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MATRICIAL MODEL FOR THE FREE MULTIPLICATIVE CONVOLUTION

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This paper investigates homomorphisms à la Bercovici–Pata between additive and multiplicative convolutions. We also consider their matricial versions which are associated with measures on the space of Hermitian matrices and on the unitary group. The previous results combined with a matricial model of Benaych–Georges and Cabanal–Duvillard allow us to define and study the large N limit of a new matricial model on the unitary group for free multiplicative Lévy processes.

1. Introduction. The classical convolution $*$ on $\mathbb{R}$ and the classical multiplicative convolution $\circledast$ on the unit circle $\mathbb{U} = \{z \in \mathbb{C} : |z| = 1\}$, which correspond, respectively, to the addition and to the product of independent random variables, have analogues in free probability. Indeed, replacing the concept of classical independence by the concept of freeness, Voiculescu defined the free additive convolution $\boxplus$ on $\mathbb{R}$, and the free multiplicative convolution $\boxtimes$ on $\mathbb{U}$ (we refer the reader to [36] for an introduction to free convolutions). A probability measure μ on $\mathbb{R}$ is said to be $*$-*infinitely divisible* if, for all $n \in \mathbb{N}^*$, there exists a probability measure μ_n such that $\mu_n^{*n} = \mu$. The set of $*$-infinitely divisible probability measures endowed with the operation $*$ is a semigroup which we will denote by $\mathcal{ID}(\mathbb{R}, *)$, and we consider analogously the sets $\mathcal{ID}(\mathbb{U}, \circledast)$, $\mathcal{ID}(\mathbb{R}, \boxplus)$ and $\mathcal{ID}(\mathbb{U}, \boxtimes)$.

In [5], Bercovici and Pata identified an isomorphism of semigroups Λ between $\mathcal{ID}(\mathbb{R}, *)$ and $\mathcal{ID}(\mathbb{R}, \boxplus)$ which has a good behavior with respect to limit theorems: for all $\mu \in \mathcal{ID}(\mathbb{R}, *)$ and all sequence $(\mu_n)_{n\in\mathbb{N}}$ of probability measures on $\mathbb{R}$,

$$\mu_n^{*n} \xrightarrow[n\to+\infty]{(w)} \mu \quad \Longleftrightarrow \quad \mu_n^{\boxplus n} \xrightarrow[n\to+\infty]{(w)} \Lambda(\mu),$$

where the convergence is the weak convergence of measures. Unfortunately, the situation is not as symmetric in the multiplicative case. Let $\mathcal{M}_*$ denote the set of probability measures μ on $\mathbb{U}$ such that $\int_{\mathbb{U}} \zeta \, \mathrm{d}\mu(\zeta) \neq 0$. In [14], Chistyakov and Götze proved that, given a sequence $(\mu_n)_{n\in\mathbb{N}}$ of probability measures on $\mathbb{U}$, the weak convergence of $\mu_n^{\boxtimes n}$ to any measure of $\mathcal{M}_*$ implies the weak convergence of $\mu_n^{\circledast n}$; but they also proved that the converse is false. It is thus only possible to define a homomorphism of semigroups Γ between $\mathcal{ID}(\mathbb{U}, \boxtimes)$ and $\mathcal{ID}(\mathbb{U}, \circledast)$ (see

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Definition 3.3) such that, for all $\mu \in \mathcal{ID}(\mathbb{U}, \boxtimes) \cap \mathcal{M}_*$ and all sequence $(\mu_n)_{n\in\mathbb{N}}$ of probability measures on $\mathbb{U}$,

$$\mu_n^{\boxtimes n} \xrightarrow[n\to+\infty]{(w)} \mu \quad \Longrightarrow \quad \mu_n^{\circledast n} \xrightarrow[n\to+\infty]{(w)} \Gamma(\mu).$$

Finally, the homomorphism $\mathbf{e} : x \mapsto e^{ix}$ from $(\mathbb{R}, +)$ to $(\mathbb{U}, \times)$ induces a homomorphism of semigroups $\mathbf{e}_*$ between $\mathcal{ID}(\mathbb{R}, *)$ and $\mathcal{ID}(\mathbb{U}, \circledast)$, given by the push-forward of measures, which enjoys a similar property: for all $\mu \in \mathcal{ID}(\mathbb{R}, *)$ and all sequence $(\mu_n)_{n\in\mathbb{N}}$ of probability measures on $\mathbb{R}$,

$$\mu_n^{*n} \xrightarrow[n\to+\infty]{(w)} \mu \quad \Longrightarrow \quad \mathbf{e}_*(\mu_n)^{\circledast n} \xrightarrow[n\to+\infty]{(w)} \mathbf{e}_*(\mu).$$

The first aim of this work is to complete the picture which we just sketched. In Definition 3.2, we shall introduce a new homomorphism of semigroups $\mathbf{e}_{\boxplus}$ between $\mathcal{ID}(\mathbb{R}, \boxplus)$ and $\mathcal{ID}(\mathbb{U}, \boxtimes)$, and which is linked to the previous homomorphisms in the following way.

THEOREM 1 (see Proposition 3.4 and Theorem 3.9). *The map* $\mathbf{e}_{\boxplus} : \mathcal{ID}(\mathbb{R}, \boxplus) \to \mathcal{ID}(\mathbb{U}, \boxtimes)$ *is such that*:

1. *For all* $\mu \in \mathcal{ID}(\mathbb{R}, \boxplus)$ *and all sequence* $(\mu_n)_{n\in\mathbb{N}}$ *of probability measures on* $\mathbb{R}$,

$$\mu_n^{\boxplus n} \xrightarrow[n\to+\infty]{(w)} \mu \quad \Longrightarrow \quad \mathbf{e}_*(\mu_n)^{\boxtimes n} \xrightarrow[n\to+\infty]{(w)} \mathbf{e}_{\boxplus}(\mu).$$

2. *The following diagram commutes*:

$$\begin{array}{ccc} \mathcal{ID}(\mathbb{R}, *) & \xrightarrow{\ \Lambda\ } & \mathcal{ID}(\mathbb{R}, \boxplus) \\ \mathbf{e}_* \downarrow & & \downarrow \mathbf{e}_{\boxplus} \\ \mathcal{ID}(\mathbb{U}, \circledast) & \xleftarrow[\ \Gamma\ ]{} & \mathcal{ID}(\mathbb{U}, \boxtimes). \end{array} \tag{1.1}$$

Another contribution to the question of going from $\boxplus$ to $\boxtimes$ is given by the map EXP of Friedrich and McKay in [21–23]. Their approach differs from ours in that they consider general normalized linear functionals on the set of polynomials. We consider only linear functionals which come from concrete probability measures. There is a question of positivity which prevents one to consider the restriction of EXP on $\mathcal{ID}(\mathbb{R}, \boxplus)$ because EXP does not map a priori a measure on $\mathbb{R}$ to a measure on $\mathbb{U}$. The question of finding a nontrivial homomorphism from the set of probability measures on $\mathbb{R}$ to the set of probability measures on $\mathbb{U}$ (e.g., as a restriction of EXP or as an extension of $\mathbf{e}_{\boxplus}$) still remains unsolved.

In the highly noncommutative theory of Lie groups, there is a well-known process which connects additive infinitely divisible laws with multiplicative ones. It consists in passing to the limit the product of multiplicative little increments which are built from additive increments using the exponential map (see [20]). A natural question is whether there exists a matrix approximation of $\mathbf{e}_{\boxplus}$ which arises from this procedure.

Our starting point is a matricial model for $\mathcal{ID}(\mathbb{R},\boxplus)$ which has been constructed simultaneously by Benaych–Georges and Cabanal–Duvillard in [4, 12]. For all $N\in\mathbb{N}$, let us consider the classical convolution $*$ on the set of Hermitian matrices $\mathcal{H}_N$, and denote by $\mathcal{ID}_{\mathrm{inv}}(\mathcal{H}_N,*)$ the set of infinitely divisible probability measures on $\mathcal{H}_N$ which are invariant under conjugation by unitary matrices. For all $\mu\in\mathcal{ID}(\mathbb{R},\boxplus)$, Benaych–Georges and Cabanal–Duvillard proved that there exists an element of $\mathcal{ID}_{\mathrm{inv}}(\mathcal{H}_N,*)$, which we shall denote by $\Pi_N(\mu)$ (see Section 7.1), such that:

1. For all $\mu\in\mathcal{ID}(\mathbb{R},\boxplus)$, the spectral measure of a random matrix with distribution $\Pi_N(\mu)$ converges weakly almost surely to μ as N tends to infinity.
2. $\Pi_N:\mathcal{ID}(\mathbb{R},\boxplus)\to\mathcal{ID}_{\mathrm{inv}}(\mathcal{H}_N,*)$ is a homomorphism of semigroups.

On the other hand, the map $\mathbf{e}:H\mapsto e^{iH}$ from $\mathcal{H}_N$ to the unitary group $U(N)$ induces, with some care, a homomorphism of semigroups from $\mathcal{ID}_{\mathrm{inv}}(\mathcal{H}_N,*)$ to the set $\mathcal{ID}_{\mathrm{inv}}(U(N),\circledast)$ of infinitely divisible measures on $U(N)$ which are invariant under conjugation. Indeed, for all $\mu\in\mathcal{ID}_{\mathrm{inv}}(\mathcal{H}_N,*)$, the sequence $(\mathbf{e}_*(\mu^{*1/n})^{\circledast n})_{n\in\mathbb{N}^*}$ converges weakly to a measure $\mathcal{E}_N(\mu)\in\mathcal{ID}_{\mathrm{inv}}(U(N),\circledast)$ (see Proposition–Definition 6.2). The situation can be summed up in the following diagram:

$$
\begin{array}{ccc}
\mathcal{ID}(\mathbb{R},\boxplus) & \xrightarrow{\ \Pi_N\ } & \mathcal{ID}_{\mathrm{inv}}(\mathcal{H}_N,*) \\
\mathbf{e}_{\boxplus}\big\downarrow & & \big\downarrow \mathcal{E}_N \\
\mathcal{ID}(\mathbb{U},\boxtimes) & & \mathcal{ID}_{\mathrm{inv}}\big(U(N),\circledast\big).
\end{array}
\tag{1.2}
$$

When $N=1$, we have $\Pi_1=\Lambda^{-1}$, $\mathcal{E}_1=\mathbf{e}_*$, and consequently the diagram (1.2) is exactly the top part of the diagram (1.1). The second main result of this work is the definition of a homomorphism of semigroups $\Gamma_N:\mathcal{ID}(\mathbb{U},\boxtimes)\to\mathcal{ID}_{\mathrm{inv}}(U(N),\circledast)$ which completes the picture as follows (see Section 7.2).

THEOREM 2 (see Proposition 7.5 and Theorem 7.8). *The map Γ_N is such that*:

1. *For all $\mu\in\mathcal{ID}(\mathbb{U},\boxtimes)$, the spectral measure of a random matrix $U^{(N)}$ with distribution $\Gamma_N(\mu)$ converges weakly almost surely to μ, in the sense that, for each continuous function f on $\mathbb{U}$, one has the almost sure convergence*

$$\lim_{N\to\infty}\frac{1}{N}\operatorname{Tr}\big(f(U^{(N)})\big)=\int_{\mathbb{U}} f\,\mathrm{d}\mu.$$

2. *The following diagram commutes*:

$$
\begin{array}{ccc}
\mathcal{ID}(\mathbb{R},\boxplus) & \xrightarrow{\Pi_N} & \mathcal{ID}_{\text{inv}}(\mathcal{H}_N,*) \\
\mathbf{e}_{\boxplus}\big\downarrow & & \big\downarrow \mathcal{E}_N \\
\mathcal{ID}(\mathbb{U},\boxtimes) & \xrightarrow[\Gamma_N]{} & \mathcal{ID}_{\text{inv}}\big(U(N),\circledast\big).
\end{array}
\tag{1.3}
$$

This result can be expressed by saying that the map $\mathbf{e}_{\boxplus}$ is the limit of the map $\mathcal{E}_N$ as N tends to infinity. The first assertion of the theorem above is a generalisation of a result of Biane: in [9], he proved that the spectral measure of a Brownian motion on $U(N)$ with adequately chosen speed converges to the distribution of a free unitary Brownian motion at each fixed time. This convergence can be viewed as a particular case of Theorem 2.

The proof itself of Theorem 2 is interesting at least for two reasons. It is the first time that the free log-cumulants, originated in [30], are used for proving an asymptotic result of random matrices. Second, the proof relies upon a key object, the symmetric group $\mathfrak{S}_n$, which is linked to both the combinatorics of free probability theory, and the computation of conjugate-invariant measures on $U(N)$. More precisely, in [25], Lévy established that the asymptotic distribution of a Brownian motion on the unitary group is closely related to the counting of paths in the Caley graph of $\mathfrak{S}_n$. Similarly, for all $\mu \in \mathcal{ID}(\mathbb{U},\boxtimes)$, the asymptotic distribution of a random matrix with law $\Gamma_N(\mu)$ involves the counting of paths in $\mathfrak{S}_n$, each step of which is given by the following generator (see Lemma 7.9):

$$T(\sigma) = nL\kappa_1(\mu)\cdot\sigma + \sum_{\substack{2\leq m\leq n \\ c\ m\text{-cycle of }\mathfrak{S}_n \\ c\sigma\preceq\sigma}} L\kappa_m(\mu)\cdot c\sigma,$$

where $(L\kappa_n(\mu))_{n\in\mathbb{N}^*}$ are the free log-cumulants of μ.

In fact, the full Brownian motion on $U(N)$ converges to the free unitary Brownian motion in noncommutative distribution (see [9]). In our framework, as a $\Gamma_N(\mu)$-distributed matrix is invariant by conjugation by unitary matrices, the classical result of asymptotic freeness of Voiculescu induces immediately a similar convergence, which is stated in Theorem 3 (see Section 7.4 for details).

THEOREM 3. *Let $(U_t)_{t\in\mathbb{R}_+}$ be a free unitary multiplicative Lévy process with marginal distributions $(\mu_t)_{t\in\mathbb{R}_+}$ in $\mathcal{M}_*$. For all $N\in\mathbb{N}^*$, let $(U_t^{(N)})_{t\in\mathbb{R}_+}$ be a Lévy process with marginal distributions $(\Gamma_N(\mu_t))_{t\in\mathbb{R}_+}$. Then $(U_t^{(N)})_{t\in\mathbb{R}_+}$ converges to $(U_t)_{t\in\mathbb{R}_+}$ in noncommutative $*$-distribution. In other words, for each integer $n\geq 1$, for each noncommutative polynomial P in n variables, each choice of n*

nonnegative reals $t_1, \dots, t_n$, *and each choice of* $\varepsilon_1, \dots, \varepsilon_n \in \{1, *\}$, *one has the almost sure convergence*

$$\lim_{N\to\infty} \frac{1}{N} \operatorname{Tr}\big(P\big((U_{t_1}^{(N)})^{\varepsilon_1}, \dots, (U_{t_n}^{(N)})^{\varepsilon_n}\big)\big) = \tau\big(P\big(U_{t_1}^{\varepsilon_1}, \dots, U_{t_n}^{\varepsilon_n}\big)\big).$$

Moreover, independent copies of $(U_t^{(N)})_{t\in\mathbb{R}_+}$ *converge to freely independent copies of* $(U_t)_{t\in\mathbb{R}_+}$.

The rest of the paper is organized as follows. In Section 2, we give an overview of the theory of infinitely divisible measures. In Section 3, we define $\mathbf{e}_{\boxplus}$ and Γ and we prove Theorem 1. Section 4 is devoted to the notion of free log-cumulants, which is an important tool for the proof of the asymptotic results of this paper. Section 5 presents a description of convolution semigroups on the unitary group, and studies more precisely those which are invariant by conjugation. Section 6 links together the measures on the Hermitian matrices with the measures on the unitary matrices through the stochastic exponentiation $\mathcal{E}_N$. Finally, Section 7 provides the definition of the random matrix models Π_N and Γ_N, and the proof of Theorems 2 and 3.

2. Infinite divisibility for uni-dimensional convolutions.

In this section, we give the necessary background concerning $\mathcal{ID}(\mathbb{R}, *)$, $\mathcal{ID}(\mathbb{U}, \circledast)$, $\mathcal{ID}(\mathbb{R}, \boxplus)$ and $\mathcal{ID}(\mathbb{U}, \boxtimes)$. In particular, we give a description of the characteristic pair and the characteristic triplet of an infinitely divisible measure in each case.

We say that a sequence of finite measures $(\mu_n)_{n\in\mathbb{N}}$ on $\mathbb{C}$ *converges weakly* to a measure μ if for all continuous and bounded complex function f,

$$\lim_{n\to\infty} \int_{\mathbb{C}} f \,\mathrm{d}\mu_n = \int_{\mathbb{C}} f \,\mathrm{d}\mu.$$

2.1. *Classical infinite divisibility on* $\mathbb{R}$.

Let $\mu \in \mathcal{ID}(\mathbb{R}, *)$. There exists a sequence $(\mu_n)_{n\in\mathbb{N}^*}$ of probability measures such that, for all $n \in \mathbb{N}^*$, $\mu_n^{*n} = \mu$. The important fact is that the measures

$$\mathrm{d}\sigma_n(x) = n \frac{x^2}{x^2+1} \mu_n(\mathrm{d}x)$$

converge weakly to a measure σ and the reals

$$\gamma_n = n \int_{\mathbb{R}} \frac{x}{x^2+1} \mu_n(\mathrm{d}x)$$

converge to a constant $\gamma \in \mathbb{R}$. The pair (γ, σ) is known as the $*$-*characteristic pair* for μ and it is uniquely determined by μ. More generally, we have the following characterization.

THEOREM 2.1 ([5], Theorem 3.3). *Let $\mu \in \mathcal{ID}(\mathbb{R}, *)$ with $*$-characteristic pair (γ, σ). Let $k_1 < k_2 < \cdots$ be natural numbers and $(\mu_n)_{n\in\mathbb{N}^*}$ be a sequence of probability measures on $\mathbb{R}$. The following assertions are equivalent*:

1. *The measures* $\underbrace{\mu_n * \cdots * \mu_n}_{k_n \text{ times}}$ *converge weakly to* μ.
2. *The measures*

$$\mathrm{d}\sigma_n(x) = k_n \frac{x^2}{x^2+1} \mu_n(\mathrm{d}x)$$

converge weakly to σ *and*

$$\lim_{n\to\infty} k_n \int_{\mathbb{R}} \frac{x}{x^2+1} \mu_n(\mathrm{d}x) = \gamma.$$

In addition to [5], we refer the reader to the very complete lecture notes [3]. We present now two additional properties of the $*$-characteristic pairs. First, there is a one-to-one correspondence between $*$-infinitely divisible probability measures and pairs (γ, σ). Indeed, for all finite measure σ on $\mathbb{R}$, and all constant $\gamma \in \mathbb{R}$, there exists a unique $*$-infinitely divisible probability measure such that (γ, σ) is the $*$-characteristic pair for μ. Second, the $*$-characteristic pairs linearize the convolution: let μ_1 and μ_2 be two $*$-infinitely divisible measures with respective $*$-characteristic pairs (γ_1, σ_1) and (γ_2, σ_2). The measure $\mu_1 * \mu_2$ is a $*$-infinitely divisible measure with $*$-characteristic pair $(\gamma_1 + \gamma_2, \sigma_1 + \sigma_2)$.

Let us review another, perhaps more classical, characterization of infinitely divisible measures. Let μ be $*$-infinitely divisible and (γ, σ) be its $*$-characteristic pair. We set

$$\begin{aligned} &a = \sigma(\{0\}), \qquad \rho(\mathrm{d}x) = \frac{1+x^2}{x^2} \cdot 1_{\mathbb{R}\setminus\{0\}}(x)\sigma(\mathrm{d}x) \quad \text{and} \\ &\eta = \gamma + \int_{\mathbb{R}} x\left(1_{[-1,1]}(x) - \frac{1}{1+x^2}\right)\rho(\mathrm{d}x). \end{aligned} \tag{2.1}$$

The triplet (η, a, ρ) is called the *$*$-characteristic triplet* for μ. Observe that ρ is such that the function $x \mapsto \min(1, x^2)$ is ρ-integrable and $\rho(\{0\}) = 0$. Such a measure is called a *Lévy measure* on $\mathbb{R}$. Conversely, for all (η, a, ρ) with $\eta \in \mathbb{R}$, $a \geq 0$ and ρ a Lévy measure on $\mathbb{R}$, there exists a unique $*$-infinitely divisible probability measure such that (η, a, ρ) is the $*$-characteristic triplet for μ.

EXAMPLE 2.2. Here are three important classes of $*$-infinitely divisible measures:

1. For any constant η in $\mathbb{R}$, the Dirac distribution δ_η is in $\mathcal{ID}(\mathbb{R}, *)$, and its $*$-characteristic triplet is $(\eta, 0, 0)$.

2. For any constant $a > 0$, the Gaussian distribution of variance a is

$$\mathcal{N}_a(\mathrm{d}x) = \frac{1}{\sqrt{2\pi a}} e^{-x^2/(2a)}\,\mathrm{d}x \in \mathcal{ID}(\mathbb{R}, *)$$

whose $*$-characteristic triplet is $(0, a, 0)$.

3. For any constant $\lambda > 0$ and any probability measure $\rho \in \mathcal{P}(\mathbb{R})$, the compound Poisson distribution with rate λ and jump distribution ρ is

$$\mathrm{Poiss}^*_{\lambda,\rho} = e^{-\lambda} \sum_{n\in\mathbb{N}} \frac{\lambda^n}{n!} \rho^{*n} \in \mathcal{ID}(\mathbb{R}, *)$$

whose $*$-characteristic triplet is $(\lambda \int_{[-1,1]} x\rho(\mathrm{d}x), 0, \lambda\rho_{|\mathbb{R}\setminus\{0\}})$. One important particular case is when $\rho = \delta_1$: the Poisson distribution Poiss^*_λ of mean λ is

$$\mathrm{Poiss}^*_\lambda(\mathrm{d}x) = \mathrm{Poiss}^*_{\lambda,\delta_1}(\mathrm{d}x) = e^{-\lambda} \sum_{n\in\mathbb{N}} \frac{\lambda^n}{n!} \delta_n \in \mathcal{ID}(\mathbb{R}, *).$$

2.2. *The Bercovici–Pata bijection.* In [5], Bercovici and Pata proved that all results of the previous section stay true if one replaces the classical convolution $*$ by the free additive convolution $\boxplus$. This leads to the *Bercovici–Pata bijection* Λ from $\mathcal{ID}(\mathbb{R}, *)$ to $\mathcal{ID}(\mathbb{R}, \boxplus)$ which maps a $*$-infinitely divisible measure with $*$-characteristic pair (γ, σ) to the $\boxplus$-infinitely divisible measure with $\boxplus$-characteristic pair (γ, σ). Its importance is due to the following theorem.

THEOREM 2.3 ([5]). *The Bercovici–Pata bijection Λ has the following properties*:

1. *For all $\mu, \nu \in \mathcal{ID}(\mathbb{R}, *)$, $\Lambda(\mu * \nu) = \Lambda(\mu) \boxplus \Lambda(\nu)$.*

2. *For all natural numbers $k_1 < k_2 < \cdots$, all sequence $(\mu_n)_{n\in\mathbb{N}^*}$ of probability measures on $\mathbb{R}$ and all $*$-infinitely divisible measure μ, the measures $\mu_n^{*k_n}$ converge weakly to μ if and only if the measures $\mu_n^{\boxplus k_n}$ converge weakly to $\Lambda(\mu)$.*

EXAMPLE 2.4. Here are the free analogues of the measures presented in Example 2.2:

1. For any constant η in $\mathbb{R}$, we have $\Lambda(\delta_\eta) = \delta_\eta \in \mathcal{ID}(\mathbb{R}, \boxplus)$, and its $\boxplus$-characteristic triplet is $(\eta, 0, 0)$.

2. For any constant $a > 0$, the semi-circular distribution of variance a is

$$\mathcal{S}_a(\mathrm{d}x) = \frac{1}{2\pi a}\sqrt{4a - x^2} \cdot 1_{[-2\sqrt{a}, 2\sqrt{a}]}(x)\,\mathrm{d}x \in \mathcal{ID}(\mathbb{R}, \boxplus)$$

whose characteristic triplet is $(0, a, 0)$. We have $\Lambda(\mathcal{N}_a) = \mathcal{S}_a$.

