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THE DETERMINANT OF THE ITERATED MALLIAVIN MATRIX AND THE DENSITY OF A PAIR OF MULTIPLE INTEGRALS

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Dedicated to the memory of Professor Donald Burkholder

The aim of this paper is to show an estimate for the determinant of the covariance of a two-dimensional vector of multiple stochastic integrals of the same order in terms of a linear combination of the expectation of the determinant of its iterated Malliavin matrices. As an application, we show that the vector is not absolutely continuous if and only if its components are proportional.

1. Introduction. A basic result in Malliavin calculus says that if the Malliavin matrix $\Lambda = (\langle DF_i, DF_j\rangle_H)_{1\le i,j\le d}$ of a d-dimensional random vector $\mathbf{F} = (F_1, \dots, F_d)$ is nonsingular almost surely, then this vector has an absolutely continuous law with respect to the Lebesgue measure on $\mathbb{R}^d$. In the special case of vectors whose components belong to a finite sum of Wiener chaos, Nourdin, Nualart and Poly proved in [1] that the following conditions are equivalent:

(a) The law of $\mathbf{F}$ is not absolutely continuous.
(b) $E \det \Lambda = 0$.

A natural question is the relation between $E\det\Lambda$ and the determinant of the covariance matrix C of the random vector $\mathbf{F}$. Clearly, if $\det C = 0$, then the components of $\mathbf{F}$ are linearly dependent and the law of $\mathbf{F}$ is not absolutely continuous, which implies $E\det\Lambda = 0$. The converse is not true if $d \ge 3$. For instance, the vector $(X_1^2 - 1, X_2^2 - 1, X_1X_2)$, where X_1 and X_2 are two independent random variables in the first chaos, with unit variance, satisfies $\det\Lambda = 0$ but $\det C \ne 0$. Note that the components of this vector are multiple integrals in the second Wiener chaos.

The purpose of this paper is to show the equivalence between $E\det\Lambda = 0$ and $\det C = 0$ in the particular case of a two-dimensional random vector (F, G) whose

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components are multiple stochastic integrals of the same order n. This implies that the random vector (F, G) has an absolutely continuous law with respect to the Lebesgue measure on $\mathbb{R}^2$ if and only if its components are proportional, as in the Gaussian case. This result was established for $n = 2$ in [1], and for $n = 3, 4$ in [5]. Our proof in the general case $n \geq 2$ is based on the notion of iterated Malliavin matrix and the computation of the expectation of its determinant.

In connection with this equivalence, we will derive an inequality relating $E \det \Lambda$ and $\det C$, which has its own interest. In the case of double stochastic integrals, that is, if $n = 2$, it was proved in [1] that

$$E \det \Lambda \geq 4 \det C.$$

We extend this inequality proving that

$$E \det \Lambda \geq c_n \det C$$

holds for $n = 3, 4$ with $c_3 = \frac{9}{4}$ and $c_4 = \frac{16}{9}$. For $n \geq 5$, we obtain a more involved inequality, where in the left-hand side we have a linear combination (with positive coefficients) of the expectation of the iterated Malliavin matrices of (F, G) of order k for $1 \leq k \leq [\frac{n-1}{2}]$ (see Theorem 2 below).

The paper is organized as follows. In Section 2, we present some preliminary results and notation. Section 3 contains a general decomposition of the determinant of the iterated Malliavin matrix of a two-dimensional random vector into a sum of squares. In Section 4, we prove our main result which is based on a further decomposition of the determinant of the iterated Malliavin matrix of a vector whose components are multiple stochastic integrals. Finally, the application to the characterization of absolute continuity is obtained in Section 5.

2. Preliminaries. We briefly describe the tools from the analysis on Wiener space that we will need in our work. For complete presentations, we refer to [4] or [2]. Let H be a real and separable Hilbert space and consider an *isonormal Gaussian process* $(W(h), h \in H)$. That is, $(W(h), h \in H)$ is a Gaussian family of centered random variables on a probability space $(\Omega, \mathcal{F}, P)$ such that $EW(h)W(g) = \langle f, g\rangle_H$ for every $h, g \in H$. Assume that the σ-algebra $\mathcal{F}$ is generated by W.

For any integer $n \geq 1$ we denote by $\mathcal{H}_n$ the nth Wiener chaos generated by W. That is, $\mathcal{H}_n$ is the vector subspace of $L^2(\Omega)$ generated by the random variables $(H_n(W(h)), h \in H, \|h\|_H = 1)$ where H_n is the Hermite polynomial of degree n. We denote by H_0 the space of constant random variables. Let $H^{\otimes n}$ and $H^{\odot n}$ denote, respectively, the nth tensor product and the nth symmetric tensor product of H. For any $n \geq 1$, the mapping $I_n(h^{\otimes n}) = H_n(W(h))$ can be extended to an isometry between the symmetric tensor product $H^{\odot n}$ endowed with the norm $\sqrt{n!}\|\cdot\|_{H^{\otimes n}}$ and the nth Wiener chaos $\mathcal{H}_n$. For any $f \in H^{\odot n}$, the random variable $I_n(f)$ is called the multiple Wiener–Itô integral of f with respect to W.

Consider $(e_j)_{j\geq 1}$ a complete orthonormal system in H and let $f \in H^{\odot n}$, $g \in H^{\odot m}$ be two symmetric tensors with $n, m \geq 1$. Then

$$f = \sum_{j_1,\dots,j_n \geq 1} f_{j_1,\dots,j_n} e_{j_1} \otimes \cdots \otimes e_{j_n} \tag{2.1}$$

and

$$g = \sum_{k_1,\dots,k_m \geq 1} g_{k_1,\dots,k_m} e_{k_1} \otimes \cdots \otimes e_{k_m}, \tag{2.2}$$

where the coefficients are given by $f_{j_1,\dots,j_n} = \langle f, e_{j_1} \otimes \cdots \otimes e_{j_n} \rangle$ and $g_{k_1,\dots,k_m} = \langle g, e_{k_1} \otimes \cdots \otimes e_{k_m} \rangle$. These coefficients are symmetric, that is, they satisfy $f_{j_{\sigma(1)},\dots,j_{\sigma(n)}} = f_{j_1,\dots,j_n}$ and $g_{k_{\pi(1)},\dots,k_{\pi(m)}} = g_{k_1,\dots,k_m}$ for every permutation σ of the set $\{1,\dots,n\}$ and for every permutation π of the set $\{1,\dots,m\}$.

