}^d$ (the standard unit coordinate vectors), and define $\Omega = (\mathcal{M}_d)^{\mathbb{Z}^d}$. An *environment* is an element $\omega \in \Omega$. For $x \in \mathbb{Z}^d$ and $e \in \mathcal{E}_d$, we denote by $\omega(x, e)$ the probability that the measure $\omega(x)$ gives to e. Let P be an i.i.d. measure on Ω, in the sense that $P = \nu^{\mathbb{Z}^d}$ for some probability measure ν on $\mathcal{M}_d$. Throughout this paper, we assume that P is uniformly elliptic, that is, there exists some $\eta > 0$ such that for every $e \in \mathcal{E}_d$

$$(1.1) \qquad P\big(\{\omega \in \Omega : \omega(0, e) \geq \eta\}\big) = \nu\big(\{\omega \in \mathcal{M}_d : \omega(e) \geq \eta\}\big) = 1.$$

For a given, fixed environment $\omega \in \Omega$ and $x \in \mathbb{Z}^d$, the *quenched* random walk on it (or the *quenched law*) is a time homogeneous Markov chain on $\mathbb{Z}^d$ with transition probabilities

$$P_\omega^x(X_{n+1} = y + e | X_n = y) = \omega(y, e) \qquad \forall y \in \mathbb{Z}^d, e \in \mathcal{E}_d$$

and initial distribution $P_\omega^x(X_0 = x) = 1$. We let $\mathbf{P}^x = P \otimes P_\omega^x$ be the joint law of the environment and the walk, and define the *annealed* (or *averaged*) law as its marginal on the space of trajectories

$$\mathbb{P}^x(\cdot) = \int_\Omega P_\omega^x(\cdot) \, dP(\omega).$$

We use the notation E, E_ω^x and $\mathbb{E}^x$ for the expectations of the measures P, P_ω^x and $\mathbb{P}^x$, respectively.

In [27, 29], Sznitman and Zerner proved that the limiting velocity of the random walk

$$\mathbb{v} = \lim_{n \to \infty} \frac{X_n}{n}$$

exists for P-almost every environment and P_ω^0-almost every trajectory of the random walk on it. A question to remain open, which in fact is one the most important open questions in the field, is whether the limiting velocity is an almost sure constant.

An important family of measures P for the model is given by the following definition.

DEFINITION 1.1.   The RWRE is said to be ballistic if the limiting velocity is a nonzero almost sure constant.

1.2. *Conditions for ballisticity.*   In [25, 26], Sznitman introduced two criteria for ballisticity of RWRE, called conditions (T) and (T'). In order to give a formal definition of these conditions, some preliminary definitions are needed.

DEFINITION 1.2.   Let $\ell \in S^{d-1} := \{x \in \mathbb{R}^d : \|x\|_2 = 1\}$ be a direction in $\mathbb{R}^d$.

(1) For $L > 0$ and a sequence $\{X_n\}$ (in $\mathbb{Z}^d$), define

$$T_L = T_L^{(\ell)}(\{X_n\}) = \inf\{n \geq 0 : \langle X_n, \ell \rangle \geq L\},$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^d$.

(2) Similarly, for a set $A \subset \mathbb{Z}^d$ and a sequence $\{X_n\}$ (in $\mathbb{Z}^d$), denote

$$T_A = T_A(\{X_n\}) = \inf\{n \geq 0 : X_n \in A\}.$$

We can now state the definition of Sznitman's ballisticity conditions.

DEFINITION 1.3.   (1) Given $0 < \gamma \leq 1$, we say that P satisfies condition (T_γ) in direction $\ell_0 \in S^{d-1}$ if for every $\ell \in S^{d-1}$ in some neighborhood of ℓ_0 there exists a finite constant C such that

$$\mathbb{P}^0(T_L^{(-\ell)} < T_L^{(\ell)}) < Ce^{-L^\gamma}.$$

(2) P is said to satisfy condition (T) if it satisfies condition (T_1).

(3) P is said to satisfy condition (T') if it satisfies condition (T_γ) for some $\frac{1}{2} < \gamma < 1$.

REMARK 1.4.   It was shown in [26] that all the conditions (T_γ) for $\frac{1}{2} < \gamma < 1$ are equivalent.

The relation between ballisticity and the above definition is given by the following theorem and conjecture.

THEOREM 1.5 (Sznitman [26]). *If condition (T') holds for some direction $\ell_0 \in S^{d-1}$, then the RWRE is ballistic, and the limiting velocity v satisfies $\langle v, \ell_0 \rangle > 0$. In addition, under this assumption, condition (T') holds for all $\ell \in S^{d-1}$ satisfying $\langle v, \ell \rangle > 0$.*

CONJECTURE (Sznitman). *Condition (T') is equivalent to ballisticity.*

In recent years, several improvements of Theorem 1.5 have been proved: in [10], Drewitz and Ramírez showed that for some constant $\gamma_d \in (0.366, 0.388)$ which is dimension dependent (T_γ) for $\gamma \in (\gamma_d, 1)$ are all equivalent. In [2], Theorem 1.4, Berger showed that in dimension $d \geq 4$ the condition (T_γ) for $\gamma \in (0, 1)$ implies ballisticity. In an additional work [11], Drewitz and Ramírez showed that in dimension $d \geq 4$ all the conditions (T_γ) for $\gamma \in (0, 1)$ are equivalent. In [5], Berger, Drewitz and Ramírez showed that in fact (fast enough) polynomial decay (see Definition 1.6 below) is equivalent to condition (T_γ) for any $0 < \gamma < 1$. Finally, in [9] Campos and Ramírez proved ballisticity for some nonuniformly elliptic environments satisfying (fast enough) polynomial decay.

DEFINITION 1.6 [Condition $(\mathscr{P})$]. Let N_0 be an even integer. For a coordinate direction $\ell = \ell_1$, let $\ell_2, \ldots, \ell_d$ be any fixed completion of ℓ_1 to an orthonormal basis of $\mathbb{R}^d$ and define

$$\text{Box}_x = \left\{ y \in \mathbb{Z}^d : -\frac{N_0}{2} < \langle y - x, \ell \rangle < N_0, \langle y - x, \ell_j \rangle < 25N_0^3 \ \forall 2 \leq j \leq d \right\},$$

$$\widetilde{\text{Box}}_x = \left\{ y \in \mathbb{Z}^d : \tfrac{1}{3} N_0 \leq \langle y - x, \ell \rangle < N_0, \langle y - x, \ell_j \rangle < N_0^3 \ \forall 2 \leq j \leq d \right\},$$

$$\partial \text{Box}_x = \left\{ y \in \mathbb{Z}^d \setminus \text{Box}_x : \exists z \in \text{Box}_x \text{ such that } \|y - z\|_1 = 1 \right\}$$

and

$$\partial_+ \text{Box}_x = \left\{ y \in \partial \text{Box}_x : \langle y - x, \ell \rangle \geq N_0, |\langle y - x, \ell_j \rangle| < 25N_0^3 \ \forall 2 \leq j \leq d \right\}.$$

Fix $M > 0$ and $\ell \in S^{d-1}$. We say that condition $\mathscr{P}_M|\ell$ is fulfilled if

$$\sup_{x \in \widetilde{\text{Box}}_0} \mathbb{P}^x(T_{\partial \text{Box}_0} \neq T_{\partial_+ \text{Box}_0}) < \frac{1}{N_0^M}$$

holds for some $N_0 \geq \exp(100 + 4d(\ln \eta)^2)$, where η is the ellipticity constant defined in (1.1). We say that condition $(\mathscr{P})$ holds in direction ℓ if condition $\mathscr{P}_M|\ell$ holds for some $M > 15d + 5$.

DEFINITION 1.7. Throughout this paper, we denote by $(\mathscr{P})$ the following equivalent conditions:

(1) (T').
(2) (T_γ) for some $\gamma \in (0, 1)$.
(3) (T_γ) for all $\gamma \in (0, 1)$.
(4) $(\mathscr{P})$.

1.3. *The environment viewed from the particle.* Let $\{X_n\}$ be a RWRE. The *environment viewed from the particle* is the discrete time process $\{\overline{\omega}_n\}$ defined on Ω by

$$\overline{\omega}_n = \sigma_{X_n}\omega,$$

where for $x \in \mathbb{Z}^d$ we denote by σ_x the shift in direction x of ω, that is, $\sigma_x\omega(y, \cdot) = \omega(x + y, \cdot)$ for every $y \in \mathbb{Z}^d$.

Beside the fact that the environment viewed from the particle process takes values in a compact space, it has the advantage of being Markovian, cf. [7], with respect to the transition kernel

$$(1.2) \qquad \Re g(\omega) = \sum_{e \in \mathcal{E}_d} \omega(0, e) g(\sigma_e \omega),$$

defined for every bounded measurable function $g : \Omega \to \mathbb{R}$.

It is natural to ask what are the invariant measures of the Markov chain $\{\overline{\omega}_n\}$.

DEFINITION 1.8. A probability measure Q on Ω is said to be invariant (or invariant with respect to the point of view of the particle), if for every bounded continuous function $g : \Omega \to \mathbb{R}$

$$(1.3) \qquad \int_\Omega \Re g(\omega) \, dQ(\omega) = \int_\Omega g(\omega) \, dQ(\omega).$$

One can find many examples for invariant measures with respect to the process $\{\overline{\omega}_n\}$. For example, every Dirac probability measure of any translation invariant environment provides such an example. One additional method to obtain invariant measures is by taking any sub-sequential limit of the Céasro means $\{\frac{1}{n}\sum_{k=0}^{n-1}\Re^k\nu\}$, where ν is any probability measure on Ω and $\Re\nu$ is the measure defined by the identity $\int_\Omega f(\omega) \, d(\Re\nu)(\omega) = \int_\Omega \Re f(\omega) \, d\nu(\omega)$ for every bounded measurable function $f : \Omega \to \mathbb{R}$.

As it turns out, an invariant measure Q is particularly useful when it is also equivalent to the original measure P. In this case, we say that the static point of view (the one related to P) is equivalent to the dynamic point of view (the one related to Q). If such a measure exists, it can be used to prove law of large numbers and central limit theorem type results; see, for example, [3, 8, 12, 14, 15, 17, 19, 24, 28] and the references therein.

The existence of an equivalent invariant measure was proved in several cases. In the one-dimensional case, the existence of an equivalent measure was proved by Alili [1]. In the reversible case, also known as random conductance model, the existence of an invariant equivalent measure is a well-known fact for most cases. For balanced RWRE, the existence of such a measure was proved by Lawler in [16]. Later on, this was strengthened to the case of balanced elliptic RWRE by Guo and Zeitouni in [13] and even further to the nonelliptic case (for genuinely d-dimensional measures) by Berger and Deuschel in [4]. For Dirichlet random walks,

a classification for the cases where such a measure exists was proved by Sabot in [23]. Finally, partial results in the ballistic case are also known; see Section 1.5 below.

The following result was proved by Kozlov in [15] (for the proof see also [8, 12]).

THEOREM 1.9 (Kozlov [15]). *Assume P is elliptic[3] and ergodic with respect to $\{\sigma_x\}_{x \in \mathbb{Z}^d}$. Assume there exists an invariant probability measure Q for the environment seen from the point of view of the particle which is absolutely continuous with respect to P. Then the following hold*:

(1) *Q is equivalent to P.*
(2) *The environment viewed from the particle with initial law Q is ergodic.*
(3) *Q is the unique invariant probability measure for the point of view of the particle which is absolutely continuous with respect to P.*
(4) *The Céasro means $\{\frac{1}{N+1} \sum_{k=0}^{N} \mathfrak{R}^k P\}$ converge weakly to Q.*

1.4. *Main goal.* This paper has two purposes. The first is to prove the equivalence of the dynamic and static point of views under condition ($\mathscr{P}$), uniform ellipticity and the additional assumption that $d \geq 4$. The second purpose of this paper is to prove a certain type of local limit theorem relating the quenched and annealed laws by a prefactor.

1.5. *Known results in the strongly ballistic case.* Let $d \geq 2$. In [25], Sznitman proved an annealed CLT under condition (T'). The ideas he presented may also be used to prove an annealed local CLT. For completeness, we present a proof of the annealed local CLT in the Appendix. In [7], Bolthausen and Sznitman proved the equivalence of the static and dynamic point of views for certain (nonnestling) ballistic random walks in random environment, when $d \geq 4$ and the disorder is low. In [19], Rassoul-Agha proved the existence of an equivalent invariant measure on half spaces under Kalikow's condition, mixing and uniform ellipticity. In [6], Berger and Zeitouni and in [20–22] Rassoul-Agha and Seppäläinen proved quenched invariance principle under moments assumptions for the first regeneration time. In particular, a quenched CLT holds under condition ($\mathscr{P}$).

1.6. *Main results.* Our two main results are the following.

THEOREM 1.10. *Let $d \geq 4$ and assume P is uniformly elliptic, i.i.d. and satisfies condition ($\mathscr{P}$). Then there exists a unique probability measure Q on the space of environments which is invariant with respect to the point of view of the particle and is equivalent to the original measure P. In addition, $E[(\frac{dQ}{dP})^k] < \infty$ for every $k \in \mathbb{N}$.*

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[3] P is called elliptic if $P(\min_{e \in \mathcal{E}_d} \omega(x, e) > 0) = 1 \ \forall x \in \mathbb{Z}^d$.

THEOREM 1.11. *Let $d \geq 4$ and assume P is uniformly elliptic, i.i.d. and satisfies condition ($\mathscr{P}$). Then there exists a unique measurable, nonnegative function $f \in L^1(\Omega, P)$ such that for P-almost every $\omega \in \Omega$*

$$(1.4) \qquad \lim_{n \to \infty} \sum_{x \in \mathbb{Z}^d} \left| P_\omega^0(X_n = x) - \mathbb{P}^0(X_n = x) f(\sigma_x \omega) \right| = 0.$$

This unique function f is the Radon–Nikodym derivative $\frac{dQ}{dP}$ of the probability measure Q constructed in Theorem 1.10.

1.7. *Remarks about lower dimensions.* In this paper, we only prove Theorem 1.10 and Theorem 1.11 in dimension 4 or higher. Here, we wish to remark about the situation in lower dimensions.

For $d = 1$, the existence of an equivalent measure which is invariant with respect to the point of view of the particle was proved by Alili; see [1].

We conjecture that similar results should hold in dimension 3. In fact, the only place in the proof where we directly use the condition $d \geq 4$ is in [2], Lemma 4.10; see also Lemma 2.12 below. On the other hand, we believe that in dimension 2 an equivalent probability measure which is invariant with respect to the point of view of the particle does not exist.

1.8. *Structure of the paper and general remarks.* In Section 2, we recall some of the notation from [2] as well as some of the result obtained there. In addition, we prove a slightly different version of [2], Lemma 4.2, thus giving annealed estimations for a fixed time. In Section 3, we generalize [2], Proposition 4.5, which gives an upper bound on the difference between the annealed and quenched distribution, to include estimations on the exit time of the box. Section 4 is devoted to converting the estimation obtained in Section 3 for $(d - 1)$-dimensional cubes in a time interval into a result about d-dimensional cubes in a fixed time. In Section 5, we bootstrap the result for large d-dimensional cubes obtained in Section 4 all the way to boxes of finite size. Section 6 is devoted to the proof of the first main result, the existence of an equivalent probability measure on the space of environments which is invariant with respect to the point of view of the particle. Finally, in Section 7 we prove the second main result regarding the existence of a prefactor.

Throughout this paper, the value of constants c and C may change from one line to the next. Numbered constants, such as $c_1, c_2, \ldots$ are fixed according to their first appearance in the text. Expectation with respect to a measure μ which is not P, P_ω or $\mathbb{P}$ is denoted by E_μ. Finally, some of the inequalities may only hold for large enough values of N, n and M.

2. Notation and other preliminary results. We start by recalling some of the notation and results from [2] to be used throughout the paper. In addition, we cite an inequality by McDiarmid for future use and state analogous result to [2], Lemma 4.2, for the annealed measure in a fixed time.

For $k, N \geq 0$, define $R_k(N) = \lfloor e^{(\log N)^{(k+2)/(k+3)}} \rfloor$ and denote $R(N) = R_1(N)$. Note that $R_0(N) = \lfloor \log N \rfloor$ and that for every $k, n \geq 0$, and every large enough N,

$$R_k^n(N) := (R_k(N))^n \leq R_{k+1}(N) < N.$$

Let

$$\vartheta = \lim_{n \to \infty} \frac{X_n}{\|X_n\|_2}$$

be the direction of the speed. We assume without loss of generality that $\langle \vartheta, e_1 \rangle > 0$ and note that due to the results of [25, 26], this implies that $(\mathscr{P})$ holds both in direction ϑ and in direction e_1.

DEFINITION 2.1. For $k \in \mathbb{N}$, define H_k to be the hyperplane $H_k = \{x \in \mathbb{Z}^d : \langle x, e_1 \rangle = k\}$.

DEFINITION 2.2. By the term $N^{-\xi(1)}$, we mean a nonnegative function of $N \in \mathbb{N}$ which decays faster than any polynomial, that is, if $f(N) = N^{-\xi(1)}$, then for every $k \in \mathbb{N}$

$$\lim_{N \to \infty} N^k f(N) = 0.$$

Note that $N^{-\xi(1)}$ is independent of the environment unless otherwise stated.

DEFINITION 2.3. For two nonempty sets $A, B \subset \mathbb{Z}^d$, we define $\text{dist}(A, B) = \min\{\|x - y\|_1 : x \in A, y \in B\}$. If $A = \{x\}$ we write $\text{dist}(x, B)$ instead of $\text{dist}(\{x\}, B)$.

DEFINITION 2.4. For $x = (x_1, \ldots, x_d) \in \mathbb{Z}^d$ and $n \in \mathbb{N}$, we denote $x \leftrightarrow n$ if x and n have the same parity, that is, $\sum_{i=1}^d x_i + n$ is even. In a similar way for $x, y \in \mathbb{Z}^d$, we denote $x \leftrightarrow y$ if $\sum_{i=1}^d (x_i + y_i)$ is even.

DEFINITION 2.5. Recall that for $x \in \mathbb{Z}^d$ we denote by σ_x the shift in direction x in ω, that is, $\sigma_x \omega(y, \cdot) = \omega(x + y, \cdot)$ for every $y \in \mathbb{Z}^d$.

DEFINITION 2.6. For $z \in \mathbb{Z}^d$ and $N \in \mathbb{N}$, we define (see also Figure 1):

(1) the parallelogram of size N and center z to be

$$\mathcal{P}(z, N) = \left\{ x \in \mathbb{Z}^d : \left| \langle x - z, e_1 \rangle \right| < N^2, \right.$$

$$\left. \left\| x - z - \vartheta \cdot \frac{\langle x - z, e_1 \rangle}{\langle \vartheta, e_1 \rangle} \right\|_\infty < N R_5(N) \right\}.$$
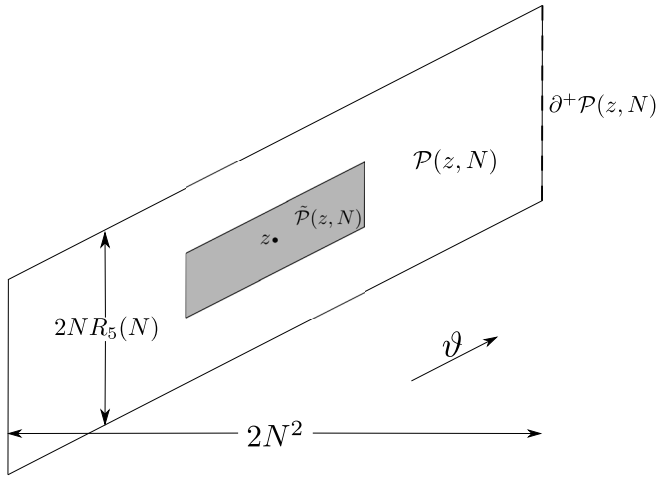
FIG. 1. *The basic block $\mathcal{P}(z, N)$: the box $\tilde{\mathcal{P}}(z, N)$ is in gray and the right boundary $\partial^+\mathcal{P}(z, N)$ is the dashed line.*

(2) The middle third of $\mathcal{P}(z, N)$
$$\tilde{\mathcal{P}}(z, N) = \left\{ x \in \mathbb{Z}^d : |\langle x - z, e_1 \rangle| < \frac{1}{3}N^2, \right.$$
$$\left. \left\| x - z - \vartheta \cdot \frac{\langle x - z, e_1 \rangle}{\langle \vartheta, e_1 \rangle} \right\|_\infty < \frac{1}{3}NR_5(N) \right\}.$$

(3) The boundary of $\mathcal{P}(z, N)$
$$\partial\mathcal{P}(z, N) = \left\{ x \in \mathbb{Z}^d \setminus \mathcal{P}(z, N) : \exists y \in \mathcal{P}(z, N) \text{ s.t. } \|x - y\|_1 = 1 \right\}.$$

(4) The right boundary of $\mathcal{P}(z, N)$
$$\partial^+\mathcal{P}(z, N) = \left\{ x \in \partial\mathcal{P}(z, N) : \langle x - z, e_1 \rangle = N^2 \right\}.$$

2.1. *Regeneration times.*

DEFINITION 2.7.   Let $\{X_n\}$ be a nearest-neighbor sequence in $\mathbb{Z}^d$, and let $\ell \in S^{d-1}$ be a direction. We say that t is a regeneration time for $\{X_n\}$ in direction ℓ if the following holds:

(1) $\langle X_s, \ell \rangle < \langle X_t, \ell \rangle$ for every $s < t$.
(2) $\langle X_{t+1}, \ell \rangle > \langle X_t, \ell \rangle$.
(3) $\langle X_s, \ell \rangle > \langle X_{t+1}, \ell \rangle$ for every $s > t + 1$.

The following theorem summarize the results on the regeneration time structure.

THEOREM 2.8 ([26, 27]).   *Assume that P satisfies (T_γ) in direction ℓ_0 for some $\gamma > 0$. Then*:

(1) *With probability one, there exist infinitely many regeneration times, which we denote by $\tau_1 < \tau_2 < \cdots$.*

(2) *The ensemble*

$$\{(\tau_{n+1} - \tau_n, X_{\tau_{n+1}} - X_{\tau_n})\}_{n \geq 1}$$

is an i.i.d. ensemble under the annealed measure.

(3) *There exists $C > 0$ such that for every $n \in \mathbb{N}$*

$$\mathbb{P}(\tau_2 - \tau_1 = n) \leq C\mathbb{P}(\tau_1 = n),$$

and for every $y \in \mathbb{Z}^d$

$$\mathbb{P}(X_{\tau_2} - X_{\tau_1} = y) \leq C\mathbb{P}(X_{\tau_1} = y).$$

(4) *There exists $c > 0$ such that for every n,*

$$\mathbb{P}\big(\exists k \leq \tau_1 : \|X_k\|_\infty > n\big) \leq e^{-cn^\gamma}.$$

The following is the main technical statement from [2].

THEOREM 2.9 ([2], Proposition 2.2). *If $d \geq 4$, and P satisfies condition $(\mathscr{P})$ in one of the $2d$-principal directions, then for every $\alpha < d$*

$$\mathbb{P}(\tau_1 > k) \leq \exp(-(\log t)^\alpha).$$

COROLLARY 2.10. *For $N \in \mathbb{N}$ denote by $B_N = B_N(\{X_n\})$, the event*

$$B_N(\{X_n\}) = \{\forall 1 \leq k \leq N^2 : \tau_k - \tau_{k-1} \leq R(N)\},$$

where $\tau_0 = 0$. Then $\mathbb{P}(B_N) \geq 1 - N^{-\xi(1)}$.

REMARK 2.11. Note that the event $B_N(\{X_n\})$ implies the event that the distance traveled between two regeneration times is bounded by $R(N)$ as well, that is,

$$A_N(\{X_n\}) = \{\forall 1 \leq k \leq N^2 : \max\{\|X_t - X_{\tau_{k-1}}\|_\infty : \tau_{k-1} \leq t \leq \tau_k\} \leq R(N)\}$$

satisfies $B_N(\{X_n\}) \subset A_N(\{X_n\})$ and in particular $\mathbb{P}(A_N) \geq 1 - N^{-\xi(1)}$.

2.2. *Intersections of paths of random walks.* The following lemma estimates the number of intersections of two independent random walks in dimension $d \geq 4$. This is in fact the only place in the proof where the assumption $d \geq 4$ is used explicitly. Denote by $P_{\omega,\omega}^{z,z}$, $E_{\omega,\omega}^{z,z}$ the law (resp., expectation) of two random walks on the same environment ω, which conditioned on ω evolve independently according to the quenched law of ω starting from z.

LEMMA 2.12 ([2], Lemma 4.10).   *Let $d \geq 4$ and assume P is uniformly elliptic, i.i.d. and satisfies $(\mathscr{P})$. Let $X^{(1)} = \{X_n^{(1)}\}$ and $X^{(2)} = \{X_n^{(2)}\}$ be two independent random walks running in the same environment ω. For $i \in \{1, 2\}$, let $[X^{(i)}]$ be the set of points visited by $X^{(i)}$. Then*

$$P\big(\{\omega \in \Omega : E_{\omega,\omega}^{0,0}\big[|[X^{(1)}] \cap [X^{(2)}] \cap \mathcal{P}(0, N)|\mathbb{1}_{A_N(\{X_n^{(1)}\}) \cap A_N(\{X_n^{(2)}\})}\big] \geq R_2(N)\}\big)$$

$$= N^{-\xi(1)}.$$

For future use, we denote

(2.1)
$$J(N) = \big\{\omega \in \Omega : E_{\omega,\omega}^{z,z}\big[|[X^{(1)}] \cap [X^{(2)}] \cap \mathcal{P}(0, N)|$$
$$\times \mathbb{1}_{A_N(\{X_n^{(1)}\}) \cap A_N(\{X_n^{(2)}\})}\big] \leq R_2(N), \ \forall z \in \tilde{\mathcal{P}}(0, N)\big\}.$$

Therefore, due to the last lemma, we have $P(J(N)) = 1 - N^{-\xi(1)}$.

2.3. *McDiarmid's inequality.*   The following Azuma type inequality, proved by McDiarmid in [18], is used in Section 3.

THEOREM 2.13 ([18], Theorem 3.14).   *Let $\{M_k\}_{k=0}^n$ be a martingale with respect to a probability measure $\mathbb{P}$, given by $M_k = E_{\mathbb{P}}[X|\mathscr{F}_k]$, with $M_0 = E_{\mathbb{P}}[X]$. For $1 \leq k \leq n$ let $U_k = \mathrm{esssup}(|M_k - M_{k-1}||\mathscr{F}_{k-1})$ and define $U = \sum_{k=1}^n U_k^2$. Then*

$$\mathbb{P}\big(|M_n - M_0| > \alpha, U \leq c\big) \leq 2e^{-\alpha^2/(2c)}.$$

2.4. *Annealed estimation for a fixed time.*   In this subsection, we state some standard estimations on the annealed measure of the random walk. The proof is a standard and straightforward use of Fourier transform techniques applied to the regeneration structure described in Section 2.1. The first three claims are proved in a very similar way to the proof of [2], Lemma 4.2 (see also Lemma 3.3 for another version). The formal statement is the following.

LEMMA 2.14.   *Assume that P is uniformly elliptic, i.i.d. and satisfies $(\mathscr{P})$. Then for large enough $n \in \mathbb{N}$ and $x, y, z, w \in \mathbb{Z}^d$ such that $\|x - y\|_1 = 1$, $\|z - w\|_1 = 1$*

(2.2)
$$\mathbb{P}^z(X_n = x) \leq Cn^{-d/2},$$

(2.3)
$$\big|\mathbb{P}^z(X_n = x) - \mathbb{P}^z(X_{n+1} = y)\big| \leq Cn^{-(d+1)/2},$$

(2.4)
$$\big|\mathbb{P}^z(X_n = x) - \mathbb{P}^w(X_{n+1} = x)\big| \leq Cn^{-(d+1)/2}.$$

In addition, for every $\varepsilon > 0$ and every partition $\Pi_n^{(\varepsilon)}$ of $\mathbb{Z}^d$ into boxes of side length n^ε.

(2.5)
$$\sum_{\Delta \in \Pi_n^{(\varepsilon)}} \sum_{\substack{x \in \Delta \\ x \leftrightarrow n}} \Big[\max_{y \in \Delta} \mathbb{P}^0(X_n = y) - \mathbb{P}^0(X_n = x)\Big] \leq Cn^{-1/2 + 3d\varepsilon}.$$

The proof of Lemma 2.14 can be found in Appendix A.2.

Before turning to the last estimation of this subsection, we state here a very simple claim to be used in several places.

CLAIM 2.15. *Let A be an event in the σ-algebra of $(\mathbb{Z}^d)^{\mathbb{N}}$ and assume that $\mathbb{P}(A) \leq \varepsilon$, then $P(\{\omega \in \Omega : P_\omega(A) \geq \sqrt{\varepsilon}\}) \leq \sqrt{\varepsilon}$. In particular, if a sequence of events $\{A_N\}$ satisfies $\mathbb{P}(A_N) = 1 - N^{-\xi(1)}$, then $P(\{\omega \in \Omega : P_\omega(A_N) = 1 - N^{-\xi(1)}\}) = 1 - N^{-\xi(1)}$.*

PROOF. Define the random variable $X : \Omega \to [0, 1]$ by $X(\omega) = P_\omega(A)$. By the Markov inequality, $P(X(\omega) \geq \sqrt{\varepsilon}) \leq \frac{E[X(\omega)]}{\sqrt{\varepsilon}} = \frac{\mathbb{P}(A)}{\sqrt{\varepsilon}} \leq \frac{\varepsilon}{\sqrt{\varepsilon}} = \sqrt{\varepsilon}$. $\square$

Next, we show that the location of the walk at time n is concentrated in a box which is a bit larger than $\sqrt{n}$. More formally, we have the following.

LEMMA 2.16. *Assume that P is uniformly elliptic, i.i.d. and satisfies $(\mathscr{P})$. Then*:

(1) $\mathbb{P}^z(\|X_n - \mathbb{E}^z[X_n]\|_\infty > \sqrt{n}R_5(n)) \leq e^{-R_5(n)} = n^{-\xi(1)}$,

(2) $P(\{\omega \in \Omega : P_\omega^z(\|X_n - \mathbb{E}^z[X_n]\|_\infty > \sqrt{n}R_5(n)) \leq e^{-(1/2)R_5(n)}\}) = 1 - n^{-\xi(1)}$,

(3) *For every $\delta > 0$ there exists $C > 0$ such that $\mathbb{P}^z(\|X_n - \mathbb{E}^z[X_n]\|_\infty > C\sqrt{n}) < \delta$.*

The proof of Lemma 2.16 can be found in Appendix A.2.

3. Adding time estimation. The goal of this section is to prove a generalized version of [2], Proposition 4.5. The original lemma gives a bound on the difference between the probability measures $\mathbb{P}^z(X_{T_{\partial\mathcal{P}(0,N)}} \in \cdot)$ and $P_\omega^z(X_{T_{\partial\mathcal{P}(0,N)}} \in \cdot)$ to hit any cube in a partition of $\partial^+\mathcal{P}(0, N)$ into cubes of side length N^θ, for any $0 < \theta \leq 1$. This estimation immediately implies that the total variation of the two measures goes to zero as N goes to infinity. Here, we show that if an estimation on the hitting time $T_{\partial\mathcal{P}(0,N)}$ is added, then a similar estimation can be derived. More formally, we have the following.

PROPOSITION 3.1. *Let $d \geq 4$ and assume P is uniformly elliptic, i.i.d. and satisfies $(\mathscr{P})$. For every $0 < \theta \leq 1$, let $F(N) = F(N, \theta)$ be the event that for every $z \in \tilde{\mathcal{P}}(0, N)$, every cube Δ of side length N^θ which is contained in $\partial^+\mathcal{P}(0, N)$ and every interval I of length N^θ*

$$\left| P_\omega^z(X_{T_{\partial\mathcal{P}(0,N)}} \in \Delta, T_{\partial\mathcal{P}(0,N)} \in I) - \mathbb{P}^z(X_{T_{\partial\mathcal{P}(0,N)}} \in \Delta, T_{\partial\mathcal{P}(0,N)} \in I) \right|$$

$$\leq CN^{-d(1-\theta)-((d-2)/(d+2))\theta}.$$

Then $P(F(N)) = 1 - N^{-\xi(1)}$.

The proof of Proposition 3.1 follows the one of [2], Proposition 4.5 (see also [2], Section 4, and in particular Lemma 4.15). Here are the main steps of the proof: the proof starts with another version for annealed derivatives bounds (see Lemma 3.3). Next, in Lemma 3.4 we prove an annealed concentration inequality for the hitting time $T_{\partial \mathcal{P}(0,N)}$. Lemma 3.5 provides a first weak estimation for the difference between the quenched and annealed hitting probabilities for large enough boxes, that is, $\theta > \frac{d}{d+1}$. Using induction and the estimation from the last lemma, we prove an upper bound on the probability to hit a given box of side length N^θ in a time interval of length N^θ for every $0 < \theta \leq 1$ (see Lemma 3.6). In Lemma 3.7, we use the upper quenched estimations in order to show that the difference between the quenched and annealed hitting probabilities, in a slightly further hyperplane are as required. Finally, in the proof of Proposition 3.1, we show how to translate the estimations from the further hyperplane back to the original hyperplane. The first main tool used in the proof is an environment exposure procedure, which in the context of ballistic RWRE already appeared in the work of Bolthausen and Sznitman [7]. This exposure procedure defines a martingale and allows the use of Azuma's and McDiarmid's inequalities. The second main tool is the intersection estimate for two independent random walks from Lemma 2.12.

REMARK 3.2.   In Section 4, we use Proposition 3.1 for boxes whose side length is only asymptotic to N^θ (for some $0 < \theta < 1$), that is, the side length is $N^\theta + o(N^\theta)$. One can verify that the same proof holds for such boxes as well.

We start by stating another version for the estimation on the annealed measure (see Lemma 2.14 and [2], Lemma 4.2).

LEMMA 3.3 (Annealed derivative estimations).   *Assume P is uniformly elliptic, i.i.d. and satisfies $(\mathscr{P})$. Fix $z_1 \in \mathbb{Z}^d$, $N \in \mathbb{N}$ and let $z \in \tilde{\mathcal{P}}(z_1, N)$. Let $\{X_n\}$ be an RWRE starting at z. Then for large enough N:*

(1) *For every $m \in \mathbb{N}$ and every $x \in \partial^+ \mathcal{P}(z_1, N)$*

$$(3.1) \qquad\qquad \mathbb{P}^z(T_{\partial \mathcal{P}(0,N)} = m, X_{T_{\partial \mathcal{P}(0,N)}} = x) < CN^{-d}.$$

(2) *For every $m \in \mathbb{N}$ and every $x, y \in \partial^+ \mathcal{P}(z_1, N)$ such that $\|x - y\|_1 = 1$*

$$(3.2) \quad \begin{aligned} &\big|\mathbb{P}^z(T_{\partial \mathcal{P}(0,N)} = m, X_{T_{\partial \mathcal{P}(0,N)}} = x) - \mathbb{P}^z(T_{\partial \mathcal{P}(0,N)} = m+1, X_{T_{\partial \mathcal{P}(0,N)}} = y)\big| \\ &\quad < CN^{-d-1}. \end{aligned}$$

(3) *For every $m \in \mathbb{N}$, every $x \in \partial^+ \mathcal{P}(z_1, N)$ and every $1 \leq j \leq d$*

$$(3.3) \quad \begin{aligned} &\big|\mathbb{P}^z(T_{\partial \mathcal{P}(0,N)} = m, X_{T_{\partial \mathcal{P}(0,N)}} = x) - \mathbb{P}^{z+e_j}(T_{\partial \mathcal{P}(0,N)} = m+1, X_{T_{\partial \mathcal{P}(0,N)}} = x)\big| \\ &\quad < CN^{-d-1}. \end{aligned}$$

The proof of Lemma 3.3 can be found in Appendix A.2.

Next, we prove an annealed concentration inequality for the hitting time $T_{\partial \mathcal{P}}$.

LEMMA 3.4. *Let $d \geq 4$ and assume P is uniformly elliptic, i.i.d. and satisfies* $(\mathscr{P})$. *Then*

$$(3.4) \qquad \mathbb{P}^z(T_{\partial \mathcal{P}(0,N)} \neq T_{\partial^+ \mathcal{P}(0,N)}) = N^{-\xi(1)}$$

and for every $z \in \tilde{\mathcal{P}}(0, N)$

$$(3.5) \qquad \mathbb{P}^z(|T_{\partial \mathcal{P}(0,N)} - \mathbb{E}^z[T_{\partial \mathcal{P}(0,N)}]| > N R_2(N)) = N^{-\xi(1)}.$$

PROOF. The fact that (3.4) holds was proved in [2], Lemma 4.2(1). For (3.5), we first show that $|\mathbb{E}^z[\tau_k] - \mathbb{E}^z[\tau_k | B_N]| = N^{-\xi(1)}$ for every $1 \leq k \leq N^2$, where B_N is as defined in Corollary 2.10. Indeed, using the notation $\tau_0 = 0$, for every $1 \leq k \leq N^2$

$$\mathbb{E}^z[|\tau_k - \tau_r k - 1| \cdot \mathbb{1}_{B_N^c}]$$
$$\leq \mathbb{E}^z[|\tau_k - \tau_{k-1}| \cdot \mathbb{1}_{\exists j \neq k |\tau_j - \tau_{j-1}| \geq R(N)}] + \mathbb{E}^z[|\tau_k - \tau_{k-1}| \cdot \mathbb{1}_{|\tau_k - \tau_{k-1}| \geq R(N)}]$$
$$\leq \mathbb{E}^z[|\tau_k - \tau_{k-1}|]\mathbb{P}^z(B_N^c) + \sum_{t > R(N)} t \cdot \mathbb{P}^z(|\tau_k - \tau_{k-1}| = t)$$
$$\leq \mathbb{E}^z[|\tau_k - \tau_{k-1}|]\mathbb{P}^z(B_N^c) + \sum_{t > R(N)} t \cdot \exp(-(\log t)^\alpha) = N^{-\xi(1)},$$

where for the last inequality we used Theorem 2.8 and for the last equality we used Corollary 2.10. Therefore, for every $1 \leq k \leq N^2$

$$|\mathbb{E}^z[\tau_k - \tau_{k-1} | B_N] - \mathbb{E}^z[\tau_k - \tau_{k-1}]|$$
$$\leq |\mathbb{E}^z[\tau_k - \tau_{k-1} | B_N] - \mathbb{E}^z[(\tau_k - \tau_{k-1})\mathbb{1}_{B_N}]|$$
$$\quad + |\mathbb{E}^z[(\tau_k - \tau_{k-1})\mathbb{1}_{B_N}] - \mathbb{E}^z[\tau_k - \tau_{k-1}]|$$
$$= |(1 - \mathbb{P}^z(B_N))\mathbb{E}^z[(\tau_k - \tau_{k-1}) | B_N]| + |\mathbb{E}^z[(\tau_k - \tau_{k-1})\mathbb{1}_{B_N^c}]|$$
$$= \mathbb{P}^z(B_N^c)\mathbb{E}^z[(\tau_k - \tau_{k-1}) | B_N] + \mathbb{E}^z[(\tau_k - \tau_{k-1})\mathbb{1}_{B_N^c}]$$
$$\leq R(N)\mathbb{P}^z(B_N^c) + N^{-\xi(1)} = N^{-\xi(1)}.$$

Summing the differences $\{\mathbb{E}^z[\tau_j - \tau_{j-1} | B_N] - \mathbb{E}^z[\tau_j - \tau_{j-1}]\}_{j=1}^k$ gives

$$|\mathbb{E}^z[\tau_k | B_N] - \mathbb{E}^z[\tau_k]| = N^{-\xi(1)}, \qquad 1 \leq k \leq N^2.$$

Since we know that $\mathbb{P}^z(B_N) = 1 - N^{-\xi(1)}$ (see Corollary 2.10), it is enough to show that

$$\mathbb{P}^z(|T_{\partial \mathcal{P}(0,N)} - \mathbb{E}^z[T_{\partial \mathcal{P}(0,N)}]| > N R_2(N) | B_N) = N^{-\xi(1)}.$$

Under the event B_N, there exist some $1 \leq k \leq N^2$ such that $\tau_k \leq T_{\partial P} \leq \tau_k + R(N)$ and thus (using the first estimation)

$$\mathbb{P}^z\big(\big|T_{\partial \mathcal{P}(0,N)} - \mathbb{E}^z[T_{\partial \mathcal{P}(0,N)}]\big| > N R_2(N)|B_N\big)$$

$$\leq \sum_{k=1}^{N^2} \mathbb{P}^z\bigg(\big|\tau_k - \mathbb{E}^z[\tau_k]\big| > \frac{1}{2} N R_2(N)\Big|B_N\bigg)$$

$$\leq \sum_{k=1}^{N^2} \mathbb{P}^z\bigg(\big|\tau_k - \mathbb{E}^z[\tau_k|B_N]\big| > \frac{1}{4} N R_2(N)\Big|B_N\bigg) + N^{-\xi(1)}.$$

Note that conditioned on B_N the first N regenerations are still i.i.d., so by Azuma's inequality this can be bounded by

$$\sum_{k=1}^{N^2} 2\exp\bigg(-\frac{N^2 R_2^2(N)}{32 k R^2(N)}\bigg) + N^{-\xi(1)} \leq e^{-R_2(N)} + N^{-\xi(1)} = N^{-\xi(1)}. \qquad \square$$

LEMMA 3.5. *Let $d \geq 4$ and assume P is uniformly elliptic, i.i.d. and satisfies ($\mathscr{P}$). Fix $0 < \theta \leq 1$. Let $L(N) = L(\theta, N)$ be the event that for every $\frac{2}{5}N^2 \leq M \leq N^2$, every $z \in \tilde{\mathcal{P}}(0,N)$, every $(d-1)$-dimensional cube Δ of size N^θ which is contained in H_M and every interval $I \subset \mathbb{N}$ of length N^θ*

$$\big|P_\omega^z(X_{T_M} \in \Delta, T_M \in I, B_N) - \mathbb{P}^z(X_{T_M} \in \Delta, T_M \in I, B_N)\big| \leq N^{d(\theta-1)}.$$

Then for $\theta > \frac{d}{d+1}$, $P(L(\theta, N)) = 1 - N^{-\xi(1)}$.