3. For any constant $\lambda > 0$, the free Poisson distribution with mean λ, also called the Marçenko–Pastur distribution, is

$$\mathrm{Poiss}^{\boxplus}_{\lambda,\delta_1}(\mathrm{d}x) = \begin{cases} (1-\lambda)\delta_0 + \dfrac{1}{2\pi x}\sqrt{(x-a)(b-x)}1_{a\leq x\leq b}\,\mathrm{d}x, & \text{if } 0 \leq \lambda \leq 1, \\ \dfrac{1}{2\pi x}\sqrt{(x-a)(b-x)}1_{a\leq x\leq b}\,\mathrm{d}x, & \text{if } \lambda > 1, \end{cases}$$

where $a = (1-\sqrt{\lambda})^2$ and $b = (1+\sqrt{\lambda})^2$. Its $\boxplus$-characteristic triplet is $(\lambda, 0, \lambda\delta_1)$. More generally, for any constant $\lambda > 0$ and probability measure $\rho \in \mathcal{P}(\mathbb{R})$, the free compound Poisson distribution with rate λ and jump distribution ρ is the measure $\mathrm{Poiss}^{\boxplus}_{\lambda,\rho} \in \mathcal{ID}(\mathbb{R}, \boxplus)$ whose $\boxplus$-characteristic triplet is $(\lambda \int_{[-1,1]} x\rho(\mathrm{d}x), 0, \lambda\rho)$. We have $\Lambda(\mathrm{Poiss}^{*}_{\lambda,\rho}) = \mathrm{Poiss}^{\boxplus}_{\lambda,\rho}$.

We finish this section with a technical lemma, which is a straightforward reformulation of Theorem 2.1, using the relation given by (2.1).

LEMMA 2.5. *Let $\mu \in \mathcal{ID}(\mathbb{R}, \boxplus)$ and (η, a, ρ) be its $\boxplus$-characteristic triplet. Let $k_1 < k_2 < \cdots$ be natural numbers and $(\mu_n)_{n\in\mathbb{N}^*}$ a sequence of probability measures on $\mathbb{R}$ such that the measures $\mu_n^{\boxplus k_n}$ converge weakly to μ. Then, for all $f : \mathbb{R} \to \mathbb{C}$ continuous, bounded, and such that $f(x) \sim_{x\to 0} f_0 x^2$, we have*

$$\lim_{n\to\infty} k_n \int_{\mathbb{R}} f\,\mathrm{d}\mu_n = \int_{\mathbb{R}} f\,\mathrm{d}\rho + af_0 \quad \textit{and} \quad \lim_{n\to\infty} k_n \int x 1_{[-1,1]}(x)\,\mathrm{d}\mu_n(x) = \eta.$$

2.3. *Classical infinite divisibility on $\mathbb{U}$.* As we will now see, the particularity of $\mathcal{ID}(\mathbb{U}, \circledast)$ is the existence of idempotent measures, an infinite class which has no equivalent in $\mathcal{ID}(\mathbb{R}, *)$, $\mathcal{ID}(\mathbb{R}, \boxplus)$ or $\mathcal{ID}(\mathbb{U}, \boxtimes)$. Our references in this section are [14, 31, 33].

A probability measure μ on $\mathbb{U}$ is said to be idempotent if $\mu \circledast \mu = \mu$. Each compact subgroup of $\mathbb{U}$ leads to an idempotent measure given by its Haar measure. More concretely, let $m \in \mathbb{N}$. The mth roots of unity form a subgroup of $\mathbb{U}$, whose Haar measure is denoted by λ_m. We have $\lambda_m \circledast \lambda_m = \lambda_m$ and consequently $\lambda_m \in \mathcal{ID}(\mathbb{U}, \circledast)$. We denote by λ, or λ_∞, the Haar measure on $\mathbb{U}$, which is also $\circledast$-infinitely divisible. Fortunately, the measures $(\lambda_m)_{m\in\mathbb{N}\cup\{\infty\}}$ are the unique measures on $\mathbb{U}$ which are idempotent.

How can we identify measures of $\mathcal{ID}(\mathbb{U}, \circledast)$ which are not idempotent? Recall that $\mathcal{M}_*$ is the set of probability measures μ on $\mathbb{U}$ such that $\int_{\mathbb{U}} \zeta\,\mathrm{d}\mu(\zeta) \neq 0$. It is easy to see that measures in $\mathcal{M}_*$ are not idempotent, with the exception of δ_1. In fact, every measure in $\mathcal{ID}(\mathbb{U}, \circledast)$ factorizes into the product of an idempotent measure with a measure in $\mathcal{ID}(\mathbb{U}, \circledast) \cap \mathcal{M}_*$. For the study of $\mathcal{ID}(\mathbb{U}, \circledast) \cap \mathcal{M}_*$, it is useful to introduce the *characteristic function*: for all probability measure μ on $\mathbb{U}$, it is the function $\widehat{\mu} : \mathbb{Z} \to \mathbb{C}$ defined for all $k \in \mathbb{Z}$ by

$$\widehat{\mu}(k) = \int_{\mathbb{U}} \zeta^k\,\mathrm{d}\mu(\zeta).$$

It is multiplicative for the convolution $\circledast$ in the sense that, for all μ, ν probability measures on $\mathbb{U}$, and all $k \in \mathbb{Z}$, we have

$$\widehat{\mu \circledast \nu}(k) = \widehat{\mu}(k) \cdot \widehat{\nu}(k). \tag{2.2}$$

For all $m \in \mathbb{N}^*$ and $k \in \mathbb{Z}$, we obviously have $\widehat{\lambda_m}(k) = 1$ if k is divisible by m and 0 if not. Using the characteristic function, we can now characterize the measures in $\mathcal{ID}(\mathbb{U}, \circledast) \cap \mathcal{M}_*$. Let $\mu \in \mathcal{ID}(\mathbb{U}, \circledast) \cap \mathcal{M}_*$. There exists a finite measure ν on $\mathbb{U}$ and a real $\alpha \in \mathbb{R}$ such that, for all $k \in \mathbb{Z}$,

$$\widehat{\mu}(k) = e^{ik\alpha} \exp\left(\int_{\mathbb{U}} \underbrace{\frac{\zeta^k - 1 - ik\Im(\zeta)}{1 - \Re(\zeta)}}_{=-k^2 \text{ if } \zeta=1} \, d\nu(\zeta) \right).$$

Unfortunately, the pair $(e^{i\alpha}, \nu)$ is not unique (see the end of the current section). We say that $(e^{i\alpha}, \nu)$ is *a $\circledast$-characteristic pair* for μ. Conversely, for all pair (ω, ν) such that $\omega \in \mathbb{U}$ and ν is a finite measure on $\mathbb{U}$, there exists a unique $\circledast$-infinitely divisible measure μ which admits (ω, ν) as a $\circledast$-characteristic pair.

Similar to the additive case, we introduce now the characteristic triplet. Let $\mu \in \mathcal{ID}(\mathbb{U}, \circledast) \cap \mathcal{M}_*$ and let (ω, ν) be a $\circledast$-characteristic pair for μ. We set

$$b = 2\nu(\{1\}) \quad \text{and} \quad \upsilon(d\zeta) = \frac{1}{1 - \Re\zeta} \cdot 1_{\mathbb{U}\setminus\{1\}}(\zeta)\nu(d\zeta). \tag{2.3}$$

We have, for all $k \in \mathbb{Z}$,

$$\widehat{\mu}(k) = \omega^k \exp\left(-\frac{1}{2}bk^2 + \int_{\mathbb{U}} (\zeta^k - 1 - ik\Im(\zeta)) \, d\upsilon(\zeta) \right).$$

We say that (ω, b, υ) is *a $\circledast$-characteristic triplet* for μ. Let us remark that $\upsilon(\{1\}) = 0$ and $\int_{\mathbb{U}}(1 + \Re(\zeta)) \, d\upsilon(\zeta) < +\infty$. Such a measure is called a *Lévy measure* on $\mathbb{U}$. As expected, for all (ω, b, υ) with $\omega \in \mathbb{U}$, $b \geq 0$ and υ a Lévy measure on $\mathbb{U}$, there exists a unique $\circledast$-infinitely divisible probability measure such that (ω, b, υ) is a $\circledast$-characteristic triplet for μ. Moreover, for all μ_1 and μ_2 $\circledast$-infinitely divisible measures with $\circledast$-characteristic triplets $(\omega_1, b_1, \upsilon_1)$ and $(\omega_2, b_2, \upsilon_2)$, we see thanks to (2.2) that $\mu_1 \circledast \mu_2 \in \mathcal{ID}(\mathbb{U}, \circledast) \cap \mathcal{M}_*$ with $\circledast$-characteristic triplet $(\omega_1\omega_2, b_1 + b_2, \upsilon_1 + \upsilon_2)$.

To sum up the previous discussion, for all $\mu \in \mathcal{ID}(\mathbb{U}, \circledast)$, there exist $m \in \mathbb{N} \cup \{\infty\}$, $\omega \in \mathbb{U}$ and ν a finite measure on $\mathbb{U}$ such that, for all $k \in \mathbb{Z}$,

$$\widehat{\mu}(k) = \widehat{\lambda_m}(k) \cdot \omega^k \exp\left(\int_{\mathbb{U}} \underbrace{\frac{\zeta^k - 1 - ik\Im(\zeta)}{1 - \Re(\zeta)}}_{=-k^2 \text{ if } \zeta=1} \, d\nu(\zeta) \right).$$

EXAMPLE 2.6. Here again, we can distinguish three classes of $\circledast$-infinitely divisible measures:

1. For any constant $\omega \in \mathbb{U}$, $(\omega, 0, 0)$ is a $\circledast$-characteristic triplet of the Dirac distribution $\delta_\omega \in \mathcal{ID}(\mathbb{U}, \circledast)$.

2. For any constant $b > 0$, the wrapped Gaussian distribution of parameter b is $\mathbf{e}_*(\mathcal{N}_b) \in \mathcal{ID}(\mathbb{U}, \circledast)$ whose one $\circledast$-characteristic triplet is $(1, b, 0)$.

3. For any constant $\lambda > 0$ and any probability measure υ on $\mathbb{U}$, the compound Poisson distribution with rate λ and jump distribution υ is

$$\mathrm{Poiss}^{\circledast}_{\lambda,\upsilon} = e^{-\lambda} \sum_{n \in \mathbb{N}} \frac{\lambda^n}{n!} \upsilon^{\circledast n} \in \mathcal{ID}(\mathbb{U}, \circledast)$$

whose one $\circledast$-characteristic triplet is $(\exp(i\lambda \int_{\mathbb{U}} \Im \,\mathrm{d}\upsilon), 0, \lambda \upsilon_{|\mathbb{U}\setminus\{1\}})$.

We give now a case of $\circledast$-infinitely divisible measure which admits two different $\circledast$-characteristic pairs. Set

$$\mu = e^{-\pi} \left(\frac{\cosh(\pi) + 1}{2} \delta_1 + \frac{\cosh(\pi) - 1}{2} \delta_{-1} + \frac{\sinh(\pi)}{2} \delta_i + \frac{\sinh(\pi)}{2} \delta_{-i} \right).$$

For all $n \in \mathbb{Z}$, we have $\widehat{\mu}(4n) = 1$, $\widehat{\mu}(4n+1) = \widehat{\mu}(4n+3) = e^{-\pi}$ and $\widehat{\mu}(4n+2) = e^{-2\pi}$. It is immediate that, for $\upsilon = \pi\delta_i$ or $\upsilon = \pi\delta_{-i}$, we have

$$\widehat{\mu}(n) = \exp\left(\int_{\mathbb{U}} \big(\zeta^n - 1 - in\Im(\zeta)\big) \,\mathrm{d}\upsilon(\zeta) \right).$$

Thus, the measure μ admits $(1, 0, \pi\delta_i)$ and $(1, 0, \pi\delta_{-i})$ as $\circledast$-characteristic triplets. One can also see [14] for others examples.

2.4. *The convolution $\boxtimes$ and the S-transform.* The free multiplicative convolution $\boxtimes$ can be described succinctly in terms of the S-transform. Let us explain how it works.

Let μ be a finite measure on $\mathbb{U}$. For all $k \in \mathbb{N}$, we set $m_k(\mu) = \int_{\mathbb{C}} \zeta^k \,\mathrm{d}\mu(\zeta)$, which is finite, and we call $(m_k(\mu))_{k \in \mathbb{N}}$ the *moments* of μ. We consider the formal power series

$$M_\mu(z) = \sum_{k=0}^{\infty} m_k(\mu) z^k.$$

Let us assume that $\mu \in \mathcal{M}_*$. We define S_μ, the *S-transform* of μ, to be the formal power series such that $zS_\mu(z)/(1+z)$ is the inverse under composition of $M_\mu(z) - 1$. The S-transform is a $\boxtimes$-homomorphism (see [6]): for all μ and $\nu \in \mathcal{M}_*$,

$$S_{\mu \boxtimes \nu} = S_\mu \cdot S_\nu.$$

For all $\mu \in \mathcal{M}_*$, the series $S_\mu(z)$ is convergent in a neighborhood of 0, and we can therefore identify S_μ with a function which is analytic in a neighborhood of zero. Sometimes it will be convenient to use the function

$$\Sigma_\mu(z) = S_\mu\big(z/(1-z)\big)$$

which is also analytic in a neighborhood of 0.

2.5. *Free infinite divisibility on* $\mathbb{U}$. For the free multiplicative convolution, the existence of different proper subgroups of $\mathbb{U}$ does not imply the existence of different idempotent measures. Indeed, the Haar measure λ and δ_1 are the unique probability measures on $\mathbb{U}$ which are idempotent. Moreover, λ is an absorbing element for $\boxtimes$ and it is the unique $\boxtimes$-infinitely divisible measure in $\mathcal{ID}(\mathbb{U},\boxtimes)\setminus\mathcal{M}_*$ according to [6]. Consequently, we will focus our study on $\mathcal{ID}(\mathbb{U},\boxtimes)\cap\mathcal{M}_*$.

Let $\mu\in\mathcal{M}_*$ be a $\boxtimes$-infinitely divisible measure. From Theorem 6.7. of [6], there exists a unique finite measure $\nu\in\mathcal{M}_{\mathbb{U}}$ and a real $\alpha\in\mathbb{R}$ such that

$$\Sigma_\mu(z)=\exp\left(-i\alpha+\int_{\mathbb{U}}\frac{1+\zeta z}{1-\zeta z}\,\mathrm{d}\nu(\zeta)\right).$$

The pair $(e^{i\alpha},\nu)$ is called the $\boxtimes$-*characteristic pair* for μ, and, on the contrary to the classical case, it is uniquely determined by μ. We have

$$S_\mu(z)=e^{-i\alpha}\exp\left(\int_{\mathbb{U}}\frac{1+z+\zeta z}{1+z-\zeta z}\,\mathrm{d}\nu(\zeta)\right). \tag{2.4}$$

We observe that, for $\zeta\neq 1$, we have

$$\frac{1+z+\zeta z}{1+z-\zeta z}=\frac{1}{1-\Re\zeta}\left(i\Im(\zeta)+\frac{1-\zeta}{1+z(1-\zeta)}\right),$$

which implies that, defining $\omega=e^{i\alpha}$, $b=2\nu(\{1\})$ and $\upsilon(\mathrm{d}\zeta)=\frac{1}{1-\Re\zeta}\times 1_{\mathbb{U}\setminus\{1\}}(\zeta)\nu(\mathrm{d}\zeta)$, we have

$$S_\mu(z)=\omega^{-1}\exp\left(\frac{b}{2}+bz+\int_{\mathbb{U}}i\Im(\zeta)+\frac{1-\zeta}{1+z(1-\zeta)}\,\mathrm{d}\upsilon(\zeta)\right). \tag{2.5}$$

We will call (ω,b,υ) the $\boxtimes$-*characteristic triplet* for μ. Conversely, for all triplet (ω,b,υ) such that $\omega\in\mathbb{U}$, $b\in\mathbb{R}^+$ and υ is a Lévy measure on $\mathbb{U}$, there exists a unique $\boxtimes$-infinitely divisible measure μ whose $\boxtimes$-characteristic triplet is (ω,b,υ). Indeed, according to Theorem 6.7 of [6], if we define

$$v(z)=-\operatorname{Log}(\omega)+\frac{b}{2}+bz+\int_{\mathbb{U}}i\Im(\zeta)+\frac{1-\zeta}{1+z(1-\zeta)}\,\mathrm{d}\upsilon(\zeta)$$

using the principal value Log, then the function $S(z)=\exp(v(z))$ is the S-transform of a unique $\boxtimes$-infinitely measure $\mu\in\mathcal{M}_*$.

Let $\mu_1,\mu_2\in\mathcal{M}_*$ be two $\boxtimes$-infinitely divisible measures with respective $\boxtimes$-characteristic triplets $(\omega_1,b_1,\upsilon_1)$ and $(\omega_2,b_2,\upsilon_2)$. The measure $\mu_1\boxtimes\mu_2\in\mathcal{M}_*$ is a $\boxtimes$-infinitely divisible measure with $\boxtimes$-characteristic triplet $(\omega_1\omega_2,b_1+b_2,\upsilon_1+\upsilon_2)$.

EXAMPLE 2.7. The three classes of $\boxtimes$-infinitely divisible measures are:

1. For any constant $\omega\in\mathbb{U}$, $(\omega,0,0)$ is a $\boxtimes$-characteristic triplet of the Dirac distribution $\delta_\omega\in\mathcal{ID}(\mathbb{U},\boxtimes)$.

2. For any constant $b > 0$, the measure on $\mathbb{U}$ analogous to the Gaussian distribution law is the measure $\mathcal{B}_b \in \mathcal{ID}(\mathbb{U}, \boxtimes)$ whose $\boxtimes$-characteristic triplet is $(1, b, 0)$; it is the law of a free unitary Brownian motion at time b.

3. For any constant $\lambda > 0$ and any probability measure υ on $\mathbb{U}$, the free compound Poisson distribution with rate λ and jump distribution υ is the measure $\mathrm{Poiss}^{\boxtimes}_{\lambda,\upsilon} \in \mathcal{ID}(\mathbb{U}, \boxtimes)$ whose $\boxtimes$-characteristic triplet is $(\exp(i\lambda \int_{\mathbb{U}} \Im \, \mathrm{d}\upsilon), 0, \lambda\upsilon_{|\mathbb{U}\setminus\{1\}})$.

3. Homomorphisms between $\mathcal{ID}(\mathbb{R}, *)$, $\mathcal{ID}(\mathbb{U}, \circledast)$, $\mathcal{ID}(\mathbb{R}, \boxplus)$ and $\mathcal{ID}(\mathbb{U}, \boxtimes)$. In this section, we define $\mathbf{e}_\boxplus$ and Γ and prove Theorem 1. The definitions and the commutativity of (1.1) is a routine program. The very difficulty consists in proving the first item of Theorem 1, or equivalently Theorem 3.9. We shall do it in Section 3.2.

3.1. *Definitions of* $\mathbf{e}_\boxplus$ *and* Γ. In order to motivate the definition of $\mathbf{e}_\boxplus$, we start by indicating how a $*$-characteristic triplet is transformed by the homomorphism $\mathbf{e}_*$.

Let us recall that, for all measure μ on $\mathbb{R}$, $\mathbf{e}_*(\mu)$ denotes the push-forward measure of μ by the map $\mathbf{e} : x \mapsto e^{ix}$. Let us denote by $\mathbf{e}_*(\mu)_{|\mathbb{U}\setminus\{1\}}$ the measure induced by $\mathbf{e}_*(\mu)$ on $\mathbb{U} \setminus \{1\}$.

PROPOSITION 3.1. *For all $\mu \in \mathcal{ID}(\mathbb{R}, *)$ with $*$-characteristic triplet (η, a, ρ),*

$$(\omega, b, \upsilon) = \left(\exp\left(i\eta + i\int_{\mathbb{R}} (\sin(x) - 1_{[-1,1]}(x)x)\rho(\mathrm{d}x)\right), a, \mathbf{e}_*(\rho)_{|\mathbb{U}\setminus\{1\}}\right)$$

is a $\circledast$-characteristic triplet of $\mathbf{e}_*(\mu)$.

PROOF. First of all, the Fourier transform of a $*$-infinitely divisible measure is well known (see [5, 32]): for all $\theta \in \mathbb{R}$, we have

$$\int_{\mathbb{R}} e^{i\theta x} \, \mathrm{d}\mu(x) = \exp\left(i\eta\theta - \frac{1}{2}a\theta^2 + \int_{\mathbb{R}} (e^{i\theta x} - 1 - i\theta x 1_{[-1,1]}(x)) \, \mathrm{d}\rho(x)\right).$$

Let $n \in \mathbb{N}$. We have

$$\begin{aligned}
\widehat{\mathbf{e}_*(\mu)}(n) &= \int_{\mathbb{U}} \zeta^n \, \mathrm{d}(\mathbf{e}_*(\mu))(\zeta) = \int_{\mathbb{R}} e^{inx} \, \mathrm{d}\mu(x) \\
&= \exp\left(i\eta n - \frac{1}{2}an^2 + \int_{\mathbb{R}} (e^{inx} - 1 - inx 1_{[-1,1]}(x)) \, \mathrm{d}\rho(x)\right) \\
&= \exp\left(i\eta n + in\int_{\mathbb{R}} (\sin(x) - 1_{[-1,1]}(x)x)\rho(\mathrm{d}x) - \frac{1}{2}an^2 \right. \\
&\qquad \left. + \int_{\mathbb{R}} (e^{inx} - 1 - in\sin(x)) \, \mathrm{d}\rho(x)\right) \\
&= \omega^n \exp\left(-\frac{1}{2}bn^2 + \int_{\mathbb{U}} (\zeta^n - 1 - in\Im(\zeta)) \, \mathrm{d}\upsilon(\zeta)\right),
\end{aligned}$$

which proves that (ω, b, υ) is a $\circledast$-characteristic triplet of $\mathbf{e}_*(\mu)$. $\square$

We define $\mathbf{e}_{\boxplus} : \mathcal{ID}(\mathbb{R}, \boxplus) \to \mathcal{ID}(\mathbb{U}, \boxtimes)$ by analogy with the previous proposition.

DEFINITION 3.2. For all $\mu \in \mathcal{ID}(\mathbb{R}, \boxplus)$ with $\boxplus$-characteristic triplet (η, a, ρ), we define $\mathbf{e}_{\boxplus}(\mu)$ to be the $\boxtimes$-infinitely divisible measure on $\mathbb{U}$ with $\boxtimes$-characteristic triplet

$$(\omega, b, \upsilon) = \left(\exp\left(i\eta + i\int_{\mathbb{R}} (\sin(x) - 1_{[-1,1]}(x)x)\rho(\mathrm{d}x)\right), a, \mathbf{e}_*(\rho)_{|\mathbb{U}\setminus\{1\}}\right). \tag{3.1}$$

The definition of $\Gamma : \mathcal{ID}(\mathbb{U}, \boxtimes) \to \mathcal{ID}(\mathbb{U}, \circledast)$ is even simpler.

DEFINITION 3.3. For all $\mu \in \mathcal{ID}(\mathbb{U}, \boxtimes) \cap \mathcal{M}_*$ with characteristic triplet (ω, b, υ), we define $\Gamma(\mu)$ to be the $\circledast$-infinitely divisible measure on $\mathbb{U}$ with characteristic triplet (ω, b, υ). Moreover, for λ being the Haar measure of $\mathbb{U}$, we set $\Gamma(\lambda) = \lambda$.

PROPOSITION 3.4. *The maps $\mathbf{e}_{\boxplus}$ and Γ have the following properties*:

1. *For all μ and $\nu \in \mathcal{ID}(\mathbb{R}, \boxplus)$, we have $\mathbf{e}_{\boxplus}(\mu \boxplus \nu) = \mathbf{e}_{\boxplus}(\mu) \boxtimes \mathbf{e}_{\boxplus}(\nu)$.*
2. *For all μ and $\nu \in \mathcal{ID}(\mathbb{U}, \boxtimes)$, we have $\Gamma(\mu \boxtimes \nu) = \Gamma(\mu) \circledast \Gamma(\nu)$.*
3. *We have $\Gamma \circ \mathbf{e}_{\boxplus} \circ \Lambda = \mathbf{e}_*$.*

PROOF. The results follow from the comparison of the different characteristic triplets. □

We summarize here the successive action of $\Lambda, \mathbf{e}_{\boxplus}, \Gamma$ and $\mathbf{e}_*$ on, respectively, a Dirac measure δ_η ($\eta \in \mathbb{R}$), a Gaussian measure $\mathcal{N}_b$ ($b > 0$), and a compound Poisson distribution with rate $\lambda > 0$ and jump distribution ρ (Example 2.2). As expected, their images are, respectively, their free analogues on $\mathbb{R}$ (Example 2.4), their free analogues on $\mathbb{U}$ (Example 2.7) and their multiplicative analogues on $\mathbb{U}$ (Example 2.6):

$$\begin{array}{ccccccc} & \Lambda & & \mathbf{e}_{\boxplus} & & \Gamma & \\ \delta_\eta & \longmapsto & \delta_\eta & \longmapsto & \delta_{e^{i\eta}} & \longmapsto & \delta_{e^{i\eta}} \\ \mathcal{N}_b & \longmapsto & \mathcal{S}_b & \longmapsto & \mathcal{B}_b & \longmapsto & \mathbf{e}_*(\mathcal{N}_b) \\ \mathrm{Poiss}^*_{\lambda,\rho} & \longmapsto & \mathrm{Poiss}^{\boxplus}_{\lambda,\rho} & \longmapsto & \mathrm{Poiss}^{\boxtimes}_{\lambda,\mathbf{e}_*(\rho)} & \longmapsto & \mathrm{Poiss}^{\circledast}_{\lambda,\mathbf{e}_*(\rho)}. \end{array}$$

3.2. *A limit theorem.* The definition of Γ is justified, if needed, by the following result of Chistyakov and Götze.