Note that, throughout the paper, we will usually omit the subindex $H^{\otimes k}$ in the notation for the norm and the scalar product in $H^{\otimes k}$, for any $k \geq 1$.

If $f \in H^{\odot n}$, $g \in H^{\odot m}$ are symmetric tensors given by (2.1) and (2.2), respectively, then the contraction of order r of f and g is given by

$$\begin{aligned} f \otimes_r g = \sum_{i_1,\dots,i_r \geq 1} \sum_{j_1,\dots,j_{n-r} \geq 1} \sum_{k_1,\dots,k_{m-r} \geq 1} & f_{i_1,\dots,i_r,j_1,\dots,j_{n-r}} g_{i_1,\dots,i_r,k_1,\dots,k_{m-r}} \\ & \times (e_{j_1} \otimes \cdots \otimes e_{j_{n-r}}) \\ & \otimes (e_{k_1} \otimes \cdots \otimes e_{k_{m-r}}) \end{aligned} \tag{2.3}$$

for every $r = 0,\dots,m \wedge n$. In particular, $f \otimes_0 g = f \otimes g$. Note that $f \otimes_r g$ belongs to $H^{\otimes(m+n-2r)}$ for every $r = 0,\dots,m \wedge n$ and it is not in general symmetric. We will denote by $f \tilde{\otimes}_r g$ the symmetrization of $f \otimes_r g$. In the particular case when $H = L^2(T, \mathcal{B}, \mu)$ where μ is a sigma-finite measure without atoms, (2.3) becomes

$$\begin{aligned} (f \otimes_r g)&(t_1,\dots,t_{m+n-2r}) \\ &= \int_{T^r} d\mu(u_1) \cdots d\mu(u_r) f(u_1,\dots,u_r,t_1,\dots,t_{n-r}) \\ &\qquad \times g(u_1,\dots,u_r,t_{n-r+1},\dots,t_{m+n-2r}). \end{aligned} \tag{2.4}$$

An important role will be played by the following product formula for multiple Wiener–Itô integrals: if $f \in H^{\odot n}$, $g \in H^{\odot m}$ are symmetric tensors, then

$$I_n(f) I_m(g) = \sum_{r=0}^{m \wedge n} r! C_m^r C_n^r I_{m+n-2r}(f \tilde{\otimes}_r g). \tag{2.5}$$

We will need some elements of the Malliavin calculus with respect to the isonormal Gaussian process W. Let $\mathcal{S}$ be the set of all smooth and cylindrical random variables of the form

$$F = \varphi\big(W(h_1), \dots, W(h_n)\big), \tag{2.6}$$

where $n \geq 1$, $\varphi : \mathbb{R}^n \to \mathbb{R}$ is an infinitely differentiable function with compact support, and $h_i \in H$ for $i = 1, \dots, n$. If F is given by (2.6), the *Malliavin derivative* of F with respect to W is the element of $L^2(\Omega; H)$ defined as

$$DF = \sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i}\big(W(h_1), \dots, W(h_n)\big) h_i.$$

By iteration, for every $k \geq 2$, one can define the kth derivative $D^{(k)}F$ which is an element of $L^2(\Omega; H^{\odot k})$. For $k \geq 1$, $\mathbb{D}^{k,2}$ denotes the closure of $\mathcal{S}$ with respect to the norm $\|\cdot\|_{\mathbb{D}^{k,2}}$, defined by the relation

$$\|F\|_{\mathbb{D}^{k,2}}^2 = E|F|^2 + \sum_{i=1}^{k} E\big\|D^{(i)}F\big\|_{H^{\otimes i}}^2.$$

If $F = I_n(f)$, where $f \in H^{\odot n}$ and $I_n(f)$ denotes the multiple integral of order n with respect to W, then

$$DI_n(f) = n \sum_{j=1}^{\infty} I_{n-1}(f \otimes_1 e_j) e_j.$$

More generally, for any $1 \leq k \leq n$, the iterated Malliavin derivative of $I_n(f)$ is given by

$$D^{(k)} I_n(f) = \frac{n!}{(n-k)!} \sum_{j_1, \dots, j_k \geq 1} I_{n-k}(f_{j_1, \dots, j_k}) e_{j_1} \otimes \cdots \otimes e_{j_k},$$

where

$$f_{j_1, \dots, j_k} = f \otimes_k (e_{j_1} \otimes \cdots \otimes e_{j_k}). \tag{2.7}$$

We denote by δ the adjoint of the operator D, also called the *divergence operator* or *Skorohod integral*. A random element $u \in L^2(\Omega; H)$ belongs to the domain of δ, denoted $\operatorname{Dom} \delta$, if and only if it verifies

$$\big|E\langle DF, u\rangle_H\big| \leq c_u \sqrt{E\big(F^2\big)}$$

for any $F \in \mathbb{D}^{1,2}$, where c_u is a constant depending only on u. If $u \in \operatorname{Dom} \delta$, then the random variable $\delta(u)$ is defined by the duality relationship

$$E\big(F\delta(u)\big) = E\langle DF, u\rangle_H,$$

which holds for every $F \in \mathbb{D}^{1,2}$. If $F = I_n(f)$ is a multiple stochastic integral of order n, with $f \in H^{\odot n}$, then DF belongs to the domain of δ and

$$\delta DF = nF. \tag{2.8}$$

3. Decomposition of the determinant of the iterated Malliavin matrix. In this section, we obtain a decomposition into a sum of squares for the determinant of the iterated Malliavin matrix of a 2-dimensional random vector. We recall that if F, G are two random variables in the space $\mathbb{D}^{1,2}$, *the Malliavin matrix* of the random vector (F, G) is defined as the following 2×2 random matrix:

$$\Lambda = \begin{pmatrix} \|DF\|_H^2 & \langle DF, DG\rangle_H \\ \langle DF, DG\rangle_H & \|DF\|_H^2 \end{pmatrix}.$$

More generally, fix $k \geq 2$ and suppose that F, G are two random variables in $\mathbb{D}^{k,2}$. The *kth iterated Malliavin matrix of the vector* (F, G) is defined as

$$\Lambda^{(k)} = \begin{pmatrix} \|D^{(k)}F\|_{H^{\otimes k}}^2 & \langle D^{(k)}F, D^{(k)}G\rangle_{H^{\otimes k}} \\ \langle D^{(k)}F, D^{(k)}G\rangle_{H^{\otimes k}} & \|D^{(k)}G\|_{H^{\otimes k}}^2 \end{pmatrix}.$$

We set $\Lambda^{(1)} = \Lambda$. For every $j_1, \dots, j_k \geq 1$, we will write

$$D^{(k)}_{j_1,\dots,j_k}F = \langle D^{(k)}F, e_{j_1} \otimes \cdots \otimes e_{j_k}\rangle_{H^{\otimes k}}.$$

The next proposition provides an expression of the determinant of the iterated Malliavin matrix of a random vector as a sum of squared random variables.

