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SMOLUCHOWSKI–KRAMERS APPROXIMATION AND LARGE DEVIATIONS FOR INFINITE-DIMENSIONAL NONGRADIENT SYSTEMS WITH APPLICATIONS TO THE EXIT PROBLEM[1]

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In this paper, we study the quasi-potential for a general class of damped semilinear stochastic wave equations. We show that as the density of the mass converges to zero, the infimum of the quasi-potential with respect to all possible velocities converges to the quasi-potential of the corresponding stochastic heat equation, that one obtains from the zero mass limit. This shows in particular that the Smoluchowski–Kramers approximation is not only valid for small time, but in the zero noise limit regime, can be used to approximate long-time behaviors such as exit time and exit place from a basin of attraction.

1. Introduction. In the present paper, we are dealing with the following stochastic wave equation in a bounded regular domain $D \subset \mathbb{R}^d$, with $d \geq 1$:

$$
(1.1)\quad \begin{cases} \mu \dfrac{\partial^2 u_\varepsilon^\mu}{\partial t^2}(t,\xi) = \Delta u_\varepsilon^\mu(t,\xi) - \dfrac{\partial u_\varepsilon^\mu}{\partial t}(t,\xi) + B\big(u_\varepsilon^\mu(t)\big)(\xi) + \sqrt{\varepsilon}\dfrac{\partial w^Q}{\partial t}(t,\xi), \\ \qquad \xi \in D, \\ u_\varepsilon^\mu(0,\xi) = u_0(\xi), \qquad \dfrac{\partial u_\varepsilon^\mu}{\partial t}(0,\xi) = v_0(\xi), \\ \qquad \xi \in D, u_\varepsilon^\mu(t,\xi) = 0, \xi \in \partial D. \end{cases}
$$

Here B is a Lipschitz continuous mapping, whose Lipschitz norm is dominated by the first eigenvalue of the Laplacian. This means in particular that the identically zero solution is globally asymptotically stable in the absence of noise. $\partial w^Q/\partial t$ is a cylindrical Wiener process, white in time and colored in space, with covariance Q^2, and μ and ε are small positive constants.

As a consequence of the Newton law, we may interpret the solution $u_\varepsilon^\mu(t,\xi)$ of equation (1.1) as the displacement field of the particles of a material continuum in the domain D, subject to a random external force field $\sqrt{\varepsilon}\,\partial w^Q/\partial t(t,\xi)$ and a damping force proportional to the velocity field $\partial u_\varepsilon^\mu/\partial t(t,\xi)$. The Laplacian describes interaction forces between neighboring particles, in the presence of a

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nonlinear reaction described by B. The constant μ represents the constant density of the particles.

In [2] and [3], it has been proven that, for fixed $\varepsilon > 0$, as the density μ converges to 0, the solution $u_\varepsilon^\mu(t)$ of problem (1.1) converges to the solution $u_\varepsilon(t)$ of the stochastic first-order equation

$$
(1.2)\qquad \begin{cases} \dfrac{\partial u_\varepsilon}{\partial t}(t,\xi) = \Delta u_\varepsilon(t,\xi) + B\big(u_\varepsilon(t)\big)(\xi) + \sqrt{\varepsilon}\dfrac{\partial w^Q}{\partial t}(t,\xi), \\ \qquad \xi \in D, \\ u_\varepsilon(0,\xi) = u_0(\xi), \\ \qquad \xi \in D, u_\varepsilon(t,\xi) = 0, \xi \in \partial D, \end{cases}
$$

uniformly for t on fixed intervals. More precisely, we have shown that for any $\eta > 0$ and $T > 0$,

$$
(1.3)\qquad \lim_{\mu \to 0} \mathbb{P}\Big(\sup_{t \in [0,T]} \big| u_\varepsilon^\mu(t) - u_\varepsilon(t) \big|_H > \eta \Big) = 0.
$$

Such an approximation is known as the Smoluchowski–Kramers approximation and goes back to the seminal works by Smoluchowski [25] and Kramers [21]. In physical and chemical applications, it is important to be able to replace the second-order system (1.1), with the first-order system (1.2), which is considered to be simpler to treat. This is what is usually done, when, under the assumption that the mass is negligible, the damped Langevin equation is replaced by a standard stochastic differential equation. In recent years, the validity of the Smoluchowski–Kramers approximation has attracted considerable interest, especially for finite-dimensional systems (see, e.g., [8, 15, 16, 19, 20, 24] and [22]), but also several results have been obtained in infinite dimension; see [2, 3, 6]. Moreover, we would like to mention that there exists a large literature where a similar type of approximation is used in applications, when, instead of the small mass asymptotics, the large damping asymptotics are considered; see the works by E. Vanden-Eijnden and coauthors.

Once one has proved the validity of (1.3), an important question arises: how do some relevant asymptotic properties of the second and the first-order systems compare, with respect to the small mass asymptotic? In [15] and [8], the case of systems with a finite number of degrees of freedom have been studied, and the large deviation estimates, with the exit problem from a domain, various averaging procedures, the Wong–Zakai approximation and homogenization have been compared. It has been proven that in some cases the two asymptotics do match together properly, and in other cases they do not.

In [2], where the validity of the Smoluchowski–Kramers approximation for SPDEs has been approached for the first time, the long time behavior of equations (1.1) and (1.2) has been compared, under the assumption that the two systems are of gradient type. Actually, in the case of white noise in space and time (i.e., $Q = I$) and dimension $d = 1$, an explicit expression for the Boltzman distribution of the process $z_\varepsilon^\mu(t) := (u_\varepsilon^\mu(t), \partial u_\varepsilon^\mu/\partial t(t))$ in the phase space

$\mathcal{H} := L^2(0,1) \times H^{-1}(0,1)$ has been given. Of course, since in the functional space $\mathcal{H}$ there is no analogous of the Lebesgue measure, an auxiliary Gaussian measure has been introduced, with respect to which the density of the Boltzman distribution has been written down. This auxiliary Gaussian measure is the stationary measure of the linear wave equation related to problem (1.1). In particular, it has been shown that the first marginal of the invariant measure associated with the process $z_\varepsilon^\mu(t)$ does not depend on μ and coincides with the invariant measure of the process $u_\varepsilon(t)$, defined as the unique solution of the heat equation (1.2).

In the present paper, we are interested in comparing the small noise asymptotics, as $\varepsilon \downarrow 0$, for system (1.1) and system (1.2). Actually, we want to show that the Smoluchowski–Kramers approximation, that works on finite time intervals, is good also in the large deviations regime.

Large deviation principles and exit problems have been studied for a variety of SPDEs in the small noise regime (see [1, 9, 11, 18, 23] and [26]). We want to compare the quasi-potential $V^\mu(x,y)$ associated with (1.1), with the quasi-potential $V(x)$ associated with (1.2), and we want to show that for any closed set $N \subset L^2(D)$, it holds that

$$\lim_{\mu\to 0}\inf_{x\in N} \bar{V}_\mu(x) := \lim_{\mu\to 0}\inf_{x\in N}\inf_{y\in H^{-1}(D)} V^\mu(x,y) = \inf_{x\in N} V(x). \tag{1.4}$$

In the above formulas, x and y are functions of ξ, and the quasi-potential $V^\mu(x,y)$ denotes the infimum value of the large-deviation rate function over all $L^2(D)$-valued paths satisfying appropriate boundary conditions; for details, see page 4 and Section 5.

This means that taking first the limit as $\varepsilon \downarrow 0$ (large deviation) and then taking the limit as $\mu \downarrow 0$ (Smoluchowski–Kramers approximation) is the same as first taking the limit as $\mu \downarrow 0$ and then as $\varepsilon \downarrow 0$. In particular, this result provides a rigorous mathematical justification of what is done in applications, when, in order to study rare events and transitions between metastable states for the more complicated system (1.1), as well as exit times from basins of attraction and the corresponding exit places, the relevant quantities associated with the large deviations for system (1.2) are considered.

In our previous paper [7], we addressed this problem in the particular case when system (1.1) is of gradient type, that is,

$$B(x) = -Q^2 F_x(x), \qquad x \in L^2(D), \tag{1.5}$$

where $F_x(x)$ denotes the Frechet derivative for some $F : L^2(D) \to \mathbb{R}$, where Q^2 is the covariance of the Gaussian random perturbation. This applies, for example, to the linear case (i.e., $B = 0$) in any space dimension or to the case

$$B(x)(\xi) = b\big(\xi, x(\xi)\big), \qquad \xi \in D,$$

when $D = [0, L]$ and $Q = I$. In [7] we showed that if (1.5) holds, then for any $\mu > 0$,

$$V^\mu(x,y) = \big|(-\Delta)^{1/2} Q^{-1} x\big|^2_{L^2(D)} + 2F(x) + \mu\big|Q^{-1} y\big|^2_{L^2(D)}, \tag{1.6}$$

for any $(x, y) \in \mathrm{Dom}((-\Delta)^{1/2}Q^{-1}) \times \mathrm{Dom}(Q^{-1})$. Therefore, as

$$V(x) = \left|(-\Delta)^{1/2}Q^{-1}x\right|_H^2 + 2F(x), \qquad x \in \mathrm{Dom}\big((-\Delta)^{1/2}Q^{-1}\big),$$

from (1.6) we have concluded that for any $\mu > 0$,

$$\bar{V}_\mu(x) := \inf_{y \in H^{-1}(D)} V^\mu(x, y) = V^\mu(x, 0) = V(x), \qquad x \in \mathrm{Dom}\big((-\Delta)^{1/2}Q^{-1}\big). \tag{1.7}$$

In particular, this means that $\bar{V}_\mu(x)$ does not just coincide with $V(x)$ at the limit, as in (1.4), but for any fixed $\mu > 0$.

In the general nongradient case that we are considering in the present paper, the situation is considerably more delicate, and we cannot expect anything explicit, as in (1.6). The lack of an explicit expression for $V^\mu(x, y)$ and $V(x)$ makes the proof of (1.4) much more difficult and requires the introduction of new arguments and techniques.

The first key idea in order to prove (1.4) is to characterize $V^\mu(x, y)$ as the minimum value for a suitable functional. We recall that the quasi-potential $V^\mu(x, y)$ is defined as the minimum energy required to the system to go from the asymptotically stable equilibrium 0 to the point $(x, y) \in \mathcal{H}$, in any time interval. Namely,

$$V^\mu(x, y) = \inf\big\{I^\mu_{0,T}(z); z(0) = 0, z(T) = (x, y), T > 0\big\},$$

where

$$I^\mu_{0,T}(z) = \tfrac{1}{2}\inf\big\{|\psi|^2_{L^2((0,T);H)} : z = z^\mu_\psi\big\},$$

is the large deviation action functional, and $z^\mu_\psi = (u^\mu_\psi, \partial u^\mu_\psi/\partial t)$ is a mild solution of the skeleton equation associated with equation (1.1), with control $\psi \in L^2((0, T); H)$,

$$\mu\frac{\partial^2 u^\mu_\psi}{\partial t^2}(t) = \Delta u^\mu_\psi(t) - \frac{\partial u^\mu_\psi}{\partial t}(t) + B\big(u^\mu_\psi(t)\big) + Q\psi(t), \qquad t \in [0, T]. \tag{1.8}$$

By working thoroughly with the skeleton equation (1.8), we show that, for small enough $\mu > 0$,

$$V^\mu(x, y) = \min\Big\{I^\mu_{-\infty,0}(z) : \lim_{t\to-\infty}|z(t)|_{\mathcal{H}} = 0, z(0) = (x, y)\Big\}, \tag{1.9}$$

where the minimum is taken over all $z \in C((-\infty, 0]; \mathcal{H})$. In particular, we get that the level sets of V^μ and $\bar{V}_\mu$ are compact in $\mathcal{H}$ and $L^2(D)$, respectively. Moreover, we show that both V^μ and $\bar{V}_\mu$ are well defined and continuous in suitable Sobolev spaces of functions. We would like to stress that in [5] a result analogous to (1.9) has been proved for equation (1.2) and $V(x)$, in terms of the corresponding functional $I_{-\infty,0}$. In both cases, the proof is highly nontrivial, due to the degeneracy of the associated control problems, and requires a detailed analysis of the optimal regularity of the solution of the skeleton equation (1.8).

The second key idea is based on the fact that as in [8], where the finite-dimensional case is studied, for all functions $z \in C((-\infty, 0]; \mathcal{H})$ that are regular enough,

$$
\begin{aligned}
I^{\mu}_{-\infty}(z) &= I_{-\infty}(\varphi) + \frac{\mu^2}{2}\int_{-\infty}^{0}\left|Q^{-1}\frac{\partial^2\varphi}{\partial t^2}(t)\right|_H^2 dt \\
&\quad + \mu\int_{-\infty}^{0}\left\langle Q^{-1}\frac{\partial^2\varphi}{\partial t^2}(t), Q^{-1}\left(\frac{\partial\varphi}{\partial t}(t) - A\varphi(t) - B\big(\varphi(t)\big)\right)\right\rangle_H dt \qquad (1.10)\\
&=: I_{-\infty}(\varphi) + J^{\mu}_{-\infty}(z),
\end{aligned}
$$

where $\varphi(t) = \Pi_1 z(t)$. Thus if $\bar{z}^{\mu}$ is the minimizer of $\bar{V}_{\mu}(x)$, whose existence is guaranteed by (1.9), and if $\bar{z}^{\mu}$ has enough regularity to guarantee that all terms in (1.10) are meaningful, we obtain

$$
\bar{V}_{\mu}(x) = I_{-\infty}(\bar{\varphi}_{\mu}) + J^{\mu}_{-\infty}\big(\bar{z}^{\mu}\big) \geq V(x) + J^{\mu}_{-\infty}\big(\bar{z}^{\mu}\big). \qquad (1.11)
$$

In the same way, if $\bar{\varphi}$ is a minimizer for $V(x)$ and is regular enough, then

$$
\bar{V}_{\mu}(x) \leq I^{\mu}_{-\infty}(\bar{\varphi}, \partial\bar{\varphi}/\partial t) = V(x) + J^{\mu}_{-\infty}\big((\bar{\varphi}, \partial\bar{\varphi}/\partial t)\big). \qquad (1.12)
$$

If we could prove that

$$
\liminf_{\mu\to 0} J^{\mu}_{-\infty}\big(\bar{z}^{\mu}\big) = \limsup_{\mu\to 0} J^{\mu}_{-\infty}\big((\bar{\varphi}, \partial\bar{\varphi}/\partial t)\big) = 0, \qquad (1.13)
$$

from (1.11) and (1.12) we could conclude that (1.4) holds true. But unfortunately, neither $\bar{z}^{\mu}$ nor $\bar{\varphi}$ have the required regularity to justify (1.13). Thus we have to proceed with suitable approximations, which among other things, require us to prove the continuity of the mappings $\bar{V}_{\mu} : \mathrm{Dom}((-\Delta)^{1/2}Q^{-1}) \to \mathbb{R}$, uniformly with respect to $\mu \in (0, 1]$.

In the second part of the paper we want to apply (1.4) to the study of the exit time and of the exit place of u^{μ}_{ε} from a given domain in $L^2(D)$. For any open and bounded domain $G \subset L^2(D)$, containing the asymptotically stable equilibrium 0, and for any $z_0 \in G \times H^{-1}(D)$, we define the exit time

$$
\tau^{\mu,\varepsilon}_{z_0} := \inf\big\{t \geq 0 : u^{\mu}_{\varepsilon,z_0}(t) \in \partial G\big\}.
$$

Our first goal is to show that for fixed $\mu > 0$ and $z_0 \in G$,

$$
\lim_{\varepsilon\to 0} \varepsilon \log \mathbb{E}\tau^{\mu,\varepsilon}_{z_0} = \inf_{x\in\partial G} \bar{V}_{\mu}(x), \qquad (1.14)
$$

and

$$
\lim_{\varepsilon\to 0} \varepsilon \log\big(\tau^{\mu,\varepsilon}_{z_0}\big) = \inf_{x\in\partial G} \bar{V}_{\mu}(x) \qquad \text{in probability.} \qquad (1.15)
$$

We also want to prove that if $N \subset \partial G$ has the property that $\inf_{x\in N} \bar{V}_{\mu}(x) > \inf_{x\in\partial G} \bar{V}_{\mu}(x)$, then

$$
\lim_{\varepsilon\to 0} \mathbb{P}\big(u^{\mu}_{\varepsilon,z_0}\big(\tau^{\mu,\varepsilon}_{z_0}\big) \in N\big) = 0. \qquad (1.16)
$$

We would like to stress that the method we are using here in our infinite-dimensional setting has several considerable differences compared to the classical finite-dimensional argument developed in [17]; see also [13]. The most fundamental difference between the two settings is that unlike the finite-dimensional case, in the infinite-dimensional case the quasi-potentials $\bar{V}_\mu$ are not continuous in $L^2(D)$. Nevertheless, we show here that the lower-semi-continuity of $\bar{V}_\mu$ in $L^2(D)$, along with a convex type regularity assumption for the domain G, are sufficient to prove our results. Another important difference is that u_ε^μ is not a Markov process, but the pair $(u_\varepsilon^\mu, \partial u_\varepsilon^\mu/\partial t)$ in the phase space $\mathcal{H}$ is. For this reason, the exit time problem should be considered as the exit from the cylinder $G \times H^{-1} \subset \mathcal{H}$. Unfortunately, this is an unbounded domain, and as we show in Section 3, the unperturbed trajectories are not uniformly attracted to zero from this cylinder. The methods we use to prove the exit time and exit place results should be applicable to most stochastic equations with second-order time derivatives.

In a similar manner, one can show that if

$$\tau_{u_0}^\varepsilon = \inf\{t > 0 : u_\varepsilon(t) \notin G\}$$

is the exit time from G for the solution of (1.2), and $V(x)$ is the quasi-potential associated with this system, the exit time and exit place results for the first-order system are analogous to (1.14), (1.15) and (1.16).

As a consequence of (1.7), in the gradient case, (1.14) and (1.15) imply that for any fixed $\mu > 0$, the exit time and exit place asymptotics of (1.1) match those of (1.2). In particular, for any $\mu > 0$,

$$\lim_{\varepsilon\to 0} \varepsilon \log \mathbb{E}\tau_{z_0}^{\mu,\varepsilon} = \inf_{x\in\partial G} V(x) = \lim_{\varepsilon\to 0} \varepsilon \log \mathbb{E}\tau_{u_0}^\varepsilon, \tag{1.17}$$

and

$$\lim_{\varepsilon\to 0} \varepsilon \log \tau_{z_0}^{\mu,\varepsilon} = \inf_{x\in\partial G} V(x) = \lim_{\varepsilon\to 0} \varepsilon \log \tau_{u_0}^\varepsilon \qquad \text{in probability.} \tag{1.18}$$

Additionally, if there exists a unique $x^* \in \partial G$ such that $V(x^*) = \inf_{x\in\partial G} V(x)$, (1.16) implies that

$$\lim_{\varepsilon\to 0} u_\varepsilon^\mu(\tau^{\mu,\varepsilon}) = x^* = \lim_{\varepsilon\to 0} u_\varepsilon(\tau^\varepsilon) \qquad \text{in probability.}$$

In the general nongradient case, we cannot have (1.17) and (1.18). Nevertheless, in view of (1.4), the exit time and exit place asymptotics of (1.1) can be approximated by V. Namely,

$$\lim_{\mu\to 0}\lim_{\varepsilon\to 0} \varepsilon \log \mathbb{E}\tau_{z_0}^{\mu,\varepsilon} = \inf_{x\in\partial G} V(x) = \lim_{\varepsilon\to 0} \varepsilon \log \mathbb{E}\tau_{u_0}^\varepsilon,$$

and

$$\lim_{\mu\to 0}\lim_{\varepsilon\to 0} \varepsilon \log \tau_{z_0}^{\mu,\varepsilon} = \inf_{x\in\partial G} V(x) = \lim_{\varepsilon\to 0} \varepsilon \log \tau_{u_0}^\varepsilon \qquad \text{in probability.}$$

Furthermore, if there exists a unique $x^* \in \partial G$ such that $V(x^*) = \inf_{x \in \partial G} V(x)$, then

$$\lim_{\mu \to 0} \lim_{\varepsilon \to 0} u_\varepsilon^\mu(\tau^{\mu,\varepsilon}) = x^* = \lim_{\varepsilon \to 0} u_\varepsilon(\tau^\varepsilon) \qquad \text{in probability.}$$

The paper is organized as follows. In Section 2, we introduce the main notation and assumptions. In Section 3, we prove results about the equation in the absence of random forcing. In Section 4, we consider the skeleton equation, where the random forcing has been replaced by a deterministic control. In Section 5, we introduce the functionals $I^\mu_{-\infty,0}$ and $I_{-\infty,0}$, which act on trajectories on the negative half-line, and we show that the level sets of these functionals are compact. We then characterize the quasi-potentials V^μ and V in terms of $I^\mu_{-\infty,0}$ and $I_{-\infty,0}$. The compactness results of Section 5 are essential for the proofs in the remaining sections. In Section 6, we prove that the quasi-potentials are continuous on certain Sobolev spaces. Sections 7 and 8 contain the proofs of the convergence of $V^\mu \to V$ as $\mu \to 0$. In Section 9, we prove that the limit of the exit place and the exponential rate of divergence for exit time from a region in $L^2(D)$ are characterized by the quasi-potentials.

In what follows we shall denote by c (without any index) any positive constant appearing in inequalities, whose dependence on some parameters is not important. Such a constant may change, even in the same chain of inequalities. When we want to emphasize the dependence of the constant c on some parameters $p_1, \dots, p_n$, we will denote it by $c_{p_1,\dots,p_n}$.