THEOREM 3.5 [14]. *For all $\mu \in \mathcal{ID}(\mathbb{U}, \boxtimes) \cap \mathcal{M}_*$, all natural numbers $k_1 < k_2 < \cdots$ and all sequence $(\mu_n)_{n\in\mathbb{N}^*}$ of probability measures in $\mathcal{M}_*$ such that the measures $\mu_n^{\boxtimes k_n}$ converge weakly to μ, the measures $\mu_n^{\circledast k_n}$ converge weakly to $\Gamma(\mu)$.*

The rest of this section is devoted to proving an analogous theorem for $\mathbf{e}_{\boxplus}$. This goal is achieved in Theorem 3.9. Let us start by a key result, interesting in its own, about the convergence toward a $\boxtimes$-infinitely divisible measure. The following proposition is the analogue of Theorem 2.1 for the convolution $\boxtimes$. We refer the reader to Theorem 4.3 of [7] and Theorem 2.3 of [14] for other similar criterions. The major difference between these results and ours is the shift of μ_n considered: in Proposition 3.6, we consider the angular part $\omega_n = m_1(\mu_n)/|m_1(\mu_n)|$ of the mean of μ_n.

For all measure μ_n on $\mathbb{U}$, all $\omega_n \in \mathbb{U}$ and all $k_n \in \mathbb{N}$, we denote by $k_n(1 - \Re(\zeta))\,\mathrm{d}\mu_n(\omega_n\zeta)$ the measure such that, for all bounded Borel functions f on $\mathbb{U}$,

$$\int_{\mathbb{U}} f(\zeta) k_n\big(1-\Re(\zeta)\big)\,\mathrm{d}\mu_n(\omega_n\zeta) = k_n \int_{\mathbb{U}} f\big(\omega_n^{-1}\zeta\big)\big(1-\Re\big(\omega_n^{-1}\zeta\big)\big)\,\mathrm{d}\mu_n(\zeta).$$

PROPOSITION 3.6. *Let $\mu \in \mathcal{ID}(\mathbb{U}, \boxtimes)$ with $\boxtimes$-characteristic pair (ω, ν). Let $k_1 < k_2 < \cdots$ be a sequence of natural numbers. Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of measures in $\mathcal{M}_*$ and $(\omega_n)_{n\in\mathbb{N}}$ a sequence of elements of $\mathbb{U}$ such that, for all $n \in \mathbb{N}$, $\omega_n = m_1(\mu_n)/|m_1(\mu_n)|$. The following assertions are equivalent*:

1. *The measures $\underbrace{\mu_n \boxtimes \cdots \boxtimes \mu_n}_{k_n \text{ times}}$ converge weakly to μ.*
2. *The measures*

$$\mathrm{d}\nu_n(x) = k_n\big(1-\Re(\zeta)\big)\,\mathrm{d}\mu_n(\omega_n\zeta)$$

converge weakly to ν and

$$\lim_{n\to\infty} \omega_n^{k_n} = \omega.$$

In concrete cases, the second item is often easier to verify. For example, it allows us to infer that, for any constant $\lambda > 0$ and any probability measure υ on $\mathbb{U}$, the measure $\mathrm{Poiss}^{\boxtimes}_{\lambda,\upsilon}$ is the weak limit of $((1-\lambda/n)\delta_1 + (\lambda/n)\upsilon)^{\boxtimes n}$ as n tends to ∞.

We would point out the recent work [1] which proves that the convergence of Proposition 3.6 above implies local convergences of the probability densities.

PROOF OF PROPOSITION 3.6. Let $n \in \mathbb{N}$. We set $r_n = |m_1(\mu_n)|$, so that $m_1(\mu_n) = r_n\omega_n$. We define also $\mu_n^\circ \in \mathcal{M}_*$ such that $\mathrm{d}\mu_n^\circ(\zeta) = \mathrm{d}\mu_n(\omega_n\zeta)$. The measure μ_n° will be the link between μ_n and ν_n. Observe that $M_{\mu_n^\circ}(z) = M_{\mu_n}(\omega_n^{-1}z)$, which implies that $S_{\mu_n^\circ}(z) = \omega_n S_{\mu_n}(z)$. The first step of the proof is to write M_{ν_n} with the help of $M_{\mu_n^\circ}$. For all $\zeta \in \mathbb{U}$ and $z \in \mathbb{C}$ sufficiently small, we have

$$2\frac{1-\Re\zeta}{1-\zeta z} = (z-1)\left[(1-z)\frac{\zeta}{1-\zeta z} - 1\right] + 1 - \bar{\zeta}.$$

Integrating with respect to μ_n°, and remarking that

$$\int_{\mathbb{U}} \bar{\zeta}\, \mathrm{d}\mu_n^\circ(\zeta) = \overline{\int_{\mathbb{U}} \zeta\, \mathrm{d}\mu_n(\zeta)/\omega_n} = \bar{r}_n = r_n,$$

we deduce that

$$\frac{2}{k_n} M_{\nu_n} = (z-1)\left[\frac{1-z}{z}\left(M_{\mu_n^\circ} - 1 - \frac{z}{1-z}\right)\right] + (1-r_n). \tag{3.2}$$

Let us recall the useful information about the S-transform: it is a $\boxtimes$-homomorphism and $zS(z)/(1+z)$ is the inverse under composition of $M(z)-1$ (see Section 2.4). Moreover, we need also the following general result.

PROPOSITION 3.7 ([6]). *Let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of measures in $\mathcal{M}_*$ and $\mu \in \mathcal{M}_*$. Then the weak convergence of $(\mu_n)_{n\in\mathbb{N}}$ to μ is equivalent to the uniform convergence of S_{μ_n} in some neighborhood of* 0.

Let us suppose that the measures $\mu_n^{\boxtimes k_n}$ converge weakly to μ. Then the series $S_{\mu_n^{\boxtimes k_n}} = (S_{\mu_n})^{k_n}$ converges uniformly to S_μ in some neighborhood of 0. Therefore, we have $\lim_{n\to\infty} m_1(\mu_n)^{-k_n} = \lim_{n\to\infty} S_{\mu_n}^{k_n}(0) = S_\mu(0)$. Thanks to (2.4), we know that $S_\mu(0) = \omega^{-1} e^{\nu(\mathbb{U})}$, which implies that $\lim_{n\to\infty} \omega_n^{k_n} = \omega$ and $\lim_{n\to\infty} r_n^{k_n} = e^{-\nu(\mathbb{U})}$. Of course, it implies that $\lim_{n\to\infty} k_n(r_n - 1) = -\nu(\mathbb{U})$. It remains now to prove that ν_n converges weakly to ν. Let us denote by $u(z)$ the series

$$u(z) = \int_{\mathbb{U}} \frac{1+z+\zeta z}{1+z-\zeta z}\, \mathrm{d}\nu(\zeta).$$

Thanks to (2.4), we have $\omega S_\mu(z) = \exp(u(z))$, and consequently, $S_{\mu_n^\circ}(z)^{k_n} = \omega_n^{k_n} S_{\mu_n}(z)^{k_n}$ converges uniformly to $\omega S_\mu(z) = \exp(u(z))$ in a neighborhood of 0. Thus, in some neighborhood of 0, the only possible limit of the sequence $S_{\mu_n^\circ}(z)$ is 1, the S-transform of δ_1. From the compactness of the set of probability measures on $\mathbb{U}$ with respect to the weak convergence and Proposition 3.7, we deduce that the measures μ_n° converge weakly to this unique cluster point δ_1, and that $S_{\mu_n^\circ}(z)$ converges uniformly to 1 in a neighborhood of 0. For sufficiently large n, the principal branch $\log(S_{\mu_n^\circ}(z))$ is defined for z in a neighborhood of 0, and $k_n \log(S_{\mu_n^\circ}(z))$ converges uniformly to $u(z)$. Since $\log(w) \sim_{\omega\to 1} w-1$, we conclude that

$$\lim_{n\to\infty} k_n\big(S_{\mu_n}^\circ(z) - 1\big) = u(z) \tag{3.3}$$

uniformly in a neighborhood of 0. At this stage of the proof, we need to inverse formal series, at least asymptotically, in order to have the relation (3.3) in terms of $M_{\mu_n^\circ}(z)$. Let us set $\chi_n(z) = zS_{\mu_n^\circ}/(z+1)$ and $\psi_n(z) = M_{\mu_n^\circ}(z) - 1$, in such a way that, in a neighborhood of 0, we have $z = \chi_n(\psi_n(z)) = \psi_n(\chi_n(z))$. Observe that $\lim_{n\to\infty} \chi_n(z) = z/(1+z)$, $\lim_{n\to\infty} \chi_n'(z) = 1/(1+z)^2$ in a neighborhood of 0, while $\lim_{n\to\infty} \psi_n(z) = z/(1-z)$ (this follows from the weak convergence of μ_n° to 1). We denote, respectively, by $\chi_\infty(z)$, $\chi'_\infty(z)$ and $\psi_\infty(z)$ those three limits.

One hand, using the mean value theorem for the real part and the imaginary part of $t \mapsto \chi_n(t\psi_n(z) + (1-t)\psi_\infty(z))/(\psi_n(z) - \psi_\infty(z))$, we observe that $\Delta_n(z)$ defined as

$$\Delta_n(z) = 1_{\psi_n(z)\neq\psi_\infty(z)} \frac{\chi_n(\psi_n(z)) - \chi_n(\psi_\infty(z))}{\psi_n(z) - \psi_\infty(z)} + 1_{\psi_n(z)=\psi_\infty(z)} \chi'_n(\psi_\infty(z))$$

converges to $\chi'_\infty(\psi_\infty(z)) = (1-z)^2$, as n tends to ∞, for z in some neighborhood of 0 (at this stage of the proof, the uniform convergence is no longer necessary). On the other hand, we can rewrite (3.3) as $\lim_{n\to\infty} k_n(\chi_n(\psi_\infty(z)) - \chi_\infty(\psi_\infty(z))) = zu(\psi_\infty(z))$. Now,

$$\begin{aligned}
0 &= \lim_{n\to\infty} k_n(z-z) \\
&= \lim_{n\to\infty} k_n\bigl(\chi_n(\psi_n(z)) - \chi_\infty(\psi_\infty(z))\bigr) \\
&= \lim_{n\to\infty} \Delta_n(z) \cdot k_n\bigl(\psi_n(z) - \psi_\infty(z)\bigr) + \lim_{n\to\infty} k_n\bigl(\chi_n(\psi_\infty(z)) - \chi_\infty(\psi_\infty(z))\bigr) \\
&= \chi'_\infty(\psi_\infty(z)) \cdot \lim_{n\to\infty} k_n\bigl(\psi_n(z) - \psi_\infty(z)\bigr) + zu\bigl(\psi_\infty(z)\bigr)
\end{aligned}$$

from which we deduce that $\lim_{n\to\infty} k_n(\psi_n(z) - \psi_\infty(z)) = -zu(\psi_\infty(z))/(1-z)^2$, or equivalently, that

$$\lim_{n\to\infty} k_n\left(M_{\mu_n^\circ}(z) - 1 - \frac{z}{1-z}\right) = \frac{-z}{(1-z)^2} u\left(\frac{z}{1-z}\right)$$

pointwise in some neighborhood of 0. This limit and the limit $-\nu(\mathbb{U}) = \lim_{n\to\infty} k_n(r_n - 1)$ put together in (3.2) yields

$$\lim_{n\to\infty} 2M_{\nu_n} = u\left(\frac{z}{1-z}\right) + \nu(\mathbb{U}).$$

But we have $u(z/(1-z)) + \nu(\mathbb{U}) = \int_{\mathbb{U}} \frac{2}{1-\zeta z}\, \mathrm{d}\nu(\zeta) = 2M_\nu$, which implies that $\lim_{n\to\infty} M_{\nu_n} = M_\nu$ pointwise in some neighborhood of 0. Consequently, ν is the unique cluster point of $\{\nu_n\}_{n\in\mathbb{N}}$. Because $\sup_{n\in\mathbb{N}} \nu_n(\mathbb{U}) = \sup_{n\in\mathbb{N}} M_{\nu_n}(0) < \infty$, the set $\{\nu_n\}_{n\in\mathbb{N}}$ is compact for the weak convergence, and finally, the measures ν_n converge weakly to ν.

Conversely, let us suppose the weak convergence of ν_n to ν and $\lim_{n\to\infty} \omega_n^{k_n} = \omega$. We can basically retrace our steps in order to arrive at $\lim_{n\to\infty} S_{\mu_n^{\boxtimes k_n}} = S_\mu$ pointwise in some neighborhood of 0. Proposition 3.7 and the compactness of the set of probability measures on $\mathbb{U}$ with respect to the weak convergence allow us to conclude that the sequence $\mu_n^{\boxtimes k_n}$ converges weakly to its unique cluster point μ. □

COROLLARY 3.8. *Let* $\mu \in \mathcal{ID}(\mathbb{U}, \boxtimes)$ *with* $\boxtimes$*-characteristic triplet* (ω, b, υ). *Let* $k_1 < k_2 < \cdots$ *be a sequence of natural numbers. Let* $(\mu_n)_{n\in\mathbb{N}}$ *be a sequence of measures in* $\mathcal{M}_*$ *and* $(\omega_n)_{n\in\mathbb{N}}$ *be such that, for all* $n \in \mathbb{N}$, $\omega_n = m_1(\mu_n)/|m_1(\mu_n)|$. *The following assertions are equivalent*:

1. *The measures* $\underbrace{\mu_n \boxtimes \cdots \boxtimes \mu_n}_{k_n \text{ times}}$ *converge weakly to* μ.

2. $\lim_{n\to\infty} \omega_n^{k_n} = \omega$ *and the measures* $\mathrm{d}\nu_n(x) = k_n(1 - \Re(\zeta))\,\mathrm{d}\mu_n(\omega_n\zeta)$ *converge weakly to* $(1 - \Re(\zeta))\,\mathrm{d}\upsilon(\zeta) + \frac{b}{2}\delta_1$.

We are now ready to prove the first main theorem of this paper.

THEOREM 3.9. *For all* $\mu \in \mathcal{ID}(\mathbb{R}, \boxplus)$, *all natural numbers* $k_1 < k_2 < \cdots$ *and all sequence* $(\mu_n)_{n\in\mathbb{N}^*}$ *of probability measures on* $\mathbb{R}$ *such that the measures* $\mu_n^{\boxplus k_n}$ *converge weakly to* μ, *the measures* $\mathbf{e}_*(\mu_n)^{\boxtimes k_n}$ *converge weakly to* $\mathbf{e}_\boxplus(\mu)$.

Let us derive right now some consequences of this theorem. It allows us to transfer limit theorems about $\boxplus$ into limit theorem about $\boxtimes$. For example, for all $b > 0$, the semi-circular measure is such that $\mathcal{S}_{b/n}^{\boxplus n} = \mathcal{S}_b$. We deduce that $\mathcal{B}_b = \mathbf{e}_\boxplus(\mathcal{S}_b)$, which is the law of a free unitary Brownian motion at time b, is the weak limit of the measures $\mathbf{e}_*(\mathcal{S}_{b/n})^{\boxtimes n}$. Using Theorem 2.3, we know also that the measures $\mathcal{N}_{b/n}^{\boxtimes n}$ converge weakly to $\mathcal{S}_b$. By consequence, $\mathcal{B}_b$ is also the weak limit of $\mathbf{e}_*(\mathcal{N}_{b/n})^{\boxtimes n}$ as n tends to ∞.

PROOF OF THEOREM 3.9. Let (η, a, ρ) be the $\boxplus$-characteristic triplet of μ, and (ω, b, υ) be the $\boxtimes$-characteristic triplet of $\mathbf{e}_\boxplus(\mu)$ given by (3.1). In order to use Corollary 3.8, we first prove that $\mathbf{e}_*(\mu_n) \in \mathcal{M}_*$ for n sufficiently large.

Because $e^{ix} - 1 = ix1_{[-1,1]}(x) + (e^{ix} - 1 - ix1_{[-1,1]}(x))$, we have

$$\left(\int_\mathbb{R} e^{ix}\,\mathrm{d}\mu_n(x) - 1\right) = i\int_\mathbb{R} x1_{[-1,1]}(x)\,\mathrm{d}\mu_n(x) + \int_\mathbb{R}\big(e^{ix} - 1 - ix1_{[-1,1]}(x)\big)\,\mathrm{d}\mu_n(x).$$

We use Lemma 2.5, and the fact that $e^{ix} - 1 - ix1_{[-1,1]}(x) \sim_{x\to 0} -\frac{1}{2}x^2$, to deduce that

$$\lim_{n\to\infty} k_n\left(\int_\mathbb{R} e^{ix}\,\mathrm{d}\mu_n(x) - 1\right) = i\eta - \frac{a}{2} + \int \big(e^{ix} - 1 - 1_{[-1,1]}(x)ix\big)\rho(\mathrm{d}x). \tag{3.4}$$

Consequently, $m_1(\mathbf{e}_*(\mu_n)) = \int_\mathbb{R} e^{ix}\,\mathrm{d}\mu_n(x)$ tends to 1 as n tends to ∞, and $\mathbf{e}_*(\mu_n) \in \mathcal{M}_*$ for n sufficiently large. Without loss of generality, we assume that $\mathbf{e}_*(\mu_n)$ is in $\mathcal{M}_*$ for all $n \in \mathbb{N}$. We set $(r_n, \omega_n)_{n\in\mathbb{N}}$ the sequence of $[0, 1] \times \mathbb{U}$ such that, for all $n \in \mathbb{N}$, we have $m_1(\mathbf{e}_*(\mu_n)) = r_n\omega_n$. Thanks to Corollary 3.8, it suffices to prove that $\lim_{n\to\infty}\omega_n^{k_n}=\omega$ and to prove that the measure $k_n(1-\Re(\zeta))\,\mathrm{d}(\mathbf{e}_*(\mu_n))(\omega_n\zeta)$ converge weakly to $(1-\Re(\zeta))\,\mathrm{d}\upsilon+\frac{b}{2}\delta_1$ to conclude.

From (3.4), we deduce that

$$\lim_{n\to\infty} r_n^{k_n}\omega_n^{k_n}=\lim_{n\to\infty}\left(\int_{\mathbb{R}} e^{ix}\,\mathrm{d}\mu_n(x)\right)^{k_n}$$
$$=\exp\left(i\eta-\frac{a}{2}+\int\left(e^{ix}-1-1_{[-1,1]}(x)ix\right)\rho(\mathrm{d}x)\right),$$

and this result can be split into

$$\lim_{n\to\infty} r_n^{k_n}=\exp\left(-\frac{a}{2}+\int\left(\cos(x)-1\right)\rho(\mathrm{d}x)\right)$$

and

$$\lim_{n\to\infty}\omega_n^{k_n}=\exp\left(i\eta+i\int\left(\sin(x)-1_{[-1,1]}(x)x\right)\rho(\mathrm{d}x)\right)=\omega.$$

Using the real logarithm, we deduce that, as n tends to ∞,

$$r_n^{-1}=1+\frac{1}{k_n}\left(\frac{a}{2}-\int\left(\cos(x)-1\right)\rho(\mathrm{d}x)\right)+o\left(\frac{1}{k_n}\right). \tag{3.5}$$

Using $\omega_n=r_n^{-1}\int_{\mathbb{R}}e^{ix}\,\mathrm{d}\mu_n(x)$, (3.4) and (3.5), it follows that, as n tends to ∞,

$$\omega_n=1+\frac{i}{k_n}\left(\eta+\int\left(\sin(x)-1_{[-1,1]}(x)x\right)\rho(\mathrm{d}x)\right)+o\left(\frac{1}{k_n}\right). \tag{3.6}$$

In order to prove that the measures $k_n(1-\Re(\zeta))\,\mathrm{d}\mathbf{e}_*\mu_n(\omega_n\zeta)$ converge weakly to $(1-\Re(\zeta))\,\mathrm{d}\upsilon+\frac{b}{2}\delta_1$, we shall use the method of moments and prove that, for all $m\in\mathbb{N}$,

$$\lim_{n\to\infty}k_n\int_{\mathbb{U}}\zeta^m\left(1-\Re(\zeta)\right)\mathrm{d}\left(\mathbf{e}_*(\mu_n)\right)(\omega_n\zeta)=\int_{\mathbb{U}}\zeta^m\left(1-\Re(\zeta)\right)\mathrm{d}\upsilon(\zeta)+\frac{b}{2}.$$

Let $n\in\mathbb{N}$. We have

$$k_n\int_{\mathbb{U}}\zeta^m\left(1-\Re(\zeta)\right)\mathrm{d}\left(\mathbf{e}_*(\mu_n)\right)(\omega_n\zeta)$$
$$=k_n\int_{\mathbb{U}}\omega_n^{-m}\zeta^m\left(1-\Re\left(\omega_n^{-1}\zeta\right)\right)\mathrm{d}\left(\mathbf{e}_*(\mu_n)\right)(\zeta)$$
$$=k_n\omega_n^{-m}\int_{\mathbb{R}}e^{imx}\left(1-\Re\left(\omega_n^{-1}e^{ix}\right)\right)\mathrm{d}\mu_n(x)$$
$$=k_n\omega_n^{-m}\int_{\mathbb{R}}e^{imx}\left(1-\Re(\omega_n)\cos(x)-\Im(\omega_n)\sin(x)\right)\mathrm{d}\mu_n(x).$$

Let us decompose the integral under study into four terms:

$$\begin{aligned}
&k_n \int_{\mathbb{U}} \zeta^m \big(1 - \Re(\zeta)\big) \,\mathrm{d}\big(\mathbf{e}_*(\mu_n)\big)(\omega_n \zeta) \\
&\quad = k_n \omega_n^{-m} \Re(\omega_n) \int_{\mathbb{R}} e^{imx} \big(1 - \cos(x)\big) \,\mathrm{d}\mu_n(x) \\
&\qquad + k_n \big(1 - \Re(\omega_n)\big) \omega_n^{-m} \int_{\mathbb{R}} e^{imx} \,\mathrm{d}\mu_n(x) \\
&\qquad + k_n \Im(\omega_n) \omega_n^{-m} \int_{\mathbb{R}} \big(x 1_{[-1,1]}(x) - e^{imx} \sin(x)\big) \,\mathrm{d}\mu_n(x) \\
&\qquad - k_n \Im(\omega_n) \omega_n^{-m} \int_{\mathbb{R}} x 1_{[-1,1]}(x) \,\mathrm{d}\mu_n(x).
\end{aligned}$$

Thanks to Lemma 2.5, and because $\lim_{n\to\infty} \omega_n = 1$, we know the limit of the first term:

$$\begin{aligned}
\lim_{n\to\infty} k_n \omega_n^{-m} \Re(\omega_n) \int_{\mathbb{R}} e^{imx} \big(1 - \cos(x)\big) \,\mathrm{d}\mu_n(x) &= \int_{\mathbb{R}} e^{imx} \big(1 - \cos(x)\big) \,\mathrm{d}\rho(x) + \frac{b}{2} \\
&= \int_{\mathbb{U}} \zeta^n \big(1 - \Re(\zeta)\big) \,\mathrm{d}\upsilon(\zeta) + \frac{b}{2}.
\end{aligned}$$

The three others terms tend to 0. Indeed, (3.6) implies that $k_n(1 - \Re(\omega_n)) = o(1/k_n)$ and $\Im(\omega_n) = O(1/k_n)$ when n tends to ∞. We know that $\omega_n^{-m} = O(1)$ and $\int_{\mathbb{R}} e^{imx} \,\mathrm{d}\mu_n(x) = O(1)$ when n tends to ∞. Finally, Lemma 2.5 tells us that $\int_{\mathbb{R}} (x 1_{[-1,1]}(x) - e^{imx} \sin(x)) \,\mathrm{d}\mu_n(x) = O(1/k_n)$ and $\int_{\mathbb{R}} x 1_{[-1,1]}(x) \,\mathrm{d}\mu_n(x) = O(1/k_n)$ as n tends to ∞. Thus,

$$k_n \big(1 - \Re(\omega_n)\big) \omega_n^{-m} \int_{\mathbb{R}} e^{imx} \,\mathrm{d}\mu_n(x),$$

$$k_n \Im(\omega_n) \omega_n^{-m} \int_{\mathbb{R}} \big(x 1_{[-1,1]}(x) - e^{imx} \sin(x)\big) \,\mathrm{d}\mu_n(x)$$

and

$$-k_n \Im(\omega_n) \omega_n^{-m} \int_{\mathbb{R}} x 1_{[-1,1]}(x) \,\mathrm{d}\mu_n(x)$$

are $o(1)$ as n tends to ∞, and the result follows. □

4. Free log-cumulants. We are at the beginning of the second part of the paper, the aim of which is to prove Theorems 2 and 3. This goal is achieved in Section 7. While Sections 5 and 6 investigates the distributions of certain classes of random matrices, the current section is devoted to establish Proposition 4.1 which is the result of free probability needed for the asymptotic theorems proved in the last section of the paper. As a consequence, Section 4 can be read independently of Sections 5 and 6.