PROOF.   Fix θ, and let $\frac{d}{d+1} < \theta' < \theta$. Let $V = [N^{2\theta'}]$. Fix $\frac{2}{5}N^2 \leq M \leq N^2$, $v \in H_{M+V}$ and $m \in \mathbb{N}$. Finally denote by $\mathcal{G}$ the σ-algebra determined by the configuration on

$$\mathcal{P}^M(0, N) := \mathcal{P}(0, N) \cap \{x : \langle x, e_1 \rangle \leq M\}.$$

We are interested in the quantity (see also Figure 2)

$$J^{(M)}(v, m) = E\big[P_\omega^z(X_{T_{M+V}} = v, T_{M+V} = m, B_N)|\mathcal{G}\big].$$

In order to estimate $J^{(M)}(v, m)$ we order the vertices of $\mathcal{P}^M(0, N)$ lexicographically, $x_1, x_2, \ldots$ with the first coordinate being the most significant and let $\{\mathcal{F}_k\}$ be the σ-algebra determined by $\omega(x_1, \cdot), \ldots, \omega(x_k, \cdot)$, so in particular for every $-N^2 + 1 \leq l \leq M$ the vertices in $H_l \cap \mathcal{P}^M(0, N)$ are exposed after those in $H_{l-1} \cap \mathcal{P}^M(0, N)$.

Consider the martingale $M_k = E[P_\omega^z(X_{T_{M+V}} = v, T_{M+V} = m|B_N)|\mathcal{F}_k]$. In order to use McDiarmid's inequality, we first bound $U_k := \text{esssup}(|M_k - M_{k-1}||\mathcal{F}_{k-1})$. We claim that for an event of environments with P probability $\geq 1 - N^{-\xi(1)}$

$$U_k \leq C R(N) E\big[P_\omega^z(x_k \text{ is visited }|B_N)|\mathcal{F}_{k-1}\big]V^{-(d+1)/2}.$$
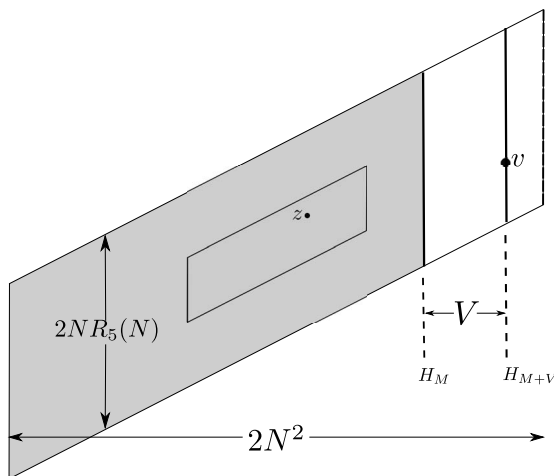
FIG. 2. *The quantity $J^{(M)}(v, m)$ is the probability of hitting the point v at time m, conditioned on the environments in the gray area, and averaged (annealed) elsewhere. The small parallelogram indicates the middle third $\widetilde{\mathcal{P}}(0, N)$ in which the random walk starts.*

Indeed, let ω' be an environment that agrees with ω everywhere except possibly in x_k. Let P be the probability measure under which the random walk has quenched transition probabilities given by ω in $\{x_i : i \leq k\}$ and averaged (annealed) transition probabilities in $\mathbb{Z}^d \setminus \{x_i : i \leq k\}$ conditioned the event B_N. Similarly, let P' be the probability measure defined like P with ω' instead of ω. More formally for an event $A \subset (\mathbb{Z}^d)^{\mathbb{N}}$, we have $P(A) = E[P_\omega^z(A|B_N)|\mathcal{F}_k]$ and equivalently for P'. Then

$$U_k \leq \sup_{\omega'} |P'(X_{T_{M+V}} = v, T_{M+V} = m) - P(X_{T_{M+V}} = v, T_{M+V} = m)|$$

$$\leq \sup_{\omega'} |P'(X_{T_{M+V}} = v, T_{M+V} = m, \{x_k \text{ is not visited}\})$$

$$- P(X_{T_{M+V}} = v, T_{M+V} = m, \{x_k \text{ is not visited}\})|$$

$$+ \sup_{\omega'} |P'(X_{T_{M+V}} = v, T_{M+V} = m, \{x_k \text{ is visited}\})$$

$$- P(X_{T_{M+V}} = v, T_{M+V} = m, \{x_k \text{ is visited}\})|,$$

where the supremum is taken over all environments ω' that agree with ω on $\mathbb{Z}^d \setminus \{x_k\}$. Note that on the event $\{x_k$ is not visited$\}$, the distributions P and P' are the same and, therefore, the difference of the probabilities equals zero. On the other hand, on the event $\{x_k$ is visited$\}$, we can couple both walks together until the first hitting time of x_k (which in particular implies that the hitting time of x_k is the same). Since we conditioned on the event B_N, the next regeneration time after hitting x_k is at most $R(N)$ steps later. Therefore, from Lemma 3.3 it follows that

$$|P^z(X_{T_{M+V}} = v, T_{M+V} = m | x_k \text{ is visited}) - \mathbb{P}^{x_k}(X_{T_{M+V}} = v, T_{M+V} = m)|$$

$$< CR(N)V^{-(d+1)/2}$$

and

$$\left| \mathbb{P}'^z(X_{T_{M+V}} = v, T_{M+V} = m | x_k \text{ is visited}) - \mathbb{P}^{x_k}(X_{T_{M+V}} = v, T_{M+V} = m) \right|$$
$$< CR(N)V^{-(d+1)/2}.$$

Consequently, we get

$$U_k \leq CR(N)V^{-(d+1)/2}\mathbb{P}^z(x_k \text{ is visited})$$

as required.

Next, we show that $U := \sum_k U_k^2$ is bounded by $CR_2^2(N)V^{-d-1}$ provided $\omega \in J(N)$. Indeed, noting that if x is visited and B_N holds, then the first visit to the layer $H_{\langle x,e_1\rangle-1}$ is in the box $B(x) = \{y : y \in H_{\langle x,e_1\rangle-1}, \|y - x\|_\infty \leq R(N)\}$ it follows that

$$U_k \leq CR(N)V^{-(d+1)/2}\mathbf{P}^z(x_k \in [X]|\mathcal{F}_{k-1}, B_N)$$
$$\leq CR(N)V^{-(d+1)/2} \sum_{y\in B(x_k)} \mathbf{P}^z(T_{\langle x,e_1\rangle-1} = y|\mathcal{F}_{k-1})$$
$$= CR(N)V^{-(d+1)/2} \sum_{y\in B(x_k)} P_\omega^z(T_{\langle x,e_1\rangle-1} = y)$$
$$\leq CR(N)V^{-(d+1)/2} \sum_{y\in B(x_k)} P_\omega^z(y \in [X]).$$

Since $|B(x_k)| \leq C \cdot 2^d \cdot R^d(N)$ and every $y \in \mathbb{Z}^d$ is in $B(x)$ for at most $2^d R^d(N)$ points $x \in \mathbb{Z}^d$, we get

$$U := \sum_{k=1}^n U_k^2 \leq C \sum_{k=1}^n R^2(N)V^{-d-1} \cdot \left[\sum_{y\in B(x_k)} P_\omega^z(y \in [X]) \right]^2$$
$$\leq R^2(N) \cdot 2^d R^d(N)V^{-d-1} \sum_{k=1}^n \sum_{y\in B(x_k)} P_\omega^z(y \in [X])^2$$
$$\leq C \cdot 2^{2d} \cdot R^{2d+2}(N)V^{-d-1} \sum_{y\in\mathcal{P}(0,N)} P_\omega^z(y \in [X], B_N(\{X_n\}))^2$$
$$\overset{(1)}{\leq} C \cdot 2^{2d} \cdot R^{2d+2}(N) \cdot R_2(N)V^{-d-1} \leq C \cdot R_2^2(N)V^{-d-1},$$

where for (1) we used the assumption $\omega \in J(N)$. Thus, by McDiarmid's inequality (see Theorem 2.13) for every $\delta > 0$

$$P\big(\big|E\big[P_\omega^z(X_{T_{M+V}} = v, T_{M+V} = m, B_N|\mathcal{G}\big]$$
$$- \mathbb{P}^z(X_{T_{M+V}} = v, T_{M+V} = m, B_N\big| > \delta\big)$$
$$\leq P\big(J(N)^c\big) + 2\exp\left(-\frac{\delta^2}{2CR_2^2(N)V^{-d-1}}\right).$$

In particular, for $\delta = \frac{1}{4}N^{-d} = \frac{1}{4}V^{-(d+1)/2}V^{\eta}$, with $\eta = \frac{(d+1)\theta'-d}{2\theta'} > 0$ we get

$$P\Big(\big|E\big[P_{\omega}^z(X_{T_{M+V}} = v, T_{M+V} = m, B_N|\mathcal{G})\big]$$

$$- \mathbb{P}^z(X_{T_{M+V}} = v, T_{M+V} = m, B_N)\big| > \frac{1}{4}N^{-d}\Big)$$

$$\leq N^{-\xi(1)} + 2\exp\Big(-\frac{[N^{2\theta'}]^{2\eta}}{32CR_2^2(N)}\Big) = N^{-\xi(1)}.$$

Using Corollary 2.10, this also gives

$$P\big(\big|E\big[P_{\omega}^z(X_{T_{M+V}} = v, T_{M+V} = m|\mathcal{G})\big] - \mathbb{P}^z(X_{T_{M+V}} = v, T_{M+V} = m)\big| > \frac{1}{2}N^{-d}\big)$$

$$= N^{-\xi(1)}.$$

Let $W_1(N) \subset \Omega$ be the event that

$$\big|E\big[P_{\omega}^z(X_{T_{M+V}} = v, T_{M+V} = m, B_N|\mathcal{G})\big] - \mathbb{P}^z(X_{T_{M+V}} = v, T_{M+V} = m, B_N)\big|$$
$$\leq \frac{1}{2}N^{-d}$$

for every $\frac{2}{5}N^2 \leq M \leq N^2$, every $v \in H_{M+V} \cap \mathcal{P}(0, 2N)$, every $z \in \tilde{\mathcal{P}}(0, N)$ and every $m \in [0, N^3]$. Then by the above argument $P(W_1(N)) = 1 - N^{-\xi(1)}$. Consider now $\omega \in W_1(N)$, $\frac{2}{5}N^2 \leq M \leq N^2$ a cube Δ of side length N^θ which is contained in H_M and an interval I of length N^θ. We wish to estimate

$$\big|P_{\omega}^z(X_{T_M} \in \Delta, T_M \in I, B_N) - \mathbb{P}^{z_1}(X_{T_M} \in \Delta, T_M \in I, B_N)\big|.$$

Let $c(\Delta)$ and $c(I)$ be the centers of the cube Δ and the interval I, respectively. Let $c'(\Delta) = c(\Delta) + V\frac{\vartheta}{\langle\vartheta, e_1\rangle}$, $c'(I) = c(I) + V\frac{1}{\langle v, e_1\rangle}$ and define (see also Figure 3)

$$\Delta^{(1)} = \big\{v \in H_{M+V} : \|v - c'(\Delta)\|_\infty < \tfrac{1}{2} \cdot \tfrac{9}{10} \cdot N^\theta\big\},$$
$$\Delta^{(2)} = \big\{v \in H_{M+V} : \|v - c'(\Delta)\|_\infty < \tfrac{1}{2} \cdot \tfrac{11}{10} \cdot N^\theta\big\},$$
$$I^{(1)} = \big\{t \in \mathbb{N} : |t - c'(I)| < \tfrac{1}{2} \cdot \tfrac{9}{10} \cdot N^\theta\big\},$$

and

$$I^{(2)} = \big\{t \in \mathbb{N} : |t - c'(I)| < \tfrac{1}{2} \cdot \tfrac{11}{10} \cdot N^\theta\big\}.$$

Annealed estimations (for the proof see Appendix A.3) yields

$$(3.6) \qquad \mathbb{P}^z\big(X_{T_{M+V}} \in \Delta^{(1)}, T_{M+V} \in I^{(1)}\big) < \mathbb{P}^z\big(X_{T_M} \in \Delta, T_M \in I\big) + N^{-\xi(1)},$$

$$(3.7) \qquad \mathbb{P}^z\big(X_{T_{M+V}} \in \Delta^{(2)}, T_{M+V} \in I^{(2)}\big) > \mathbb{P}^z\big(X_{T_M} \in \Delta, T_M \in I\big) - N^{-\xi(1)}$$
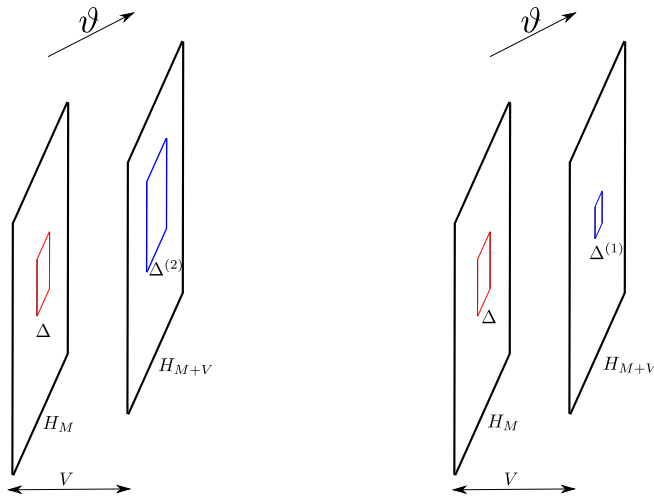
FIG. 3. *Using the annealed walk in the area between the hyperplane H_M and H_{M+V}, we can turn the estimates on the hitting probabilities of the small box $\Delta^{(1)}$ and big box $\Delta^{(2)}$ in the hyperplane H_{M+V} into both quenched and annealed estimations for the hitting probabilities of the box Δ in the hyperplane H_M. The probability to hit Δ and not to hit $\Delta^{(2)}$ as well as the probability not to hit Δ but to hit $\Delta^{(1)}$ are of order $N^{-\tilde{\xi}(1)}$.*

and also, due to Claim 2.15, for an event $W_2(N)$ of P probability $\geq 1 - N^{-\xi(1)}$

$$(3.8) \qquad \begin{aligned} E\big[P_\omega^z (X_{T_{M+V}} \in \Delta^{(1)}, T_{M+V} \in I^{(1)}) | \mathcal{G} \big] \\ < P_\omega^z (X_{T_M} \in \Delta, T_M \in I) + N^{-\xi(1)}, \end{aligned}$$

$$(3.9) \qquad \begin{aligned} E\big[P_\omega^z (X_{T_{M+V}} \in \Delta^{(2)}, T_{M+V} \in I^{(2)}) | \mathcal{G} \big] \\ > P_\omega^z (X_{T_M} \in \Delta, T_M \in I) - N^{-\xi(1)}. \end{aligned}$$

Thus, from the definition of $W_1(N)$, $W_2(N)$ and the last 4 estimations, it follows that $W_1(N) \cap W_2(N) \subset L(\theta, N)$ and the proof is complete. $\quad\square$

LEMMA 3.6. *Let $d \geq 4$ and assume P is uniformly elliptic, i.i.d. and satisfies $(\mathscr{P})$. For every $0 < \theta \leq 1$ and $h \in \mathbb{N}$ let $D^{(\theta,h)}(N)$ be the event that for every $z \in \tilde{\mathcal{P}}(0, N)$, every $\frac{1}{2}N^2 \leq M \leq N^2$, every $(d-1)$-dimensional cube of side length N^θ which is contained in H_M and every interval $I \subset \mathbb{N}$ of length N^θ*

$$(3.10) \qquad P_\omega^z (X_{T_M} \in \Delta, T_M \in I) \leq R_h(N) N^{(\theta-1)d}$$

and

$$(3.11) \qquad P_\omega^z (X_{T_M} \in \Delta) \leq R_h(N) N^{(\theta-1)(d-1)}.$$

Then for every $0 < \theta \leq 1$ there exists $h = h(\theta)$ such that $P(D^{(\theta,h)}(N)) = 1 - N^{-\xi(1)}$.

PROOF. The proof of (3.11) is the content of [2], Lemma 4.13, and, therefore, we restrict attention to the proof of (3.10). We prove the lemma by a descending induction on θ. From Lemma 3.5 together with Lemma 3.3(1), $P(D^{(\theta,1)}) \geq P(L(\theta, N)) = 1 - N^{-\xi(1)}$ for every $\frac{d}{d+1} < \theta \leq 1$. For the induction step, fix θ and assume that the statement of the lemma holds for some θ' such that $\theta > \frac{d}{d+1}\theta'$. Define $h' = h(\theta')$ and $\rho = \frac{\theta}{\theta'} > \frac{d}{d+1}$. Let

$$S(N) = D^{(\rho,1)}(N) \cap \bigcap_{\substack{z \in \mathcal{P}(0,2N) \\ s \in [-2NR_5(N), 2NR_5(N)]}} \sigma_z \varrho_s\left(D^{(\theta',h')}([N^\rho])\right) \cap T(N, \rho),$$

where ϱ is the time shift for the random walk, defined by $\varrho(X_1, X_2, \ldots) = (X_2, X_3, \ldots)$ and

$$T(N, \rho) = \Big\{\omega \in \Omega : \forall v \in \mathcal{P}(0, N), P_\omega^v\left(X_{T_{\partial\mathcal{P}(v,[N^\rho])}} \notin \partial^+\mathcal{P}\left(v, [N^\rho]\right)\right) < e^{-R(N)}$$

$$\text{and } P_\omega^v\left(\left|T_{\partial\mathcal{P}(v,[N^\rho])} - \mathbb{E}^v[T_{\partial\mathcal{P}(v,[N^\rho])}]\right| > NR_2(N)\right) = N^{-\xi(1)}\Big\}.$$

From the definition of $S(N)$, Lemma 3.4 and the induction assumption, we know that $P(S(N)) = 1 - N^{-\xi(1)}$. Therefore, we need to show that for some h and all N large enough, we have $S(N) \subset D^{(\theta,h)}(N)$. To this end, fix $\omega \in S(N)$, $z \in \tilde{\mathcal{P}}(0, N)$, $\frac{1}{2}N^2 \leq M \leq N^2$, a $(d-1)$-dimensional cube Δ of size length N^θ in $\mathcal{P}(0, N) \cap H_M$ and an interval $I \subset [\mathbb{E}^z[T_M] - NR_2(N), \mathbb{E}^z[T_M] + NR_2(N)]$ of length N^θ. As before, we denote by $c(\Delta)$ and $c(I)$ the centers of Δ and I, respectively. Let $V = [N^\rho]^2$, $c(\Delta)' = c(\Delta) - V\frac{\vartheta}{\langle\vartheta,e_1\rangle}$ and $c(I)' = c(I) - V\frac{1}{\langle v,e_1\rangle}$. Since $\omega \in \bigcap_{z \in \mathcal{P}(0,2N)} \bigcap_{s \in [-2NR_5(N), 2NR_5(N)]} \sigma_z \varrho_s(D^{(\theta',h')}([N^\rho]))$ it follows that for every $v \in H_{M-V}$ and every $t \in \mathbb{N}$

$$P_\omega^v(X_{T_M} \in \Delta, T_M \in I - t) < R_{h'}(N)N^{\rho(\theta'-1)d} = R_{h'}(N)N^{(\theta-\rho)d}.$$

In addition, due to the Markov property of the quenched law

$$P_\omega^z(X_{T_M} \in \Delta, T_M \in I)$$

$$= \sum_{\substack{v \in H_{M-V} \cap \mathcal{P}(x',[N^\rho]) \\ |t-c'(I)| \leq N^\rho R_5(N^\rho)}} P_\omega^z(X_{T_{M-V}} = v, T_{M-V} = t)P_\omega^v(X_{T_M} \in \Delta, T_M \in I - t)$$

$$+ N^{-\xi(1)}$$

$$\leq \sum_{\substack{v \in H_{M-V} \cap \mathcal{P}(x',[N^\rho]) \\ |t-c'(I)| \leq N^\rho R_5(N^\rho)}} P_\omega^z(X_{T_{M-V}} = v, T_{M-V} = t)R_{h'}(N)N^{(\theta-\rho)d} + N^{-\xi(1)}.$$

Now, the last sum can be separated into the sum over $2^{d-1}R_5(N^\rho)^{d-1}$ $(d-1)$-dimensional cubes of side length N^ρ and $2R_2(N^\rho)$ intervals of length N^ρ. Since

$\omega \in D^{(\rho,1)}(N)$ the probability to hit each of these cubes in any of these time intervals is bounded by $R_1(N)N^{(\rho-1)d}$. Thus,

$$P_\omega^z(X_{T_M} \in \Delta, T_M \in I) < 2^d R_5(N^\rho)^d R_1(N)N^{(\rho-1)d} R_{h'}(N)N^{(\theta-\rho)d} + N^{-\xi(1)}$$

$$\le R_{\max\{6,h'\}+1}(N)N^{(\theta-1)d},$$

and the proof is complete by taking $h = \max\{6, h'\} + 1$. $\quad\square$

LEMMA 3.7. *Let $d \ge 4$ and assume P is uniformly elliptic, i.i.d. and satisfies* ($\mathscr{P}$). *Let $\mathcal{G}$ be the σ-algebra generated by $\{\omega(z) : \langle z, e_1 \rangle \le N^2\}$. Let $\eta > 0$, $V = [N^\eta]$ and define $R(N, \eta)$ to be the event that for every $z \in \tilde{\mathcal{P}}(0, N)$, every $v \in H_{N^2+V}$ and every $m \in \mathbb{N}$*

$$\left| E\big[P_\omega^z(X_{T_{N^2+V}} = v, T_{N^2+V} = m)|\mathcal{G}\big] - \mathbb{P}^z(X_{T_{N^2+V}} = v, T_{N^2+V} = m) \right|$$

$$\le N^{-d}V^{(1-d)/6}.$$

Then $P(R(N, \eta)) = 1 - N^{-\xi(1)}$.

PROOF.   Let $v \in H_{N^2+V}$, $m \in \mathbb{N}$ and let $\theta > 0$ be such that $\theta < \frac{1}{20}\eta$. Let K be an integer such that $2^{-K-1}N^2 \le V < 2^{-K}N^2$, and for $0 \le k \le K$ define (see also Figure 4)

$$\mathcal{P}^{(k)} = \mathcal{P}(0, N) \cap \{x \in \mathbb{Z}^d : 0 \le N^2 - \langle x, e_1 \rangle \le 2^{-k}N^2\} \qquad \forall 1 \le k \le K,$$

$$\mathcal{P}^{(0)} = \mathcal{P}(0, N) \cap \left\{x \in \mathbb{Z}^d : \frac{N^2}{2} \le N^2 - \langle x, e_1 \rangle\right\},$$

$$F(v) = \left\{x \in \mathcal{P}(0, N) : \left\|x - v - \vartheta\frac{\langle x - v, e_1 \rangle}{\langle \vartheta, e_1 \rangle}\right\|_\infty \le |\langle v - x, e_1 \rangle|^{1/2}R_2(N)\right\},$$

$$\mathcal{P}^{(k)}(v) = \mathcal{P}^{(k)} \cap F(v),$$

and

$$\widehat{\mathcal{P}}^{(k)}(v) = \{y \in \mathbb{Z}^d : \exists x \in \mathcal{P}^{(k)}(v) \text{ such that } \|x - y\|_\infty < R_2(N)\}.$$

Condition on the event $D^{(\theta,h)}$ from Lemma 3.6, with h such that $P(D^{(\theta,h)}(N)) = 1 - N^{-\xi(1)}$. As in [2], Lemma 4.14, for $0 \le k \le K$ and $\omega \in D^{(\theta,h)}$, we have the estimation

$$V(k) = E_{\omega,\omega}^{z,z}[|[X^{(1)}] \cap [X^{(2)}] \cap \mathcal{P}^{(k)}(v)|]$$

$$\le \begin{cases} R_2(N), & k = 0, \\ R_{h+1}(N)N^{2((d+1)/2+(1-\theta)(1-d))}2^{-k\lfloor(d+1)/2\rfloor}, & 1 \le k \le K, \end{cases}$$

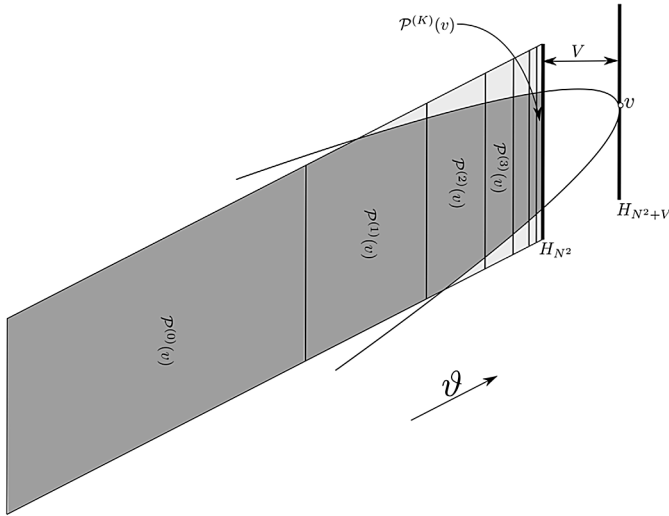where $E_{\omega,\omega}^{z,z}$ is as defined in Section 2.2.

FIG. 4. *The dark gray areas are $\mathcal{P}^{(k)}(v)$ for different values of k. The environment in the light gray area has negligible influence on the probability of hitting v. (The picture is not to scale.)*

Indeed, for $k = 0$ this follows from Lemma 2.12 while for $k > 0$

$$V(k) = \sum_{x \in \mathcal{P}^{(k)}(v)} \left[P_\omega^z(x \text{ is visited}) \right]^2$$

$$\leq \sum_{x \in \mathcal{P}^{(k)}(v)} \left[\sum_{y: \|y-x\|_\infty < R(N)} P_\omega^z(X_{T_{\langle y,e_1 \rangle}} = y) \right]^2 + N^{-\xi(1)}$$

$$\overset{(1)}{\leq} \sum_{x \in \mathcal{P}^{(k)}(v)} C \cdot R^d(N) \cdot \sum_{y: \|y-x\|_\infty < R(N)} \left[P_\omega^z(X_{T_{\langle y,e_1 \rangle}} = y) \right]^2 + N^{-\xi(1)}$$

$$\overset{(2)}{\leq} C \cdot R^{2d}(N) \sum_{y \in \widehat{\mathcal{P}}^{(k)}(v)} \left[P_\omega^z(X_{T_{\langle y,e_1 \rangle}} = y) \right]^2 + N^{-\xi(1)}$$

$$\overset{(3)}{\leq} R_2(N) \sum_{y \in \widehat{\mathcal{P}}^{(k)}(v)} R_h^2(N) N^{2(1-\theta)(1-d)}$$

$$\leq R_{h+1}(N) N^{2((d+1)/2+(1-\theta)(1-d))} 2^{-k\lfloor (d+1)/2 \rfloor},$$

where for (1) we used Cauchy–Schwarz inequality, for (2) we used the fact that each point is counted at most $R^d(N)$ times and for (3) the assumption $\omega \in D^{(\theta,h)}(N)$.

We now use again the filtration $\{\mathcal{F}_i\}$ from Lemma 3.5, and consider the martingale

$$M_i = E\left[P_\omega^z(X_{T_{N^2+V}} = v, T_{N^2+V} = m | B_N) | \mathcal{F}_i \right].$$

In order to use McDiarmid's inequality, we need to bound $U_i = \text{esssup}(|M_i - M_{i-1}||\mathcal{F}_{i-1})$ under the assumption $\omega \in D^{(\theta,h)}(N)$. Let x be such that ω_x is measurable with respect to $\mathcal{F}_i$ but not with respect to $\mathcal{F}_{i-1}$. By a similar argument as in the proof of Lemma 3.5, we have $U_i = N^{-\xi(1)}$ if $x \notin F(v)$, while for $x \in F(v)$

$$U_i \leq R(N)E\big[P_\omega^z(x \text{ is hit}|B_N)|\mathcal{F}_{i-1}\big]\text{Der}\big(N^2 + V - \langle x, e_1\rangle\big),$$

where $\text{Der}(N^2 + V - \langle x, e_1\rangle)$ is the maximal derivative of the annealed distribution with respect to both place and time with distance $N^2 + V - \langle x, e_1\rangle$ to the hitting hyperplane. By Lemma 3.3, this derivative is bounded by $CN^{-d-1}2^{k(d/2)}$ and, therefore, whenever $\omega \in D^{(\theta,h)}$

$$U = \sum_i U_i^2 \leq C \sum_{k=0}^{K} V(k)N^{-2(d+1)}2^{kd} + N^{-\xi(1)}$$

$$\leq CR_{h+1}(N)N^{-2(d+1)}$$

$$+ CR_{h+1}(N)N^{2((d+1)/2+(1-\theta)(1-d))-2(d+1)}\sum_{k=1}^{K}2^{kd-k((d+1)/2)} + N^{-\xi(1)}$$

$$\leq CR_{h+1}(N)\big[N^{-2(d+1)} + N^{2((d+1)/2+(1-\theta)(1-d))-2(d+1)}c2^{((d-1)/2)K}\big]$$

$$+ N^{-\xi(1)}$$

$$\leq CR_{h+1}(N)\big[N^{-2(d+1)} + N^{2((d+1)/2+(1-\theta)(1-d))-2(d+1)}2^{((d-1)/2)K}\big]$$

$$+ N^{-\xi(1)}.$$

Recalling that K was chosen so that $2^K < N^2V^{-1}$ we can bound the last sum term by

$$CR_{h+1}(N)\big[N^{-2(d+1)} + N^{2((d+1)/2+(1-\theta)(1-d))-2(d+1)}N^{(d-1)}V^{-(d-1)/2}\big]$$

$$+ N^{-\xi(1)}$$

$$= CR_{h+1}(N)\big[N^{-2(d+1)} + N^{-2d-2\theta(1-d)}V^{-(d-1)/2}\big] + N^{-\xi(1)}$$

$$\leq CN^{-2d}N^{-((d-1)/6)+\varepsilon}$$

for some small enough $\varepsilon > 0$.

Using McDiarmid's inequality (see Theorem 2.13),

$$P\big(\big|E\big[P_\omega^z(X_{T_{N^2+V}} = v, T_{N^2+V} = m, B_N)|\mathcal{G}\big]$$

$$- \mathbb{P}^z(X_{T_{N^2+V}} = v, T_{N^2+V} = m, B_N)\big| > N^{-d}V^{(1-d)/6}\big)$$

$$\leq P\big(\big|E\big[P_\omega^z(X_{T_{N^2+V}} = v, T_{N^2+V} = m, B_N)|\mathcal{G}\big]$$

$$- \mathbb{P}^z(X_{T_{N^2+V}} = v, T_{N^2+V} = m, B_N)\big| > N^{-d}V^{(1-d)/6}, D^{(\theta,h)}(N)\big)$$

$$+ N^{-\xi(1)}$$

$$\leq P\big(\big|E\big[P_\omega^z(X_{T_{N^2+V}} = v, T_{N^2+V} = m, B_N)|\mathcal{G}\big]$$

$$- \mathbb{P}^z(X_{T_{N^2+V}} = v, T_{N^2+V} = m, B_N)\big| > N^{-d}V^{(1-d)/6},$$

$$U \leq CN^{-2d-((d-1)/6)+\varepsilon}\big) + N^{-\xi(1)}$$

$$\leq C\exp\big(-cN^{-((d+1)/6)(1-2\eta)-\varepsilon}\big) + N^{-\xi(1)} = N^{-\xi(1)}.$$

Since $P(B_N) = 1 - N^{-\xi(1)}$ by Corollary 2.10, this completes the proof. $\quad\square$

We are finally ready to prove Proposition 3.1.

PROOF OF PROPOSITION 3.1. Let $\eta > 0$ and define $V = [N^\eta]$. By Lemma 3.7, we know that $P(R(N,\eta)) = 1 - N^{-\xi(1)}$. As before, all we need to show is that $R(N,\eta) \cap S(N,\eta) \subseteq F(N,\theta)$ for an appropriate choice of $\eta > 0$ and some event $S(N,\eta)$ satisfying $P(S(N,\eta)) = 1 - N^{-\xi(1)}$. This is done identically as in the last step of the proof of Lemma 3.5. Let $\omega \in R(N,\eta)$, let Δ be a cube of side length N^θ which is contained in $\partial^+\mathcal{P}(0,N)$ and let I be an interval of length N^θ in $\mathbb{N}$. As in Lemma 3.5 (see also Figure 3), we denote by $c(\Delta)$ and $c(I)$ the center of Δ and I, respectively, and let $c'(\Delta) = c(\Delta) + V\frac{\vartheta}{\langle\vartheta,e_1\rangle}$, $c'(I) = c(I) + \mathbb{E}^0[T_V]$.

Let $\Delta^{(1)}$ and $\Delta^{(2)}$ be $(d-1)$-dimensional cubes that are contained in H_{N^2+V}, centered at $c'(\Delta)$ and are of side lengths $N^\theta - R_3(N)\sqrt{V}$ and $N^\theta + R_3(N)\sqrt{V}$, respectively. In a similar fashion, let $I^{(1)}$ and $I^{(2)}$ be intervals centered at $c'(I)$ which are of lengths $N^\theta - R_3(N)\sqrt{V}$ and $N^\theta + R_3(N)\sqrt{V}$, respectively.

As in the proof of Lemma 3.5 (the proof can be found in Appendix A.3), we know that

$$(3.12) \quad \begin{aligned} &\mathbb{P}^z\big(X_{T_{N^2+V}} \in \Delta^{(1)}, T_{N^2V} \in I^{(1)}\big) \\ &\quad < \mathbb{P}^z(X_{T_{N^2}} \in \Delta, T_{N^2} \in I) + N^{-\xi(1)}, \end{aligned}$$

$$(3.13) \quad \begin{aligned} &\mathbb{P}^z\big(X_{T_{N^2+V}} \in \Delta^{(2)}, T_{N^2+V} \in I^{(2)}\big) \\ &\quad > \mathbb{P}^z(X_{T_{N^2}} \in \Delta, T_{N^2} \in I) - N^{-\xi(1)} \end{aligned}$$

and using Claim 2.15 for an event $S(N,\eta)$ such that $P(S(N,\eta)) = 1 - N^{-\xi(1)}$ we have

$$(3.14) \quad \begin{aligned} &E\big[P_\omega^z(X_{T_{N^2+V}} \in \Delta^{(1)}, T_{N^2+V} \in I^{(1)})|\mathcal{G}\big] \\ &\quad < P_\omega^z(X_{T_{N^2}} \in \Delta, T_{N^2} \in I) + N^{-\xi(1)}, \end{aligned}$$

$$(3.15) \quad \begin{aligned} &E\big[P_\omega^z(X_{T_{N^2+V}} \in \Delta^{(2)}, T_{N^2+V} \in I^{(2)})|\mathcal{G}\big] \\ &\quad > P_\omega^z(X_{T_{N^2}} \in \Delta, T_{N^2} \in I) - N^{-\xi(1)}. \end{aligned}$$

In addition, on the event $R(N, \eta)$, for $i = 1, 2$

$$\left| E\big[P_\omega^z(X_{T_{N^2+V}} \in \Delta^{(i)}, T_{N^2+V} \in I^{(i)}) | \mathcal{G} \big] - \mathbb{P}^z(X_{T_{N^2+V}} \in \Delta^{(i)}, T_{N^2+V} \in I^{(i)}) \right|$$

$$\leq |\Delta^{(i)}| \cdot |I^{(i)}| \cdot N^{-d} V^{(1-d)/6}.$$

Therefore, for $\omega \in R(N, \eta) \cap S(N, \eta)$ we have

$$\left| P_\omega^z(X_{T_{\partial\mathcal{P}(0,N)}} \in \Delta, T_{\partial\mathcal{P}(0,N)} \in I) - \mathbb{P}^z(X_{T_{\partial\mathcal{P}(0,N)}} \in \Delta, T_{\partial\mathcal{P}(0,N)} \in I) \right|$$

$$\leq (|\Delta^{(1)}||I^{(1)}| + |\Delta^{(2)}||I^{(2)}|)N^{-d} V^{(1-d)/6}$$

$$\quad + (|\Delta^{(2)}||I^{(2)}| - |\Delta^{(1)}||I^{(1)}|)CN^{-d} + N^{-\xi(1)}$$

$$\leq C\big[(N^\theta + R_3(N)\sqrt{V})^d N^{-d} V^{(1-d)/6} + R_3(N)\sqrt{V} N^{\theta(d-1)-d} \big].$$

Taking $\eta < 2\theta$ we can bound the last term by

$$C\big[N^{\theta d - d + \eta((1-d)/6)} + R_3(N) N^{\theta(d-1)-d+\eta/2} \big].$$

Notice that the exponents of the powers of N are the same when $\eta = \frac{6\theta}{d+2} < 2\theta$, in which case the last bound equals $C(1 + R_3(N)) \cdot N^{-d(1-\theta)-((d-1)/(d+2))\theta} \leq CN^{-d(1-\theta)-((d-2)/(d+2))\theta}$. Thus, the proof is complete. $\quad\square$

4. From $(d-1)$-dimensional boxes and time intervals to d-dimensional boxes in a fixed time. The goal of this section is to use the estimation proved in Section 3, for the difference between the quenched and annealed probabilities to hit boxes in a hyperplane within a time interval, in order to achieve similar estimation for the difference between the quenched and annealed probabilities to hit a d-dimensional box in a specific time. Formally, we have the following.

PROPOSITION 4.1. *Let $d \geq 4$ and assume P is uniformly elliptic, i.i.d. and satisfies $(\mathcal{P})$. For every $0 < \theta \leq \frac{1}{2}$, let $H(N) = H(N, \theta)$ be the event that for every $z \in \tilde{\mathcal{P}}(0, N)$ and every d-dimensional cube Δ of side length N^θ*

$$\left| P_\omega^z(X_N \in \Delta) - \mathbb{P}^z(X_N \in \Delta) \right| \leq CN^{-d(1-\theta)-(1/3)\theta}.$$

Then $P(H(N)) = 1 - N^{-\xi(1)}$.

REMARK 4.2. The constant $\frac{1}{3}$ can in fact be replaced by any number which is smaller than $\min\{\frac{1}{2}, \frac{d-2}{d+2}\}$.

The idea of the proof is to exploit the estimation of Proposition 3.1 and the fact that regeneration times occur quite often. More precisely, we show that the event of hitting a box Δ at time N is bounded both from below and from above by the event of hitting a certain hyperplane in a specific $(d-1)$-dimensional box within a specific time interval. This implies that the difference between the probabilities is roughly the same as in Proposition 3.1, and thus gives the required result.

PROOF OF PROPOSITION 4.1. Due to Lemma 2.16, we may restrict ourselves to boxes Δ whose center $c(\Delta)$ satisfies $\|c(\Delta) - \mathbb{E}^z[X_N]\|_\infty < \sqrt{N} R_5(N)$. Given a cube Δ of side length N^θ such that $c(\Delta)$ satisfies $\|c(\Delta) - \mathbb{E}^z[X_N]\|_\infty < \sqrt{N} R_5(N)$ let $\Delta^{(1)}$ and $\Delta^{(2)}$ be the $(d-1)$-dimensional cubes in the hyperplane $H_{\langle c(\Delta), e_1 \rangle - N^\theta}$ with center $c(\Delta) - \frac{N^\theta}{\langle \vartheta, e_1 \rangle} \vartheta$ and side length $N^\theta - R_5(N) N^{\theta/2}$ and $N^\theta + R_5(N) N^{\theta/2}$, respectively. Noting that

$$L := \langle c(\Delta), e_1 \rangle - N^\theta \geq \langle \mathbb{E}^z[X_N], e_1 \rangle - 2\sqrt{N} R_5(N)$$
$$\geq cN - 2\sqrt{N} R_5(N),$$

it follows from Proposition 3.1 [for every $\omega \in F(N, \theta)$ and every $z \in \tilde{\mathcal{P}}(0, N)$] that

$$
(4.1) \quad \begin{aligned} &\big| P_\omega^z \big(X_{T_{\partial \mathcal{P}(0, \sqrt{L})}} \in \Delta^{(1)}, T_{\partial \mathcal{P}(0, \sqrt{L})} \in I^{(1)} \big) \\ &\qquad - \mathbb{P}^z \big(X_{T_{\partial \mathcal{P}(0, \sqrt{L})}} \in \Delta^{(1)}, T_{\partial \mathcal{P}(0, \sqrt{L})} \in I^{(1)} \big) \big| \\ &\qquad\qquad \leq C N^{-d(1-\theta) - ((d-2)/(d+2))\theta}, \end{aligned}
$$

with $I^{(1)} = N - \frac{N^\theta}{\langle \mathtt{v}, e_1 \rangle} + [-\frac{1}{2}(N^\theta - R_5(N) N^{\theta/2}), \frac{1}{2}(N^\theta - R_5(N) N^{\theta/2})]$, and

$$
(4.2) \quad \begin{aligned} &\big| P_\omega^z \big(X_{T_{\partial \mathcal{P}(0, \sqrt{L})}} \in \Delta^{(2)}, T_{\partial \mathcal{P}(0, \sqrt{L})} \in I^{(2)} \big) \\ &\qquad - \mathbb{P}^z \big(X_{T_{\partial \mathcal{P}(0, \sqrt{L})}} \in \Delta^{(2)}, T_{\partial \mathcal{P}(0, \sqrt{L})} \in I^{(2)} \big) \big| \\ &\qquad\qquad \leq C N^{-d(1-\theta) - ((d-2)/(d+2))\theta}, \end{aligned}
$$

with $I^{(2)} = N - \frac{N^\theta}{\langle \mathtt{v}, e_1 \rangle} + [-\frac{1}{2}(N^\theta + R_5(N) N^{\theta/2}), \frac{1}{2}(N^\theta + R_5(N) N^{\theta/2})]$.