2. Preliminaries and assumptions. Let D be an open, bounded, regular domain in $\mathbb{R}^d$, with $d \geq 1$, and let H denote the Hilbert space $L^2(D)$. In what follows, we shall denote by A the realization in H of the Laplace operator, endowed with Dirichlet boundary conditions, and we shall denote by $\{e_k\}_{k\in\mathbb{N}}$ and $\{-\alpha_k\}_{k\in\mathbb{N}}$ the corresponding sequence of eigenfunctions and eigenvalues, with $0 < \alpha_1 \leq \cdots \leq \alpha_k \leq \cdots$, for any $k \in \mathbb{N}$. Here, we assume that the domain D is regular enough so that

$$\alpha_k \sim k^{2/d}, \qquad k \in \mathbb{N}. \tag{2.1}$$

This happens, for example, in the case of the Laplace operator Δ in *strongly regular* open sets, both with Dirichlet and with Neumann boundary conditions; see [10], Theorem 1.9.6.

For any $\delta \in \mathbb{R}$, we shall define the fractional Sobolev space H^δ as the completion of $C_0^\infty(D)$ with respect to the norm

$$|x|^2_{H^\delta} = \sum_{k=1}^{+\infty} \alpha_k^\delta \langle x, e_k\rangle_H^2 = \sum_{k=1}^{\infty} \alpha_k^\delta x_i^2.$$

H^δ is a Hilbert space, endowed with the scalar product

$$\langle x, y\rangle_{H^\delta} = \sum_{k=1}^{+\infty} \alpha_k^\delta x_k y_k, \qquad x, y \in H^\delta(D).$$

Finally, we shall denote by $\mathcal{H}_\delta$ the Hilbert space $H^\delta \times H^{\delta-1}$, and in the case $\delta = 0$ we shall set $\mathcal{H} = \mathcal{H}_0$. Moreover, we shall denote the projections

$$\Pi_1 : \mathcal{H}_\delta \to H^\delta, \qquad (u, v) \mapsto u, \qquad \Pi_2 : \mathcal{H}_\delta \to H^{\delta-1}, \qquad (u, v) \mapsto v.$$

Sometimes, for the sake of simplicity, we will denote for any $\mu > 0$ and $\delta \in \mathbb{R}$, the scaling

$$\mathcal{I}_\mu(u, v) = (u, \sqrt{\mu} v), \qquad (u, v) \in \mathcal{H}_\delta. \tag{2.2}$$

The stochastic perturbation is given by a cylindrical Wiener process $w^Q(t, \xi)$, for $t \geq 0$ and $\xi \in \mathcal{O}$, which is assumed to be white in time and colored in space, in the case of space dimension $d > 1$. Formally, it is defined as the infinite sum

$$w^Q(t, \xi) = \sum_{k=1}^{+\infty} Q e_k(\xi) \beta_k(t), \tag{2.3}$$

where $\{e_k\}_{k\in\mathbb{N}}$ is the complete orthonormal basis in $L^2(D)$ which diagonalizes A and $\{\beta_k(t)\}_{k\in\mathbb{N}}$ is a sequence of mutually independent standard Brownian motions defined on the same complete stochastic basis $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$.

HYPOTHESIS 1. *The linear operator Q is bounded in H and diagonal with respect to the basis $\{e_k\}_{k\in\mathbb{N}}$ which diagonalizes A. Moreover, if $\{\lambda_k\}_{k\in\mathbb{N}}$ is the corresponding sequence of eigenvalues, we have*

$$\frac{1}{c}\alpha_k^{-\beta} \leq \lambda_k \leq c\alpha_k^{-\beta}, \qquad k \in \mathbb{N}, \tag{2.4}$$

for some $c > 0$ and $\beta > (d-2)/4$.

REMARK 2.1. (1) If $d = 1$, according to Hypothesis 1 we can consider space–time white noise ($Q = I$).

(2) Thanks to (2.1), condition (2.4) implies that if $d \geq 2$, then there exists $\gamma < 2d/(d-2)$ such that

$$\sum_{k=1}^{\infty} \lambda_k^\gamma < \infty.$$

Moreover,

$$\sum_{k=1}^{\infty} \frac{\lambda_k^2}{\alpha_k} < \infty.$$

(3) As a consequence of (2.4), for any $\delta \in \mathbb{R}$,

$$\mathrm{Dom}\big((-A)^{\delta/2} Q^{-1}\big) = H^{\delta+2\beta},$$

and there exists $c_\delta > 0$ such that for any $x \in H^{\delta+2\beta}$,

$$\frac{1}{c_\delta}\big|(-A)^{\delta/2} Q^{-1} x\big|_H \le |x|_{\delta+2\beta} \le c_\delta\big|(-A)^{\delta/2} Q^{-1} x\big|_H.$$

Concerning the nonlinearity B, we shall assume the following conditions:

HYPOTHESIS 2. *For any $\delta \in [0, 1+2\beta]$, the mapping $B : H^\delta \to H^\delta$ is Lipschitz continuous, with*

$$[B]_{\mathrm{Lip}(H^\delta)} =: \gamma_\delta < \alpha_1.$$

Moreover, $B(0) = 0$. We also assume that B is differentiable in the space $H^{2\beta}$, with

$$\sup_{z \in \mathcal{H}} \|B_z(z)\|_{L(H^{2\beta})} = \gamma_{2\beta}.$$

REMARK 2.2. (1) The assumption that B is differentiable is made for convenience to simplify the proof of lower bounds in Theorem 8.2. We believe that by approximating the Lipschitz continuous B with a sequence of differentiable functions whose C^1 semi-norm is controlled by the Lipschitz semi-norm of B, the results proved in Theorem 8.2 should remain true.

(2) If we define for any $x \in H$,

$$B(x)(\xi) = b\big(\xi, x(\xi)\big), \qquad \xi \in D,$$

and we assume that $b(\xi, \cdot) \in C^{2k}(\mathbb{R})$, for $k \in [\beta + \delta/2 - 5/4, \beta + \delta/2 - 1/4]$, and

$$\frac{\partial^j b}{\partial \sigma^j}(\xi, \sigma)\bigg|_{\sigma=0} = 0, \qquad \xi \in \bar{D},$$

then B maps H^δ into itself, for any $\delta \in [0, 1+2\beta]$. The Lipschitz continuity of B in H^δ and the bound on the Lipschitz norm are satisfied if the derivatives of $b(\xi, \cdot)$ are small enough.

With this notation, equation (1.2) can be written as the following abstract evolution equation in H:

$$du_\varepsilon(t) = \big[Au_\varepsilon(t) + B\big(u_\varepsilon(t)\big)\big]\,dt + \sqrt{\varepsilon}\,dw^Q(t), \qquad u(0) = u_0. \tag{2.5}$$

DEFINITION 2.3. A predictable process $u_\varepsilon \in L^2(\Omega; C([0,T]; H))$ is a *mild solution* to equation (2.5) if

$$u_\varepsilon(t) = e^{tA}u_0 + \int_0^t e^{(t-s)A} B\big(u_\varepsilon(s)\big)\,ds + \sqrt{\varepsilon}\int_0^t e^{(t-s)A}\,dw^Q(s).$$

Now, for each $\mu > 0$ and $\delta \in \mathbb{R}$, we define $A_\mu : \mathrm{Dom}(A_\mu) \subset \mathcal{H}_\delta \to \mathcal{H}_\delta$ by setting

$$A_\mu(u, v) = \left(v, \frac{1}{\mu} Au - \frac{1}{\mu} v\right), \qquad (u, v) \in \mathrm{Dom}(A_\mu) = \mathcal{H}_{1+\delta}, \tag{2.6}$$

and we denote by $S_\mu(t)$ the semigroup on $\mathcal{H}_\delta$ generated by A_μ. In [2], Proposition 2.4, it is proved that for each $\mu > 0$ there exist $\omega_\mu > 0$ and $M_\mu > 0$ such that

$$\|S_\mu(t)\|_{\mathcal{L}(\mathcal{H})} \le M_\mu e^{-\omega_\mu t}, \qquad t \ge 0. \tag{2.7}$$

Notice that since for any $\delta \in \mathbb{R}$ and $(u, v) \in \mathcal{H}_\delta$,

$$\big((-A)^\delta \Pi_1 S_\mu(t)(u, v), (-A)^\delta \Pi_2 S_\mu(t)(u, v)\big) = S_\mu(t)\big((-A)^\delta u, (-A)^\delta v\big), \qquad t \ge 0,$$

(2.7) implies that for any $\delta \in \mathbb{R}$,

$$\|S_\mu(t)\|_{\mathcal{L}(\mathcal{H}_\delta)} \le M_\mu e^{-\omega_\mu t}, \qquad t \ge 0. \tag{2.8}$$

Next, for any $\mu > 0$ we denote

$$B_\mu(u, v) = \frac{1}{\mu}\big(0, B(u)\big), \qquad (u, v) \in \mathcal{H},$$

and

$$Q_\mu u = \frac{1}{\mu}(0, Qu), \qquad u \in H.$$

With this notation, equation (1.1) can be written as the following abstract evolution equation in the space $\mathcal{H}$:

$$dz(t) = \big[A_\mu z(t) + B_\mu\big(z(t)\big)\big]\,dt + \sqrt{\varepsilon} Q_\mu\, dw(t), \qquad z(0) = (u_0, v_0). \tag{2.9}$$

DEFINITION 2.4. A predictable process u_ε^μ is a *mild solution* of (2.9) if

$$u_\varepsilon^\mu \in L^2\big(\Omega; C\big([0, T]; H\big)\big), \qquad v_\varepsilon^\mu =: \frac{\partial u_\varepsilon^\mu}{\partial t} \in L^2\big(\Omega; C\big([0, T]; H^{-1}\big)\big),$$

for any $T > 0$, and

$$\begin{aligned} z_\varepsilon^\mu(t) = {} & S_\mu(t) z(0) + \int_0^t S_\mu(t - s) B_\mu\big(z_\varepsilon^\mu(s)\big)\, ds \\ & + \sqrt{\varepsilon} \int_0^t S_\mu(t - s) Q_\mu\, dw(s), \end{aligned} \tag{2.10}$$

where $z(0) = (u_0, v_0)$ and $z_\varepsilon^\mu = (u_\varepsilon^\mu, v_\varepsilon^\mu)$.

In view of Hypothesis 1 and of the fact that $B : H \to H$ is Lipschitz continuous, for any $\mu > 0$ and any initial condition $z_0 = (u_0, v_0) \in \mathcal{H}$, there exists a unique mild solution u^μ_ε for equation (1.1); for a proof, see, for example, [2]. In [2], Theorem 4.6, we proved that for any fixed $\varepsilon > 0$ and $T > 0$, the solution u^μ_ε of equation (1.1) converges in $C([0,T]; H)$, in probability sense, to the solution u_ε of equation (1.2), as $\mu \downarrow 0$. Namely, for any $\eta > 0$,

$$\lim_{\mu\to 0} \mathbb{P}\Big(\sup_{t\in[0,T]} \big|u^\mu_\varepsilon(t) - u_\varepsilon(t)\big|_H > \eta\Big) = 0.$$

3. The unperturbed equation. We consider here equation (2.9) with $\varepsilon = 0$. Namely,

$$\frac{dz}{dt}(t) = A_\mu z(t) + B_\mu\big(z(t)\big), \qquad z(0) = z_0 = (u_0, v_0). \tag{3.1}$$

The solution to (3.1) will be denoted by $z^\mu_{z_0}(t)$. We recall here that γ_0 denotes the Lipschitz constant of B in H; see Hypothesis 2.

LEMMA 3.1. *If $\mu < (\alpha_1 - \gamma_0)\gamma_0^{-2}$, there exists a constant $c_1(\mu) > 0$ such that*

$$\sup_{t\geq 0} \big|z^\mu_{z_0}(t)\big|_{\mathcal{H}} + \big|z^\mu_{z_0}\big|_{L^2((0,+\infty);\mathcal{H})} \leq c_1(\mu)|z_0|_{\mathcal{H}}, \qquad z_0 \in \mathcal{H}. \tag{3.2}$$

PROOF. If $\varphi(t) = \Pi_1 z^\mu_{z_0}(t)$, then

$$\mu \frac{\partial^2\varphi}{\partial t^2}(t) + \frac{\partial\varphi}{\partial t}(t) = A\varphi(t) + B\big(\varphi(t)\big). \tag{3.3}$$

By taking the inner product of (3.3) with $\frac{\partial\varphi}{\partial t}$ in H^{-1}, and by using the Lipschitz continuity of B in H, we see that

$$\begin{aligned} &\mu \frac{d}{dt}\left|\frac{\partial\varphi}{\partial t}(t)\right|^2_{H^{-1}} + 2\left|\frac{\partial\varphi}{\partial t}(t)\right|^2_{H^{-1}} \\ &\qquad \leq -\frac{d}{dt}|\varphi(t)|^2_H + \left|\frac{\partial\varphi}{\partial t}(t)\right|^2_{H^{-1}} + \frac{\gamma_0^2}{\alpha_1}|\varphi(t)|^2_H. \end{aligned} \tag{3.4}$$

By integrating this expression in time, we see that

$$\begin{aligned} &\mu\left|\frac{\partial\varphi}{\partial t}(t)\right|^2_{H^{-1}} + |\varphi(t)|^2_H + \int_0^t \left|\frac{\partial\varphi}{\partial s}(s)\right|^2_{H^{-1}} ds \\ &\qquad \leq \mu|v_0|^2_{H^{-1}} + |u_0|^2_H + \frac{\gamma_0^2}{\alpha_1}\int_0^t |\varphi(s)|^2_H\, ds. \end{aligned} \tag{3.5}$$

Next, by taking the inner product of (3.3) with $\varphi(t)$ in H^{-1}, since

$$\left\langle \frac{\partial^2\varphi}{\partial t^2}(t), \varphi(t)\right\rangle_{H^{-1}} = \frac{1}{2}\frac{d^2}{dt^2}|\varphi(t)|^2_{H^{-1}} - \left|\frac{\partial\varphi}{\partial t}(t)\right|^2_{H^{-1}},$$

we have

$$\mu \frac{d^2}{dt^2}|\varphi(t)|^2_{H^{-1}} + \frac{d}{dt}|\varphi(t)|^2_{H^{-1}} \leq -2|\varphi(t)|^2_H + \frac{2\gamma_0}{\alpha_1}|\varphi(t)|^2_H + 2\mu\left|\frac{\partial\varphi}{\partial t}(t)\right|^2_{H^{-1}}.$$

By (3.4), this yields

$$\begin{aligned}
&\mu \frac{d^2}{dt^2}|\varphi(t)|^2_{H^{-1}} + \frac{d}{dt}|\varphi(t)|^2_{H^{-1}} \\
&\quad\leq -2|\varphi(t)|^2_H + \frac{2\gamma_0}{\alpha_1}|\varphi(t)|^2_H - 2\mu^2\frac{d}{dt}\left|\frac{\partial\varphi}{\partial t}(t)\right|^2_{H^{-1}} - 2\mu\frac{d}{dt}|\varphi(t)|^2_H \\
&\qquad + \frac{2\gamma_0^2\mu}{\alpha_1}|\varphi(t)|^2_H.
\end{aligned} \tag{3.6}$$

Now, if $\mu < (\alpha_1 - \gamma_0)\gamma_0^{-2}$, it follows

$$\rho_\mu := 2 - \frac{2\gamma_0}{\alpha_1} - \frac{2\mu\gamma_0^2}{\alpha_1} > 0.$$

Then, by integrating both sides in (3.6), we see

$$\begin{aligned}
&\mu\frac{d}{dt}|\varphi(t)|^2_{H^{-1}} + |\varphi(t)|^2_{H^{-1}} + \rho_\mu\int_0^t|\varphi(s)|^2_H\,ds \\
&\quad\leq 2\mu\langle v_0, u_0\rangle_{H^{-1}} + |u_0|^2_{H^{-1}} + 2\mu^2|v_0|^2_{H^{-1}} + 2\mu|u_0|^2_H,
\end{aligned} \tag{3.7}$$

and this implies that

$$\begin{aligned}
&\int_0^\infty|\varphi(t)|^2_H\,ds \\
&\quad\leq \frac{1}{\rho_\mu}\big(2\mu\langle v_0, u_0\rangle_{H^{-1}} + |u_0|^2_{H^{-1}} + 2\mu^2|v_0|^2_{H^{-1}} + 2\mu|u_0|^2_H\big).
\end{aligned} \tag{3.8}$$

Actually, if there exists $t_0 > 0$ and $\delta > 0$ such that

$$\int_0^{t_0}|\varphi(t)|^2_H\,ds > \frac{1}{\rho_\mu}\big(2\mu\langle v_0, u_0\rangle_{H^{-1}} + |u_0|^2_{H^{-1}} + 2\mu^2|v_0|^2_{H^{-1}} + 2\mu|u_0|^2_H\big) + \delta,$$

then, in view of (3.7), for any $t > t_0$,

$$\mu\frac{d}{dt}|\varphi(t)|^2_{H^{-1}} < -\delta.$$

This would imply that for any $t > t_0$,

$$|\varphi(t)|^2_{H^{-1}} < |\varphi(t_0)|^2_{H^{-1}} - (t - t_0)\delta,$$

and in particular, that for large values of t,

$$|\varphi(t)|^2_{H^{-1}} < 0,$$

which is impossible.

We conclude the proof by combining (3.5) and (3.8), to see that

$$\mu\left|\frac{\partial\varphi}{\partial t}(t)\right|^2_{H^{-1}}+|\varphi(t)|^2_H+\int_0^t\left|\frac{\partial\varphi}{\partial s}(s)\right|^2_{H^{-1}}ds+\int_0^t|\varphi(s)|_H\,ds\le c|z_0|^2_{\mathcal{H}}. \qquad \square$$

LEMMA 3.2. *Assume $\mu<(\alpha_1-\gamma_0)\gamma_0^{-2}$. Then for any $R>0$,*

$$\lim_{t\to+\infty}\sup_{|z_0|_{\mathcal{H}}\le R}|z^\mu_{z_0}(t)|_{\mathcal{H}}=0. \tag{3.9}$$

PROOF. Let us fix $R,\rho>0$ and for any $\mu>0$ let us define

$$T=\frac{(c_1(\mu))^4R^2}{\rho^2}.$$

Let $|z_0|_{\mathcal{H}}\le R$. Since

$$|z^\mu_{z_0}|_{L^2((0,T);\mathcal{H})}\ge\sqrt{T}\min_{s\le T}|z^\mu_{z_0}(s)|_{\mathcal{H}},$$

according to (3.2) there must exist $t_0<T$ such that

$$|z^\mu_{z_0}(t_0)|_{\mathcal{H}}\le\frac{\rho}{c_1(\mu)}.$$

By using again (3.2), this implies

$$\sup_{t\ge T}|z^\mu_{z_0}(t)|_{\mathcal{H}}=\sup_{t\ge T}|z^\mu_{z^\mu_{z_0}(t_0)}(t-t_0)|_{\mathcal{H}}\le\rho.$$

Notice that T is independent of our choice of z_0, so we can conclude that

$$\sup_{t\ge T}\sup_{|z_0|_{\mathcal{H}}\le R}|z^\mu_{z_0}(t)|_{\mathcal{H}}\le\rho. \qquad \square$$

Now that we have shown that the unperturbed system is uniformly attracted to 0 from any bounded set in $\mathcal{H}$, we show that if the initial velocity is large enough, $\Pi_1z^\mu_{z_0}$ will leave any bounded set.