Mastnak and Nica explain in [30] that, in order to treat the multi-dimensional free multiplicative convolution, it is preferable to work with a logarithmic version of the S-transform. This leads to a sequence of coefficients which in [13] are called the free log-cumulants. In this section, we use the theory of free log-cumulants to establish Proposition 4.1 which links in an explicit formula the moments of a $\boxtimes$-infinitely divisible measure to its $\boxtimes$-characteristic triplet. We start by stating Proposition 4.1, after which we introduce the free log-cumulants, which will be used only in the proof of Proposition 4.1.

4.1. *Moments of a $\boxtimes$-infinitely divisible measure.* Proposition 4.1 involves combinatorics on the symmetric group $\mathfrak{S}_n$. We first present the poset structure of $\mathfrak{S}_n$.

Let $n \in \mathbb{N}^*$. Let $\mathfrak{S}_n$ be the group of permutations of $\{1, \dots, n\}$. For all permutation $\sigma \in \mathfrak{S}_n$, we denote by $\ell(\sigma)$ the numbers of cycles of σ and we set $|\sigma| = n - \ell(\sigma)$. The minimal number of transpositions required to write σ is $|\sigma|$ and we have $|\sigma| = 0$ if and only if σ is the identity $1_{\mathfrak{S}_n}$. We define a distance on $\mathfrak{S}_n$ by $d(\sigma_1, \sigma_2) = |\sigma_1^{-1}\sigma_2|$. The set $\mathfrak{S}_n$ can be endowed with a partial order by the relation $\sigma_1 \preceq \sigma_2$ if $d(1_{\mathfrak{S}_n}, \sigma_1) + d(\sigma_1, \sigma_2) = d(1_{\mathfrak{S}_n}, \sigma_2)$, or similarly if σ_1 is on a geodesic between $1_{\mathfrak{S}_n}$ and σ_2. The minimal element of $\mathfrak{S}_n$ is thus $1_{\mathfrak{S}_n}$.

For all $\sigma \in \mathfrak{S}_n$, we denote by $[1_{\mathfrak{S}_n}, \sigma]$ the segment between $1_{\mathfrak{S}_n}$ and the σ, that is, the set $\{\pi \in \mathfrak{S}_n : \pi \preceq \sigma\}$. It is a lattice with respect to the partial order. A $(l+1)$-tuple $\Gamma = (\sigma_0, \dots, \sigma_l)$ of $[1_{\mathfrak{S}_n}, \sigma]$ such that

$$\sigma_0 \prec \sigma_1 \prec \cdots \prec \sigma_l \preceq \sigma$$

is called a *simple chain* if and only if, for all $1 \leq i \leq l$, $\sigma_{i-1}^{-1}\sigma_i$ is a nontrivial cycle. The length k of a k-cycle c will be denoted by $\sharp c$. We are now ready to state the main result of this section.

PROPOSITION 4.1. *Let $\mu \in \mathcal{ID}(\mathbb{U}, \boxtimes)$ with $\boxtimes$-characteristic triplet (ω, b, υ). For all $n \in \mathbb{N}^*$ and all $\sigma \in \mathfrak{S}_n$, we have*

$$\prod_{c \text{ cycle of } \sigma} m_{\sharp c}(\mu) = e^{nL\kappa_1(\mu)} \cdot \sum_{\substack{\Gamma \text{ simple chain in } [1_{\mathfrak{S}_n}, \sigma] \\ \Gamma = (\sigma_0, \dots, \sigma_{|\Gamma|}), \sigma_{|\Gamma|} = \sigma}} \frac{1}{|\Gamma|!} \prod_{i=1}^{|\Gamma|} L\kappa_{d(\sigma_i, \sigma_{i-1})+1}(\mu),$$

where:

1. $L\kappa_1(\mu) = \mathrm{Log}(\omega) - b/2 + \int_{\mathbb{U}} (\Re(\zeta) - 1)\, \mathrm{d}\upsilon(\zeta)$,
2. $L\kappa_2(\mu) = -b + \int_{\mathbb{U}} (\zeta - 1)^2\, \mathrm{d}\upsilon(\zeta)$
3. *and* $L\kappa_n(\mu) = \int_{\mathbb{U}} (\zeta - 1)^n\, \mathrm{d}\upsilon(\zeta)$ *for all* $n \geq 3$.

The proof of Proposition 4.1 requires the notion of free log-cumulants and we postpone it until Section 4.4. In the mean time, we review the properties of the free log-cumulants that we shall use.

4.2. *The noncrossing partitions.* The definition of the free log-cumulants involves combinatorial formulae which are related to noncrossing partitions. We describe here the poset structure of the set of noncrossing partitions $NC(n)$, and we shall see that it is intimately linked to the poset structure of $\mathfrak{S}_n$.

A partition of the set $\{1,\dots,n\}$ is said to have a crossing if there exist $1 \le i < j < k < l \le n$, such that i and k belong to some block of the partition and j and l belong to another block. If a partition has no crossings, it is called noncrossing. The set of all noncrossing partitions of $\{1,\dots,n\}$ is denoted by $NC(n)$. It is a lattice with respect to the relation of fineness defined as follows: for all π_1 and $\pi_2 \in NC(n)$, we declare that $\pi_1 \preceq \pi_2$ if every block of π_1 is contained in a block of π_2. We denote, respectively, by 0_n and 1_n the minimal element $\{\{1\},\dots,\{n\}\}$ of $NC(n)$, and the maximal element $\{\{1,\dots,n\}\}$ of $NC(n)$.

In [10], Biane describes an isomorphism between the posets $NC(n)$ and $[1_{\mathfrak{S}_n},(1\cdots n)] \subset \mathfrak{S}_n$. It consists simply in defining, from every partition $\pi \in NC(n)$, the permutation σ_π which is the product, over all blocks $\{i_1 < \cdots < i_k\}$ of π, of the k-cycle $(i_1 \cdots i_k)$. In other words, take the cycles of σ_π to be the blocks of π with the cyclic order induced by the natural order of $\{1,\dots,n\}$. Note that $\sigma_{0_n} = 1_{\mathfrak{S}_n}$ and $\sigma_{1_n} = (1\cdots n)$.

LEMMA 4.2. *The function $\pi \mapsto \sigma_\pi$ is a poset isomorphism between $NC(n)$ and $[1_{\mathfrak{S}_n},(1\cdots n)]$.*

Let $\pi \in NC(n)$. It is immediate that the map $\sigma \mapsto \sigma^{-1}\sigma_\pi$ is an order-reversing bijection of $[1_{\mathfrak{S}_n},\sigma_\pi]$. The corresponding decreasing bijection K_π of $\{\pi' \in NC(n) : \pi' \preceq \pi\}$ is called the Kreweras complementation map with respect to π. If $\pi = 1_n$, we set $K(\sigma) = K_{1_n}(\sigma)$.

Let $n \in \mathbb{N}$. A chain in the lattice $NC(n)$ is a $(l+1)$-tuple of the form $\Gamma = (\pi_0,\dots,\pi_l)$ with $\pi_0,\dots,\pi_l \in NC(n)$ such that $\pi_0 \prec \pi_1 \prec \cdots \prec \pi_l$ (notice that we do not impose $\pi_0 = 0_n$ nor $\pi_l = 1_n$, unlike in [30]). The positive integer l appearing is called the length of the chain, and is denoted by $|\Gamma|$. If, for all $1 \le i \le l$, $K_{\pi_i}(\pi_{i-1})$ has exactly one block which has more than two elements, we say that Γ is a simple chain in $NC(n)$. This way, we have a one-to-one correspondence between simple chains in $NC(n)$ and simple chains in $[1_{\mathfrak{S}_n},(1\cdots n)]$ via the isomorphism of Lemma 4.2.

4.3. *Free log-cumulants.* Let $\mu \in \mathcal{M}_*$. We denote by $W_\mu(z)$ the inverse under composition of $zM_\mu(z)$, and we denote by $C_\mu(z)$ the formal power series $M_\mu(W_\mu(z))$. The coefficients $(\kappa_k(\mu))_{k\in\mathbb{N}^*}$ of

$$C_\mu(z) = 1 + \sum_{k=1}^{\infty} \kappa_k(\mu) z^k$$

are known as the *free cumulants* of μ. Let $\pi \in NC(n)$. We set

$$\kappa[\pi](\mu) = \prod_{B \text{ block of } \pi} \kappa_{|B|}(\mu).$$

For all $n \geq 2$, we set

$$L\kappa_n(\mu) = m_1(\mu)^{-n} \sum_{\substack{\Gamma \text{ chain in } NC(n) \\ \Gamma=(\pi_0,\dots,\pi_{|\Gamma|}) \\ \pi_0=0_n,\pi_{|\Gamma|}=1_n}} \frac{(-1)^{1+|\Gamma|}}{|\Gamma|} \prod_{i=1}^{|\Gamma|} \kappa\bigl[K_{\pi_i}(\pi_{i-1})\bigr](\mu).$$

We shall call the coefficients $(L\kappa_k(\mu))_{n\leq 2}$ the *free log-cumulants* of μ. We define also the LS-*transform* of μ by

$$LS_\mu(z) = \sum_{n=2}^{\infty} L\kappa_n(\mu) z^n.$$

Let us define also $L\kappa_1(\mu)$, or $L\kappa(\mu)$, the free log-cumulant of order 1 of μ, by $\mathrm{Log}(m_1(\mu))$, where Log is the principal logarithm.

REMARK 4.3. From Proposition 4.5 of [30], we see that this definition of LS_μ extends the definition of the LS-transform of μ given by Definition 1.4 of [30] in the case $m_1(\mu) \neq 1$. The definition of the free log-cumulants $(L\kappa_n(\mu))_{n\in\mathbb{N}^*}$ follows [13], but we observe that $L\kappa_n(\mu)$ would be denoted by $L\kappa_n(A)$ in [13], where A would be a random variable whose law is μ.

As the free cumulants linearize $\boxplus$, the free log-cumulants linearize $\boxtimes$.

PROPOSITION 4.4 (Corollary 1.5 of [30], Proposition 2.11 of [13]). *For all* $\mu, \nu \in \mathcal{M}_*$, *we have* $L\kappa_1(\mu \boxtimes \nu) \equiv L\kappa_1(\mu) + L\kappa_1(\nu) \pmod{2i\pi}$ *and*, *for all* $n \geq 2$,

$$L\kappa_n(\mu \boxtimes \nu) = L\kappa_n(\mu) + L\kappa_n(\nu).$$

For concrete calculations, one would prefer to have an analytical description of the free log-cumulants. We have $S_\mu(0) = 1/m_1(\mu)$ and by consequence, we can define the formal logarithm of $m_1(\mu) \cdot S_\mu$ as the formal series $\log(m_1(\mu) \cdot S_\mu) = -\sum_{n=1}^{\infty} \frac{1}{n}(1 - m_1(\mu)S_\mu(z))^n$.

PROPOSITION 4.5 (Corollary 6.12 of [30]). *Let* $\mu \in \mathcal{M}_*$. *We have*

$$LS_\mu(z) = -z \log\bigl(m_1(\mu) \cdot S_\mu(z)\bigr).$$

REMARK 4.6. Technically, Corollary 6.12 of [30] only deals with measures, or more precisely linear functionals on $\mathbb{C}[X]$, such that $m_1(\mu)=1$. One can adapt the proof presented in [30]. Alternatively, argue as follows. From a measure $\mu \in \mathcal{M}_*$, we can define $\varphi_\mu : \mathbb{C}[X] \to \mathbb{C}$ such that $\varphi_\mu(X^k) = m_1(\mu)^{-k} m_k(\mu)$. Then we observe that $S_{\varphi_\mu} = m_1(\mu)\cdot S_\mu(z)$ and $LS_{\varphi_\mu} = LS_\mu$. As a consequence, $LS_\mu(z) = LS_{\varphi_\mu} = -z\log(S_{\varphi_\mu}) = -z\log(m_1(\mu)\cdot S_\mu(z))$.

Let $\pi \in NC(n)$ be such that π has exactly one block which has at least two elements. Let $\{j_1,\dots,j_N\}$ be this block of π, with $j_1 < \cdots < j_N$. Let us denote by $L\kappa[\pi](\mu)$ the free log-cumulant $L\kappa_N(\mu)$.

PROPOSITION 4.7 (Corollary 2.9 of [13]). *Let $\mu \in \mathcal{M}_*$ and $n \in \mathbb{N}^*$. We have*

$$m_n(\mu) = e^{nL\kappa_1(\mu)} \cdot \sum_{\substack{\Gamma \text{ simple chain in } NC(n) \\ \Gamma=(\pi_0,\dots,\pi_{|\Gamma|}),\pi_0=0_n}} \frac{1}{|\Gamma|!} \prod_{i=1}^{|\Gamma|} L\kappa\big[K_{\pi_i}(\pi_{i-1})\big](\mu). \tag{4.1}$$

4.4. *Proof of Proposition* 4.1. Let us formulate a more general formula than (4.1) with the help of the symmetric group.

LEMMA 4.8. *Let $\mu \in \mathcal{M}_*$ and $n \in \mathbb{N}^*$. For all $\sigma \in \mathfrak{S}_n$, we have*

$$\begin{aligned}&\prod_{c \text{ cycle of } \sigma} m_{\sharp c}(\mu) \\ &\quad = e^{nL\kappa_1(\mu)} \cdot \sum_{\substack{\Gamma \text{ simple chain in } [1_{\mathfrak{S}_n},\sigma] \\ \Gamma=(\sigma_0,\dots,\sigma_{|\Gamma|}),\sigma_{|\Gamma|}=\sigma}} \frac{1}{|\Gamma|!} \prod_{i=1}^{|\Gamma|} L\kappa_{d(\sigma_i,\sigma_{i-1})+1}(\mu).\end{aligned} \tag{4.2}$$

PROOF. The analogue formula of (4.1) for simple chains in $[1_{\mathfrak{S}_n}, (1\cdots n)]$ is obtained via the isomorphism of Lemma 4.2, remarking that, for a l-cycle $\sigma_1^{-1}\sigma_2$ of $[1_{\mathfrak{S}_n}, (1\cdots n)]$, we have $l = n - \ell(\sigma_1^{-1}\sigma_2) + 1 = d(\sigma_1,\sigma_2)+1$. By consequence, we have

$$m_n(\mu) = e^{nL\kappa_1(\mu)} \cdot \sum_{\substack{\Gamma \text{ simple chain in } [1_{\mathfrak{S}_n},(1\cdots n)] \\ \Gamma=(\sigma_0,\dots,\sigma_{|\Gamma|}),\sigma_0=1}} \frac{1}{|\Gamma|!} \prod_{i=1}^{|\Gamma|} L\kappa_{d(\sigma_i,\sigma_{i-1})+1}(\mu).$$

Applying the Kreweras complementation $\sigma \mapsto \sigma^{-1}(1\cdots n)$ which is an isomorphism and preserves simple chains, we obtain

$$m_n(\mu) = e^{nL\kappa_1(\mu)} \cdot \sum_{\substack{\Gamma \text{ simple chain in } [1_{\mathfrak{S}_n},(1\cdots n)] \\ \Gamma=(\sigma_0,\dots,\sigma_{|\Gamma|}),\sigma_{|\Gamma|}=(1\cdots n)}} \frac{1}{|\Gamma|!} \prod_{i=1}^{|\Gamma|} L\kappa_{d(\sigma_i,\sigma_{i-1})+1}(\mu).$$

We now use the fact that for a cycle c of length $\sharp c$, the segment $[1_{\mathfrak{S}_n}, c] \subset \mathfrak{S}_n$ is isomorphic as a lattice to $[1_{\mathfrak{S}_{\sharp c}}, (1 \cdots \sharp c)] \subset \mathfrak{S}_{\sharp c}$, and by consequence, (4.2) is true if σ is a cycle.

For an arbitrary permutation σ, we decompose it into cycles $c_1, \ldots, c_{\ell(\sigma)}$. Constructing a simple chain of length k ending at σ is equivalent to constructing $\ell(\sigma)$ simple chains ending, respectively, at $c_1, \ldots, c_{\ell(\sigma)}$, whose lengths $l_1, \ldots, l_{\ell(\sigma)}$ add up to k, and shuffling the steps of these paths, that is choosing a sequence $(C_1, \ldots, C_{\ell(\sigma)})$ of subsets of $\{1, \ldots, k\}$ which partition $\{1, \ldots, k\}$ and whose cardinals are $l_1, \ldots, l_{\ell(\sigma)}$, respectively. Using the formula (4.2) for cycles, this remark leads to (4.2) for an arbitrary $\sigma \in \mathfrak{S}_n$. $\square$

In order to complete the proof of Proposition 4.1, it suffices to compute explicitly the free log-cumulants of a $\boxtimes$-infinitely divisible measure.

PROPOSITION 4.9. *Let $\mu \in \mathcal{ID}(\mathbb{U}, \boxtimes)$ with $\boxtimes$-characteristic triplet (ω, b, υ). We have*:

1. $L\kappa_1(\mu) = \mathrm{Log}(\omega) - b/2 + \int_{\mathbb{U}} (\Re(\zeta) - 1)\, \mathrm{d}\upsilon(\zeta)$,
2. $L\kappa_2(\mu) = -b + \int_{\mathbb{U}} (\zeta - 1)^2\, \mathrm{d}\upsilon(\zeta)$
3. *and* $L\kappa_n(\mu) = \int_{\mathbb{U}} (\zeta - 1)^n\, \mathrm{d}\upsilon(\zeta)$ *for all* $n \geq 3$.

PROOF. The data of $S_\mu(z)$ is given by (2.5). We first remark that

$$m_1(\mu) = S_\mu(0)^{-1} = \omega e^{-b/2 - \int_{\mathbb{U}} (i\Im(\zeta) + 1 - \zeta)\, \mathrm{d}\upsilon(\zeta)},$$

from which we deduce that $L\kappa_1(\mu) = \mathrm{Log}(m_1(\mu)) = \mathrm{Log}(\omega) - b/2 + \int_{\mathbb{U}} (\Re(\zeta) - 1)\, \mathrm{d}\upsilon(\zeta)$. We also have $m_1(\mu) S_\mu(z) = S_\mu(z)/S_\mu(0) = \exp(bz + \int_{\mathbb{U}} \frac{1-\zeta}{1+z(1-\zeta)} - (1 - \zeta)\, \mathrm{d}\upsilon(\zeta))$. Therefore,

$$LS_\mu(z) = -z \log\bigl(m_1(\mu) \cdot S_\mu(z)\bigr) = -bz^2 + \int_{\mathbb{U}} \frac{z^2(\zeta - 1)^2}{1 - z(\zeta - 1)}\, \mathrm{d}\upsilon(\zeta).$$

We identify $(L\kappa_n(\mu))_{n \geq 2}$ as the coefficients of $LS_\mu(z) = \sum_{n=2}^{\infty} L\kappa_n(\mu) z^n$. $\square$

5. Convolution semigroups on $U(N)$. In this section, we define and study the convolution semigroups on the unitary group $U(N)$. More precisely, we are interested in computing $\int_{U(N)} g^{\otimes n}\, \mathrm{d}\mu(g)$ for μ arising from a convolution semigroup. In Propositions 5.2 and 5.8, we shall express this quantity in two different ways. The technique of proof is in the spirit of [25]. It relies on a detailed comprehension of the generator of a convolution semigroup on $U(N)$ (see [29]), and on the Schur–Weyl duality (see Section 5.4, and [15, 16]).

Let $N \in \mathbb{N}$ and let $M_N(\mathbb{C})$ be the space of matrices of dimension N. If $M \in M_N(\mathbb{C})$, we denote by M^* the adjoint of M. Let us denote by $\mathrm{Tr} : M_N(\mathbb{C}) \to \mathbb{C}$

the unnormalized trace. The identity matrix is denoted by I_N. We consider the unitary group

$$U(N) = \{U \in M_N(\mathbb{C}) : U^*U = I_N\}.$$

The $\circledast$-convolution of two probability measures μ and ν on $U(N)$ is defined to be the unique probability measure $\mu \circledast \nu$ on $U(N)$ such that $\int_{U(N)} f \, \mathrm{d}(\mu \circledast \nu) = \int_{U(N)} f(gh)\mu(\mathrm{d}g)\nu(\mathrm{d}h)$ for all bounded Borel function f on $U(N)$. Let us denote by $\mathcal{ID}(U(N), \circledast)$ the space of infinitely divisible probability measures on $U(N)$ and by $\mathcal{ID}_{\mathrm{inv}}(U(N), \circledast)$ the subspace of measures μ in $\mathcal{ID}(U(N), \circledast)$ which are invariant by unitary conjugation, that is, such that for all bounded Borel function f on $U(N)$ and all $g \in U(N)$, we have

$$\int_{U(N)} f \, \mathrm{d}\mu = \int_{U(N)} f(ghg^*) \, \mathrm{d}\mu(h).$$

5.1. *Generators of semigroups.* Let $\mu = (\mu_t)_{t \in \mathbb{R}^+}$ be a weakly continuous convolution semigroup on $U(N)$ starting at $\mu_0 = \delta_e$. We define the transition semigroup $(P_t)_{t \in \mathbb{R}^+}$ as follows: for all $t \in \mathbb{R}^+$, all bounded Borel function f on $U(N)$ and all $h \in U(N)$, we set $P_t f(h) = \int_{U(N)} f(hg)\mu_t(\mathrm{d}g)$. The *generator* of μ, is defined to be the linear operator L on $C(U(N))$ such as $Lf = \lim_{t \to 0}(P_t f - f)/t$ whenever this limit exists.

In order to describe the generator of a semigroup, we shall successively introduce in the three next paragraphs the Lie algebra $\mathfrak{u}(N)$ of $U(N)$, a scalar product on $\mathfrak{u}(N)$ and the notion of Lévy measure on $U(N)$.

The unitary group $U(N)$ is a compact real Lie group of dimension N^2, whose Lie algebra $\mathfrak{u}(N)$ is the real vector space of skew-Hermitian matrices: $\mathfrak{u}(N) = \{M \in M_N(\mathbb{C}) : M^* + M = 0\}$. We consider also the special unitary group $SU(N) = \{U \in U(N) : \det U = 1\}$, whose Lie algebra is $\mathfrak{su}(N) = \{M \in \mathfrak{u}(N) : \mathrm{Tr}(U) = 0\}$. We remark that $\mathfrak{u}(N) = \mathfrak{su}(N) \oplus (i\mathbb{R}I_N)$. Any $Y \in \mathfrak{u}(N)$ induces a *left invariant vector field* Y^l on $U(N)$ defined for all $g \in U(N)$ by $Y^l(g) = DL_g(Y)$ where DL_g is the differential map of $h \mapsto gh$.

We consider the following *inner product* on $\mathfrak{u}(N)$:

$$(X, Y) \mapsto \langle X, Y \rangle_{\mathfrak{u}(N)} = \mathrm{Tr}(X^*Y) = -\mathrm{Tr}(XY).$$

It is a real scalar product on $\mathfrak{u}(N)$ which is invariant under the adjoint action of $U(N)$, and its restriction to $\mathfrak{su}(N)$ is also a real scalar product which is invariant by unitary conjugation on both argument. Let us fix an orthonormal basis $\{Y_1, \dots, Y_{N^2-1}\}$ of $\mathfrak{su}(N)$ and set $Y_{N^2} = \frac{i}{\sqrt{N}} I_N$. This way, $\{Y_1, \dots, Y_{N^2}\}$ is an orthonormal basis of $\mathfrak{u}(N)$.