PROPOSITION 1. *Suppose that (F, G) is a 2-dimensional random vector whose components belong to $\mathbb{D}^{k,2}$ for some $k \geq 1$. Let $\Lambda^{(k)}$ be the kth iterated Malliavin matrix of (F, G). Then*

$$\det \Lambda^{(k)} = \frac{1}{2} \sum_{i_1,\dots,i_k,l_1,\dots,l_k \geq 1} \left(D^{(k)}_{i_1,\dots,i_k}F D^{(k)}_{l_1,\dots,l_k}G - D^{(k)}_{l_1,\dots,l_k}F D^{(k)}_{i_1,\dots,i_k}G\right)^2. \tag{3.1}$$

PROOF. For every $k \geq 1$, we have

$$\|D^{(k)}F\|_{H^{\otimes k}}^2 = \sum_{i_1,\dots,i_k \geq 1} \left(D^{(k)}_{i_1,\dots,i_k}F\right)^2,$$

$$\|D^{(k)}G\|_{H^{\otimes k}}^2 = \sum_{i_1,\dots,i_k \geq 1} \left(D^{(k)}_{i_1,\dots,i_k}G\right)^2$$

and

$$\langle D^{(k)}F, D^{(k)}G\rangle_{H^{\otimes k}} = \sum_{i_1,\dots,i_k \geq 1} D^{(k)}_{i_1,\dots,i_k}F D^{(k)}_{i_1,\dots,i_k}G.$$

Thus,

$$\begin{aligned}\det\Lambda^{(k)} &= \sum_{i_1,\dots,i_k\geq 1}\big(D^{(k)}_{i_1,\dots,i_k}F\big)^2 \sum_{i_1,\dots,i_k\geq 1}\big(D^{(k)}_{i_1,\dots,i_k}G\big)^2 \\ &\quad -\Big(\sum_{i_1,\dots,i_k\geq 1} D^{(k)}_{i_1,\dots,i_k}F D^{(k)}_{i_1,\dots,i_k}G\Big)^2 \\ &= \frac{1}{2}\sum_{i_1,\dots,i_k,l_1,\dots,l_k\geq 1}\big(D^{(k)}_{i_1,\dots,i_k}F D^{(k)}_{l_1,\dots,l_k}G - D^{(k)}_{l_1,\dots,l_k}F D^{(k)}_{i_1,\dots,i_k}G\big)^2. \qquad \square\end{aligned}$$

4. The iterated Malliavin matrix of a two-dimensional vector of multiple integrals. Throughout this section, we assume that the components of the random vector (F,G) are multiple Wiener–Itô integrals. More precisely, we fix $n,m\geq 1$ and we consider the vector

$$(F,G) = \big(I_n(f), I_m(g)\big),$$

where $f\in H^{\odot n}$ and $g\in H^{\odot m}$. Since for every $1\leq k\leq \min(n,m)$,

$$D^{(k)}_{i_1,\dots,i_k}F = \frac{n!}{(n-k)!}I_{n-k}(f_{i_1,\dots,i_k})$$

[with $f_{i_1,\dots,i_k}$ defined by (2.7)] and

$$D^{(k)}_{i_1,\dots,i_k}G = \frac{m!}{(m-k)!}I_{m-k}(g_{i_1,\dots,i_k})$$

formula (3.1) reduces to

$$\begin{aligned}\det\Lambda^{(k)} &= \frac{1}{2}\bigg(\frac{n!}{(n-k)!}\frac{m!}{(m-k)!}\bigg)^2 \\ &\quad \times \sum_{i_1,\dots,i_k,l_1,\dots,l_k\geq 1}\big[I_{n-k}(f_{i_1,\dots,i_k})I_{m-k}(g_{l_1,\dots,l_k}) \\ &\qquad\qquad - I_{n-k}(f_{l_1,\dots,l_k})I_{m-k}(g_{i_1,\dots,i_k})\big]^2.\end{aligned}$$

By the product formula for multiple integrals (2.5), we can write

$$\begin{aligned}\det\Lambda^{(k)} &= \frac{1}{2}\bigg(\frac{n!}{(n-k)!}\frac{m!}{(m-k)!}\bigg)^2 \\ &\quad \times \sum_{i_1,\dots,i_k,l_1,\dots,l_k\geq 1}\Bigg(\sum_{r=0}^{(n-k)\wedge(m-k)} r!C^r_{n-k}C^r_{m-k}I_{m+n-2k-2r} \\ &\qquad\qquad \times [f_{i_1,\dots,i_k}\otimes_r g_{l_1,\dots,l_k} \\ &\qquad\qquad - f_{l_1,\dots,l_k}\otimes_r g_{i_1,\dots,i_k}]\Bigg)^2.\end{aligned}$$

Taking the mathematical expectation, the isometry property of multiple stochastic integrals implies that

$$
\begin{aligned}
E \det \Lambda^{(k)} &= \frac{1}{2}\left(\frac{n!}{(n-k)!}\frac{m!}{(m-k)!}\right)^2 \\
&\qquad \times \sum_{i_1,\dots,i_k,l_1,\dots,l_k \ge 1} \sum_{r=0}^{(n-k)\wedge(m-k)} \left(r! C_{n-k}^r C_{m-k}^r\right)^2 (m+n-2k-2r)! \\
&\qquad\qquad \times \| f_{i_1,\dots,i_k} \tilde{\otimes}_r g_{l_1,\dots,l_k} \\
&\qquad\qquad\qquad - f_{l_1,\dots,l_k} \tilde{\otimes}_r g_{i_1,\dots,i_k} \|^2 \qquad (4.1) \\
&:= \sum_{r=0}^{(n-k)\wedge(m-k)} T_r^{(k)},
\end{aligned}
$$

where

$$
T_r^{(k)} = \frac{1}{2}\alpha_{k,r} \sum_{i_1,\dots,i_k,l_1,\dots,l_k \ge 1} \| f_{i_1,\dots,i_k} \tilde{\otimes}_r g_{l_1,\dots,l_k} - f_{l_1,\dots,l_k} \tilde{\otimes}_r g_{i_1,\dots,i_k} \|^2, \tag{4.2}
$$

with

$$
\alpha_{k,r} = \left(\frac{n!m!}{(n-k-r)!(m-k-r)!r!}\right)^2 (m+n-2k-2r)!.
$$

We will explicitly compute the terms $T_r^{(k)}$ in (4.2). To do this, we will need several auxiliary lemmas. The first one is an immediate consequence of the definition of contraction.