LEMMA 3.3. *For any $\mu>0$ and $t>0$, there exists $c_2(\mu,t)>0$ such that*

$$\sup_{s\le t}|\Pi_1S_\mu(s)(0,v_0)|_H\ge c_2(\mu,t)|v_0|_{H^{-1}},\qquad v_0\in H^{-1}. \tag{3.10}$$

PROOF. Let $\varphi(t)=\Pi_1S_\mu(t)(0,v_0)$. Then

$$\mu\frac{\partial^2\varphi}{\partial t^2}(t)+\frac{\partial\varphi}{\partial t}(t)=A\varphi(t),\qquad\varphi(0)=0,\qquad\frac{\partial\varphi}{\partial t}(0)=v_0.$$

By taking the inner product of this equation with $\frac{\partial\varphi}{\partial t}(t)$ in H^{-1}, we see that

$$\mu\frac{d}{dt}\left|\frac{\partial\varphi}{\partial t}(t)\right|^2_{H^{-1}}+2\left|\frac{\partial\varphi}{\partial t}(t)\right|^2_{H^{-1}}=-\frac{d}{dt}|\varphi(t)|^2_H.$$

Therefore, by standard calculations,

$$\begin{aligned}\left|\frac{\partial\varphi}{\partial t}(t)\right|^2_{H^{-1}} &= e^{-(2t)/\mu}|v|^2_{H^{-1}} - \frac{1}{\mu}\int_0^t e^{-(2(t-s))/\mu}\frac{d}{ds}|\varphi(s)|^2_H\,ds \\ &= e^{-(2t)/\mu}|v_0|^2_{H^{-1}} - \frac{1}{\mu}|\varphi(t)|^2_H + \frac{2}{\mu^2}\int_0^t e^{-(2(t-s))/\mu}|\varphi(s)|^2_H\,ds,\end{aligned}$$

so that

$$\left|\frac{\partial\varphi}{\partial t}(t)\right|^2_{H^{-1}} \le e^{-(2t)/\mu}|v_0|^2_{H^{-1}} + \frac{1}{\mu}\sup_{s\le t}|\varphi(s)|^2_H. \tag{3.11}$$

Next, since

$$\begin{aligned}\frac{d}{dt}\left|\varphi(t)+\mu\frac{\partial\varphi}{\partial t}(t)\right|^2_{H^{-1}} &= 2\left\langle\varphi(t)+\mu\frac{\partial\varphi}{\partial t}(t), \frac{\partial\varphi}{\partial t}(t)+\mu\frac{\partial^2\varphi}{\partial t^2}(t)\right\rangle_{H^{-1}} \\ &= 2\left\langle\varphi(t)+\mu\frac{\partial\varphi}{\partial t}(t), A\varphi(t)\right\rangle_{H^{-1}} \\ &= -2|\varphi(t)|^2_H - \mu\frac{d}{dt}|\varphi(t)|_H,\end{aligned} \tag{3.12}$$

if we integrate in time we get

$$\mu^2|v_0|^2_{H^{-1}} = \left|\varphi(t)+\mu\frac{\partial\varphi}{\partial t}(t)\right|^2_{H^{-1}} + 2\int_0^t|\varphi(s)|^2_H\,ds + \mu|\varphi(t)|^2_H.$$

For any $a > 0$ to be chosen later, we have

$$\left|\varphi(t)+\mu\frac{\partial\varphi}{\partial t}(t)\right|^2_{H^{-1}} \le (1+a^{-1})\frac{1}{\alpha_1}|\varphi(t)|^2_H + \mu^2(1+a)\left|\frac{\partial\varphi}{\partial t}(t)\right|^2_{H^{-1}}$$

and therefore,

$$\mu^2|v_0|^2_{H^{-1}} \le \left(\mu + 2t + (1+a^{-1})\frac{1}{\alpha_1}\right)\sup_{s\le t}|\varphi(s)|^2_H + \mu^2(1+a)\left|\frac{\partial\varphi}{\partial t}(t)\right|^2_{H^{-1}}.$$

Thanks to (3.11), this yields

$$\begin{aligned}&\mu^2(1-(1+a)e^{-(2t)/\mu})|v_0|^2_{H^{-1}} \\ &\quad\le \left(\mu + 2t + (1+a^{-1})\frac{1}{\alpha_k} + (1+a)\mu\right)\sup_{s\le t}|\varphi(s)|^2_H,\end{aligned}$$

and our conclusion follows if we pick $a < e^{(2t)/\mu} - 1$. $\square$

As a consequence of the previous lemma, we can conclude that the following lower bound estimate holds for the solution of (3.1):

LEMMA 3.4. *For any $\mu > 0$ and $t > 0$, there exists $c(\mu, t) > 0$ such that*

$$\sup_{s\leq t}\left|\Pi_1 z^\mu_{z_0}(s)\right|_H \geq c(\mu,t)|\Pi_2 z_0|_{H^{-1}}, \qquad z_0\in\mathcal{H}. \tag{3.13}$$

PROOF. Let $z_0 = (u_0, v_0)$. Since

$$\Pi_1 z^\mu_{z_0}(t) = \Pi_1 S_\mu(t)(u_0,0) + \Pi_1 S_\mu(t)(0,v_0) + \Pi_1\int_0^t S_\mu(t-s)B_\mu\big(z^\mu_{z_0}(s)\big)\,ds,$$

from the Hypothesis 2 and (2.7), for any $s > 0$,

$$\left|\Pi_1 S_\mu(s)(0,v_0)\right|_H \leq \left(2M_\mu + \frac{\gamma_0 M_\mu}{\omega_\mu \mu}\right)\sup_{r\leq s}\left|\Pi_1 z^\mu_{z_0}(r)\right|_H.$$

According to (3.10), this implies that for any $t > 0$,

$$c_2(\mu,t)|v_0|_{H^{-1}} \leq \sup_{s\leq t}\left|\Pi_1 S_\mu(t)(0,v_0)\right|_H \leq \left(2M_\mu + \frac{\gamma_0 M_\mu}{\omega_\mu\mu}\right)\sup_{s\leq t}\left|\Pi_1 z^\mu_{z_0}(s)\right|_H.$$

Therefore, the result follows with

$$c(\mu,t) = c_1(\mu,t)\left(2M_\mu + \frac{\gamma_0 M_\mu}{\omega_\mu\mu}\right)^{-1}. \qquad \square$$

4. The skeleton equation. For any $\mu > 0$ and $s < t$ and for any $\psi \in L^2((s,t); H)$, we define

$$L^\mu_{s,t}\psi = \int_s^t S_\mu(t-r)Q_\mu\psi(r)\,dr.$$

Clearly $L^\mu_{s,t}$ is a continuous bounded linear operator from $L^2([s,t]; H)$ into $\mathcal{H}$. If we define the pseudo-inverse of $L^\mu_{s,t}$ as

$$\big(L^\mu_{s,t}\big)^{-1}(x) = \arg\min\left\{\left|\big(L^\mu_{s,t}\big)^{-1}(\{x\})\right|_{L^2([s,t];H)}\right\}, \qquad x\in \operatorname{Im}\big(L^\mu_{s,t}\big),$$

we have the following bounds.

THEOREM 4.1. *For any $\mu > 0$ and $s < t$, it holds that*

$$\left|\big(L^\mu_{s,t}\big)^{-1}z\right|_{L^2((s,t);H)} = \sqrt{2}\left|\big(C_\mu - S_\mu(t-s)C_\mu S^\star_\mu(t-s)\big)^{-1/2}z\right|_{\mathcal{H}}, \qquad z\in\operatorname{Im}\big(L^\mu_{s,t}\big), \tag{4.1}$$

where

$$C_\mu(u,v) = \left(Q^2(-A)^{-1}u, \frac{1}{\mu}Q^2(-A)^{-1}v\right), \qquad (u,v)\in\mathcal{H}. \tag{4.2}$$

Moreover, for every $\mu > 0$ there exists $T_\mu > 0$ such that

$$\operatorname{Im}\big(L^\mu_{s,t}\big) = \operatorname{Im}\big((C_\mu)^{1/2}\big) = \mathcal{H}_{1+2\beta}, \qquad t-s\geq T_\mu \tag{4.3}$$

and

$$\big|(L^\mu_{s,t})^{-1}z\big|_{L^2((s,t);H)} \le c(\mu, t-s)|z|_{\mathcal{H}_{1+2\beta}}, \qquad z \in \mathcal{H}_{1+2\beta}, \tag{4.4}$$

for some constant $c(\mu, r) > 0$, *with* $r \ge T_\mu$.

PROOF. It is immediate to check that for any $z \in \mathcal{H}$,

$$\big|(L^\mu_{s,t})^\star z\big|^2_{L^2((s,t);H)} = \frac{1}{\mu^2}\int_0^{t-s}\big|Q(-A)^{-1}\Pi_2 S^\star_\mu(r)z\big|^2_H\,dr. \tag{4.5}$$

Now, if we expand $S^\star_\mu(t)(u,v)$ in Fourier series, we have (see [2], Proposition 2.3)

$$S^\star_\mu(t)(u,v) = \sum_{k=1}^\infty\big(\hat{f}^\mu_k(t)e_k, \hat{g}^\mu_k(t)e_k\big),$$

where $\hat{f}^\mu_k$ and $\hat{g}^\mu_k$ solve the system

$$\begin{cases} \mu\big(\hat{f}^\mu_k\big)'(t) = -\hat{g}^\mu_k(t), & \hat{f}^\mu_k(0) = u_k, \\ \mu\big(\hat{g}^\mu_k\big)'(t) = \mu\alpha_k\hat{f}^\mu_k(t) - \hat{g}^\mu_k(t), & \hat{g}^\mu_k(0) = v_k. \end{cases} \tag{4.6}$$

In particular,

$$\big|\hat{g}^\mu_k(t)\big|^2 = -\frac{\mu^2\alpha_k}{2}\frac{d}{dt}\big|\hat{f}^\mu_k(t)\big|^2 - \frac{\mu}{2}\frac{d}{dt}\big|\hat{g}^\mu_k(t)\big|^2. \tag{4.7}$$

Due to (4.5), we get

$$\begin{aligned}
&\big|(L^\mu_{s,t})^\star z\big|^2_{L^2([s,t];H)} \\
&\quad= \frac{1}{2}\sum_{k=1}^\infty\int_0^{t-s}\Big(-\frac{\lambda_k^2}{\alpha_k}\frac{d}{dr}\big|\hat{f}^\mu_k(r)\big|^2 - \frac{\lambda_k^2}{\mu\alpha_k^2}\frac{d}{dr}\big|\hat{g}^\mu_k(r)\big|^2\Big)\,dr \\
&\quad= \frac{1}{2}\sum_{k=1}^\infty\Big(-\frac{\lambda_k^2}{\alpha_k}\big|\hat{f}^\mu_k(t-s)\big|^2 - \frac{\lambda_k^2}{\alpha_k^2\mu}\big|\hat{g}^\mu_k(t-s)\big|^2 + \frac{\lambda_k^2}{\alpha_k}|u_k|^2 + \frac{\lambda_k^2}{\alpha_k^2\mu}|v_k|^2\Big) \qquad (4.8)\\
&\quad= \frac{1}{2}\big(\big|C_\mu^{1/2}z\big|^2_{\mathcal{H}} - \big|C_\mu^{1/2}S^\star_\mu(t-s)z\big|^2_{\mathcal{H}}\big) \\
&\quad= \frac{1}{2}\big\langle\big(C_\mu - S_\mu(t-s)C_\mu S^\star_\mu(t-s)\big)z, z\big\rangle_{\mathcal{H}}.
\end{aligned}$$

This implies that

$$\mathrm{Im}\big(L^\mu_{s,t}\big) = \mathrm{Im}\big(\big(C_\mu - S_\mu(t-s)C_\mu S^\star_\mu(t-s)\big)^{1/2}\big),$$

and (4.1) follows.

Next, in order to prove (4.3), we notice that

$$C_1^{1/2}S^\star_\mu(t) = S^\star_\mu(t)C_1^{1/2}, \qquad t \ge 0 \tag{4.9}$$

and that

$$(1 \wedge \sqrt{\mu})|C_\mu^{1/2} z|_{\mathcal{H}} \leq |C_1^{1/2} z|_{\mathcal{H}} \leq (1 + \sqrt{\mu})|C_\mu^{1/2} z|_{\mathcal{H}}$$

so that, due to (2.7), we have

$$|C_\mu^{1/2} S_\mu^\star(t) z|_{\mathcal{H}} \leq c_\mu M_\mu e^{-\omega_\mu t} |C_\mu^{1/2} z|_{\mathcal{H}}, \qquad t \geq 0.$$

According to (4.8), this implies

$$\begin{aligned} |(L_{s,t}^\mu)^* z|_{\mathcal{H}}^2 &= \tfrac{1}{2}|C_\mu^{1/2} z|_{\mathcal{H}}^2 - \tfrac{1}{2}|C_\mu^{1/2} S_\mu(t-s) z|_{\mathcal{H}} \\ &\geq \tfrac{1}{2}\big(1 - c_\mu^2 M_\mu^2 e^{-2\omega_\mu(t-s)}\big)|C_\mu^{1/2} z|_{\mathcal{H}}^2. \end{aligned}$$

Therefore, if we pick $T_\mu > 0$ large enough so that $c_\mu^2 M_\mu e^{-\omega_\mu T_\mu} < 1$, we obtain that

$$\operatorname{Im}(L_{s,t}^\mu) = \operatorname{Im}\big((C_\mu)^{1/2}\big),$$

and

$$|(L_{s,t}^\mu)^{-1} z|_{L^2((s,t);H)} \leq \sqrt{2}\big(1 - c_\mu^2 M_\mu^2 e^{-2\omega_\mu r}\big)^{-1/2} |(C_\mu)^{-1/2} z|_{\mathcal{H}}.$$

Now, for any $\mu > 0$, we have $\operatorname{Im}((C_\mu)^{1/2}) = \mathcal{H}_{1+2\beta}$, and

$$(1 \wedge \mu)|z|_{\mathcal{H}_{1+2\beta}} \leq |(C_\mu)^{-1/2} z|_{\mathcal{H}} \leq (1+\mu)|z|_{\mathcal{H}_{1+2\beta}}, \tag{4.10}$$

(4.3) and (4.4) follow immediately, with

$$c(\mu, r) = (1+\mu)\sqrt{2}\big(1 - c_\mu^2 M_\mu^2 e^{-2\omega_\mu r}\big)^{-1/2}. \qquad \square$$

REMARK 4.2. (1) In fact, it is possible to show that $\operatorname{Im}(L_{s,t}^\mu) = \operatorname{Im}((C_\mu)^{1/2})$, for all $t - s > 0$, by using the explicit representation of $S_\mu^\star(t)$.

(2) From (2.7) and (4.1), it easily follows that

$$|(L_{-\infty,t}^\mu)^{-1} z|_{L^2((-\infty,t);H)} = \sqrt{2}|C_\mu^{-1/2} z|_{\mathcal{H}}, \qquad z \in \operatorname{Im}(L_{-\infty,t}^\mu). \tag{4.11}$$

LEMMA 4.3. *Let us fix $\psi \in L^2((-\infty,0); H^{2\alpha})$, with $\alpha \in [0, 1/2]$ and $\mu > 0$, and let $z_\psi^\mu \in C((-\infty,0);\mathcal{H})$ solve the equation*

$$z_\psi^\mu(t) = \int_{-\infty}^t S_\mu(t-s) B_\mu\big(z_\psi^\mu(s)\big)\, ds + \int_{-\infty}^t S_\mu(t-s) Q_\mu \psi(s)\, ds, \qquad t \in \mathbb{R}. \tag{4.12}$$

Then if

$$\lim_{t \to -\infty} |z_\psi^\mu(t)|_{\mathcal{H}} = 0, \tag{4.13}$$

we have $z_\psi^\mu \in C((-\infty,0);\mathcal{H}_{1+2(\alpha+\beta)})$ and

$$\lim_{t \to -\infty} |z_\psi^\mu(t)|_{\mathcal{H}_{1+2(\alpha+\beta)}} = 0. \tag{4.14}$$

PROOF. According to (2.8), for any $\delta > 0$, we have

$$\begin{aligned}\left|\int_{-\infty}^t S_\mu(t-s)B_\mu\big(z_\psi^\mu(s)\big)\,ds\right|_{\mathcal{H}_\delta} &\leq \frac{M_\mu}{\mu}\sup_{s\leq t}\big|B\big(\Pi_1 z_\psi^\mu(s)\big)\big|_{H^{\delta-1}}\int_{-\infty}^t e^{-\omega_\mu(t-s)}\,ds \\ &\leq \frac{M_\mu}{\mu\omega_\mu}\sup_{s\leq t}\big|B\big(\Pi_1 z_\psi^\mu(s)\big)\big|_{H^{\delta-1}}.\end{aligned}$$

Therefore, due to Hypothesis 2, if we take $\delta = 1$,

$$\left|\int_{-\infty}^t S_\mu(t-s)B_\mu\big(z_\psi^\mu(s)\big)\,ds\right|_{\mathcal{H}_1} \leq \frac{M_\mu\gamma_0}{\mu\omega_\mu}\sup_{s\leq t}\big|\Pi_1 z_\psi^\mu(s)\big|_H. \tag{4.15}$$

For the second term in (4.12), if $\psi \in L^2(-\infty, 0; H^{2\alpha})$, then $Q_\mu\psi \in L^2((-\infty, 0); \mathcal{H}_{1+2(\alpha+\beta)})$, with

$$|Q_\mu\psi|_{L^2((-\infty,t);\mathcal{H}_{1+2(\alpha+\beta)})} \leq \frac{c}{\mu}|\psi|_{L^2((-\infty,t);H^{2\alpha})}, \qquad t \leq 0.$$

Due to (2.8), this yields

$$\begin{aligned}&\left|\int_{-\infty}^t S_\mu(t-s)Q_\mu\psi(s)\,ds\right|_{\mathcal{H}_{1+2(\alpha+\beta)}} \\ &\quad\leq \frac{M_\mu}{\mu}\left(\int_0^\infty e^{-2\omega_\mu s}\,ds\right)^{1/2}|\psi|_{L^2((-\infty,t);H^{2\alpha})}.\end{aligned} \tag{4.16}$$

Therefore, from (4.12), (4.15) and (4.16), we get

$$\big|z_\psi^\mu(t)\big|_{\mathcal{H}_1} \leq c_\mu\Big(\sup_{s\leq t}\big|\Pi_1 z_\psi^\mu(s)\big|_H + |\psi|_{L^2((-\infty,t);H^{2\alpha})}\Big).$$

In particular, we have $z_\psi^\mu \in L^\infty((-\infty, 0); \mathcal{H}_1)$ and

$$\lim_{t\to-\infty}\big|z_\psi^\mu(t)\big|_{\mathcal{H}_1} = 0.$$

Now, by repeating the same arguments, we can prove that for any $n \in \mathbb{N}$, with $n \leq [1 + 2\beta]$, if

$$z_\psi^\mu \in L^\infty\big((-\infty, 0); \mathcal{H}_n\big) \quad \text{and} \quad \lim_{t\to-\infty}\big|z_\psi^\mu(t)\big|_{\mathcal{H}_n} = 0,$$

then

$$z_\psi^\mu \in L^\infty\big((-\infty, 0); \mathcal{H}_{n+1}\big) \quad \text{and} \quad \lim_{t\to-\infty}\big|z_\psi^\mu(t)\big|_{\mathcal{H}_{n+1}} = 0.$$

Since there exists $\bar{n} \in \mathbb{N}$ such that $\mathcal{H}_{1+2(\alpha+\beta)} \supset \mathcal{H}_{\bar{n}}$, we can conclude that z_ψ^μ belongs to $L^\infty((-\infty, 0); \mathcal{H}_{1+2(\alpha+\beta)})$, and (4.14) holds. Continuity follows easily by standard arguments; for continuity of convolutions, see [12], Appendix A. □

REMARK 4.4. (1) From the previous lemma, we have that if $z^\mu_\psi \in C((-\infty,0);\mathcal{H})$ solves equation (4.12) and limit (4.13) holds, then $z^\mu_\psi(t) \in \mathcal{H}_{1+2\beta}$, for any $t \leq 0$. In particular $z^\mu_\psi(0) \in \mathcal{H}_{1+2\beta}$.

(2) In [5], Lemma 3.5, it has been proven that the same holds for equation (2.5). Actually, if $\varphi_\psi \in C((-\infty,0);H)$ is the solution to

$$\varphi_\psi(t) = \int_{-\infty}^t e^{(t-s)A} B\big(\varphi_\psi(s)\big)\,ds + \int_{-\infty}^t e^{(t-s)A} Q\psi(s)\,ds,$$

for $\psi \in L^2((-\infty,0);H)$, and

$$\lim_{t\to-\infty} |\varphi_\psi(t)|_H = 0,$$

then $\varphi_\psi \in C((-\infty,0);H^{1+2\beta})$, and there exists a constant such that for all $t \leq 0$,

$$|\varphi_\psi(t)|_{H^{1+2\beta}} \leq c|\psi|_{L^2((-\infty,0;H)}. \tag{4.17}$$

Moreover,

$$\lim_{t\to-\infty} |\varphi_\psi(t)|_{H^{1+2\beta}} = 0. \tag{4.18}$$

LEMMA 4.5. *Let $\alpha \in [0,1/2]$, and let $\psi_1,\psi_2 \in L^2((-\infty,0);H^{2\alpha})$. In correspondence of each ψ_i, let $z^\mu_{\psi_i} \in C((-\infty,0);\mathcal{H}_{1+2(\alpha+\beta)})$ be a solution of equation* (4.12), *verifying* (4.13). *Then $z^\mu_{\psi_i} \in L^2((-\infty,0);\mathcal{H}_{1+2(\alpha+\beta)})$, for $i=1,2$, and there exist $\mu_0>0$ and $c>0$ such that for any $\mu \leq \mu_0$ and $\tau \leq 0$,*

$$\begin{aligned} &|z^\mu_{\psi_1} - z^\mu_{\psi_2}|^2_{L^2((-\infty,\tau);\mathcal{H}_{1+2(\alpha+\beta)})} + \sup_{t\leq\tau} |\mathcal{I}_\mu(z^\mu_{\psi_1}(t) - z^\mu_{\psi_2}(t))|^2_{\mathcal{H}_{1+2(\alpha+\beta)}} \\ &\qquad \leq c|\psi_1-\psi_2|^2_{L^2((-\infty,\tau);H^{2\alpha})}, \end{aligned} \tag{4.19}$$

where $\mathcal{I}_\mu$ is defined in (2.2).