It is convenient now to introduce an arbitrary auxiliary set of local coordinates around I_N. Let $\Re, \Im : U(N) \to M_N(\mathbb{C})$ be such that for all $U \in U(N)$, we have $\Re(U) = (U + U^*)/2$ and $\Im(U) = (U - U^*)/2i$. Note that $i\Im$ takes its values in $\mathfrak{u}(N)$. A *Lévy measure* Π on $U(N)$ is a measure on $U(N)$ such that $\Pi(\{I_N\}) = 0$, for all neighborhood V of I_N, we have $\Pi(V^c) < +\infty$ and $\int_{U(N)} \|i\Im(x)\|^2_{\mathfrak{u}(N)} \Pi(\mathrm{d}x) < \infty$.

The following theorem gives us a characterization of the generator of such semigroups.

THEOREM 5.1 ([2, 29]). *Let $\mu = (\mu_t)_{t\in\mathbb{R}^+}$ be a weakly continuous convolution semigroup on $U(N)$ starting at $\mu_0 = \delta_e$. There exist an element $Y_0 \in \mathfrak{u}(N)$, a symmetric positive semidefinite matrix $(y_{i,j})_{1\leq i,j\leq N^2}$ and a Lévy measure Π on $U(N)$ such that the generator L of μ is the left-invariant differential operator given, for all $f \in C^2(U(N))$ and all $h \in U(N)$, by*

$$\begin{aligned} Lf(h) &= Y_0^l f(h) + \frac{1}{2}\sum_{i,j=1}^{N^2} y_{i,j} Y_i^l Y_j^l f(h) \\ &\quad + \int_{U(N)} f(hg) - f(h) - \big(i\Im(g)\big)^l f(h)\Pi(\mathrm{d}g). \end{aligned} \tag{5.1}$$

Conversely, given such a triplet $(Y_0, (y_{i,j})_{1\leq i,j\leq N^2}, \Pi)$, it exists a unique weakly continuous convolution semigroup on $U(N)$ starting at δ_e whose generator is given by (5.1).

The triplet $(Y_0, (y_{i,j})_{1\leq i,j\leq N^2}, \Pi)$ is called the *characteristic triplet* of $(\mu_t)_{t\in\mathbb{R}^+}$, or of L. Let $\mu \in \mathcal{ID}(U(N), \circledast)$ be such that it exists a weakly continuous convolution semigroup $(\mu_t)_{t\in\mathbb{R}^+}$ with $\mu_1 = \mu$ and $\mu_0 = \delta_{I_N}$. In this case, we say that the characteristic triplet of $(\mu_t)_{t\in\mathbb{R}^+}$ is *a characteristic triplet* of μ. It is not unique but it completely characterizes the measure μ. Conversely, every triplet of this form is a characteristic triplet of a unique measure in $\mathcal{ID}(U(N), \circledast)$.

5.2. *Expected values of polynomials of the entries*. Let $n \in \mathbb{N}^*$. In this section, we give a formula for $\int_{U(N)} g^{\otimes n}\,\mathrm{d}\mu(g)$ when μ arises from a convolution semigroup. Consider the representation $\rho^n_{U(N)}$ of the Lie group $U(N)$ on the vector space $(\mathbb{C}^N)^{\otimes n}$ given by

$$\rho^n_{U(N)}(g) = \underbrace{g \otimes \cdots \otimes g}_{n \text{ times}} \in U(N)^{\otimes n} \subset \mathrm{End}\big((\mathbb{C}^N)^{\otimes n}\big).$$

Set $d\rho^n_{U(N)}(L) = L(\rho^n_{U(N)})(I_N) \in \mathrm{End}((\mathbb{C}^N)^{\otimes n})$, where $\rho^n_{U(N)}$ is seen as an element of $C^2(U(N)) \otimes \mathrm{End}((\mathbb{C}^N)^{\otimes n})$.

PROPOSITION 5.2. *Let $(\mu_t)_{t\in\mathbb{R}^+}$ be a weakly continuous convolution semigroup on $U(N)$ starting at $\mu_0 = \delta_e$ with generator L and characteristic triplet $(Y_0, (y_{i,j})_{1\leq i,j\leq N^2}, \Pi)$. For all $t \in \mathbb{R}_+$, we have the equality in* $\mathrm{End}((\mathbb{C}^N)^{\otimes n})$

$$\int_{U(N)} g^{\otimes n}\,\mathrm{d}\mu_t(g) = \exp\big(t\,d\rho^n_{U(N)}(L)\big)$$

with

$$d\rho^n_{U(N)}(L) = \sum_{1\le k\le n} \mathrm{Id}_N^{\otimes k-1} \otimes Y_0 \otimes \mathrm{Id}_N^{\otimes n-k} + \frac{1}{2}\sum_{i,j=1}^{N^2} y_{i,j} \times \sum_{1\le k,l\le n} \big(\mathrm{Id}_N^{\otimes k-1} \otimes Y_i \otimes \mathrm{Id}_N^{\otimes n-k}\big) \circ \big(\mathrm{Id}_N^{\otimes l-1} \otimes Y_j \otimes \mathrm{Id}_N^{\otimes n-l}\big) + \int_{U(N)} \Big(g^{\otimes n} - \mathrm{Id}_N^{\otimes n} - \sum_{1\le k\le n} \mathrm{Id}_N^{\otimes k-1} \otimes i\Im(g) \otimes \mathrm{Id}_N^{\otimes n-k}\Big)\Pi(\mathrm{d}g).$$

PROOF. Let denote by $U : U(N) \to M_N(\mathbb{C})$ the identity function of $U(N)$. We compute

$$L\big(\rho^n_{U(N)}\big) = Y_0^l\big(U^{\otimes n}\big) + \frac{1}{2}\sum_{i,j=1}^{N^2} y_{i,j} Y_i^l Y_j^l\big(U^{\otimes n}\big) + \int_{U(N)} (Ug)^{\otimes n} - U^{\otimes n} - \big(i\Im(g)\big)^l\big(U^{\otimes n}\big)\Pi(\mathrm{d}g)$$

and using that, for all $Y \in \mathfrak{u}(N)$, we have $Y^l(U^{\otimes n}) = U^{\otimes n} \cdot \sum_{1\le k\le n} \mathrm{Id}_N^{\otimes k-1} \otimes Y \otimes \mathrm{Id}_N^{\otimes n-k}$,

$$\begin{aligned} L\big(\rho^n_{U(N)}\big) &= U^{\otimes n} \cdot \sum_{1\le k\le n} \mathrm{Id}_N^{\otimes k-1} \otimes Y_0 \otimes \mathrm{Id}_N^{\otimes n-k} + \frac{1}{2}U^{\otimes n} \\ &\quad \times \sum_{i,j=1}^{N^2} y_{i,j} \sum_{1\le k,l\le n} \big(\mathrm{Id}_N^{\otimes k-1} \otimes Y_i \otimes \mathrm{Id}_N^{\otimes n-k}\big) \cdot \big(\mathrm{Id}_N^{\otimes l-1} \otimes Y_j \otimes \mathrm{Id}_N^{\otimes n-l}\big) \\ &\quad + U^{\otimes n} \cdot \int_{U(N)} \Big(g^{\otimes n} - \mathrm{Id}_N^{\otimes n} - \sum_{1\le k\le n} \mathrm{Id}_N^{\otimes k-1} \otimes i\Im(g) \otimes \mathrm{Id}_N^{\otimes n-k}\Big)\Pi(\mathrm{d}g). \end{aligned}$$

Hence, $d\rho^n_{U(N)}(L) = L(\rho^n_{U(N)})(e)$ leads to the expression of $d\rho^n_{U(N)}(L)$ given above. We conclude by remarking that

$$t \to \int_{U(N)} g^{\otimes n}\,\mathrm{d}\mu_t(g) = \int_{U(N)} \rho^n_{U(N)}(g)\,\mathrm{d}\mu_t(g)$$

and $t \to \exp(t\,d\rho^n_{U(N)}(L))$ are both the unique solution to the differential equation

$$\begin{cases} y(0) = I_N^{\otimes n}, \\ y' = y \cdot d\rho^n_{U(N)}(L). \end{cases} \qquad \square$$

We now give an alternative expression of $d\rho^n_{U(N)}(L)$. Let $m \geq 0$. For all $1 \leq k_1 < \cdots < k_m \leq n$, let us denote by $\iota^{M_N(\mathbb{C})^{\otimes n}}_{k_1,\dots,k_m} : M_N(\mathbb{C})^{\otimes m} \to M_N(\mathbb{C})^{\otimes n}$ (or more simply $\iota_{k_1,\dots,k_m}$) the mapping defined by

$$\iota_{k_1,\dots,k_m}(X_1 \otimes \cdots \otimes X_m) = I_N^{\otimes k_1-1} \otimes X_1 \otimes I_N^{\otimes k_2-k_1-1} \otimes X_2 \otimes \cdots \otimes X_m \otimes I_N^{\otimes n-k_m},$$

that is to say in words that $\iota_{k_1,\dots,k_m}(X_1 \otimes \cdots \otimes X_m)$ is the tensor product of $X_1, \dots, X_m$ at the places $k_1, \dots, k_m$ and I_N at the other places.

PROPOSITION 5.3. *Let L be a generator with characteristic triplet $(Y_0, (y_{i,j})_{1\leq i,j\leq N^2}, \Pi)$. We have*

$$\begin{aligned} d\rho^n_{U(N)}(L) = &\sum_{1\leq k\leq n} \iota_k\left(Y_0 + \int_{U(N)} (\Re(g) - I_N)\Pi(\mathrm{d}g)\right) \\ &+ \frac{1}{2}\sum_{i,j=1}^{N^2} y_{i,j} \cdot \sum_{1\leq k,l\leq n} \iota_k(Y_i)\circ\iota_l(Y_j) \\ &+ \sum_{\substack{2\leq m\leq n\\ 1\leq k_1<\cdots<k_m\leq n}} \iota_{k_1,\dots,k_m}\left(\int_{U(N)} (g - I_N)^{\otimes m}\Pi(\mathrm{d}g)\right). \end{aligned}$$

PROOF. Our starting point is the expression of $d\rho^n_{U(N)}(L)$ given by Proposition 5.8. Let us remark that

$$\begin{aligned} g^{\otimes n} &= (g - I_N + I_N)^{\otimes n} \\ &= I_N^{\otimes n} + \sum_{\substack{1\leq m\leq n\\ 1\leq k_1<\cdots<k_m\leq n}} \iota_{k_1,\dots,k_m}\big((g - I_N)^{\otimes m}\big), \end{aligned}$$

from which we deduce that

$$\begin{aligned} &\int_{U(N)} \left(g^{\otimes n} - \mathrm{Id}_N^{\otimes n} - \sum_{1\leq k\leq n} \mathrm{Id}_N^{\otimes k-1} \otimes i\Im(g) \otimes \mathrm{Id}_N^{\otimes n-k}\right)\Pi(\mathrm{d}g) \\ &\quad = \int_{U(N)} \Bigg(\sum_{1\leq k\leq n} \iota_k\big(g - I_N - i\Im(g)\big) \\ &\qquad + \sum_{\substack{2\leq m\leq n\\ 1\leq k_1<\cdots<k_m\leq n}} \iota_{k_1,\dots,k_m}\big((g - I_N)^{\otimes m}\big)\Bigg)\Pi(\mathrm{d}g) \\ &\quad = \sum_{1\leq k\leq n} \iota_k\left(\int_{U(N)} (\Re(g) - I_N)\Pi(\mathrm{d}g)\right) \\ &\qquad + \sum_{\substack{2\leq m\leq n\\ 1\leq k_1<\cdots<k_m\leq n}} \iota_{k_1,\dots,k_m}\left(\int_{U(N)} (g - I_N)^{\otimes m}\Pi(\mathrm{d}g)\right) \end{aligned}$$

because all the integrand are equivalent to $\|i\Im(g)\|^2_{\mathfrak{u}(N)}$ in a neighborhood of I_N, and hence integrable with respect to Π. Replacing the last term by this new expression in Proposition 5.2 yields to the result. $\square$

5.3. *Conjugate invariant semigroups on $U(N)$.* A weakly continuous convolution semigroup $(\mu_t)_{t\in\mathbb{R}^+}$ on $U(N)$ starting at $\mu_0=\delta_e$ is said *conjugate invariant* if all μ_t belong to $\mathcal{ID}_{\text{inv}}(U(N),\circledast)$.

PROPOSITION 5.4. *Let $(\mu_t)_{t\in\mathbb{R}^+}$ be a weakly continuous convolution semigroup starting at $\mu_0=\delta_e$ which is conjugate invariant. Let $(Y_0,(y_{i,j})_{1\le i,j\le N},\Pi)$ be its characteristic triplet. The differential operator $\frac{1}{2}\sum_{i,j=1}^{N^2} y_{i,j}Y_i^lY_j^l$ and the measure Π are both conjugate invariant. Moreover, there exists three constants y_0,α and $\beta\in\mathbb{R}$ such that $Y_0=iy_0I_N$ and*

$$(y_{i,j})_{1\le i,j\le N^2}=\begin{pmatrix}\alpha & & & 0\\ & \ddots & & \\ & & \alpha & \\ 0 & & & \beta\end{pmatrix}.$$

PROOF. Thanks to [29], if we denote by $(Y_0,(y_{i,j})_{1\le i,j\le N^2},\Pi)$ the characteristic triplet of μ, the differential operator $\frac{1}{2}\sum_{i,j=1}^{N^2} y_{i,j}Y_i^lY_j^l$ and the measure Π are both conjugate invariant. The map $i\Im$ is equivariant by a unitary conjugation and following the proof of Proposition 4.2.2 of [26], we deduce that Y_0 is in the center of $\mathfrak{u}(N)$: there exists $y_0\in\mathbb{R}$ such that $Y_0=iy_0\mathrm{Id}_N$.

Because $\{Y_1,\dots,Y_{N^2-1}\}$ is a basis of the conjugate invariant Lie sub-algebra $\mathfrak{su}(N)$, $\{y_{i,N},y_{N,i}:1\le i\le N^2\}=\{0\}$, and because $\mathfrak{su}(N)$ is simple, there exists $\alpha\in\mathbb{R}$ such that $(y_{i,j})_{1\le i,j\le(N-1)^2}=\alpha I_{N-1}$. We set $\beta=y_{N,N}$. $\square$

Thus, the invariance by conjugation of μ implies that its generator L is a bi-invariant pseudo-differential operator. In this particular case, the expression of $d\rho^n_{U(N)}(L)$ can be described with the help of the symmetric group. It is the object of the next section to use the Schur–Weyl duality in order to formulate a new expression of $d\rho^n_{U(N)}(L)$.

5.4. *Schur–Weyl duality.* The Schur–Weyl duality is a deep relation between the actions of $U(N)$ and $\mathfrak{S}_n$ on $(\mathbb{C}^N)^{\otimes n}$ which allows one to transfer some elements relative to $U(N)$ to elements relative to $\mathfrak{S}_n$ (see [15–17, 25]). Let us spell out this fruitful duality.

Let $n\in\mathbb{N}$. Define the action $\rho_N^{\mathfrak{S}_n}$ of $\mathfrak{S}_n$ on $(\mathbb{C}^N)^{\otimes n}$ as follows: for all $\sigma\in\mathfrak{S}_n$ and $x_1,\dots,x_n\in\mathbb{C}^N$, we set

$$\big(\rho_N^{\mathfrak{S}_n}(\sigma)\big)(x_1\otimes\cdots\otimes x_n)=x_{\sigma^{-1}(1)}\otimes\cdots\otimes x_{\sigma^{-1}(n)}.$$

Let us denote by $\mathbb{C}[\mathfrak{S}_n]$ the group algebra of $\mathfrak{S}_n$. The action $\rho_N^{\mathfrak{S}_n}$ determines a homomorphism of associative algebra $d\rho_N^{\mathfrak{S}_n} : \mathbb{C}[\mathfrak{S}_n] \to \mathrm{End}((\mathbb{C}^N)^{\otimes n})$. The Schur–Weyl duality asserts that the sub-algebras of $\mathrm{End}((\mathbb{C}^N)^{\otimes n})$ generated by the action of $U(N)$ and $\mathfrak{S}_n$ are each other's commutant. In particular, any element of $\mathrm{End}((\mathbb{C}^N)^{\otimes n})$ which commutes with $\rho^n_{U(N)}(g)$ for all $g \in U(N)$ is an element of the algebra generated by $\rho_N^{\mathfrak{S}_n}(\mathfrak{S}_n)$, that is to say an element of $d\rho_N^{\mathfrak{S}_n}(\mathbb{C}[\mathfrak{S}_n])$.

For all $\mathbf{A} \in \mathrm{End}((\mathbb{C}^N)^{\otimes n})$, we define

$$E(\mathbf{A}) = \int_{U(N)} g^{\otimes n} \circ \mathbf{A} \circ (g^*)^{\otimes n} \,\mathrm{d}g \in \mathrm{End}\big((\mathbb{C}^N)^{\otimes n}\big),$$

where the integration is taken with respect to the Haar measure of $U(N)$. Obviously, $E(\mathbf{A})$ commutes with $\rho^n_{U(N)}(g)$ for all $g \in U(N)$, and due to the Schur–Weyl duality, $E(\mathbf{A})$ has to lie in $d\rho_N^{\mathfrak{S}_n}(\mathbb{C}[\mathfrak{S}_n])$. In Proposition 2.4 of [16], Collins and Śniady answered the question of determining an element of $\mathbb{C}[\mathfrak{S}_n]$ which is mapped on $E(\mathbf{A})$, as follows. Set

$$\Phi(\mathbf{A}) = \sum_{\sigma \in \mathfrak{S}_n} \mathrm{Tr}\big(\mathbf{A} \circ \rho_N^{\mathfrak{S}_n}(\sigma^{-1})\big) \cdot \sigma \in \mathbb{C}[\mathfrak{S}_n]$$

and define $\mathrm{Wg} = \sum_{\sigma \in \mathfrak{S}_n} \mathrm{Wg}(\sigma) \cdot \sigma \in \mathbb{C}[\mathfrak{S}_n]$ such that $d\rho_N^{\mathfrak{S}_n}(\Phi(\mathrm{Id}_N^{\otimes n}) \cdot \mathrm{Wg}) = \mathrm{Id}_N^{\otimes n}$. If $n \le N$, the element $\Phi(\mathrm{Id}_N^{\otimes n})$ is invertible and Wg must be $\Phi(\mathrm{Id}_N^{\otimes n})^{-1}$. If $N < n$, one can choose any pseudo-inverse of the symmetric element $\Phi(\mathrm{Id}_N^{\otimes n})$ to be Wg. Let us insist on the fact that Wg depends on both n and N, even if for convenience, this dependence is not explicit in the notation.

PROPOSITION 5.5 ([16]). *For all* $\mathbf{A} \in \mathrm{End}((\mathbb{C}^N)^{\otimes n})$, *we have* $E(\mathbf{A}) = d\rho_N^{\mathfrak{S}_n}(\Phi(\mathbf{A})\mathrm{Wg})$.

Very succinctly, the argument is as follows:

$$\rho_N^{\mathfrak{S}_n}(\Phi(\mathbf{A})) = \rho_N^{\mathfrak{S}_n}\big(\Phi\big(E(\mathbf{A})\big)\big) = \rho_N^{\mathfrak{S}_n}\big(\Phi\big(E(\mathbf{A}) \cdot \mathrm{Id}_N^{\otimes n}\big)\big) = E(\mathbf{A}) \cdot \rho_N^{\mathfrak{S}_n}\big(\Phi\big(\mathrm{Id}_N^{\otimes n}\big)\big).$$

It allows us to write explicitly elements of the commutant of the algebra generated by $\rho^n_{U(N)}$ as elements of $d\rho_N^{\mathfrak{S}_n}(\mathbb{C}[\mathfrak{S}_n])$. Indeed, if $\mathbf{A}$ commutes with $\rho^n_{U(N)}(g)$ for all $g \in U(N)$, we have

$$\mathbf{A} = E(\mathbf{A}) = d\rho_N^{\mathfrak{S}_n}\big(\Phi(\mathbf{A})\mathrm{Wg}\big).$$

Moreover, they give an asymptotic of the Weingarten function.

PROPOSITION 5.6 ([16]). *For all* $\sigma \in \mathfrak{S}_n$, *we have* $\mathrm{Wg}(\sigma) = O(N^{-n-|\sigma|})$ *when* N *tends to* ∞. *We have also* $\mathrm{Wg}(1_{\mathfrak{S}_n}) = N^{-n} + O(N^{-n-2})$ *when* N *tends to* ∞.

EXAMPLE 5.7. 1. For $n = 1$: we have $\mathrm{Wg} = 1_{\mathfrak{S}_n}/N$ and therefore, for all $A \in \mathrm{End}(\mathbb{C}^N)$,

$$E(A) = \frac{1}{N}\,\mathrm{Tr}(A) I_N.$$

2. For $n = 2$: we have $\mathrm{Wg} = \frac{1}{N^2-1}(1_{\mathfrak{S}_n} - \frac{1}{N}(1,2))$, and thus, for all $A, B \in \mathrm{End}(\mathbb{C}^N)$,

$$\begin{aligned} E(A \otimes B) = \frac{1}{N^2-1}\Big(&\big(\mathrm{Tr}(A)\,\mathrm{Tr}(B) - \mathrm{Tr}(AB)/N\big) \cdot I_N^{\otimes 2} \\ &+ \big(\mathrm{Tr}(AB) - \mathrm{Tr}(A)\,\mathrm{Tr}(B)/N\big) \cdot d\rho_N^{\mathfrak{S}_n}\big((1,2)\big)\Big). \end{aligned}$$

The generator L of a conjugate invariant convolution semigroup is a pseudo-differential operator which is bi-invariant, and by consequence the element $d\rho^n_{U(N)}(L)$ commutes with $\rho^n_{U(N)}(g)$ for all $g \in U(N)$. Thus, it is an element of $d\rho_N^{\mathfrak{S}_n}(\mathbb{C}[\mathfrak{S}_n])$. Let $\mathcal{T}_n$ be the subset of $\mathfrak{S}_n$ consisting of all the transpositions. For all $1 \le k_1 < \cdots < k_m \le n$, let us denote by $\iota^{\mathfrak{S}_n}_{k_1,\dots,k_m} : \mathfrak{S}_m \to \mathfrak{S}_n$ (or more simply $\iota_{k_1,\dots,k_m}$) the mapping defined by

$$\iota_{k_1,\dots,k_m}(\sigma) : \left| \begin{array}{ll} k_i \mapsto k_{\sigma(i)}, & \\ i \mapsto i, & \text{for } i \notin \{k_1,\dots,k_m\}. \end{array}\right.$$

This map is such that $\rho_N^{\mathfrak{S}_n} \circ \iota^{\mathfrak{S}_n}_{k_1,\dots,k_m} = \iota^{M_N(\mathbb{C})^{\otimes n}}_{k_1,\dots,k_m} \circ \rho_N^{\mathfrak{S}_m}$. We are now ready to state the main result of this section.