In addition, by a standard CLT type arguments, as the one in Lemma 3.5 (see Appendix A.3 for the proof), we have the following annealed estimations:

$$(4.3) \quad \mathbb{P}^z \big(X_{T_{\partial \mathcal{P}(0, \sqrt{L})}} \in \Delta^{(1)}, T_{\partial \mathcal{P}(0, \sqrt{L})} \in I^{(1)}, X_N \notin \Delta \big) \leq N^{-\xi(1)},$$

$$(4.4) \quad \mathbb{P}^z \big(\big(X_{T_{\partial \mathcal{P}(0, \sqrt{L})}} \in \Delta^{(2)}, T_{\partial \mathcal{P}(0, \sqrt{L})} \in I^{(2)} \big)^c, X_N \in \Delta \big) \leq N^{-\xi(1)}.$$

Using Claim 2.15 again, this also implies that

$$
(4.5) \quad \begin{aligned} &P \big(\{ \omega \in \Omega : P_\omega^z \big(X_{T_{\partial \mathcal{P}(0, \sqrt{L})}} \in \Delta^{(1)}, T_{\partial \mathcal{P}(0, \sqrt{L})} \in I^{(1)}, X_N \notin \Delta \big) \leq N^{-\xi(1)} \} \big) \\ &\qquad \geq 1 - N^{-\xi(1)} \end{aligned}
$$

and

$$
(4.6) \quad \begin{aligned} &P \big(\{ \omega \in \Omega : P_\omega^z \big(\big(X_{T_{\partial \mathcal{P}(0, \sqrt{L})}} \in \Delta^{(2)}, T_{\partial \mathcal{P}(0, \sqrt{L})} \in I^{(2)} \big)^c, X_N \in \Delta \big) \leq N^{-\xi(1)} \} \big) \\ &\qquad \geq 1 - N^{-\xi(1)}. \end{aligned}
$$

Combining all of the above, we get for an event with P probability $\geq 1 - N^{-\xi(1)}$ that

$$P_\omega^z(X_N \in \Delta)$$

$$\overset{(4.6)}{\leq} P_\omega^z\big(X_{T_{\partial\mathcal{P}(0,\sqrt{L})}} \in \Delta^{(2)}, T_{\partial\mathcal{P}(0,\sqrt{L})} \in I^{(2)}, X_N \in \Delta\big) + N^{-\xi(1)}$$

$$\leq P_\omega^z\big(X_{T_{\partial\mathcal{P}(0,\sqrt{L})}} \in \Delta^{(2)}, T_{\partial\mathcal{P}(0,\sqrt{L})} \in I^{(2)}\big) + N^{-\xi(1)}$$

$$\overset{(4.2)}{\leq} \mathbb{P}^z\big(X_{T_{\partial\mathcal{P}(0,\sqrt{L})}} \in \Delta^{(2)}, T_{\partial\mathcal{P}(0,\sqrt{L})} \in I^{(2)}\big) + CN^{-d(1-\theta)-((d-2)/(d+2))\theta}$$

$$\quad + N^{-\xi(1)}$$

$$\leq \mathbb{P}^z\big(X_{T_{\partial\mathcal{P}(0,\sqrt{L})}} \in \Delta^{(1)}, T_{\partial\mathcal{P}(0,\sqrt{L})} \in I^{(1)}\big)$$

$$\quad + \mathbb{P}^z\big(X_{T_{\partial\mathcal{P}(0,\sqrt{L})}} \in \Delta^{(2)} \setminus \Delta^{(1)}, T_{\partial\mathcal{P}(0,\sqrt{L})} \in I^{(2)}\big)$$

$$\quad + \mathbb{P}^z\big(X_{T_{\partial\mathcal{P}(0,\sqrt{L})}} \in \Delta^{(2)}, T_{\partial\mathcal{P}(0,\sqrt{L})} \in I^{(2)} \setminus I^{(1)}\big)$$

$$\quad + CN^{-d(1-\theta)-((d-2)/(d+2))\theta}$$

$$\overset{(4.3)}{\leq} \mathbb{P}^z\big(X_{T_{\partial\mathcal{P}(0,\sqrt{L})}} \in \Delta^{(1)}, T_{\partial\mathcal{P}(0,\sqrt{L})} \in I^{(1)}, X_N \in \Delta\big)$$

$$\quad + \mathbb{P}^z\big(X_{T_{\partial\mathcal{P}(0,\sqrt{L})}} \in \Delta^{(2)} \setminus \Delta^{(1)}, T_{\partial\mathcal{P}(0,\sqrt{L})} \in I^{(2)}\big)$$

$$\quad + \mathbb{P}^z\big(X_{T_{\partial\mathcal{P}(0,\sqrt{L})}} \in \Delta^{(2)}, T_{\partial\mathcal{P}(0,\sqrt{L})} \in I^{(2)} \setminus I^{(1)}\big)$$

$$\quad + CN^{-d(1-\theta)-((d-2)/(d+2))\theta}$$

$$\leq \mathbb{P}^z(X_N \in \Delta) + \mathbb{P}^z\big(X_{T_{\partial\mathcal{P}(0,\sqrt{L})}} \in \Delta^{(2)} \setminus \Delta^{(1)}, T_{\partial\mathcal{P}(0,\sqrt{L})} \in I^{(2)}\big)$$

$$\quad + \mathbb{P}^z\big(X_{T_{\partial\mathcal{P}(0,\sqrt{L})}} \in \Delta^{(2)}, T_{\partial\mathcal{P}(0,\sqrt{L})} \in I^{(2)} \setminus I^{(1)}\big)$$

$$\quad + CN^{-d(1-\theta)-((d-2)/(d+2))\theta}$$

$$\overset{(1)}{\leq} \mathbb{P}^z(X_N \in \Delta) + CR_5(N) \cdot N^{\theta(d-1/2)} \cdot N^{-d} + CN^{-d(1-\theta)-((d-2)/(d+2))\theta}$$

$$= \mathbb{P}^z(X_N \in \Delta) + CN^{-d(1-\theta)-(1/3)\theta},$$

whereas for (1), we used the annealed derivative estimation proved in Lemma 3.3 as well as the fact that the number of pairs (x, t) such that $(x, t) \in (\Delta^{(2)} \setminus \Delta^{(1)}, I^{(2)})$ or $(x, t) \in (\Delta^{(2)}, I^{(2)} \setminus I^{(1)})$ is bounded by $CR_5(N)N^{\theta(d-1/2)}$. The other direction

$$P_\omega^z(X_N \in \Delta) \geq \mathbb{P}^z(X_N \in \Delta) - CN^{-d(1-\theta)-(1/3)\theta}$$

follows via the same argument except we use (4.1), (4.4) and (4.5) instead of (4.2), (4.3) and (4.6). $\quad\square$

5. Total variation bound for finite boxes. In the previous section, it was shown that for every $0 < \theta \leq 1$ the difference between the quenched and annealed probabilities to hit a d-dimensional box of side length N^θ at time N is bounded by $CN^{-d(1-\theta)-((d-2)/(d+2))\theta}$. Since a must be inside a the box of side length $2N+1$ around its starting point at time N, it in particular implies that the total variation between the quenched and annealed distribution over any partition of $\mathbb{Z}^d$ into d-dimensional boxes of side length N^θ goes to zero as N goes to infinity. The goal of this section is to strengthen this result and prove that the same result hold for partitions of $\mathbb{Z}^d$ into boxes whose side length is of constant size, independent of N. More formally, we have the following.

THEOREM 5.1. *Let $d \geq 4$ and assume P is uniformly elliptic, i.i.d. and satisfies ($\mathscr{P}$). For $N, M \in \mathbb{N}$ denote by $G(N) = G(N, M)$ the set of environments $\omega \in \Omega$ such that for every $z \in \mathbb{Z}^d$ satisfying $\|z\|_\infty \leq N$*

$$\sum_{\Delta \in \Pi} \left| P_\omega^z(X_N \in \Delta) - \mathbb{P}^z(X_N \in \Delta) \right| \leq \frac{C_2}{M^{c_1}} + \frac{C_2}{N^{c_1}},$$

where Π is any partition of $\mathbb{Z}^d$ into boxes of side length M. Then for an appropriate $0 < c_1, C_2 < \infty$, $P(G(N)) = 1 - N^{-\xi(1)}$.

The idea of the proof is to shrink the size of the boxes repeatedly, each time by a constant factor from the previous step. This is done as follows: first, we fix some factor, say $\theta = \frac{1}{200}$. Then, in the kth step of the process, we let the random walk run for $N^{1/2^k}$ steps and ask for the difference between the annealed and quenched measures hit the same box of side $N^{\theta/2^k}$. Repeating the last procedure roughly $\log \log N$ times, we get boxes of constant side length M. The idea is to bound the total variation of the $(k+1)$th step of this process by the one of the kth step. Denoting by λ_k the total variation of the kth step, we show that $\lambda_k \leq \lambda_{k-1} + CN_k^{-\alpha}$ for some $C, \alpha > 0$. An additional short calculation then yields the result.

PROOF OF THEOREM 5.1. We start by introducing some notation to be used throughout the proof. Let $\theta = \frac{1}{200}$. For $j \in \mathbb{N}$ denote $N_j = \lfloor N^{1/2^j} \rfloor$ and let $r(N) = \lceil \log_2(\frac{\log N}{\theta \log M}) \rceil$ (which is the minimal natural number such that $N_{r(N)}^\theta \leq M$). Moreover, denote $n_0 = n - \sum_{j=1}^{r(N)} N_j$ and $n_k = \sum_{j=1}^k N_j$, $\forall 1 \leq k \leq r(N)$. For $0 \leq k \leq r(n)$, let Π_k be a partition of $\mathbb{Z}^d$ into boxes of side length $\lfloor N_k^\theta \rfloor$. Finally, for $0 \leq k \leq r(N)$ let

$$\lambda_k = \sum_{\Delta \in \Pi_k} \left| P_\omega^z(X_{n_k} \in \Delta) - \mathbb{P}^z(X_{n_k} \in \Delta) \right|.$$

Note that in particular $\lambda_{r(N)}$ is the total variation between the quenched and annealed measures on cubes of side length $\leq M$ which is the term we wish to bound

from above. If one wish to be slightly more precise, one should replace $N_{r(N)}$ by M, and thus obtaining total variation for boxes of side length M, this however does not influence the estimates to follow.

As stated before the main idea of the proof is to prove an inequality of the form

$$\lambda_k \le \lambda_{k-1} + C N_k^{-\alpha} \qquad \forall 1 \le k \le r(N)$$

for some $\alpha > 0$, which immediately implies $\lambda_{r(N)} \le \lambda_1 + C \sum_{k=1}^{r(N)} N_k^{-\alpha}$. As it turns out the last term is bounded by $C_2 M^{-c_1}$ for some constants $0 < c_1, C_2 < \infty$, while the first term, that is, λ_1, is bounded (due to Proposition 4.1) by $C N^{-((d-2)/(d+2))\theta}$, and the result follows.

We now turn to the estimation of λ_k. By the triangle inequality and the Markov property of P_ω, we have

$$\lambda_k = \sum_{\Delta \in \Pi_k} \left| P_\omega^z(X_{n_k} \in \Delta) - \mathbb{P}^z(X_{n_k} \in \Delta) \right|$$

$$= \sum_{\Delta \in \Pi_k} \left| \sum_{\Delta' \in \Pi_{k-1}} \left[P_\omega^z(X_{n_k} \in \Delta, X_{n_{k-1}} \in \Delta') - \mathbb{P}^z(X_{n_k} \in \Delta, X_{n_{k-1}} \in \Delta') \right] \right|$$

$$\le \sum_{\Delta \in \Pi_k} \sum_{\Delta' \in \Pi_{k-1}} \left| P_\omega^z(X_{n_k} \in \Delta, X_{n_{k-1}} \in \Delta') - \mathbb{P}^z(X_{n_k} \in \Delta, X_{n_{k-1}} \in \Delta') \right|$$

$$\le \sum_{\Delta \in \Pi_k} \sum_{\Delta' \in \Pi_{k-1}} \left| \sum_{u \in \Delta'} P_\omega^u(X_{n_k - n_{k-1}} \in \Delta) \right.$$

(5.1)

$$\left. \times \left[P_\omega^z(X_{n_{k-1}} = u) - \mathbb{P}^z(X_{n_{k-1}} \in \Delta') P_\omega^z(X_{n_{k-1}} = u | X_{n_{k-1}} \in \Delta') \right] \right|$$

$$+ \sum_{\Delta \in \Pi_k} \sum_{\Delta' \in \Pi_{k-1}} \left| \sum_{u \in \Delta'} \mathbb{P}^z(X_{n_{k-1}} \in \Delta') P_\omega^z(X_{n_{k-1}} = u | X_{n_{k-1}} \in \Delta') \right.$$

(5.2)

$$\left. \times \left[P_\omega^u(X_{n_k - n_{k-1}} \in \Delta) - \mathbb{P}^u(X_{n_k - n_{k-1}} \in \Delta) \right] \right|$$

$$+ \sum_{\Delta \in \Pi_k} \sum_{\Delta' \in \Pi_{k-1}} \left| \sum_{u \in \Delta'} \mathbb{P}^u(X_{n_k - n_{k-1}} \in \Delta) \right.$$

(5.3)

$$\left. \times \left[\mathbb{P}^z(X_{n_{k-1}} \in \Delta') P_\omega^z(X_{n_{k-1}} = u | X_{n_{k-1}} \in \Delta') - \mathbb{P}^z(X_{n_{k-1}} = u) \right] \right|$$

$$+ \sum_{\Delta \in \Pi_k} \sum_{\Delta' \in \Pi_{k-1}} \left| \sum_{u \in \Delta'} \mathbb{P}^u(X_{n_k - n_{k-1}} \in \Delta) \mathbb{P}^z(X_{n_{k-1}} = u) \right.$$

(5.4)

$$\left. - \mathbb{P}^z(X_{n_k} \in \Delta, X_{n_{k-1}} \in \Delta') \right|.$$

We turn to estimate each of the terms (5.1)–(5.4) separately.

For the first term (5.1), we have

$$
\begin{aligned}
(5.1) = \sum_{\Delta \in \Pi_k} \sum_{\Delta' \in \Pi_{k-1}} \bigg| \sum_{u \in \Delta'} P_\omega^u(X_{n_k - n_{k-1}} \in \Delta) \\
\times \big[P_\omega^z(X_{n_{k-1}} = u) - \mathbb{P}^z(X_{n_{k-1}} \in \Delta') P_\omega^z(X_{n_{k-1}} = u | X_{n_{k-1}} \in \Delta') \big] \bigg|
\end{aligned}
$$

$$
\begin{aligned}
\leq \sum_{\Delta \in \Pi_k} \sum_{\Delta' \in \Pi_{k-1}} \sum_{u \in \Delta'} P_\omega^u(X_{n_k - n_{k-1}} \in \Delta) \\
\times \big| P_\omega^z(X_{n_{k-1}} = u) - \mathbb{P}^z(X_{n_{k-1}} \in \Delta') P_\omega^z(X_{n_{k-1}} = u | X_{n_{k-1}} \in \Delta') \big|
\end{aligned}
$$

$$
\begin{aligned}
= \sum_{\Delta' \in \Pi_{k-1}} \sum_{u \in \Delta'} \big| P_\omega^z(X_{n_{k-1}} = u) \\
- \mathbb{P}^z(X_{n_{k-1}} \in \Delta') P_\omega^z(X_{n_{k-1}} = u | X_{n_{k-1}} \in \Delta') \big|
\end{aligned}
$$

$$
\begin{aligned}
= \sum_{\Delta' \in \Pi_{k-1}} \sum_{u \in \Delta'} P_\omega^z(X_{n_{k-1}} = u | X_{n_{k-1}} \in \Delta') \\
\times \big| P_\omega^z(X_{n_{k-1}} \in \Delta') - \mathbb{P}^z(X_{n_{k-1}} \in \Delta') \big|
\end{aligned}
$$

$$
= \sum_{\Delta' \in \Pi_{k-1}} \big| P_\omega^z(X_{n_{k-1}} \in \Delta') - \mathbb{P}^z(X_{n_{k-1}} \in \Delta') \big| = \lambda_{k-1}.
$$

For the second term (5.2), the triangle inequality yields

$$
\begin{aligned}
(5.2) = \sum_{\Delta \in \Pi_k} \sum_{\Delta' \in \Pi_{k-1}} \bigg| \sum_{u \in \Delta'} \mathbb{P}^z(X_{n_{k-1}} \in \Delta') P_\omega^z(X_{n_{k-1}} = u | X_{n_{k-1}} \in \Delta') \\
\times \big[P_\omega^u(X_{n_k - n_{k-1}} \in \Delta) - \mathbb{P}^u(X_{n_k - n_{k-1}} \in \Delta) \big] \bigg|
\end{aligned}
$$

$$
\begin{aligned}
\leq \sum_{\Delta' \in \Pi_{k-1}} \sum_{u \in \Delta'} \mathbb{P}^z(X_{n_{k-1}} \in \Delta') P_\omega^z(X_{n_{k-1}} = u | X_{n_{k-1}} \in \Delta') \\
\times \sum_{\Delta \in \Pi_k} \big| P_\omega^u(X_{n_k - n_{k-1}} \in \Delta) - \mathbb{P}^u(X_{n_k - n_{k-1}} \in \Delta) \big|.
\end{aligned}
$$

By Lemma 2.16, this can be bounded by

$$
\sum_{\Delta' \in \Pi_{k-1}} \sum_{u \in \Delta'} \mathbb{P}^z(X_{n_{k-1}} \in \Delta') P_\omega^z(X_{n_{k-1}} = u | X_{n_{k-1}} \in \Delta')
$$

$$
(5.5) \qquad \times \sum_{\substack{\Delta \in \Pi_k \\ \mathrm{dist}(\Delta, u) \leq \sqrt{n_k - n_{k-1}} R_5(n_k - n_{k-1})}} \big| P_\omega^u(X_{n_k - n_{k-1}} \in \Delta) - \mathbb{P}^u(X_{n_k - n_{k-1}} \in \Delta) \big|
$$

$$
+ (n_k - n_{k-1})^{-\xi(1)}.
$$

We say that a cube $\Delta' \in \Pi_{k-1}$ is good if for every $u \in \Delta'$ and every $\Delta \in \Pi_k$

$$\left| P_\omega^u(X_{n_k-n_{k-1}} \in \Delta) - \mathbb{P}^u(X_{n_k-n_{k-1}} \in \Delta) \right| \leq C N_k^{(\theta-1)(d-1)-(1/3)\theta},$$

otherwise we say that Δ' is bad. Note that the condition holds trivially for all Δ such that $\mathrm{dist}(u, \Delta) > n_k - n_{k-1}$. Noting that:

- For every u we only need to consider boxes Δ such that $\mathrm{dist}(\Delta, u) \leq \sqrt{n_k - n_{k-1}} R_5(n_k - n_{k-1}) = \sqrt{N_k} R_5(N_k)$ whose number is bounded by $\frac{N_k^{d/2} R_5^d(N_k)}{|\Delta|}$.
- We only need to consider boxes Δ' such that $\mathrm{dist}(z, \Delta') \leq n_{k-1}$.
- The event $G_N = \{$all boxes Δ' such that $\mathrm{dist}(\Delta', z) \leq n_{k-1}$ are good$\}$ satisfies $P(G_N^c) \leq n_{k-1}^d \cdot N^{-\xi(1)} = N^{-\xi(1)}$ due to Proposition 4.1.

We conclude that (5.5) is bounded by

$$C N_k^{(\theta-1)(d-1)-(1/3)\theta} \cdot \frac{N_k^{d/2} R_5(N_k)}{\lfloor N_k^\theta \rfloor^d} + P(G_N^c)$$

$$\leq C N_k^{1-(4/3)\theta-d/2} R_5^d(N_k) + N^{-\xi(1)} \leq C N_k^{1-(4/3)\theta-d/2} R_5^d(N_k).$$

Turning to deal with (5.3), notice that

$$(5.3) = \sum_{\Delta \in \Pi_k} \sum_{\Delta' \in \Pi_{k-1}} \left| \sum_{u \in \Delta'} \mathbb{P}^u(X_{n_k-n_{k-1}} \in \Delta) \right.$$

$$\times \left. \left[\mathbb{P}^z(X_{n_{k-1}} \in \Delta') P_\omega^z(X_{n_{k-1}} = u | X_{n_{k-1}} \in \Delta') - \mathbb{P}^z(X_{n_{k-1}} = u) \right] \right|$$

$$\leq \sum_{\Delta \in \Pi_k} \sum_{\Delta' \in \Pi_{k-1}} \mathbb{P}^z(X_{n_{k-1}} \in \Delta')$$

$$\times \left| \max_{u \in \Delta'} \mathbb{P}^u(X_{n_k-n_{k-1}} \in \Delta) - \min_{u \in \Delta'} \mathbb{P}^u(X_{n_k-n_{k-1}} \in \Delta) \right|$$

$$= \sum_{\Delta' \in \Pi_{k-1}} \mathbb{P}^z(X_{n_{k-1}} \in \Delta')$$

$$\times \sum_{\Delta \in \Pi_k} \left| \max_{u \in \Delta'} \mathbb{P}^u(X_{n_k-n_{k-1}} \in \Delta) - \min_{u \in \Delta'} \mathbb{P}^u(X_{n_k-n_{k-1}} \in \Delta) \right|$$

$$\overset{(1)}{\leq} \sum_{\Delta' \in \Pi_{k-1}} \mathbb{P}^z(X_{n_{k-1}} \in \Delta')$$

$$\times \sum_{\substack{\Delta \in \Pi_k \text{ s.t. } \exists u \in \Delta' \\ \mathrm{dist}(\Delta, \mathbb{E}^u[X_{n_k-n_{k-1}}]) \\ \leq \sqrt{n_k-n_{k-1}} R_5(n_k-n_{k-1})}} \left| \max_{u \in \Delta'} \mathbb{P}^u(X_{n_k-n_{k-1}} \in \Delta) \right.$$

$$\left. - \min_{u \in \Delta'} \mathbb{P}^u(X_{n_k-n_{k-1}} \in \Delta) \right| + (n_k - n_{k-1})^{-\xi(1)},$$

where for (1) we used Lemma 2.16. Due to the annealed derivative estimation from Lemma 2.14, we can bound the last term by

$$\sum_{\Delta' \in \Pi_{k-1}} \mathbb{P}^z(X_{n_{k-1}} \in \Delta') \underbrace{\left(\frac{dN_{k-1}^\theta + \sqrt{n_k - n_{k-1}} R_5(n_k - n_{k-1})}{\lfloor N_k^\theta \rfloor} \right)^d}_{\text{number of relevant boxes}} \cdot \underbrace{\lfloor N_k^\theta \rfloor^d}_{\text{size of each box}}$$

$$\times \underbrace{\frac{C}{(n_k - n_{k-1})^{(d+1)/2}}}_{\text{derivative estimation}} + (n_k - n_{k-1})^{-\xi(1)}$$

$$= \frac{C(dN_{k-1}^\theta + \sqrt{N_k} R_5(N_k))^d}{N_k^{(d+1)/2}} + N_k^{-\xi(1)} \le CR_6(N_k)N_k^{-1/2}.$$

Finally, for (5.4) we have

$$(5.4) = \sum_{\Delta \in \Pi_k} \sum_{\Delta' \in \Pi_{k-1}} \left| \sum_{u \in \Delta'} \mathbb{P}^u(X_{n_k - n_{k-1}} \in \Delta) \mathbb{P}^z(X_{n_{k-1}} = u) \right.$$

$$\left. - \mathbb{P}^z(X_{n_k} \in \Delta, X_{n_{k-1}} \in \Delta') \right|$$

$$= \sum_{\Delta \in \Pi_k} \sum_{\Delta' \in \Pi_{k-1}} \left| \sum_{u \in \Delta'} \mathbb{P}^z(X_{n_{k-1}} = u) \right.$$

$$\left. \times \left[\mathbb{P}^u(X_{n_k - n_{k-1}} \in \Delta) - \mathbb{P}^z(X_{n_k} \in \Delta | X_{n_{k-1}} = u) \right] \right|$$

$$\le \sum_{\Delta' \in \Pi_{k-1}} \sum_{u \in \Delta'} \mathbb{P}^z(X_{n_{k-1}} = u)$$

$$\times \sum_{\Delta \in \Pi_k} \left| \mathbb{P}^u(X_{n_k - n_{k-1}} \in \Delta) - \mathbb{P}^z(X_{n_k} \in \Delta | X_{n_{k-1}} = u) \right|.$$

Notice that under the event B_N, which by Corollary 2.10 satisfies $P(B_N) \ge 1 - N^{-\xi(1)}$, the first regeneration time after hitting u is after no more than $R(N)$ steps. Therefore, the distance between the regeneration times of both annealed walks started in u and started in z conditioned to hit u is at most $2R(N)$ of one another. Using the annealed derivative estimation from Lemma 2.14 for the annealed walks after the regeneration times, we get

$$\left| \mathbb{P}^u(X_{n_k - n_{k-1}} \in \Delta) - \mathbb{P}^z(X_{n_k} \in \Delta | X_{n_{k-1}} = u) \right| \le \frac{CR(N) \cdot N_k^{d\theta}}{(n_k - n_{k-1} - R(N))^{(d+1)/2}}$$

$$\le \frac{CR(N) \cdot N_k^{d\theta}}{N_k^{(d+1)/2}},$$

recalling that due to Lemma 2.16 we only need to consider boxes Δ at distance $\leq \sqrt{n_k - n_{k-1}} R_5(n_k - n_{k-1})$ from the annealed expectation $\mathbb{E}^u[X_{n_k - n_{k-1}}]$, it follows that

$$(5.4) \leq \sum_{\Delta' \in \Pi_{k-1}} \sum_{u \in \Delta'} \mathbb{P}^z(X_{n_{k-1}} = u) \cdot \left(\frac{dN_{k-1}^\theta + \sqrt{n_k - n_{k-1}} R_5(n_k - n_{k-1})}{\lfloor N_k^\theta \rfloor} \right)^d$$

$$\times \frac{CR(N) \cdot N_k^{d\theta}}{N_k^{(d+1)/2}} + N_k^{-\xi(1)}$$

$$= \left(\frac{dN_{k-1}^\theta + \sqrt{N_k} R_5(N_k)}{\lfloor N_k^\theta \rfloor} \right)^d \cdot \frac{CR(N) \cdot N_k^{d\theta}}{N_k^{(d+1)/2}} + N_k^{-\xi(1)}$$

$$\leq CR_6(N) N_k^{-1/2}.$$

Combining all of the above we conclude that under the event $G_N \cap B_N$ (whose probability is $\geq 1 - N^{-\xi(1)}$) for every $k \geq 1$

$$\lambda_k \leq \lambda_{k-1} + CN_k^{1-(4/3)\theta - d/2} R_5^d(N_k) + CN_k^{-1/2} + CR(N) N_k^{-1/2}$$

$$\leq \lambda_{k-1} + CN_k^{-1/2} R_5(N_k) \leq \lambda_{k-1} + CN_k^{-1/3}.$$

Consequently,

$$\lambda_{r(N)} \leq \lambda_1 + C \sum_{k=1}^{r(N)} N_k^{-1/3} = \lambda_1 + C \sum_{k=1}^{r(N)} \frac{1}{\lfloor N^{1/2^k} \rfloor^{1/3}}$$

$$\leq \lambda_1 + C \sum_{k=1}^{r(N)} N^{-1/(3 \cdot 2^k)} \leq \lambda_1 + C \int_1^{r(N)+1} e^{-1/(3 \cdot 2^t) \cdot \log N} dt$$

$$\overset{u = 1/(3 \cdot 2^t) \cdot \log N}{=} \lambda_1 + C \int_{\alpha_N}^{\beta_N} \frac{e^{-u}}{-\ln 3 \cdot u} du,$$

where $\alpha_N = \frac{1}{6} \cdot \log N$ and $\beta_N = \frac{1}{3 \cdot 2^{r(N)+1}} \cdot \log N$. Since for large enough N, we have $\beta_N \geq \frac{\theta}{\sqrt[6]{M}} \log N \geq 1$ we get

$$\lambda_k \leq \lambda_1 + C \int_{\alpha_N}^{\beta_N} -e^{-u} du = \lambda_1 + C[e^{-u}]_{\alpha_N}^{\beta_N} \leq \lambda_1 + Ce^{-\beta_N}$$

$$= \lambda_1 + \frac{C}{N^{1/(3 \cdot 2^{r(n)+1})}} \leq \lambda_1 + \frac{C}{M^{1/6}}.$$

Finally, recalling the definition λ_1 and the fact that $n_0 \geq cN$ it follows from Proposition 4.1, that $\lambda_1 \leq CN^{-(1/3)\theta}$, which completes the proof. $\square$

6. Proof of Theorem 1.10. In this section, we prove our first main result, that is, the existence of a probability measure on the space of environments, which is equivalent to the original i.i.d. measure and is invariant with respect to the point of view of the particle. The proof is divided into two parts. In the first and main part of the proof the existence of an invariant measure which is not singular with respect to the original i.i.d. measure is proved. In the second part, we show that the existence of such a measure guarantees the existence of an equivalent invariant measure.

In order to prove the existence of a nonsingular invariant measure, we exploit the result from the last section which allows us to construct a coupling of the annealed and quenched law of the walk at time N such that for most environments, that is, with P probability $\geq 1 - N^{-\xi(1)}$, will keep them at distance at most M of one another with positive probability independent of N. Using the uniform ellipticity, the last coupling can be strengthen to guarantee the walks will coincide at time N with positive probability, which again is uniform in N. Defining now, two random environments ω_N, ω'_N which are the original environment shifted according to the location of the annealed and quenched random walks at time N, respectively, we get a coupling of the two such that $\omega_N = \omega'_N$ with positive probability. Taking a Cesaro partial limit of the laws of ω_N and ω'_N, we get two probability measures on environments which are the original i.i.d. measure and an invariant measure with respect to the point of view of the particle, respectively. By taking the above coupling to the limit, we can conclude that both measures will give the same environment with positive probability, and, therefore, in particular that they are not singular.

In the second part of the proof (see Lemma 6.2), we use general properties of probability measures which are invariant with respect to the point of view of the particle in order to show that the existence of a nonsingular invariant probability measure guarantees the existence of an equivalent invariant one. Recently, we learned that the method of obtaining an absolutely continuous probability measure from a nonsingular one already appeared in [19], Lemma 5. For the readers convenience and in order to keep the section self-contained, we include a proof below.

In Section 6.2, we discuss several properties of the Radon–Nikodym derivative of the invariant measure with respect to the i.i.d. measure. This includes estimation on its average on a box as well as the existence of all of its moments.

6.1. *Existence of an equivalent measure.*

LEMMA 6.1. *Let $d \geq 4$ and assume P is uniformly elliptic, i.i.d. and satisfies ($\mathscr{P}$). Then there exists a measure Q on the space of environments which is invariant with respect to the point of view of the particle and is not singular with respect to the original i.i.d. measure P.*

PROOF.   Fix $0 < \varepsilon < 1$, a large $M \in \mathbb{N}$ and denote by $K(N) = K(N, M, \varepsilon)$ the set of environments $\omega \in \Omega$ such that

$$(6.1) \qquad \sum_{\Delta \in \Pi(M)} \left| P_\omega^0 (X_N \in \Delta) - \mathbb{P}^0 (X_N \in \Delta) \right| < \varepsilon,$$

where $\Pi(M)$ is a partition of $\mathbb{Z}^d$ into d-dimensional boxes of side length M. By Theorem 5.1, for every $\varepsilon > 0$ there exists $M \in \mathbb{N}$ (independent of N) such that $P(K(N)) \geq 1 - N^{-\xi(1)}$. Equation (6.1) tells us that the total variation distance of the respective distributions $\mathbb{P}^0 (X_N \in \cdot)$ and $P_\omega^0 (X_N \in \cdot)$ on $\Pi(M)$ is less than ε and that therefore there exists a coupling $\tilde{\Theta}_{\omega, N, M}$ on $\Pi(M) \times \Pi(M)$ of both measures such that $\tilde{\Theta}_{\omega, N, M}(\Lambda_\Pi) > 1 - \varepsilon$, where $\Lambda_\Pi = \{(\Delta, \Delta) : \Delta \in \Pi(M)\}$.

Next, using the last coupling, we show how to construct a new coupling of $\mathbb{P}^0 (X_N = \cdot)$ and $P_\omega^0 (X_N = \cdot)$ on $\mathbb{Z}^d \times \mathbb{Z}^d$ which gives a positive (independent of N) probability to the event $\Lambda = \{(x, x) : x \in \mathbb{Z}^d\}$. Define $\Theta_{\omega, N}$ on $\mathbb{Z}^d \times \mathbb{Z}^d$ by

$$\Theta_{\omega, N}(x, y) := \sum_{\Delta, \Delta' \in \Pi(M)} \tilde{\Theta}_{\omega, N-dM, M}(\Delta, \Delta') \mathbb{P}^0 (X_N = x | X_{N-dM} \in \Delta) P_\omega^0$$

$$\times (X_N = y | X_{N-dM} \in \Delta').$$

Note that due to the law of total probability $\Theta_{\omega, N}$ is indeed a coupling of $\mathbb{P}^0 (X_N = \cdot)$ and $P_\omega^0 (X_N = \cdot)$.

For $x \in \mathbb{Z}^d$, let Δ_x be the unique cube that contains x in the partition $\Pi(M)$. Since the side length of each box in the partition $\Pi(M)$ is M it follows that the random walk can reach from each point in the box Δ_x to x in less than dM steps. Recalling also that the law of P is uniformly elliptic with elliptic constant η [see (1.1)] we conclude that

$$\Theta_{\omega, N}(x, x) \geq \tilde{\Theta}_{\omega, N-dM, M}(\Delta_x, \Delta_x) \mathbb{P}^0 (X_N = x | X_{N-dM} \in \Delta_x) P_\omega^0$$

$$\times (X_N = x | X_{N-dM} \in \Delta_x)$$

$$\geq \tilde{\Theta}_{\omega, N-dM, M}(\Delta_x, \Delta_x) \eta^{2dM}.$$

Summing over x, we get

$$\Theta_{\omega, N}(\Lambda) = \sum_{x \in \mathbb{Z}^d} \Theta_{\omega, N}(x, x) \geq \sum_{x \in \mathbb{Z}^d} \tilde{\Theta}_{\omega, N, M}(\Delta_x, \Delta_x) \eta^{2dM}$$

$$= \sum_{\Delta \in \Pi(M)} \tilde{\Theta}_{\omega, N, M}(\Delta, \Delta) M^d \eta^{2dM} > (1 - \varepsilon) M^d \eta^{2dM}.$$

The last coupling allows us to construct for every N two probability measures on Ω that coincide with positive probability (independent of N). Indeed, for $N \in \mathbb{N}$ let Q_N and P_N be defined by

$$P_N(A) = E \left[\sum_{x \in \mathbb{Z}^d} \mathbb{P}^0 (X_N = x) \mathbb{1}_{\sigma_x \omega \in A} \right]$$

and

$$Q_N(A) = E\left[\sum_{x \in \mathbb{Z}^d} P_\omega^0(X_N = x)\mathbb{1}_{\sigma_x \omega \in A}\right].$$

Note that for every $N \in \mathbb{N}$ the measure P_N is in fact the i.i.d. measure P since the annealed walk is independent of the environment distribution. Indeed, for every measurable event $A \subset \Omega$

$$P_N(A) = E\left[\sum_{x \in \mathbb{Z}^d} \mathbb{P}^0(X_N = x)\mathbb{1}_{\sigma_x \omega \in A}\right] = \sum_{x \in \mathbb{Z}^d} \mathbb{P}^0(X_N = x)E[\mathbb{1}_{\sigma_x \omega \in A}]$$

$$= \sum_{x \in \mathbb{Z}^d} \mathbb{P}^0(X_N = x)P(\sigma_{-x}A) = \sum_{x \in \mathbb{Z}^d} \mathbb{P}^0(X_{T_N} = x)P(A) = P(A).$$

Also note that using the coupling $\Theta_{\omega,N}$ we have for every measurable event A

$$\left|Q_N(A) - P_N(A)\right| = \left|E\left[\sum_{x \in \mathbb{Z}^d}[\mathbb{P}^0(X_N = x) - P_\omega^0(X_N = x)]\mathbb{1}_{\sigma_x \omega \in A}\right]\right|$$

$$= \left|E\left[\sum_{x \in \mathbb{Z}^d}\left[\sum_{y \in \mathbb{Z}^d}\Theta_{\omega,N}(x,y) - \sum_{z \in \mathbb{Z}^d}\Theta_{\omega,N}(z,x)\right]\mathbb{1}_{\sigma_x \omega \in A}\right]\right|$$

$$= \left|E\left[\sum_{x \in \mathbb{Z}^d}\left[\sum_{y \neq x}\Theta_{\omega,N}(x,y) - \sum_{z \neq x}\Theta_{\omega,N}(z,x)\right]\mathbb{1}_{\sigma_x \omega \in A}\right]\right|$$

$$\leq \max\left\{\sum_{x \in \mathbb{Z}^d}\sum_{y \neq x}\Theta_{\omega,N}(x,y), \sum_{x \in \mathbb{Z}^d}\sum_{z \neq x}\Theta_{\omega,N}(z,x)\right\}$$

$$< 1 - (1 - \varepsilon)M^d \eta^{2dM}.$$

Let $\{n_k\}$ be a subsequence such that the weak limits of the Cesaro sequences $\{\frac{1}{n_k}\sum_{N=0}^{n_k-1} Q_N\}_{k \geq 1}$, $\{\frac{1}{n_k}\sum_{N=0}^{n_k-1} P_N\}_{k \geq 1}$ and $\{\frac{1}{n_k}\sum_{N=0}^{n_k-1} \Theta_{\omega,N}\}_{k \geq 1}$ exists. Since for every $N \in \mathbb{N}$ the measure P_N equals P it follows that the limit of $\{\frac{1}{n_k}\sum_{N=0}^{n_k-1} P_N\}_{k \geq 1}$ is P as well. Next, notice that the weak limit of $\{\frac{1}{n_k}\sum_{N=0}^{n_k-1} Q_N\}_{k \geq 1}$ which we denote by Q is invariant with respect to the point of view of the particle [see (1.2) and (1.3) for the definition]. Indeed, for every bounded continuous function $f : \Omega \to \mathbb{R}$

$$\int_\Omega \mathfrak{R}f(\omega)\,dQ(\omega) = \lim_{k \to \infty} \frac{1}{n_k}\sum_{N=0}^{n_k-1}\int_\Omega \mathfrak{R}f(\omega)\,dQ_N(\omega)$$

$$= \lim_{k \to \infty} \frac{1}{n_k}\sum_{N=0}^{n_k-1}\int_\Omega \sum_{e \in \mathcal{E}_d} \omega(0,e)f(\sigma_e \omega)\,dQ_N(\omega)$$

$$= \lim_{k \to \infty} \frac{1}{n_k} \sum_{N=0}^{n_k-1} \int_\Omega f(\omega) \, dQ_{N+1}(\omega)$$

$$= \lim_{k \to \infty} \frac{1}{n_k} \sum_{N=1}^{n_k} \int_\Omega f(\omega) \, dQ_N(\omega)$$

$$= \int_\Omega f(\omega) \, dQ(\omega),$$

where $\mathfrak{R}$ is as in (1.2). Finally, we show that Q and P are not singular. Using the coupling of P_N and Q_N, for every event $A \subset \Omega$ we have

$$\big|P(A) - Q(A)\big| = \lim_{k \to \infty} \frac{1}{n_k} \left| \sum_{N=0}^{n_k-1} (P_N(A) - Q_N(A)) \right|$$

$$\leq \lim_{k \to \infty} \frac{1}{n_k} \sum_{N=0}^{n_k-1} \big|P_N(A) - Q_N(A)\big|$$

$$\leq 1 - (1-\varepsilon)M^d \eta^{2dM}.$$

Since this holds for all events, it follows that $\|P - Q\|_{\mathrm{TV}} \leq 1 - (1-\varepsilon)M^d \eta^{2dM}$, and thus P and Q are not singular. $\quad\square$

LEMMA 6.2. *Assume P is uniformly elliptic and i.i.d. If there exists a probability measure Q on the space of environments which is invariant with respect to the point of view of the particle and is not singular with respect to P, then there exists a probability measure $\tilde{Q}$ which is invariant with respect to the point of view of the particle and is also equivalent to P.*

PROOF. Denote by $Q = Q_c + Q_s$ the Lebesgue decomposition of Q to an absolutely continuous part Q_c (w.r.t. P) and a singular part Q_s (w.r.t. P). Let $f = \frac{dQ_c}{dP}$ and define $A = \{\omega \in \Omega : f(\omega) = 0\}$. From the invariance with respect to the point of view of the particle and the uniform ellipticity, we have

$$Q = \sum_{e \in \mathcal{E}_d} \omega(e)\sigma_e \circ Q \geq \eta \sum_{e \in \mathcal{E}_d} \sigma_e \circ Q$$

and, therefore, $\sigma_e \circ Q \ll Q$ for every $e \in \mathcal{E}_d$. Since in addition we have $(\sigma_e \circ Q)_s = \sigma_e \circ Q_s$, $(\sigma_e \circ Q)_c = \sigma_e \circ Q_c$ and $\frac{d(\sigma_e \circ Q)_c}{dP}(\cdot) = f(\sigma_e(\cdot))$ we get that

$$(6.2) \qquad Q_c = \sum_{e \in \mathcal{E}_d} \omega(e)\sigma_e \circ Q_c \geq \eta \sum_{e \in \mathcal{E}_d} \sigma_e \circ Q_c$$

and thus

$$f(\omega) \geq \eta \sum_{e \in \mathcal{E}_d} f\big(\sigma_e(\omega)\big).$$

Consequently, $\omega \in A$ implies $\sigma_e \omega \in A$ for every $e \in \mathcal{E}_d$, P-a.s.