PROOF. If we define

$$u(t) = (-A)^{\alpha+\beta}\Pi_1\big(z^\mu_{\psi_1}(t) - z^\mu_{\psi_2}(t)\big), \qquad t \leq 0,$$

and

$$\psi(t) = (-A)^{\alpha+\beta} Q\big(\psi_1(t) - \psi_2(t)\big), \qquad t \leq 0,$$

we have

$$\begin{aligned} &\mu\frac{\partial^2 u}{\partial t^2}(t) + \frac{\partial u}{\partial t}(t) \\ &\qquad = Au(t) + (-A)^{\alpha+\beta}\big(B\big(\Pi_1 z^\mu_{\psi_1}(t)\big) - B\big(\Pi_1 z^\mu_{\psi_2}(t)\big)\big) + \psi(t). \end{aligned} \tag{4.20}$$

According to Hypothesis 2, $B : H^{2(\alpha+\beta)} \to H^{2(\alpha+\beta)}$ is Lipschitz-continuous, and then

$$\begin{aligned}
&\big|(-A)^{\alpha+\beta}\big(B\big(\Pi_1 z^{\mu}_{\psi_1}(t)\big) - B\big(\Pi_1 z^{\mu}_{\psi_2}(t)\big)\big)\big|_H \\
&\quad = \big|B\big(\Pi_1 z^{\mu}_{\psi_1}(t)\big) - B\big(\Pi_1 z^{\mu}_{\psi_2}(t)\big)\big|_{H^{2(\alpha+\beta)}} \\
&\quad \le \gamma_{2(\alpha+\beta)}\big|\Pi_1\big(z^{\mu}_{\psi_1}(t)\big) - z^{\mu}_{\psi_2}(t)\big)\big|_{H^{2(\alpha+\beta)}} = \gamma_{2(\alpha+\beta)}\big|u(t)\big|_H.
\end{aligned}$$

Therefore, by taking the scalar product of both sides with $\partial u/\partial t$, we get

$$\begin{aligned}
&\left|\frac{\partial u}{\partial t}(t)\right|^2_H + \frac{\mu}{2}\frac{d}{dt}\left|\frac{\partial u}{\partial t}(t)\right|^2_H + \frac{1}{2}\frac{d}{dt}\big|(-A)^{1/2}u(t)\big|^2_H \\
&\qquad \le \gamma_{2(\alpha+\beta)}\big|u(t)\big|_H\left|\frac{\partial u}{\partial t}(t)\right|_H + \big|\psi(t)\big|_H\left|\frac{\partial u}{\partial t}(t)\right|_H.
\end{aligned} \tag{4.21}$$

Now, since

$$\begin{aligned}
&\gamma_{2(\alpha+\beta)}\big|u(t)\big|_H\left|\frac{\partial u}{\partial t}(t)\right|_H + \big|\psi(t)\big|_H\left|\frac{\partial u}{\partial t}(t)\right|_H \\
&\qquad \le \frac{1}{2}\left|\frac{\partial u}{\partial t}(t)\right|^2_H + \gamma^2_{2(\alpha+\beta)}\big|u(t)\big|^2_H + \big|\psi(t)\big|^2_H,
\end{aligned}$$

(4.21) implies

$$\begin{aligned}
&\left|\frac{\partial u}{\partial t}(t)\right|^2_H + \mu\frac{d}{dt}\left|\frac{\partial u}{\partial t}(t)\right|^2_H + \frac{d}{dt}\big|(-A)^{1/2}u(t)\big|^2_H \\
&\qquad \le 2\gamma^2_{2(\alpha+\beta)}\big|u(t)\big|^2_H + 2\big|\psi(t)\big|^2_H.
\end{aligned} \tag{4.22}$$

Therefore, integrating this expression with respect to $t \in (-\infty, \tau)$, we obtain

$$\begin{aligned}
&\int_{-\infty}^{\tau}\left|\frac{\partial u}{\partial t}(t)\right|^2_H dt + \big|u(\tau)\big|^2_{H^1} + \mu\left|\frac{\partial u}{\partial t}(t)\right|^2_H \\
&\qquad \le 2\gamma^2_{2(\alpha+\beta)}\int_{-\infty}^{\tau}\big|u(t)\big|^2_H\,dt + 2\int_{-\infty}^{\tau}\big|\psi(t)\big|^2_H\,dt,
\end{aligned} \tag{4.23}$$

since, due to Lemma 4.3,

$$\begin{aligned}
&\int_{-\infty}^{\tau}\frac{d}{dt}\left(\mu\left|\frac{\partial u}{\partial t}(t)\right|^2_H + \big|(-A)^{1/2}u(t)\big|^2_H\right)dt \\
&\quad = \mu\left|\frac{\partial u}{\partial t}(\tau)\right|^2_H + \big|u(\tau)\big|^2_{H^1} - \lim_{T\to-\infty}\left(\mu\left|\frac{\partial u}{\partial t}(T)\right|^2_H + \big|u(T)\big|^2_{H^1}\right) \\
&\quad = \mu\left|\frac{\partial u}{\partial t}(\tau)\right|^2_H + \big|u(\tau)\big|^2_{H^1}.
\end{aligned}$$

Next we take the inner product of each side of (4.20) with $u(t)$ and use the fact that

$$\left\langle \frac{\partial^2 u}{\partial t^2}(t), u(t) \right\rangle_H = \frac{1}{2}\frac{d^2}{dt^2}|u(t)|_H^2 - \left|\frac{\partial u}{\partial t}(t)\right|_H^2$$

and again the Lipschitz-continuity of B in $H^{2(\alpha+\beta)}$ to get

$$\begin{aligned}
\frac{\mu}{2}\frac{d^2}{dt^2}|u(t)|_H^2 &+ \frac{1}{2}\frac{d}{dt}|u(t)|_H^2 + \hat{\gamma}|u(t)|_{H^1}^2 \\
&\leq \mu\left|\frac{\partial u}{\partial t}(t)\right|_H^2 + \langle \psi(t), u(t)\rangle_H \\
&\leq \mu\left|\frac{\partial u}{\partial t}(t)\right|_H^2 + \frac{\hat{\gamma}}{2}|u(t)|_{H^1}^2 + c|\psi(t)|_H^2,
\end{aligned}$$

where $\hat{\gamma} := 1 - \gamma_{2(\alpha+\beta)})/\alpha_1 > 0$. This yields

$$\mu\frac{d^2}{dt^2}|u(t)|_H^2 + \frac{d}{dt}|u(t)|_H^2 + \hat{\gamma}|u(t)|_{H^1}^2 \leq 2\mu\left|\frac{\partial u}{\partial t}(t)\right|_H^2 + c|\psi(t)|_H^2. \tag{4.24}$$

Combining together (4.22) and (4.24), we get

$$\begin{aligned}
\mu\frac{d^2}{dt^2}|u(t)|_H^2 &+ \frac{d}{dt}|u(t)|_H^2 + \hat{\gamma}|u(t)|_{H^1}^2 + 2\mu^2\frac{d}{dt}\left|\frac{\partial u}{\partial t}(t)\right|_H^2 + 2\mu\frac{d}{dt}|u(t)|_{H^1}^2 \\
&\leq c_1\mu|u(t)|_H^2 + c_2(1+\mu)|\psi(t)|_H^2.
\end{aligned}$$

If we take

$$\mu < \frac{\hat{\gamma}\alpha_1}{2c_1},$$

and integrate both sides with respect to $t \in (-\infty, \tau)$, as a consequence of (4.14), we get

$$\frac{1}{2}\int_{-\infty}^{\tau}|u(t)|_{H^1}^2\,dt \leq -2\mu\left\langle u(\tau), \frac{\partial u}{\partial t}(\tau)\right\rangle_H + c_2(1+\mu)\int_{-\infty}^{\tau}|\psi(t)|_H^2\,dt. \tag{4.25}$$

Substituting this back into (4.23), we have

$$\begin{aligned}
\int_{-\infty}^{\tau}&\left(\left|\frac{\partial u}{\partial t}(t)\right|_H^2 + |u(t)|_{H^1}^2\right)dt + \mu\left|\frac{\partial u}{\partial t}(\tau)\right|_H^2 + |u(\tau)|_{H^1}^2\,dt \\
&\leq -c\mu\left\langle u(\tau), \frac{\partial u}{\partial t}(\tau)\right\rangle_H + c\int_{-\infty}^{\tau}|\psi(t)|_H^2\,dt \\
&\leq c\sqrt{\mu}\left(\mu\left|\frac{\partial u}{\partial t}(\tau)\right|_H^2 + |u(\tau)|_{H^1}^2\right) + c\int_{-\infty}^{\tau}|\psi(t)|_H^2\,dt.
\end{aligned}$$

Therefore, since

$$\big|\psi(t)\big|_H \le c\big|\psi_1(t)-\psi_2(t)\big|_{H^{2\alpha}},$$

and

$$\big|\mathcal{I}_\mu\big(z^\mu_{\psi_1}(\tau)-z^\mu_{\psi_2}(\tau)\big)\big|_{\mathcal{H}_{1+2(\alpha+\beta)}} = \mu\left|\frac{\partial u}{\partial t}(\tau)\right|^2_H + \big|u(\tau)\big|^2_{H^1},$$

if we choose μ_0 small enough, this yields (4.19). □

REMARK 4.6. (1) Notice that since $B(0)=0$, we have $z^\mu_0=0$, so that from (4.19), we get

$$|z^\mu_\psi|^2_{L^2((-\infty,\tau);\mathcal{H}_{1+2(\alpha+\beta)})} + \sup_{t\le\tau}|\mathcal{I}_\mu z^\mu_\psi(t)|^2_{\mathcal{H}_{1+2(\alpha+\beta)}} \le c|\psi|^2_{L^2((-\infty,\tau);H^{2\alpha})}, \tag{4.26}$$

for any $\mu\le\mu_0$ and $\tau\le 0$.

(2) By proceeding as in the proof of Lemma 4.5, we can prove that

$$\begin{aligned}|z^\mu_{\psi_1}-z^\mu_{\psi_2}|^2_{L^2((-\infty,\tau);\mathcal{H}_{2\beta})} &+ \sup_{t\le\tau}|\mathcal{I}_\mu\big(z^\mu_{\psi_1}(t)-z^\mu_{\psi_2}(t)\big)|^2_{\mathcal{H}_{2\beta}}\\ &\le c|\psi_1-\psi_2|^2_{L^2((-\infty,\tau);H^{-1})}\end{aligned} \tag{4.27}$$

and

$$|z^\mu_\psi|^2_{L^2((-\infty,\tau);\mathcal{H}_{2\beta})} + \sup_{t\le\tau}|\mathcal{I}_\mu z^\mu_\psi(t)|^2_{\mathcal{H}_{2\beta}} \le c|\psi|^2_{L^2((-\infty,\tau);H^{-1})}.$$

5. A characterization of the quasi-potential.

For any $t_1<t_2$, $\mu>0$ and $z\in C((t_1,t_2);\mathcal{H})$, we define

$$I^\mu_{t_1,t_2}(z) = \tfrac{1}{2}\inf\{|\psi|^2_{L^2((t_1,t_2);H)} : z = z^\mu_{\psi,z_0}\}, \tag{5.1}$$

where z^μ_{ψ,z_0} is a mild solution of the skeleton equation associated with equation (2.9), with deterministic control $\psi\in L^2((t_1,t_2);H)$ and initial conditions z_0, namely

$$\frac{dz^\mu_{\psi,z_0}}{dt}(t) = A_\mu z^\mu_{\psi,z_0}(t) + B_\mu\big(z^\mu_{\psi,z_0}(t)\big) + Q_\mu\psi(t), \qquad t_1\le t\le t_2. \tag{5.2}$$

As in Definition 2.4, for $\varepsilon,\mu>0$ and $z_0\in\mathcal{H}$, we denote by $z^\mu_{\varepsilon,z_0}\in L^2(\Omega;C([0,T];\mathcal{H}))$ the mild solution of equation (2.9). Since the mapping $B_\mu:\mathcal{H}\to\mathcal{H}$ is Lipschitz-continuous and the noisy perturbation in (2.9) is of additive type, as an immediate consequence of the contraction lemma, for any fixed $\mu>0$, the family $\{\mathcal{L}(z^\mu_{\varepsilon,z_0})\}_{\varepsilon>0}$ satisfies a large deviation principle in $C([t_1,t_2];\mathcal{H})$, with action functional $I^\mu_{t_1,t_2}$. In particular, for any $\delta>0$ and $T>0$,

$$\liminf_{\varepsilon\to 0}\varepsilon\log\Big(\inf_{z_0\in\mathcal{H}}\mathbb{P}\big(|z^\mu_{\varepsilon,z_0}-z^\mu_{\psi,z_0}|_{C([0,T];\mathcal{H})}<\delta\big)\Big) \ge -\frac{1}{2}|\psi|^2_{L^2((0,T);H)}, \tag{5.3}$$

and if $K^{\mu}_{0,T}(r) = \{z \in C([0,T];\mathcal{H}) : I^{\mu}_{0,T}(z) \leq r\}$,

$$\limsup_{\varepsilon \to 0} \varepsilon \log\Big(\sup_{z_0 \in \mathcal{H}} \mathbb{P}\big(\mathrm{dist}_{\mathcal{H}}\big(z^{\mu}_{\varepsilon,z_0}, K^{\mu}_{0,T}(r)\big) > \delta\big)\Big) \leq -r. \tag{5.4}$$

Analogously, if for any $\varepsilon > 0$, u_ε denotes the mild solution of equation (2.5), the family $\{\mathcal{L}(u_\varepsilon)\}_{\varepsilon>0}$ satisfies a large deviation principle in $C([t_1,t_2];H)$ with action functional

$$I_{t_1,t_2}(\varphi) = \inf\{\tfrac{1}{2}|\psi|^2_{L^2([t_1,t_2];H)} : \varphi = \varphi_\psi\}, \tag{5.5}$$

where φ_ψ is a mild solution of the skeleton equation associated with equation (2.5)

$$\frac{du}{dt}(t) = Au(t) + B\big(u(t)\big) + Q\psi(t), \qquad t_1 \leq t \leq t_2.$$

In particular, the functionals $I^{\mu}_{t_1,t_2}$ and I_{t_1,t_2} are lower semi-continuous and have compact level sets. Moreover, it is not difficult to show that for any compact sets $E \subset H$ and $\mathcal{E} \subset \mathcal{H}$, the level sets

$$K_{E,t_1,t_2}(r) = \big\{\varphi \in C\big([t_1,t_2];H\big); I_{t_1,t_2}(\varphi) \leq r, \varphi(t_1) \in E\big\}$$

and

$$K^{\mu}_{\mathcal{E},t_1,t_2}(r) = \big\{z \in C\big([t_1,t_2];\mathcal{H}\big); I^{\mu}_{t_1,t_2}(z) \leq r, z(t_1) \in \mathcal{E}\big\}$$

are compact.

In what follows, for the sake of brevity, for any $\mu > 0$ and $t \in (0,+\infty]$, we shall define $I^{\mu}_t := I^{\mu}_{0,t}$ and $I^{\mu}_{-t} := I^{\mu}_{-t,0}$, and analogously, for any $t \in (0,+\infty]$, we shall define $I_t := I_{0,t}$ and $I_{-t} := I_{-t,0}$. In particular, we shall set

$$I^{\mu}_{-\infty}(z) = \sup_{t>0} I^{\mu}_{-t}(z), \qquad I_{-\infty}(\varphi) = \sup_{t>0} I_{-t}(\varphi).$$

Moreover, for any $r > 0$ we shall set

$$K^{\mu}_{-\infty}(r) = \Big\{z \in C\big((-\infty,0];\mathcal{H}\big); \lim_{t\to-\infty}\big|z(t)\big|_{\mathcal{H}} = 0, I^{\mu}_{-\infty}(z) \leq r\Big\}$$

and

$$K_{-\infty}(r) = \Big\{\varphi \in C\big((-\infty,0];H\big); \lim_{t\to-\infty}\big|\varphi(t)\big|_{H} = 0, I_{-\infty}(\varphi) \leq r\Big\}.$$

Once we have introduced the action functionals $I^{\mu}_{t_1,t_2}$ and I_{t_1,t_2}, we can introduce the corresponding *quasi-potentials*, by setting for any $\mu > 0$ and $(x,y) \in \mathcal{H}$,

$$V^{\mu}(x,y) = \inf\big\{I^{\mu}_{0,T}(z); z(0) = 0, z(T) = (x,y), T > 0\big\},$$

and for any $x \in H$,

$$V(x) = \inf\big\{I_{0,T}(\varphi); \varphi(0) = 0, \varphi(T) = x, T \geq 0\big\}.$$

Moreover, for any $\mu > 0$ and $x \in H$, we shall define

$$\bar{V}_\mu(x) = \inf_{y \in H^{-1}} V^\mu(x, y). \tag{5.6}$$

In [5], Proposition 5.1, it has been proved that the level set $K_{-\infty}(r)$ is compact in the space $C((-\infty, 0]; H)$, endowed with the uniform convergence on bounded sets, and in [5], Proposition 5.4, it has been proven that

$$V(x) = \min\Big\{I_{-\infty}(\varphi); \varphi \in C\big((-\infty, 0]; H\big), \lim_{t \to -\infty} \big|\varphi(t)\big|_H = 0, \varphi(0) = x\Big\}.$$

In what follows we want to prove an analogous result for $K^\mu_{-\infty}$, $V^\mu(x, y)$ and $\bar{V}_\mu(x)$.

THEOREM 5.1. *For small enough $\mu > 0$, the level sets $K^\mu_{-\infty}(r)$ are compact in the topology of uniform convergence on bounded intervals.*

PROOF. Suppose that z_n is a sequence in $K^\mu_{-\infty}(r)$ where $\mu \leq \mu_0$ and μ_0 is the constant introduced in Lemma 4.5. Let c be the constant from that lemma, and let

$$\mathcal{E} := \big\{z \in \mathcal{H} : \big|C_\mu^{-1/2} z\big|_{\mathcal{H}} \leq \sqrt{2cr}\big\}.$$

By Lemma 4.5, $z_n \in K^\mu_{\mathcal{E}, -N, 0}(r)$, for any $N \in \mathbb{N}$. Since $\mathcal{E}$ is compact in $\mathcal{H}$, in view of what we have seen above, $K^\mu_{\mathcal{E}, -N, 0}(r) \subset C([-N, 0]; \mathcal{H})$ is compact, for each $N \in \mathbb{N}$. Then, by using a diagonalization procedure, we can find a subsequence of $\{z_n\}$ that converges uniformly to a limit $z^\mu \in C((-\infty, 0]; \mathcal{H})$, uniformly on $[-N, 0]$ for all N. This means that there exist controls ψ_N such that for $t \in [-N, 0]$,

$$\begin{aligned} z^\mu(t) = S_\mu(t + N) z^\mu(-N) &+ \int_{-N}^t S_\mu(t - s) B_\mu\big(z^\mu(s)\big)\, ds \\ &+ \int_{-N}^t S_\mu(t - s) Q_\mu \psi_N(s)\, ds \end{aligned}$$

and

$$\tfrac{1}{2} |\psi_N|^2_{L^2([-N, 0]; H)} \leq r.$$

All of these ψ_N's coincide because if $\varphi = \Pi_1 z^\mu$ satisfies the above equation,

$$\psi_N(t) = Q^{-1}\bigg(\mu \frac{\partial^2 \varphi}{\partial t^2}(t) + \frac{\partial \varphi}{\partial t}(t) - A\varphi(t) - B\big(\varphi(t)\big)\bigg)$$

weakly. Therefore, we can let $\psi = \psi_N$ and notice that

$$\tfrac{1}{2} |\psi|^2_{L^2((-\infty, 0); H)} \leq r.$$

This implies that for each $N_0 \in \mathbb{N}$,

$$z^\mu(t) = S_\mu(t+N_0)z^\mu(-N_0) + \int_{-N_0}^t S_\mu(t-s)B_\mu\big(z^\mu(s)\big)\,ds + \int_{-N_0}^t S_\mu(t-s)Q_\mu\psi(s)\,ds.$$

Thus, by taking the limit as $N_0 \to +\infty$, we conclude that

$$z^\mu(t) = \int_{-\infty}^t S_\mu(t-s)B_\mu\big(z^\mu(s)\big)\,ds + \int_{-\infty}^t S_\mu(t-s)Q_\mu\psi(s)\,ds, \qquad t \le 0.$$

Finally, we need to show that

$$\lim_{t\to-\infty} |z^\mu(t)|_{\mathcal{H}} = 0.$$

By (4.26), each z_n has the property that

$$|z_n|_{L^2((-\infty,0);\mathcal{H})} \le c\sqrt{r}.$$

Since $z_n \to z^\mu$ uniformly in $C((-N,0);\mathcal{H})$ for each N,

$$|z^\mu|_{L^2((-\infty,0);\mathcal{H})} = \lim_{N\to+\infty} |z^\mu|_{L^2((-N,0);\mathcal{H})} \le c\sqrt{r}.$$

Next, by (4.16) and Hypothesis 2,

$$\begin{aligned}|z^\mu(t)|_{\mathcal{H}_1} &= \left|\int_{-\infty}^t S_\mu(t-s)\big(B_\mu\big(z^\mu(s)\big) + Q\mu\psi(s)\big)\,ds\right|_{H_1} \\ &\le c|z^\mu|_{L^2((-\infty,t);\mathcal{H})} + c|\psi|_{L^2((-\infty,t);H^{-2\beta})}.\end{aligned}$$

Because $z^\mu \in L^2((-\infty,0);\mathcal{H})$, and $\psi \in L^2((-\infty,0);H)$,

$$\lim_{t\to-\infty} |z^\mu(t)|_{\mathcal{H}_1} = 0. \qquad \square$$

COROLLARY 5.2. *There exists $\mu_0 > 0$ such that for any $\psi \in L^2((-\infty,0);H)$ and $\mu \le \mu_0$ there exists $z_\psi^\mu \in C((-\infty,0];\mathcal{H})$ such that*

$$z_\psi^\mu(t) = \int_{-\infty}^t S_\mu(t-s)B_\mu\big(z_\psi^\mu(s)\big)\,ds + \int_{-\infty}^t S_\mu(t-s)Q_\mu\psi(s)\,ds, \qquad t \le 0. \tag{5.7}$$

Moreover,

$$\lim_{t\to-\infty} |z_\psi^\mu(t)|_{\mathcal{H}} = 0.$$

PROOF. A standard fixed point argument shows that for any $\mu > 0$ and $N \in \mathbb{N}$, there exists $z_N^\mu \in C([-N,0];\mathcal{H})$ satisfying

$$z_N^\mu(t) = \int_{-N}^t S_\mu(t-s) B_\mu\big(z_N^\mu(s)\big)\,ds + \int_{-N}^t S_\mu(t-s) Q_\mu \psi(s)\,ds.$$

Each z_N^μ can be seen as an element of $C((-\infty,0];H)$, just by extending it to $z_N^\mu(t) = 0$, for all $t < -N$. According to Theorem 5.1, there exists a subsequence $\{z_{N_k}^\mu\}$ converging to some $z^\mu \in K_{-\infty}^\mu(\frac{1}{2}|\psi|^2_{L^2((-\infty,0);H)})$, uniformly on compact sets. We notice that for any fixed $N_0 \in \mathbb{N}$ and $t \geq -N_0$,

$$z_N^\mu(t) = S_\mu(t+N_0) z_N^\mu(-N_0) + \int_{-N_0}^t S_\mu(t-s) B_\mu\big(z_N^\mu(s)\big)\,ds$$
$$+ \int_{-N_0}^t S_\mu(t-s) Q_\mu \psi(s)\,ds.$$

Therefore, by taking the limit as $N \to +\infty$, we obtain

$$z^\mu(t) = S_\mu(t+N_0) z^\mu(-N_0) + \int_{-N_0}^t S_\mu(t-s) B_\mu\big(z^\mu(s)\big)\,ds$$
$$+ \int_{-N_0}^t S_\mu(t-s) Q_\mu \psi(s)\,ds.$$

Finally, if we let $N_0 \to +\infty$, we see that z^μ solves equation (5.7). $\square$

As $K_{-\infty}(r)$ is compact in $C((-\infty,0];H)$ with respect to the uniform convergence on bounded intervals, we have analogously that for any $\varphi \in L^2((-\infty,0)$, there exists $\varphi_\psi \in C((-\infty,0];H)$ such that

$$\varphi_\psi(t) = \int_{-\infty}^t e^{(t-s)A} B\big(\varphi(s)\big)\,ds + \int_{-\infty}^t e^{(t-s)A} Q\psi(s)\,ds,$$

and

$$\lim_{t \to -\infty} \big|\varphi_\psi(t)\big|_H = 0.$$

In [5], it has been proved that the $V(x)$ can be characterized as

$$V(x) = \inf\Big\{ I_{-\infty}(\varphi) : \lim_{t \to -\infty} \varphi(t) = 0, \varphi(0) = x \Big\}.$$

Here, we want to prove that an analogous result holds for $V^\mu(x,y)$ and $\bar{V}_\mu(x)$, at least for μ sufficiently small.