PROPOSITION 5.8. *Let $y_0, \alpha, \beta \in \mathbb{R}$ and Π be a Lévy measure on $U(N)$ which is conjugate invariant. Let $\mu \in \mathcal{ID}(U(N), \circledast)$ with characteristic triplet*

$$\left(iy_0 I_N, \begin{pmatrix} \alpha & & & 0 \\ & \ddots & & \\ & & \alpha & \\ 0 & & & \beta \end{pmatrix}, \Pi \right).$$

We have $\int_{U(N)} g^{\otimes n}\,\mathrm{d}\mu(g) = d\rho_N^{\mathfrak{S}_n}(e^{\tilde{L}})$, where

$$\begin{aligned} \tilde{L} = &\left(niy_0 - \frac{n^2}{N}\frac{\beta}{2} + \left(\frac{n^2}{N} - nN\right)\frac{\alpha}{2} + \frac{n}{N}\int_{U(N)} \mathrm{Tr}\big(\Re(g) - 1\big)\Pi(\mathrm{d}g)\right) 1_{\mathfrak{S}_n} \\ &- \alpha \sum_{\tau \in \mathcal{T}_n} \tau + \sum_{\substack{2 \le m \le n \\ 1 \le k_1 < \cdots < k_m \le n}} \sum_{\sigma,\pi \in \mathfrak{S}_m} \mathrm{Wg}(\sigma^{-1}\pi) \\ &\times \int_{U(N)} \prod_{c \text{ cycle of } \sigma} \mathrm{Tr}\big((g-1)^{\sharp c}\big)\Pi(\mathrm{d}g) \cdot \iota_{k_1,\dots,k_m}(\pi). \end{aligned}$$

PROOF. Let $(\mu_t)_{t\in\mathbb{R}^+}$ be the weakly continuous convolution semigroup whose characteristic triplet is

$$\left(iy_0 I_N, \begin{pmatrix} \alpha & & 0 & \\ & \ddots & & \\ & & \alpha & 0 \\ & & 0 & \beta \end{pmatrix}, \Pi\right),$$

and let L be its generator. By definition, $\mu = \mu_1$, and thanks to Proposition 5.2, we know that

$$\int_{U(N)} g^{\otimes n}\, \mathrm{d}\mu(g) = \exp\bigl(d\rho^n_{U(N)}(L)\bigr).$$

To conclude, it suffices to prove that $d\rho^n_{U(N)}(L) = d\rho_N^{\mathfrak{S}_n}(\tilde{L})$. We start from Proposition 5.3. We have

$$\begin{aligned} d\rho^n_{U(N)}(L) = \sum_{1\le k\le n} \iota_k\biggl(iny_0 + \int_{U(N)} \bigl(\Re(g) - I_N\bigr)\Pi(\mathrm{d}g)\biggr) \\ + \frac{\alpha}{2}\sum_{i=1}^{N^2-1} \cdot \sum_{1\le k,l\le n} \iota_k(Y_i)\circ\iota_l(Y_i) + \frac{\beta}{2}\sum_{1\le k,l\le n} \iota_k(Y_{N^2})\circ\iota_l(Y_{N^2}) \\ + \sum_{\substack{2\le m\le n \\ 1\le k_1<\cdots<k_m\le n}} \iota_{k_1,\dots,k_m}\biggl(\int_{U(N)} (g - I_N)^{\otimes m}\,\Pi(\mathrm{d}g)\biggr). \end{aligned}$$

Thanks to the invariance under conjugation of Π and $\sum_{i=1}^{N^2-1} Y_i\otimes Y_i$, we know from Example 5.7 that

$$\begin{aligned} \int_{U(N)} \bigl(\Re(g) - I_N\bigr)\Pi(\mathrm{d}g) &= \int_{U(N)} E\bigl(\Re(g) - I_N\bigr)\Pi(\mathrm{d}g) \\ &= \int_{U(N)} \frac{1}{N}\operatorname{Tr}\bigl(\Re(g) - 1\bigr)\Pi(\mathrm{d}g) \end{aligned}$$

and

$$\sum_{i=1}^{N^2-1} Y_i\otimes Y_i = E\biggl(\sum_{i=1}^{N^2-1} Y_i\otimes Y_i\biggr) = \frac{1}{N} I_N^{\otimes 2} - \rho_N^{\mathfrak{S}_n}\bigl((1,2)\bigr).$$

We also deduce from Proposition 5.5 that

$$\begin{aligned} &\int_{U(N)} (g - I_N)^{\otimes m}\,\Pi(\mathrm{d}g) \\ &\quad = \int_{U(N)} E\bigl((g - I_N)^{\otimes m}\bigr)\Pi(\mathrm{d}g) \\ &\quad = \int_{U(N)} \sum_{\sigma,\pi\in\mathfrak{S}_m} \mathrm{Wg}(\sigma^{-1}\pi) \prod_{c \text{ cycle of } \sigma} \operatorname{Tr}\bigl((g-1)^{\sharp c}\bigr)\cdot d\rho_N^{\mathfrak{S}_m}(\pi)\Pi(\mathrm{d}g). \end{aligned}$$

Thus, we have

$$\begin{aligned}
&\mathrm{d}\rho^n_{U(N)}(L)\\
&\quad=\left(niy_0-\frac{n^2}{N}\frac{\beta}{2}+\left(\frac{n^2}{N}-nN\right)\frac{\alpha}{2}+\frac{n}{N}\int_{U(N)}\mathrm{Tr}\big(\Re(g)-1\big)\Pi(\mathrm{d}g)\right)I_N^{\otimes n}\\
&\qquad-\alpha\sum_{i=1}^{N^2-1}\cdot\sum_{1\le k<l\le n}\iota_{k,l}\circ\rho_N^{\mathfrak{S}_2}\big((1,2)\big)\\
&\qquad+\sum_{\substack{2\le m\le n\\ 1\le k_1<\cdots<k_m\le n}}\sum_{\sigma,\pi\in\mathfrak{S}_m}\mathrm{Wg}\big(\sigma^{-1}\pi\big)\cdot\int_{U(N)}\prod_{c\text{ cycle of }\sigma}\mathrm{Tr}\big((g-1)^{\sharp c}\big)\Pi(\mathrm{d}g)\\
&\qquad\times\iota_{k_1,\dots,k_m}\circ\rho_N^{\mathfrak{S}_m}(\pi),
\end{aligned}$$

from which we deduce that $d\rho^n_{U(N)}(L)=d\rho_N^{\mathfrak{S}_n}(\tilde{L})$. □

6. The stochastic exponential $\mathcal{E}_N$. In this section, we shall describe $\mathcal{E}_N$, a map which connects the infinitely divisible measures on the space of Hermitian matrices $\mathcal{H}_N$ and the infinitely divisible measures on $U(N)$. We start by presenting $\mathcal{E}_N$ in Proposition–Definition 6.2, and the rest of the section is devoted to the proof of Proposition–Definition 6.2.

We consider the Hilbert space of Hermitian matrices

$$\mathcal{H}_N=\{x\in M_N(\mathbb{C}):x^*=x\}.$$

We denote by $*$ the classical convolution on the vector space $\mathcal{H}_N$: given two probability measures μ and ν on $\mathcal{H}_N$, the convolution $\mu*\nu$ is such that $\int_{\mathcal{H}_N}f\,\mathrm{d}(\mu*\nu)=\int_{\mathcal{H}_N}\int_{\mathcal{H}_N}f(x+y)\mu(\mathrm{d}x)\nu(\mathrm{d}y)$ for all bounded Borel function f on $\mathcal{H}_N$. Let us denote by $\mathcal{ID}(\mathcal{H}_N,*)$ the space of infinitely divisible probability measures on $\mathcal{H}_N$ and by $\mathcal{ID}_{\mathrm{inv}}(\mathcal{H}_N,*)$ the subspace of measures μ in $\mathcal{ID}(\mathcal{H}_N,*)$ which are invariant by unitary conjugation, that is, such that for all bounded Borel function f on $\mathcal{H}_N$ and all $g\in U(N)$, we have

$$\int_{\mathcal{H}_N}f\,\mathrm{d}\mu=\int_{\mathcal{H}_N}f\big(gxg^*\big)\,\mathrm{d}\mu(x).$$

6.1. *Infinite divisibility on $\mathcal{H}_N$.* The advantage of $\mathcal{ID}(\mathcal{H}_N,*)$ is that each infinitely divisible measure arises from a unique convolution semigroup, and by consequence, is characterized by a unique generator. In order to describe this generator, we introduce now an inner product on $\mathcal{H}_N$ and we define the notion of Lévy measure.

We endow $\mathcal{H}_N$ with the following inner product:

$$(x,y)\mapsto\langle x,y\rangle_{\mathcal{H}_N}=\mathrm{Tr}\big(x^*y\big)=\mathrm{Tr}(xy).$$

It is a real scalar product on $\mathcal{H}_N$ which is invariant by unitary conjugation. We remark that $i\mathcal{H}_N = \mathfrak{u}(N)$. Thus, the family $\{X_1, \dots, X_{N^2}\} = \{-iY_1, \dots, -iY_{N^2}\}$ is an orthonormal basis of $\mathcal{H}_N$ such that $X_{N^2} = \frac{1}{\sqrt{N}} I_N$. It is now useful to fix one compact neighborhood B of 0: we choose to set $B = B(0,1)$, the closed unit ball of $\mathcal{H}_N$. A *Lévy measure* Π on $\mathcal{H}_N$ is a measure on $\mathcal{H}_N$ such that both $\Pi(\{0\}) = 0$ and such that $\int_B \|x\|^2_{\mathcal{H}_N} \Pi(\mathrm{d}x)$ and $\Pi(B^c)$ are finite.

Let $C_b^2(\mathcal{H}_N)$ be the space of function $f \in C^2(\mathcal{H}_N)$ with bounded first- and second-order partial derivatives.

THEOREM 6.1 [29, 32]. *Let $\mu \in \mathcal{ID}(\mathcal{H}_N, *)$. There exists a unique weakly continuous semigroup $(\mu^{*t})_{t\in\mathbb{R}^+}$ such that $\mu^{*0} = \delta_0$ and $\mu^{*1} = \mu$. There exist an element $X_0 \in \mathcal{H}_N$, a symmetric positive semidefinite matrix $(y_{i,j})_{1\le i,j\le N^2}$ and a Lévy measure Π on $\mathcal{H}_N$ such that the generator L of $(\mu^{*t})_{t\in\mathbb{R}^+}$ is given for all $f \in C_b^2(\mathcal{H}_N)$ and all $y \in \mathcal{H}_N$ by*

$$\begin{aligned} Lf(y) &= \partial_{X_0} f(y) + \frac{1}{2}\sum_{i,j=1}^{N^2} y_{i,j}\, \partial_{X_i}\partial_{X_j} f(y) \\ &\quad + \int_{\mathcal{H}_N} f(y+x) - f(y) - 1_B(x)\,\partial_x f(y)\Pi(\mathrm{d}x). \end{aligned} \tag{6.1}$$

The triplet $(X_0, (y_{i,j})_{1\le i,j\le N^2}, \Pi)$ is called the *characteristic triplet* of μ, and its associated generator L is called the generator of μ. Conversely, given such a triplet $(X_0, (y_{i,j})_{1\le i,j\le N^2}, \Pi)$, there exists a unique infinitely divisible measure μ whose generator is given by (6.1).

Let us remark that the functions $\mathbf{e}$ and sin make sense on $\mathcal{H}_N$. For all $x \in \mathcal{H}_N$, we have

$$\mathbf{e}(x) = \exp(ix) \in U(N) \quad \text{and} \quad \sin(x) = \Im \circ \mathbf{e} = \left(e^{ix} - e^{-ix}\right)/2i \in \mathcal{H}_N.$$

As previously, for all measure Π on $\mathcal{H}_N$, the measure $\mathbf{e}_*(\Pi)$ denotes the push-forward Π on $\mathcal{H}_N$ by the mapping $\mathbf{e} : \mathcal{H}_N \to U(N)$, and the measure $\mathbf{e}_*(\Pi)_{|U(N)\setminus\{I_N\}}$ is the measure on $U(N) \setminus \{I_N\}$ induced by $\mathbf{e}_*(\Pi)$. We are now able to formulate the main result of this section.

PROPOSITION–DEFINITION 6.2. *For all $\mu \in \mathcal{ID}(\mathcal{H}_N, *)$ with characteristic triplet*

$$\left(X_0, (y_{i,j})_{1\le i,j\le N^2}, \Pi\right),$$

we define $\mathcal{E}_N(\mu)$ to be the measure of $\mathcal{ID}(U(N), \circledast)$ with characteristic triplet

$$\left(iX_0 + i\int_{\mathcal{H}_N} \left(\sin(x) - 1_B(x)x\right)\Pi(\mathrm{d}x), (y_{i,j})_{1\le i,j\le N^2}, \mathbf{e}_*(\Pi)_{|U(N)\setminus\{I_N\}}\right).$$

The map $\mathcal{E}_N : \mathcal{ID}(\mathcal{H}_N, *) \to \mathcal{ID}(U(N), \circledast)$ is called the stochastic exponential and has the following properties:

1. *For all* $\mu \in \mathcal{ID}(\mathcal{H}_N, *)$, *the measures* $(\mathbf{e}_*(\mu^{*1/n}))^{\circledast n}$ *converge weakly to* $\mathcal{E}_N(\mu)$.

2. *The stochastic exponential maps* $\mathcal{ID}_{\text{inv}}(\mathcal{H}_N, *)$ *to* $\mathcal{ID}_{\text{inv}}(U(N), \circledast)$, *and for all* μ, ν *measures of* $\mathcal{ID}_{\text{inv}}(\mathcal{H}_N, *)$, *we have*

$$\mathcal{E}_N(\mu * \nu) = \mathcal{E}_N(\mu) \circledast \mathcal{E}_N(\nu).$$

The tool used to prove this proposition is the Fourier transform of a measure on $U(N)$. Before proving Proposition 6.2 in Section 6.3, let us introduce this notion.

6.2. *Fourier transform on* $U(N)$. The set $\widehat{U(N)}$ of isomorphism classes of irreducible representations of $U(N)$ is in bijection with the set $\mathbb{Z}^N_\downarrow$ of nonincreasing sequences of integers $\alpha = (\alpha_1 \geq \cdots \geq \alpha_N)$. For all $\alpha \in \mathbb{Z}^N_\downarrow$, let $\pi^\alpha \in \widehat{U(N)}$ be a unitary representation in the corresponding class, acting on a vector space E_α, and let χ_α be its character, that is to say the function $\operatorname{Tr} \circ \pi^\alpha$. We will also consider the normalized character $\psi_\alpha(\cdot) = \chi_\alpha(\cdot)/\chi_\alpha(I_N)$.

Let μ be a probability measure on $U(N)$. The Fourier transform $\widehat{\mu}$ of μ is defined for all $\alpha \in \mathbb{Z}^N_\downarrow$ by $\widehat{\mu}(\alpha) = \int_{U(N)} \pi^\alpha(g)\mu(\mathrm{d}g) \in \operatorname{End}(E_\alpha)$. Here are three properties of the Fourier transform (see [24, 34]).

1. For all probability measures μ and ν, and for all $\alpha \in \mathbb{Z}^N_\downarrow$ we have $\widehat{\mu \circledast \nu}(\alpha) = \widehat{\mu}(\alpha)\widehat{\nu}(\alpha)$.

2. A sequence of probability measures $(\mu_n)_{n\in\mathbb{N}}$ converges weakly to a measure μ if and only if for all $\alpha \in \mathbb{Z}^N_\downarrow$, the sequence $(\widehat{\mu_n}(\alpha))_{n\in\mathbb{N}}$ converges to $\widehat{\mu}(\alpha)$.

3. A probability measure μ is central, or conjugate invariant, if and only if for all $\alpha \in \mathbb{Z}^N_\downarrow$, $\widehat{\mu}(\alpha)$ is a homogeneous dilation, and in this case $\widehat{\mu}(\alpha) = (\int_{U(N)} \psi_\alpha(g)\mu(\mathrm{d}g))\mathrm{Id}_{E_\alpha}$.

The following proposition gives the Fourier transform of a measure arising from a convolution semigroup.

PROPOSITION 6.3. *Let* $(\mu_t)_{t\in\mathbb{R}^+}$ *be a weakly continuous convolution semigroup on* $U(N)$ *starting at* $\mu_0 = \delta_e$ *with generator* L. *For all* $t \geq 0$, *and all* $\alpha \in \mathbb{Z}^N_\downarrow$, *we have* $\widehat{\mu_t}(\alpha) = e^{tL\pi^\alpha(I_N)}$. *Moreover, if* μ *is conjugate invariant, we have* $\widehat{\mu_t}(\alpha) = e^{tL\psi_\alpha(I_N)}\mathrm{Id}_{E_\alpha}$.

PROOF. For all $\alpha \in \mathbb{Z}^N_\downarrow$, we have $\widehat{\mu_t}(\alpha) = \int_{U(N)} \pi^\alpha(g)\mu_t(\mathrm{d}g) = \mathrm{Id}_{E_\alpha} + t \cdot L\pi^\alpha(I_N) + o_{t\to 0}(t)$, which implies that $\widehat{\mu_t}(\alpha) = \lim_{s\to 0} \widehat{\mu_s}(\alpha)^{t/s} = e^{tL\pi^\alpha(I_N)}$. If μ is conjugate invariant, then, for all $t \in \mathbb{R}^+$, μ_t is conjugate invariant, and we can replace π^α by ψ_α in the previous computation. □

COROLLARY 6.4. *Let $(\mu_t)_{t\in\mathbb{R}^+}$ and $(\nu_t)_{t\in\mathbb{R}^+}$ be two weakly continuous conjugate invariant convolution semigroups on $U(N)$ starting at $\mu_0=\delta_e$, with respective characteristic triplets*

$$\big(Y_0,(y_{i,j})_{1\le i,j\le N^2},\Pi\big)\quad \textit{and}\quad \big(Y_0',(y_{i,j}')_{1\le i,j\le N^2},\Pi'\big).$$

Then $(\mu_t\circledast\nu_t)_{t\in\mathbb{R}^+}$ is a weakly continuous convolution semigroup on $U(N)$ starting at $\mu_0=\delta_e$, with characteristic triplet

$$\big(Y_0+Y_0',(y_{i,j}+y_{i,j}')_{1\le i,j\le N^2},\Pi+\Pi'\big).$$

PROOF. Remark that $(y_{i,j}+y_{i,j}')_{1\le i,j\le N^2}$ is a symmetric positive semidefinite matrix and that $\Pi+\Pi'$ is a Lévy measure. Let L and L' be the respective generators of $(\mu_t)_{t\in\mathbb{R}^+}$ and $(\nu_t)_{t\in\mathbb{R}^+}$ given by (5.1). Thanks to Proposition 6.3 and to the conjugation invariance, for all $\alpha\in\mathbb{Z}^N_\downarrow$, we have

$$\widehat{\mu_t*\nu_t}(\alpha)=\widehat{\mu_t}(\alpha)\cdot\widehat{\nu_t}(\alpha)=e^{tL\psi_\alpha(I_N)}e^{tL'\psi_\alpha(I_N)}\mathrm{Id}_{E_\alpha}=e^{t(L+L')\psi_\alpha(I_N)}\mathrm{Id}_{E_\alpha}.$$

To conclude, observe that, for each time $t\in\mathbb{R}_+$, the measure at time t of the weakly continuous semigroup whose characteristic triplet is $(Y_0+Y_0',(y_{i,j}+y_{i,j}')_{1\le i,j\le N^2},\Pi+\Pi')$ has the same Fourier transform as $\mu_t\circledast\nu_t$. □

LEMMA 6.5. *Let μ and $\nu\in\mathcal{ID}_{\mathrm{inv}}(U(N),\circledast)$ with characteristic triplet $(Y_0,(y_{i,j})_{1\le i,j\le N^2},\Pi)$ and $(Y_0',(y_{i,j}')_{1\le i,j\le N^2},\Pi')$. Then, $(Y_0+Y_0',(y_{i,j}+y_{i,j}')_{1\le i,j\le N^2},\Pi+\Pi')$ is a characteristic triplet of $\mu\circledast\nu$. In particular, for all $k\in\mathbb{Z}$, $(Y_0+2ik\pi I_N,(y_{i,j})_{1\le i,j\le N^2},\Pi)$ is also a characteristic triplet of μ.*

PROOF. The first assertion follows from Corollary 6.4. For the second assertion, we remark that $(\delta_{e^{2ikt\Pi}I_N})_{t\in\mathbb{R}^+}$ is a weakly continuous convolution semigroup with characteristic triplet $(2ik\pi,0,0)$. By consequence, $(Y_0+2ik\pi I_N,(y_{i,j})_{1\le i,j\le N^2},\Pi)$ is a characteristic triplet of $\mu\circledast\delta_{e^{2ik\pi}I_N}=\mu$. □

We are now ready to prove Proposition–Definition 6.2.

6.3. *Proof of Proposition–Definition* 6.2. First of all, we remark that the sine function is bounded and $\sin(x)-x\sim_{x\to0}x^3/6$, which implies that $\int_{\mathcal{H}_N}(\sin(x)-1_B(x)x)\Pi(\mathrm{d}x)$ exists.

We start by proving the first item. Let $\mu\in\mathcal{ID}(\mathcal{H}_N,*)$. Let us denote by L_μ the generator of μ and by $L_{\mathcal{E}_N(\mu)}$ the generator of $\mathcal{E}_N(\mu)$. Let $\alpha\in\mathbb{Z}^N_\downarrow$. We have

$$\begin{aligned}\mathbf{e}_*\widehat{(\mu^{*1/n})}(\alpha)&=\int_{\mathcal{H}_N}\pi^\alpha\big(\mathbf{e}(x)\big)\mu^{*1/n}(\mathrm{d}x)\\&=\mathrm{Id}_{E_\alpha}+L_\mu\big(\pi^\alpha\circ\mathbf{e}\big)(0)/n+o_{n\to\infty}(1/n),\end{aligned}$$

which implies that $\lim_{n\to\infty}(\mathbf{e}_*(\widehat{\mu^{*1/n}}))^{\circledast n}(\alpha) = \lim_{n\to\infty}(\mathbf{e}_*\widehat{(\mu^{*1/n}})(\alpha))^n = e^{L_\mu(\pi^\alpha\circ\mathbf{e})(0)}$. Let us compute

$$L_\mu(\pi^\alpha\circ\mathbf{e})(0) = \partial_{X_0}(\pi^\alpha\circ\mathbf{e})(0) + \frac{1}{2}\sum_{i,j=1}^{N^2} y_{i,j}\,\partial_{X_i}\,\partial_{X_j}(\pi^\alpha\circ\mathbf{e})(0)$$
$$+\int_{\mathcal{H}_N}\pi^\alpha(e^{i(x+0)}) - \pi^\alpha(e^{i0}) - 1_B(x)\,\partial_x(\pi^\alpha\circ\mathbf{e})(0)\Pi(\mathrm{d}x).$$

Recall that, for all $Y\in\mathfrak{u}(N)$, Y^l is the left invariant vector field on $U(N)$ induced by Y. Using the fact that, for all $x\in\mathcal{H}_N$, $\partial_x(\pi^\alpha\circ\mathbf{e})(0) = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\pi^\alpha(e^{itx}) = (ix)^l\pi^\alpha(I_N)$, we infer

$$\begin{aligned}
L_\mu(\pi^\alpha\circ\mathbf{e})(0) &= (iX_0)^l(\pi^\alpha)(I_N) + \frac{1}{2}\sum_{i,j=1}^{N^2} y_{i,j}Y_i^lY_j^l(\pi^\alpha)(I_N)\\
&\qquad + \int_{\mathcal{H}_N}\pi^\alpha(e^{ix}) - \mathrm{Id}_{E_\alpha} - 1_B(x)(ix)^l\pi^\alpha(I_N)\Pi(\mathrm{d}x)\\
&= (iX_0)^l(\pi^\alpha)(I_N) + \int_{\mathcal{H}_N}(i\sin(x) - i1_B(x)x)^l\pi^\alpha(I_N)\Pi(\mathrm{d}x)\\
&\qquad + \frac{1}{2}\sum_{i,j=1}^{N^2} y_{i,j}Y_i^lY_j^l(\pi^\alpha)(I_N)\\
&\qquad + \int_{\mathcal{H}_N}\pi^\alpha(\mathbf{e}(x)) - \mathrm{Id}_{E_\alpha} - (i\Im(\mathbf{e}(x)))^l\pi^\alpha(I_N)\Pi(\mathrm{d}x)\\
&= L_{\mathcal{E}_N(\mu)}\pi^\alpha(I_N).
\end{aligned}$$

Finally, for all $\alpha\in\mathbb{Z}_\downarrow^N$, the sequence $(\mathbf{e}_*(\widehat{\mu^{*1/n}}))^{*n}(\alpha)$ converges to $e^{L_{\mathcal{E}_N(\mu)}\pi^\alpha(I_N)} = \widehat{\mathcal{E}_N(\mu)}(\alpha)$ and consequently the sequence $(\mathbf{e}_*(\mu^{*1/n}))^{\circledast n}$ converges to $\mathcal{E}_N(\mu)$.

For the proof of the second item, we use the Fourier transform of a measure in $\mathcal{ID}(\mathcal{H}_N,*)$, which is given by the following proposition.

PROPOSITION 6.6 ([32]). *Let $\mu\in\mathcal{ID}(\mathcal{H}_N,*)$ with characteristic triplet $(X_0,(y_{i,j})_{1\le i,j\le N^2},\Pi)$. We have $\int_{\mathcal{H}_N}e^{i\,\mathrm{Tr}(xy)}\mu^{*t}(\mathrm{d}x) = \exp(t\varphi_\mu(y))$ with*

$$\varphi_\mu(y) = i\,\mathrm{Tr}(X_0y) - \frac{1}{2}\sum_{i,j=1}^{N^2} y_{i,j}\,\mathrm{Tr}(X_iy)\,\mathrm{Tr}(X_jy)$$
$$+\int_{\mathcal{H}_N}e^{i\,\mathrm{Tr}(xy)} - 1 - i1_B(x)\,\mathrm{Tr}(xy)\Pi(\mathrm{d}x).$$

Let $\mu \in \mathcal{ID}_{\text{inv}}(\mathcal{H}_N, *)$. We claim that, for all $t \geq 0$, $\mu^{*t} \in \mathcal{ID}_{\text{inv}}(\mathcal{H}_N, *)$. Assuming for a moment that this claim is proved, let us explain how it leads to the result: in this case, each measure $(\mathbf{e}_*(\mu^{*1/n}))^{\circledast n}$ is conjugate invariant and so is the limit $\mathcal{E}_N(\mu)$. In addition, for all $\mu, \nu \in \mathcal{ID}_{\text{inv}}(\mathcal{H}_N, *)$, the characteristic triplets of $\mathcal{E}_N(\mu * \nu)$ and of $\mathcal{E}_N(\mu) \circledast \mathcal{E}_N(\nu)$ coincide thanks to Corollary 6.4, and thus $\mathcal{E}_N(\mu * \nu) = \mathcal{E}_N(\mu) \circledast \mathcal{E}_N(\nu)$.