In particular, we get that A is σ_{e_1} invariant and, therefore, by ergodicity that it is a 0–1 event. This immediately implies that if Q is not singular with respect to P, that is, $P(A) \neq 1$, then $P(A) = 0$ and thus $P \ll Q_c$. Taking $\tilde{Q} = \frac{Q_c}{Q_c(\Omega)}$, we get that $\tilde{Q}$ is equivalent to the i.i.d. measure and is invariant with respect to the point of view of the particle [by the first equality in (6.2)]. $\quad\square$

REMARK 6.3. Note that the sequence of probability measures $\{\sum_{N=0}^{n-1} Q_N\}_{n \geq 1}$ equals to $\{\sum_{N=0}^{n-1} \mathfrak{R}^N P\}_{n \geq 1}$. Recalling Theorem 1.9 it follows that the measure $\sum_{N=0}^{n-1} Q_N$ converges (without taking a subsequence) to the equivalent measure Q which is the *unique* probability measure invariant with respect to the point of view of the particle. In particular, there is no need to restrict ourselves to the absolutely continuous part as done in Lemma 6.2.

6.2. *Some properties of the Radon–Nikodym derivative.* In this subsection, we discuss some properties of the equivalent probability measure Q and its Radon–Nikodym derivative. The next definition will be useful in the statement of the lemmas.

DEFINITION 6.4. Given two environments $\omega, \omega' \in \Omega$ define their distance by

$$\mathrm{dist}(\omega, \omega') = \inf\{\|x\|_1 : \omega' = \sigma_x \omega\},$$

where the infimum over an empty set is defined to be infinity.

For future use, we denote by Ψ and Ψ_N the couplings of P and Q and of P_N and Q_N, respectively, on $\Omega \times \Omega$, that is,

$$(6.3) \qquad \Psi_N(A) = E\left[\sum_{x,y \in \mathbb{Z}^d} \Theta_{\omega,N}(x,y) \mathbb{1}_{(\sigma_x \omega, \sigma_y \omega) \in A}\right],$$

and Ψ is the weak limit of the Cesaro sequence $\{\frac{1}{n} \sum_{N=0}^{n-1} \Psi_N\}_{n=1}^{\infty}$ along any converging sub-sequence which we denote from here on by $\{n_k\}_{k \geq 1}$.

Our main goal is to prove the following concentration inequality for the average of the Radon–Nikodym derivative on a box.

LEMMA 6.5. *Let $M \in \mathbb{N}$ and denote by Δ_0 a d-dimensional cube of side length M in $\mathbb{Z}^d$. Then for every $\varepsilon > 0$,*

$$P\left(\left|\frac{1}{|\Delta_0|} \sum_{x \in \Delta_0} \frac{dQ}{dP}(\sigma_x \omega) - 1\right| > \varepsilon\right) \leq M^{-\xi(1)}.$$

As a first step toward the proof of Lemma 6.5, we prove the following.

LEMMA 6.6.   *For $M \in \mathbb{N}$ let $D_M^{(1)} : \Omega \to [0, \infty]$ and $D_M^{(2)} : \Omega \to [0, \infty]$ be defined by*

$$D_M^{(1)}(\omega) = E_\Psi[\mathbb{1}_{\mathrm{dist}(\omega_1, \omega_2) > dM} | \mathfrak{F}_{\omega_1}](\omega)$$

and

$$D_M^{(2)}(\omega) = E_\Psi[\mathbb{1}_{\mathrm{dist}(\omega_1, \omega_2) > dM} | \mathfrak{F}_{\omega_2}](\omega),$$

where $\mathfrak{F}_{\omega_1}, \mathfrak{F}_{\omega_2}$ are the σ-algebras generated by the first, respectively, second coordinate in $\Omega \times \Omega$ and Ψ is as defined below (6.3). For every $M \in \mathbb{N}$, there exists an event $\mathbf{F}_M$ with the following properties:

(1) $P(\mathbf{F}_M) = 1 - M^{-\xi(1)}$.

(2) *For every $\varepsilon > 0$, if M is large enough, then $D_M^{(1)}(\omega) \leq \varepsilon \mathbb{1}_{\mathbf{F}_M}(\omega) + \mathbb{1}_{\mathbf{F}_M^c}(\omega)$ and $\frac{dQ}{dP}(\omega) D_M^{(2)}(\omega) \leq \varepsilon \mathbb{1}_{\mathbf{F}_M}(\omega) + \mathbb{1}_{\mathbf{F}_M^c}(\omega)$.*

PROOF.   Let

$$\mathbf{F}_M = \bigcap_{k=M}^{\infty} \left\{ \omega \in \Omega : \forall x \in [-k, k]^d \cap \mathbb{Z}^d, \right.$$

$$\left. \sum_{\Delta \in \Pi_M} |\mathbb{P}^x(X_k \in \Delta) - P_\omega^x(X_k \in \Delta)| \leq \frac{C_2}{M^{c_1}} + \frac{C_2}{k^{c_1}} \right\},$$

where Π_M is a partition of $\mathbb{Z}^d$ into boxes of side length M and $0 < c_1, C_2 < \infty$ are the constants from Theorem 5.1. Thus, by the same theorem, we have $P(\mathbf{F}_M) = 1 - M^{-\xi(1)}$. Fix some $\varepsilon > 0$. The definition of $\mathbf{F}_M$ together with the definition of the couplings $\tilde{\Theta}_{\omega, k, M}$ constructed in the proof of Lemma 6.1 implies that for every $\omega \in \mathbf{F}_M$, every $k \geq M$ and every $x \in [-k, k]^d \cap \mathbb{Z}^d$ we have $\tilde{\Theta}_{\sigma_x \omega, k, M}(\Lambda_{\Pi_M}) > 1 - \frac{2C_2}{M^{c_1}} > 1 - \varepsilon$ for large enough M, where as before $\Lambda_{\Pi_M} = \{(\Delta, \Delta) : \Delta \in \Pi_M\}$.

Before turning to prove the estimates for $\{D_M^{(i)}(\omega)\}_{i \in \{1, 2\}}$, we prove a similar results for the conditional expectations of Ψ_N. For $N, M \in \mathbb{N}$ and $i \in \{1, 2\}$, define $D_{M,N}^{(i)} : \Omega \to [0, \infty]$ by $D_{M,N}^{(i)}(\omega) = E_{\Psi_N}[\mathbb{1}_{\mathrm{dist}(\omega_1, \omega_2) > dM} | \mathfrak{F}_{\omega_i}](\omega)$. Note that for P-almost every environment $\omega \in \Omega$ we have

$$(6.4) \qquad D_{M,N}^{(1)}(\omega) = \sum_{x, y \in \mathbb{Z}^d} \Theta_{\sigma_{-x}\omega, N}(x, y) \mathbb{1}_{\|x - y\|_1 > dM}$$

and

$$(6.5) \qquad D_{M,N}^{(2)}(\omega) = \left(\frac{dQ_N}{dP}(\omega) \right)^{-1} \sum_{x, y \in \mathbb{Z}^d} \Theta_{\sigma_{-y}\omega, N}(x, y) \mathbb{1}_{\|x - y\|_1 > dM}.$$

Indeed, using (6.3) we have for every measurable event $A \subset \Omega$

$$E_{\Psi_N}[\mathbb{1}_{A \times \Omega} \mathbb{1}_{\text{dist}(\omega_1, \omega_2) > dM}]$$
$$= \Psi_N (A \times \Omega \cap \{(\omega_1, \omega_2) : \text{dist}(\omega_1, \omega_2) > dM\})$$
$$= E\left[\sum_{x, y \in \mathbb{Z}^d} \Theta_{\omega, N}(x, y) \mathbb{1}_{(\sigma_x \omega, \sigma_y \omega) \in A \times \Omega} \mathbb{1}_{\text{dist}(\sigma_x \omega, \sigma_y \omega) > dM} \right]$$
$$= \sum_{x, y \in \mathbb{Z}^d} E[\Theta_{\omega, N}(x, y) \mathbb{1}_{\sigma_x \omega \in A} \mathbb{1}_{\|x - y\|_1 > dM}],$$

which by translation invariance of P equals

$$\sum_{x, y \in \mathbb{Z}^d} E[\Theta_{\sigma_{-x} \omega, N}(x, y) \mathbb{1}_{\omega \in A} \mathbb{1}_{\|x - y\|_1 > dM}]$$
$$= E\left[\sum_{x, y \in \mathbb{Z}^d} \Theta_{\sigma_{-x} \omega, N}(x, y) \mathbb{1}_{\omega \in A} \mathbb{1}_{\|x - y\|_1 > dM} \right].$$

Due to the fact that the first marginal of Ψ_N is P the last term equals

$$E_{\Psi_N}\left[\mathbb{1}_{(\omega, \omega') \in A \times \Omega} \cdot \sum_{x, y \in \mathbb{Z}^d} \Theta_{\sigma_{-x} \omega, N}(x, y) \mathbb{1}_{\|x - y\|_1 > dM} \right],$$

which by the definition of conditional expectation implies (6.4). A similar argument shows that

$$E_{\Psi_N}[\mathbb{1}_{\Omega \times A} \mathbb{1}_{\text{dist}(\omega_1, \omega_2) > dM}]$$
$$= E\left[\sum_{x, y \in \mathbb{Z}^d} \Theta_{\sigma_{-x} \omega', N}(x, y) \mathbb{1}_{\omega' \in A} \mathbb{1}_{\|x - y\|_1 > dM} \right]$$
$$= E_{Q_N}\left[\left(\frac{dQ_N}{dP}(\omega') \right)^{-1} \sum_{x, y \in \mathbb{Z}^d} \Theta_{\sigma_{-x} \omega', N}(x, y) \mathbb{1}_{\|x - y\|_1 > dM} \mathbb{1}_{\omega' \in A} \right]$$
$$= E_{\Psi_N}\left[\left(\frac{dQ_N}{dP}(\omega_2) \right)^{-1} \sum_{x, y \in \mathbb{Z}^d} \Theta_{\sigma_{-x} \omega_2, N}(x, y) \mathbb{1}_{\|x - y\|_1 > dM} \cdot \mathbb{1}_{\Omega \times A}(\omega_1, \omega_2) \right]$$

and thus that (6.5) holds as well.

Since $\Theta_{\sigma_{-x} \omega, N}(x, y) > 0$ implies $x \in [-N, N]^d \cap \mathbb{Z}^d$, it follows that for large enough M, every $\omega \in \mathbf{F}_M$ and every $N \geq M$

$$\sum_{x, y \in \mathbb{Z}^d} \Theta_{\sigma_{-x} \omega, N}(x, y) \mathbb{1}_{\|x - y\|_1 > dM}$$
$$= 1 - \sum_{x, y \in \mathbb{Z}^d} \Theta_{\sigma_{-x} \omega, N}(x, y) \mathbb{1}_{\|x - y\|_1 \leq dM}$$

$$\leq 1 - \min_{z \in [-N,N]^d \cap \mathbb{Z}^d} \sum_{x,y \in \mathbb{Z}^d} \Theta_{\sigma_{-z}\omega,N}(x,y)\mathbb{1}_{\|x-y\|_1 \leq dM}$$

$$\leq 1 - \min_{z \in [-N,N]^d \cap \mathbb{Z}^d} \sum_{\Delta \in \Pi_M} \sum_{x,y \in \Delta} \Theta_{\sigma_{-z}\omega,N}(x,y)$$

$$= 1 - \min_{z \in [-N,N]^d \cap \mathbb{Z}^d} \left(\sum_{\Delta \in \Pi_M} \tilde{\Theta}_{\sigma_{-z}\omega,N,M}(\Delta,\Delta) \right)$$

$$= 1 - \min_{z \in [-N,N]^d \cap \mathbb{Z}^d} \tilde{\Theta}_{\sigma_{-z}\omega,N,M}(\Lambda_{\Pi_M}) < \varepsilon.$$

Thus,

$$D_{M,N}^{(1)}(\omega) = \sum_{x,y \in \mathbb{Z}^d} \Theta_{\sigma_{-x}\omega,N}(x,y)\mathbb{1}_{\|x-y\|_1 > dM} \leq \varepsilon \mathbb{1}_{\mathbf{F}_M}(\omega) + \mathbb{1}_{\mathbf{F}_M^c}(\omega)$$

and similarly,

$$\frac{dQ_N}{dP}(\omega)D_{M,N}^{(2)}(\omega) = \sum_{x,y \in \mathbb{Z}^d} \Theta_{\sigma_{-y}\omega,N}(x,y)\mathbb{1}_{\|x-y\|_1 > dM} \leq \varepsilon \mathbb{1}_{\mathbf{F}_M}(\omega) + \mathbb{1}_{\mathbf{F}_M^c}(\omega).$$

Next, we turn to prove the estimate for $\{D_M^{(i)}\}_{i \in \{1,2\}}$. It is enough to show that along some sub-sequence of $\{n_k\}_{k \geq 1}$ (which for simplicity we still denote by $\{n_k\}_{k \geq 1}$)

$$D_M^{(1)}(\omega) = \lim_{k \to \infty} \frac{1}{n_k} \sum_{N=0}^{n_k-1} D_{M,N}^{(1)}(\omega) \quad \text{and}$$

(6.6)

$$D_M^{(2)}(\omega) = \left(\frac{dQ}{dP}(\omega) \right)^{-1} \lim_{k \to \infty} \frac{1}{n_k} \sum_{N=0}^{n_k-1} \frac{dQ_N}{dP}(\omega)D_{M,N}^{(2)}(\omega), \qquad P\text{-a.s.}$$

Indeed, if (6.6) holds, then for P-almost every ω we have

$$D_M^{(1)}(\omega) = \lim_{k \to \infty} \frac{1}{n_k} \sum_{N=0}^{n_k-1} D_{M,N}^{(1)}(\omega)$$

$$= \lim_{k \to \infty} \frac{1}{n_k} \left[\sum_{N=0}^{M-1} D_{M,N}^{(1)}(\omega) + \sum_{N=M}^{n_k-1} D_{M,N}^{(1)}(\omega) \right]$$

$$\leq \lim_{k \to \infty} \frac{1}{n_k} \left[M + \sum_{N=M}^{n_k-1} D_{M,N}^{(1)}(\omega) \right]$$

$$\leq \lim_{k \to \infty} \frac{1}{n_k} \left[M + \sum_{N=M}^{n_k-1} \left(\varepsilon \mathbb{1}_{\mathbf{F}_M}(\omega) + \mathbb{1}_{\mathbf{F}_M^c}(\omega) \right) \right]$$

$$= \varepsilon \mathbb{1}_{\mathbf{F}_M}(\omega) + \mathbb{1}_{\mathbf{F}_M^c}(\omega)$$

and similarly

$$\frac{dQ}{dP}(\omega)D_M^{(2)}(\omega)$$

$$= \lim_{k\to\infty} \frac{1}{n_k} \sum_{N=0}^{n_k-1} \frac{dQ_N}{dP}(\omega) D_{M,N}^{(2)}(\omega)$$

$$= \lim_{k\to\infty} \frac{1}{n_k} \left[\sum_{N=0}^{M-1} \frac{dQ_N}{dP}(\omega) D_{M,N}^{(2)}(\omega) + \sum_{N=M}^{n_k-1} \frac{dQ_N}{dP}(\omega) D_{M,N}^{(2)}(\omega) \right]$$

$$\leq \lim_{k\to\infty} \frac{1}{n_k} \left[\sum_{N=0}^{M-1} \frac{dQ_N}{dP}(\omega) + \sum_{N=M}^{n_k-1} \frac{dQ_N}{dP} D_{M,N}^{(1)}(\omega) \right]$$

$$\leq \lim_{k\to\infty} \frac{1}{n_k} \left[\sum_{N=0}^{M-1} \frac{dQ_N}{dP}(\omega) + \sum_{N=M}^{n_k-1} \left(\varepsilon \mathbb{1}_{\mathbf{F}_M}(\omega) + \mathbb{1}_{\mathbf{F}_M^c}(\omega) \right) \right]$$

$$= \varepsilon \mathbb{1}_{\mathbf{F}_M}(\omega) + \mathbb{1}_{\mathbf{F}_M^c}(\omega).$$

Turning to prove (6.6), for every measurable event $A \subset \Omega$ we have

$$E\big[D_M^{(1)}(\omega)\mathbb{1}_A(\omega)\big]$$

$$= E_\Psi\big[\mathbb{1}_{\text{dist}(\omega_1,\omega_2)>dM} \cdot \mathbb{1}_{A\times\Omega}(\omega_1,\omega_2)\big]$$

$$= \Psi\big(\{(\omega_1,\omega_2) : \text{dist}(\omega_1,\omega_2) > dM\} \cap A \times \Omega\big)$$

$$\stackrel{(1)}{=} \lim_{k\to\infty} \frac{1}{n_k} \sum_{N=0}^{n_k-1} \Psi_N\big(\{(\omega_1,\omega_2) : \text{dist}(\omega_1,\omega_2) > dM\} \cap A \times \Omega\big)$$

$$= \lim_{k\to\infty} \frac{1}{n_k} \sum_{N=0}^{n_k-1} E_{\Psi_N}\big[\mathbb{1}_{\text{dist}(\omega_1,\omega_2)>dM} \cdot \mathbb{1}_{A\times\Omega}(\omega_1,\omega_2)\big]$$

$$\stackrel{(2)}{=} \lim_{k\to\infty} \frac{1}{n_k} \sum_{N=0}^{n_k-1} E_{\Psi_N}\big[D_{M,N}^{(1)}(\omega_1) \cdot \mathbb{1}_{A\times\Omega}(\omega_1,\omega_2)\big]$$

$$= \lim_{k\to\infty} \frac{1}{n_k} \sum_{N=0}^{n_k-1} E\big[D_{M,N}^{(1)}(\omega_1) \cdot \mathbb{1}_A(\omega_1)\big]$$

$$= \lim_{k\to\infty} E\left[\frac{1}{n_k} \sum_{N=0}^{n_k-1} D_{M,N}^{(1)}(\omega_1) \cdot \mathbb{1}_A(\omega_1) \right],$$

where (1) is due to the definition of Ψ below (6.3) and (2) uses the definition of $D_{M,N}^{(1)}$ as the conditional expectation. This implies that $\frac{1}{n_k} \sum_{N=0}^{n_k-1} D_{M,N}^{(1)}$ converges in $L^1(P)$ to $D_M^{(1)}$ and thus by standard arguments contains a subsequence that

converges P-almost surely. Similarly, for $D_M^{(2)}$

$$E_Q[D_M^{(2)}(\omega)\mathbb{1}_A(\omega)]$$

$$= E_\Psi[\mathbb{1}_{\text{dist}(\omega_1,\omega_2)>dM} \cdot \mathbb{1}_{\Omega \times A}(\omega_1, \omega_2)]$$

$$= \Psi(\{(\omega_1, \omega_2) : \text{dist}(\omega_1, \omega_2) > dM\} \cap \Omega \times A)$$

$$= \lim_{k \to \infty} \frac{1}{n_k} \sum_{N=0}^{n_k-1} \Psi_N(\{(\omega_1, \omega_2) : \text{dist}(\omega_1, \omega_2) > dM\} \cap \Omega \times A)$$

$$= \lim_{k \to \infty} \frac{1}{n_k} \sum_{N=0}^{n_k-1} E_{\Psi_N}[\mathbb{1}_{\text{dist}(\omega_1,\omega_2)>dM} \cdot \mathbb{1}_{\Omega \times A}(\omega_1, \omega_2)]$$

$$= \lim_{k \to \infty} \frac{1}{n_k} \sum_{N=0}^{n_k-1} E_{\Psi_N}[D_{M,N}^{(2)}(\omega_2) \cdot \mathbb{1}_{\Omega \times A}(\omega_1, \omega_2)]$$

$$= \lim_{k \to \infty} \frac{1}{n_k} \sum_{N=0}^{n_k-1} E_{Q_N}[D_{M,N}^{(2)}(\omega_2) \cdot \mathbb{1}_A(\omega_2)]$$

$$= \lim_{k \to \infty} \frac{1}{n_k} \sum_{N=0}^{n_k-1} E_Q\left[\left(\frac{dQ}{dP}(\omega_2)\right)^{-1} \cdot \frac{dQ_N}{dP}(\omega_2) \cdot D_{M,N}^{(2)}(\omega_2) \cdot \mathbb{1}_A(\omega_2)\right]$$

$$= \lim_{k \to \infty} E_Q\left[\left(\frac{dQ}{dP}(\omega_2)\right)^{-1} \cdot \frac{1}{n_k} \sum_{N=0}^{n_k-1} \frac{dQ_N}{dP}(\omega_2) \cdot D_{M,N}^{(2)}(\omega_2) \cdot \mathbb{1}_A(\omega_2)\right].$$

This proves the second quality in (6.6), Q (and thus P)-almost surely for an appropriate sub-sequence. □

PROOF OF LEMMA 6.5. The proof deals separately with the events $B_\varepsilon^- = \{\omega \in \Omega : \frac{1}{|\Delta_0|} \sum_{x \in \Delta_0} \frac{dQ}{dP}(\sigma_x \omega) < 1 - \varepsilon\}$ and $B_\varepsilon^+ = \{\omega \in \Omega : \frac{1}{|\Delta_0|} \sum_{x \in \Delta_0} \frac{dQ}{dP}(\sigma_x \omega) > 1 + \varepsilon\}$. We start with the event B_ε^-. The idea is to separate the event B_ε^- into two events the first with probability $M^{-\xi(1)}$ and the second, denoted S_ε^-, which will turn out to be with P probability zero measure. To this end, assume without loss of generality that Δ_0 is centered at the zero, denote $M_\varepsilon = \frac{\varepsilon}{6d^2}M$, define $\Delta_0^- = \{x \in \mathbb{Z}^d : \|x\|_\infty < M - dM_\varepsilon\}$ and let

$$S_\varepsilon^- = \{\omega \in B_\varepsilon^- : \sigma_x \omega \in \mathbf{F}_{M_\varepsilon}, \forall x \in \Delta_0\},$$

where $\mathbf{F}_{M_\varepsilon}$ is the event from Lemma 6.6. Due to property (1) of $\mathbf{F}_{M_\varepsilon}$ from Lemma 6.6

$$P(S_\varepsilon^-) \geq P(B_\varepsilon^-) - |\Delta_0|P(\mathbf{F}_{M_\varepsilon}^c) = P(B_\varepsilon^-) - M^d \cdot (M_\varepsilon)^{-\xi(1)}$$

$$= P(B_\varepsilon^-) - M^{-\xi(1)},$$

and, therefore, it is enough to show that $P(S_\varepsilon^-) = 0$. We claim that there exists an event $K \subset S_\varepsilon^-$ such that (1) $P(K) \geq P(S_\varepsilon^-) \cdot ((4d)^d |\Delta_0|)^{-1}$ and (2) if $\omega, \omega' \in K$ and $\omega \neq \omega'$, then $\mathrm{dist}(\omega, \omega') > 4dM$. Indeed, for every $x \in \mathbb{Z}^d$ let U_x be an independent (of everything defined so far) random variable uniformly distributed on $[0, 1]$, and define[4]

$$K = \{\omega \in S_\varepsilon^- : \forall x \in 4d\Delta_0 \text{ if } \sigma_x \omega \in B_\varepsilon^- \text{ then } U_x < U_0\}.$$

Informally, from each family of environments whose distance is smaller than $4dM$ we choose one uniformly. This immediately implies that for two distinct points in K property (2) holds. Property (1) on the other hand holds due to translation invariance of P.

Now, let

$$H = \bigcup_{x \in \Delta_0} \sigma_x K \quad \text{and} \quad H^- = \bigcup_{x \in \Delta_0^-} \sigma_x K.$$

By property (2) of K, in both cases this is a disjoint union and, therefore, recalling once more the translation invariance of the measure P, we have

(6.7)
$$P(H) = |\Delta_0| P(K) \quad \text{and}$$
$$P(H^-) = |\Delta_0^-| P(K) = |\Delta_0| \left(1 - \frac{\varepsilon}{6d^2}\right)^d P(K) > \left(1 - \frac{\varepsilon}{6}\right) P(H).$$

Going back to the definition of the event B_ε^- and recalling that $K \subset S_\varepsilon^- \subset B_\varepsilon^-$ we get

$$Q(H) = \int_H \frac{dQ}{dP}(\omega) \, dP(\omega) = \sum_{x \in \Delta_0} \int_{\sigma_x K} \frac{dQ}{dP}(\omega) \, dP(\omega)$$

$$= \int_K \sum_{x \in \Delta_0} \frac{dQ}{dP}(\sigma_x \omega) \, dP(\omega)$$

$$\leq \int_K (1 - \varepsilon) |\Delta_0| \, dP(\omega) = (1 - \varepsilon) |\Delta_0| P(K)$$

$$= (1 - \varepsilon) P(H).$$

Combining with (6.7), for small enough $\varepsilon > 0$ this yields

(6.8)
$$Q(H) \leq (1 - \varepsilon) P(H) = \frac{1 - \varepsilon}{1 - \varepsilon/6} \left(1 - \frac{\varepsilon}{6}\right) P(H) < \frac{1 - \varepsilon}{1 - \varepsilon/6} P(H^-)$$
$$< \left(1 - \frac{\varepsilon}{3}\right) P(H^-).$$

---

[4]The event K is not measurable in the σ-algebra of Ω. However, using Fubini's theorem we can find a section in Ω which is measurable and have the desired properties.

Let $A = \{(\omega, \omega') : \omega \in H^-, \omega' \notin H\}$. Then by (6.7) and (6.9)

$$\Psi(A) \geq P(H^-) - Q(H) \geq P(H) - \left(1 - \frac{\varepsilon}{3}\right) P(H^-)$$

(6.9)
$$\geq \frac{\varepsilon}{3} P(H^-) > \frac{\varepsilon}{3}\left(1 - \frac{\varepsilon}{6}\right) P(H) > \frac{\varepsilon}{4} P(H).$$

By the construction of K, for every $(\omega, \omega') \in A$ we have $\operatorname{dist}(\omega, \omega') > dM_\varepsilon$ and, therefore,

$$
\int_H D_{M_\varepsilon}^{(1)}(\omega) \, dP(\omega)
$$

(6.10)
$$
= \int_{H \times \Omega} D_{M_\varepsilon}^{(1)}(\omega) \, d\Psi(\omega, \omega') \geq \int_{H^- \times \Omega} D_{M_\varepsilon}^{(1)}(\omega) \, d\Psi(\omega, \omega')
$$

$$
= \int_{\Omega \times \Omega} E_\Psi[\mathbb{1}_{\operatorname{dist}(\omega, \omega') > dM_\varepsilon} | \mathfrak{F}_\omega](\omega) \mathbb{1}_{H^- \times \Omega}(\omega, \omega') \, d\Psi(\omega, \omega')
$$

$$
= \int_{\Omega \times \Omega} E_\Psi[\mathbb{1}_{\operatorname{dist}(\omega, \omega') > dM_\varepsilon} \mathbb{1}_{H^- \times \Omega}(\omega, \omega') | \mathfrak{F}_\omega](\omega) \, d\Psi(\omega, \omega')
$$

$$
= \int_{\Omega \times \Omega} \mathbb{1}_{\operatorname{dist}(\omega, \omega') > dM_\varepsilon} \mathbb{1}_{H^- \times \Omega}(\omega, \omega') \, d\Psi(\omega, \omega')
$$

$$
\geq \int_{\Omega \times \Omega} \mathbb{1}_{\operatorname{dist}(\omega, \omega') > dM_\varepsilon} \mathbb{1}_A(\omega, \omega') \, d\Psi(\omega, \omega')
$$

$$
= \int_{\Omega \times \Omega} \mathbb{1}_A(\omega, \omega') \, d\Psi(\omega, \omega')
$$

$$
= \Psi(A) > \frac{\varepsilon}{4} P(H),
$$

where for the last inequality we used (6.9). However, recalling that $H \subset \mathbf{F}_{M_\varepsilon}$ by definition, and using Lemma 6.6 with M_ε and $\frac{\varepsilon}{5}$ instead of M and ε we get

$$
\int_H D_{M_\varepsilon}^{(1)}(\omega) \, dP(\omega) \leq \int_H \frac{\varepsilon}{5} \mathbb{1}_{\mathbf{F}_{M_\varepsilon}}(\omega) + \mathbb{1}_{\mathbf{F}_{M_\varepsilon}^c}(\omega) \, dP(\omega)
$$

(6.11)
$$
= \int_H \frac{\varepsilon}{5} \, dP(\omega) = \frac{\varepsilon}{5} P(H).
$$

Combining (6.10) and (6.11) we must conclude that $P(H) = 0$ and, therefore $P(K) = 0$. This however, by property (1) of K, implies that $P(S_\varepsilon^-) = 0$ and, therefore, finally that $P(B_\varepsilon^-) = M^{-\xi(1)}$.

Next, we turn to deal with the event B_ε^+. As in the proof for B_ε^- for $\varepsilon > 0$ we denote $M_\varepsilon = \frac{\varepsilon}{6d^2} M$. Also assume without loss of generality that Δ_0 is centered in zero, define $\Delta_0^+ = \{x \in \mathbb{Z}^d : \|x\|_\infty < M + dM_\varepsilon\}$ and let

$$
S_\varepsilon^+ = \{\omega \in B_\varepsilon^+ : \sigma_x \omega \in \mathbf{F}_{M_\varepsilon}, \forall x \in \Delta_0^+\},
$$

where $\mathbf{F}_{M_\varepsilon}$ is the event from Lemma 6.6. Due to property (1) of $\mathbf{F}_{M_\varepsilon}$ from Lemma 6.6

$$P(S_\varepsilon^+) \geq P(B_\varepsilon^+) - |\Delta_0^+| P(\mathbf{F}_{M_\varepsilon}^c) = P(B_\varepsilon^+) - \left(1 + \frac{\varepsilon}{6d}\right)^d M^d \cdot (M_\varepsilon)^{-\xi(1)}$$

$$= P(B_\varepsilon^+) - M^{-\xi(1)}$$

and, therefore, it is enough to show that $P(S_\varepsilon^+) = 0$. As for S_ε^- we claim that there exists an event $K \subset S_\varepsilon^+$ such that (1) $P(K) \geq P(S_\varepsilon^+)/((4d)^d |\Delta_0^+|)^{-1}$ and (2) if $\omega, \omega' \in K$ and $\omega \neq \omega'$, then $\text{dist}(\omega, \omega') > 4d(M + M_\varepsilon)$.

Now, let

$$H = \bigcup_{x \in \Delta_0} \sigma_x K \quad \text{and} \quad H^+ = \bigcup_{x \in \Delta_0^+} \sigma_x K.$$

By property (2) of K, in both cases this is a disjoint union and, therefore, recalling once more the translation invariance of the measure P, we have for small enough $\varepsilon > 0$

(6.12)
$$P(H) = |\Delta_0| P(K) \quad \text{and}$$

$$P(H^+) = |\Delta_0^+| P(K) = \left(1 + \frac{\varepsilon}{6d^2}\right)^d |\Delta_0| P(K) < \left(1 + \frac{\varepsilon}{6}\right) P(H).$$

Going back to the definition of the event B_ε^+ and recalling that $K \subset S_\varepsilon^+ \subset B_\varepsilon^+$ we get

$$Q(H) = \int_H \frac{dQ}{dP}(\omega) \, dP(\omega) = \sum_{x \in \Delta_0} \int_{\sigma_x K} \frac{dQ}{dP}(\omega) \, dP(\omega)$$

$$= \int_K \sum_{x \in \Delta_0} \frac{dQ}{dP}(\sigma_x \omega) \, dP(\omega)$$

(6.13)

$$> \int_K (1 + \varepsilon) |\Delta_0| \, dP(\omega) = (1 + \varepsilon) |\Delta_0| P(K)$$

$$= (1 + \varepsilon) P(H),$$

and, therefore, combining with (6.12), for small enough $\varepsilon > 0$ this yields

(6.14)
$$Q(H) > (1 + \varepsilon) P(H) = \frac{1 + \varepsilon}{1 + \varepsilon/6} \left(1 + \frac{\varepsilon}{6}\right) P(H) > \frac{1 + \varepsilon}{1 + \varepsilon/6} P(H^+)$$

$$> \left(1 + \frac{\varepsilon}{3}\right) P(H^+).$$

Let $A = \{(\omega, \omega') : \omega \notin H^+, \omega' \in H\}$. Then by (6.14)

(6.15)
$$\Psi(A) \geq Q(H) - P(H^+) > Q(H) - \frac{1}{1 + \varepsilon/3} Q(H) = \frac{\varepsilon/3}{1 + \varepsilon/3} Q(H)$$

$$\geq \frac{\varepsilon}{4} Q(H).$$

By the construction of K, for every $(\omega, \omega') \in A$ we have $\text{dist}(\omega, \omega') > dM_\varepsilon$ and, therefore,

$$
\begin{aligned}
\int_H D_{M_\varepsilon}^{(2)}(\omega) \, dQ(\omega) \\
&= \int_{\Omega \times H} D_{M_\varepsilon}^{(2)}(\omega') \, d\Psi(\omega, \omega') \\
&= \int_{\Omega \times \Omega} E_\Psi[\mathbb{1}_{\text{dist}(\omega,\omega')>dM_\varepsilon}|\mathfrak{F}_{\omega'}](\omega')\mathbb{1}_{\Omega \times H}(\omega, \omega') \, d\Psi(\omega, \omega') \\
&= \int_{\Omega \times \Omega} E_\Psi[\mathbb{1}_{\text{dist}(\omega,\omega')>dM_\varepsilon}\mathbb{1}_{\Omega \times H}(\omega, \omega')|\mathfrak{F}_{\omega'}](\omega') \, d\Psi(\omega, \omega') \\
&= \int_{\Omega \times \Omega} \mathbb{1}_{\text{dist}(\omega,\omega')>dM_\varepsilon}\mathbb{1}_{\Omega \times H}(\omega, \omega') \, d\Psi(\omega, \omega') \\
&\geq \int_{\Omega \times \Omega} \mathbb{1}_{\text{dist}(\omega,\omega')>dM_\varepsilon}\mathbb{1}_A(\omega, \omega') \, d\Psi(\omega, \omega') \\
&= \int_{\Omega \times \Omega} \mathbb{1}_A(\omega, \omega') \, d\Psi(\omega, \omega') \\
&= \Psi(A) \geq \frac{\varepsilon}{4} Q(H),
\end{aligned}
$$

(6.16)

where for the last inequality we used (6.15). However, recalling that $H \subset \mathbf{F}_{M_\varepsilon}$ by definition, that $P(H) \leq Q(H)$ by (6.13) and using Lemma 6.6 with M_ε and $\frac{\varepsilon}{5}$ instead of M and ε we get

$$
\begin{aligned}
\int_H D_{M_\varepsilon}^{(2)} \, dQ(\omega) &\leq \int_H \frac{1}{f(\omega)}\left[\frac{\varepsilon}{5}\mathbb{1}_{\mathbf{F}_{M_\varepsilon}} + \mathbb{1}_{\mathbf{F}_{M_\varepsilon}^c}\right] dQ(\omega) \\
&= \int_H \left[\frac{\varepsilon}{5}\mathbb{1}_{\mathbf{F}_{M_\varepsilon}} + \mathbb{1}_{\mathbf{F}_{M_\varepsilon}^c}\right] dP(\omega) \\
&= \int_H \frac{\varepsilon}{5} \, dP(\omega) = \frac{\varepsilon}{5} P(H) \leq \frac{\varepsilon}{5} Q(H).
\end{aligned}
$$

(6.17)

Combining (6.16) and (6.17), we must conclude that $Q(H) = 0$. Therefore, by (6.13) we have $P(H) = 0$, and thus that $P(K) = 0$. This however, by property (1) of K, implies that $P(S_\varepsilon^+) = 0$ and, therefore, finally that $P(B_\varepsilon^+) = M^{-\xi(1)}$. $\quad\square$

COROLLARY 6.7. *Let $d \geq 4$ and assume P is uniformly elliptic, i.i.d. and satisfies $(\mathscr{P})$. Then $E[(\frac{dQ}{dP})^k] < \infty$ for every $k \in \mathbb{N}$.*

PROOF. For every $M \in \mathbb{N}$ large enough, Lemma 6.5 implies

$$
P\left(\frac{dQ}{dP}(\omega) \geq 2(2M+1)^d\right) \leq P\left(\frac{1}{(2M+1)^d}\sum_{x \in [-M.M]^d \cap \mathbb{Z}^d} \frac{dQ}{dP}(\sigma_x \omega) \geq 2\right)
$$

$$
= M^{-\xi(1)}.
$$

Thus, $\frac{dQ}{dP}$ has super polynomial decay and the result follows. $\quad\square$

PROOF OF THEOREM 1.10. The proof is the content of Lemmas 6.1, 6.2 and Corollary 6.7. □

7. Proof of Theorem 1.11. In this section, we prove our second main result, the prefactor local limit theorem. The uniqueness of the prefactor follows from its definition quite easily and most of the work is concentrated into the proof of existence. Our candidate for the prefactor is the Radon–Nikodym derivative of the equivalent measure Q constructed in the previous section. The proof proceeds as follows: instead of directly comparing the quenched measure $P_\omega^0(X_n = \cdot)$ and the annealed times the prefactor measure $\mathbb{P}^0(X_n = \cdot) f(\sigma.\omega)$, we introduce two new measures (denoted temporarily by ρ_1 and ρ_2) and show that the total variation of the pairs $(\mathbb{P}^0(X_n = \cdot) f(\sigma.\omega), \rho_1)$, (ρ_1, ρ_2), $(\rho_2, P_\omega^0(X_n = \cdot))$ goes to zero as n goes to infinity for P-almost every environment. Both measures ρ_1 and ρ_2 are constructed in a way that allows us to exploit the previous results on the connection between the quenched and annealed measures in the total variations estimations. More formally, we fix two parameters $0 < \delta < \varepsilon < \frac{1}{2}$ and define the measures ρ_1 and ρ_2 as follows: for ρ_1, we choose a point by first choosing a point according to the annealed law at time $n - n^\varepsilon$ times the prefactor and then letting it "evolve" according to the quenched law for n^ε additional steps. For ρ_2 we fix some partition of $\mathbb{Z}^d$ to boxes of side length n^δ, choose a box according to the quenched measure at time $n - n^\varepsilon$, choose a point inside the box proportionally to its prefactor and then let it "evolve" into a new point according to the quenched law for n^ε additional steps. For a more precise definition of the measures, see Definition 7.2.

7.1. *Uniqueness.* We start with a proof that the prefactor, if exists, is unique. Assume both f and g satisfy (1.4) and denote $h = f - g$. By the triangle inequality for P-almost every $\omega \in \Omega$

$$(7.1) \qquad \lim_{n \to \infty} \sum_{x \in \mathbb{Z}^d} \mathbb{P}^0(X_n = x) |h(\sigma_x \omega)| = 0,$$

that is, $\lim_{n \to \infty} \mathbb{E}^0[|h(\sigma_{X_n} \omega)|] = 0$, P-a.s. If $h \neq 0$, then there exists a measurable subset A of Ω such that $P(A) > 0$ and $|h| > c > 0$ on A. Thus, for every $n \in \mathbb{N}$

$$E[\mathbb{E}^0[|h(\sigma_{X_n} \omega)|]] \geq E[\mathbb{E}^0[|h(\sigma_{X_n} \omega)| \mathbb{1}_{\sigma_{X_n} \omega \in A}]] \geq cE[\mathbb{E}^0[\mathbb{1}_{\sigma_{X_n} \omega \in A}]]$$

$$= cE[\mathbb{P}^0(\sigma_{X_n} \omega \in A)] = c \sum_{y \in \mathbb{Z}^d} \mathbb{P}^0(X_n = y) E[\mathbb{1}_{\sigma_y \omega \in A}]$$

$$(7.2) \qquad = c \sum_{y \in \mathbb{Z}^d} \mathbb{P}^0(X_n = y) P(\sigma_y \omega \in A)$$

$$= cP(A) \sum_{y \in \mathbb{Z}^d} \mathbb{P}^0(X_n = y)$$

$$= cP(A) > 0.$$

Since

$$E\big[\mathbb{E}^0\big[|h(\sigma_{X_n}\omega)|\big]\big] = E\bigg[\sum_{y\in\mathbb{Z}^d}|h(\sigma_y\omega)|\mathbb{P}^0(X_n=y)\bigg]$$

$$= \sum_{y\in\mathbb{Z}^d}\mathbb{P}^0(X_n=y)\cdot E\big[|h(\sigma_y\omega)|\big]$$

$$= \sum_{y\in\mathbb{Z}^d}\mathbb{P}^0(X_n=y)\cdot E\big[|h(\omega)|\big] = E\big[|h|\big],$$

the sequence $\{|h(\sigma_{X_n}\omega)|\}_{n\in\mathbb{N}}$ is tight and, therefore, by (7.2) it follows that $\lim_{n\to\infty}\mathbb{E}^0[|h(\sigma_{X_n}\omega)|] > 0$ a contradiction to (7.1).

7.2. *Existence.* Let $f(\omega)$ be the Radon–Nikodym derivative of Q defined in Theorem 1.10. We will show that f satisfies Theorem 1.11 starting with the following simple proposition.