LEMMA 1. *Let $f \in H^{\odot n}$, $g \in H^{\odot m}$. Then for every $k, r \ge 0$ such that $k + r \le m \wedge n$,*

$$
\sum_{i_1,\dots,i_k \ge 1} f_{i_1,\dots,i_k} \otimes_r g_{i_1,\dots,i_k} = f \otimes_{r+k} g.
$$

The next lemma summarizes the results in Lemmas 3 and 4 in [5] (see also Lemma 2.2 in [3]).

LEMMA 2. *Assume $f, h \in H^{\odot n}$ and $g, \ell \in H^{\odot m}$.*

(i) *For every $r = 0, \dots, (m-1) \wedge (n-1)$ we have*

$$
\langle f \otimes_{n-r} h, g \otimes_{m-r} \ell \rangle = \langle f \otimes_r g, h \otimes_r \ell \rangle.
$$

(ii) *The following equality holds*:

$$
\langle f \tilde{\otimes} g, \ell \tilde{\otimes} h \rangle = \frac{m!n!}{(m+n)!} \sum_{r=0}^{m\wedge n} C_n^r C_m^r \langle f \otimes_r \ell, h \otimes_r g \rangle.
$$

We are now ready to calculate the term $T_0^{(k)}$.

PROPOSITION 2. *Let $f \in H^{\odot n}$, $g \in H^{\odot m}$. Let $T_0^{(k)}$ be given by* (4.2). *Then for every* $1 \le k \le \min(m, n)$

$$T_0^{(k)} = \frac{m!^2 n!^2}{(m-k)!(n-k)!} \sum_{s=0}^{(m-k)\wedge(n-k)} C_{m-k}^s C_{n-k}^s \left[\|f \otimes_s g\|^2 - \|f \otimes_{s+k} g\|^2\right].$$

PROOF. From (4.2), we can write

$$\begin{aligned} T_0^{(k)} &= \frac{1}{2}\alpha_{k,0} \sum_{i_1,\dots,i_k,l_1,\dots,l_k \ge 1} \|f_{i_1,\dots,i_k} \tilde{\otimes} g_{l_1,\dots,l_k} - f_{l_1,\dots,l_k} \tilde{\otimes} g_{i_1,\dots,i_k}\|^2 \\ &= \alpha_{k,0} \sum_{i_1,\dots,i_k,l_1,\dots,l_k \ge 1} \left[\|f_{i_1,\dots,i_k} \tilde{\otimes} g_{l_1,\dots,l_k}\|^2 \right. \\ &\qquad \left. - \langle f_{i_1,\dots,i_k} \tilde{\otimes} g_{l_1,\dots l_k}, g_{i_1,\dots,i_k} \tilde{\otimes} f_{l_1,\dots,l_k} \rangle \right]. \end{aligned} \tag{4.3}$$

By Lemma 2, point (ii) and point (i),

$$\begin{aligned} &\|f_{i_1,\dots,i_k} \tilde{\otimes} g_{l_1,\dots,l_k}\|^2 \\ &\quad = \langle f_{i_1,\dots,i_k} \tilde{\otimes} g_{l_1,\dots,l_k}, f_{i_1,\dots,i_k} \tilde{\otimes} g_{l_1,\dots,l_k} \rangle \\ &\quad = \frac{(m-k)!(n-k)!}{(m+n-2k)!} \\ &\qquad \times \sum_{s=0}^{(m-k)\wedge(n-k)} C_{m-k}^s C_{n-k}^s \langle f_{i_1,\dots,i_k} \otimes_s g_{l_1,\dots,l_k}, f_{i_1,\dots,i_k} \otimes_s g_{l_1,\dots,l_k} \rangle \\ &\quad = \frac{(m-k)!(n-k)!}{(m+n-2k)!} \\ &\qquad \times \sum_{s=0}^{(m-k)\wedge(n-k)} C_{m-k}^s C_{n-k}^s \\ &\qquad\qquad \times \langle f_{i_1,\dots,i_k} \otimes_{n-k-s} f_{i_1,\dots,i_k}, g_{l_1,\dots,l_k} \otimes_{m-k-s} g_{l_1,\dots,l_k} \rangle. \end{aligned} \tag{4.4}$$

Also, Lemma 1 and Lemma 2 point (i) imply

$$\begin{aligned} &\sum_{i_1,\dots,i_k,l_1,\dots,l_k \ge 1} \langle f_{i_1,\dots,i_k} \otimes_{n-s} f_{i_1,\dots,i_k}, g_{l_1,\dots,l_k} \otimes_{m-s} g_{l_1,\dots,l_k} \rangle \\ &\qquad = \langle f \otimes_{n-s} f, g \otimes_{m-s} g \rangle = \|f \otimes_s g\|^2. \end{aligned} \tag{4.5}$$

On the other hand, using again Lemma 2 point (ii),

$$\begin{aligned} &\langle f_{i_1,\dots,i_k} \tilde{\otimes} g_{l_1,\dots,l_k}, g_{i_1,\dots,i_k} \tilde{\otimes} f_{l_1,\dots,l_k} \rangle \\ &\quad = \frac{(m-k)!(n-k)!}{(m+n-2k)!} \\ &\qquad \times \sum_{s=0}^{(m-k)\wedge(n-k)} C_{m-k}^s C_{n-k}^s \langle f_{i_1,\dots,i_k} \otimes_s g_{i_1,\dots,i_k}, f_{l_1,\dots,l_k} \otimes_s g_{l_1,\dots,l_k} \rangle. \end{aligned} \tag{4.6}$$