Thus, it remains to prove that, for all $t \geq 0$, $\mu^{*t} \in \mathcal{ID}_{\text{inv}}(\mathcal{H}_N, *)$. For this, we prove that the Fourier transform of μ^{*t} is conjugate invariant. First, φ_μ is conjugate invariant. Indeed, for all $g \in U(N)$, we have

$$\begin{aligned} \exp \circ \varphi_\mu(gyg^*) &= \int_{\mathcal{H}_N} e^{i \operatorname{Tr}(xgyg^*)} \, \mathrm{d}\mu(x) = \int_{\mathcal{H}_N} e^{i \operatorname{Tr}(g^*xgy)} \, \mathrm{d}\mu(x) \\ &= \int_{\mathcal{H}_N} e^{i \operatorname{Tr}(xy)} \, \mathrm{d}\mu(x) = \exp \circ \varphi_\mu(y). \end{aligned}$$

We deduce that φ_μ is conjugate invariant since it is continuous and $\exp \circ \varphi_\mu$ is conjugate invariant. Consequently, $\int_{\mathcal{H}_N} e^{i \operatorname{Tr}(xg \cdot g^*)} \, \mathrm{d}\mu^{*t}(x) = \exp(t\varphi_\mu(\cdot))$ is conjugate invariant, which is sufficient to conclude.

7. Random matrices. In this last section, we shall define the mappings Π_N and Γ_N. Then we prove Theorem 2, and in particular our main result, the convergence of the empirical spectral measures of random matrices distributed over $\Gamma_N(\mu)$ for some $\mu \in \mathcal{ID}(\mathbb{U}, \boxtimes)$ (see Theorem 7.8). We finish the section by the proof of Theorem 3.

7.1. *The matrix model* Π_N. Recall that the covariance matrix, which corresponds to the diffuse part of an infinitely divisible measure, depends on the choice of a basis of $\mathcal{H}_N$ (see Section 6.1). In this article, we fixed an orthonormal basis $\{X_1, \ldots, X_{N^2}\}$ of $\mathcal{H}_N$ such that $X_{N^2} = \frac{1}{\sqrt{N}} I_N$.

DEFINITION 7.1. Let $\mu \in \mathcal{ID}(\mathbb{R}, \boxplus)$ and let (η, a, ρ) be its $\boxplus$-characteristic triplet. The distribution $\Pi_N(\mu) \in \mathcal{ID}_{\text{inv}}(\mathcal{H}_N, *)$ is defined to be the infinitely divisible measure with characteristic triplet $(\eta I_N, a_N, \rho_N)$, where a_N is the $N^2 \times N^2$-matrix

$$a_N = \begin{pmatrix} \frac{a}{N+1} & & & 0 \\ & \ddots & & \\ & & \frac{a}{N+1} & \\ 0 & & & a \end{pmatrix},$$

and ρ_N is the Lévy measure on $\mathcal{H}_N$ which is the push-forward measure of $N\rho \otimes$ Haar by the mapping from $\mathbb{R} \times U(N)$ to $\mathcal{H}_N$ defined by

$$(x, g) \mapsto g \begin{pmatrix} x & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix} g^*.$$

The application $\Pi_N : \mathcal{ID}(\mathbb{R}, \boxplus) \to \mathcal{ID}_{\text{inv}}(\mathcal{H}_N, *)$ is obviously a homomorphism of semigroups and we have $\Pi_1 = \Lambda^{-1}$. Moreover, Π_N is a matricial model for $\mathcal{ID}(\mathbb{R}, \boxplus)$ in the sense of the following theorem.

THEOREM 7.2 ([4, 12]). *Let $\mu \in \mathcal{ID}(\mathbb{R}, \boxplus)$. For all $N \in \mathbb{N}^*$, let H_N be a random matrix whose law is $\Pi_N(\mu)$, and let μ_{H_N} be its empirical spectral measure, that is to say*

$$\mu_{H_N} = \frac{1}{N} \sum_{\substack{\text{eigenvalue } \lambda \text{ of } H_N \\ \text{(with multiplicity)}}} \delta_\lambda.$$

Then the measures μ_{H_N} converge weakly almost surely to μ when N tends to ∞.

In [4, 12], the model is in fact defined starting from a measure $\mu \in \mathcal{ID}(\mathbb{R}, *)$. More precisely, for all $\mu \in \mathcal{ID}(\mathbb{R}, *)$ with $*$-characteristic triplet (η, a, ρ) and Lévy exponent

$$\varphi_\mu(\theta) = \left(i\eta\theta - \frac{1}{2}a\theta^2 + \int_{\mathbb{R}} \left(e^{i\theta x} - 1 - i\theta x 1_{[-1,1]}(x)\right) \mathrm{d}\rho(x)\right),$$

Benaych–Georges and Cabanal–Duvillard defined $\Lambda_N(\mu) \in \mathcal{ID}_{\text{inv}}(\mathcal{H}_N, *)$ by its Fourier transform: for $x, y \in \mathcal{H}_N$, we have

$$\int_{\mathcal{H}_N} e^{i \operatorname{Tr}(xy)} \Lambda_N(\mu)(\mathrm{d}x) = \exp\left(\varphi_{\Lambda_N(\mu)}(y)\right),$$

where $\varphi_{\Lambda_N(\mu)}(y) = N\mathbb{E}[\varphi_\mu(\langle u, yu\rangle)]$, with u uniformly distributed on the unit sphere of $\mathbb{C}^N$. More explicitly,

$$\begin{aligned} \varphi_{\Lambda_N(\mu)}(y) = {} & i\eta \operatorname{Tr}(y) - \frac{a}{2(N+1)}\left(\operatorname{Tr}(y)\operatorname{Tr}(y) + \operatorname{Tr}\left(y^2\right)\right) \\ & + \int_{\mathcal{H}_N} e^{i \operatorname{Tr}(xy)} - 1 - i 1_B(x) \operatorname{Tr}(xy) \Pi(\mathrm{d}x). \end{aligned}$$

Using Proposition 6.6, we see that it is exactly the Fourier transform of the infinitely divisible measure of $\mathcal{ID}_{\text{inv}}(\mathcal{H}_N, *)$ with characteristic triplet $(\eta I_N, a_N,$

ρ_N). Consequently, we have $\Lambda_N = \Pi_N \circ \Lambda$, or $\Pi_N = \Lambda_N \circ \Lambda^{-1}$ which can be expressed as the commutativity of the following diagram:

$$\begin{array}{ccccc} & \Lambda_N & & \Pi_N & \\ \mathcal{ID}(\mathbb{R},*) & \longrightarrow & \mathcal{ID}_{\mathrm{inv}}(\mathcal{H}_N,*) & \longleftarrow & \mathcal{ID}(\mathbb{R},\boxplus). \\ & & \underbrace{\qquad\qquad\qquad}_{\Lambda} & & \end{array}$$

Nevertheless, we prefer to use Π_N which turns out to be more suitable for our present purposes (see Theorem 2). One can consult also [18, 19] for further information about this model.

7.2. *The matrix model* Γ_N. Here again, observe that the data of a covariance matrix of $\mathfrak{u}(N)$ depends on the basis chosen, and recall that we fixed an orthonormal basis $\{Y_1, \dots, Y_{N^2}\}$ of $\mathfrak{u}(N)$ such that $Y_{N^2} = \frac{i}{\sqrt{N}} I_N$ (see Section 5.1).

DEFINITION 7.3. Let $\mu \in \mathcal{ID}(\mathbb{U}, \boxtimes)$ and let (ω, b, υ) be its $\boxtimes$-characteristic triplet. The distribution $\Gamma_N(\mu) \in \mathcal{ID}_{\mathrm{inv}}(U(N), \circledast)$ is defined to be the infinitely divisible measure with characteristic triplet $(\mathrm{Log}(\omega) I_N, b_N, \upsilon_N)$, where Log is the principal logarithm, b_N is the $N^2 \times N^2$-matrix

$$b_N = \begin{pmatrix} \dfrac{b}{N+1} & & & 0 \\ & \ddots & & \\ & & \dfrac{b}{N+1} & \\ 0 & & & b \end{pmatrix},$$

and υ_N is the Lévy measure on $U(N)$ which is the push-forward measure of $N\upsilon \otimes$ Haar by the mapping from $\mathbb{U} \times U(N)$ to $U(N)$ defined by

$$(\zeta, g) \mapsto g \begin{pmatrix} \zeta & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} g^*.$$

If λ is the Haar measure on $\mathbb{U}$, then we agree to define $\Gamma_N(\lambda)$ to be the Haar measure of $U(N)$.

From this definition, we deduce right now the second half of Theorem 2, as a consequence of the following propositions.

PROPOSITION 7.4. *For all μ and $\nu \in \mathcal{ID}(\mathbb{U}, \boxtimes)$, we have $\Gamma_N(\mu \boxtimes \nu) = \Gamma_N(\mu) \circledast \Gamma_N(\nu)$.*

PROOF. Let μ and $\nu \in \mathcal{ID}(\mathbb{U}, \boxtimes)$. If μ or ν is equal to λ, we have $\mu \boxtimes \nu = \lambda$. In this case, $\Gamma_N(\mu)$ or $\Gamma_N(\nu)$ is the Haar measure on $U(N)$ and consequently, $\text{Haar} = \Gamma_N(\mu \boxtimes \nu) = \Gamma_N(\mu) \circledast \Gamma_N(\nu)$.

If $\mu, \nu \in \mathcal{ID}(\mathbb{U}, \boxtimes) \cap \mathcal{M}_*$, with respective $\boxtimes$-characteristic triplets $(\omega_1, b_1, \upsilon_1)$ and $(\omega_2, b_2, \upsilon_2)$, the measure $\mu \boxtimes \nu \in \mathcal{M}_*$ is a $\boxtimes$-infinitely divisible measure with $\boxtimes$-characteristic triplet $(\omega_1\omega_2, b_1 + b_2, \upsilon_1 + \upsilon_2)$. We denote by $(Y_0, (y_{i,j})_{1\le i,j\le N^2}, \Pi)$ and $(Y_0', (y_{i,j}')_{1\le i,j\le N^2}, \Pi')$ the respective characteristic triplets of $\Gamma(\mu \boxtimes \nu)$ and $\Gamma(\mu) \circledast \Gamma(\nu)$. It is straightforward to verify that $((y_{i,j})_{1\le i,j\le N^2}, \Pi) = ((y_{i,j}')_{1\le i,j\le N^2}, \Pi')$, and it remains to compare Y_0 and Y_0'. We have $Y_0 = \text{Log}(\omega_1\omega_2) I_N$ and $Y_0 = (\text{Log}(\omega_1) + \text{Log}(\omega_2)) I_N$. As a consequence, Y_0 and Y_0' differ by a multiple of $2i\pi I_N$. Using Lemma 6.5, we deduce that $(Y_0, (y_{i,j})_{1\le i,j\le N^2}, \Pi)$ and $(Y_0', (y_{i,j}')_{1\le i,j\le N^2}, \Pi')$ are characteristic triplets of the same measure. In other words, $\Gamma(\mu \boxtimes \nu) = \Gamma(\mu) \circledast \Gamma(\nu)$. ☐

PROPOSITION 7.5. *For all* $\mu \in \mathcal{ID}(\mathbb{R}, \boxplus)$, *we have* $\Gamma_N \circ \mathbf{e}_\boxplus(\mu) = \mathcal{E}_N \circ \Pi_N(\mu)$.

PROOF. Let (η, a, ρ) be the $\boxplus$-characteristic triplet of μ. We denote by $(Y_0, (y_{i,j})_{1\le i,j\le N^2}, \Pi)$ and $(Y_0', (y_{i,j}')_{1\le i,j\le N^2}, \Pi')$ the respective characteristic triplets of $\Gamma_N \circ \mathbf{e}_\boxplus(\mu)$ and $\mathcal{E}_N \circ \Pi_N(\mu)$. We remark first that, following the definitions,

$$
\begin{aligned}
(y_{i,j})_{1\le i,j\le N^2} &= (y_{i,j}')_{1\le i,j\le N^2} \\
&= \begin{pmatrix} \frac{a}{N+1} & & & 0 \\ & \ddots & & \\ & & \frac{a}{N+1} & \\ 0 & & & a \end{pmatrix}
\end{aligned}
$$

and $\Pi = \Pi' = M_{|U(N)\setminus\{I_N\}}$ where M is the push-forward measure of $N\rho \otimes \text{Haar}$ by the mapping from $\mathbb{R} \times U(N)$ to $U(N)$ given by

$$
(x, g) \mapsto g \begin{pmatrix} e^{ix} & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix} g^*.
$$

To conclude, it remains to compare Y_0 and Y_0'. We have

$$
Y_0 = \text{Log} \circ \exp\left(i\eta + i \int_{\mathbb{R}} (\sin(x) - 1_{[-1,1]}(x)x)\rho(\mathrm{d}x)\right) I_N
$$

and

$$
\begin{aligned}
Y_0' &= i\eta I_N + i\int_{\mathcal{H}_N} \big(\sin(x) - 1_U(x)x\big)\,\mathrm{d}\rho_N(x) \\
&= i\eta I_N + iN\int_{\mathbb{R}}\int_{U(N)} g \begin{pmatrix} \big(\sin(x) - 1_{[-1,1]}(x)x\big) & & 0 \\ & 0 & \\ & & \ddots & \\ 0 & & & 0 \end{pmatrix} g^*\,\mathrm{d}g\rho(\mathrm{d}x) \\
&= i\eta I_N + iN\int_{\mathbb{R}} \frac{1}{N}\big(\sin(x) - 1_{[-1,1]}(x)x\big)\rho(\mathrm{d}x) \\
&= \left(i\eta + i\int_{\mathbb{R}}\big(\sin(x) - 1_{[-1,1]}(x)x\big)\rho(\mathrm{d}x)\right) I_N,
\end{aligned}
$$

where we have used that $E(A) = \frac{1}{N}\operatorname{Tr}(A)I_N$ (see Example 5.7) for the integration with respect to the Haar measure of $U(N)$. The difference between Y_0 and Y_0' is a multiple of $2i\pi I_N$. Using Lemma 6.5, we deduce that $(Y_0, (y_{i,j})_{1\le i,j\le N^2}, \Pi)$ and $(Y_0', (y'_{i,j})_{1\le i,j\le N^2}, \Pi')$ are characteristic triplets of the same measure. In other words, $\Gamma_N \circ \mathbf{e}_{\boxplus}(\mu) = \mathcal{E}_N \circ \Pi_N(\mu)$. □

7.3. *The large-N limit.* We are now ready to prove the first half of Theorem 2, and we begin with a concentration result, very similar to [12], Theorem III.4. For all $f : \mathbb{U} \to \mathbb{R}$, the map $\operatorname{tr} f : U \to \operatorname{tr}(f(U))$ is defined on $U(N)$ by spectral calculus.

THEOREM 7.6. *Let $\mu \in \mathcal{ID}(\mathbb{U}, \boxtimes) \cap \mathcal{M}_*$. For all $N \in \mathbb{N}^*$, let U_N be a random matrix whose law is $\Gamma_N(\mu)$. Let us consider $f : \mathbb{U} \to \mathbb{R}$ a Lipschitz continuous function. Then, for all $\varepsilon > 0$, there exists $K > 0$ such that, for all $N \in \mathbb{N}^*$,*

$$
\mathbb{P}\big[\big|\operatorname{tr} f(U_N) - \mathbb{E}[\operatorname{tr} f(U_N)]\big| \ge \varepsilon\big] \le 2e^{-NK}.
$$

PROOF. We adapt the proof of [12], Theorem III.4, in our multiplicative case. First of all, f is almost everywhere differentiable. The derivative f' of f and the Lipschitz norm $\|f\|_{\mathrm{Lip}}$ of f are defined as follows:

$$
f'(z) = \lim_{h\to 0} \frac{|f(ze^{ih}) - f(z)|}{h} \quad \text{and} \quad \|f\|_{\mathrm{Lip}} = \sup_{x\neq y\in\mathbb{R}} \frac{|f(e^{ix}) - f(e^{iy})|}{|x-y|}.
$$

Let (ω, b, υ) be the $\boxtimes$-characteristic triplet of μ. The distribution $\Gamma_N(\mu)$ is the infinitely divisible measure with characteristic triplet $(\operatorname{Log}(\omega)I_N, b_N, \upsilon_N)$ where b_N and υ_N are obtained from b and υ as in Definition 7.3. We consider a weakly continuous convolution semigroup $(\mu_t)_{t\in\mathbb{R}^+}$ with characteristic triplet $(\operatorname{Log}(\omega)I_N, b_N, \upsilon_N)$, in such a way that $\mu_1 = \Gamma_N(\mu)$ and $\mu_0 = \delta_{I_N}$. We consider the transition semigroup $(P_t)_{t\in\mathbb{R}^+}$ of $(\mu_t)_{t\in\mathbb{R}^+}$: for all $t \in \mathbb{R}^+$, all bounded Borel function f on $U(N)$, and all $h \in U(N)$, $P_t f(h) = \int_{U(N)} f(hg)\mu_t(\mathrm{d}g)$. The generator L of $(P_t)_{t\in\mathbb{R}^+}$ is explicitly given by (5.1).

We can assume without loss of generality that $\mathbb{E}[\operatorname{tr} f(U_N)] = 0$. Let us set, for any $\lambda \in \mathbb{R}$ and $t \in [0, 1]$,

$$\phi_\lambda(t) = P_t\big(\exp\big(\lambda P_{1-t}(\operatorname{tr} f)\big)\big)(I_N).$$

We have in particular $\phi_\lambda(0) = 1$ and $\phi_\lambda(1) = \mathbb{E}[\exp(\lambda \operatorname{tr} f(U_N))]$. Using the commutativity of L and P_t, we write

$$\phi'_\lambda(t) = P_t \circ L\big(e^{\lambda P_{1-t}(\operatorname{tr} f)}\big)(I_N) - \lambda P_t\big(L\big(P_{1-t}(\operatorname{tr} f)\big) \cdot e^{\lambda P_{1-t}(\operatorname{tr} f)}\big)(I_N).$$

We replace L by its definition (5.1). Denoting $P_{1-t}(\operatorname{tr} f)$ by F_t and $V \mapsto F_t(VU)$ by $F_t(\cdot U)$, it leads to the noncancelable terms

$$\begin{aligned}
\phi'_\lambda(t) &= P_t\left(\frac{\lambda^2 b}{2(N+1)}\sum_{i=1}^{N^2-1}\big(Y_i^l F_t\big)^2 \cdot e^{\lambda F_t}\right)(I_N)\\
&\quad + P_t\left(\frac{\lambda^2 b}{2}\big(Y_{N^2}^l F_t\big)^2 \cdot e^{\lambda F_t}\right)(I_N) \qquad (7.1)\\
&\quad + P_t\left(\int_{U(N)}\big(e^{\lambda F_t(\cdot U)-\lambda F_t} - 1 - \lambda F_t(\cdot U) - \lambda F_t\big)\,\mathrm{d}\upsilon_N(U) \cdot e^{\lambda F_t}\right)(I_N).
\end{aligned}$$

Let us examine the first two terms. In [28], Proof of the second assertion of Proposition 3.1, Lévy and Maïda proved that the quantity $\sum_{i=1}^{N^2}(Y_i^l F_t)^2$ is less than $\|f\|_{\mathrm{Lip}}^2/N$. We retrace briefly the structure of their proof, and refer to [28] for the detailed steps (note that the norm on $\mathfrak{u}(N)$ we choose differs from the norm in [28] by a $1/N$ factor). We have almost everywhere on $U(N)$

$$\begin{aligned}
\sum_{i=1}^{N^2}\big(Y_i^l F_t\big)^2 &= \sum_{i=1}^{N^2}\big(P_{1-t} \circ Y_i^l(\operatorname{tr} f)\big)^2 \le \sum_{i=1}^{N^2} P_{1-t}\big(\big(Y_i^l(\operatorname{tr} f)\big)^2\big)\\
&= P_{1-t}\left(\sum_{i=1}^{N^2}\big(\operatorname{tr}\big(f' Y_i\big)\big)^2\right) = \frac{1}{N}P_{1-t}\big(\operatorname{tr}\big(f'^2\big)\big) \le \frac{1}{N}\|f\|_{\mathrm{Lip}}^2.
\end{aligned}$$

They used the nonobvious facts that $Y_i^l(\operatorname{tr} f) = \operatorname{tr}(f' Y_i)$ and that $\sum_{i=1}^{N^2}(\operatorname{tr}(XY_i))^2 = \operatorname{tr}(X^2)/N$. The same argument with $Y_{n^2} = -iI_N/\sqrt{N}$ shows that, almost everywhere,

$$\begin{aligned}
\big(Y_{N^2}^l F_t\big)^2 &= \big(P_{1-t} \circ Y_{N^2}^l(\operatorname{tr} f)\big)^2 \le \sum_{i=1}^{N^2} P_{1-t}\big(\big(Y_{N^2}^l(\operatorname{tr} f)\big)^2\big)\\
&= \frac{1}{N}P_{1-t}\big(\big(\operatorname{tr} f'\big)^2\big) \le \frac{1}{N}\|f\|_{\mathrm{Lip}}^2.
\end{aligned}$$

Finally, by continuity, we infer that

$$\left|\frac{\lambda^2 b}{2(N+1)}\sum_{i=1}^{N^2-1}(Y_i^l F_t)^2+\frac{\lambda^2 b}{2}(Y_{N^2}^l F_t)^2\right|\leq \frac{\lambda^2 b}{2N}\|f\|_{\mathrm{Lip}}^2. \tag{7.2}$$

For the third term, we will need the following result. The nuclear norm, or Schatten 1-norm, is defined, for all $A\in M_N(\mathbb{C})$, as $\|A\|_1=\mathrm{Tr}(\sqrt{A^*A})=\sum_{\sigma \text{ singular value of } A}|\sigma|$.

LEMMA 7.7. *For all Lipschitz continuous function $f:\mathbb{U}\to\mathbb{R}$, and all $U,V\in U(N)$, we have*

$$\left|\mathrm{tr}(f(U))-\mathrm{tr}(f(V))\right|\leq\frac{1}{N}\left(\frac{\pi}{2}\right)^2\|f\|_{\mathrm{Lip}}\|U-V\|_1.$$

PROOF. Let us state the Hoffman–Wiedlandt inequality for unitary matrices (see [8], Theorem 5.2, for a proof): there exists a way of ordering the eigenvalues $\lambda_1(U),\dots,\lambda_n(U)$ of U and the eigenvalues $\lambda_1(V),\dots,\lambda_n(V)$ such that $\sum_{i=1}^N|\lambda_i(U)-\lambda_i(V)|\leq(\pi/2)\|U-V\|_1$.