THEOREM 5.3. *For small enough $\mu > 0$, we have the following representation for the quasi-potentials $V^\mu(x,y)$:*

$$V^\mu(x,y) = \min\Big\{ I_{-\infty}^\mu(z) : \lim_{t \to -\infty} \big|z(t)\big|_{\mathcal{H}} = 0, z(0) = (x,y) \Big\}, \tag{5.8}$$

and for $\bar{V}_\mu(x)$,

$$\bar{V}_\mu(x) = \min\Big\{ I^\mu_{-\infty}(z) : \lim_{t\to-\infty} |z(t)|_{\mathcal{H}} = 0, \Pi_1 z(0) = x \Big\}, \tag{5.9}$$

whenever these quantities are finite.

PROOF. From the definitions of $I^\mu_{t_1,t_2}$, it is clear that

$$V^\mu(x,y) = \inf\{ I^\mu_{t_1,0}(z) : z(t_1) = 0, z(0) = (x,y), t_1 \le 0 \}.$$

Now, if we define

$$M^\mu(x,y) = \inf\Big\{ I^\mu_{-\infty}(\varphi) : \lim_{t\to-\infty} |z(t)|_{\mathcal{H}} = 0, z(0) = (x,y) \Big\}, \tag{5.10}$$

it is immediate to check that $M^\mu(x,y) \le V^\mu(x,y)$, for any $(x,y)\in\mathcal{H}$. To see this, we observe that if $z \in C([t_1,0];\mathcal{H})$, with $z(t_1)=0$ and $z(0)=(x,y)$, then

$$\hat{z}(t) = \begin{cases} 0, & t \le t_1, \\ z(t), & t_1 < t \le 0 \end{cases} \tag{5.11}$$

has the property that $\hat{z}(0) = (x,y)$, and $|\hat{z}(t)|_{\mathcal{H}} \to 0$, as $t \to -\infty$. Moreover,

$$I^\mu_{-\infty}(\hat{z}) = I^\mu_{t_1,0}(z).$$

Therefore, we need to show that $V^\mu(x,y) \le M^\mu(x,y)$, for all $(x,y)\in\mathcal{H}$.

If $M^\mu(x,y) = +\infty$, there is nothing to prove. So, assume that $M^\mu(x,y) < +\infty$. In view of Theorem 5.1, there is a minimizer $z^\mu \in C((-\infty,0];\mathcal{H}_{1+2\beta})$, with $z^\mu(0) = (x,y)$ such that

$$M^\mu(x,y) = I^\mu_{-\infty}(z^\mu).$$

Moreover, thanks to (4.14),

$$\lim_{t\to-\infty} |z^\mu(t)|_{\mathcal{H}_{1+2\beta}} = 0.$$

This means that for $\varepsilon > 0$ fixed, there exists $t_\varepsilon < 0$ such that

$$|z^\mu(t)|_{\mathcal{H}_{1+2\beta}} < \varepsilon, \qquad t \le t_\varepsilon.$$

Now, let us denote $z_\varepsilon = z^\mu(t_\varepsilon)$, and let us define

$$\psi_\varepsilon = \big(L^\mu_{t_\varepsilon - T_\mu, t_\varepsilon}\big)^{-1} z_\varepsilon,$$

where $T_\mu > 0$ is the time introduced in Theorem 4.1. Then, by Theorem 4.1,

$$|\psi_\varepsilon|_{L^2((t_\varepsilon - T_\mu, t_\varepsilon);H)} \le c(\mu, T_\mu)|z_\varepsilon|_{\mathcal{H}_{1+2\beta}} \le \varepsilon c(\mu, T_\mu). \tag{5.12}$$

Next, for $t \in [t_\varepsilon - T_\mu, t_\varepsilon]$, we define

$$\zeta^\mu_\varepsilon(t) = \int_{t_\varepsilon - T_\mu}^t S_\mu(t-s) Q_\mu \psi_\varepsilon(s)\, ds.$$

Clearly we have $\zeta_\varepsilon^\mu(t_\varepsilon - T_\mu) = 0$ and $\zeta_\varepsilon^\mu(t_\varepsilon) = z_\varepsilon$. Moreover, thanks to (2.8), we have

$$\begin{aligned}|\zeta_\varepsilon^\mu(t)|_{\mathcal{H}_{1+2\beta}} &\leq \frac{M_\mu}{\mu}\int_{t_\varepsilon - T_\mu}^t e^{-\omega_\mu(t-s)}|Q\psi_\varepsilon(s)|_{H^{2\beta}}\,ds\\ &\leq \frac{cM_\mu}{\mu}\int_{t_\varepsilon - T_\mu}^t e^{-\omega_\mu(t-s)}|\psi_\varepsilon(s)|_H\,ds,\end{aligned}$$

so that, due to (5.12),

$$\begin{aligned}&\int_{t_\varepsilon - T_\mu}^{t_\varepsilon}|\zeta_\varepsilon^\mu(t)|^2_{\mathcal{H}_{1+2\beta}}\,dt\\ &\quad\leq \left(\frac{cM_\mu}{\mu}\right)^2\int_{t_\varepsilon - T_\mu}^{t_\varepsilon}\left(\int_{t_\varepsilon - T_\mu}^t M_\mu e^{-\omega_\mu(t-s)}|\psi_\varepsilon(s)|_H\,ds\right)^2 dt \qquad (5.13)\\ &\quad\leq \left(\frac{M_\mu}{\omega_\mu\mu}\right)^2|\psi_\varepsilon|^2_{L^2((t_\varepsilon - T_\mu, t_\varepsilon);H)} \leq \left(\frac{M_\mu}{\omega_\mu\mu}\right)^2 c(\mu, T_\mu)^2\varepsilon^2.\end{aligned}$$

Since

$$\begin{aligned}\zeta_\varepsilon^\mu(t) &= \int_{t_\varepsilon - T_\mu}^t S_\mu(t-s)B_\mu\big(z_\varepsilon^\mu(s)\big)\,ds\\ &\quad+ \int_{t_\varepsilon - T_\mu}^t S_\mu(t-s)Q_\mu\big(\psi_\varepsilon(s) - Q^{-1}B\big(\Pi_1 z_\varepsilon^\mu(s)\big)\big)\,ds,\end{aligned}$$

we have

$$I^\mu_{t_\varepsilon - T_\mu, t_\varepsilon}\big(\zeta_\varepsilon^\mu\big) \leq 2|\psi_\varepsilon|^2_{L^2((t_\varepsilon - T_\mu, t_\varepsilon);H)} + 2\big|Q^{-1}B\big(\Pi_1 z_\varepsilon^\mu\big)\big|^2_{L^2((t_\varepsilon - T_\mu, t_\varepsilon);H)}.$$

Then, due to Hypothesis 2,

$$\begin{aligned}\big|Q^{-1}B\big(\Pi_1\zeta_\varepsilon^\mu(s)\big)\big|_H &\leq c\big|B\big(\Pi_1\zeta_\varepsilon^\mu(s)\big)\big|_{H^{2\beta}} \leq c\gamma_{2\beta}\big|\Pi_1\zeta_\varepsilon^\mu(s)\big|_{H^{2\beta}}\\ &\leq c\gamma_{2\beta}\big|\zeta_\varepsilon^\mu(s)\big|_{\mathcal{H}_{2\beta}},\end{aligned}$$

and thanks to (5.12) and (5.13), we can conclude

$$I^\mu_{t_\varepsilon - T_\mu, t_\varepsilon}\big(\zeta_\varepsilon^\mu\big) \leq c_\mu\varepsilon^2. \qquad (5.14)$$

Finally, we define

$$\hat{\zeta}_\varepsilon^\mu(t) = \begin{cases}\zeta_\varepsilon^\mu(t), & t_\varepsilon - T_\mu \leq t \leq t_\varepsilon,\\ z^\mu(t), & t > t_\varepsilon.\end{cases} \qquad (5.15)$$

It is immediate to check that $\hat{\zeta}_\varepsilon^\mu \in C([t_\varepsilon - T_\mu, 0]; \mathcal{H})$, $\hat{\zeta}_\varepsilon^\mu = 0$ and $\hat{\zeta}_\varepsilon^\mu(0) = (x, y)$. Moreover, thanks to (5.14),

$$\begin{aligned}I^\mu_{t_\varepsilon - T_\mu, 0}\big(\hat{\zeta}_\varepsilon^\mu\big) &\leq I^\mu_{-\infty}\big(z^\mu\big) + I^\mu_{t_\varepsilon - T_\mu, t_\varepsilon}\big(\zeta_\varepsilon^\mu\big)\\ &= M^\mu(x, y) + I^\mu_{t_\varepsilon - T, t_\varepsilon}\big(\zeta_\varepsilon^\mu\big) \leq M^\mu(x, y) + c_\mu\varepsilon^2.\end{aligned} \qquad (5.16)$$

Due to the arbitrariness of $\varepsilon > 0$, this implies

$$V^{\mu}(x, y) \leq M^{\mu}(x, y),$$

and then (5.8) follows.

Finally, in order to prove (5.9), we just notice that there exists $\{y_n\} \subset H^{-1}$ such that

$$\bar{V}_{\mu}(x) = \lim_{n\to\infty} V^{\mu}(x, y_n)$$

and

$$V^{\mu}(x, y_n) = I^{\mu}_{-\infty}(z_n),$$

for some $\{z_n\} \subset C((-\infty, 0]; \mathcal{H})$ such that $z_n(0) = (x, y_n)$ and

$$\lim_{t\to-\infty} |z_n(t)|_{\mathcal{H}} = 0.$$

As

$$\sup_{n\in\mathbb{N}} I^{\mu}_{-\infty}(z_n) < \infty,$$

due to Theorem 5.1, we have that there exists a subsequence $\{z_{n_k}\}$ which is uniformly convergent on bounded sets to some $z \in C((-\infty, 0]; \mathcal{H})$. In particular, $\Pi_1 z(0) = x$ and $|z(t)|_{\mathcal{H}} \to 0$, as $t \to -\infty$. Since $I^{\mu}_{-\infty}$ is lower semi-continuous, we have

$$I^{\mu}_{-\infty}(z) \leq \liminf_{k\to\infty} I^{\mu}_{-\infty}(z_{n_k}) = \bar{V}_{\mu}(x),$$

and then $\bar{V}_{\mu}(x) = I^{\mu}_{-\infty}(z)$, so that (5.9) holds true. $\square$

The characterization of $V^{\mu}(x, y)$ and $\bar{V}_{\mu}(x)$ given in Theorem 5.3, implies that V^{μ} and $\bar{V}_{\mu}$ have compact level sets.

THEOREM 5.4. *For any $\mu > 0$ and $r \geq 0$, the level sets*

$$K^{\mu}(r) = \{(x, y) \in \mathcal{H} : V^{\mu}(x, y) \leq r\}$$

and

$$K_{\mu}(r) = \{x \in H : \bar{V}_{\mu}(x) \leq r\}$$

are compact, in $\mathcal{H}$ and H, respectively.

PROOF. We prove this result for V^{μ} and K^{μ}, as the proof for $\bar{V}_{\mu}$ and K_{μ} is completely analogous. Let $\{(x_n, y_n)\}_{n\in\mathbb{N}} \subset K^{\mu}(r)$. In view of Theorem 5.3, for each $n \in \mathbb{N}$ there exists $z^n \in C((-\infty, 0]; \mathcal{H})$, with $z^n(0) = (x_n, y_n)$, and $|z^n(t)|_H \to 0$, as $t \downarrow -\infty$, such that $V^{\mu}(x_n, y_n) = I^{\mu}_{-\infty}(z^n)$. As $I^{\mu}_{-\infty}(z^n) \leq r$, and the level sets of $I^{\mu}_{-\infty}$ are compact in $C((-\infty, 0]; \mathcal{H})$, as shown in Theorem 5.1, there exists a subsequence $\{z^{n_k}\} \subseteq \{z^n\}$ converging to some $\hat{z} \in C((-\infty, 0]; \mathcal{H})$, with $I^{\mu}_{-\infty}(\hat{z}) \leq r$. Since

$$\lim_{k\to\infty} (x_{n_k}, y_{n_k}) = \lim_{k\to\infty} z^{n_k}(0) = \hat{z}(0) =: (\hat{x}, \hat{y}) \qquad \text{in } \mathcal{H},$$

due to Theorem 5.3, we have

$$V^{\mu}(\hat{x}, \hat{y}) \leq I^{\mu}_{-\infty}(\hat{z}) \leq r,$$

so that $(\hat{x}, \hat{y}) \in K^{\mu}(r)$. □

6. Continuity of V^{μ} and $\bar{V}_{\mu}$.

As a consequence of Theorem 5.4, the mappings $V^{\mu} : \mathcal{H} \to [0, +\infty]$ and $\bar{V}_{\mu} : H \to [0, +\infty]$ are lower semicontinuous. Our purpose here is to prove that the mappings

$$V^{\mu} : \mathcal{H}_{1+2\beta} \to [0, +\infty), \qquad \bar{V}_{\mu} : H^{1+2\beta} \to [0, +\infty)$$

are well defined and continuous, uniformly in $0 < \mu < 1$.

LEMMA 6.1. *Let us fix $(x, y) \in \mathcal{H}_{1+2\beta}$ and $\mu > 0$, and let $z(t) = S_{\mu}(-t)(x, -y)$, $t \leq 0$. Then if we denote $\varphi(t) = \Pi_1 z(t)$, we have that φ is a weak solution to*

$$\begin{cases} \mu \dfrac{\partial^2 \varphi}{\partial t^2}(t) = A\varphi(t) + \dfrac{\partial \varphi}{\partial t}(t), \qquad t \leq 0, \\ \varphi(0) = x, \qquad \dfrac{\partial \varphi}{\partial t}(0) = y \end{cases} \tag{6.1}$$

and

$$\begin{aligned} &\frac{1}{2}\int_{-\infty}^{0} \left| Q^{-1}\left(\mu \frac{\partial^2 \varphi}{\partial t^2}(t) + \frac{\partial \varphi}{\partial t}(t) - A\varphi(t)\right)\right|_H^2 dt \\ &\qquad = \left|(-A)^{1/2} Q^{-1} x\right|_H^2 + \mu \left|Q^{-1} y\right|_H^2. \end{aligned} \tag{6.2}$$

Moreover, $\varphi \in L^2((-\infty, 0); H^{1+2\beta})$ and

$$\int_{-\infty}^{0} |\varphi(t)|^2_{H^{1+2\beta}}\, dt \leq c\left(1 + \mu + \mu^2\right)\left|(x, y)\right|^2_{\mathcal{H}_{1+2\beta}}. \tag{6.3}$$

PROOF. The weak formulation (6.1) is clear because for $t < 0$,

$$\frac{\partial z}{\partial t}(t) = -A_{\mu} S_{\mu}(-t)(x, -y) = \left(-\Pi_2 z(t), -\frac{1}{\mu} A\varphi(t) + \frac{1}{\mu}\Pi_2 z(t)\right),$$

so that

$$\mu \frac{\partial^2 \varphi}{\partial t^2}(t) = A\varphi(t) + \frac{\partial \varphi}{\partial t}(t).$$

Moreover,

$$\frac{\partial \varphi}{\partial t}(0) = -\Pi_2 z(0) = y.$$

Now, property (6.2) can be proven by noticing that

$$
\begin{aligned}
&\frac{1}{2}\int_{-\infty}^{0}\left|Q^{-1}\left(\mu\frac{\partial^2\varphi}{\partial t^2}(t)+\frac{\partial\varphi}{\partial t}(t)-A\varphi(t)\right)\right|_H^2 dt\\
&\quad=\frac{1}{2}\int_{-\infty}^{0}\left|Q^{-1}\left(\mu\frac{\partial^2\varphi}{\partial t^2}(t)-\frac{\partial\varphi}{\partial t}(t)-A\varphi(t)\right)\right|_H^2 dt\\
&\qquad+2\int_{-\infty}^{0}\left\langle Q^{-1}\frac{\partial\varphi}{\partial t}(t), Q^{-1}\left(\mu\frac{\partial^2\varphi}{\partial t^2}(t)-A\varphi(t)\right)\right\rangle_H dt\\
&\quad=|Q^{-1}(-A)^{1/2}x|_H^2+\mu|Q^{-1}y|_H^2-\lim_{t\to-\infty}|C_\mu^{-1/2}z(t)|_H^2.
\end{aligned}
$$

Then (6.2) follows from (2.8), as

$$
\left|C_\mu^{-1/2}z(t)\right|_H\le\left|z(t)\right|_{\mathcal{H}_{1+2\beta}}\le M_\mu e^{-\omega_\mu t}\left|(x,y)\right|_{\mathcal{H}_{1+2\beta}}\to 0 \qquad \text{as } t\downarrow-\infty.
$$

Finally, to obtain estimate (6.3), we notice that if

$$
\varphi(t)=\Pi_1 S_\mu(-t)(x,-y),
$$

then by (3.12),

$$
|\varphi(t)|_{H^{1+2\beta}}^2=\frac{1}{2}\frac{d}{dt}\left|\varphi(t)-\mu\frac{\partial\varphi}{\partial t}(t)\right|_{H^{2\beta}}^2+\frac{\mu}{2}\frac{d}{dt}|\varphi(t)|_{H^{1+2\beta}}^2.
$$

Integrating, we obtain

$$
\int_{-\infty}^{0}|\varphi(t)|_{H^{1+2\beta}}^2\,dt=\frac{1}{2}|x+\mu y|_{H^{2\beta}}^2+\frac{\mu}{2}|x|_{H^{1+2\beta}}^2,
$$

which yields (6.3). □

As a consequence of the previous lemma, we obtain the following bounds for $V^\mu(x,y)$ and $\bar{V}_\mu(x)$:

COROLLARY 6.2. *There exists $c>0$ such that for any $\mu>0$ and $(x,y)\in\mathcal{H}_{1+2\beta}$, we have*

$$
V^\mu(x,y)\le c\left(1+\mu+\mu^2\right)\left|(x,y)\right|_{\mathcal{H}_{1+2\beta}}^2 \tag{6.4}
$$

and

$$
\bar{V}_\mu(x)\le c(1+\mu)|x|_{H^{1+2\beta}}^2. \tag{6.5}
$$

PROOF. The proof is based on the fact that

$$
V^\mu(x,y)\le I_{-\infty}^{\mu}\left(\Pi_1 S_\mu(-\cdot)(x,-y)\right)
$$

and

$$\bar{V}_\mu(x) \le I^\mu_{-\infty}\big(\Pi_1 S_\mu(-\cdot)(x,0)\big).$$

Now, if we set $z(t) = S_\mu(-t)(x,-y)$ and $\varphi(t) = \Pi_1 z(t)$, due to Hypothesis 2, we have

$$\begin{aligned} I^\mu_{-\infty}(z) &= \frac{1}{2}\int_{-\infty}^0 \left| Q^{-1}\left(\mu \frac{\partial^2\varphi}{\partial t^2}(t) + \frac{\partial \varphi}{\partial t}(t) - A\varphi(t) - B\big(\varphi(t)\big)\right)\right|_H^2 dt \\ &\le \int_{-\infty}^0 \left| Q^{-1}\left(\frac{\partial^2\varphi}{\partial t^2}(t) + \frac{\partial \varphi}{\partial t}(t) - A\varphi(t)\right)\right|_H^2 dt + c\gamma_{2\beta}^2 \int_{-\infty}^0 |\varphi(t)|^2_{H^{2\beta}}\,dt. \end{aligned}$$

From (6.2) and (6.3), this gives (6.4). Finally, (6.5) is a consequence of (6.4) and because of the way $\bar{V}_\mu(x)$ has been defined. □

Now, we can prove the continuity of V^μ and $\bar{V}_\mu$.