Again, Lemma 1 and Lemma 2 point (i) imply

$$\begin{aligned}\sum_{i_1,\dots,i_k,l_1,\dots,l_k\ge 1} &\langle f_{i_1,\dots,i_k}\otimes_s g_{i_1,\dots,i_k}, f_{l_1,\dots,l_k}\otimes_s g_{l_1,\dots,l_k}\rangle \\ &= \langle f\otimes_{s+k} g, f\otimes_{s+k} g\rangle = \|f\otimes_{s+k} g\|^2.\end{aligned}\tag{4.7}$$

Then, substituting (4.4), (4.5), (4.6) and (4.7) into (4.3) yields the desired result. □

It is also possible to compute the terms $T_r^{(k)}$ for every $1\le r\le (n-k)\wedge(m-k)$ but the corresponding expressions are more complicated, involving some kind of contractions of contractions. In order to obtain this type of formula, we need the following generalization of point (ii) in Lemma 2.

For $f,h\in H^{\odot n}$ and $g,\ell\in H^{\odot m}$ and for $r,s\ge 0$ such that $r+s\le m\wedge n$ we denote by $(f\otimes_r g)\widehat{\otimes}_s(\ell\otimes_r h)$ the contraction of r coordinates between f and g and between ℓ and h, s coordinates between f and ℓ and between g and h, $n-r-s$ coordinates between f and h and $m-r-s$ coordinates between g and ℓ. That is,

$$\begin{aligned}(f\otimes_r g)\widehat{\otimes}_s(\ell\otimes_r h) = \sum &\langle f_{i_1,\dots,i_r,j_1,\dots,j_s,k_1,\dots,k_{n-r-s}}\rangle\langle g_{i_1,\dots,i_r,l_1,\dots,l_s,p_1,\dots,p_{m-r-s}}\rangle\\ &\times\langle \ell_{p_1,\dots,p_r,j_1,\dots,j_s,p_1,\dots,p_{m-r-s}}\rangle\\ &\times\langle h_{p_1,\dots,p_r,l_1,\dots,l_s,k_1,\dots,k_{n-r-s}}\rangle,\end{aligned}$$

where the sum runs over all indices greater or equal than one. Notice that

$$(f\otimes_r g)\widehat{\otimes}_s(\ell\otimes_r h) = (f\otimes_s \ell)\widehat{\otimes}_r(g\otimes_s h).$$

LEMMA 3. *Assume $f,h\in H^{\odot n}$ and $g,\ell\in H^{\odot m}$. Then for every $r=0,\dots,(m-1)\wedge(n-1)$ we have*

$$\langle f\tilde{\otimes}_r g, \ell\tilde{\otimes}_r h\rangle = \frac{(n-r)!(m-r)!}{(m+n-2r)!}\sum_{s=0}^{(m-r)\wedge(n-r)} C_{n-r}^s C_{m-r}^s (f\otimes_r g)\widehat{\otimes}_s(\ell\otimes_r h).$$

PROOF. We can write

$$f\tilde{\otimes}_r g = \sum_{i_1,\dots,i_r} f_{i_1,\dots,i_r}\tilde{\otimes} g_{i_1,\dots,i_r}$$

and

$$\ell\tilde{\otimes}_r h = \sum_{i_1,\dots,i_r} \ell_{i_1,\dots,i_r}\tilde{\otimes} h_{i_1,\dots,i_r}.$$

Then Lemma 2 point (ii) gives

$$\begin{aligned}\langle f \tilde{\otimes}_r g, \ell \tilde{\otimes}_r h\rangle &= \sum_{i_1,\dots,i_r,l_1,\dots l_r} \langle f_{i_1,\dots,i_r} \tilde{\otimes} g_{i_1,\dots,i_r}, \ell_{l_1,\dots,l_r} \tilde{\otimes} h_{l_1,\dots,l_r}\rangle \\ &= \frac{(n-r)!(m-r)!}{(m+n-2r)!} \\ &\quad \times \sum_{s=0}^{(m-r)\wedge(n-r)} C_{n-r}^s C_{m-r}^s \\ &\qquad \times \langle f_{i_1,\dots,i_r} \otimes_s \ell_{l_1,\dots,l_r}, h_{l_1,\dots,l_r} \otimes_s g_{i_1,\dots,i_r}\rangle,\end{aligned}$$

which implies the desired result. □

Notice that for $r=0$,

$$(f \otimes g) \widehat{\otimes}_s (\ell \otimes h) = \langle f \otimes_s \ell, h \otimes_s g\rangle,$$

so Lemma 2 point (ii) is a particular case of Lemma 3 when $r=0$.

PROPOSITION 3. *Let $(F,G) = (I_n(f), I_m(g))$ with $f \in H^{\odot n}$ and $g \in H^{\odot m}$. Then, for every $r = 1,\dots,(n-k)\wedge(m-k)$*

$$\begin{aligned}T_r^{(k)} = \beta_{k,r} &\sum_{s=0}^{(n-k-r)\wedge(m-k-r)} C_{n-k-r}^s C_{m-k-r}^s \\ &\times \big((f \otimes_r g) \widehat{\otimes}_s (g \otimes_r f) \\ &\quad - (f \otimes_r g) \widehat{\otimes}_{s+k} (g \otimes_r f)\big),\end{aligned} \tag{4.8}$$

where

$$\beta_{k,r} = \frac{n!^2 m!^2}{(n-k-r)!(m-k-r)!(r!)^2}.$$

PROOF. From (4.2), we can write

$$\begin{aligned}T_r^{(k)} = \alpha_{k,r} &\sum_{i_1,\dots,i_k,l_1,\dots,l_k \geq 1} \big[\| f_{i_1,\dots,i_k} \tilde{\otimes}_r g_{l_1,\dots,l_k}\|^2 \\ &- \langle f_{i_1,\dots,i_k} \tilde{\otimes}_r g_{l_1,\dots,l_k}, f_{l_1,\dots,l_k} \tilde{\otimes}_r g_{i_1,\dots,i_k}\rangle\big].\end{aligned} \tag{4.9}$$