PROPOSITION 7.1.   *For P-almost every ω every $n \in \mathbb{N}$ and every $x \in \mathbb{Z}^d$*

$$f(\sigma_x\omega) = \sum_{y\in\mathbb{Z}^d}P_\omega^y(X_n=x)f(\sigma_y\omega).$$

PROOF.   For $n=1$, this follows from the definition of $f=\frac{dQ}{dP}$ as the Radon–Nikodym derivative of the measure Q which is invariant with respect to the point of view of the particle. Indeed, using (1.2) and (1.3) and the translation invariance of P for every bounded measurable function $g:\Omega\to\mathbb{R}$ we have

$$\int_\Omega g(\omega)f(\omega)\,dP(\omega) = \int_\Omega g(\omega)\,dQ(\omega) = \int_\Omega \mathfrak{R}g(\omega)\,dQ(\omega)$$

$$= \int_\Omega (\mathfrak{R}g(\omega))f(\omega)\,dP(\omega)$$

$$= \int_\Omega \sum_{e\in\mathcal{E}_d}\omega(0,e)g(\sigma_e\omega)f(\omega)\,dP(\omega)$$

$$= \int_\Omega \sum_{e\in\mathcal{E}_d}\omega(-e,0)g(\omega)f(\sigma_{-e}\omega)\,dP(\omega)$$

$$= \int_\Omega g(\omega)\sum_{e\in\mathcal{E}_d}\omega(e,0)f(\sigma_e\omega)\,dP(\omega)$$

and, therefore,

$$f(\omega) = \sum_{e\in\mathcal{E}_d}\omega(e,0)f(\sigma_e\omega) = \sum_{e\in\mathcal{E}_d}P_\omega^e(X_1=0)f(\sigma_e\omega).$$

Applying the last equality for $\sigma_x\omega$ gives the result in the case $n=1$.

For $n > 1$, the proof follows by induction. Indeed,

$$\sum_{y \in \mathbb{Z}^d} P_\omega^y(X_n = x) f(\sigma_y \omega) = \sum_{y \in \mathbb{Z}^d} \sum_{z \in \mathbb{Z}^d} P_\omega^y(X_{n-1} = z) P_\omega^z(X_1 = x) f(\sigma_y \omega)$$

$$= \sum_{z \in \mathbb{Z}^d} P_\omega^z(X_1 = x) \sum_{y \in \mathbb{Z}^d} P_\omega^y(X_{n-1} = z) f(\sigma_y \omega)$$

$$\overset{(1)}{=} \sum_{z \in \mathbb{Z}^d} P_\omega^z(X_1 = x) f(\sigma_z \omega) \overset{(2)}{=} f(\sigma_x \omega),$$

where for (1) we used the induction assumption and in (2) we used the case $n = 1$.
$\square$

As stated at the beginning of the section, the proof of Theorem 1.11 uses comparison with two additional probability measures which we now define.

DEFINITION 7.2. For $n \in \mathbb{N}$ and $\omega \in \Omega$, define the following probability measures on $\mathbb{Z}^d$:

(1) $\nu_\omega^{\mathrm{ann} \times \mathrm{pre}, n}$—the annealed at time n times the prefactor (normalized)

$$\nu_\omega^{\mathrm{ann} \times \mathrm{pre}, n}(x) = \frac{1}{Z_{\omega,n}} \mathbb{P}^0(X_n = x) f(\sigma_x \omega),$$

where $Z_{\omega,n} = \sum_{x \in \mathbb{Z}^d} \mathbb{P}^0(X_n = x) f(\sigma_x \omega)$ is a normalizing constant. In Lemma 7.3, we show that $\lim_{n \to \infty} Z_{\omega,n} = 1$, P-almost surely.

(2) $\nu_\omega^{\mathrm{que}, n}$—the quenched measure at time n

$$\nu_\omega^{\mathrm{que}, n}(x) = P_\omega^0(X_n = x).$$

(3) $\nu_\omega^{\mathrm{box\text{-}que} \times \mathrm{pre}, n} = \nu_{\omega, \Pi}^{\mathrm{box\text{-}que} \times \mathrm{pre}, n}$—the quenched measure on boxes with a choice of a point in the box proportional to the prefactor. Given a partition Π of $\mathbb{Z}^d$ into boxes of side length l, we choose a box according to the quenched measure at time n and then choose a point inside of the box proportionally to the value of the Radon–Nikodym derivative there.

$$\nu_{\omega, \Pi}^{\mathrm{box\text{-}que} \times \mathrm{pre}, n}(x) = \begin{cases} P_\omega^0(X_n \in \Delta_x) \dfrac{f(\sigma_x \omega)}{\sum_{y \in \Delta_x, y \leftrightarrow n} f(\sigma_y \omega)}, & x \leftrightarrow n, \\ 0, & \text{otherwise,} \end{cases}$$

where Δ_x is the unique d-dimensional box that contains x in the partition Π.

Before turning to the proof of Theorem 1.11, we wish to study the normalization constant $Z_{\omega,n}$ of the measure $\nu_\omega^{\mathrm{ann} \times \mathrm{pre}, n}$.

LEMMA 7.3. *With the notation as in Definition 7.2 for P-almost every ω, we have $\lim_{n \to \infty} Z_{\omega,n} = 1$.*

PROOF. Fix $\varepsilon > 0$, $0 < \delta < \frac{1}{6d}$ and let Π be a partition of $\mathbb{Z}^d$ into boxes of side length n^δ. If $x, y \in \Delta$ for some $\Delta \in \Pi$, then the annealed derivative estimation (see Lemma 2.14) gives

$$
(7.3) \qquad \left| \mathbb{P}^0(X_n = x) - \mathbb{P}^0(X_n = y) \right| \le C \|x - y\|_1 n^{-(d+1)/2}
$$
$$
\le C n^{-(d+1)/2+\delta}.
$$

Denoting $\Pi_n = \{ \Delta \in \Pi : \Delta \cap [-n, n]^d \neq \varnothing \}$ we have

$$
|Z_{\omega,n} - 1| = \left| \sum_{x \in \mathbb{Z}^d} \mathbb{P}(X_n = x) [f(\sigma_x \omega) - 1] \right|
$$
$$
= \left| \sum_{\Delta \in \Pi_n} \sum_{\substack{x \in \Delta \\ x \leftrightarrow n}} \mathbb{P}^0(X_n = x) [f(\sigma_x \omega) - 1] \right|.
$$

By Lemma 2.16, there exists $C_\varepsilon > 0$ such that $\mathbb{P}^0(\|X_n - \mathbb{E}^0[X_n]\|_1 > C_\varepsilon \sqrt{n}) < \varepsilon$. Separating the sum into boxes in $\widehat{\Pi}_n = \{ \Delta \in \Pi_n : \Delta \cap \{ x \in \mathbb{Z}^d : \|x - \mathbb{E}^0[X_n]\| \le C_\varepsilon \sqrt{n} \} \neq \varnothing \}$ and in $\Pi_n \setminus \widehat{\Pi}_n$ we can bound the last term by

$$
(7.4) \qquad \le \left| \sum_{\Delta \in \Pi_n \setminus \widehat{\Pi}_n} \sum_{\substack{x \in \Delta \\ x \leftrightarrow n}} \mathbb{P}^0(X_n = x) [f(\sigma_x \omega) - 1] \right|
$$

$$
(7.5) \qquad + \left| \sum_{\Delta \in \widehat{\Pi}_n} \sum_{\substack{x \in \Delta \\ x \leftrightarrow n}} \left(\frac{1}{|\Delta|} \sum_{\substack{y \in \Delta \\ y \leftrightarrow n}} [\mathbb{P}^0(X_n = y) - \mathbb{P}^0(X_n = x)] \right) [f(\sigma_x \omega) - 1] \right|
$$

$$
(7.6) \qquad + \left| \sum_{\Delta \in \widehat{\Pi}_n} \sum_{\substack{x \in \Delta \\ x \leftrightarrow n}} \frac{1}{|\Delta|} \sum_{\substack{y \in \Delta \\ y \leftrightarrow n}} \mathbb{P}^0(X_n = y) [f(\sigma_x \omega) - 1] \right|.
$$

We start by evaluating the term (7.4). By Lemma 6.5, there exists some constant C, such that with P probability $\ge 1 - n^{-\xi(1)}$ for every $\Delta \in \Pi_n$ (and in particular in $\Pi_n \setminus \widehat{\Pi}_n$) we have $\sum_{\substack{y \in \Delta \\ y \leftrightarrow n}} [f(\sigma_y \omega) + 1] \le C|\Delta|$. Therefore, under the above event, we can bound (7.4) by

$$
\sum_{\Delta \in \Pi_n \setminus \widehat{\Pi}_n} \sum_{\substack{x \in \Delta \\ x \leftrightarrow n}} \mathbb{P}^0(X_n = x) [f(\sigma_x \omega) + 1]
$$

$$
\le \sum_{\Delta \in \Pi_n \setminus \widehat{\Pi}_n} \max_{\substack{x \in \Delta \\ x \leftrightarrow n}} \mathbb{P}^0(X_n = x) \sum_{\substack{y \in \Delta \\ x \leftrightarrow n}} [f(\sigma_y \omega) + 1]
$$

$$
\le C \sum_{\Delta \in \Pi_n \setminus \widehat{\Pi}_n} |\Delta| \cdot \max_{\substack{x \in \Delta \\ x \leftrightarrow n}} \mathbb{P}^0(X_n = x).
$$

Using Lemma 2.14, (2.5) and the definition of $\widehat{\Pi}_n$, we thus have

$$
\begin{aligned}
(7.4) &\le C \sum_{\Delta \in \Pi_n \setminus \widehat{\Pi}_n} |\Delta| \cdot \max_{\substack{x \in \Delta \\ x \leftrightarrow n}} \mathbb{P}^0(X_n = x) \\
&\le C \sum_{\substack{\Delta \in \Pi_n \setminus \widehat{\Pi}_n \\ y \leftrightarrow n}} \max_{\substack{y \in \Delta}} \Big[\max_{\substack{x \in \Delta \\ x \leftrightarrow n}} \mathbb{P}^0(X_n = x) - \mathbb{P}^0(X_n = y) \Big] \\
&\quad + C \sum_{\substack{\Delta \in \Pi_n \setminus \widehat{\Pi}_n \\ y \leftrightarrow n}} \max_{\substack{y \in \Delta}} \mathbb{P}^0(X_n = y) \\
&\le \frac{C}{n^{1/2 - 3d\delta}} + \varepsilon.
\end{aligned}
$$

Recalling that $\delta < \frac{1}{6d}$ and taking $n \to \infty$ this gives (by an application of the Borel–Cantelli lemma)

$$
\limsup_{n \to \infty} (7.4) \le \limsup_{n \to \infty} \sum_{\Delta \in \Pi_n \setminus \widehat{\Pi}_n} \sum_{\substack{x \in \Delta \\ x \leftrightarrow n}} \mathbb{P}^0(X_n = x) [f(\sigma_x \omega) + 1] \le \varepsilon, \qquad P\text{-a.s.}
$$

Next, we deal with the term (7.5). Due to (7.3), this is bounded by

$$
\begin{aligned}
&\sum_{\Delta \in \widehat{\Pi}_n} \sum_{\substack{x \in \Delta \\ x \leftrightarrow n}} \Big| \frac{1}{|\Delta|} \sum_{\substack{y \in \Delta \\ y \leftrightarrow n}} [\mathbb{P}(X_n = y) - \mathbb{P}(X_n = x)] \Big| [f(\sigma_x \omega) + 1] \\
&\le \sum_{\Delta \in \widehat{\Pi}_n} \sum_{\substack{x \in \Delta \\ x \leftrightarrow n}} \frac{C}{n^{((d+1)/2) - \delta}} [f(\sigma_x \omega) + 1] \\
&\le \frac{C}{n^{1/2 - \delta}} \cdot \Big(\frac{1}{n^{d/2}} \sum_{\|x - \mathbb{E}^0[X_n]\| \le 2C_\varepsilon \sqrt{n}} f(\sigma_x \omega) \Big) + \frac{C}{n^{1/2 - \delta}}.
\end{aligned}
$$

By Lemma 6.5 and an application of Borel–Cantelli for P-almost every ω once n is large enough, we have $\frac{1}{n^{d/2}} \sum_{\|x - \mathbb{E}^0[X_n]\| \le 2C_\varepsilon \sqrt{n}} f(\sigma_x \omega) \le (8C_\varepsilon)^d$, and thus the last term tends to zero as n goes to infinity P-almost surely.

Finally, for (7.6), we recall that Lemma 2.14 also ensures $\mathbb{P}(X_n = x) \le C n^{-d/2}$ for every $x \in \mathbb{Z}^d$ and, therefore,

$$
\begin{aligned}
(7.6) &= \Big| \sum_{\Delta \in \widehat{\Pi}_n} \sum_{x \in \Delta} \frac{1}{|\Delta|} \sum_{y \in \Delta} \mathbb{P}(X_n = y) [f(\sigma_x \omega) - 1] \Big| \\
&\le \sum_{\Delta \in \widehat{\Pi}_n} \frac{1}{|\Delta|} \sum_{y \in \Delta} \mathbb{P}(X_n = y) \Big| \sum_{x \in \Delta} [f(\sigma_x \omega) - 1] \Big| \\
&\le \frac{C}{n^{d/2}} \sum_{\Delta \in \widehat{\Pi}_n} \Big| \sum_{x \in \Delta} [f(\sigma_x \omega) - 1] \Big| \\
&= \frac{C}{n^{d(1/2 - \delta)}} \sum_{\Delta \in \widehat{\Pi}_n} \Big| \frac{1}{|\Delta|} \sum_{x \in \Delta} [f(\sigma_x \omega) - 1] \Big|.
\end{aligned}
$$

Lemma 6.5 now implies that

$$P\left(\frac{C}{n^{d(1/2-\delta)}} \sum_{\Delta \in \widehat{\Pi}_n} \left|\frac{1}{|\Delta|} \sum_{x \in \Delta} [f(\sigma_x \omega) - 1]\right| > \varepsilon\right)$$

$$\leq P\left(\exists \Delta \in \widehat{\Pi}_n : \left|\frac{1}{|\Delta|} \sum_{x \in \Delta} [f(\sigma_x \omega) - 1]\right| > \frac{\varepsilon}{C \cdot C_\varepsilon^d}\right)$$

$$\leq n^{d(1/2-\delta)} P\left(\left|\frac{1}{|\Delta_0|} \sum_{x \in \Delta_0} [f(\sigma_x \omega) - 1]\right| > \frac{\varepsilon}{C \cdot C_\varepsilon^d}\right)$$

$$= n^{d(1/2-\delta)} \cdot n^{-\xi(1)} = n^{-\xi(1)},$$

where Δ_0 is any choice for $\Delta_0 \in \widehat{\Pi}_n$. Therefore, by Borel–Cantelli, we have

$$\limsup_{n \to \infty}(7.6) \leq \limsup_{n \to \infty} \frac{C}{n^{d(1/2-\delta)}} \sum_{\Delta \in \widehat{\Pi}_n} \left|\frac{1}{|\Delta|} \sum_{x \in \Delta} [f(\sigma_x \omega) - 1]\right| \leq \varepsilon.$$

Combining all of the above, we see that P-almost surely

$$\limsup_{n \to \infty} |Z_{\omega,n} - 1| \leq 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, the result follows. $\quad\square$

Before turning to the main lemma in the proof of Theorem 1.11, we give two additional preliminary definitions needed in order to construct the intermediate measures:

DEFINITION 7.4. Let ν_ω^1 and ν_ω^2 be two probability measures on $\mathbb{Z}^d$, which may depend on $\omega \in \Omega$.

(1) The L^1 distance of ν_ω^1 and ν_ω^2 is given by $\|\nu_\omega^1 - \nu_\omega^2\|_1 = \sum_{x \in \mathbb{Z}^d} |\nu_\omega^1(x) - \nu_\omega^2(x)|$ (note that this equals twice the total variation between ν_ω^1 and ν_ω^2).

(2) The environment-convolution of ν_ω^1 and ν_ω^2 is a new probability measure on $\mathbb{Z}^d$, denoted $(\nu^1 * \nu^2)_\omega$, given by

$$\left(\nu^1 * \nu^2\right)_\omega(x) = \sum_{y \in \mathbb{Z}^d} \nu_\omega^1(y)\nu_{\sigma_y \omega}^2(x - y).$$

We can now state the main lemma in the proof of Theorem 1.11. As already stated above, instead of comparing directly the L^1 distance of $\nu_\omega^{\mathrm{ann} \times \mathrm{pre},n}$ and $\nu_\omega^{\mathrm{que},n}$, that is, the annealed times the prefactor and the quenched probability measures, appearing in Theorem 1.11, we take a more indirect approach and use two other measures as intermediaries. This allows us to use previous results on the Radon–Nikodym derivative and other relations between the quenched and annealed measures in the evaluation of the L^1 distances.

LEMMA 7.5. *Fix $0 < \delta < \varepsilon < \frac{1}{4}$, and for $n \in \mathbb{N}$ abbreviate $k = \lceil n^\varepsilon \rceil$ and $l = \lceil n^\delta \rceil$. Fix a partition Π of $\mathbb{Z}^d$ into boxes of side length l. With the notation as in Definitions 7.2 and 7.4, we have for P-almost every $\omega \in \Omega$:*

(1) $\lim_{n\to\infty} \|\nu_\omega^{\text{ann}\times\text{pre},n} - (\nu^{\text{ann}\times\text{pre},n-k} * \nu^{\text{que},k})_\omega\|_1 = 0.$
(2) $\lim_{n\to\infty} \|(\nu^{\text{ann}\times\text{pre},n-k} * \nu^{\text{que},k})_\omega - (\nu_\Pi^{\text{box-que}\times\text{pre},n-k} * \nu^{\text{que},k})_\omega\|_1 = 0.$
(3) $\lim_{n\to\infty} \|(\nu_\Pi^{\text{box-que}\times\text{pre},n-k} * \nu^{\text{que},k})_\omega - (\nu^{\text{que},n-k} * \nu^{\text{que},k})_\omega\|_1 = 0.$

REMARK 7.6. (1) In the temporary notation from the beginning of this section, we have $\rho_1 = (\nu^{\text{ann}\times\text{pre},n-k} * \nu^{\text{que},k})_\omega$ and $\rho_2 = (\nu^{\text{box-que}\times\text{pre},n-k} * \nu^{\text{que},k})_\omega$.
(2) Note that by the Markov property of the quenched walk $\nu_\omega^{\text{que},\cdot}$ we have $(\nu^{\text{que},n-k} * \nu^{\text{que},k})_\omega = \nu_\omega^{\text{que},n}$.

PROOF OF LEMMA 7.5 PART (1). We need to show that

$$\lim_{n\to\infty} \sum_{x\in\mathbb{Z}^d} \left| \frac{1}{Z_{n,\omega}} \mathbb{P}^0(X_n = x) f(\sigma_x\omega) \right.$$

$$\left. - \frac{1}{Z_{n-k,\omega}} \sum_{y\in\mathbb{Z}^d} \mathbb{P}^0(X_{n-k} = y) f(\sigma_y\omega) P_\omega^y(X_k = x) \right| = 0,$$

which by Lemma 7.3 (and the fact that we can restrict attention to $x \in \mathbb{Z}^d$ such that $\|x\|_1 \leq n$) is equivalent to showing

$$\lim_{n\to\infty} \sum_{x\in[-n,n]^d\cap\mathbb{Z}^d} \left| \mathbb{P}^0(X_n = x) f(\sigma_x\omega) \right.$$

$$\left. - \sum_{y\in[-n,n]^d\cap\mathbb{Z}^d} \mathbb{P}^0(X_{n-k} = y) f(\sigma_y\omega) P_\omega^y(X_k = x) \right| = 0.$$

Denote $B_n = \{x \in [-n, n]^d \cap \mathbb{Z}^d : \|x - \mathbb{E}^0[X_n]\|_1 \leq R_5(n)\sqrt{n}\}$. By the triangle inequality,

$$\sum_{x\in[-n,n]^d\cap\mathbb{Z}^d} \left| \mathbb{P}^0(X_n = x) f(\sigma_x\omega) \right.$$

$$\left. - \sum_{y\in[-n,n]^d\cap\mathbb{Z}^d} \mathbb{P}^0(X_{n-k} = y) f(\sigma_y\omega) P_\omega^y(X_k = x) \right|$$

$$(7.7) \quad \leq \sum_{x\in B_n} \left| \sum_{y\in[-n,n]^d\cap\mathbb{Z}^d} [\mathbb{P}^0(X_n = x) - \mathbb{P}^0(X_{n-k} = y)] f(\sigma_y\omega) P_\omega^y(X_k = x) \right|$$

$$(7.8) \quad + \sum_{x\in B_n} \mathbb{P}^0(X_n = x) \left| f(\sigma_x\omega) - \sum_{y\in[-n,n]^d\cap\mathbb{Z}^d} f(\sigma_y\omega) P_\omega^y(X_k = x) \right|$$

$$(7.9) \quad + \sum_{x \in [-n,n]^d \cap \mathbb{Z}^d \setminus B_n} \left| \mathbb{P}^0(X_n = x) f(\sigma_x \omega) \right.$$

$$- \sum_{y \in [-n,n]^d \cap \mathbb{Z}^d} \mathbb{P}^0(X_{n-k} = y) f(\sigma_y \omega) P_\omega^y(X_k = x) \bigg|.$$

Dealing with each of the terms separately [starting with (7.7)], by the annealed derivative estimation from Lemma 2.14

$$(7.7) \leq \sum_{x \in B_n} \frac{Ck}{n^{(d+1)/2}} \left(\sum_{\text{dist}(y, B_n) \leq k} f(\sigma_y \omega) P_\omega^y(X_k = x) \right)$$

$$\leq \frac{Ck}{n^{(d+1)/2}} \sum_{\text{dist}(y, B_n) \leq k} f(\sigma_y \omega)$$

$$= \frac{Ck}{n^{1/2}} \cdot \frac{1}{n^{d/2}} \sum_{\text{dist}(y, B_n) \leq k} f(\sigma_y \omega).$$

By Lemma 6.5 for P-almost every ω and large enough n, we have

$$\frac{1}{n^{d/2}} \sum_{\text{dist}(y, B_n) \leq k} f(\sigma_y \omega) \leq 2R_6(n).$$

Thus, using the fact that $k = n^\varepsilon \ll n^{1/4}$, it follows that the last term tends to zero P-almost surely as n tends to ∞.

Turning to deal with (7.8), we recall that by Proposition 7.1 we have $f(\sigma_x \omega) - \sum_{y \in [-n,n]^d \cap \mathbb{Z}^d} f(\sigma_y \omega) P_\omega^y(X_k = x) = 0$ for every $x \in \mathbb{Z}^d$ such that $x + [-k,k]^d \cap \mathbb{Z}^d \subset [-n,n]^d \cap \mathbb{Z}^d$. In particular, denoting $\tilde{B}_n = B_n \setminus \{x \in \mathbb{Z}^d : x + [-k,k]^d \cap \mathbb{Z}^d \subset [-n,n]^d \cap \mathbb{Z}^d\}$ and using the annealed estimations from Lemma 2.14

$$(7.8) = \sum_{x \in \tilde{B}_n} \mathbb{P}^0(X_n = x) \left| f(\sigma_x \omega) - \sum_{y \in [-n,n]^d \cap \mathbb{Z}^d} f(\sigma_y \omega) P_\omega^y(X_k = x) \right|$$

$$\leq \sum_{x \in \tilde{B}_n} \mathbb{P}^0(X_n = x) f(\sigma_x \omega)$$

$$+ \sum_{x \in \tilde{B}_n} \mathbb{P}^0(X_n = x) \sum_{y \in [-n,n]^d \cap \mathbb{Z}^d} f(\sigma_y \omega) P_\omega^y(X_k = x)$$

$$\leq \frac{C|\tilde{B}_n|}{n^{d/2}} \cdot \frac{1}{|\tilde{B}_n|} \sum_{x \in \tilde{B}_n} f(\sigma_x \omega) + \frac{C|\tilde{B}_n|}{n^{d/2}} \cdot \frac{1}{|\tilde{B}_n|} \sum_{x \in \tilde{B}_n} \sum_{\|y - x\|_1 \leq k} f(\sigma_y \omega),$$

where $|\tilde{B}_n|$ is the size of $\tilde{B}_n$. From the definition of $\tilde{B}_n$, it follows that $|\tilde{B}_n| \leq Ck(\sqrt{n}R_5(n))^{d-1}$ and, therefore,

$$(7.8) = \frac{C|\tilde{B}_n|}{n^{d/2}} \cdot \frac{1}{|\tilde{B}_n|} \sum_{x \in \tilde{B}_n} f(\sigma_x \omega) + \frac{C|\tilde{B}_n|}{n^{d/2}} \cdot \frac{1}{|\tilde{B}_n|} \sum_{x \in \tilde{B}_n} \sum_{\|y-x\|_1 \leq k} f(\sigma_y \omega)$$

$$\leq \frac{CkR_5^{d-1}(n)}{n^{1/2}} \cdot \frac{1}{|\tilde{B}_n|} \sum_{x \in \tilde{B}_n} f(\sigma_x \omega)$$

$$+ \frac{CkR_5^{d-1}(n)}{n^{1/2}} \cdot \frac{1}{|\tilde{B}_n|} \sum_{x \in \tilde{B}_n} \sum_{\|y-x\|_1 \leq k} f(\sigma_y \omega).$$

Using again Lemma 6.5, and the choice $k = n^\varepsilon = o(n^{1/4})$, it follows that both sums tends to zero as n goes to infinity, P-almost surely.

Finally, we turn to deal with (7.9). Using Lemma 2.16, we have $\mathbb{P}(X_n \notin B_n) = n^{-\xi(1)}$. Recalling also that $k = \lceil n^\varepsilon \rceil = o(n)$, we note that if $P_\omega^y(X_k = x) > 0$ then $\|x - y\|_1 \leq k$, and thus for $x \in [-n,n]^d \cap \mathbb{Z}^d \setminus B_n$ and large enough n

$$\|y - \mathbb{E}^0[X_{n-k}]\|_1 \geq \|x - \mathbb{E}^0[X_n]\|_1 - \|\mathbb{E}^0[X_n] - \mathbb{E}^0[X_{n-k}]\|_1 - \|x - y\|_1$$

$$\geq \sqrt{n}R_5(n) - 2k \geq \tfrac{1}{2}\sqrt{n}R_5(n).$$

This, however, due to Lemma 2.16, yields $\mathbb{P}^0(X_{n-k} = y) = n^{-\xi(1)}$ and, therefore,

$$(7.9) \leq \sum_{x \in [-n,n]^d \cap \mathbb{Z}^d \setminus B_n} \mathbb{P}^0(X_n = x) f(\sigma_x \omega)$$

$$+ \sum_{x \in [-n,n]^d \cap \mathbb{Z}^d \setminus B_n} \sum_{y \in [-n,n]^d \cap \mathbb{Z}^d} \mathbb{P}^0(X_{n-k} = y) f(\sigma_y \omega) P_\omega^y(X_k = x)$$

$$\leq n^{-\xi(1)} \sum_{x \in [-n,n]^d \cap \mathbb{Z}^d \setminus B_n} f(\sigma_x \omega)$$

$$+ n^{-\xi(1)} \sum_{x \in [-n,n]^d \cap \mathbb{Z}^d \setminus B_n} \sum_{y \in [-n,n]^d \cap \mathbb{Z}^d} f(\sigma_y \omega) P_\omega^y(X_k = x)$$

$$\leq 2 \cdot n^{-\xi(1)} \sum_{x \in [-n,n]^d \cap \mathbb{Z}^d} f(\sigma_x \omega).$$

By Lemma 6.5, we have $P(\sum_{x \in [-n,n]^d \cap \mathbb{Z}^d} f(\sigma_x \omega) \leq 2n^d) > 1 - n^{-\xi(1)}$ and, therefore, by the Borel–Cantelli lemma for large enough n

$$(7.9) \leq 4n^{-\xi(1)} \cdot n^d = n^{-\xi(1)} \xrightarrow[n \to \infty]{} 0. \qquad \square$$

PROOF OF LEMMA 7.5 PART (2). Since

$$\|(\nu^{\text{ann} \times \text{pre}, n-k} * \nu^{\text{que}, k})_\omega - (\nu^{\text{box-que} \times \text{pre}, n-k} * \nu^{\text{que}, k})_\omega\|_1$$

$$\leq \|\nu_\omega^{\text{ann} \times \text{pre}, n-k} - \nu_\omega^{\text{box-que} \times \text{pre}, n-k}\|_1,$$

it is enough to deal with the right-hand side and show that

$$\lim_{n\to\infty} \sum_{\substack{x\in\mathbb{Z}^d \\ x\leftrightarrow n-k}} \left| \frac{1}{Z_{n-k,\omega}} \mathbb{P}^0(X_{n-k}=x) f(\sigma_x\omega) \right.$$

$$\left. - \frac{f(\sigma_x\omega)}{\sum_{y\in\Delta_x, y\leftrightarrow n-k} f(\sigma_y\omega)} P_\omega^0(X_{n-k}\in\Delta_x) \right| = 0, \qquad P\text{-a.s.}$$

Using Lemma 7.3 once more, this is equivalent to showing

$$\lim_{n\to\infty} \sum_{\substack{x\in\mathbb{Z}^d \\ x\leftrightarrow n-k}} f(\sigma_x\omega) \left| \mathbb{P}^0(X_{n-k}=x) \right.$$

(7.10)

$$\left. - P_\omega^0(X_{n-k}\in\Delta_x) \frac{1}{\sum_{y\in\Delta_x, y\leftrightarrow n-k} f(\sigma_y\omega)} \right| = 0, \qquad P\text{-a.s.}$$

Denoting $B_n = \{x\in[-n,n]^d\cap\mathbb{Z}^d : \|x-\mathbb{E}^0[X_n]\|_1 \leq C_\varepsilon\sqrt{n}\}$ [with C_ε such that $\mathbb{P}^0(\|X_n-\mathbb{E}^0[X_n]\|_1 > \frac{C_\varepsilon}{2}\sqrt{n}) < \varepsilon$ for large enough n] and using the triangle inequality the last sum is bounded by

$$\sum_{\substack{x\in[-n,n]\cap\mathbb{Z}^d\setminus B_n \\ x\leftrightarrow n-k}} f(\sigma_x\omega) \left| \mathbb{P}^0(X_{n-k}=x) \right.$$

(7.11)

$$\left. - \frac{1}{\sum_{y\in\Delta_x, y\leftrightarrow n-k} f(\sigma_y\omega)} P_\omega^0(X_{n-k}\in\Delta_x) \right|$$

(7.12)
$$+ \sum_{\substack{x\in B_n \\ x\leftrightarrow n-k}} f(\sigma_x\omega) \left| \mathbb{P}^0(X_{n-k}=x) - \frac{2}{|\Delta_x|}\mathbb{P}^0(X_{n-k}\in\Delta_x) \right|$$

$$+ \sum_{\substack{x\in B_n \\ x\leftrightarrow n-k}} f(\sigma_x\omega) \left| \frac{2}{|\Delta_x|}\mathbb{P}^0(X_{n-k}\in\Delta_x) \right.$$

(7.13)

$$\left. - \frac{1}{\sum_{y\in\Delta_x, y\leftrightarrow n-k} f(\sigma_y\omega)}\mathbb{P}^0(X_{n-k}\in\Delta_x) \right|$$

$$+ \sum_{\substack{x\in B_n \\ x\leftrightarrow n-k}} f(\sigma_x\omega) \left| \frac{1}{\sum_{y\in\Delta_x, y\leftrightarrow n-k} f(\sigma_y\omega)}\mathbb{P}^0(X_{n-k}\in\Delta_x) \right.$$

(7.14)

$$\left. - \frac{1}{\sum_{y\in\Delta_x, y\leftrightarrow n-k} f(\sigma_y\omega)} P_\omega^0(X_{n-k}\in\Delta_x) \right|.$$

Dealing with each of the terms separately, and starting with (7.11), we have the following estimate:

$$(7.11) \leq \sum_{x \in [-n,n]^d \cap \mathbb{Z}^d \setminus B_n} \mathbb{P}^0(X_{n-k} = x) f(\sigma_x \omega)$$

$$+ P_\omega^0(\|X_{n-k} - \mathbb{E}^0[X_n]\|_1 > C_\varepsilon \sqrt{n}).$$

The term $\sum_{x \in [-n,n]^d \cap \mathbb{Z}^d \setminus B_n} \mathbb{P}^0(X_{n-k} = x) f(\sigma_x \omega)$ goes to zero as n goes to infinity by the same argument used to bound (7.4) in Lemma 7.3. For the second term, Claim 2.15 implies that for a set of environments, with P probability $> 1 - \sqrt{\varepsilon}$, for large enough n

$$P_\omega^0(\|X_{n-k} - \mathbb{E}^0[X_n]\|_1 > C_\varepsilon \sqrt{n}) \leq P_\omega^0\left(\|X_n - \mathbb{E}^0[X_n]\|_1 > \frac{C_\varepsilon}{2}\sqrt{n}\right) \leq \sqrt{\varepsilon}.$$

Since $\varepsilon > 0$ was arbitrary, this proves that the first term goes to zero as n goes to infinity.

Turning to (7.12), the annealed derivative estimations from Lemma 2.14 yields

$$(7.12) \leq C \cdot \sum_{\substack{x \in B_n \\ x \leftrightarrow n-k}} f(\sigma_x \omega) \frac{2}{|\Delta_x|} \sum_{\substack{y \in \Delta_x \\ y \leftrightarrow n-k}} \left|\mathbb{P}(X_{n-k} = x) - \mathbb{P}(X_{n-k} = y)\right|$$

$$\leq C \cdot \sum_{\substack{x \in B_n \\ x \leftrightarrow n-k}} f(\sigma_x \omega) \frac{2}{|\Delta_x|} \sum_{\substack{y \in \Delta_x \\ y \leftrightarrow n-k}} \frac{C}{(n-k)^{(d+1)/2}} \|x - y\|_1$$

$$\overset{(1)}{\leq} C \cdot \sum_{\substack{x \in B_n}} f(\sigma_x \omega) \frac{1}{|\Delta_x|} \sum_{y \in \Delta_x} \frac{C}{(n-k)^{(d+1)/2}} \cdot dn^\delta$$

$$= \frac{C' \cdot n^{d/2+\delta}}{(n-k)^{(d+1)/2}} \cdot \left(\frac{1}{n^{d/2}} \sum_{x \in B_n} f(\sigma_x \omega)\right) \overset{n \to \infty}{\longrightarrow} 0, \qquad P\text{-a.s.,}$$

where for (1) we used the fact that the side length of every cube Δ is n^δ and for the limit we used Lemma 6.5, the fact that $k = n^\varepsilon = o(n^{1/4})$ and also that $\delta < \varepsilon < \frac{1}{2}$.

Next, we deal with (7.13). Denoting $\widehat{\Pi}_n = \{\Delta \in \Pi : \Delta \cap B_n \neq \varnothing\}$ and using the annealed derivative estimations from Lemma 6.5 we conclude that

$$(7.13) = \sum_{\substack{x \in B_n \\ x \leftrightarrow n-k}} f(\sigma_x \omega) \frac{2}{|\Delta_x|} \mathbb{P}(X_{n-k} \in \Delta_x) \left|1 - \frac{1}{2/|\Delta_x| \sum_{y \in \Delta_x, y \leftrightarrow n-k} f(\sigma_y \omega)}\right|$$

$$\leq \frac{C}{(n-k)^{d/2}} \sum_{\substack{x \in B_n \\ x \leftrightarrow n-k}} f(\sigma_x \omega) \left|1 - \frac{1}{2/|\Delta_x| \sum_{y \in \Delta_x, y \leftrightarrow n-k} f(\sigma_y \omega)}\right|$$

$$\leq C\left(1 - \frac{k}{n}\right)^{-d/2}\frac{1}{n^{d/2}}$$

$$\times \sum_{\Delta \in \widehat{\Pi}_n}\sum_{x \in \Delta}f(\sigma_x\omega)\left|1 - \frac{1}{2/|\Delta|\sum_{y\in\Delta_x,\,y\leftrightarrow n-k}f(\sigma_y\omega)}\right|$$

$$= C\left(1 - \frac{k}{n}\right)^{-d/2}\frac{1}{n^{(d/2)(1-2\delta)}}\sum_{\Delta \in \widehat{\Pi}_n}\left|\frac{1}{|\Delta|}\sum_{\substack{x \in \Delta \\ x \leftrightarrow n-k}}f(\sigma_x\omega) - 1\right|.$$

Using the same argument that was used to bound (7.6), we get that the last term goes to zero as n goes to infinity P-a.s. Finally, we estimate (7.14).

$$(7.14) \leq \sum_{\substack{x \in B_n \\ x \leftrightarrow n-k}}\frac{f(\sigma_x\omega)}{\sum_{y\in\Delta_x,\,y\leftrightarrow n-k}f(\sigma_y\omega)}\left|\mathbb{P}^0(X_{n-k}\in\Delta_x) - P^0_\omega(X_{n-k}\in\Delta_x)\right|$$

$$= \sum_{\Delta \in \widehat{\Pi}_n}\left|\mathbb{P}^0(X_{n-k}\in\Delta) - P^0_\omega(X_{n-k}\in\Delta)\right|.$$

The last term, however, is bounded by $Cn^{-(1/3)\delta}$ by Proposition 4.1 for P-almost every ω and large enough n, and thus goes to zero as n goes to infinity. $\quad\square$

Part (3) of Lemma 7.5 will follow from the following more general lemma.

LEMMA 7.7. *Let* $x, y \in \mathbb{Z}^d$ *satisfy* $\|x - y\|_1 \leq n^\theta$ *for some* $\theta < \frac{1}{2}$. *Then the set of environments for which*

$$\left|P^x_\omega(X_n = z) - P^y_\omega(X_n = z)\right| = n^{-\xi(1)} \qquad \forall z \in \mathbb{Z}^d$$

has P *probability* $\geq 1 - n^{-\xi(1)}$.