THEOREM 6.3. *For each $\mu > 0$, the mappings $V^\mu : \mathcal{H}_{1+2\beta} \to [0,+\infty)$ and $\bar{V}_\mu : H^{1+2\beta} \to [0,+\infty)$ are well defined and continuous. Moreover,*

$$\begin{aligned} &\lim_{n\to\infty} \big|(x,y) - (x_n,y_n)\big|_{\mathcal{H}_{1+2\beta}} = 0 \\ &\qquad \Longrightarrow \quad \lim_{n\to\infty} \sup_{0<\mu<1} \big|V^\mu(x,y) - V^\mu(x_n,y_n)\big| = 0 \end{aligned} \tag{6.6}$$

and

$$\lim_{n\to\infty} |x - x_n|_{H^{1+2\beta}} = 0 \quad \Longrightarrow \quad \lim_{n\to\infty} \sup_{0<\mu<1} \big|\bar{V}_\mu(x) - \bar{V}_\mu(x_n)\big| = 0. \tag{6.7}$$

PROOF. In view of Corollary 6.2, if $(x,y) \in \mathcal{H}_{1+2\beta}$, then $V^\mu(x,y) < +\infty$, and if $x \in H^{1+2\beta}$, then $\bar{V}_\mu(x) < +\infty$. On the other hand, if $V^\mu(x,y) < +\infty$, thanks to Theorem 5.3, there exists $z^\mu \in C((-\infty,0];\mathcal{H})$ such that

$$V^\mu(x,y) = I^\mu_{-\infty}\big(z^\mu\big), \qquad z^\mu(0) = (x,y).$$

According to Lemma 4.3, this implies that $z^\mu \in C((-\infty,0];\mathcal{H}_{1+2\beta})$, so that $(x,y) = z^\mu(0) \in \mathcal{H}_{1+2\beta}$. Analogously, if $\bar{V}_\mu(x) < +\infty$, we can prove that $x \in H^{1+2\beta}$, so that we can conclude that the mappings V^μ and $\bar{V}_\mu$ are well defined in $\mathcal{H}_{1+2\beta}$ and $H^{1+2\beta}$, respectively.

Now, in order to prove (6.6), by using again Theorem 5.3, for each $n \in \mathbb{N}$ we can find $z^\mu_n \in C((-\infty,0];\mathcal{H})$ such that

$$V^\mu(x_n,y_n) = I^\mu_{-\infty}\big(z^\mu_n\big), \qquad z^\mu_n(0) = (x_n,y_n).$$

Then if we define

$$\hat{z}^\mu_n(t) = S_\mu(-t)(x - x_n, y - y_n)$$

and

$$\varphi_n^\mu(t) = \Pi_1 z_n^\mu(t), \qquad \hat{\varphi}_n^\mu(t) = \Pi_1 \hat{z}_n^\mu(t), \qquad t \le 0,$$

we have $\hat{z}_n^\mu(0) = (x - x_n, y - y_n)$, and for any $\varepsilon > 0$,

$$\begin{aligned}
V^\mu(x, y) &\le I_{-\infty}^\mu\big(z_n^\mu + \hat{z}_n^\mu\big) \\
&\le \frac{1}{2}\int_{-\infty}^0 \bigg| Q^{-1}\bigg(\mu \frac{\partial^2 \varphi_n^\mu}{\partial t^2}(t) + \frac{\partial \varphi_n^\mu}{\partial t}(t) - A\varphi_n^\mu(t) + B\big(\Pi_1 \varphi_n^\mu(t)\big)\bigg) \\
&\qquad\qquad + Q^{-1}\bigg(\mu \frac{\partial^2 \hat{\varphi}_n^\mu}{\partial t^2}(t) + \frac{\partial \hat{\varphi}_n^\mu}{\partial t}(t) - A\hat{\varphi}_n^\mu(t)\bigg) \\
&\qquad\qquad\qquad\qquad + Q^{-1}\big(B\big(\varphi_n^\mu + \hat{\varphi}_n^\mu(t)\big) - B\big(\varphi_n^\mu(t)\big)\big)\bigg|_H^2 \, dt \\
&\le (1+\varepsilon) I_{-\infty}^\mu\big(z_n^\mu\big) \\
&\quad + \bigg(1 + \frac{1}{\varepsilon}\bigg)\int_{-\infty}^0 \bigg| Q^{-1}\bigg(\frac{\partial^2 \hat{\varphi}_n^\mu}{\partial t^2}(t) + \frac{\partial \hat{\varphi}_n^\mu}{\partial t}(t) - A\hat{\varphi}_n^\mu(t)\bigg)\bigg|_H^2 \, dt \\
&\quad + c\bigg(1 + \frac{1}{\varepsilon}\bigg)\int_{-\infty}^0 \big|\hat{\varphi}_n^\mu(t)\big|_{H^{2\beta}}^2 \, dt.
\end{aligned}$$

Now, by (6.2) and (6.3), we see that for $0 < \mu < 1$,

$$\begin{aligned}
V^\mu(x, y) \le (1+\varepsilon) V^\mu(x_n, y_n) & \\
+ c\bigg(1 + \frac{1}{\varepsilon}\bigg)&\big|(x - x_n, y - y_n)\big|_{\mathcal{H}_{1+2\beta}}^2 \\
+ c^2\bigg(1 + \frac{1}{\varepsilon}\bigg)&\big|(x - x_n, y - y_n)\big|_{\mathcal{H}_{2\beta}}.
\end{aligned}$$

If we follow the same procedure with z^μ as the minimizer of $V^\mu(x, y)$ and

$$\hat{z}_n^\mu(t) = S_\mu(-t)(x_n - x, y - y_n),$$

we see that for $0 < \mu < 1$,

$$\begin{aligned}
V^\mu(x_n, y_n) \le (1+\varepsilon) V^\mu(x, y) & \\
+ c\big(1 + \varepsilon^{-1}\big)&\big|(x - x_n, y - y_n)\big|_{\mathcal{H}_{1+2\beta}}^2 \\
+ c\big(1 + \varepsilon^{-1}\big)&\big|(x - x_n, y - y_n)\big|_{\mathcal{H}_{2\beta}}.
\end{aligned}$$

From these two estimates and Corollary 6.2, we see that

$$\begin{aligned}
&\sup_{0<\mu<1} \big|V^\mu(x, y) - V^\mu(x_n, y_n)\big| \\
&\qquad \le c\varepsilon\big|(x, y)\big|_{\mathcal{H}_{1+2\beta}}^2 + c\big(1 + \varepsilon^{-1}\big)\big|(x - x_n, y - y_n)\big|_{\mathcal{H}_{1+2\beta}}^2,
\end{aligned}$$

so that

$$\limsup_{n\to\infty}\sup_{0<\mu<1}\big|V^\mu(x,y)-V^\mu(x_n,y_n)\big|\leq c\varepsilon\big|(x,y)\big|^2_{\mathcal{H}_{1+2\beta}}.$$

Due to the arbitrariness of $\varepsilon>0$, (6.6) follows. The proof of (6.7) is completely analogous to the proof of (6.6), and for this reason we omit it. □

7. Upper bound. In this section we show that for any closed set $N\subset H$,

$$\limsup_{\mu\downarrow 0}\inf_{x\in N}\bar{V}_\mu(x)\leq\inf_{x\in N}V(x). \tag{7.1}$$

First of all, we notice that if $I_{-\infty}(\varphi)<\infty$, then

$$\varphi\in L^2\big((-\infty,0);H^{2(1+\beta)}\big),\qquad \frac{\partial\varphi}{\partial t}\in L^2\big((-\infty,0);H^{2\beta}\big), \tag{7.2}$$

and

$$I_{-\infty}(\varphi)=\frac{1}{2}\int_{-\infty}^0\left|Q^{-1}\left(\frac{\partial\varphi}{\partial t}(t)-A\varphi(t)-B\big(\varphi(t)\big)\right)\right|_H^2dt. \tag{7.3}$$

Actually, if φ solves

$$\varphi(t)=\int_{-\infty}^t e^{(t-s)A}B\big(\varphi(s)\big)\,ds+\int_{-\infty}^t e^{(t-s)A}Q\psi(s)\,ds,$$

then we can check that (7.2) holds and

$$\psi(t)=Q^{-1}\left(\frac{\partial}{\partial t}\varphi(t)-A\varphi(t)-B\big(\varphi(t)\big)\right),$$

so that (7.3) follows. Moreover, if

$$\varphi\in L^2\big((-\infty,0);H^{2(1+\beta)}\big),\qquad \frac{\partial\varphi}{\partial t},\frac{\partial^2\varphi}{\partial t^2}\in L^2\big((-\infty,0);H^{2\beta}\big),$$

then

$$I^\mu_{-\infty}(z)=\frac{1}{2}\int_{-\infty}^0\left|Q^{-1}\left(\mu\frac{\partial^2\varphi}{\partial t^2}\varphi(t)+\frac{\partial}{\partial t}\varphi(t)-A\varphi(t)-B\big(\varphi(t)\big)\right)\right|_H^2dt,$$

where

$$z(t)=\left(\varphi(t),\frac{\partial\varphi}{\partial t}(t)\right).$$

Actually, if $I^\mu_{-\infty}(z)<\infty$, then z solves

$$z(t)=\int_{-\infty}^t S_\mu(t-s)B_\mu\big(z(s)\big)\,ds+\int_{-\infty}^t S_\mu(t-s)Q_\mu\psi(s)\,ds$$

so that

$$\psi(t) = Q^{-1}\Big(\mu \frac{\partial^2 \varphi}{\partial t^2}(t) + \frac{\partial \varphi}{\partial t}(t) - A\varphi(t) - B\big(\varphi(t)\big)\Big)$$

weakly.

In particular, as in [8], where the finite-dimensional case is studied, this means

$$\begin{aligned} I^{\mu}_{-\infty}(z) &= I_{-\infty}(\varphi) + \frac{\mu^2}{2}\int_{-\infty}^0 \left| Q^{-1}\frac{\partial^2\varphi}{\partial t^2}(t)\right|_H^2 dt \\ &\quad + \mu \int_{-\infty}^0 \Big\langle Q^{-1}\frac{\partial^2\varphi}{\partial t^2}(t), Q^{-1}\Big(\frac{\partial\varphi}{\partial t}(t) - A\varphi(t) - B\big(\varphi(t)\big)\Big)\Big\rangle_H dt, \end{aligned} \tag{7.4}$$

where $\varphi(t) = \Pi_1 z(t)$, as long as all of these terms are finite.

Now, for any $\mu > 0$, let us define

$$\rho_\mu(t) = \frac{1}{\mu^\alpha}\rho\Big(\frac{t}{\mu^\alpha}\Big), \qquad t \in \mathbb{R}, \tag{7.5}$$

for some $\alpha > 0$ to be chosen later, where $\rho \in C^\infty(\mathbb{R})$ is the usual mollifier function such that

$$\operatorname{supp}(\rho) \subset\subset [0,2], \qquad \int_{\mathbb{R}} \rho(s)\,ds = 1, \qquad 0 \le \rho \le 1.$$

This scaling ensures that

$$\int_{\mathbb{R}} \rho_\mu(s)\,ds = 1.$$

Next, we define φ_μ as the convolution

$$\varphi_\mu(t) = \int_{-\infty}^0 \rho_\mu(t-s)\varphi(s)\,ds. \tag{7.6}$$

LEMMA 7.1. *Assume that*

$$\varphi \in L^2\big((-\infty,0); H^{2(1+\beta)}\big) \cap C\big((-\infty,0]; H^{1+2\beta}\big), \qquad \frac{\partial\varphi}{\partial t} \in L^2\big((-\infty,0); H^{2\beta}\big)$$

with

$$\varphi(0) = x \in H^{1+2\beta}, \qquad \lim_{t\to-\infty} |\varphi(t)|_{H^{1+2\beta}} = 0.$$

Then

$$\begin{aligned} &\varphi_\mu \in L^2\big((-\infty,0); H^{2(1+\beta)}\big) \cap C\big((-\infty,0]; H^{1+2\beta}\big), \\ &\qquad\qquad \frac{\partial\varphi_\mu}{\partial t} \in L^2\big((-\infty,0); H^{2\beta}\big), \end{aligned} \tag{7.7}$$

and

$$\lim_{t\to-\infty}\sup_{\mu>0}|\varphi_\mu(t)|_{H^{1+2\beta}}=0. \tag{7.8}$$

Moreover,

$$\frac{\partial^2\varphi_\mu}{\partial t^2}\in L^2\big((-\infty,0);H^{2\beta}\big),$$

and for all $\mu>0$,

$$\left|\frac{\partial^2\varphi_\mu}{\partial t^2}\right|_{L^2((-\infty,0);H^{2\beta})}\le\frac{c}{\mu^\alpha}\left|\frac{\partial\varphi_\mu}{\partial t}\right|_{L^2((-\infty,0);H^{2\beta})}. \tag{7.9}$$

PROOF. Recall that $\operatorname{supp}(\rho)\subset\subset[0,2]$. Therefore,

$$\varphi_\mu(t)=\int_{t-2\mu^\alpha}^t\rho_\mu(t-s)\varphi(s)\,ds,$$

and it follows by Cauchy–Schwarz that

$$\begin{aligned}&\int_{-\infty}^0\left|\int_{t-2\mu^\alpha}^t\rho_\mu(t-s)\varphi(s)\,ds\right|^2_{H^{2(1+\beta)}}dt\\&\quad\le\int_{-\infty}^0\left(\int_0^{2\mu^\alpha}\rho_\mu^2(s)\,ds\right)\left(\int_{t-2\mu^\alpha}^t|\varphi(s)|^2_{H^{2(1+\beta)}}\,ds\right)dt.\end{aligned}$$

Therefore, as

$$\int_0^{2\mu^\alpha}\rho_\mu^2(s)\,ds\le\frac{2}{\mu^\alpha},$$

we get

$$\begin{aligned}|\varphi_\mu|^2_{L^2((-\infty,0);H^{2(1+\beta)})}\,dt&\le\frac{2}{\mu^\alpha}\int_{-\infty}^0\int_{t-2\mu^\alpha}^t|\varphi(s)|^2_{H^{2(1+\beta)}}\,ds\,dt\\&\le\frac{2\mu^\alpha}{\mu^\alpha}\int_{-\infty}^0|\varphi(s)|^2_{H^{2(1+\beta)}}\,ds\\&=2|\varphi|^2_{L^2((-\infty,0);H^{2(1+\beta)})}.\end{aligned} \tag{7.10}$$

Next, since

$$\lim_{t\to-\infty}|\varphi(t)|_{H^{1+2\beta}}=0,$$

we have that $\varphi:(-\infty,0]\to H^{1+2\beta}$ is uniformly continuous. Therefore, as

$$\begin{aligned}&\left|\int_{-\infty}^{t_1}\rho_\mu(t_1-s)\varphi(s)\,ds-\int_{-\infty}^{t_2}\rho_\mu(t_2-s)\varphi(s)\,ds\right|_{H^{1+2\beta}}\\&\quad=\left|\int_0^\infty\rho_\mu(s)\big(\varphi(t_1-s)-\varphi(t_2-s)\big)\,ds\right|_{H^{1+2\beta}},\end{aligned}$$

we can conclude that φ_μ is uniformly continuous too, with values in $H^{1+2\beta}$. Finally, since

$$\frac{\partial \varphi_\mu}{\partial t}(t) = \int_0^\infty \rho_\mu(s) \frac{\partial \varphi}{\partial t}(t-s)\, ds,$$

by proceeding as above we get

$$\frac{\partial \varphi_\mu}{\partial t} \in L^2\big((-\infty, 0); H^{2\beta}\big),$$

so that, thanks to (7.10), we can conclude that (7.7) holds true.

Concerning (7.8), let us fix $\varepsilon > 0$. Then there exists $T_\varepsilon > 0$ such that

$$|\varphi(t)|_{H^{1+2\beta}} < \varepsilon, \qquad t < -T_\varepsilon.$$

Then, for $t < -T_\varepsilon$, we have

$$\begin{aligned} \left| \int_{-\infty}^t \rho_\mu(t-s)\varphi(s)\, ds \right|_{H^{1+2\beta}} &\le \int_{-\infty}^t \rho_\mu(t-s)|\varphi(s)|_{H^{1+2\beta}}\, ds \\ &\le \varepsilon \int_0^\infty \rho_\mu(s)\, ds = \varepsilon, \end{aligned}$$

and this yields (7.8).

Finally, let us prove (7.9). As

$$\frac{\partial \varphi_\mu}{\partial t}(t) = \int_{-\infty}^0 \rho_\mu(t-s) \frac{\partial \varphi}{\partial s}(s)\, ds,$$

we have

$$\frac{\partial^2 \varphi_\mu}{\partial t^2}(t) = \int_{-\infty}^0 \frac{d}{dt}\rho_\mu(t-s)\frac{\partial \varphi}{\partial s}(s)\, ds = \frac{1}{\mu^{2\alpha}} \int_{-\infty}^0 \rho'\left(\frac{t-s}{\mu^\alpha}\right) \frac{\partial \varphi}{\partial s}(s)\, ds.$$

This yields

$$\begin{aligned} &\int_{-\infty}^0 \left| \frac{\partial^2 \varphi_\mu}{\partial t^2}(t) \right|_{H^{2\beta}}^2 dt \\ &\quad = \frac{1}{\mu^{4\alpha}} \int_{-\infty}^0 \left| \int_{t-2\mu^\alpha}^t \rho'\left(\frac{t-s}{\mu^\alpha}\right) \frac{\partial}{\partial s}\varphi(s)\, ds \right|_{H^{2\beta}}^2 dt \\ &\quad \le \frac{1}{\mu^{4\alpha}} \int_{-\infty}^0 \left(\int_{t-2\mu^\alpha}^t \left(\rho'\left(\frac{t-s}{\mu^\alpha}\right)\right)^2 ds \right) \left(\int_{t-2\mu^\alpha}^t \left| \frac{\partial \varphi}{\partial s}(s)\, ds \right|_{H^{2\beta}}^2 ds \right) dt \\ &\quad \le \frac{2|\rho'|_\infty^2}{\mu^{3\alpha}} \int_{-\infty}^0 \int_{t-2\mu^\alpha}^t \left| \frac{\partial \varphi}{\partial s}(s) \right|_{H^{2\beta}}^2 ds\, dt \le \frac{c}{\mu^{2\alpha}} \int_{-\infty}^0 \left| \frac{\partial \varphi}{\partial s}(s) \right|_{H^{2\beta}}^2 ds. \qquad \square \end{aligned}$$

The following approximation results hold:

LEMMA 7.2. *Under the same assumptions of Lemma 7.1, we have*

$$\lim_{\mu\to 0}\big|x-\varphi_\mu(0)\big|_{H^{1+2\beta}}=0 \tag{7.11}$$

and

$$\lim_{\mu\to 0}\sup_{t\leq 0}\big|\varphi_\mu(t)-\varphi(t)\big|_{H^{1+2\beta}}=0. \tag{7.12}$$

Moreover,

$$\lim_{\mu\to 0}|\varphi_\mu-\varphi|_{L^2((-\infty,0);H^{2(1+\beta)})}=0, \tag{7.13}$$

and

$$\lim_{\mu\to 0}\left|\frac{\partial\varphi_\mu}{\partial t}-\frac{\partial\varphi}{\partial t}\right|_{L^2((-\infty,0);H^{2\beta})}=0. \tag{7.14}$$

PROOF. We have

$$\varphi_\mu(0)-x=\int_{-\infty}^0\rho_\mu(-s)\big(\varphi(s)-\varphi(0)\big)\,ds,$$

so that, by the continuity of φ in $H^{1+2\beta}$, (7.11) follows.

In order to prove (7.12), we have

$$\big|\varphi_\mu(t)-\varphi(t)\big|_{H^{1+2\beta}}\leq\int_{-\infty}^t\rho_\mu(t-s)\big|\varphi(s)-\varphi(t)\big|_{H^{1+2\beta}}\,ds.$$

Now, as $\varphi:(-\infty,0]\to H^{1+2\beta}$ is uniformly continuous, for any fixed $\varepsilon>0$, there exists $\delta_\varepsilon>0$ such that

$$|t-s|<\delta_\varepsilon\quad\Longrightarrow\quad\big|\varphi(s)-\varphi(t)\big|_{H^{1+2\beta}}<\frac{\varepsilon}{2}.$$

Then if we pick μ small enough so that $\mu^\alpha<\delta_\varepsilon/2$,

$$\big|\varphi_\mu(t)-\varphi(t)\big|_{H^{1+2\beta}}\leq\int_{-\infty}^t\rho_\mu(t-s)\big|\varphi(s)-\varphi(t)\big|_{H^{1+2\beta}}\,ds\leq\int_{t-2\mu^\alpha}^t\frac{1}{\mu^\alpha}\frac{\varepsilon}{2}=\varepsilon,$$

uniformly in t. This proves (7.12).

Limit (7.13) can be proved using the fact that

$$\begin{aligned}
&|\varphi_\mu-\varphi|_{L^2((-\infty,0);H^{2(1+\beta)})}\\
&\quad=\sup_{|h|_{L^2((-\infty,0);H)}\leq 1}\int_{-\infty}^0\big\langle(-A)^{1+\beta}\big((\varphi_\mu)(t)-\varphi(t)\big),h(t)\big\rangle_H\,dt\\
&\quad=\sup_{|h|_{L^2((-\infty,0);H)}\leq 1}\int_{-\infty}^0\int_0^{2\mu^\alpha}\rho_\mu(s)\big\langle(-A)^{1+\beta}\big(\varphi(t-s)-\varphi(t)\big),h(t)\big\rangle_H\,ds\,dt\\
&\quad\leq\int_0^{2\mu^\alpha}\rho_\mu(s)\big|\varphi(\cdot-s)-\varphi(\cdot)\big|_{L^2((-\infty,0);H^{2(1+\beta)})}\,ds.
\end{aligned}$$

Because translation is continuous in L^2, this converges to 0 as $\mu \downarrow 0$. The same argument will show that (7.14) holds true. $\square$

Using these estimates we can prove the main result of this section.

THEOREM 7.3. *For any $x \in H^{1+2\beta}$, we have*

$$\limsup_{\mu \downarrow 0} \bar{V}_\mu(x) \leq V(x). \tag{7.15}$$

PROOF. Let φ be the minimizer of $V(x)$. This means $\varphi(0) = x$, (7.3) holds and $I_{-\infty}(\varphi) = V(x)$. For each $\mu > 0$, let φ_μ be the convolution given by (7.6), and let $x_\mu = \varphi_\mu(0)$.