For all $\zeta_1,\zeta_2\in\mathbb{U}$, we have $|f(\zeta_1)-f(\zeta_2)|\leq\|f\|_{\mathrm{Lip}}d(\zeta_1,\zeta_2)\leq(\pi/2)\times\|f\|_{\mathrm{Lip}}|\zeta_1-\zeta_2|$. Therefore,

$$\begin{aligned}
&\left|\mathrm{tr}(f(U))-\mathrm{tr}(f(V))\right|\\
&\quad=\frac{1}{N}\left|\sum_{i=1}^N f(\lambda_i(U))-f(\lambda_i(V))\right|\leq\frac{1}{N}\sum_{i=1}^N\left|f(\lambda_i(U))-f(\lambda_i(V))\right|\\
&\quad\leq\frac{1}{N}\frac{\pi}{2}\|f\|_{\mathrm{Lip}}\sum_{i=1}^N\left|\lambda_i(U)-\lambda_i(V)\right|\leq\frac{1}{N}\left(\frac{\pi}{2}\right)^2\|f\|_{\mathrm{Lip}}\|U-V\|_1.
\end{aligned}$$

□

Set $K_1=(\pi/2)^2\|f\|_{\mathrm{Lip}}\geq 0$. Because P_{1-t} does not increase the uniform norm, the lemma above implies that $|F_t(\cdot U)-F_t|\leq K_1\|U-I_N\|_1/N$. Since $|e^u-1-u|\leq\frac{u^2}{2}e^{|u|}$ for all $u\in\mathbb{R}$, we get

$$\left|e^{\lambda F_t(\cdot U)-\lambda F_t}-1-\lambda F_t(\cdot U)-\lambda F_t\right|\leq\frac{\lambda^2}{2N^2}K_1^2\|U-I_N\|_1^2e^{\lambda K_1\|U-I_N\|_1/N}.$$

Let us integrate with respect to $\mathrm{d}\upsilon_N(U)$ thanks to Definition 7.3. Remark first that, for all $(\zeta,g)\in\mathbb{U}\times U(N)$, we have

$$\left\|g\begin{pmatrix}\zeta & 0 & \cdots & 0\\ 0 & 1 & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ 0 & \cdots & 0 & 1\end{pmatrix}g^*-I_N\right\|_1=|\zeta-1|\leq 2.$$

Consequently,

$$\begin{aligned}(7.3)\qquad &\int_{U(N)} \left|e^{\lambda F_t(\cdot U)-\lambda F_t} - 1 - \lambda F_t(\cdot U) - \lambda F_t\right| \mathrm{d}\upsilon_N(U) \\ &\quad \leq \frac{\lambda^2}{2N} K_1^2 \int_{\mathbb{U}} |\zeta - 1|^2 \,\mathrm{d}\upsilon(\zeta) e^{2\lambda K_1/N}.\end{aligned}$$

Finally, denoting $K_2 = b\|f\|_{\mathrm{Lip}}/2 \geq 0$ and $K_3 = K_1^2 \int_{\mathbb{U}} |\zeta - 1|^2 \,\mathrm{d}\upsilon(\zeta)/2 \geq 0$, we deduce from (7.1), (7.2) and (7.3) that

$$\phi_\lambda'(t) \leq \frac{\lambda^2}{N}\left(K_2 + K_3 e^{2\lambda K_1/N}\right)\phi_\lambda(t).$$

Integrating this inequality leads to

$$\mathbb{E}\left[\exp\left(\lambda \operatorname{tr} f(U_N)\right)\right] = \phi_\lambda(1) \leq \exp\left(\frac{\lambda^2}{N}\left(K_2 + K_3 e^{2\lambda K_1/N}\right)\right).$$

We deduce that, for all $\varepsilon > 0$,

$$\begin{aligned}\mathbb{P}\left[\operatorname{tr} f(U_N) \geq \varepsilon\right] &\leq \mathbb{E}\left[\exp\left(\lambda \operatorname{tr} f(U_N) - \lambda\varepsilon\right)\right] \\ &\leq \exp\left(-N\left(\frac{\lambda}{N}\varepsilon - \frac{\lambda^2}{N^2}\left(K_2 + K_3 e^{2\lambda K_1/N}\right)\right)\right).\end{aligned}$$

Notice that $\sup_{\lambda\in\mathbb{R}}(\lambda\varepsilon - \lambda^2(K_2 + K_3 e^{2\lambda K_1})) > 0$, so if we define $K > 0$ to be this supremum, we have

$$\mathbb{P}\left[\operatorname{tr} f(U_N) \geq \varepsilon\right] \leq e^{-NK}.$$

To complete the proof, we apply this inequality to $-f$, and the result follows. $\square$

THEOREM 7.8. *Let $\mu \in \mathcal{ID}(\mathbb{U}, \boxtimes)$. For all $N \in \mathbb{N}^*$, let U_N be a random matrix whose law is $\Gamma_N(\mu)$, and whose empirical spectral measure is*

$$\mu_{U_N} = \frac{1}{N} \sum_{\substack{\text{eigenvalue } \lambda \text{ of } U^N \\ \text{(with multiplicity)}}} \delta_\lambda.$$

Then the measures μ_{U_N} converge weakly almost surely to μ when N tends to ∞.

PROOF. If μ is the Haar measure λ of $\mathbb{U}$, then $\Gamma_N(\mu)$ is the Haar measure on $U(N)$ for which the result is well known. Let us assume that $\mu \in \mathcal{ID}(\mathbb{U}, \boxtimes) \cap \mathcal{M}_*$, and let (ω, b, υ) be its $\boxtimes$-characteristic triplet. Due to the concentration result of Theorem 7.6, it remains to establish the convergence of the deterministic measure $\mathbb{E}[\mu_{U_N}]$ to μ. We verify the convergence of moments. As in Section 2, for all $n \in \mathbb{N}$, $m_n(\mu)$ is the nth moment of μ. Let $n \in \mathbb{N}$. We want to prove that $m_n(\mathbb{E}[\mu_{U_N}]) =$

$\mathbb{E}[\mathrm{Tr}((U_N)^n)]/N$ tends to $m_n(\mu)$ as N tends to ∞. We will prove the result under the following stronger form: for all $\sigma \in \mathfrak{S}_n$,

$$\lim_{N\to\infty} \mathbb{E}\Big[N^{-\ell(\sigma)} \prod_{c \text{ cycle of } \sigma} \mathrm{Tr}\big(U_N^{\sharp c}\big)\Big] = \prod_{c \text{ cycle of } \sigma} m_{\sharp c}(\mu).$$

Thanks to Definition 7.3, we know that a characteristic triplet of $\Gamma_N(\mu)$ is given by

$$\left(iy_0 I_N, \begin{pmatrix} \alpha & & & 0 \\ & \ddots & & \\ & & \alpha & \\ 0 & & & \beta \end{pmatrix}, \Pi \right),$$

where $y_0 = -i\,\mathrm{Log}(\omega)$, $\alpha = b/(N+1)$, $\beta = b$ and Π is the Lévy measure obtained from υ as in Definition 7.3.

Fix $\sigma \in \mathfrak{S}_n$. Using the notation of Section 5, we observe that, for all $U \in U(N)$ and $\sigma \in \mathfrak{S}_n$, we have

$$\prod_{c \text{ cycle of } \sigma} \mathrm{Tr}\big(U^{\sharp c}\big) = \mathrm{Tr}_{(\mathbb{C}^N)^{\otimes n}}\big(U^{\otimes n} \circ \rho_N^{\mathfrak{S}_n}(\sigma)\big). \tag{7.4}$$

In order to use Proposition 5.8, we define $\tilde{L}_N \in \mathbb{C}[\mathfrak{S}_n]$ by

$$\begin{aligned}\tilde{L}_N = {} & \left(niy_0 - \frac{n^2}{N}\frac{\beta}{2} + \left(\frac{n^2}{N} - nN\right)\frac{\alpha}{2} + \frac{n}{N}\int_{U(N)} \mathrm{Tr}\big(\Re(g) - 1\big)\Pi(\mathrm{d}g)\right)1_{\mathfrak{S}_n} \\ & - \alpha \sum_{\tau \in \mathcal{T}_n} \tau + \sum_{\substack{2 \le m \le n \\ 1 \le k_1 < \cdots < k_m \le n}} \sum_{\pi', \pi \in \mathfrak{S}_m} \mathrm{Wg}(\pi'^{-1}\pi) \\ & \times \int_{U(N)} \prod_{c \text{ cycle of } \sigma} \mathrm{Tr}\big((g-1)^{\sharp c}\big)\Pi(\mathrm{d}g) \cdot \iota_{k_1,\dots,k_m}(\pi) \\ = {} & \left(n\,\mathrm{Log}(\omega) - \frac{n^2}{N}b + \left(\frac{n^2}{N} - nN\right)\frac{b}{2(N+1)} + n\int_{\mathbb{U}} \big(\Re(\zeta) - 1\big)\upsilon(\mathrm{d}\zeta)\right)1_{\mathfrak{S}_n} \\ & - \frac{b}{N+1}\sum_{\tau \in \mathcal{T}_n} \tau \\ & + \sum_{\substack{2 \le m \le n \\ 1 \le k_1 < \cdots < k_m \le n}} \sum_{\pi', \pi \in \mathfrak{S}_m} \mathrm{Wg}(\pi'^{-1}\pi) N \int_{\mathbb{U}} (\zeta - 1)^m \upsilon(\mathrm{d}\zeta) \cdot \iota_{k_1,\dots,k_m}(\pi).\end{aligned}$$

Using Proposition 5.8, we have

$$\begin{aligned}\mathbb{E}\Big[N^{-\ell(\sigma)} \prod_{c \text{ cycle of } \sigma} \mathrm{Tr}\big(U_N^{\sharp c}\big)\Big] &= N^{-\ell(\sigma)}\,\mathrm{Tr}_{(\mathbb{C}^N)^{\otimes n}}\big(\mathbb{E}[U_N^{\otimes n}] \circ \rho_N^{\mathfrak{S}_n}(\sigma)\big) \\ &= N^{-\ell(\sigma)}\,\mathrm{Tr}_{(\mathbb{C}^N)^{\otimes n}}\big(\rho_N^{\mathfrak{S}_n}(e^{\tilde{L}_N}\sigma)\big).\end{aligned}$$

From (7.4), we deduce also that, for all $\sigma \in \mathfrak{S}_n$, we have

$$\mathrm{Tr}_{(\mathbb{C}^N)^{\otimes n}}(\rho_N^{\mathfrak{S}_n}(\sigma)) = N^{\ell(\sigma)}.$$

We denote by N^ℓ (resp., $N^{-\ell}$) the linear operator on $\mathbb{C}[\mathfrak{S}_n]$ defined by $N^\ell(\sigma) = N^{\ell(\sigma)}\sigma$ [resp., $N^{-\ell}(\sigma) = N^{-\ell(\sigma)}\sigma$] and by ϕ the linear functional defined by $\phi(\sigma) = 1$. This way, we have $\mathrm{Tr}_{(\mathbb{C}^N)^{\otimes n}} \circ \rho_N^{\mathfrak{S}_n} = \phi \circ N^\ell$. Let us also denote by T_N the linear operator on $\mathbb{C}[\mathfrak{S}_n]$ of multiplication by $\tilde{L}_N$, defined by $T_N(\sigma) = \tilde{L}_N\sigma$. We can rewrite

$$\begin{aligned}\mathbb{E}\Big[N^{-\ell(\sigma)} \prod_{c \text{ cycle of } \sigma} \mathrm{Tr}(U_N^{\sharp c})\Big] &= \mathrm{Tr}_{(\mathbb{C}^N)^{\otimes n}}(\rho_N^{\mathfrak{S}_n}(e^{\tilde{L}_N} N^{-\ell(\sigma)}\sigma)) \\ &= \phi(N^\ell e^{T_N} N^{-\ell}(\sigma)) \\ &= \phi(e^{N^\ell T_N N^{-\ell}}(\sigma)).\end{aligned}$$

We take the limit with the help of the following lemma. Recall that $(L\kappa_n(\mu))_{n\in\mathbb{N}^*}$ are the free log-cumulants of μ (see Section 4), which are given by:

1. $L\kappa_1(\mu) = \mathrm{Log}(\omega) - b/2 + \int_{\mathbb{U}}(\Re(\zeta) - 1)\, \mathrm{d}\upsilon(\zeta)$,
2. $L\kappa_2(\mu) = -b + \int_{\mathbb{U}}(\zeta - 1)^2\, \mathrm{d}\upsilon(\zeta)$
3. and $L\kappa_n(\mu) = \int_{\mathbb{U}}(\zeta - 1)^n\, \mathrm{d}\upsilon(\zeta)$ for all $n \geq 2$.

LEMMA 7.9. *When N tends to ∞, the operator $N^\ell T_N N^{-\ell}$ converges to an operator T which is such that, for all $\sigma \in \mathfrak{S}_n$,*

$$T(\sigma) = nL\kappa_1(\mu)\cdot\sigma + \sum_{\substack{2\le m\le n \\ c\ m\text{-cycle of } \mathfrak{S}_n \\ c\sigma \preceq \sigma}} L\kappa_m(\mu)\cdot c\sigma.$$

PROOF. We shall prove that, for a fixed $\sigma \in \mathfrak{S}_n$, $\lim_{N\to\infty} N^\ell T_N N^{-\ell}(\sigma) = T(\sigma)$. Let us compute

$$N^\ell T_N N^{-\ell}(\sigma) = N^{\ell(\sigma)} N^\ell(\tilde{L}\sigma).$$

Replacing $\tilde{L}$ by its value gives us $N^{\ell(\sigma)} N^\ell(\tilde{L}\sigma) = (I + II + III)\sigma$, with

$$I = \left(n\,\mathrm{Log}(\omega) - \frac{n^2}{N}b + \left(\frac{n^2}{N} - nN\right)\frac{b}{2(N+1)} + n\int_{\mathbb{U}}(\Re(\zeta) - 1)\upsilon(\mathrm{d}\zeta)\right)1_{\mathfrak{S}_n},$$

$$II = -\frac{b}{N+1}\sum_{\tau\in\mathcal{T}_n} N^{\ell(\tau\sigma)-\ell(\sigma)}\tau,$$

and

$$\begin{aligned}III = &\sum_{\substack{2\le m\le n \\ 1\le k_1<\cdots<k_m\le n}} \sum_{\pi\in\mathfrak{S}_m} \int_{\mathbb{U}}(\zeta - 1)^m\ \upsilon(\mathrm{d}\zeta)\cdot\left(\sum_{\pi'\in\mathfrak{S}_m} \mathrm{Wg}(\pi'^{-1}\pi)\right) \\ &\times N^{1+\ell(\iota_{k_1,\dots,k_m}(\pi)\sigma)-\ell(\sigma)}\cdot \iota_{k_1,\dots,k_m}(\pi).\end{aligned}$$

The first limit is immediate:

$$\lim_{N\to\infty} I = \left(n\operatorname{Log}(\omega) - \frac{n}{2}b + n\int_{\mathbb{U}}(\Re(\zeta)-1)\upsilon(\mathrm{d}\zeta)\right)1_{\mathfrak{S}_n} = nL\kappa_1(\mu)1_{\mathfrak{S}_n}.$$

For the second and the third term, we recall that for all $\pi \in \mathfrak{S}_n$, we have

$$d(1,\sigma) \le d(1,\pi\sigma) + d(\pi\sigma,\sigma)$$

with equality if and only if $\pi\sigma \preceq \sigma$ (see Section 4.1).

Let us focus on II. We fix $\tau \in \mathcal{T}_n$. We know that $d(1,\sigma) \le d(1,\tau\sigma) + d(\tau\sigma,\sigma)$. In term of numbers of cycles, it means that $n - \ell(\sigma) \le n - \ell(\tau\sigma) + n - \ell(\tau)$. Because $\ell(\tau) = n-1$, we have $\ell(\tau\sigma) - \ell(\sigma) \le 1$ with equality if and only if $\tau\sigma \preceq \sigma$. By consequence,

$$\lim_{N\to\infty} II = -b \sum_{\substack{\tau\in\mathcal{T}_n\\ \tau\sigma\preceq\sigma}} \tau\sigma.$$

A similar reasoning can be made for III. Let us fix $2 \le m \le n$, $1 \le k_1 < \cdots < k_m \le n$ and $\pi \in \mathfrak{S}_m$. We denote by c the permutation $\iota_{k_1,\dots,k_m}(\pi)$. On one hand, Proposition 5.6 gives us $\mathrm{Wg}(\pi'^{-1}\pi) = O(N^{-m-1})$ if $\pi \ne \pi'$ and $\mathrm{Wg}(\pi'^{-1}\pi) = N^{-n} + O(N^{-n-1})$ if $\pi = \pi'$, and by consequence,

$$\sum_{\pi'\in\mathfrak{S}_m} \mathrm{Wg}(\pi'^{-1}\pi) = N^{-m} + O(N^{-m-1}).$$

On the other hand, we know that $d(1,\sigma) \le d(1,c\sigma) + d(c\sigma,\sigma)$. In terms of numbers of cycles, it means that $n - \ell(\sigma) \le n - \ell(c\sigma) + n - \ell(c)$. Because $\ell(c) = \ell(\iota_{k_1,\dots,k_m}(\pi)) = n - m + \ell(\pi)$, we have $1 + \ell(c\sigma) - \ell(\sigma) \le 1 + m - \ell(\pi)$. Thus, we have

$$1 + \ell(c\sigma) - \ell(\sigma) \le m$$

with equality if and only if we have both $c\sigma \preceq \sigma$ and $\ell(\pi) = 1$. Consequently, the term

$$\sum_{\pi'\in\mathfrak{S}_m} \mathrm{Wg}(\pi'^{-1}\pi) N^{1+\ell(\iota_{k_1,\dots,k_m}(\pi)\sigma)-\ell(\sigma)}$$

is equal to $1 + O(N^{-1})$ if we have both $c\sigma \preceq \sigma$ and $\ell(\pi) = 1$, but it is $O(N^{-1})$ if not. Finally,

$$\begin{aligned}\lim_{N\to\infty} III &= \sum_{\substack{2\le m\le n\\ 1\le k_1<\cdots<k_m\le n}} \sum_{\substack{\pi\ m\text{-cycle of } \mathfrak{S}_m\\ \iota_{k_1,\dots,k_m}(\pi)\sigma\preceq\sigma}} \int_{\mathbb{U}} (\zeta-1)^m \upsilon(\mathrm{d}\zeta)\cdot \iota_{k_1,\dots,k_m}(\pi)\sigma\\ &= \sum_{2\le m\le n} \sum_{\substack{c\ m\text{-cycle of } \mathfrak{S}_n\\ c\sigma\preceq\sigma}} \int_{\mathbb{U}} (\zeta-1)^m \upsilon(\mathrm{d}\zeta)\cdot c\sigma.\end{aligned}$$

Thus, we have

$$\lim_{N\to\infty} I + II + III = nL\kappa_1(\mu)\cdot\sigma + \sum_{\substack{2\le m\le n \\ c\ m\text{-cycle of } \mathfrak{S}_n \\ c\sigma\preceq\sigma}} L\kappa_m(\mu)\cdot c\sigma = T(\sigma).$$

□

As a consequence, we have

$$\begin{aligned}
&\lim_{N\to\infty}\mathbb{E}\Big[N^{-\ell(\sigma)}\prod_{c \text{ cycle of } \sigma}\mathrm{Tr}\big(U_N^{\sharp c}\big)\Big] \\
&\quad= \phi\big(e^T(\sigma)\big) = \phi\big(e^{nL\kappa_1(\mu)}e^{T-nL\kappa_1(\mu)}(\sigma)\big) \\
&\quad= \phi\Bigg(e^{nL\kappa_1(\mu)}\sum_{\substack{\Gamma \text{ simple chain in } [1,\sigma] \\ \Gamma=(\sigma_0,\dots,\sigma_{|\Gamma|}),\sigma_{|\Gamma|}=\sigma}}\frac{1}{|\Gamma|!}\prod_{i=1}^{|\Gamma|}L\kappa_{d(\sigma_i,\sigma_{i-1})+1}(\mu)\cdot\sigma_0\Bigg) \\
&\quad= e^{nL\kappa_1(\mu)}\sum_{\substack{\Gamma \text{ simple chain in } [1,\sigma] \\ \Gamma=(\sigma_0,\dots,\sigma_{|\Gamma|}),\sigma_{|\Gamma|}=\sigma}}\frac{1}{|\Gamma|!}\prod_{i=1}^{|\Gamma|}L\kappa_{d(\sigma_i,\sigma_{i-1})+1}(\mu).
\end{aligned}$$

Using (4.1) on the right-hand side, we conclude that

$$\lim_{N\to\infty}\mathbb{E}\Big[N^{-\ell(\sigma)}\prod_{c \text{ cycle of } \sigma}\mathrm{Tr}\big(U_N^{\sharp c}\big)\Big] = \prod_{c \text{ cycle of } \sigma} m_{\sharp c}(\mu).$$

□

REMARK 7.10. In fact, the proof can be easily extended to a more general situation. Let $\mu \in \mathcal{ID}(\mathbb{U}, \boxtimes)$ and let (ω, b, υ) be its $\boxtimes$-characteristic triplet. For all $N \in \mathbb{N}^*$, let $y_0^{(N)}, \alpha^{(N)}, \beta^{(N)} \in \mathbb{R}$ and $\Pi^{(N)}$ be a Lévy measure on $U(N)$ which is conjugate invariant. We suppose that:

1. $\lim_{N\to\infty} e^{iy_0^{(N)}} = \omega$, $\alpha^{(N)} \sim_{N\to\infty} b/N$ and $\beta^{(N)} = O(1)$ as N tends to ∞.
2. For all $k \ge 2$,

$$\lim_{N\to\infty}\frac{1}{N}\int_{U(N)}\mathrm{Tr}\big((g - I_N)^k\big)\Pi^{(N)}(\mathrm{d}g) = \int_{\mathbb{U}}(\zeta - 1)^k\upsilon(\mathrm{d}\zeta).$$

3. For all $k_1, \dots, k_n \in \mathbb{N}^*$ such that $k_1 + \cdots + k_n \ge 2$, we have, as N tends to infinity,

$$\frac{1}{N}\int_{U(N)}\mathrm{Tr}\big((g - I_N)^{k_1}\big)\cdots\mathrm{Tr}\big((g - I_N)^{k_n}\big)\Pi^{(N)}(\mathrm{d}g) = O(1).$$

Then the conclusion of Theorem 7.8 is still true whenever U_N is a random matrix whose law is an infinitely divisible measure which admits

$$\left(iy_0^{(N)} I_N, \begin{pmatrix} \alpha^{(N)} & & & 0 \\ & \ddots & & \\ & & \alpha^{(N)} & \\ 0 & & & \beta^{(N)} \end{pmatrix}, \Pi^{(N)} \right)$$

as a characteristic triplet.

7.4. *Proof of Theorem* 3. We refer the reader to [36] for the main definitions of free probability spaces. We call *free unitary multiplicative Lévy process* a family $(U_t)_{t\in\mathbb{R}_+}$ of unitary elements of a noncommutative probability space $(\mathcal{A},\tau)$ such that:

1. $U_0 = 1_{\mathcal{A}}$.
2. For all $0 \le s \le t$, the distribution of $U_t U_s^{-1}$ depends only on $t-s$.
3. For all $0 \le t_1 < \cdots < t_n$, the elements $U_{t_1}, U_{t_2}U_{t_1}^{-1}, \dots U_{t_n}U_{t_{n-1}}^{-1}$ are freely independent.
4. The distribution of U_t converge weakly to δ_1 as t tends to 0.

Notice that this definition differs from the definition in [11] by the first and the fourth items.

Let $(U_t)_{t\in\mathbb{R}_+}$ be a free unitary multiplicative Lévy process with marginal distributions $(\mu_t)_{t\in\mathbb{R}_+}$ in $\mathcal{M}_*$. Then, $(\mu_t)_{t\in\mathbb{R}_+}$ is a weakly continuous semigroup of measures for the convolution $\boxtimes$ on $\mathbb{U}$. Moreover, there exists $\alpha\in\mathbb{R}$ and $b\ge 0$ and υ a Lévy measure on $\mathbb{U}$ such that, for all $t\ge 0$, $(e^{i\alpha t}, tb, t\upsilon)$ is a $\boxtimes$-characteristic triplet of U_t (see [6]). Using Lemma 6.5, it is straightforward to verify that the weakly continuous semigroup whose characteristic triplet is $(i\alpha I_N, b_N, \upsilon_N)$ coincides with $(\Gamma_N(\mu_t))_{t\in\mathbb{R}_+}$. Therefore, there exists a Lévy process $(U_t^{(N)})_{t\in\mathbb{R}_+}$ in $U(N)$ such that $\Gamma_N(\mu_t)$ is the distribution of $U_t^{(N)}$ for each $t\in\mathbb{R}_+$ (see [29]). We already know that, for each fixed $t\in\mathbb{R}_+$, the element $U_t^{(N)}$ converges almost surely to U_t in noncommutative $*$-distribution, in the sense that, for each noncommutative polynomial P in two variables, one has the almost sure convergence

$$\lim_{N\mapsto\infty} \frac{1}{N} \operatorname{Tr}\big(P\big(U_t^{(N)}, U_t^{(N)*}\big)\big) = \tau\big(P\big(U_t, U_t^*\big)\big).$$

Since the increments of $(U_t)_{t\in\mathbb{R}_+}$ are freely independent, to prove the convergence of the whole process, it suffices to prove that the increments of $(U_t^{(N)})_{t\in\mathbb{R}_+}$ are asymptotically free. This is a well-known consequence of the fact that the increments of $(U_t^{(N)})_{t\in\mathbb{R}_+}$ are independent and invariant under conjugation by unitary matrices (see, e.g., [15, 35–37], or the Appendix of [27] for a concise treatment).

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