PROOF. Fix $\theta < \theta' < 1$ such that $\theta' < \frac{d+1}{2}\theta$, $M \in \mathbb{N}$ and a partition Π of $\mathbb{Z}^d$ into boxes of side length M. By Theorem 5.1, if M is large enough, then the event

$$G(n, M) = \left\{\omega \in \Omega : \sum_{\Delta \in \Pi}\left|P^w_\omega(X_{\lceil n^\theta\rceil}\in\Delta) - \mathbb{P}^w(X_{\lceil n^\theta\rceil}\in\Delta)\right| < \frac{1}{8},\right.$$

$$\left.\forall w \text{ s.t. } \|w - x\|_1 \leq n^2\right\}$$

satisfies $P(G(n, M)) = 1 - n^{-\xi(1)}$. In particular, using Lemma 2.14, whenever $\|y - x\|_1 \leq n^{\theta'}$, for large enough n we have

$$\left|P^x_\omega(X_{\lceil n^\theta\rceil}\in\Delta) - P^y_\omega(X_{\lceil n^\theta\rceil}\in\Delta)\right|$$

$$\leq \left|P^x_\omega(X_{\lceil n^\theta\rceil}\in\Delta) - \mathbb{P}^x(X_{\lceil n^\theta\rceil}\in\Delta)\right| + \left|\mathbb{P}^x(X_{\lceil n^\theta\rceil}\in\Delta) - \mathbb{P}^y(X_{\lceil n^\theta\rceil}\in\Delta)\right|$$

$$+ \left| P_\omega^y(X_{\lceil n^\theta \rceil} \in \Delta) - \mathbb{P}^y(X_{\lceil n^\theta \rceil} \in \Delta) \right|$$

$$\leq \frac{1}{4} + \frac{Cn^{\theta'}}{n^{((d+1)/2)\theta}} = \frac{1}{4} + Cn^{\theta' - ((d+1)/2)\theta} < \frac{1}{2}.$$

Consequently, there exists a coupling of $P_\omega^x(X_{\lceil n^\theta \rceil} \in \cdot)$ and $P_\omega^y(X_{\lceil n^\theta \rceil} \in \cdot)$ on $\Pi \times \Pi$ denoted $\tilde{\Xi}^{x,y} = \tilde{\Xi}_{n,\theta,\omega}^{x,y}$ such that $\tilde{\Xi}^{x,y}(\{(\Delta, \Delta) : \Delta \in Q\}) > \frac{1}{2}$. Using the uniform ellipticity, and the last coupling we can construct a new coupling $\Xi_1^{x,y}$ of $P_\omega^x(X_{\lceil n^\theta \rceil} = \cdot)$ and $P_\omega^y(X_{\lceil n^\theta \rceil} = \cdot)$ on $\mathbb{Z}^d \times \mathbb{Z}^d$ such that $\Xi_1^{x,y}(\Lambda) \geq \frac{1}{2}\eta^{2dM}$, where $\Lambda = \{(z, z) : z \in \mathbb{Z}^d\}$ (for a more detailed explanation on the construction, see the proof of Lemma 6.1). Next, for $k \geq 2$ we construct inductively a new coupling of $P_\omega^x(X_{k\lceil n^\theta \rceil} = \cdot)$ and $P_\omega^y(X_{k\lceil n^\theta \rceil} = \cdot)$ on $\mathbb{Z}^d \times \mathbb{Z}^d$ such that $\Xi_{\lceil \log^2 n \rceil \cdot \lceil n^{\theta'} \rceil}^{x,y}(\Lambda) = 1 - n^{-\xi(1)}$. The construction goes as follows: first, note that if $a, b \in \mathbb{Z}^d$ are any pair of points such that $\|a - x\|_1, \|b - x\|_1 \leq n^2$, then by the same reasoning, we can construct a coupling of $P_\omega^a(X_{\lceil n^\theta \rceil} = \cdot)$ and $P_\omega^b(X_{\lceil n^\theta \rceil} = \cdot)$ on $\mathbb{Z}^d \times \mathbb{Z}^d$, denoted $\Xi_1^{a,b}$, such that $\Xi_1^{a,b}(\Lambda) > \frac{1}{2}\eta^{2dM}$. Next, assuming the coupling $\Xi_{k-1}^{x,y}$ was constructed we define $\Xi_k^{x,y}$ via the following procedure: choose a pair of points (a, b) according to the previous coupling $\Xi_{k-1}^{x,y}$. If $a = b$, couple the random walks together (to go along the same path) for additional $\lceil n^\theta \rceil$ steps. If $a \neq b$ and $\|a - b\| \leq n^{\theta'}$, couple the random walks using the coupling $\Xi_1^{a,b}$. Finally, if $\|a - b\| > n^{\theta'}$ we let the random walks evolve independently. Formally, this is given by

$$\Xi_k^{x,y}(w_1, w_2) = \sum_{a,b \in \mathbb{Z}^d} \Xi_{k-1}^{x,y}(a, b) \big[\mathbb{1}_{a=b} \mathbb{1}_{w_1=w_2} P_\omega^a(X_{\lceil n^\theta \rceil} = w_1)$$

$$+ \mathbb{1}_{0 < \|a-b\|_1 \leq n^{\theta'}} \cdot \Xi_1^{a,b}(w_1, w_2)$$

$$+ \mathbb{1}_{\|a-b\|_1 > n^{\theta'}} \cdot P_\omega^a(X_{n^\theta} = w_1) P_\omega^b(X_{n^\theta} = w_2) \big].$$

It is not hard to verify that this indeed defines a coupling of $P_\omega^x(X_{k\lceil n^\theta \rceil} = \cdot)$ and $P_\omega^y(X_{k\lceil n^\theta \rceil} = \cdot)$, and that in fact by the definition of θ' and the assumption $\|x - y\|_1 \leq n^\theta$ that $\Xi_k^{x,y}(a, b) = 0$ whenever $\|a - b\|_1 \geq n^{\theta'}$ and n is large enough. Therefore,

$$\Xi_k^{x,y}(\Lambda) = \sum_{w \in \mathbb{Z}^d} \sum_{a,b \in \mathbb{Z}^d} \Xi_{k-1}^{x,y}(a, b) \big[\mathbb{1}_{a=b} \mathbb{1}_{w=w} P_\omega^a(X_{\lceil n^\theta \rceil} = w)$$

$$+ \mathbb{1}_{0 < \|a-b\|_1 \leq n^{\theta'}} \cdot \Xi_1^{a,b}(w, w)$$

$$+ \mathbb{1}_{\|a-b\| > n^{\theta'}} \cdot P_\omega^a(X_{\lceil n^\theta \rceil} = w) P_\omega^b(X_{\lceil n^\theta \rceil} = w) \big]$$

$$\geq \sum_{w \in \mathbb{Z}^d} \sum_{a,b \in \mathbb{Z}^d} \Xi_{k-1}^{x,y}(a, b) \big[\mathbb{1}_{a=b} P_\omega^a(X_{\lceil n^\theta \rceil} = w)$$

$$+ \mathbb{1}_{0 < \|a-b\|_1 \leq n^{\theta'}} \cdot \Xi_1^{a,b}(w, w) \big]$$

$$= \Xi_{k-1}^{x,y}(\Lambda) + \sum_{a,b\in\mathbb{Z}^d} \Xi_{k-1}^{x,y}(a,b)\mathbb{1}_{0<\|a-b\|_1\leq n^{\theta'}} \cdot \Xi_1^{a,b}(\Delta)$$

$$\geq \Xi_{k-1}^{x,y}(\Lambda) + \sum_{a,b\in\mathbb{Z}^d} \Xi_{k-1}^{x,y}(a,b)\mathbb{1}_{0<\|a-b\|_1\leq n^{\theta'}} \cdot \frac{1}{2}\eta^{2dM}$$

$$= \Xi_{k-1}^{x,y}(\Lambda) + \frac{1}{2}\eta^{2dM}\,\Xi_{k-1}^{x,y}\big(\{(a,b): 0<\|a-b\|_1\leq n^{\theta'}\}\big)$$

$$= \Xi_{k-1}^{x,y}(\Lambda) + \frac{1}{2}\eta^{2dM}\big(1 - \Xi_{k-1}^{x,y}(\Lambda) - \Xi_{k-1}^{x,y}\big(\{(a,b): \|a-b\|_1 > n^{\theta'}\}\big)\big)$$

$$\geq \Xi_{k-1}^{x,y}(\Lambda) + \frac{1}{2}\eta^{2dM}\big(1 - \Xi_{k-1}^{x,y}(\Lambda)\big).$$

Fixing $r > 0$, as long as $\Xi_j^{x,y}(\Lambda) < 1 - n^{-r}$ for $j \leq k$ the last inequality gives

$$\frac{\Xi_k^{x,y}(\Lambda)}{\Xi_{k-1}^{x,y}(\Lambda)} \geq 1 + \frac{1}{2}\eta^{2dM}\cdot\frac{n^{-r}}{1-n^{-r}},$$

which implies that $\Xi_k^{x,y}(\Lambda)$ grows exponentially in this regime. Hence, for some $C = C(r) < \infty$ we have $\Xi_{\lceil C\log n\rceil}^{x,y}(\Lambda) > 1 - n^{-r}$ and in particular since $\{\Xi_k^{x,y}(\Lambda)\}_{k\geq 1}$ is nondecreasing $\Xi_{\lceil\log^2 n\rceil}^{x,y}(\Lambda) > \Xi_{\lceil C\log n\rceil}^{x,y}(\Lambda) > 1 - n^{-r}$ for every $r \in \mathbb{N}$ and large enough n, that is, $\Xi_{\lceil\log^2 n\rceil}^{x,y}(\Lambda) = 1 - n^{-\xi(1)}$. We can now construct a coupling of $P_\omega^x(X_n = \cdot)$ and $P_\omega^y(X_n = \cdot)$ on $\mathbb{Z}^d$, by using the coupling $\Xi_{\lceil\log^2 n\rceil}^{x,y}$ until time $\lceil\log^2 n\rceil \cdot \lceil n^\theta\rceil$. Formally, if they coincided until time $\lceil\log^2 n\rceil \cdot \lceil n^\theta\rceil$ we couple them together (to go along the same path) until time n, or if not to move independently until time n. This yields a coupling such that $\sum_{z\in\mathbb{Z}} |P_\omega^x(X_n = z) - P_\omega^y(X_n = z)| < n^{-\xi(1)}$ as required. □

PROOF OF LEMMA 7.5 PART (3).   Written explicitly (after some manipulations)

$$\big\|\big(\nu^{\mathrm{box\text{-}que}\times\mathrm{pre},n-k} * \nu^{\mathrm{que},k}\big)_\omega - \big(\nu^{\mathrm{que},n-k} * \nu^{\mathrm{que},k}\big)_\omega\big\|$$

$$= \sum_{\substack{x\in\mathbb{Z}^d \\ x\leftrightarrow n}} \bigg|\sum_{\Delta\in\Pi}\sum_{\substack{y\in\Delta \\ y\leftrightarrow n-k}} P_\omega^0(X_{n-k}\in\Delta)P_\omega^y(X_k = x)$$

(7.15)
$$\times\bigg[\frac{f(\sigma_y\omega)}{\sum_{z\in\Delta}f(\sigma_z\omega)} - P_\omega^0(X_{n-k} = y|X_{n-k}\in\Delta)\bigg]\bigg|$$

$$\leq \sum_{\substack{x\in\mathbb{Z}^d \\ x\leftrightarrow n}}\sum_{\Delta\in\Pi} P_\omega^0(X_{n-k}\in\Delta)\bigg|\sum_{\substack{y\in\Delta \\ y\leftrightarrow n-k}} P_\omega^y(X_k = x)\bigg[\frac{f(\sigma_y\omega)}{\sum_{z\in\Delta}f(\sigma_z\omega)}$$

$$- P_\omega^0(X_{n-k} = y|X_{n-k}\in\Delta)\bigg]\bigg|.$$

Noticing that for every $\Delta \in \Pi$ and $x \in \mathbb{Z}^d$ we have

$$\sum_{\substack{y \in \Delta \\ y \leftrightarrow n-k}} \left(\frac{1}{|\Delta|} \sum_{w \in \Delta} P_\omega^w(X_k = x) \right) \left[\frac{f(\sigma_y \omega)}{\sum_{z \in \Delta} f(\sigma_z \omega)} - P_\omega^0(X_{n-k} = y | X_{n-k} \in \Delta) \right] = 0,$$

it follows that (7.15) equals

$$\sum_{\substack{x \in \mathbb{Z}^d \\ x \leftrightarrow n}} \sum_{\Delta \in \Pi} P_\omega^0(X_{n-k} \in \Delta) \left| \sum_{\substack{y \in \Delta \\ y \leftrightarrow n-k}} \left[P_\omega^y(X_k = x) - \left(\frac{2}{|\Delta|} \sum_{\substack{w \in \Delta \\ w \leftrightarrow n}} P_\omega^w(X_k = x) \right) \right] \right.$$

$$\times \left. \left[\frac{f(\sigma_y \omega)}{\sum_{z \in \Delta, z \leftrightarrow n-k} f(\sigma_z \omega)} - P_\omega^0(X_{n-k} = y | X_{n-k} \in \Delta) \right] \right|$$

$$= \sum_{\substack{x \in \mathbb{Z}^d \\ x \leftrightarrow n}} \sum_{\Delta \in \Pi} P_\omega^0(X_{n-k} \in \Delta) \left| \sum_{\substack{y \in \Delta \\ y \leftrightarrow n-k}} \left[\frac{2}{|\Delta|} \sum_{\substack{w \in \Delta \\ w \leftrightarrow n}} (P_\omega^y(X_k = x) - P_\omega^w(X_k = x)) \right] \right.$$

$$\times \left. \left[\frac{f(\sigma_y \omega)}{\sum_{z \in \Delta, z \leftrightarrow n-k} f(\sigma_z \omega)} - P_\omega^0(X_{n-k} = y | X_{n-k} \in \Delta) \right] \right|$$

$$\leq \sum_{\substack{x \in \mathbb{Z}^d \\ x \leftrightarrow n}} \sum_{\Delta \in \Pi} P_\omega^0(X_{n-k} \in \Delta) \sum_{\substack{y \in \Delta \\ y \leftrightarrow n-k}} \left| \frac{2}{|\Delta|} \sum_{\substack{w \in \Delta \\ w \leftrightarrow n}} (P_\omega^y(X_k = x) - P_\omega^w(X_k = x)) \right|$$

$$\times \left| \frac{f(\sigma_y \omega)}{\sum_{z \in \Delta, z \leftrightarrow n-k} f(\sigma_z \omega)} - P_\omega^0(X_{n-k} = y | X_{n-k} \in \Delta) \right|$$

$$= \sum_{\Delta \in \Pi} \sum_{\substack{x \in \mathbb{Z}^d \\ \mathrm{dist}(x,\Delta) \leq k \\ x \leftrightarrow n}} P_\omega^0(X_{n-k} \in \Delta) \sum_{\substack{y \in \Delta \\ y \leftrightarrow n-k}} \left| \frac{2}{|\Delta|} \sum_{\substack{w \in \Delta \\ w \leftrightarrow n}} (P_\omega^y(X_k = x) \right.$$

$$\left. - P_\omega^w(X_k = x)) \right| \left| \frac{f(\sigma_y \omega)}{\sum_{z \in \Delta, z \leftrightarrow n-k} f(\sigma_z \omega)} - P_\omega^0(X_{n-k} = y | X_{n-k} \in \Delta) \right|.$$

By Lemma 7.7 applied with k as n and $\theta = \frac{\delta}{\varepsilon}$, this is bounded by

$$\sum_{\Delta \in \Pi} \sum_{\substack{x \in \mathbb{Z}^d \\ \mathrm{dist}(x,\Delta) \leq k \\ x \leftrightarrow n}} P_\omega^0(X_{n-k} \in \Delta)$$

$$\times \sum_{\substack{y \in \Delta \\ y \leftrightarrow n-k}} k^{-\xi(1)} \left| \left[\frac{f(\sigma_y \omega)}{\sum_{z \in \Delta, z \leftrightarrow n-k} f(\sigma_z \omega)} - P_\omega^0(X_{n-k} = y | X_{n-k} \in \Delta) \right] \right|$$

$$\leq \sum_{\Delta \in \Pi} \sum_{\substack{x \in \mathbb{Z}^d \\ \mathrm{dist}(x,\Delta) \leq k}} P_\omega^0(X_{n-k} \in \Delta) 2k^{-\xi(1)}$$

$$\leq 2k^{-\xi(1)} \cdot (2k+l)^d \xrightarrow[n\to\infty]{} 0. \qquad \square$$

PROOF OF THEOREM 1.11. Combining all the claims of Lemma 7.5 and using the triangle inequality, we conclude that

$$\lim_{n\to\infty} \sum_{x \in \mathbb{Z}^d} \left| \frac{1}{Z_{\omega,n}} \mathbb{P}^0(X_n = x) f(\sigma_x \omega) - \sum_{y \in \mathbb{Z}^d} P_\omega^0(X_{n-k} = y) P_{\sigma_y \omega}^0(X_k = x - y) \right| = 0.$$

Using the Markov property of the quenched law and Lemma 7.3, this implies

$$\lim_{n\to\infty} \sum_{x \in \mathbb{Z}^d} \left| \mathbb{P}^0(X_n = x) f(\sigma_x \omega) - P_\omega^0(X_n = x) \right| = 0,$$

and completes the proof. $\square$

APPENDIX

A.1. Annealed local CLT.

PROPOSITION A.1. *Let $d \geq 2$ and assume P is uniformly elliptic, i.i.d. and satisfies $(\mathscr{P})$. Then*

$$\lim_{n\to\infty} \sum_{\substack{x \in \mathbb{Z}^d \\ x \leftrightarrow n}} \left| \mathbb{P}^0(X_n = x) \right.$$

$$\left. - \frac{2}{(2\pi n)^{d/2} \sqrt{\det \Sigma}} \exp\left(-\frac{1}{2n} (x - \mathbb{E}^0[X_n])^T \Sigma^{-1} (x - \mathbb{E}^0[X_n]) \right) \right| = 0,$$

where Σ is the covariance matrix of $X_{\tau_2 - \tau_1}$.

The crucial ingredient in the proof is the annealed CLT proved by Sznitman in [25] for uniformly elliptic i.i.d. random walks in random environments satisfying condition $(\mathscr{P})$. Then standard annealed derivative estimations to approximate the value of the annealed in a given point by its average on a box of side length $\varepsilon n^{1/2}$ gives the required result.

PROOF OF PROPOSITION A.1. Fix $\varepsilon, \delta > 0$ and let $\Pi_n^{(\varepsilon)}$ be a partition of $\mathbb{Z}^d$ into boxes of side length $\lceil \varepsilon n^{1/2} \rceil$. Let $C_\delta > 0$ be a constant such that (due to Lemma 2.16) $\mathbb{P}^0(\|X_n - \mathbb{E}^0[X_n]\|_\infty > C_\delta \sqrt{n}) < \delta$ and denote by $\widehat{\Pi}_n^{(\varepsilon,\delta)}$ the family

of boxes in $\Pi_n^{(\varepsilon)}$ intersecting $\{x \in \mathbb{Z}^d : \|x - \mathbb{E}^0[X_n]\|_\infty \le C_\delta \sqrt{n}\}$. Then

$$
\sum_{\substack{x \in \mathbb{Z}^d \\ x \leftrightarrow n}} \Bigg| \mathbb{P}^0(X_n = x)
$$

$$
- \frac{2}{(2\pi n)^{d/2} \sqrt{\det \Sigma}} \exp\left(-\frac{1}{2n}(x - \mathbb{E}^0[X_n])^T \Sigma^{-1}(x - \mathbb{E}^0[X_n])\right) \Bigg|
$$

$$
= \sum_{\Delta \in \Pi_n^{(\varepsilon)} \setminus \widehat{\Pi}_n^{(\varepsilon,\delta)}} \sum_{\substack{x \in \mathbb{Z}^d \\ x \leftrightarrow n}} \Bigg| \mathbb{P}^0(X_n = x)
$$

(A.1)
$$
- \frac{2}{(2\pi n)^{d/2} \sqrt{\det \Sigma}} \exp\left(-\frac{1}{2n}(x - \mathbb{E}^0[X_n])^T \Sigma^{-1}(x - \mathbb{E}^0[X_n])\right) \Bigg|
$$

$$
+ \sum_{\Delta \in \widehat{\Pi}_n^{(\varepsilon,\delta)}} \sum_{\substack{x \in \mathbb{Z}^d \\ x \leftrightarrow n}} \Bigg| \mathbb{P}^0(X_n = x)
$$

(A.2)
$$
- \frac{2}{(2\pi n)^{d/2} \sqrt{\det \Sigma}} \exp\left(-\frac{1}{2n}(x - \mathbb{E}^0[X_n])^T \Sigma^{-1}(x - \mathbb{E}^0[X_n])\right) \Bigg|.
$$

We estimate each of the term separately starting with (A.1). Due to the choice of C_δ

$$
(A.1) \le \sum_{x: \|x - \mathbb{E}^0[X_n]\|_\infty > C_\delta \sqrt{n}} \mathbb{P}^0(X_n = x)
$$

$$
+ \frac{2}{(2\pi n)^{d/2} \sqrt{\det \Sigma}} \exp\left(-\frac{1}{2n}(x - \mathbb{E}^0[X_n])^T \Sigma^{-1}(x - \mathbb{E}^0[X_n])\right)
$$

$$
\le \delta + C \cdot \exp\left(-\frac{c}{2}C_\delta^2\right).
$$

Thus, by increasing C_δ we can ensure that (A.1) is bounded by 2δ.

Turning to deal with (A.2), we estimate each of the terms inside the absolute value by an average on the appropriate box containing it. Due to the annealed derivative estimations from Lemma 2.14, we have

$$
\left| \mathbb{P}^0(X_n = x) - \frac{2}{(\lceil \varepsilon n^{1/2} \rceil)^d} \mathbb{P}^0(X_n \in \Delta) \right| \le \frac{C \varepsilon n^{1/2}}{n^{(d+1)/2}} = \frac{C\varepsilon}{n^{d/2}},
$$

for every $\Delta \in \Pi_n^{(\varepsilon)}$ and every $x \in \Delta$ such that $x \leftrightarrow n$. In addition, for every $\Delta \in \Pi_n^{(\varepsilon)}$ and every $x \in \Delta$

$$
\left| \exp\left(-\frac{1}{2n}(x - \mathbb{E}^0[X_n])^T \Sigma^{-1}(x - \mathbb{E}^0[X_n])\right) \right.
$$

$$
\left. - \frac{1}{(\lceil \varepsilon n^{1/2} \rceil)^d} \int_\Delta \exp\left(-\frac{1}{2n}(y - \mathbb{E}^0[X_n])^T \Sigma^{-1}(y - \mathbb{E}^0[X_n])\right) dy \right|
$$

$$\leq \exp\left(-\frac{1}{2n}(x - \mathbb{E}^0[X_n])^T \Sigma^{-1}(x - \mathbb{E}^0[X_n])\right)$$

$$\times \frac{1}{(\lceil \varepsilon n^{1/2} \rceil)^d}\left|\int_\Delta 1 - \exp\left(-\frac{1}{2n}(y - x)^T \Sigma^{-1}(y - x)\right)dy\right|$$

$$\leq \exp\left(-\frac{1}{2n}(x - \mathbb{E}^0[X_n])^T \Sigma^{-1}(x - \mathbb{E}^0[X_n])\right) \cdot C\varepsilon^2 \leq C\varepsilon^2.$$

Combining the last two estimation gives

$$(A.1) \leq \sum_{\Delta \in \widehat{\Pi}_n^{(\varepsilon,\delta)}} \sum_{\substack{x \in \mathbb{Z}^d \\ x \leftrightarrow n}} \left|\mathbb{P}^0(X_n = x) - \frac{2}{(\lceil \varepsilon n^{1/2} \rceil)^d}\mathbb{P}^0(X_n \in \Delta)\right|$$

$$+ \sum_{\Delta \in \widehat{\Pi}_n^{(\varepsilon,\delta)}} \sum_{\substack{x \in \mathbb{Z}^d \\ x \leftrightarrow n}} \left|\frac{2}{(\lceil \varepsilon n^{1/2} \rceil)^d}\mathbb{P}^0(X_n \in \Delta) - \frac{2}{(2\pi n)^{d/2}\sqrt{\det \Sigma}}\right.$$

$$\times \frac{1}{(\lceil \varepsilon n^{1/2} \rceil)^d}\int_\Delta \exp\left(-\frac{1}{2n}(y - \mathbb{E}^0[X_n])^T \Sigma^{-1}(y - \mathbb{E}^0[X_n])\right)dy\bigg|$$

$$+ \sum_{\Delta \in \widehat{\Pi}_n^{(\varepsilon,\delta)}} \sum_{\substack{x \in \mathbb{Z}^d \\ x \leftrightarrow n}} \frac{2}{(2\pi n)^{d/2}\sqrt{\det \Sigma}}$$

$$\times \left|\frac{1}{(\lceil \varepsilon n^{1/2} \rceil)^d}\int_\Delta \exp\left(-\frac{1}{2n}(y - \mathbb{E}^0[X_n])^T \Sigma^{-1}(y - \mathbb{E}^0[X_n])\right)dy\right.$$

$$- \exp\left(-\frac{1}{2n}(x - \mathbb{E}^0[X_n])^T \Sigma^{-1}(x - \mathbb{E}^0[X_n])\right)\bigg|$$

$$\leq \sum_{\Delta \in \widehat{\Pi}_n^{(\varepsilon,\delta)}} \sum_{\substack{x \in \mathbb{Z}^d \\ x \leftrightarrow n}} \frac{C\varepsilon}{n^{d/2}} + \sum_{\Delta \in \widehat{\Pi}_n^{(\varepsilon,\delta)}} \sum_{\substack{x \in \Delta \\ x \leftrightarrow n}} \frac{C\varepsilon^2}{(2\pi n)^{d/2}\sqrt{\det \Sigma}}$$

$$+ \sum_{\Delta \in \widehat{\Pi}_n^{(\varepsilon,\delta)}} \sum_{\substack{x \in \mathbb{Z}^d \\ x \leftrightarrow n}} \left|\frac{2}{(\lceil \varepsilon n^{1/2} \rceil)^d}\mathbb{P}^0(X_n \in \Delta) - \frac{1}{(2\pi n)^{d/2}\sqrt{\det \Sigma}}\right.$$

$$\times \frac{2}{(\lceil \varepsilon n^{1/2} \rceil)^d}\int_\Delta \exp\left(-\frac{1}{2n}(y - \mathbb{E}^0[X_n])^T \Sigma^{-1}(y - \mathbb{E}^0[X_n])\right)dy\bigg|.$$

The total number of vertices in the boxes in $\widehat{\Pi}_n^{(\varepsilon,\delta)}$ is $(C_\delta n^{1/2})^d$ and, therefore, the first two sums are bounded by $C_\delta^d \cdot C\varepsilon$. As for the last term, we have

$$\sum_{\Delta \in \widehat{\Pi}_n^{(\varepsilon,\delta)}} \sum_{\substack{x \in \mathbb{Z}^d \\ x \leftrightarrow n}} \left|\frac{2}{(\lceil \varepsilon n^{1/2} \rceil)^d}\mathbb{P}^0(X_n \in \Delta) - \frac{1}{(2\pi n)^{d/2}\sqrt{\det \Sigma}} \cdot \frac{2}{(\lceil \varepsilon n^{1/2} \rceil)^d}\right.$$

$$\times \int_\Delta \exp\left(-\frac{1}{2n}(y - \mathbb{E}^0[X_n])^T \Sigma^{-1}(y - \mathbb{E}^0[X_n])\right)dy\bigg|$$

$$= \sum_{\Delta \in \widehat{\Pi}_n^{(\varepsilon,\delta)}} \left| \mathbb{P}^0(X_n \in \Delta) - \frac{2}{(2\pi n)^{d/2}\sqrt{\det \Sigma}} \right.$$

$$\left. \times \int_\Delta \exp\left(-\frac{1}{2n}(y - \mathbb{E}^0[X_n])^T \Sigma^{-1}(y - \mathbb{E}^0[X_n])\right) dy \right|.$$

Apply the functional CLT proved by Sznitman in [25] and nothing that for a fixed ε and δ the sum is finite gives that the last term goes to zero as n goes to infinity.

Combining all of the above, we conclude that

$$\limsup_{n\to\infty} \sum_{\substack{x\in\mathbb{Z}^d \\ x\leftrightarrow n}} \left| \mathbb{P}^0(X_n = x) - \frac{2}{(2\pi n)^{d/2}\sqrt{\det \Sigma}} \right.$$

$$\left. \times \exp\left(-\frac{1}{2n}(x - \mathbb{E}^0[X_n])^T \Sigma^{-1}(x - \mathbb{E}^0[X_n])\right) \right|$$

$$\leq C_\delta^d \cdot C\varepsilon + 2\delta.$$

By first taking $\delta > 0$ arbitrary small and then choosing $\varepsilon > 0$ even smaller so that $C_\delta^d \cdot C\varepsilon < \delta$ the result follows. $\square$

A.2. Annealed derivative estimations. In this part of the Appendix, we prove Lemma 3.3 and Lemma 2.14 regarding annealed derivative estimations.

A.2.1. *General estimations.* We start with the following claim which is a general result for i.i.d. random variables on a lattice:

CLAIM A.2. *Let $\{Y_i\}_{i=1}^\infty$ and $\{Z_i\}_{i=1}^\infty$ be a sequence of d-dimensional random variables and a sequence of 1-dimensional nonnegative integer valued random variables, respectively, such that $\{(Y_i, Z_i)\}_{i=1}^\infty$ are independent and have joint distribution P. Assume in addition that $\{(Y_i, Z_i)\}_{i=2}^\infty$ are i.i.d. and there exists $v \in \mathbb{Z}^d$, $k \in \mathbb{N}$ such that $\mathsf{P}((Y_2, Z_2) = (v, k)) > 0$ and $\mathsf{P}((Y_2, Z_2) = (v + e_i, k + 1)) > 0$ for every $1 \leq i \leq d$. Let $S_n = \sum_{i=1}^n Y_i$ and $T_n = \sum_{i=1}^n Z_i$. Then there exists $C < \infty$ which is determined by distribution P such that for every $n \in \mathbb{N}$, $m \in \mathbb{N}$, every $x, y \in \mathbb{Z}^d$ such that $\|x - y\|_1 = 1$ and every $1 \leq k \leq d$*

$$(A.3) \qquad\qquad \mathsf{P}((S_n, T_n) = (x, m)) < Cn^{-(d+1)/2},$$

$$(A.4) \quad \left| \mathsf{P}((S_n, T_n) = (x, m)) - \mathsf{P}((S_n, T_n) = (y, m+1)) \right| < Cn^{-(d+2)/2}$$

and

$$(A.5) \quad \begin{aligned} &\left| \mathsf{P}((S_n, T_n) = (x, m)) + \mathsf{P}((S_n, T_n) = (y + e_k, m)) - \mathsf{P}((S_n, T_n) \right. \\ &\left. = (x + e_k, m+1)) - \mathsf{P}((S_n, T_n) = (y, m+1)) \right| < Cn^{-(d+3)/2}. \end{aligned}$$

In addition, if $\{Y_i\}_{i=1}^{\infty}$ and $\{Z_i\}_{i=1}^{\infty}$ have finite moments. Then for every $\varepsilon > 0$, every $m \in \mathbb{N}$ and every partition Π_n of $\mathbb{Z}^d$ into boxes of side length n^{ε}

$$\sum_{\Delta \in \Pi_n} \sum_{\substack{x \in \Delta \\ x \leftrightarrow m}} \left[\max_{y \in \Delta} \mathsf{P}((S_n, T_n) = (y, m)) - \mathsf{P}((S_n, T_n) = (x, m)) \right]$$

(A.6)
$$\leq C n^{-1+3d\varepsilon}.$$

PROOF. Let χ be the characteristic function of (Y_2, Z_2). Since (Y_2, Z_2) is concentrated on a lattice χ is periodic. The existence of v, k as above implies that the period is 2π in every coordinate. The existence of v and k also implies that there are $D > 0$ and $\delta > 0$ such that:

(1) $|\chi(\xi, s)| < 1 - D$ for every $(\xi, s) \in [-\pi, \pi]^{d+1}$ such that $\|(\xi, s)\|_1 \geq \delta$,
(2) $|\chi(\xi, s)| < 1 - D\|(\xi, s)\|_1^2$ for every $(\xi, s) \in [-\pi, \pi]^{d+1}$ such that $\|(\xi, s)\|_1 < \delta$.

The last two facts implies [(A.3)–(A.5)]. Indeed,

$$\mathsf{P}\left(\sum_{i=2}^{n} (Y_i, Z_i) = (x, m) \right)$$

$$= \frac{1}{(2\pi)^{d+1}} \int_{[-\pi,\pi]^{d+1}} e^{-i\langle \xi, x \rangle - i \langle s, m \rangle} \chi^{n-1}(\xi, s) \, d\xi \, ds$$

$$\leq \int_{[-\pi,\pi]^{d+1}} |\chi^{n-1}(\xi, s)| \, d\xi \, ds$$

$$= \int_{\|(\xi,s)\|_1 > \delta} |\chi^{n-1}(\xi, s)| \, d\xi \, ds + \int_{\|(\xi,s)\|_1 \leq \delta} |\chi^{n-1}(\xi, s)| \, d\xi \, ds$$

$$\leq (2\pi)^{d+1} (1-D)^{n-1} + \int_{\|(\xi,s)\|_1 \leq \delta} (1 - D\|(\xi, s)\|_1^2)^{n-1} \, d\xi \, ds$$

$$< C n^{-(d+1)/2}$$

and convolution with the distribution of (Y_1, Z_1) only decreases the supremum.

To see (A.4), note that $y = x \pm e_j$ for some $1 \leq j \leq d$ and, therefore,

$$\left| \mathsf{P}\left(\sum_{i=2}^{n} (Y_i, Z_i) = (x, m) \right) - \mathsf{P}\left(\sum_{i=2}^{n} (Y_i, Z_i) = (x + e_j, m+1) \right) \right|$$

$$= \frac{1}{(2\pi)^{d+1}}$$

$$\times \left| \int_{[-\pi,\pi]^{d+1}} \left(e^{-i\langle \xi, x \rangle - i \langle s, m \rangle} - e^{-i\langle \xi, x \pm e_j \rangle - i \langle s, m+1 \rangle} \right) \chi(\xi, s)^{n-1} \, d\xi \, ds \right|$$

$$\leq \frac{1}{(2\pi)^{d+1}} \int_{[-\pi,\pi]^{d+1}} \left| e^{-i\langle \xi, x \rangle - i \langle s, m \rangle} - e^{-i\langle \xi, x \pm e_j \rangle - i \langle s, m+1 \rangle} \right|$$

$$\times |\chi(\xi, s)|^{n-1} \, d\xi \, ds.$$

Recalling that $|e^{-i\langle\xi,x\rangle-i\langle s,m\rangle} - e^{-i\langle\xi,x\pm e_j\rangle-i\langle s,m+1\rangle}| \leq |\langle(\xi,s),\langle\pm e_j,1\rangle\rangle|$, we can bound the last term by

$$\frac{1}{(2\pi)^{d+1}} \int_{[-\pi,\pi]^{d+1}} |s \pm \langle\xi,e_j\rangle| \cdot |\chi(\xi,s)| \, d\xi \, ds$$

$$\leq (1-D)^{n+1} + C \int_{\|(\xi,s)\|_1 \leq \delta} |s \pm \langle\xi,e_j\rangle|(1 - D\|(\xi,s)\|_1^2)^{n-1} \, d\xi \, ds$$

$$\leq (1-D)^{n+1} + C \int_{\|(\xi,s)\|_1 \leq \delta} |s \pm \langle\xi,e_j\rangle| e^{-Dn\|(\xi,s)\|_1^2} \, d\xi \, ds.$$

Substituting $\zeta = \xi\sqrt{n}$ and $t = s\sqrt{n}$ the last integral equals

$$\frac{C}{\sqrt{n}^{d+2}} \int_{\|(\zeta,t)\|_1 \leq \delta\sqrt{n}} |t \pm \langle\zeta,e_j\rangle| e^{-D\|(\zeta,t)\|_1^2} \, d\zeta \, dt$$

$$\leq \frac{C}{n^{(d+2)/2}} \int_{\mathbb{R}^{d+1}} |t \pm \langle\zeta,e_j\rangle| e^{-D\|(\zeta,t)\|_1^2} \, d\zeta \, dt = O(n^{-(d+2)/2}).$$

For (A.5), note that

$$\left| P\left(\sum_{i=2}^{n}(Y_i,Z_i) = (x,m)\right) + P\left(\sum_{i=2}^{n}(Y_i,Z_i) = (y+e_k,m)\right)\right.$$

$$\left. - P\left(\sum_{i=2}^{n}(Y_i,Z_i) = (y,m+1)\right) - P\left(\sum_{i=2}^{n}(Y_i,Z_i) = (x+e_k,m+1)\right)\right|$$

$$= \left| P\left(\sum_{i=2}^{n}(Y_i,Z_i) = (x,m)\right) + P\left(\sum_{i=2}^{n}(Y_i,Z_i) = (x+e_j+e_k,m)\right)\right.$$

$$- P\left(\sum_{i=2}^{n}(Y_i,Z_i) = (x+e_j,m+1)\right)$$

$$\left. - P\left(\sum_{i=2}^{n}(Y_i,Z_i) = (x+e_k,m+1)\right)\right|$$

$$= \frac{1}{(2\pi)^{d+1}} \left| \int_{[-\pi,\pi]^{d+1}} (e^{-i\langle\xi,x\rangle-i\langle s,m\rangle} + e^{-i\langle\xi,x+e_j+e_k\rangle-i\langle s,m\rangle} \right.$$

$$\left. - e^{-i\langle\xi,x+e_j\rangle-i\langle s,m+1\rangle} - e^{-i\langle\xi,x+e_k\rangle-i\langle s,m+1\rangle})\chi(\xi,s)^{n-1} \, d\xi \, ds \right|$$

$$\leq \left| \int_{[-\pi,\pi]^{d+1}} |1 + e^{-i\langle\xi,e_j+e_k\rangle} - e^{-i\langle\xi,e_j\rangle-i\langle s,1\rangle} - e^{-i\langle\xi,e_k\rangle-i\langle s,1\rangle}| \right.$$

$$\left. \times |\chi(\xi,s)|^{n-1} \, d\xi \, ds \right|$$

$$\leq \left| \int_{[-\pi,\pi]^{d+1}} |\langle(\xi,s),(e_j,1)\rangle||\langle(\xi,s),(e_k,1)\rangle||\chi(\xi,s)|^{n-1} \, d\xi \, ds \right|$$

and the proof continuous now as before except that we gained an additional factor of $n^{-1/2}$.

Finally, we turn to the proof of (A.6). For every $\Delta \in \Pi$ denote by x_Δ a point in Δ such that $P(S_n = x_\Delta) = \max_{y \in \Delta} P(S_n = y)$. As a first step, we show that

$$(A.7) \qquad \sum_{\Delta \in \Pi_n} \sum_{x \in \Delta} \big[P(S_n = x_\Delta) - P(S_n = x) \big] \leq C n^{-1/2 + 3d\varepsilon}.$$

By [2], Claim 4.3, for every $x, y \in \mathbb{Z}^d$ such that $\|x - y\|_1 = 1$, we have $|P(S_n = y) - P(S_n = x)| \leq C n^{-(d+1)/2}$ and, therefore, for every $\Delta \in \Pi$

$$\sum_{x \in \Delta} \big[P(S_n = x_\Delta) - P(S_n = x) \big]$$

$$(A.8) \qquad\qquad \leq \sum_{x \in \Delta} \|x_\Delta - x\|_\infty C n^{-(d+1)/2}$$

$$\leq C \sum_{x \in \Delta} n^{d\varepsilon} \cdot n^{-(d+1)/2} = C n^{-(d+1)/2 + 2d\varepsilon}.$$

Splitting the sum over the boxes to those boxes whose distance from $\mathsf{E}[S_n]$ is greater or smaller than $n^{1/2+\varepsilon}$, we can the first with by an Azuma inequality which yields $P(\|S_n - \mathsf{E}[S_n]\|_1 > n^{1/2+\varepsilon}) = n^{-\xi(1)}$ and the second with (A.8) we get

$$\sum_{\Delta \in \Pi_n} \sum_{x \in \Delta} \big[P(S_n = x_\Delta) - P(S_n = x) \big]$$

$$\leq \sum_{\substack{\Delta \in \Pi_n \\ \mathrm{dist}(\Delta, \mathsf{E}[S_n]) \leq n^{1/2+\varepsilon}}} C n^{-(d+1)/2 + 2d\varepsilon} + n^{-\xi(1)} \leq C n^{-1/2 + 3d\varepsilon}$$

Finally, we turn to prove (A.6). Denote

$$g(m) = \sum_{\Delta \in \Pi_n} \sum_{\substack{x \in \Delta \\ x \leftrightarrow m}} \Big[\max_{y \in \Delta} P\big((S_n, T_n) = (y, m)\big) - P\big((S_n, T_n) = (x, m)\big) \Big].$$

By Azuma's inequality $P(|T_n - \mathsf{E}[T_n]| > n^{1/2+\varepsilon}) = n^{-\xi(1)}$ and, therefore, it is enough to deal with $m \in \mathbb{N}$ such that $|m - \mathsf{E}[T_n]| \leq n^{1/2+\varepsilon}$. By the same estimation and (A.7), we also know that $n^{1/2+\varepsilon} \min_{m:|m-\mathsf{E}[T_n]| \leq n^{1/2+\varepsilon}} g(m) \leq C n^{-1/2 + 3d\varepsilon}$ and, therefore,

$$(A.9) \qquad \min_{m:|m-\mathsf{E}[T_n]| \leq n^{1/2+\varepsilon}} g(m) \leq C n^{-1 + (3d-1)\varepsilon}.$$

However, using (A.5), for every $m \in \mathbb{N}$ and $x, z \in \Delta$ such that $x \leftrightarrow m$, $z \leftrightarrow m + 1$ we have

$$\max_{y \in \Delta} P\big((S_n, T_n) = (y, m)\big) - P\big((S_n, T_n) = (x, m)\big)$$

$$- \max_{w \in \Delta} P\big((S_n, T_n) = (w, m+1)\big) - P\big((S_n, T_n) = (z, m)\big)$$

$$\leq \sum_{k=1}^{d} \sum_{\substack{x,y \in \Delta \\ x \leftrightarrow m \\ \|y-x\|_1 = 1}} \left| P\big((S_n, T_n) = (x, m)\big) + P\big((S_n, T_n) = (y + e_k, m)\big) \right.$$

$$\left. - P\big((S_n, T_n) = (x + e_k, m+1)\big) - P\big((S_n, T_n) = (y, m+1)\big) \right|$$

$$\leq C n^{-(d+3)/2 + d\varepsilon}$$

and, therefore, by separating the sum into boxes which are at distance $\leq n^{1/2+\varepsilon}$ and those who at distance $> n^{1/2+\varepsilon}$ we get

(A.10)
$$\begin{aligned} \left| g(m) - g(m+1) \right| &\leq C n^{(1/2+\varepsilon)d} \cdot n^{-(d+3)/2 + d\varepsilon} + n^{-\xi(1)} \\ &= C n^{-3/2 + 2d\varepsilon}. \end{aligned}$$

Using (A.9) and (A.10) gives

$$g(m) \leq C n^{-1+(3d-1)\varepsilon} + C n^{-3/2 + 2d\varepsilon} \cdot n^{1/2+\varepsilon} \leq C n^{-1+3d\varepsilon}$$

for every $m \in \mathbb{N}$ such that $|m - \mathsf{E}[T_n]| \leq n^{1/2+\varepsilon}$, and thus completes the proof. □

A.2.2. *Proof of Lemma* 3.3. Before turning to the proof of the lemmas, we give the following estimations on hitting point of an hyperplane conditioned to contain a regeneration time. More formally, we have the following.