It is clear that

$$\bar{V}_\mu(x_\mu) \leq I^\mu_{-\infty}(z_\mu), \tag{7.16}$$

where

$$z_\mu(t) = \left(\varphi_\mu(t), \frac{\partial \varphi_\mu}{\partial t}(t)\right), \qquad t \leq 0.$$

According to Lemma 7.1, we can apply (7.4), and we have

$$\begin{aligned}
I^\mu_{-\infty}(z_\mu) \leq{}& \frac{c\mu^2}{2} \int_{-\infty}^0 \left|\frac{\partial^2 \varphi_\mu}{\partial t^2}(t)\right|^2_{H^{2\beta}} dt + I_{-\infty}(\varphi_\mu) \\
&+ \mu \int_{-\infty}^0 \left\langle Q^{-1} \frac{\partial^2 \varphi_\mu}{\partial t^2}(t), Q^{-1}\left(\frac{\partial \varphi_\mu}{\partial t}(t) - A\varphi_\mu(t) - B\big(\varphi_\mu(t)\big)\right)\right\rangle_H dt \\
\leq{}& \frac{\mu^2}{2} \int_{-\infty}^0 \left|\frac{\partial^2 \varphi_\mu}{\partial t^2}(t)\right|^2_{H^{2\beta}} + I_{-\infty}(\varphi_\mu) \\
&+ \mu \left(\int_{-\infty}^0 \left|\frac{\partial^2 \varphi_\mu}{\partial t^2}(t)\right|^2_{H^{2\beta}} dt\right)^{1/2} \big(I_{-\infty}(\varphi_\mu)\big)^{1/2}.
\end{aligned}$$

By (7.9), this implies

$$\begin{aligned}
I^\mu_{-\infty}(z_\mu) \leq I_{-\infty}(\varphi_\mu) + c\mu^{2-2\alpha} \left|\frac{\partial \varphi}{\partial t}\right|^2_{L^2((-\infty,0);H^{2\beta})} \\
+ c\mu^{1-\alpha} \left|\frac{\partial \varphi}{\partial t}\right|_{L^2((-\infty,0);H^{2\beta})} \big(I_{-\infty}(\varphi_\mu)\big)^{1/2},
\end{aligned}$$

and by (7.13) and (7.14),

$$\lim_{\mu \downarrow 0} I_{-\infty}(\varphi_\mu) = I_{-\infty}(\varphi) = V(x).$$

Therefore, if we pick $\alpha < 1$ in (7.5), we get

$$\limsup_{\mu \downarrow 0} \bar{V}_\mu(x_\mu) \leq \limsup_{\mu \downarrow 0} I^\mu_{-\infty}(z_\mu) \leq V(x). \tag{7.17}$$

Since, in view of (7.11) and Theorem 6.3,

$$\limsup_{\mu\downarrow 0} \bar{V}_\mu(x_\mu) = \limsup_{\mu\downarrow 0} \bar{V}_\mu(x),$$

we can conclude that (7.15) holds. □

COROLLARY 7.4. *For any closed set $N \subset H$,*

$$\limsup_{\mu\to 0} \inf_{x\in N} \bar{V}_\mu(x) \le \inf_{x\in N} V(x). \tag{7.18}$$

PROOF. If $\inf_{x\in N} V(x) = +\infty$, then the theorem is trivially true, so we assume that this is not the case. Then by the compactness of the level sets of V and the closedness of N, there exists $x_0 \in N$ such that $V(x_0) = \inf_{x\in N} V(x)$. By (7.15), we can conclude, as

$$\limsup_{\mu\to 0} \inf_{x\in N} \bar{V}_\mu(x) \le \limsup_{\mu\downarrow 0} \bar{V}_\mu(x_0) \le V(x_0) = \inf_{x\in N} V(x). \qquad \square$$

8. Lower bound. Let $N \subset H$ be a closed set with $N \cap H^{1+2\beta} \neq \varnothing$. In particular, by Theorem 6.3 we have $\inf_{x\in N} \bar{V}_\mu(x) < +\infty$. Due to (5.9) and Theorem 5.1, there exists $z^\mu \in C((-\infty, 0]; \mathcal{H})$ such that

$$x^\mu := \Pi_1 z^\mu(0) \in N, \qquad I^\mu_{-\infty}(z^\mu) = \bar{V}_\mu(x^\mu) = \inf_{x\in N} \bar{V}_\mu(x).$$

Now, let $\psi^\mu \in L^2((-\infty, 0); H)$ be the minimal control such that

$$z^\mu(t) = \int_{-\infty}^t S_\mu(t-s) B_\mu(z^\mu(s))\, ds + \int_{-\infty}^t S_\mu(t-s) Q_\mu \psi^\mu(s)\, ds,$$

and

$$\inf_{x\in N} \bar{V}_\mu(x) = \bar{V}_\mu(x^\mu) = \frac{1}{2}|\psi^\mu|^2_{L^2((-\infty,0);H)}. \tag{8.1}$$

In what follows, we shall denote $y^\mu = \Pi_2 z^\mu(0)$. For any $\delta > 0$, we define the approximate control

$$\psi^{\mu,\delta}(t) = (I - \delta A)^{-1/2} \psi^\mu(t), \qquad t \le 0,$$

and in view of Corollary 5.2 we can define $z^{\mu,\delta}$ to be the solution to the corresponding control problem

$$z^{\mu,\delta}(t) = \int_{-\infty}^t S_\mu(t-s) B_\mu(z^{\mu,\delta}(s))\, ds + \int_{-\infty}^t S_\mu(t-s) Q_\mu \psi^{\mu,\delta}(s)\, ds.$$

Notice that, according to (4.14),

$$\lim_{t\to-\infty} |z^{\mu,\delta}|_{\mathcal{H}_{1+2\beta}} = 0.$$

Moreover, as $\psi^{\mu,\delta} \in L^2((-\infty,0); H^1)$, thanks to (4.14) we have

$$\lim_{t\to-\infty} |z^{\mu,\delta}|_{\mathcal{H}_{2(1+\beta)}} = 0.$$

In what follows, we shall denote $(x^{\mu,\delta}, y^{\mu,\delta}) = z^{\mu,\delta}(0)$.

LEMMA 8.1. *There exists $\mu_0 > 0$ such that*

$$\lim_{\delta\to 0} \sup_{\mu\le\mu_0} |x^\mu - x^{\mu,\delta}|^2_{H^{2\beta}} = 0. \tag{8.2}$$

PROOF. By (4.27), there exists $\mu_0 > 0$ such that for $\mu < \mu_0$,

$$|x^\mu - x^{\mu,\delta}|_{H^{2\beta}} \le c|\psi^\mu - \psi^{\mu,\delta}|_{L^2((-\infty,0);H^{-1})}.$$

Now, since for any $h \in H$,

$$|(-A)^{-1/2}(I-\delta A)^{-1/2}h - (-A)^{-1/2}h|^2_H = \sum_{k=1}^\infty \frac{1}{\alpha_k}\left(1 - \frac{1}{(1+\delta\alpha_k)^{1/2}}\right)^2 h_k^2,$$

and

$$\left(1 - \frac{1}{(1+\delta\alpha_k)^{1/2}}\right)^2 \le \alpha_k\delta,$$

we have

$$|(-A)^{-1/2}(I-\delta A)^{-1/2}h - (-A)^{-1/2}h|^2_H \le \delta|h|^2_H.$$

This implies

$$|x^\mu - x^{\mu,\delta}|^2_{H^{2\beta}} \le c\delta\int_{-\infty}^0 |\psi^\mu(s)|^2_H\,ds = c\delta \inf_{x\in N} \bar V_\mu(x).$$

In Corollary 7.4 we have proved

$$\limsup_{\mu\downarrow 0} \inf_{x\in N} \bar V_\mu(x) \le \inf_{x\in N} V(x),$$

and then we obtain

$$\sup_{\mu\le\mu_0} |x^\mu - x^{\mu,\delta}|_{H^{2\beta}} \le c\sqrt{\delta}, \tag{8.3}$$

which implies (8.2). $\square$

Now we can prove the main result of this section.

THEOREM 8.2. *For any closed $N \subset H$, we have*

$$\inf_{x\in N} V(x) \le \liminf_{\mu\downarrow 0} \inf_{x\in N} \bar V_\mu(x). \tag{8.4}$$

PROOF. If the right-hand side of (8.4) is infinite, the theorem is trivially true. Therefore, in what follows we can assume that

$$\liminf_{\mu\to 0}\inf_{x\in N}\bar{V}_\mu(x)<+\infty. \tag{8.5}$$

We first observe that if we define

$$\varphi^{\mu,\delta}(t)=\Pi_1 z^{\mu,\delta}(t),\qquad t\leq 0,$$

in view of (7.4),

$$\begin{aligned}V\big(x^{\mu,\delta}\big)\leq I_{-\infty}\big(\varphi^{\mu,\delta}\big)&=I^\mu_{-\infty}\big(z^{\mu,\delta}\big)-\frac{\mu^2}{2}\int_{-\infty}^0\left|Q^{-1}\frac{\partial^2\varphi^{\mu,\delta}}{\partial t^2}(t)\right|_H^2 dt\\&\quad-\mu\int_{-\infty}^0\Big\langle Q^{-1}\frac{\partial^2\varphi^{\mu,\delta}}{\partial t^2}(t),\\&\qquad Q^{-1}\frac{\partial\varphi^{\mu,\delta}}{\partial t}(t)-Q^{-1}A\varphi^{\mu,\delta}(t)-Q^{-1}B\big(\varphi^{\mu,\delta}(t)\big)\Big\rangle_H dt.\end{aligned}\tag{8.6}$$

Since

$$\big|\psi^{\mu,\delta}(t)\big|_H=\big|(I-\delta A)^{-1/2}\psi^\mu(t)\big|_H\leq\big|\psi^\mu(t)\big|_H,\qquad t\leq 0,\tag{8.7}$$

we have

$$I^\mu_{-\infty}\big(z^{\mu,\delta}\big)\leq I^\mu_{-\infty}\big(z^\mu\big)=\inf_{x\in N}\bar{V}_\mu(x),$$

so that

$$\begin{aligned}V\big(x^{\mu,\delta}\big)&\leq\inf_{x\in N}\bar{V}_\mu(x)\\&\quad-\mu\int_{-\infty}^0\Big\langle Q^{-1}\frac{\partial^2\varphi^{\mu,\delta}}{\partial t^2}(t),\\&\qquad Q^{-1}\frac{\partial\varphi^{\mu,\delta}}{\partial t}(t)-Q^{-1}A\varphi^{\mu,\delta}(t)-Q^{-1}B\big(\varphi^{\mu,\delta}(t)\big)\Big\rangle_H dt.\end{aligned}$$

Thanks to (4.14) and Hypothesis 2, by integrating by parts,

$$\begin{aligned}V\big(x^{\mu,\delta}\big)&\leq\inf_{x\in N}\bar{V}_\mu(x)\\&\quad-\frac{\mu}{2}\big|Q^{-1}y^{\mu,\delta}\big|_H^2-\mu\big\langle(-A)Q^{-1}x^{\mu,\delta},Q^{-1}y^{\mu,\delta}\big\rangle_H\\&\quad+\mu\big\langle Q^{-1}B\big(x^{\mu,\delta}\big),Q^{-1}y^{\mu,\delta}\big\rangle_H\\&\quad+c\mu\int_{-\infty}^0\left|\frac{\partial\varphi^{\mu,\delta}}{\partial t}(t)\right|^2_{H^{1+2\beta}}dt+c\gamma_{2\beta}\mu\int_{-\infty}^0\left|\frac{\partial\varphi^{\mu,\delta}}{\partial t}(t)\right|^2_{H^{2\beta}}dt\\&=\inf_{x\in N}\bar{V}_\mu(x)+\sum_{i=1}^5 I_i^{\mu,\delta}.\end{aligned}\tag{8.8}$$

First, we note that

$$I_1^{\mu,\delta} \leq 0. \tag{8.9}$$

Next, by (4.26) we see that

$$\begin{aligned} I_2^{\mu,\delta} + I_4^{\mu,\delta} &\leq c\sqrt{\mu}\big(|x^{\mu,\delta}|^2_{H^{2\beta+2}} + \mu|y^{\mu,\delta}|^2_{H^{2\beta+1}}\big) + c\mu\int_{-\infty}^0 |z^{\mu,\delta}(t)|^2_{\mathcal{H}^{2+2\beta}}\,dt \\ &\leq c(\mu+\sqrt{\mu})\int_{-\infty}^0 |\psi^{\mu,\delta}(t)|^2_{H^1}\,dt. \end{aligned}$$

Since for any $h \in H$, we have $(I-\delta A)^{-1/2}h \in \mathrm{Dom}(-A)^{1/2}$ and

$$\big|(-A)^{1/2}(I-\delta A)^{-1/2}h\big|_H \leq \delta^{-1/2}|h|_H,$$

we have

$$\big|\psi^{\mu,\delta}(t)\big|_{H^1} \leq \delta^{-1/2}\big|\psi^{\mu}(t)\big|_H, \qquad t \leq 0.$$

Therefore, by (8.1),

$$\begin{aligned} I_2^{\mu,\delta} + I_4^{\mu,\delta} &\leq c\delta^{-1/2}(\mu+\sqrt{\mu})\int_0^t |\psi^{\mu}(t)|^2_H \\ &= 2c\delta^{-1/2}(\mu+\sqrt{\mu})\inf_{x\in N}\bar{V}_\mu(x). \end{aligned} \tag{8.10}$$

By the same arguments, (4.26) and (8.7) give

$$I_3^{\mu,\delta} + I_5^{\mu,\delta} \leq c(\mu+\sqrt{\mu})\inf_{x\in N}\bar{V}_\mu(x). \tag{8.11}$$

Combining together (8.9), (8.10) and (8.11) with (8.8), we obtain

$$V\big(x^{\mu,\delta}\big) \leq \inf_{x\in N}\bar{V}_\mu(x) + c(\mu+\sqrt{\mu})\big(1+\delta^{-1/2}\big)\inf_{x\in N}\bar{V}_\mu(x). \tag{8.12}$$

From this, choosing $\delta = \sqrt{\mu}$, and due to (8.5), we see that

$$\liminf_{\mu\to 0} V\big(x^{\mu,\sqrt{\mu}}\big) \leq \liminf_{\mu\to 0}\inf_{x\in N}\bar{V}_\mu(x).$$

Since we are assuming (8.5), and since by [5], Proposition 5.1, the level sets of V are compact, there is a sequence $\mu_n \to 0$ and $x^0 \in H$ such that

$$\lim_{n\to\infty}\big|x^{\mu_n,\sqrt{\mu_n}} - x^0\big|_H = 0, \qquad V\big(x^0\big) \leq \liminf_{\mu\to 0} V\big(x^{\mu,\sqrt{\mu}}\big).$$

By (8.2), we have that x^{μ_n} converges to x^0 in H, so that $x_0 \in N$. This means that we can conclude, as

$$\inf_{x\in N} V(x) \leq V\big(x^0\big) \leq \liminf_{\mu\to 0} V\big(x^{\mu,\sqrt{\mu}}\big) \leq \liminf_{\mu\to 0}\inf_{x\in N}\bar{V}_\mu(x). \qquad \square$$

9. Application to the exit problem. In this section we study the problem of the exit of the solution u_ε^μ of equation (1.1) from a domain $G \subset H$, for any $\mu > 0$ fixed. Then we apply the limiting results proved in Theorems 7.3 and 8.2 to show that, when μ is small, the relevant quantities in the exit problem from G for the solution u_ε^μ of equation (1.1) can be approximated by the corresponding ones arising for equation (1.2).

First, let us give some assumptions on the set G.

HYPOTHESIS 3. *The domain $G \subset H$ is an open, bounded, connected set, such that $0 \in G$. Moreover, for any $x \in \partial G \cap H^{1+2\beta}$, there exists a sequence $\{x_n\}_{n\in\mathbb{N}} \subset \bar{G}^c \cap H^{1+2\beta}$ such that*

$$\lim_{n\to+\infty} |x_n - x|_{\mathcal{H}_{1+2\beta}} = 0. \tag{9.1}$$

Assume now that G is an open, bounded and connected set such that, for any $x \in \partial G \cap H^{1+2\beta}$, there exists a $y \in \bar{G}^c \cap H^{1+2\beta}$ such that

$$\{ty + (1-t)x : 0 < t \leq 1\} \subset \bar{G}^c. \tag{9.2}$$

Then it is immediate to check that (9.1) is satisfied. Condition (9.2) is true, for example, if G is convex, because of the Hahn–Banach separation theorem and the density of $H^{1+2\beta}$ in H.

LEMMA 9.1. *Under Hypothesis 3,*

$$\bar{V}_\mu(\partial G) := \inf_{x\in\partial G} \bar{V}_\mu(x) = \bar{V}_\mu(x_{G,\mu}) < \infty, \tag{9.3}$$

for some $x_{G,\mu} \in \partial G \cap \mathcal{H}_{1+2\beta}$.

PROOF. Since $\bar{G}^c$ is an open set, there exists $\tilde{x} \in \bar{G}^c \cap H^{1+2\beta}$. Because $0 \in G$, and the path $t \mapsto t\tilde{x}$ is continuous, there must exist $0 < t_0 < 1$ such that $t_0\tilde{x} \in \partial G$. Clearly, $t_0\tilde{x} \in H^{1+2\beta}$, so that, as $\partial G \cap H^{1+2\beta} \neq \varnothing$, according to Theorem 6.3,

$$\inf_{x\in\partial G} \bar{V}_\mu(x) < \infty.$$

Moreover, thanks to Theorem 5.4, the first equality in (9.3) implies that there exists $x_{G,\mu} \in \partial G \cap \mathcal{H}_{1+2\beta}$ such that

$$\bar{V}_\mu(x_{G,\mu}) = \bar{V}_\mu(\partial G). \tag{9.4}$$

$\square$

Now, if we denote by $z_{\varepsilon,z_0}^\mu = (u_{\varepsilon,z_0}^\mu, v_{\varepsilon,z_0}^\mu)$ the mild solution of (2.9), with initial position and velocity $z_0 = (u_0, v_0) \in \mathcal{H}$, we define the exit time

$$\tau_{z_0}^{\mu,\varepsilon} = \inf\{t > 0 : u_{\varepsilon,z_0}^\mu(t) \notin G\}. \tag{9.5}$$

Here is the main result of this section:

THEOREM 9.2. *There exists* $\mu_0 > 0$ *such that for* $\mu < \mu_0$, *the following conditions are verified. For any* $z_0 = (u_0, v_0) \in \mathcal{H}$ *such that* $u_0 \in G$ *and* $u^\mu_{0,z_0}(t) \in G$, *for* $t \geq 0$:

(1) *The exit time has the following asymptotic growth*:

$$\lim_{\varepsilon \to 0} \varepsilon \log \mathbb{E}\big(\tau^{\mu,\varepsilon}_{z_0}\big) = \inf_{x \in \partial G} \bar{V}_\mu(x), \tag{9.6}$$

and for any $\eta > 0$,

$$\lim_{\varepsilon \to 0} \mathbb{P}\bigg(\exp\bigg(\frac{1}{\varepsilon}\big(\bar{V}_\mu(\partial G) - \eta\big)\bigg) \leq \tau^{\mu,\varepsilon}_{z_0} \leq \exp\bigg(\frac{1}{\varepsilon}\big(\bar{V}_\mu(\partial G) + \eta\big)\bigg)\bigg) = 1. \tag{9.7}$$

(2) *For any closed* $N \subset \partial G$ *such that* $\inf_{x \in N} \bar{V}_\mu(x) > \inf_{x \in \partial G} \bar{V}_\mu(x)$, *it holds that*

$$\lim_{\varepsilon \to 0} \mathbb{P}\big(u^\mu_{\varepsilon,z_0}\big(\tau^{\mu,\varepsilon}_{z_0}\big) \in N\big) = 0. \tag{9.8}$$

REMARK 9.3. The requirement that $u^\mu_{0,z_0}(t) \in G$ for all $t \geq 0$ is necessary because in Lemma 3.4, we show that there exist $z_0 \in G \times H^{-1}$ such that u^μ_{0,z_0} leaves G in finite time. Of course, for these initial conditions, the stochastic processes u^μ_{ε,z_0} will also exit in finite time for small ε.

In [4] it has been proven that an analogous result to Theorem 9.2 holds for equation (2.5). If we denote by u_{ε,u_0} the mild solutions of equation (2.5), with initial condition $u_0 \in H$, we define the exit time

$$\tau^\varepsilon_{u_0} = \inf\big\{t > 0 : u_{\varepsilon,u_0}(t) \notin G\big\}.$$

In [4] it has been proven that for any $u_0 \in G$ such that $u_{0,u_0}(t) \in G$, for any $t \geq 0$, it holds that

$$\lim_{\varepsilon \to 0} \varepsilon \log \mathbb{E}\big(\tau^\varepsilon_{u_0}\big) = \inf_{x \in \partial G} V(x).$$

Similarly, as we would expect, it also holds that

$$\lim_{\varepsilon \to 0} \varepsilon \log \tau^\varepsilon_{u_0} = \inf_{x \in \partial G} V(x) \qquad \text{in probability},$$

and if $N \subset \partial G$ is closed and $\inf_{x \in N} V(x) > \inf_{x \in \partial G} V(x)$,

$$\lim_{\varepsilon \to 0} \mathbb{P}\big(u^\varepsilon_{u_0}\big(\tau^\varepsilon_{u_0}\big) \in N\big) = 0.$$

The proof of these facts is analogous to the proof of Theorem 9.2.