Applying Lemma 3 yields

$$\begin{aligned}&\| f_{i_1,\dots,i_k} \tilde{\otimes}_r g_{l_1,\dots,l_k}\|^2 \\ &\quad = \langle f_{i_1,\dots,i_k} \tilde{\otimes}_r g_{l_1,\dots,l_k}, f_{i_1,\dots,i_k} \tilde{\otimes}_r g_{l_1,\dots,l_k}\rangle \\ &\quad = \frac{(n-k-r)!(m-k-r)!}{(m+n-2k-2r)!}\end{aligned} \tag{4.10}$$

$$\times \sum_{s=0}^{(n-k-r)\wedge(m-k-r)} C_{n-k-r}^{s} C_{m-k-r}^{s} \times (f_{i_1,\dots,i_k} \otimes_r g_{l_1,\dots,l_k}) \mathbin{\widehat{\otimes}_s} (g_{l_1,\dots,l_k} \otimes_r f_{i_1,\dots,i_k}).$$

Notice that

$$\sum_{i_1,\dots,i_k,l_1,\dots,l_k \ge 1} (f_{i_1,\dots,i_k} \otimes_r g_{l_1,\dots,l_k}) \mathbin{\widehat{\otimes}_s} (g_{l_1,\dots,l_k} \otimes_r f_{i_1,\dots,i_k}) = (f \otimes_r g) \mathbin{\widehat{\otimes}_s} (g \otimes_r f). \tag{4.11}$$

Analogously, we get

$$\begin{aligned}\langle f_{i_1,\dots,i_k} \tilde{\otimes}_r g_{l_1,\dots,l_k}, f_{l_1,\dots,l_k} \tilde{\otimes}_r g_{i_1,\dots,i_k} \rangle &= \frac{(n-k-r)!(m-k-r)!}{(m+n-2k-2r)!} \\ &\quad \times \sum_{s=0}^{(n-k-r)\wedge(m-k-r)} C_{n-k-r}^{s} C_{m-k-r}^{s} \\ &\quad \times (f_{i_1,\dots,i_k} \otimes_r g_{l_1,\dots,l_k}) \mathbin{\widehat{\otimes}_s} (g_{i_1,\dots,i_k} \otimes_r f_{l_1,\dots,l_k}),\end{aligned} \tag{4.12}$$

and

$$\sum_{i_1,\dots,i_k,l_1,\dots,l_k \ge 1} (f_{i_1,\dots,i_k} \otimes_r g_{l_1,\dots,l_k}) \mathbin{\widehat{\otimes}_s} (g_{i_1,\dots,i_k} \otimes_r f_{l_1,\dots,l_k}) = (f \otimes_r g) \mathbin{\widehat{\otimes}_{s+k}} (g \otimes_r f). \tag{4.13}$$

Substituting (4.10), (4.11), (4.12) and (4.13) into (4.9) we obtain the desired formula. $\square$

In the particular case $n = m$, expression (4.8) can be written as

$$T_r^{(k)} = \sum_{s=0}^{n-k-r} T_{r,s}^{(k)}, \tag{4.14}$$

where

$$T_{r,s}^{(k)} = \frac{(n!)^4}{((n-k-r)!r!)^2} (C_{n-k-r}^{s})^2 \times \big((f \otimes_r g) \mathbin{\widehat{\otimes}_s} (g \otimes_r f) - (f \otimes_r g) \mathbin{\widehat{\otimes}_{s+k}} (g \otimes_r f)\big).$$

For $r = n - k$, expression (4.14) can be simplified as follows.

COROLLARY 1. *Let $(F,G)=(I_n(f),I_n(g))$ with $f,g\in H^{\odot n}$. Then for $k=1,\dots,n-1$*

$$T_{n-k}^{(k)}=\frac{n!^4}{(n-k)!^2}\big[\|f\otimes_{n-k}g\|^2-\langle f\otimes_{n-k}g,g\otimes_{n-k}f\rangle\big].$$

PROOF. When $r=n-k$, there is only one term in the sum (4.14), obtained for $s=0$. It is easy to see that

$$(f\otimes_{n-k}g)\,\widehat{\otimes}_0\,(g\otimes_{n-k}f)=(f\otimes g)\,\widehat{\otimes}_{n-k}\,(g\otimes f)=\|f\otimes_{n-k}g\|^2$$

and

$$(f\otimes_{n-k}g)\,\widehat{\otimes}_k\,(g\otimes_{n-k}f)=\langle f\otimes_{n-k}g,g\otimes_{n-k}f\rangle. \qquad \square$$

From (4.1) and Proposition 2, we obtain the following expression for the determinant of the kth iterated Malliavin matrix.

THEOREM 1. *Let $f\in H^{\odot n}$, $g\in H^{\odot m}$. Then for every $1\le k\le m\wedge n$,*

$$\begin{aligned}E\det\Lambda^{(k)}&\\&=\frac{m!^2n!^2}{(m-k)!(n-k)!}\sum_{s=0}^{(m-k)\wedge(n-k)}C_{m-k}^sC_{n-k}^s\\&\qquad\times\big[\|f\otimes_s g\|^2-\|f\otimes_{s+k}g\|^2\big]+R_{m,n,k},\end{aligned}$$

where $R_{m,n,k}=\sum_{r=1}^{(m-k)\wedge(n-k)}T_r^{(k)}$ and $T_r^{(k)}$ is given by (4.8). Note that $R_{m,n,k}\ge 0$ for every $n,m\ge 1$ and for every $k=1,\dots,n\wedge m$ [this follows from (4.2)].

In the case of multiple integrals of the same order (i.e., $m=n$), we have the following result.