LEMMA A.3. *Let $d \geq 4$ and assume P is uniformly elliptic, i.i.d. and satisfies* $(\mathscr{P})$. *Fix $z \in \mathbb{Z}^d$, $N \in \mathbb{N}$ and let $z_1 \in \tilde{\mathcal{P}}(0, N)$. Let $\{X_n\}$ be an RWRE starting at z_1. For $k, l \in \mathbb{N}$ let $B(l, k)$ be the event that $\langle X_{\tau_k}, e_1 \rangle = l$, $B(l) = \bigcup_k B(l, k)$ and*

$$\hat{B}(l, N^2) \equiv \hat{B}(l) = B(l) \cap \bigcap_{j=l+1}^{N^2} B^c(j).$$

Then (for a given $l \leq N^2$):

(1) *For every $n \in \mathbb{N}$ and $w \in H_l$*

(A.11)
$$\mathbb{P}^{z_1}\big((X_{T_l}, T_l) = (w, n) | \hat{B}(l)\big) \leq C l^{-d/2}.$$

(2) *For every $n \in \mathbb{N}$, and every $w, z \in H_l$ such that $\|w - z\|_1 = 1$*

(A.12)
$$\begin{aligned} \big| \mathbb{P}^{z_1}\big((X_{T_l}, T_l) = (w, n) | \hat{B}(l)\big) &- \mathbb{P}^{z_1}\big((X_{T_l}, T_l) = (z, n+1) | \hat{B}(l)\big) \big| \\ &\leq C l^{-(d+1)/2}. \end{aligned}$$

(3) *For every $n \in \mathbb{N}$ and every $w \in H_l$*

(A.13)
$$\begin{aligned} \big| \mathbb{P}^{z_1}\big((X_{T_l}, T_l) = (w, n) | \hat{B}(l)\big) &- \mathbb{P}^{z_1+e_1}\big((X_{T_l}, T_l) = (w, n+1) | \hat{B}(l)\big) \big| \\ &\leq C l^{-(d+1)/2}. \end{aligned}$$

PROOF. Due to the independence of (X_{T_l}, T_l) from $\bigcap_{j=l+1}^{N^2} B^c(j)$, we get that for every $M \in \mathbb{N}$

$$\mathbb{P}^{z_1}\big((X_{T_l}, T_l) = (w, n) | \hat{B}(l)\big)$$

$$= \mathbb{P}^{z_1}\big((X_{T_l}, T_l) = (w, n) | B(l)\big)$$

$$= \frac{1}{\mathbb{P}^{z_1}(B(l))} \sum_{k=1}^{\infty} \mathbb{P}^{z_1}\big((X_{\tau_k}, \tau_k) = (w, n)\big)$$

$$= \frac{1}{\mathbb{P}^{z_1}(B(l))} \sum_{k=1}^{M} \mathbb{P}^{z_1}\left((X_{\tau_k}, \tau_k) = (w, n), \langle X_{\tau_{\lceil k/2 \rceil}}, e_1 \rangle \geq \frac{l}{2}\right)$$

$$+ \frac{1}{\mathbb{P}^{z_1}(B(l))} \sum_{k=1}^{M} \mathbb{P}^{z_1}\left((X_{\tau_k}, \tau_k) = (w, n), \langle X_{\tau_k} - X_{\tau_{\lceil k/2 \rceil}}, e_1 \rangle \geq \frac{l}{2}\right)$$

$$+ \frac{1}{\mathbb{P}^{z_1}(B(l))} \sum_{k=M+1}^{\infty} \mathbb{P}^{z_1}\left((X_{\tau_k}, \tau_k) = (w, n), \langle X_{\tau_{\lceil k/2 \rceil}}, e_1 \rangle \leq \frac{l}{2}\right)$$

$$+ \frac{1}{\mathbb{P}^{z_1}(B(l))} \sum_{k=M+1}^{\infty} \mathbb{P}^{z_1}\left((X_{\tau_k}, \tau_k) = (w, n), \langle X_{\tau_k} - X_{\tau_{\lceil k/2 \rceil}}, e_1 \rangle \leq \frac{l}{2}\right).$$

Using Claim A.2 gives

$$\mathbb{P}^{z_1}\left((X_{\tau_k}, \tau_k) = (w, n), \langle X_{\tau_{\lceil k/2 \rceil}}, e_1 \rangle \leq \frac{l}{2}\right)$$

$$= \sum_{x : \langle x, e_1 \rangle \leq l/2} \sum_{s \in \mathbb{N}} \mathbb{P}^{z_1}\big((X_{\tau_{\lceil k/2 \rceil}}, \tau_{\lceil k/2 \rceil}) = (x, s)\big)$$

(A.14)
$$\times \mathbb{P}^{z_1}\big((X_{\tau_k}, \tau_k) = (w, n) | (X_{\tau_{\lceil k/2 \rceil}}, \tau_{\lceil k/2 \rceil}) = (x, s)\big)$$

$$\leq C k^{-(d+1)/2} \sum_{x : \langle x, e_1 \rangle \leq l/2} \sum_{s \in \mathbb{N}} \mathbb{P}^{z_1}\big((X_{\tau_{\lceil k/2 \rceil}}, \tau_{\lceil k/2 \rceil}) = (x, s)\big)$$

$$= C k^{-(d+1)/2} \mathbb{P}^{z_1}\left(\langle X_{\tau_{\lceil k/2 \rceil}}, e_1 \rangle \leq \frac{l}{2}\right),$$

and in a similar manner

$$\mathbb{P}^{z_1}\left((X_{\tau_k}, \tau_k) = (w, n), \langle X_{\tau_k} - X_{\tau_{\lceil k/2 \rceil}}, e_1 \rangle \leq \frac{l}{2}\right)$$

$$\leq C k^{-(d+1)/2} \mathbb{P}^{z_1}\left(\langle X_{\tau_k} - X_{\tau_{\lceil k/2 \rceil}}, e_1 \rangle \leq \frac{l}{2}\right)$$

$$\leq C k^{-(d+1)/2} \mathbb{P}^{z_1}\left(\langle X_{\tau_{\lceil k/2 \rceil}}, e_1 \rangle \leq \frac{l}{2}\right).$$

Repeating the same calculations while separating the sum according to the events $\langle X_{\tau_{\lfloor k/2 \rfloor}}, e_1 \rangle \geq \frac{l}{2}$ and $\langle X_{\tau_k} - X_{\tau_{\lfloor k/2 \rfloor}}, e_1 \rangle \geq \frac{l}{2}$ we get that

$$\mathbb{P}^{z_1}\left((X_{\tau_k}, \tau_k) = (w, n), \langle X_{\tau_{\lceil k/2 \rceil}}, e_1 \rangle \geq \frac{l}{2} \right)$$

$$\leq C k^{-(d+1)/2} \mathbb{P}^{z_1}\left(\langle X_{\tau_{\lceil k/2 \rceil}}, e_1 \rangle \geq \frac{l}{2} \right)$$

and

$$\mathbb{P}^{z_1}\left((X_{\tau_k}, \tau_k) = (w, n), \langle X_{\tau_k} - X_{\tau_{\lceil k/2 \rceil}}, e_1 \rangle \geq \frac{l}{2} \right)$$

$$\leq C k^{-(d+1)/2} \mathbb{P}^{z_1}\left(\langle X_{\tau_{\lceil k/2 \rceil}}, e_1 \rangle \geq \frac{l}{2} \right),$$

combining all of the above yields

$$\mathbb{P}^{z_1}((X_{T_l}, T_l) = (w, n) | \hat{B}(l)) \leq C \sum_{k=1}^{M} k^{-(d+1)/2} \mathbb{P}^{z_1}\left(\langle X_{\tau_{\lceil k/2 \rceil}}, e_1 \rangle \geq \frac{l}{2} \right)$$

$$+ C \sum_{k=M+1}^{\infty} k^{-(d+1)/2} \mathbb{P}^{z_1}\left(\langle X_{\tau_{\lceil k/2 \rceil}}, e_1 \rangle \leq \frac{l}{2} \right).$$

The argument continues now as in the proof of [2], Lemma 4.2. Choosing

$$M = \frac{l}{\mathbb{E}^{z_1}[\langle X_{\tau_2} - X_{\tau_1}, e_1 \rangle]} = \Theta(l)$$

and using Theorem 2.8 (see also Remark 2.11) we get that $\tau_k - \tau_{k-1}$ has finite $2d$ moments, and from standard estimates for the sum of i.i.d. variable [that the $2d$ moment for the sum of k i.i.d. mean zero random variables grows like $O(k^d)$], thus

$$\mathbb{P}^{z_1}\left(\langle X_{\tau_{\lceil k/2 \rceil}}, e_1 \rangle \geq \frac{l}{2} \right), \mathbb{P}^{z_1}\left(\langle X_{\tau_{\lceil k/2 \rceil}}, e_1 \rangle \leq \frac{l}{2} \right) \leq \min\left[1, \frac{C k^d}{(M-k)^{2d}} \right]$$

and, therefore,

$$\mathbb{P}^{z_1}(X_l = x | \hat{B}(l)) \leq C \sum_{k=1}^{\infty} k^{-(d+1)/2} \min\left[1, \frac{C k^d}{(M-k)^{2d}} \right] = O(l^{-d/2}).$$

To see the last equality, we have to separate the sum into four parts as in [2], Lemma 4.2, the first part is a sum over $k \in [1, \frac{M}{2}]$ which contains roughly l summands, each of them is bounded by $C l^{-d}$ and therefore the whole sum is bounded by $C l^{-d+1}$. The second part is a sum over $k \in [\frac{M}{2}, M - \sqrt{M}]$. In this case, the

sum is bounded up to a constant by

$$\int_{M/2}^{M-\sqrt{M}} x^{(d-1)/2}(M-x)^{-2d}\,dx$$

$$= \int_{\sqrt{M}}^{M/2} (M-y)^{(d-1)/2} y^{-2d}\,dy$$

$$\leq \left(\frac{M}{2}\right)^{(d-1)/2} \int_{\sqrt{M}}^{M/2} y^{-2d}\,dy$$

$$\leq C M^{(d-1)/2} \sqrt{M}^{-2d+1} = O(M^{-d/2}) = O(l^{-d/2}).$$

The third sum is over $k \in [M-\sqrt{M}, M+\sqrt{M}]$. This part contains roughly $\sqrt{M}$ summands, each of them is bounded by $M^{-(d+1)/2}$ so the sum is $O(l^{-d/2})$. Finally, the last sum is over $k \geq M + \sqrt{M}$. This case is similar to the second sum and is bounded up to a constant by

$$\int_{M+\sqrt{M}}^{\infty} x^{(d-1)/2}(x-M)^{-2d}\,dx$$

$$= \int_{M+\sqrt{M}}^{2M} x^{(d-1)/2}(x-M)^{-2d}\,dx + \int_{2M}^{\infty} x^{(d-1)/2}(x-M)^{-2d}\,dx$$

$$= \int_{\sqrt{M}}^{M} (y+M)^{(d-1)/2} y^{-2d}\,dy + \int_{M}^{\infty} (y+M)^{(d-1)/2} y^{-2d}\,dy$$

$$\leq C M^{(d-1)/2} \int_{\sqrt{M}}^{M} y^{-2d}\,dy + C \int_{M}^{\infty} y^{(d-1)/2} y^{-2d}\,dy = O(l^{-d/2}).$$

Thus,

$$\mathbb{P}^{z_1}\big((X_{T_l}, T_l) = (w,n) | \hat{B}(l)\big) \leq C l^{-d/2}.$$

The arguments for the other two inequalities are very similar and, therefore, we only discuss the proof for (A.12). Assuming without loss of generality that $y = x + e_j$ for some $1 \leq j \leq d$ we have

$$\big|\mathbb{P}^{z_1}\big((X_{T_l}, T_l) = (w,n)|\hat{B}(l)\big) - \mathbb{P}^{z_1}\big((X_{T_l}, T_l) = (z,n+1)|\hat{B}(l)\big)\big|$$

$$= \big|\mathbb{P}^{z_1}\big((X_{T_l}, T_l) = (w,n)\big|B(l)\big) - \mathbb{P}^{z_1}\big((X_{T_l}, T_l) = (z,n+1)\big|B(l)\big)\big|$$

$$\leq \frac{1}{\mathbb{P}^{z_1}(B(l))} \sum_{k=1}^{\infty} \big|\mathbb{P}^{z_1}\big((X_{\tau_k}, \tau_k) = (w,n)\big) - \mathbb{P}^{z_1}\big((X_{\tau_k}, \tau_k) = (z,n+1)\big)\big|.$$

We can now continue as in the previous case by separating the sum for $k \leq M$ and $k > M$ and also adding either the assumption $\langle X_{\tau_k} - X_{\tau_{\lceil k/2 \rceil}}, e_1 \rangle \geq \frac{l}{2}$ or $\langle X_{\tau_{\lceil k/2 \rceil}}, e_1 \rangle \geq \frac{l}{2}$. Now we can estimate each term in the same way except that

in (A.14) we have

$$
\begin{aligned}
\Bigg| \mathbb{P}^{z_1} & \Bigg((X_{\tau_k}, \tau_k) = (w, n), \langle X_{\tau_{\lceil k/2 \rceil}, e_1} \rangle \leq \frac{l}{2} \Bigg) \\
& - \mathbb{P}^{z_1} \Bigg((X_{\tau_k}, \tau_k) = (z, n+1), \langle X_{\tau_{\lceil k/2 \rceil}, e_1} \rangle \leq \frac{l}{2} \Bigg) \Bigg| \\
& \leq \sum_{x: \langle x, e_1 \rangle \leq l/2} \sum_{s \in \mathbb{N}} \mathbb{P}^{z_1} \big((X_{\tau_{\lceil k/2 \rceil}}, \tau_{\lceil k/2 \rceil}) = (x, s) \big) \\
& \quad \times \big| \mathbb{P}^{z_1} \big((X_{\tau_k}, \tau_k) = (w, n) | (X_{\tau_{\lceil k/2 \rceil}}, \tau_{\lceil k/2 \rceil}) = (x, s) \big) \\
& \quad - \mathbb{P}^{z_1} \big((X_{\tau_k}, \tau_k) = (z, n+1) | (X_{\tau_{\lceil k/2 \rceil}}, \tau_{\lceil k/2 \rceil}) = (x, s) \big) \big| \\
& \leq \sum_{x: \langle x, e_1 \rangle \leq l/2} \sum_{s \in \mathbb{N}} C k^{-(d+2)/2} \mathbb{P}^{z_1} \big((X_{\tau_{\lceil k/2 \rceil}}, \tau_{\lceil k/2 \rceil}) = (x, s) \big) \\
& = C k^{-(d+2)/2} \mathbb{P}^{z_1} \Bigg(\langle X_{\tau_{\lceil k/2 \rceil}}, e_1 \rangle \leq \frac{l}{2} \Bigg),
\end{aligned}
$$

where for the last inequality we used (A.4) instead of (A.3). Apart from that difference, the proof continues via the same lines. $\square$

Next, we turn to the proof of the annealed estimations. We follow the same ideas as in the proof of [2], Lemma 4.2.

PROOF OF LEMMA 3.3.   (1) Denote $u := (X_{T_{\partial \mathcal{P}(0,N)}}, T_{\partial \mathcal{P}(0,N)})$. Then

$$
\begin{aligned}
\mathbb{P}^z \big(u &= (x, m) \big) \\
&= \sum_{l \leq N^2} \mathbb{P}^z \big(\hat{B}(l) \big) \sum_{\substack{w \in H_l, n \in \mathbb{N} \\ w - z_1 \leftrightarrow n}} \mathbb{P}^z \big((X_{T_l}, T_l) = (w, n) | \hat{B}(l) \big) \\
& \quad \times \mathbb{P}^z \big(u = (x, m) | \hat{B}(l), (X_{T_l}, T_l) = (w, n) \big) \\
& \overset{(1)}{\leq} \sum_{l \leq N^2} \mathbb{P}^z \big(\hat{B}(l) \big) \max_{\substack{w \in H_l, n \in \mathbb{N} \\ w - z_1 \leftrightarrow n}} C l^{-d/2} \mathbb{P}^z \big(u = (x, m) | \hat{B}(l), (X_{T_l}, T_l) = (w, n) \big) \\
& \overset{(2)}{=} \sum_{l \leq N^2} \mathbb{P}^z \big(\hat{B}(l) \big) C l^{-d/2} \\
& \overset{(3)}{\leq} \sum_{l \leq N^2} C e^{-c((N^2 - l)/2)^{\gamma}} C l^{-d/2} \\
& = \sum_{l \leq N^2/2} C e^{-c((N^2 - l)/2)^{\gamma}} C l^{-d/2} + \sum_{N^2/2 \leq l \leq N^2} C e^{-c((N^2 - l)/2)^{\gamma}} C l^{-d/2} \\
& \leq C \frac{N^2}{2} e^{-c(N^2/2)^{\gamma}} + \frac{C}{N^d} \sum_{N^2/2 \leq l \leq N^2} C e^{-c((N^2 - l)/2)^{\gamma}} \leq \frac{C}{N^d},
\end{aligned}
$$

where for (1) we used Lemma 3.3, (3.1), for (2) we used the shift invariance of the annealed walk and for the first sum in (3) we used Corollary 2.10 (see also Remark 2.11).

(2) For $y \in \mathbb{Z}^d$ such that $\|y - x\|_1 = 1$, we can find $2 \le j \le d$ such that $y = x \pm e_j$ (without loss of generality assume the sign is $+$) and, therefore,

$$
\begin{aligned}
& \mathbb{P}^z\big(u = (y, m + 1)\big) \\
& \quad = \sum_{l \le N^2} \mathbb{P}^z(\hat{B}(l)) \sum_{\substack{w \in H_l, n \in \mathbb{N} \\ w - z_1 \leftrightarrow n}} \mathbb{P}^z\big((X_{T_l}, T_l) = (w + e_j, n + 1)|\hat{B}(l)\big) \\
& \qquad \times \mathbb{P}^z\big(u = (y, m + 1)|\hat{B}(l), (X_{T_l}, T_l) = (w + e_j, n + 1)\big) \\
& \quad = \sum_{l \le N^2} \mathbb{P}^z(\hat{B}(l)) \sum_{\substack{w \in H_l, n \in \mathbb{N} \\ w - z_1 \leftrightarrow n}} \mathbb{P}^z\big((X_{T_l}, T_l) = (w + e_j, n + 1)|\hat{B}(l)\big) \\
& \qquad \times \mathbb{P}^z\big(u = (x, m)|\hat{B}(l), (X_{T_l}, T_l) = (w, n)\big).
\end{aligned}
$$

Subtracting the formula for $\mathbb{P}^z(u = (y, m + 1))$ from the one for $\mathbb{P}^{z_1}(u = (x, m))$, we thus get

$$
\begin{aligned}
& \big|\mathbb{P}^z\big(u = (x, m)\big) - \mathbb{P}^z\big(u = (y, m + 1)\big)\big| \\
& \quad \le \sum_{l \le N^2} \mathbb{P}^z(\hat{B}(l)) \sum_{\substack{w \in H_l, n \in \mathbb{N} \\ w - z_1 \leftrightarrow n}} \big[\big|\mathbb{P}^z\big((X_{T_l}, T_l) = (w, n)|\hat{B}(l)\big) \\
& \qquad - \mathbb{P}^z\big((X_{T_l}, T_l) = (w + e_j, n + 1)|\hat{B}(l)\big)\big| \\
& \qquad \times \mathbb{P}^z\big(u = (x, m)|\hat{B}(l), (X_{T_l}, T_l) = (w, n)\big)\big] \\
& \quad \le \sum_{l \le N^2} \mathbb{P}^z(\hat{B}(l)) \max_{w \in H_l, 2 \le j \le d, n \in \mathbb{N}} \big|\mathbb{P}^z\big((X_{T_l}, T_l) = (w, n)|\hat{B}(l)\big) \\
& \qquad - \mathbb{P}^z\big((X_{T_l}, T_l) = (w + e_j, n + 1)|\hat{B}(l)\big)\big|,
\end{aligned}
$$

where as before we used the shift invariance. Using (A.12), we get

$$
\big|\mathbb{P}^z\big(u = (x, m)\big) - \mathbb{P}^z\big(u = (y, m + 1)\big)\big| \le \sum_{l \le N^2} \mathbb{P}^z(\hat{B}(l)) C l^{-(d+1)/2}
$$

which by the same argument as before is bounded by CN^{-d-1}.

(3) We start with the case where $w = z + e_j$ for some $2 \le j \le d$. Due to the (2), we have

$$
\begin{aligned}
& \big|\mathbb{P}^z\big(u = (x, m)\big) - \mathbb{P}^{z+e_j}\big(u = (x, m + 1)\big)\big| \\
& \quad \le \big|\mathbb{P}^z\big(u = (x, m)\big) - \mathbb{P}^{z+e_j}\big(u = (x + e_j, m)\big)\big| \\
& \qquad + \big|\mathbb{P}^{z+e_j}\big(u = (x + e_j, m)\big) - \mathbb{P}^{z+e_j}\big(u = (x, m + 1)\big)\big| \\
& \quad \le \big|\mathbb{P}^z\big(u = (x, m)\big) - \mathbb{P}^{z+e_j}\big(u = (x + e_j, m)\big)\big| + CN^{-d-1},
\end{aligned}
$$

and, therefore, it is enough to compare $\mathbb{P}^z(u = (x, m))$ with $\mathbb{P}^{z+e_j}(u = (x + e_j, m))$. In this case, we have

$$\mathbb{P}^{z+e_j}\big(u = (x + e_j, m)\big)$$

$$= \sum_{l \leq N^2} \mathbb{P}^{z+e_j}\big(\hat{B}(l)\big) \sum_{\substack{w \in H_l, n \in \mathbb{N} \\ w - z_1 \leftrightarrow n}} \mathbb{P}^{z+e_j}\big((X_{T_l}, T_l) = (w + e_j, n) | \hat{B}(l)\big)$$

$$\times \mathbb{P}^{z+e_j}\big(u = (x + e_j, m) | \hat{B}(l), (X_{T_l}, T_l) = (w + e_j, n)\big)$$

$$= \sum_{l \leq N^2} \mathbb{P}^{z+e_j}\big(\hat{B}(l)\big) \sum_{\substack{w \in H_l, n \in \mathbb{N} \\ w - z_1 \leftrightarrow n}} \mathbb{P}^{z+e_j}\big((X_{T_l}, T_l) = (w + e_j, n) | \hat{B}(l)\big)$$

$$\times \mathbb{P}^z\big(u = (x, m) | \hat{B}(l), (X_{T_l}, T_l) = (w, n)\big)$$

$$= \sum_{l \leq N^2} \mathbb{P}^z\big(\hat{B}(l)\big) \sum_{\substack{w \in H_l, n \in \mathbb{N} \\ w - z_1 \leftrightarrow n}} \mathbb{P}^{z+e_j}\big((X_{T_l}, T_l) = (w + e_j, n) | \hat{B}(l)\big)$$

$$\times \mathbb{P}^z\big(u = (x, m) | \hat{B}(l), (X_{T_l}, T_l) = (w, n)\big)$$

and, therefore,

$$\big|\mathbb{P}^z\big(u = (x, m)\big) - \mathbb{P}^{z+e_j}\big(u = (x + e_j, m)\big)\big|$$

$$\leq \sum_{l \leq N^2} \mathbb{P}^z\big(\hat{B}(l)\big) \sum_{\substack{w \in H_l, n \in \mathbb{N} \\ w - z_1 \leftrightarrow n}} \big[\big|\mathbb{P}^{z+e_j}\big((X_{T_l}, T_l) = (w + e_j, n) | \hat{B}(l)\big)$$

$$- \mathbb{P}^z\big((X_{T_l}, T_l) = (w, n) | \hat{B}(l)\big)\big| \mathbb{P}^z\big(u = (x, m) | \hat{B}(l), (X_{T_l}, T_l) = (w, n)\big)\big]$$

$$\leq \sum_{l \leq N^2} \mathbb{P}^z\big(\hat{B}(l)\big) \max_{w \in H_l, 2 \leq j \leq d, n \in \mathbb{N}} \big|\mathbb{P}^{z+e_j}\big((X_{T_l}, T_l) = (w + e_j, n) | \hat{B}(l)\big)$$

$$- \mathbb{P}^z\big((X_{T_l}, T_l) = (w, n) | \hat{B}(l)\big)\big|.$$

Using (A.12) and (A.13) gives

$$\big|\mathbb{P}^z\big(u = (x, m)\big) - \mathbb{P}^{z+e_1}\big(u = (x + e_1, m)\big)\big| \leq \sum_{l \leq N^2} \mathbb{P}^z\big(\hat{B}(l)\big) C l^{-(d+1)/2}$$

and the proof is completed in the same way as in (2). Finally, we turn to deal with the case $w = z + e_1$. One can rewrite the term for $\mathbb{P}^{z+e_1}(u = (x = m))$ as

$$\mathbb{P}^{z+e_1}\big(u = (x, m)\big)$$

$$= \sum_{l \leq N^2 - 1} \mathbb{P}^{z+e_1}\big(\hat{B}(l + 1)\big)$$

$$\times \sum_{\substack{w \in H_l, n \in \mathbb{N} \\ w - z \leftrightarrow n}} \mathbb{P}^{z+e_1}\big((X_{T_{l+1}}, T_{l+1}) = (w + e_1, n) | \hat{B}(l + 1)\big)$$

$$\times \mathbb{P}^{z+e_1}\big(u = (x,m)|\hat{B}(l+1), (X_{T_{l+1}}, T_{l+1}) = (w+e_1, n)\big)$$

$$= \sum_{l \leq N^2-1} \mathbb{P}^z\big(\hat{B}(l)\big) \sum_{\substack{w \in H_l, n \in \mathbb{N} \\ w-z \leftrightarrow n}} \mathbb{P}^z\big((X_{T_l}, T_l) = (w,n)|\hat{B}(l)\big)$$

$$\times \mathbb{P}^{z+e_1}\big(u = (x,m)|\hat{B}(l+1), (X_{T_{l+1}}, T_{l+1}) = (w+e_1, n)\big)$$

and, therefore, using Lemma A.3

$$\big|\mathbb{P}^z\big(u = (x,m)\big) - \mathbb{P}^{z+e_1}\big(u = (x,m)\big)\big|$$

$$\leq \sum_{l \leq N^2-1} \mathbb{P}^z\big(\hat{B}(l)\big) \sum_{\substack{w \in H_l, n \in \mathbb{N} \\ w-z \leftrightarrow n}} \mathbb{P}^z\big((X_{T_l}, T_l) = (w,n)|\hat{B}(l)\big)$$

$$\times \big|\mathbb{P}^z\big(u = (x,m)|\hat{B}(l), (X_{T_l}, T_l) = (w,n)\big)$$

$$- \mathbb{P}^{z+e_1}\big(u = (x,m)|\hat{B}(l+1), (X_{T_{l+1}}, T_{l+1}) = (w+e_1, n)\big)\big|$$

$$+ \mathbb{P}^z\big(\hat{B}(N^2)\big) \max_{\substack{w \in H_{N^2}, n \in \mathbb{N} \\ w-z \leftrightarrow n}} \mathbb{P}^z\big((X_{T_{N^2}}, T_{N^2}) = (w,n)|\hat{B}(N^2)\big)$$

$$\times \mathbb{P}^z\big(u = (x,m)|\hat{B}(N^2), (X_{T_{N^2}}, T_{N^2}) = (w,n)\big)$$

$$\leq \sum_{l \leq N^2-1} \mathbb{P}^z\big(\hat{B}(l)\big)$$

$$\times \sum_{\substack{w \in H_l, n \in \mathbb{N} \\ w-z \leftrightarrow n}} Cl^{-(d+1)/2} \big|\mathbb{P}^z\big(u = (x,m)|\hat{B}(l), (X_{T_l}, T_l) = (w,n)\big)$$

$$- \mathbb{P}^{z+e_1}\big(u = (x,m)|\hat{B}(l+1), (X_{T_{l+1}}, T_{l+1}) = (w+e_1, n)\big)\big|$$

$$+ \mathbb{P}^z\big(\hat{B}(N^2)\big) \cdot CN^{-(d+1)/2}$$

$$\leq 2C \sum_{l \leq N^2-1} \mathbb{P}^z\big(\hat{B}(l)\big)l^{-(d+1)/2} + \mathbb{P}^z\big(\hat{B}(N^2)\big) \cdot CN^{-(d+1)/2}$$

$$\leq CN^{-(d+1)/2},$$

as required. □

A.2.3. *Proof of Lemma* 2.14. The proof of Lemma 2.14 follows very similar lines to the one of Lemma 3.3 and is based on very similar estimations to the one obtained in Lemma A.3. Here, we need a slightly different version of it in which we replace $\hat{B}(l)$ the event that the last regeneration time is in the hyperplane l with the event $\hat{Z}(l)$ in which the last regeneration time is at time l.

LEMMA A.4 (Middle step in Lemma 3.3). *Let $d \geq 4$ and assume P is uniformly elliptic, i.i.d. and satisfies ($\mathscr{P}$). Fix $z \in \mathbb{Z}^d$ and $n \in \mathbb{N}$. For $k, l \in \mathbb{N}$ let*

$Z(l, k)$ *be the event that* $\tau_k = l$, $Z(l) = \bigcup_k Z(l, k)$ *and*

$$\hat{Z}_n(l) \equiv \hat{Z}(l) = Z(l) \cap \bigcap_{j=l+1}^n Z^c(j).$$

Then for every $z \in \mathbb{Z}^d$:

(1) *For every* $l \leq n$ *and* $x \in \mathbb{Z}^d$

(A.15) $$\mathbb{P}^z(X_l = x|\hat{Z}(l)) \leq Cl^{-d/2}.$$

(2) *For every* $l \in \mathbb{N}$, *and every* $x, y \in \mathbb{Z}^d$ *such that* $\|x - y\|_1 = 1$

(A.16) $$|\mathbb{P}^z(X_l = x|\hat{Z}(l)) - \mathbb{P}^z(X_{l+1} = y|\hat{Z}(l+1))| \leq Cl^{-(d+1)/2}.$$

(3) *For every* $l \in \mathbb{N}$ *every* $x \in \mathbb{Z}^d$ *and every* $1 \leq j \leq d$

(A.17) $$|\mathbb{P}^z(X_l = x|\hat{Z}(l)) - \mathbb{P}^{z+e_j}(X_{l+1} = x|\hat{Z}(l+1))| \leq Cl^{-(d+1)/2}.$$

(4) *For every* $\varepsilon > 0$, *every partition* Π_n *of* $\mathbb{Z}^d$ *into boxes of side length* n^ε *and any* $l \in \mathbb{N}$

(A.18) $$\sum_{\Delta \in \Pi_n} \sum_{\substack{x \in \Delta \\ x - z \leftrightarrow l}} \max_{y \in \Delta} \mathbb{P}^z(X_l = y|\hat{Z}(l)) - \mathbb{P}^z(X_l = x|\hat{Z}(l)) \leq Cl^{-1/2 + 3d\varepsilon}.$$

PROOF. Due to the independence of X_l from $\bigcap_{j=l+1}^n Z^c(j)$ conditioned on $Z(l)$ we get that for every $M \in \mathbb{N}$

$$\mathbb{P}^z(X_l = x|\hat{Z}(l)) = \mathbb{P}^z(X_l = x|Z(l))$$

$$= \frac{1}{\mathbb{P}^z(Z(l))} \sum_{k=1}^\infty \mathbb{P}^z((X_{\tau_k}, \tau_k) = (x, l))$$

$$= \frac{1}{\mathbb{P}^z(Z(l))} \sum_{k=1}^M \mathbb{P}^{z_1}\left((X_{\tau_k}, \tau_k) = (x, l), \tau_{\lceil k/2 \rceil} \geq \frac{l}{2}\right)$$

$$+ \frac{1}{\mathbb{P}^z(Z(l))} \sum_{k=1}^M \mathbb{P}^{z_1}\left((X_{\tau_k}, \tau_k) = (x, l), \tau_k - \tau_{\lceil k/2 \rceil} \geq \frac{l}{2}\right)$$

$$+ \frac{1}{\mathbb{P}^z(Z(l))} \sum_{k=M+1}^\infty \mathbb{P}^{z_1}\left((X_{\tau_k}, \tau_k) = (x, l), \tau_{\lceil k/2 \rceil} \leq \frac{l}{2}\right)$$

$$+ \frac{1}{\mathbb{P}^z(Z(l))} \sum_{k=M+1}^\infty \mathbb{P}^{z_1}\left((X_{\tau_k}, \tau_k) = (x, l), \tau_k - \tau_{\lceil k/2 \rceil} \leq \frac{l}{2}\right).$$

Claim A.2 gives

$$\mathbb{P}^z\left((X_{\tau_k}, \tau_k) = (x, l), \tau_{\lceil k/2 \rceil} \leq \frac{l}{2}\right)$$

$$= \sum_{s=0}^{l/2} \sum_{\substack{w \in \mathbb{Z}^d \\ w - z \leftrightarrow s}} \mathbb{P}^z\left((X_{\tau_{\lceil k/2 \rceil}}, \tau_{\lceil k/2 \rceil}) = (w, s)\right)$$

$$\times \mathbb{P}^z\left((X_{\tau_k}, \tau_k) = (x, l) | (X_{\tau_{\lceil k/2 \rceil}}, \tau_{\lceil k/2 \rceil}) = (w, s)\right)$$

$$\leq C k^{-(d+1)/2} \sum_{s=0}^{l/2} \sum_{\substack{w \in \mathbb{Z}^d \\ w - z \leftrightarrow s}} \mathbb{P}^z\left((X_{\tau_{\lceil k/2 \rceil}}, \tau_{\lceil k/2 \rceil}) = (w, s)\right)$$

$$= C k^{-(d+1)/2} \mathbb{P}^{z_1}\left(\tau_{\lceil k/2 \rceil} \leq \frac{l}{2}\right),$$

and similarly we have

$$\mathbb{P}^z\left((X_{\tau_k}, \tau_k) = (x, l), \tau_k - \tau_{\lceil k/2 \rceil} \leq \frac{l}{2}\right) \leq C k^{-(d+1)/2} \mathbb{P}^{z_1}\left(\tau_k - \tau_{\lceil k/2 \rceil} \leq \frac{l}{2}\right)$$

$$\leq C k^{-(d+1)/2} \mathbb{P}^{z_1}\left(\tau_{\lceil k/2 \rceil} \leq \frac{l}{2}\right).$$

Repeating the same calculations while separating the sum according to the events $\tau_{\lfloor k/2 \rfloor} \geq \frac{l}{2}$ and $\tau_k - \tau_{\lfloor k/2 \rfloor} \geq \frac{l}{2}$, we get that

$$\mathbb{P}^z\left((X_{\tau_k}, \tau_k) = (x, l), \tau_{\lceil k/2 \rceil} \geq \frac{l}{2}\right) \leq C k^{-(d+1)/2} \mathbb{P}^z\left(\tau_{\lceil k/2 \rceil} \geq \frac{l}{2}\right)$$

and

$$\mathbb{P}^z\left((X_{\tau_k}, \tau_k) = (x, l), \tau_k - \tau_{\lceil k/2 \rceil} \geq \frac{l}{2}\right) \leq C k^{-(d+1)/2} \mathbb{P}^z\left(\tau_{\lceil k/2 \rceil} \geq \frac{l}{2}\right),$$

combining all of the above yields

$$\mathbb{P}^z\left(X_l = x | \hat{Z}(l)\right) \leq C \sum_{k=1}^{M} k^{-(d+1)/2} \mathbb{P}^z\left(\tau_{\lceil k/2 \rceil} \geq \frac{l}{2}\right)$$

$$+ C \sum_{k=M+1}^{\infty} k^{-(d+1)/2} \mathbb{P}^z\left(\tau_{\lceil k/2 \rceil} \leq \frac{l}{2}\right).$$

Choosing

$$M = \frac{l}{\mathbb{E}^z[\langle \tau_2 - \tau_1, e_1 \rangle]} = O(l)$$

and using Theorem 2.9 (see also Corollary 2.10) we get that $\tau_k - \tau_{k-1}$ has finite $2d$ moments, and from standard estimates for the sum of i.i.d. variable [that the $2d$ moment for the sum of k i.i.d. mean zero random variables grows like $O(k^d)$], thus

$$\mathbb{P}^z\left(\tau_{\lceil k/2 \rceil} \geq \frac{l}{2}\right), \mathbb{P}^z\left(\tau_{\lceil k/2 \rceil} \leq \frac{l}{2}\right) \leq \min\left[1, \frac{Ck^d}{(M-k)^{2d}}\right]$$

and, therefore,

$$\mathbb{P}^{z_1}\left(X_l = x | \hat{Z}(l)\right) \leq C \sum_{k=1}^{\infty} k^{-(d+1)/2} \min\left[1, \frac{Ck^d}{(M-k)^{2d}}\right] = O(l^{-d/2}).$$

To see the last equality, we have to separate the sum into four parts as in [2], Lemma 4.2, the first part is a sum over $k \in [1, \frac{M}{2}]$ which contains roughly l summands, each of them is bounded by Cl^{-d} and therefore the whole sum is bounded by Cl^{-d+1}. The second part is a sum over $k \in [\frac{M}{2}, M - \sqrt{M}]$. In this case, the sum is bounded up to a constant by

$$\int_{M/2}^{M-\sqrt{M}} x^{(d-1)/2}(M-x)^{-2d}\, dx = \int_{\sqrt{M}}^{M/2} (M-y)^{(d-1)/2} y^{-2d}\, dy$$

$$\leq \left(\frac{M}{2}\right)^{(d-1)/2} \int_{\sqrt{M}}^{M/2} y^{-2d}\, dy$$

$$\leq C M^{(d-1)/2} \sqrt{M}^{-2d+1}$$

$$= O(M^{-d/2}) = O(l^{-d/2}).$$

The third sum is over $k \in [M - \sqrt{M}, M + \sqrt{M}]$. This part contains roughly $\sqrt{M}$ summands, each of them is bounded by $M^{-(d+1)/2}$ so the sum is $O(l^{-d/2})$. Finally, the last sum is over $k \geq M + \sqrt{M}$. This case is similar to the second sum and is bounded up to a constant by

$$\int_{M+\sqrt{M}}^{\infty} x^{(d-1)/2}(x-M)^{-2d}\, dx$$

$$= \int_{M+\sqrt{M}}^{2M} x^{(d-1)/2}(x-M)^{-2d}\, dx + \int_{2M}^{\infty} x^{(d-1)/2}(x-M)^{-2d}\, dx$$

$$= \int_{\sqrt{M}}^{M} (y+M)^{(d-1)/2} y^{-2d}\, dy + \int_{M}^{\infty} (y+M)^{(d-1)/2} y^{-2d}\, dy$$

$$\leq C M^{(d-1)/2} \int_{\sqrt{M}}^{M} y^{-2d}\, dy + C \int_{M}^{\infty} y^{(d-1)/2} y^{-2d}\, dy = O(l^{-d/2}).$$

Thus,

$$\mathbb{P}^{z_1}\left(X_l = x | \hat{Z}(l)\right) \leq C l^{-d/2}.$$

The arguments for the other two inequalities are very similar and, therefore, we only discuss the proof for A.12. Similarly, to the first case, we have

$$
\begin{aligned}
&\left|\mathbb{P}^{z_1}\big(X_l = x|\hat{Z}_n(l)\big) - \mathbb{P}^{z_1}\big(X_{l+1} = y|\hat{Z}_{n+1}(l+1)\big)\right| \\
&= \left|\mathbb{P}^{z_1}\big(X_l = x|Z(l)\big) - \mathbb{P}^{z_1}\big(X_{l+1} = y|Z(l+1)\big)\right| \\
&= \left|\frac{1}{\mathbb{P}^{z_1}(Z(l))} \sum_{k=1}^{\infty} \mathbb{P}^{z_1}\big((X_{\tau_k}, \tau_k) = (x, l)\big)\right. \\
&\qquad \left. - \frac{1}{\mathbb{P}^{z_1}(Z(l+1))} \sum_{k=1}^{\infty} \mathbb{P}^{z_1}\big((X_{\tau_k}, \tau_k) = (y, l+1)\big)\right|
\end{aligned}
$$

$$
\text{(A.19)} \quad \leq \frac{1}{\mathbb{P}^{z_1}(Z(l))} \sum_{k=1}^{\infty} \left|\mathbb{P}^{z_1}\big((X_{\tau_k}, \tau_k) = (x, l)\big) - \mathbb{P}^{z_1}\big((X_{\tau_k}, \tau_k) = (y, l+1)\big)\right|
$$

$$
\text{(A.20)} \quad + \left|\frac{1}{\mathbb{P}^{z_1}(Z(l))} - \frac{1}{\mathbb{P}^{z_1}(Z(l+1))}\right| \sum_{k=1}^{\infty} \mathbb{P}^{z_1}\big((X_{\tau_k}, \tau_k) = (y, l+1)\big).
$$

For the first term, we can now continue as in the previous case, by first separating the sum for $k \leq M$ and $k > M$ and also adding either the assumption $\langle X_{\tau_k} - X_{\tau_{\lceil k/2 \rceil}}, e_1 \rangle \geq \frac{l}{2}$ or $\langle X_{\tau_{\lceil k/2 \rceil}}, e_1 \rangle \geq \frac{l}{2}$. Now we can estimate each term in the same way except that in (A.14) we have

$$
\begin{aligned}
&\left|\mathbb{P}^{z_1}\left((X_{\tau_k}, \tau_k) = (x, l), \langle X_{\tau_{\lceil k/2 \rceil}, e_1} \rangle \leq \frac{l}{2}\right)\right. \\
&\qquad \left. - \mathbb{P}^{z_1}\left((X_{\tau_k}, \tau_k) = (y, l+1), \langle X_{\tau_{\lceil k/2 \rceil}, e_1} \rangle \leq \frac{l}{2}\right)\right| \\
&\leq \sum_{w:\langle w, e_1 \rangle \leq l/2} \sum_{s \in \mathbb{N}} \mathbb{P}^{z_1}\big((X_{\tau_{\lceil k/2 \rceil}}, \tau_{\lceil k/2 \rceil}) = (w, s)\big) \\
&\qquad \times \left|\mathbb{P}^{z_1}\big((X_{\tau_k}, \tau_k) = (x, l)|(X_{\tau_{\lceil k/2 \rceil}}, \tau_{\lceil k/2 \rceil}) = (w, s)\big)\right. \\
&\qquad \left. - \mathbb{P}^{z_1}\big((X_{\tau_k}, \tau_k) = (y, l+1)|(X_{\tau_{\lceil k/2 \rceil}}, \tau_{\lceil k/2 \rceil}) = (w, s)\big)\right| \\
&\leq \sum_{w:\langle w, e_1 \rangle \leq l/2} \sum_{s \in \mathbb{N}} C k^{-(d+2)/2} \mathbb{P}^{z_1}\big((X_{\tau_{\lceil k/2 \rceil}}, \tau_{\lceil k/2 \rceil}) = (w, s)\big) \\
&= C k^{-(d+2)/2} \mathbb{P}^{z_1}\left(\langle X_{\tau_{\lceil k/2 \rceil}}, e_1 \rangle \leq \frac{l}{2}\right),
\end{aligned}
$$

where for the last inequality we used (A.4) instead of (A.3). Apart from that difference the proof for the first term continues via the same lines. Regarding the second term, since $\{\tau_k - \tau_{k-1}\}_{k=2}^{\infty}$ are i.i.d. under $\mathbb{P}$ using the Fourier analysis of Claim A.2, one can verify that $|\mathbb{P}^{z_1}(Z(l)) - \mathbb{P}^{z_1}(Z(l+1))| \leq \frac{C}{l}$ and, there-

fore,

$$(A.20) \leq \frac{C}{l \cdot \mathbb{P}^{z_1}(l)} \cdot \mathbb{P}^{z_1}(X_{l+1} = y | Z(l+1))$$

$$\leq \frac{C}{l} \cdot \mathbb{P}^{z_1}(X_{l+1} = y | Z(l+1)) \leq C \cdot l^{(d+2)/2} = o(l^{(d+1)/2}),$$

where for the last inequality we used the first part of the lemma [see (A.16)].