In view of what we have proven in Sections 7 and 8 and of Theorem 9.2, this implies that the following Smoluchowski–Kramers approximations holds for the exit time:

THEOREM 9.4. (1) *For any initial conditions* $z_0 = (u_0, v_0)$,

$$\lim_{\mu\to 0}\lim_{\varepsilon\to 0} \varepsilon \log \mathbb{E}\big(\tau_{z_0}^{\mu,\varepsilon}\big) = \lim_{\varepsilon\to 0} \varepsilon \log \mathbb{E}\big(\tau_{u_0}^{\varepsilon}\big) = \inf_{x\in\partial G} V(x). \tag{9.9}$$

(2) *For any* $\eta > 0$, *there exists* $\mu_0 > 0$ *such that for* $\mu < \mu_0$,

$$\lim_{\varepsilon\to 0} \mathbb{P}\big(e^{1/\varepsilon(\bar{V}-\eta)} \le \tau_{z_0}^{\mu,\varepsilon} \le e^{1/\varepsilon(\bar{V}+\eta)}\big) = 1. \tag{9.10}$$

(3) *For any* $N \subset \partial G$ *such that* $\inf_{x\in N} V(x) < \inf_{x\in\partial G} V(x)$, *there exits* $\mu_0 > 0$ *such that for all* $\mu < \mu_0$,

$$\lim_{\varepsilon\to 0} \mathbb{P}_{z_0}\big(u_\varepsilon^\mu\big(\tau^{\mu,\varepsilon}\big) \in N\big) = 0.$$

We recall that in [7] we have proved that, in the case of gradient systems, for any $\mu > 0$,

$$\bar{V}_\mu(x) = V(x), \qquad x \in H.$$

This means that in this case for any $z_0 = (u_0, v_0) \in \mathcal{H}$ and $\mu > 0$,

$$\lim_{\varepsilon\to 0} \varepsilon \log \mathbb{E}\big(\tau_{z_0}^{\mu,\varepsilon}\big) = \lim_{\varepsilon\to 0} \varepsilon \log \mathbb{E}\big(\tau_{u_0}^{\varepsilon}\big) = \inf_{x\in\partial G} V(x),$$

and (9.10) holds for any $\mu > 0$.

9.1. *Proof to Theorem* 9.2. In order to prove Theorem 9.2, we will need some preliminary lemmas, whose proofs are postponed to the next subsection.

LEMMA 9.5. *For* $\mu < (\alpha_1 - \gamma_0)\gamma_0^{-2}$, *there exists a constant* $c(\mu) > 0$ *such that* $z_1, z_2 \in \mathcal{H}$

$$\sup_{\psi\in L^2((0,+\infty);H)} \sup_{t\ge 0} \big|z_{\psi,z_1}^\mu(t) - z_{\psi,z_2}^\mu(t)\big|_{\mathcal{H}} \le c(\mu)|z_1 - z_2|_{\mathcal{H}}. \tag{9.11}$$

LEMMA 9.6. *For any closed set* $N \subset H$, *and any* $A < \bar{V}_\mu(N)$, *there exists* $\rho_0 > 0$ *such that if* $z \in C((0,T); H)$, *with* $|z(0)|_{\mathcal{H}} < \rho_0$ *and* $I_{0,T}^\mu(z) < A$, *then it holds*

$$\inf_{t\le T} \operatorname{dist}_H\big(\Pi_1 z(t), N\big) > \big|z(0)\big|_{\mathcal{H}}.$$

LEMMA 9.7. *For any* $\mu, \varepsilon > 0$ *and* $z_0 \in \mathcal{H}$, *let*

$$\tau_{z_0,\rho}^{\mu,\varepsilon} := \inf\big\{t > 0 : \Pi_1 z_{\varepsilon,z_0}^\mu(t) \notin G \text{ or } \big|z_{\varepsilon,z_0}^\mu\big|_{\mathcal{H}} < \rho\big\},$$

where $\rho > 0$ *is small enough so that* $B_{\mathcal{H}}(\rho) \subset G \times H^{-1}$. *Then*

$$\lim_{t\to+\infty} \limsup_{\varepsilon\to 0} \varepsilon \log\Big(\sup_{z_0\in G\times H^{-1}} \mathbb{P}\big(\tau_{z_0,\rho}^{\mu,\varepsilon} \ge t\big)\Big) = -\infty. \tag{9.12}$$

LEMMA 9.8. *Let $\tau^{\mu,\varepsilon}_{z_0,\rho}$ be the exit time from Lemma 9.7, and let $N \subset \partial G$ be a closed set. Then*

$$\lim_{\rho\to 0}\limsup_{\varepsilon\to 0}\varepsilon\log\Big(\sup_{z_0\in B_{\mathcal{H}}((1+M_\mu)\rho)}\mathbb{P}\big(\Pi_1 z^\mu_{\varepsilon,z_0}\big(\tau^{\mu,\varepsilon}_{z_0,\rho}\big)\in N\big)\Big)\le -\bar{V}_\mu(N), \tag{9.13}$$

where $\bar{V}_\mu(N)=\inf_{x\in N}\bar{V}_\mu(x)$.

LEMMA 9.9. *For fixed $\rho>0$,*

$$\lim_{t\to 0}\limsup_{\varepsilon\to 0}\varepsilon\log\Big(\sup_{z_0\in B_{\mathcal{H}}(\rho)}\mathbb{P}\Big(\sup_{s\le t}|z^\mu_{\varepsilon,z_0}(s)|_{\mathcal{H}}\ge(1+M_\mu)\rho\Big)\Big)=-\infty.$$

PROOF OF THEOREM 9.2. As $G\subset H$ is a bounded set, there exists $R>0$ such that $G\subset B_H(R-1)$. If $c(\mu,1)$ is the constant from Lemma 3.4, for any $z_0=(u_0,v_0)\in\mathcal{H}$ such that

$$u_0\in G,\qquad |v_0|_{H^{-1}}>Rc(\mu,1)^{-1}=:\kappa,$$

we have that $\Pi_1 z^\mu_{z_0}$ leaves B_R (and therefore G) before time $t=1$. Since for any $T>0$,

$$\lim_{\varepsilon\to 0}\sup_{z_0\in\mathcal{H}}\mathbb{E}|z^\mu_{\varepsilon,z_0}-z^\mu_{z_0}|_{C([0,T];\mathcal{H})}=0, \tag{9.14}$$

this yields

$$\begin{aligned}\lim_{\varepsilon\to 0}\inf_{\substack{u_0\in G\\|v_0|_{H^{-1}}>\kappa}}\mathbb{P}\big(\tau^{\mu,\varepsilon}_{z_0}<1\big)&\ge\lim_{\varepsilon\to 0}\inf_{\substack{u_0\in G\\|v_0|_{H^{-1}}>\kappa}}\mathbb{P}\big(|z^\mu_{\varepsilon,z_0}-z^\mu_{z_0}|_{C([0,T];\mathcal{H})}\le 1\big)\\&=1.\end{aligned} \tag{9.15}$$

Now, fix $\eta>0$. According to (9.4), there exists $x_{G,\mu}\in\partial G\cap H^{1+2\beta}$ such that $\bar{V}_\mu(x_{G,\mu})=\bar{V}_\mu(\partial G)$. Now, if $\{x_n\}\subset\bar{G}^c\cap H^{1+2\beta}$ is a sequence from (9.1) such that $x_n\to x_{G,\mu}$ in $H^{1+2\beta}$, as $n\to\infty$, due to Theorem 6.3 we have that $\bar{V}_\mu(x_n)\to\bar{V}_\mu(x_{G,\mu})$. This means that there exists $\bar{n}$ such that

$$\bar{V}_\mu(x_{\bar{n}})<\bar{V}_\mu(x_{G,\mu})+\frac{\eta}{4}=\bar{V}_\mu(\partial G)+\frac{\eta}{4}.$$

In particular, there exists $T_1>0$ and $z^\mu_{\psi,0}\in C([0,T_1];\mathcal{H})$ such that $z^\mu_{\psi,0}(0)=0$ and $\Pi_1 z^\mu_{\psi,0}(T_1)=x_{\bar{n}}\in\bar{G}^c$ with

$$I^\mu_{0,T_1}\big(z^\mu_{\psi,0}\big)<\bar{V}_\mu(x_{\bar{n}})+\frac{\eta}{4}<\bar{V}_\mu(\partial G)+\frac{\eta}{2}.$$

According to (9.11), the mapping $z_0\in\mathcal{H}\mapsto z^\mu_{\psi,z_0}\in C([0,T_1];\mathcal{H})$ is continuous, and therefore, we can find $\rho>0$ such that

$$|z_0|_{\mathcal{H}}<\rho$$
$$\Longrightarrow\quad \operatorname{dist}\big(z^\mu_{\psi,z_0}(T_1),(G\times H^{-1})\big)>\tfrac{1}{2}\operatorname{dist}\big(z^\mu_{\psi,0}(T_1),(G\times H^{-1})\big)=:\alpha>0.$$

In view of (5.3), we can see that there exists $\varepsilon_1 > 0$ such that for all $\varepsilon < \varepsilon_1$, and all $|z_0|_{\mathcal{H}} < \rho$,

$$\mathbb{P}\big(\tau_{z_0}^{\mu,\varepsilon} < T_1\big) \geq \mathbb{P}\big(\big|z_{\varepsilon,z_0}^{\mu} - z_{\psi,z_0}^{\mu}\big|_{C([0,T_1];\mathcal{H})} < \alpha\big) \geq e^{-1/\varepsilon(\bar{V}_\mu(G)+\eta)}. \tag{9.16}$$

Now, by Lemma 3.2 we can find $T_2 > 0$ such that

$$\sup_{\substack{u_0 \in G \\ |v_0|_{H^{-1}} \leq \kappa}} \big|z_{z_0}^{\mu}(T_2)\big|_{\mathcal{H}} < \frac{\rho}{2}.$$

Therefore, thanks to (9.14), there exists $0 < \varepsilon_2 \leq \varepsilon_1$ such that $u_0 \in G$, and $|v_0|_{H^{-1}} \leq \kappa$,

$$\mathbb{P}\big(\big|z_{\varepsilon,z_0}^{\mu}(T_2)\big|_{\mathcal{H}} < \rho\big) > \tfrac{1}{2}, \qquad \varepsilon \leq \varepsilon_2.$$

Thanks to (9.16), by the Markov property, this implies that for $u_0 \in G$ and $|v_0|_{H^{-1}} \leq \kappa$,

$$\mathbb{P}\big(\tau_{z_0}^{\mu,\varepsilon} < T_1 + T_2\big) \geq \tfrac{1}{2} e^{-1/\varepsilon(\bar{V}_\mu(G)+\eta)}, \qquad \varepsilon < \varepsilon_2.$$

Hence, if we combine this with (9.15), we see that there exists $0 < \varepsilon_0 \leq \varepsilon_2$ such that for all $\varepsilon < \varepsilon_0$,

$$\inf_{z_0 \in G \times H^{-1}} \mathbb{P}\big(\tau_{z_0}^{\mu,\varepsilon} < 1 + T_1 + T_2\big) \geq \frac{1}{2} e^{-1/\varepsilon(\bar{V}_\mu(G)+\eta)}. \tag{9.17}$$

By using again the Markov property, for any $k \in \mathbb{N}$ and $z_0 \in G \times H^{-1}$, this gives

$$\begin{aligned}\mathbb{P}\big(\tau_{z_0}^{\mu,\varepsilon} \geq k(1 + T_1 + T_2)\big) &\leq \Big(\sup_{z_0 \in G \times H^{-1}} \mathbb{P}\big(\tau_{z_0}^{\mu,\varepsilon} \geq (1 + T_1 + T_2)\big)\Big)^k \\ &\leq \left(1 - \frac{1}{2} e^{-1/\varepsilon(\bar{V}_\mu(G)+\eta)}\right)^k,\end{aligned}$$

so that

$$\begin{aligned}\mathbb{E}\big(\tau_z^{\mu,\varepsilon}\big) &\leq (1 + T_1 + T_2) \sum_{k=0}^{\infty} \mathbb{P}\big(\tau_z^{\mu,\varepsilon} \geq k(1 + T_1 + T_2)\big) \\ &\leq 2(1 + T_1 + T_2) e^{1/\varepsilon(\bar{V}_\mu(G)+\eta)}.\end{aligned}$$

Thus the upper bound of (9.6) follows as η was chosen arbitrarily small, and the upper bound of (9.7), follows from this by using the Chebyshev inequality. □

The proofs of the lower bound for the exit time and of the exit place follow from Lemmas 9.5 to 9.9, by using the same arguments used in the finite-dimensional case; see [13] and [17]. For this reason, we omit them.

9.2. *Proofs of Lemmas* 9.5–9.9.

PROOF OF LEMMA 9.5. If we let $\varphi(t) = \Pi_1(z^\mu_{\psi,z_1}(t) - z^\mu_{\psi,z_2}(t))$, then it is a weak solution to

$$\mu\frac{\partial^2\varphi}{\partial t^2}(t) + \frac{\partial\varphi}{\partial t}(t) = A\varphi(t) + B\big(\Pi_1 z^\mu_{z_1,\psi}(t)\big) - B\big(\Pi_1 z^\mu_{z_2,\psi}(t)\big). \tag{9.18}$$

Therefore, we can conclude as in Lemma 3.1. $\square$

PROOF OF LEMMA 9.6. Fix $A < \bar V_\mu(N)$. Suppose by contradiction that there exist $\{z_n\} \subset \mathcal{H}$, $\{T_n\} \subset (0,+\infty)$ and $\{\psi_n\} \subset L^2((0,T_n);H)$ such that

$$\lim_{n\to\infty} |z_n|_{\mathcal{H}} = 0, \qquad \frac{1}{2}|\psi_n|^2_{L^2((0,T_n);H)} < A,$$

and

$$\operatorname{dist}_H\big(\Pi_1 z^\mu_{\psi_n,z_n}(T_n), N\big) \le |z_n|_{\mathcal{H}}.$$

Now, if we set $x_n := \Pi_1 z^\mu_{\psi_n,0}(T_n)$, for any $n \in \mathbb{N}$ we have, by (9.11),

$$\big|x_n - \Pi_1 z^\mu_{\psi_n,z_n}(T_n)\big|_H \le c(\mu)|z_n|_{\mathcal{H}},$$

so that

$$\operatorname{dist}_H(x_n, N) \le c(\mu)|z_n|_{\mathcal{H}} + |z_n|_{\mathcal{H}}, \qquad n \in \mathbb{N}. \tag{9.19}$$

Recalling how $\bar V_\mu$ is defined, we have

$$\bar V_\mu(x_n) \le \tfrac{1}{2}|\psi_n|^2_{L^2((0,T_n);H)} < A.$$

Now, as proven in Theorem 5.4, $\bar V_\mu$ has compact level sets. Therefore, there is a sequence $\{x_{n_k}\}_k \subset H$ such that $x_{n_k} \to x$, so that $\bar V_\mu(x) < A$. However, by (9.19), $x \in N$, and then $\bar V_\mu(N) \le \bar V_\mu(x) < \bar V_\mu(N)$, a contradiction. $\square$

PROOF OF LEMMA 9.7. Fix $R > \sup_{x\in G}|x|_H + \rho$, and by Lemma 3.4, let us take $\kappa > 0$ such that if $v_0 \in B_{H^{-1}}(\kappa)$, then $z^\mu_{z_0}$ leaves $B_R \times H^{-1}$ before time $t = 1$.

By Lemma 3.2, we can find $T_1 > 0$ such that

$$\sup_{\substack{u_0\in G\\ |v_0|_{H^{-1}}\le\kappa}} \big|z^\mu_{z_0}(T_1)\big|_{\mathcal{H}} < \frac{\rho}{2},$$

and then for any $z_0 \in G \times H^{-1}$, $z^\mu_{z_0}(t)$ leaves $(G \times H^{-1}) \setminus B_{\mathcal{H}}(\rho/2)$ in less than time $T = T_1 + 1$. This means that

$$\inf\{I^\mu_{0,T}(z) : z(t) \in \big(B_H(R) \times H^{-1}\big) \setminus B_{\mathcal{H}}(\rho/2) \text{ for } t \in [0,T]\} = a > 0 \tag{9.20}$$

because the set above contains no unperturbed trajectories. By (5.4)

$$\limsup_{\varepsilon\to 0}\varepsilon\log\Big(\sup_{z_0\in G\times H^{-1}}\mathbb{P}(\tau_0\geq T)\Big)$$
$$\leq \limsup_{\varepsilon\to 0}\varepsilon\log\bigg(\sup_{z_0\in G\times H^{-1}}\mathbb{P}\bigg(\mathrm{dist}_{C([0,T];\mathcal{H})}\big(z^{\mu}_{\varepsilon,z_0},K^{\mu}_{0,T}(a)\big)>\frac{\rho}{2}\bigg)\bigg)\leq -a.$$

By the Markov property, for any $k\in\mathbb{N}$,

$$\sup_{z_0\in G\times H^{-1}}\mathbb{P}(\tau_0\geq kT)\leq\Big(\sup_{z_0\in G\times H^{-1}}\mathbb{P}(\tau_0\geq T)\Big)^k$$

and therefore,

$$\lim_{\varepsilon\to 0}\varepsilon\log\Big(\sup_{z_0\in G\times H^{-1}}\mathbb{P}(\tau_1\geq Tk)\Big)\leq -ka.$$

□

PROOF OF LEMMA 9.8. Let $\Gamma_\rho:=B_{\mathcal{H}}((1+M_\mu)\rho)$. For any $T>0$, we have

$$\begin{aligned}&\sup_{z_0\in\Gamma_\rho}\mathbb{P}\big(\Pi_1 z^{\mu}_{\varepsilon,z_0}\big(\tau^{\mu,\varepsilon}_{z_0,\rho}\big)\in N\big)\\ &\quad\leq\sup_{z_0\in\Gamma_\rho}\mathbb{P}\big(\tau^{\mu,\varepsilon}_{z_0,\rho}>T\big)+\sup_{z_0\in\Gamma_\rho}\mathbb{P}\big(\Pi_1 z^{\mu}_{\varepsilon,z_0}(t)\in N\text{ for some }t\leq T\big).\end{aligned}\tag{9.21}$$

Next, thanks to Lemma 9.6, for any $A<\bar{V}_\mu(N)$ fixed, we can find $\rho_0>0$ such that for $\rho<\rho_0$ and any $T>0$, the set

$$\big\{z:z(0)\in\Gamma_\rho,\mathrm{dist}_{C([0,T];\mathcal{H})}\big(z,K^{\mu}_{0,T}(A)\big)\leq(1+M_\mu)\rho\big\}$$

contains no trajectories that reach N by time T. Then by (5.4), for any $\eta>0$, for small enough $\varepsilon>0$,

$$\sup_{z_0\in\Gamma_\rho}\mathbb{P}\big(\Pi_1 z^{\mu}_{\varepsilon,z_0}(t)\in N\text{ for some }t\leq T\big)$$
$$\leq\sup_{z_0\in\Gamma_\rho}\mathbb{P}\big(\mathrm{dist}_{C([0,T];\mathcal{H})}\big(z^{\mu}_{\varepsilon,z_0},K^{\mu}_{0,T}(A)\big)>(1+M_\mu)\rho\big)\leq e^{-1/\varepsilon(A-\eta)}.$$

Now, according to (9.12), we pick $T>0$ so that, for small enough $\varepsilon>0$,

$$\sup_{z_0\in\Gamma_\rho}\mathbb{P}\big(\tau^{\mu,\varepsilon}_{z_0,\rho}>T\big)\leq e^{-(1/\varepsilon)(A)}.$$

Due to (9.21), this implies our result, as $A<\bar{V}_\mu(N)$ and $\eta>0$ were arbitrary. □

PROOF OF LEMMA 9.9. If $z(t)=z^{\mu}_{\psi,z_0}(t)$, then

$$z(t)=S_\mu(t)z_0+\int_0^t S_\mu(t-s)B_\mu\big(z(s)\big)\,ds+\int_0^t S_\mu(t-s)Q_\mu\psi(s)\,ds,$$

so that, if $z_0 \in B_{\mathcal{H}}(\rho)$,

$$\sup_{s\leq t}|z(s)|_{\mathcal{H}} \leq M_\mu \rho + \frac{\gamma_0 t M_\mu}{\mu\sqrt{\alpha_1}} \sup_{s\leq t}|z(s)|_{\mathcal{H}} + \frac{M_\mu \|Q\|_{L(H)}}{\mu\sqrt{\alpha_1}}\sqrt{t}|\psi|_{L^2((0,t);H)}.$$

Therefore, if $\sup_{s\leq t}|z(s)| \geq (M_\mu + 1/2)\rho$, then we get

$$E_\mu(t) := \rho\left(\frac{1}{2} - \frac{\gamma_0 t M_\mu}{\sqrt{\alpha_1}\mu}\right)\frac{\mu\sqrt{\alpha_1}}{M_\mu\sqrt{t}} \leq |\psi|_{L^2((0,t);H)}.$$

This means that

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \varepsilon \log\Big(\sup_{z_0\in B_{\mathcal{H}}(\rho)} \mathbb{P}\Big(\sup_{s\leq t}|z^\mu_{\varepsilon,z_0}(s)|_{\mathcal{H}} \geq (1+M_\mu)\rho\Big)\Big) \\
&\quad\leq \limsup_{\varepsilon\to 0} \varepsilon \log\Big(\sup_{z_0\in B_{\mathcal{H}}(\rho)} \mathbb{P}\Big(\mathrm{dist}_{C([0,t];\mathcal{H})}\Big(z^\mu_{\varepsilon,z_0}, K^\mu_{0,t}\Big(\frac{1}{2}(E_\mu(t))^2\Big)\Big) > \frac{\rho}{2}\Big)\Big) \\
&\quad\leq -\frac{(E_\mu(t))^2}{2},
\end{aligned}$$

and our result follows as

$$\lim_{t\to 0} E_\mu(t) = +\infty. \qquad \square$$

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