COROLLARY 2. *If $f,g\in H^{\odot n}$, the determinant of the kth iterated Malliavin matrix $(1\le k\le n)$ of $(F,G)=(I_n(f),I_n(g))$ can be written as*

$$E\det\Lambda^{(k)}=\frac{n!^4}{(n-k)!^2}\sum_{s=0}^{n-k}(C_{n-k}^s)^2(\|f\otimes_s g\|^2-\|f\otimes_{s+k}g\|^2)+R_{n,n,k}.$$

EXAMPLE 1. Suppose $m=n=3$ and $k=2$. Then

$$E\det\Lambda^{(2)}=(3!)^4\big[\|f\otimes_0 g\|^2-\|f\otimes_2 g\|^2+\|f\otimes_1 g\|^2-\|f\otimes_3 g\|^2\big]+R_{3,3,2}.$$

Suppose $m=n=4$ and $k=2$. Then

$$E\det\Lambda^{(2)}=\frac{(4!)^4}{2!^2}\big[\|f\otimes_0 g\|^2-\|f\otimes_2 g\|^2+4(\|f\otimes_1 g\|^2-\|f\otimes_3 g\|^2)\big]+R_{4,4,2}.$$

Our next objective is to relate the expectation of the iterated Malliavin matrix $E\det\Lambda^{(s)}$ with the covariance matrix of the vector (F,G) in the case $n=m$. We recall that

$$\det C = n!^2\big[\|f\|^2\|g\|^2-\langle f,g\rangle^2\big].$$

Notice that, for $k=n$, the iterated Malliavin matrix $\Lambda^{(n)}$ is deterministic and $\Lambda^{(n)}=n!C$. Therefore, $\det\Lambda^{(n)}=n!^2\det C=n!^4[\|f\|^2\|g\|^2-\langle f,g\rangle^2]$.

THEOREM 2. *For any $f,g\in H^{\odot n}$, if $F=I_n(f)$ and $G=I_n(g)$, we have*

$$\sum_{s=2}^{[(n-1)/2]}\frac{n(n-2s)}{s!^2}E\det\Lambda^{(s)}+(n-1)^2E\det\Lambda^{(1)}\geq n^2\det C.$$

PROOF. From Corollary 2, taking into account that $R_{n,n,1}\geq 0$, we can write

$$\begin{aligned}E\det\Lambda^{(1)}&\geq [nn!]^2\sum_{s=0}^{n-1}(C_{n-1}^s)^2\big(\|f\otimes_s g\|^2-\|f\otimes_{s+1}g\|^2\big)\\&=n^2\det C+[nn!]^2\sum_{s=1}^{n-1}\big((C_{n-1}^s)^2-(C_{n-1}^{s-1})^2\big)\|f\otimes_s g\|^2.\end{aligned}$$

Notice that $(C_{n-1}^s)^2-(C_{n-1}^{s-1})^2=-[(C_{n-1}^{n-s})^2-(C_{n-1}^{n-1-s})^2]$. Therefore, we conclude that

$$\begin{aligned}E\det\Lambda^{(1)}&\geq n^2\det C+[nn!]^2\sum_{s=1}^{[(n-1)/2]}\big((C_{n-1}^s)^2-(C_{n-1}^{s-1})^2\big)\\&\qquad\times\big(\|f\otimes_s g\|^2-\|f\otimes_{n-s}g\|^2\big)\\&=n^2\det C+\sum_{s=1}^{[(n-1)/2]}\gamma_{n,s}\big(\|f\otimes_s g\|^2-\|f\otimes_{n-s}g\|^2\big),\end{aligned}\tag{4.15}$$

where

$$\gamma_{n,s}=\left(\frac{n!^2}{(n-s)!s!}\right)^2 n(n-2s).$$

Remark that $\gamma_{n,s}\geq 0$ if $s\leq[\frac{n-1}{2}]$. We can write, using Lemma 2 point (i) and Corollary 1,

$$\begin{aligned}\|f\otimes_s g\|^2-\|f\otimes_{n-s}g\|^2&=\|f\otimes_s g\|^2-\langle f\otimes_s g,g\otimes_s f\rangle\\&\quad-\big(\|f\otimes_{n-s}g\|^2-\langle f\otimes_{n-s}g,g\otimes_{n-s}f\rangle\big)\\&\geq-\frac{(n-s)!^2}{n!^4}T_{n-s}^{(s)}\geq-\frac{(n-s)!^2}{n!^4}E\det\Lambda^{(s)}.\end{aligned}\tag{4.16}$$

Substituting (4.16) into (4.15) yields

$$E \det \Lambda^{(1)} \geq n^2 \det C - \sum_{s=1}^{[(n-1)/2]} \frac{n(n-2s)}{s!^2} E \det \Lambda^{(s)},$$

which implies the desired result. □

REMARK 1. In the particular case $n = 2$, we obtain $E \det \Lambda^{(1)} \geq 4 \det C$, which was proved in [1]. For $n = 3$, we get $E \det \Lambda^{(1)} \geq \frac{9}{4} \det C$, and for $n = 4$, $E \det \Lambda^{(1)} \geq \frac{16}{9} \det C$. Only if $n \geq 5$ we need the expectation of the iterated Malliavin matrix to control the determinant of the covariance matrix.

5. The density of a pair of multiple integrals. In this section, we show that a random vector of dimension 2 whose components are multiple integrals in the same Wiener chaos either admits a density with respect to the Lebesque measure, or its components are proportional. We also show that a necessary and sufficient condition for such a vector not to have a density is that at least one of its iterated Malliavin matrices vanishes almost surely. In the sequel, we fix a vector $(F, G) = (I_n(f), I_n(g))$ with $f, g \in H^{\odot n}$.

In the following result, we show that, if the determinant of an iterated Malliavin matrix of a pair of multiple integrals vanishes, then the determinant of any other iterated Malliavin matrix will vanish.

PROPOSITION 4. *Let $1 \leq k, l \leq n$ with $k \neq l$. Then $E \det \Lambda^{(k)} = 0$ if and only if $E \det \Lambda^{(l)} = 0$.*

PROOF. Assume first that $k = 1$ and $l = 2$. Suppose that $E \det \Lambda^{(1)} = 0$ and let us prove that $E \det \Lambda^{(2)} = 0$. Since $\det \Lambda^{(1)} = 0$ a.s., from (3.1) we obtain

$$D_j F D_i G = D_i F D_j G \qquad \text{a.s.} \tag{5.1}$$

for any $i, j \geq 1$ (recall that $D_j F = DF \otimes_1 e_j$). That is,

$$DF D_i G = DG D_i F \qquad \text{a.s.}, \tag{5.2}$$

for any $i \geq 1$. Let us apply the divergence operator δ (the adjoint of D) to both members of equation (5.2). From (2.8), we obtain $\delta DF = nF$ and $\delta DG = nG$. Using Proposition 1.3.3 in [4], we get

$$nF D_i G - \langle DF, DD_i G\rangle_H = nG D_i F - \langle DG, DD_i F\rangle_H \qquad \text{a.s.},$$

which can be written as [using the notation (2.7)]

$$I_n(f) I_{n-1}(g_i) - (n-1) \sum_{j=1}^{\infty} I_{n-2}(g_{ij}) I_{n-1}(f_j)$$

$$= I_n(g) I_{n-1}(f_i) - (n-1) \sum_{j=1}^{\infty} I_{n-2}(f_{ij}) I_{n-1}(g_j) \qquad \text{a.s.}$$