Finally, we turn to discuss the last term. The proof is very similar to the previous ones. For every $\Delta \in \Pi_n$ denote by x_Δ a point in Δ such that $\mathbb{P}^z(X_l = x_\Delta | \hat{Z}(l)) = \max_{y \in \Delta} \mathbb{P}^z(X_l = y | \hat{Z}(l))$ we get

$$\sum_{\substack{\Delta \in \Pi_n \\ }} \sum_{\substack{x \in \Delta \\ x-z \leftrightarrow l}} \max_{y \in \Delta} \mathbb{P}^z(X_l = y | \hat{Z}(l)) - \mathbb{P}^z(X_l = x | \hat{Z}(l))$$

$$= \frac{1}{\mathbb{P}^z(Z(l))}$$

$$\times \sum_{k=1}^{\infty} \sum_{\Delta \in \Pi_n} \sum_{\substack{x \in \Delta \\ x-z \leftrightarrow l}} \mathbb{P}^z((X_{\tau_k}, \tau_k) = (x_\Delta, l)) - \mathbb{P}^z((X_{\tau_k}, \tau_k) = (x, l)).$$

Separating the sum as in the previous cases and using (A.6) in the appropriate inequality, this completes the proof. $\square$

PROOF OF LEMMA 2.14. (1) We have

$$\mathbb{P}^z(X_n = x)$$

$$= \sum_{l \leq n} \mathbb{P}^z(\hat{Z}(l)) \sum_{w \in \mathbb{Z}^d} \mathbb{P}^z(X_l = w | \hat{Z}(l)) \mathbb{P}^z(X_n = x | \hat{Z}(l), X_l = w)$$

$$\overset{(1)}{\leq} \sum_{l \leq n} \mathbb{P}^z(\hat{Z}(l)) \sum_{w \in \mathbb{Z}^d} C l^{-d/2} \mathbb{P}^z(X_n = x | \hat{Z}(l), X_l = w)$$

$$\overset{(2)}{=} \sum_{l \leq n} \mathbb{P}^z(\hat{Z}(l)) C l^{-d/2}$$

$$\overset{(3)}{\leq} \sum_{l \leq n} e^{-(\log(n-l))^2} C l^{-d/2}$$

$$= \sum_{l \leq n/2} e^{-(\log(n-l))^2} C l^{-d/2} + \sum_{n/2 \leq l \leq n} e^{-(\log(n-l))^2} C l^{-d/2}$$

$$\leq C e^{-c(\log n)^2} + C n^{-d/2} \sum_{n/2 \leq l \leq n} e^{-(\log(n-l))^2} \leq C n^{-d/2},$$

where for (1) we used Lemma A.4, (3.1), for (2) we used the shift invariance of the annealed walk, and for the first sum in (3) we used Corollary 2.10.

(2) For $y \in \mathbb{Z}^d$ such that $\|y - x\|_1 = 1$, we can find $1 \leq j \leq d$ such that $y = x + e_j$ and, therefore,

$$
\begin{aligned}
&\mathbb{P}^z(X_{n+1} = y) \\
&\quad = \sum_{l \leq n+1} \mathbb{P}^z(\hat{Z}_{n+1}(l)) \sum_{\substack{w \in \mathbb{Z}^d \\ w-z \leftrightarrow l}} \mathbb{P}^z(X_l = w | \hat{Z}_{n+1}(l)) \\
&\qquad \times \mathbb{P}^z(X_{n+1} = y | \hat{Z}_{n+1}(l), X_l = w) \\
&\quad = \sum_{l \leq n} \mathbb{P}^z(\hat{Z}_{n+1}(l+1)) \sum_{\substack{w \in \mathbb{Z}^d \\ w-z \leftrightarrow l}} \mathbb{P}^z(X_{l+1} = w + e_j | \hat{Z}_{n+1}(l)) \\
&\qquad \times \mathbb{P}^z(X_{n+1} = y | \hat{Z}_{n+1}(l), X_{l+1} = w + e_j) \\
&\quad = \sum_{l \leq n} \mathbb{P}^z(\hat{Z}_{n+1}(l+1)) \sum_{\substack{w \in \mathbb{Z}^d \\ w-z \leftrightarrow l}} \mathbb{P}^z(X_{l+1} = w + e_j | \hat{Z}_{n+1}(l+1)) \\
&\qquad \times \mathbb{P}^z(X_n = x | \hat{Z}_n(l), X_l = w).
\end{aligned}
$$

Subtracting the formula for $\mathbb{P}^z(X_{n+1} = y)$ from the one for $\mathbb{P}^z(X_n = x)$, we thus get

$$
\begin{aligned}
&\left| \mathbb{P}^z(X_n = x) - \mathbb{P}^z(X_{n+1} = y) \right| \\
&\quad \leq \sum_{l \leq n-1} \left| \mathbb{P}^z(\hat{Z}_{n+1}(l+1)) - \mathbb{P}^z(\hat{Z}_n(l)) \right| \\
&\qquad \times \sum_{w \in \mathbb{Z}^d} \mathbb{P}^z(X_{l+1} = w + e_j | \hat{Z}_{n+1}(l+1)) \mathbb{P}^z(X_n = x | \hat{Z}_n(l), X_l = w) \\
&\qquad + \sum_{l \leq n} \mathbb{P}^z(\hat{Z}_n(l)) \sum_{w \in \mathbb{Z}^d} \left| \mathbb{P}^z(X_l = w | \hat{Z}_n(l)) - \mathbb{P}^z(X_l = w + e_j | \hat{Z}_n(l)) \right| \\
&\qquad \times \mathbb{P}^z(X_n = x | \hat{Z}_n(l), X_l = w) \\
&\quad \leq \sum_{l \leq n-1} \left| \mathbb{P}^z(\hat{Z}_{n+1}(l+1)) - \mathbb{P}^z(\hat{Z}_n(l)) \right| \mathbb{P}^z(X_{n+1} = x + e_j | \hat{Z}_{n+1}(l+1)) \\
&\qquad + \sum_{l \leq n} \mathbb{P}^z(\hat{Z}_{n+1}(l)) \max_{w \in \mathbb{Z}^d, 2 \leq j \leq d} \left| \mathbb{P}^z(X_l = w | \hat{Z}_n(l)) \right. \\
&\qquad \left. - \mathbb{P}^z(X_{l+1} = w + e_j | \hat{Z}_{n+1}(l+1)) \right|,
\end{aligned}
$$

where as before we used the shift invariance. Using (A.12), the second term is bounded by $\sum_{l \leq n} \mathbb{P}^z(\hat{Z}_n(l)) C l^{-(d+1)/2}$, which by the same argument as before is

bounded by $Cn^{-(d+1)/2}$. Regarding the first term, using the first part of the lemma, see (2.2) and Theorem 2.9 gives

$$\sum_{l\leq n-1} |\mathbb{P}^z(\hat{Z}_{n+1}(l+1)) - \mathbb{P}^z_{n+1}(\hat{Z}(l))|\mathbb{P}^z(X_{n+1} = x + e_j|\hat{Z}(l+1))$$

$$\leq \sum_{l\leq n-1} \frac{|\mathbb{P}^z(\hat{Z}_{n+1}(l+1)) - \mathbb{P}^z(\hat{Z}_n(l))|}{\mathbb{P}^z(\hat{Z}_{n+1}(l+1))}\mathbb{P}^z(X_{n+1} = x + e_j)$$

$$\leq \sum_{l\leq n-1} \frac{|\mathbb{P}^z(\hat{Z}_{n+1}(l+1)) - \mathbb{P}^z(\hat{Z}_n(l))|}{\mathbb{P}^z(\hat{Z}_{n+1}(l+1))}Cn^{-d/2}$$

$$\leq \sum_{l\leq n-n^{1/4}} e^{-(\log(n-l))^2}Cn^{-d/2} + n^{-d/2}$$

$$\times \sum_{n-n^{1/4}\leq l\leq n-1} |\mathbb{P}^z(\hat{Z}_{n+1}(l+1)) - \mathbb{P}^z(\hat{Z}_n(l))|$$

$$\overset{(1)}{=} o(n^{-(d+1)/2}) + n^{-d/2}\sum_{n-n^{1/4}\leq l\leq n-1}\frac{C}{l} \leq Cn^{-d/2}\cdot n^{1/4}\cdot n^{-1}$$

$$= o(n^{-(d+1)/2}),$$

where for (1) we used the fact that for $l\geq n - n^{1/4} \geq \frac{n}{2}$

$$|\mathbb{P}^z(\hat{Z}_{n+1}(l+1)) - \mathbb{P}^z(\hat{Z}_n(l))|$$

$$= \left|\mathbb{P}^z\left(Z(l+1)\cap\bigcap_{j=l+2}^{n+1} Z(j)^c\right) - \mathbb{P}^z\left(Z(l)\cap\bigcap_{j=l+1}^{n} Z(j)^c\right)\right|$$

$$= \left|\mathbb{P}^z(Z(l+1))\mathbb{P}^z\left(\bigcap_{j=l+2}^{n+1} Z(j)^c\Big|Z(l+1)\right)\right.$$

$$\left. - \mathbb{P}^z(Z(l))\mathbb{P}^z\left(\bigcap_{j=l+1}^{n} Z(j)^c\Big|Z(l)\right)\right|$$

$$= \left|\mathbb{P}^z(Z(l+1))\mathbb{P}^z\left(\bigcap_{j=l+1}^{n} Z(j)^c\Big|Z(l)\right)\right.$$

$$\left. - \mathbb{P}^z(Z(l))\mathbb{P}^z\left(\bigcap_{j=l+1}^{n} Z(j)^c\Big|Z(l)\right)\right|$$

$$= \mathbb{P}^z\left(\bigcap_{j=l+1}^{n} Z(j)^c\Big|Z(l)\right)\cdot|\mathbb{P}^z(Z(l+1)) - \mathbb{P}^z(Z(l))|$$

$$\leq |\mathbb{P}^z(Z(l+1)) - \mathbb{P}^z(Z(l))| \leq \frac{C}{l} \leq Cn^{-1}.$$

(3) This follows exactly the same lines as the argument for the previous inequality.

(4) A similar calculation gives

$$\sum_{\substack{\Delta \in \Pi_n^{(\varepsilon)}}} \sum_{\substack{x \in \Delta \\ x \leftrightarrow n}} \left[\max_{y \in \Delta} \mathbb{P}^0(X_n = y) - \mathbb{P}^0(X_n = x) \right]$$

$$= \sum_{\substack{\Delta \in \Pi_n^{(\varepsilon)}}} \max_{\substack{x \in \Delta \\ x \leftrightarrow n}} \sum_{l \leq n} \mathbb{P}^z(\hat{Z}(l)) \sum_{\substack{w \in \mathbb{Z}^d \\ w \leftrightarrow l}} \mathbb{P}^z(X_l = w | \hat{Z}(l))$$

$$\times \left[\mathbb{P}^z(X_n = x_\Delta | \hat{Z}(l), X_l = w) - \mathbb{P}^z(X_n = x | \hat{Z}(l), X_l = w) \right]$$

$$= \sum_{l \leq n} \mathbb{P}^z(\hat{Z}(l)) \sum_{\substack{w \in \mathbb{Z}^d \\ w \leftrightarrow l}} \sum_{\substack{\Delta \in \Pi_n^{(\varepsilon)}}} \sum_{\substack{x \in \Delta \\ x \leftrightarrow n}} \left[\mathbb{P}^z(X_l = w | \hat{Z}(l)) \right.$$

$$\left. - \mathbb{P}^z(X_l = w + x - x_0 | \hat{Z}(l)) \right] \mathbb{P}^z(X_n = x_\Delta | \hat{Z}(l), X_l = w).$$

Using (A.18) and the shift invariance of the annealed measure this is bounded by

$$\sum_{l \leq n} \mathbb{P}^z(\hat{Z}(l)) C l^{-1/2 + 3d\varepsilon} \leq \sum_{l \leq n} e^{-(\log(n-l))^2} \cdot C l^{-1/2 + 3d\varepsilon} \leq C n^{-1/2 + 3d\varepsilon}. \qquad \square$$

A.2.4. *Proof of Lemma* 2.16. Recalling Corollary 2.10, we have

$$\mathbb{P}^z(\|X_n - \mathbb{E}^z[X_n]\|_\infty > \sqrt{n} R_5(n))$$

(A.21)
$$\leq \mathbb{P}^z(\|X_n - \mathbb{E}^z[X_n]\|_\infty > \sqrt{n} R_5(n) | B_n) + P(B_n^c)$$

$$\leq \mathbb{P}^z(\exists k \leq n : \|X_{\tau_k} - \mathbb{E}^z[X_{\tau_k}]\|_\infty > \tfrac{1}{3}\sqrt{n} R_5(n) | B_n) + n^{-\xi(1)}.$$

Note that conditioned on B_n the regenerations are still independent and all of them are bounded by $R(n)$. If we could show that $\|\mathbb{E}^z[X_{\tau_k}] - \mathbb{E}^z[X_{\tau_k} | B_n]\|_\infty = n^{-\xi(1)}$ then

$$(A.21) \leq \mathbb{P}^z(\exists k \leq n :, \|X_{\tau_k} - \mathbb{E}^z[X_{\tau_k} | B_n]\|_\infty > \tfrac{1}{4}\sqrt{n} R_5(n) | B_n) + n^{-\xi(1)}$$

which by Azuma's inequality is no more than

$$\sum_{k=1}^n \mathbb{P}^z\left(\|X_{\tau_k} - \mathbb{E}^z[X_{\tau_k} | B_n]\|_\infty > \frac{1}{4}\sqrt{n} R_5(n) | B_n \right) + n^{-\xi(1)}$$

$$\leq d \sum_{k=1}^n \exp\left(-\frac{n R_5^2(n)}{16k R^2(n)} \right) \leq d e^{-R_5(n)} = n^{-\xi(1)}.$$

Thus, it is left to show that $\|\mathbb{E}^z[X_{\tau_k}] - \mathbb{E}^z[X_{\tau_k} | B_n]\|_\infty = n^{-\xi(1)}$. Since $k \leq n$, by the triangle inequality it is enough to show that $\|\mathbb{E}^z[X_{\tau_k} - X_{\tau_{k-1}}] - \mathbb{E}^z[X_{\tau_k} - $

$X_{\tau_{k-1}}|B_n]\|_\infty = n^{-\xi(1)}$. However, for every $k \le n$

$$
\begin{aligned}
\big\| \mathbb{E}^z &[X_{\tau_k} - X_{\tau_{k-1}}] - \mathbb{E}^z[X_{\tau_k} - X_{\tau_{k-1}}|B_n] \big\|_\infty \\
&\le \big\| \mathbb{E}^z[X_{\tau_k} - X_{\tau_{k-1}}] - \mathbb{E}^z\big[(X_{\tau_k} - X_{\tau_{k-1}})\mathbb{1}_{B_n}\big] \big\|_\infty \\
&\quad + \big\| \mathbb{E}^z\big[(X_{\tau_k} - X_{\tau_{k-1}})\mathbb{1}_{B_n}\big] - \mathbb{E}^z[X_{\tau_k} - X_{\tau_{k-1}}|B_n] \big\|_\infty \\
&= \big\| \mathbb{E}^z\big[(X_{\tau_k} - X_{\tau_{k-1}})\mathbb{1}_{B_n^c}\big] \big\|_\infty + \mathbb{P}^z(B_n^c)\big\| \mathbb{E}^z[X_{\tau_k} - X_{\tau_{k-1}}|B_n] \big\|_\infty \\
&\le \big\| \mathbb{E}^z\big[(X_{\tau_k} - X_{\tau_{k-1}})\mathbb{1}_{B_n^c}\big] \big\|_\infty + R(n)\mathbb{P}^z(B_n^c) \\
&\le \big\| \mathbb{E}^z\big[(X_{\tau_k} - X_{\tau_{k-1}})\mathbb{1}_{\exists j \ne k, \tau_j - \tau_{j-1} > R(n)}\big] \big\|_\infty \\
&\quad + \big\| \mathbb{E}^z\big[(X_{\tau_k} - X_{\tau_{k-1}})\mathbb{1}_{\tau_k - \tau_{k-1} > R(n)}\big] \big\|_\infty + R(n)\mathbb{P}^z(B_n^c) \\
&\le \big\| \mathbb{E}^z\big[(X_{\tau_k} - X_{\tau_{k-1}})\big] \big\|_\infty \mathbb{P}^z(B_n^c) + \sum_{l=R(n)}^\infty l e^{-cl^\gamma} + R(n)\mathbb{P}^z(B_n^c) \\
&= n^{-\xi(1)},
\end{aligned}
$$

where for the last inequality we used the assumption ($\mathscr{P}$) which implies T_γ for any $0 < \gamma < 1$, and for the last equality we used Corollary 2.10.

The quenched estimation follows from the first inequality together with Claim 2.15, while the second annealed estimation follows by the exact same proof with $R_5(n)$ replaced with a large constant C.

A.3. More annealed estimations.

A.3.1. *Proof of* (3.6)–(3.9). We start with the proof of (3.6).

$$
\begin{aligned}
\mathbb{P}^z &\big(X_{T_{M+V}} \in \Delta^{(1)}, T_{M+V} \in I^{(1)}\big) \\
&= \mathbb{P}^z\big(X_{T_{M+V}} \in \Delta^{(1)}, T_{M+V} \in I^{(1)}, X_{T_M} \in \Delta, T_M \in I\big) \\
&\quad + \mathbb{P}^z\big(X_{T_{M+V}} \in \Delta^{(1)}, T_{M+V} \in I^{(1)}, (X_{T_M} \in \Delta, T_M \in I)^c\big) \\
&\le \mathbb{P}^z(X_{T_M} \in \Delta, T_M \in I) + \mathbb{P}^z\big(X_{T_{M+V}} \in \Delta^{(1)}, X_{T_M} \notin \Delta\big) \\
&\quad + \mathbb{P}^z\big(T_{M+V} \in I^{(1)}, T_M \notin I\big).
\end{aligned}
$$

Thus, it is enough to show that $\mathbb{P}^z(X_{T_{M+V}} \in \Delta^{(1)}, X_{T_M} \notin \Delta) = N^{-\xi(1)}$ and $\mathbb{P}^z(T_{M+V} \in I^{(1)}, T_M \notin I) = N^{-\xi(1)}$. Since by Lemma 3.4 $\mathbb{P}^z(X_{T_M} \notin \mathcal{P}(0, N)) = \mathbb{P}^z(X_{T_M} \notin \mathcal{P}(0, N), T_{\partial\mathcal{P}(0,N)} \ne T_{\partial^+\mathcal{P}(0,N)}) + N^{-\xi(1)} = N^{-\xi(1)}$, we have

$$
\begin{aligned}
\mathbb{P}^z &\big(X_{T_{M+V}} \in \Delta^{(1)}, X_{T_M} \notin \Delta\big) \\
&= \sum_{y \in \mathcal{P}(0,N) \cap H_M \setminus \Delta} \mathbb{P}^z\big(X_{T_M} = y, X_{T_{M+V}} \in \Delta^{(1)}\big) + N^{-\xi(1)}.
\end{aligned}
$$

However, using Lemma 3.4 once more, for every $y \in \mathcal{P}(0, N) \cap H_M \setminus \Delta$ and every $w \in \tilde{\mathcal{P}}(y, \sqrt{V})$ we have $\mathbb{P}^w(T_{\partial \mathcal{P}(y, \sqrt{V})} = T_{\partial^+ \mathcal{P}(y, \sqrt{V})}) = 1 - (\sqrt{V})^{-\xi(1)} = 1 - N^{-\xi(1)}$. Since in addition $\mathrm{dist}(\partial^+ \mathcal{P}(y, \sqrt{V}), \Delta) > \frac{1}{2} N^\theta - \frac{1}{2} \cdot \frac{9}{10} N^\theta - \sqrt{V} R_5(V) \geq \frac{1}{20} N^\theta - c N^{\theta'}$ it follows that $\mathbb{P}^w(X_{T_{M+V}} \in \Delta^{(1)}) = N^{-\xi(1)}$. To complete the argument, we note that we note that (by Corollary 2.10)

$$\mathbb{P}^z(X_{T_M} = y, X_{T_{M+V}} \in \Delta^{(1)}) = \mathbb{P}^z(X_{T_M} = y, X_{T_{M+V}} \in \Delta^{(1)}, B_N) + N^{-\xi(1)}$$

and under B_N there is a regeneration time at distance at most $R(N)$ from y. This gives a new point $w \in \mathbb{Z}^d$ [such that $\|w - y\|_\infty \leq R(N)$ and in particular $w \in \tilde{\mathcal{P}}(0, N)$] from which the probability to hit $\Delta^{(1)}$ when hitting the hyperplane H_{M+V} (conditioned to start in a regeneration time). Since the last conditioning has a positive probability this is bounded by $C \mathbb{P}^w(X_{T_{M+V}} \in \Delta^{(1)}) = N^{-\xi(1)}$. Thus,

$$\begin{aligned}
&\mathbb{P}^z(X_{T_{M+V}} \in \Delta^{(1)}, X_{T_M} \notin \Delta) \\
&\quad = \sum_{y \in \mathcal{P}(0,N) \cap H_M \setminus \Delta} \mathbb{P}^z(X_{T_M} = y, X_{T_{M+V}} \in \Delta^{(1)}) + N^{-\xi(1)} \\
&\quad \leq |\mathcal{P}(0, N)| \cdot N^{-\xi(1)} + N^{-\xi(1)} = N^{-\xi(1)}.
\end{aligned}$$

A similar argument shows that $\mathbb{P}^z(T_{M+V} \in I^{(1)}, T_M \notin I) = N^{-\xi(1)}$. Indeed, by Lemma 3.4 up to an event of probability $M^{-\xi(1)} = N^{-\xi(1)}$ the first hitting time to the hyperplane H_M is the same as the exit time of the box $\mathcal{P}(0, M)$. By the same lemma, we also know that up to an event of probability $N^{-\xi(1)}$ this time is at distance at most $N R_2(N)$ from the expectation of $\mathbb{E}^z[T_M]$. Therefore,

$$\begin{aligned}
&\mathbb{P}^z(T_{M+V} \in I^{(1)}, T_M \notin I) \\
&\quad = \sum_{\substack{t : |t - \mathbb{E}^z[T_M]| < N R_2(N) \\ t \notin I}} \mathbb{P}^z(T_m = t, T_{M+V} \in I^{(1)}) + N^{-\xi(1)}.
\end{aligned}$$

In the case $t < c(I) - N^\theta$ if $T_{M+V} \in I^{(1)}$, then the random walk crossed the distance from H_M to H_{M+V} in more then $V \frac{1}{\langle \mathrm{v}, e_1 \rangle} - \frac{1}{2} \cdot \frac{9}{10} N^\theta + N^\theta = V \frac{1}{\langle \mathrm{v}, e_1 \rangle} + \frac{1}{20} N^\theta$ which happens with probability $N^{-\xi(1)}$ by Lemma 3.4. Similarly, if $t > C(I) + N^\theta$ and $T_{M+V} \in I^{(1)}$ then the random walk crossed the distance from H_M to H_{M+V} in less than $V \frac{1}{\langle \mathrm{v}, e_1 \rangle} + \frac{1}{2} \cdot \frac{9}{10} N^\theta - \frac{1}{2} N^\theta = V \frac{1}{\langle \mathrm{v}, e_1 \rangle} - \frac{1}{20} N^\theta$ which also has probability $N^{-\xi(1)}$ by Lemma 3.4.

Thus,

$$\begin{aligned}
&\mathbb{P}^z(T_{M+V} \in I^{(1)}, T_M \notin I) \\
&\quad = \sum_{\substack{t : |t - \mathbb{E}^z[T_M]| < N R_2(N) \\ t \notin I}} \mathbb{P}^z(T_m = t, T_{M+V} \in I^{(1)}) + N^{-\xi(1)} \\
&\quad \leq C N R_2(N) \cdot N^{-\xi(1)} + N^{-\xi(1)} = N^{-\xi(1)}.
\end{aligned}$$

Turning to (3.8), we have

$$E\big[P_\omega^z(X_{T_{M+V}} \in \Delta^{(1)}, T_{M+V} \in I^{(1)})|\mathcal{G}\big]$$

$$= E\big[P_\omega^z(X_{T_{M+V}} \in \Delta^{(1)}, T_{M+V} \in I^{(1)}, X_{T_M} \in \Delta, T_M \in I)|\mathcal{G}\big]$$

$$+ E\big[P_\omega^z(X_{T_{M+V}} \in \Delta^{(1)}, T_{M+V} \in I^{(1)}, (X_{T_M} \in \Delta, T_M \in I)^c)|\mathcal{G}\big]$$

$$\leq E\big[P_\omega^z(X_{T_M} \in \Delta, T_M \in I)|\mathcal{G}\big] + E\big[P_\omega^z(X_{T_{M+V}} \in \Delta^{(1)}, X_{T_M} \notin \Delta)|\mathcal{G}\big]$$

$$+ E\big[P_\omega^z(T_{M+V} \in I^{(1)}, T_M \notin I)|\mathcal{G}\big]$$

$$= P_\omega^z(X_{T_M} \in \Delta, T_M \in I) + E\big[P_\omega^z(X_{T_{M+V}} \in \Delta^{(1)}, X_{T_M} \notin \Delta)|\mathcal{G}\big]$$

$$+ E\big[P_\omega^z(T_{M+V} \in I^{(1)}, T_M \notin I)|\mathcal{G}\big].$$

Separating to the case when B_N holds and when B_N^c (which has probability $N^{-\xi(1)}$) we can control the terms $E[P_\omega^z(X_{T_{M+V}} \in \Delta^{(1)}, X_{T_M} \notin \Delta)|\mathcal{G}]$ and $E[P_\omega^z(T_{M+V} \in I^{(1)}, T_M \notin I)|\mathcal{G}]$ by the annealed probability of the events $\{X_{T_{M+V}} \in \Delta^{(1)}, X_{T_M} = w\}$ and $\{T_{M+V} \in I^{(1)}, T_m = t\}$ with w and t the place and time of the first regeneration time after hitting the hyerplane H_M (outside of Δ). Since by the first argument those events have probability $N^{-\xi(1)}$ the proof is complete.

The proof of (3.7) and (3.9) is very similar and, therefore, is left to the reader.

A.3.2. *Proof of* (3.12)–(3.15). The proof of (3.12)–(3.15) follows the same lines as the proof of (3.6)–(3.9). The only difference is that in (3.6)–(3.9) we took boxes of side length $\frac{9}{10}N^\theta$ and $\frac{11}{10}N^\theta$ leaving a difference of wide $\frac{1}{10}N^\theta$ from the original box whose side length is N^θ. This together with the fact that the distance between the hyperplanes was $V = N^{2\theta'}$ for some $\theta' < \theta$ made it impossible to hit one box without hitting the other. Similarly in (3.12)–(3.15), we take boxes of side length $N^\theta \pm R_3(N)\sqrt{V}$. As in the previous case, we have $R_3(N)\sqrt{V} \gg \sqrt{V}$ and, therefore, the probability to hit one box without hitting the other is still of magnitude $N^{-\xi(1)}$.

A.3.3. *Proof of* (4.3)–(4.4). We start with the proof of (4.3). Denoting by $A_{t,s,w}$ the event that the first regeneration time after time t is at time s and $X_s = w$ we have

$$\mathbb{P}^z\big(X_{T_{\partial\mathcal{P}(0,\sqrt{L})}} \in \Delta^{(1)}, T_{\partial\mathcal{P}(0,\sqrt{L})} \in I^{(1)}, X_N \notin \Delta\big)$$

$$= \sum_{\substack{y \in \Delta^{(1)} \\ t \in I^{(1)}}} \mathbb{P}^z(X_{T_{\partial\mathcal{P}(0,\sqrt{L})}} = y, T_{\partial\mathcal{P}(0,\sqrt{L})} = t, X_N \notin \Delta, B_N) + N^{-\xi(1)}$$

$$= \sum_{\substack{y \in \Delta^{(1)} \\ t \in I^{(1)}}} \sum_{\substack{w:\|w-y\|_\infty \leq R(N) \\ s:|t-s| \leq R(N)}} \mathbb{P}^z(X_{T_{\partial\mathcal{P}(0,\sqrt{L})}} = y,$$

$$T_{\partial\mathcal{P}(0,\sqrt{L})} = t, B_N, A_{t,s,w}, X_N \notin \Delta) + N^{-\xi(1)}$$

$$\leq \sum_{\substack{y \in \Delta^{(1)} \; w:\|w-y\|_\infty \leq R(N) \\ t \in I^{(1)} \quad s:|t-s| \leq R(N)}} \mathbb{P}^z(X_{T_{\partial\mathcal{P}(0,\sqrt{L})}} = y, T_{\partial\mathcal{P}(0,\sqrt{L})} = t, B_N, A_{t,s,w})$$

$$\times \mathbb{P}^z(X_N \notin \Delta | X_{T_{\partial\mathcal{P}(0,\sqrt{L})}} = y, T_{\partial\mathcal{P}(0,\sqrt{L})} = t, B_N, A_{t,s,w}) + N^{-\xi(1)}$$

$$\leq \sum_{\substack{y \in \Delta^{(1)} \; w:\|w-y\|_\infty \leq R(N) \\ t \in I^{(1)} \quad s:|t-s| \leq R(N)}} \mathbb{P}^z(X_{T_{\partial\mathcal{P}(0,\sqrt{L})}} = y, T_{\partial\mathcal{P}(0,\sqrt{L})} = t, B_N, A_{t,s,w})$$

$$\times \mathbb{P}^w(X_{N-s} \notin \Delta | 0 \text{ is a regeneration time}) + N^{-\xi(1)}$$

$$\leq C \cdot \sum_{\substack{y \in \Delta^{(1)} \; w:\|w-y\|_\infty \leq R(N) \\ t \in I^{(1)} \quad s:|t-s| \leq R(N)}} \mathbb{P}^z(X_{T_{\partial\mathcal{P}(0,\sqrt{L})}} = y, T_{\partial\mathcal{P}(0,\sqrt{L})} = t, B_N, A_{t,s,w})$$

$$\times \mathbb{P}^w(X_{N-s} \notin \Delta) + N^{-\xi(1)}$$

$$\leq C R(N)^d \sum_{\substack{w:\mathrm{dist}(w,\Delta^{(1)}) \leq R(N) \\ s:\mathrm{dist}(s,I^{(1)}) \leq R(N)}} \mathbb{P}^w(X_{N-s} \notin \Delta) + N^{-\xi(1)}.$$

Since the number of pairs (w, s) satisfying the above inequalities is at most $(N^\theta R(N))^d$, it is enough to show that for every $w \in \mathbb{Z}^d$ such that $\mathrm{dist}(w, \Delta^{(1)}) \leq R(N)$ and every $s \in \mathbb{N}$ such that $\mathrm{dist}(s, I^{(1)}) \leq R(N)$ we have $\mathbb{P}^w(X_{N-s} \notin \Delta) = N^{-\xi(1)}$. To this end, fix w and s as above, and note that

$$\mathbb{P}^w(X_{N-s} \notin \Delta)$$
$$= \mathbb{P}^w(X_{N-s} \notin \Delta, T_{\partial\mathcal{P}(w,\sqrt{N^\theta})} = T_{\partial^+\mathcal{P}(w,\sqrt{N^\theta})},$$
$$T_{\partial\mathcal{P}(w,\sqrt{(1/2)N^\theta})} = T_{\partial^+\mathcal{P}(w,\sqrt{(1/2)N^\theta})}) + N^{-\xi(1)}$$
$$= \mathbb{P}^w(X_{N-s} \notin \Delta, T_{\partial\mathcal{P}(w,\sqrt{(3/2)N^\theta})} = T_{\partial^+\mathcal{P}(w,\sqrt{(3/2)N^\theta})},$$
$$T_{\partial\mathcal{P}(w,\sqrt{(1/2)N^\theta})} = T_{\partial^+\mathcal{P}(w,\sqrt{(1/2)N^\theta})},$$
$$T_{\partial\mathcal{P}(w,\sqrt{(1/2)N^\theta})} \leq N - s \leq T_{\partial\mathcal{P}(w,\sqrt{(3/2)N^\theta})})$$
$$+ \mathbb{P}^w(X_{N-s} \notin \Delta, T_{\partial\mathcal{P}(w,\sqrt{(3/2)N^\theta})} = T_{\partial^+\mathcal{P}(w,\sqrt{(3/2)N^\theta})},$$
$$N - s > T_{\partial\mathcal{P}(w,\sqrt{(3/2)N^\theta})})$$
$$+ \mathbb{P}^w(X_{N-s} \notin \Delta, T_{\partial\mathcal{P}(w,\sqrt{(1/2)N^\theta})} = T_{\partial^+\mathcal{P}(w,\sqrt{(1/2)N^\theta})},$$
$$N - s < T_{\partial\mathcal{P}(w,\sqrt{(1/2)N^\theta})}) + N^{-\xi(1)}.$$

Note, however, that if $T_{\partial \mathcal{P}(w, \sqrt{(3/2)N^\theta})} = T_{\partial^+ \mathcal{P}(w, \sqrt{(3/2)N^\theta})}$, $T_{\partial \mathcal{P}(w, \sqrt{(1/2)N^\theta})} = T_{\partial^+ \mathcal{P}(w, \sqrt{(1/2)N^\theta})}$ and $T_{\partial \mathcal{P}(w, \sqrt{(1/2)N^\theta})} \le N - s \le T_{\partial \mathcal{P}(w, \sqrt{(3/2)N^\theta})}$ then $X_{N-s} \in \Delta$ and, therefore,

$$\mathbb{P}^w(X_{N-s} \notin \Delta)$$

$$\le \mathbb{P}^w(T_{\partial \mathcal{P}(w, \sqrt{(3/2)N^\theta})} = T_{\partial^+ \mathcal{P}(w, \sqrt{(3/2)N^\theta})}, N - s > T_{\partial \mathcal{P}(w, \sqrt{(3/2)N^\theta})})$$

$$+ \mathbb{P}^w(T_{\partial \mathcal{P}(w, \sqrt{(1/2)N^\theta})} = T_{\partial^+ \mathcal{P}(w, \sqrt{(1/2)N^\theta})}, N - s \le T_{\partial \mathcal{P}(w, \sqrt{(1/2)N^\theta})})$$

$$+ N^{-\xi(1)}.$$

Since $N - s \in [\frac{N^\theta}{\langle \mathrm{v}, e_1 \rangle} - \frac{1}{2}(N^\theta - R_5(N)N^{\theta/2}), \frac{N^\theta}{\langle \mathrm{v}, e_1 \rangle} + \frac{1}{2}(N^\theta - R_5(N)N^{\theta/2})]$ it follows that

$$\mathbb{P}^w(T_{\partial \mathcal{P}(w, \sqrt{(3/2)N^\theta})} = T_{\partial^+ \mathcal{P}(w, \sqrt{(3/2)N^\theta})}, N - s > T_{\partial \mathcal{P}(w, \sqrt{(3/2)N^\theta})})$$

$$= \mathbb{P}^0(T_{\partial \mathcal{P}(0, \sqrt{(3/2)N^\theta})} = T_{\partial^+ \mathcal{P}(0, \sqrt{(3/2)N^\theta})}, N - s > T_{\partial \mathcal{P}(0, \sqrt{(3/2)N^\theta})})$$

$$\le \mathbb{P}^0\left(T_{(3/2)N^\theta} < \frac{N^\theta}{\langle \mathrm{v}, e_1 \rangle} + \frac{1}{2}(N^\theta - R_5(N)N^{\theta/2})\right)$$

$$\le \mathbb{P}^0\left(T_{(3/2)N^\theta} < \frac{(3/2)N^\theta}{\langle \mathrm{v}, e_1 \rangle} - \frac{1}{2}R_5(N)N^{\theta/2}\right) = N^{-\xi(1)}$$

and similarly

$$\mathbb{P}^w(X_{N-s} \notin \Delta, T_{\partial \mathcal{P}(w, \sqrt{(1/2)N^\theta})} = T_{\partial^+ \mathcal{P}(w, \sqrt{(1/2)N^\theta})}, N - s \le T_{\partial \mathcal{P}(w, \sqrt{(1/2)N^\theta})})$$

$$= \mathbb{P}^0(T_{\partial \mathcal{P}(0, \sqrt{(1/2)N^\theta})} = T_{\partial^+ \mathcal{P}(0, \sqrt{(1/2)N^\theta})}, N - s < T_{\partial \mathcal{P}(0, \sqrt{(1/2)N^\theta})})$$

$$\le \mathbb{P}^0\left(T_{(1/2)N^\theta} > \frac{N^\theta}{\langle \mathrm{v}, e_1 \rangle} - \frac{1}{2}(N^\theta - R_5(N)N^{\theta/2})\right)$$

$$\le \mathbb{P}^0\left(T_{(1/2)N^\theta} > \frac{(1/2)N^\theta}{\langle \mathrm{v}, e_1 \rangle} + \frac{1}{2}R_5(N)N^{\theta/2}\right) = N^{-\xi(1)}.$$

REFERENCES

[1] ALILI, S. (1999). Asymptotic behaviour for random walks in random environments. *J. Appl. Probab.* **36** 334–349. MR1724844

[2] BERGER, N. (2012). Slowdown estimates for ballistic random walk in random environment. *J. Eur. Math. Soc.* (*JEMS*) **14** 127–174. MR2862036

[3] BERGER, N. and BISKUP, M. (2007). Quenched invariance principle for simple random walk on percolation clusters. *Probab. Theory Related Fields* **137** 83–120. MR2278453

[4] BERGER, N. and DEUSCHEL, J.-D. (2014). A quenched invariance principle for non-elliptic random walk in i.i.d. balanced random environment. *Probab. Theory Related Fields* **158** 91–126. MR3152781

[5] BERGER, N., DREWITZ, A. and RAMÍREZ, A. F. (2014). Effective polynomial ballisticity conditions for random walk in random environment. *Comm. Pure Appl. Math.* **67** 1947–1973. MR3272364

[6] BERGER, N. and ZEITOUNI, O. (2008). A quenched invariance principle for certain ballistic random walks in i.i.d. environments. In *In and Out of Equilibrium*. 2. *Progress in Probability* **60** 137–160. Birkhäuser, Basel. MR2477380

[7] BOLTHAUSEN, E. and SZNITMAN, A.-S. (2002). On the static and dynamic points of view for certain random walks in random environment. *Methods Appl. Anal.* **9** 345–375. MR2023130

[8] BOLTHAUSEN, E. and SZNITMAN, A.-S. (2002). *Ten Lectures on Random Media. DMV Seminar* **32**. Birkhäuser, Basel. MR1890289

[9] CAMPOS, D. and RAMÍREZ, A. F. (2013). Ellipticity criteria for ballistic behavior of random walks in random environment. *Probab. Theory Related Fields* **160** 189–251.

[10] DREWITZ, A. and RAMÍREZ, A. F. (2011). Ballisticity conditions for random walk in random environment. *Probab. Theory Related Fields* **150** 61–75. MR2800904

[11] DREWITZ, A. and RAMÍREZ, A. F. (2012). Quenched exit estimates and ballisticity conditions for higher-dimensional random walk in random environment. *Ann. Probab.* **40** 459–534. MR2952083

[12] DREWITZ, A. and RAMÍREZ, A. F. (2014). Selected topics in random walk in random environment. *Topics in Percolative and Disordered Systems*. *Springer Proc. Math. Stat.* **69** 23–83. MR3229286

[13] GUO, X. and ZEITOUNI, O. (2012). Quenched invariance principle for random walks in balanced random environment. *Probab. Theory Related Fields* **152** 207–230. MR2875757

[14] KIPNIS, C. and VARADHAN, S. R. S. (1986). Central limit theorem for additive functionals of reversible Markov processes and applications to simple exclusions. *Comm. Math. Phys.* **104** 1–19. MR0834478

[15] KOZLOV, S. M. (1985). The averaging method and walks in inhomogeneous environments. *Uspekhi Mat. Nauk* **40** 61–120. MR0786087

[16] LAWLER, G. F. (1982/1983). Weak convergence of a random walk in a random environment. *Comm. Math. Phys.* **87** 81–87. MR0680649

[17] MATHIEU, P. and PIATNITSKI, A. (2007). Quenched invariance principles for random walks on percolation clusters. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **463** 2287–2307. MR2345229

[18] MCDIARMID, C. (1998). Concentration. In *Probabilistic Methods for Algorithmic Discrete Mathematics*. *Algorithms Combin.* **16** 195–248. Springer, Berlin. MR1678578

[19] RASSOUL-AGHA, F. (2003). The point of view of the particle on the law of large numbers for random walks in a mixing random environment. *Ann. Probab.* **31** 1441–1463. MR1989439

[20] RASSOUL-AGHA, F. and SEPPÄLÄINEN, T. (2005). An almost sure invariance principle for random walks in a space–time random environment. *Probab. Theory Related Fields* **133** 299–314. MR2198014

[21] RASSOUL-AGHA, F. and SEPPÄLÄINEN, T. (2007). Quenched invariance principle for multidimensional ballistic random walk in a random environment with a forbidden direction. *Ann. Probab.* **35** 1–31. MR2303942

[22] RASSOUL-AGHA, F. and SEPPÄLÄINEN, T. (2009). Almost sure functional central limit theorem for ballistic random walk in random environment. *Ann. Inst. Henri Poincaré Probab. Stat.* **45** 373–420. MR2521407

[23] SABOT, C. (2013). Random Dirichlet environment viewed from the particle in dimension $d \geq$ 3. *Ann. Probab.* **41** 722–743. MR3077524

[24] SIDORAVICIUS, V. and SZNITMAN, A.-S. (2004). Quenched invariance principles for walks on clusters of percolation or among random conductances. *Probab. Theory Related Fields* **129** 219–244. MR2063376

[25] SZNITMAN, A.-S. (2001). On a class of transient random walks in random environment. *Ann. Probab.* **29** 724–765. MR1849176

[26] SZNITMAN, A.-S. (2002). An effective criterion for ballistic behavior of random walks in random environment. *Probab. Theory Related Fields* **122** 509–544. MR1902189

[27] SZNITMAN, A.-S. and ZERNER, M. (1999). A law of large numbers for random walks in random environment. *Ann. Probab.* **27** 1851–1869. MR1742891

[28] ZEITOUNI, O. (2004). Random walks in random environment. In *Lectures on Probability Theory and Statistics*. *Lecture Notes in Math.* **1837** 189–312. Springer, Berlin. MR2071631

[29] ZERNER, M. P. W. (2002). A non-ballistic law of large numbers for random walks in i.i.d. random environment. *Electron. Commun. Probab.* **7** 191–197. MR1937904

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