By the product formula (2.5), the above relation becomes

$$I_{2n-1}(f\tilde{\otimes} g_i)+\sum_{k=1}^{n-1}\bigl(k!C_n^kC_{n-1}^k-(n-1)(k-1)!C_{n-1}^{k-1}C_{n-2}^{k-1}\bigr)\times I_{2n-1-2k}(f\tilde{\otimes}_k g_i)$$
$$=I_{2n-1}(g\tilde{\otimes} f_i)+\sum_{k=1}^{n-1}\bigl(k!C_n^kC_{n-1}^k-(n-1)(k-1)!C_{n-1}^{k-1}C_{n-2}^{k-1}\bigr)\times I_{2n-1-2k}(g\tilde{\otimes}_k f_i) \qquad \text{a.s.}$$

By identifying the terms in each Wiener chaos, we obtain

$$f\tilde{\otimes}_k g_i = g\tilde{\otimes}_k f_i$$

for any $i\ge 1$ and for any $k=0,\dots,n-1$. A further application of the product formula for multiple stochastic integrals yields

$$FDG=GDF \qquad \text{a.s.}$$

We differentiate the above relation in the Malliavin sense and we have

$$FD_{ij}^{(2)}G+D_iFD_jG=GD_{ij}^{(2)}F+D_iGD_jF \qquad \text{a.s.}$$

for every $i,j\ge 1$. By (5.1),

$$FD^{(2)}G=GD^{(2)}F \qquad \text{a.s.}$$

and this clearly implies that $\det\Lambda^{(2)}=0$ a.s.

Suppose now that $E\det\Lambda^{(2)}=0$. Then $\Lambda^{(2)}=0$ a.s. and from (3.1) we get

$$D_{ij}^{(2)}FD_{pq}^{(2)}G=D_{pq}^{(2)}FD_{ij}^{(2)}G \qquad \text{a.s.}$$

for any $i,j,p,q\ge 1$. This implies

$$DD_iFD_{pq}^{(2)}G=DD_iGD_{pq}^{(2)}F \qquad \text{a.s.} \tag{5.3}$$

for any $i,p,q\ge 1$ Applying the divergence operator δ to equation (5.3) yields

$$(n-1)D_iFD_{pq}^{(2)}G-\langle DD_iF,DD_{pq}^{(2)}G\rangle_H$$
$$=(n-1)D_iGD_{pq}^{(2)}F-\langle DD_iG,DD_{pq}^{(2)}F\rangle_H$$

a.s. This equality can be written as

$$I_{n-1}(f_i)I_{n-2}(g_{pq})-(n-2)\sum_{j=1}^{\infty}I_{n-3}(g_{pq})I_{n-2}(f_{ij})$$
$$=I_n(g_i)I_{n-1}(f_{pq})-(n-2)\sum_{j=1}^{\infty}I_{n-3}(f_{hl})I_{n-2}(g_{pq}) \qquad \text{a.s.}$$

By the product formula for multiple stochastic integrals, we get for every $j, p, q \geq 1$,

$$
\begin{aligned}
I_{2n-3}(f_i \tilde{\otimes} g_{pq}) + \sum_{k=1}^{n-2} \bigl[k! C_{n-2}^k C_{n-1}^k - (n-2)(k-1)! C_{n-2}^{k-1} C_{n-3}^{k-1}\bigr] \\
\times I_{2n-3-2k}(f_i \tilde{\otimes}_k g_{pq}) \\
= I_{2n-3}(g_i \tilde{\otimes} f_{pq}) + \sum_{k=1}^{n-2} \bigl[k! C_{n-2}^k C_{n-1}^k - (n-2)(k-1)! C_{n-2}^{k-1} C_{n-3}^{k-1}\bigr] \\
\times I_{2n-3-2k}(g_i \tilde{\otimes}_k f_{pq}) \qquad \text{a.s.}
\end{aligned}
$$

Identifying the coefficients of each Wiener chaos, we obtain

$$f_i \tilde{\otimes}_k g_{pq} = g_i \tilde{\otimes}_k f_{pq}$$

for any $i, p, q \geq 1$ and for any $k = 0, \dots, n-2$. This implies

$$f_i \tilde{\otimes}_k g_q = g_i \tilde{\otimes}_k f_q \tag{5.4}$$

for any $i, q \geq 1$ and for any $k = 0, \dots, n-1$. Applying again the product formula for multiple stochastic integrals, relation (5.4) leads to

$$D_i F D_q G = D_i G D_q F \qquad \text{a.s.,}$$

for any $i, q \geq 1$, which implies $\det \Lambda^{(1)} = 0$ a.s. By iterating the above argument, we easily find that $\det \Lambda^{(k)} = 0$ a.s. is equivalent to $\det \Lambda^{(l)} = 0$ a.s., for every $1 \leq k, l \leq n$ with $k \neq l$. $\square$

COROLLARY 3. *The vector $(F, G) = (I_n(f), I_n(g))$ does not admit a density if and only if there exists $k \in \{1, \dots, n\}$ such that $E \det \Lambda^{(k)} = 0$.*

PROOF. It is a consequence of Proposition 4 and Theorem 3.1 in [1]. $\square$

THEOREM 3. *Let $f, g \in H^{\odot n}$ be symmetric tensors. Then the random vector $(F, G) = (I_n(f), I_n(g))$ does not admit a density if and only if $\det C = 0$, where C denotes the covariance matrix of (F, G). In other words, the vector (F, G) does not admit a density if and only if its components are proportional.*

PROOF. If $\det C = 0$, the random variables F and G are proportional and the law of (F, G) is not absolutely continuous with respect to the Lebesgue measure. Suppose that the law of the random vector (F, G) is not absolutely continuous with respect to the Lebesque measure. Then, from the results of [1], we know that $E \det \Lambda^{(1)} = 0$. By Proposition 4, $E \det \Lambda^{(k)} = 0$ for $k = 1, \dots, n$. Then Theorem 2 implies $\det C = 0$ (notice also that $\det C = 0$ because $C = n! \Lambda^{(n)}$). $\square$

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