

## BOUNDARIES OF PLANAR GRAPHS, VIA CIRCLE PACKINGS<sup>1</sup>

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We provide a geometric representation of the Poisson and Martin boundaries of a transient, bounded degree triangulation of the plane in terms of its circle packing in the unit disc. (This packing is unique up to Möbius transformations.) More precisely, we show that any bounded harmonic function on the graph is the harmonic extension of some measurable function on the boundary of the disk, and that the space of extremal positive harmonic functions, that is, the Martin boundary, is homeomorphic to the unit circle.

All our results hold more generally for any “good”-embedding of planar graphs, that is, an embedding in the unit disc with straight lines such that angles are bounded away from 0 and  $\pi$  uniformly, and lengths of adjacent edges are comparable. Furthermore, we show that in a good embedding of a planar graph the probability that a random walk exits a disc through a sufficiently wide arc is at least a constant, and that Brownian motion on such graphs takes time of order  $r^2$  to exit a disc of radius  $r$ . These answer a question recently posed by Chelkak (2014).

**1. Introduction.** Given a Markov chain, it is natural to ask what is its “final” behavior, that is, the behavior as the time tends to infinity. For example, consider the lazy simple random walk on a rooted 3-regular tree—the path of the random walk almost surely determines a unique infinite branch of the tree. This branch is determined by the tail  $\sigma$ -field of the random walk and moreover, this  $\sigma$ -field is characterized by the set of such infinite branches. In general, it is more useful to consider the *invariant*  $\sigma$ -field  $\mathcal{I}$ , that is, all the events that are invariant under the time-shift operator. In the case of lazy Markov chains, these two  $\sigma$ -fields are equivalent [10, 19].

To any invariant event  $A$  we can associate a harmonic function  $h_A$  on the state space by  $h_A(x) = \mathbb{P}_x(A)$ , that is, the probability that  $A$  occurs starting the chain from  $x$ . (A function  $h$  is harmonic if its value at a state is the expected value of  $h$  after one step of the chain.) In fact, there is a correspondence between bounded invariant random variables and bounded harmonic functions on the state space

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(given such a random variable  $Y$ , the function is  $h_Y(x) = \mathbb{E}_x(Y)$ , see [19, 22]). Thus, the set of bounded harmonic functions on the state space characterizes all the “final” behaviors of the Markov chain.

In this paper, we consider reversible Markov chains in discrete time and space (i.e., weighted random walks on a graphs). It is not hard to see that if the chain is recurrent, then there are no nonconstant bounded harmonic functions. On the other hand, transience does not guarantee the existence of such functions, as can be seen in the simple random walk on  $\mathbb{Z}^3$ . However, in the planar case there is such a dichotomy: Benjamini and Schramm [7] proved that if  $G$  is a transient, bounded degree planar graph, then  $G$  exhibits nonconstant bounded harmonic functions.

The proof in [7] relies on the theory of circle packing. Recall that a *circle packing*  $P$  of a planar graph  $G$  is a set of circles with disjoint interiors  $\{C_v\}_{v \in G}$  such that two circles are tangent if and only if the corresponding vertices form an edge. Koebe’s circle packing theorem [20] states that any planar graph has a circle packing, and that for triangulations (graphs where all faces are triangles) the circle packing is essentially unique. Given a circle packing, we embed the graph in  $\mathbb{R}^2$ , with straight line segments between the corresponding centers of circles for edges. The *carrier* of  $P$ , denoted  $\text{carr}(P)$ , is the union of all the closed polygons corresponding to the faces. He and Schramm [16] provided an insightful connection between the probabilistic notion of recurrence or transience of  $G$  and the geometry of  $\text{carr}(P)$ . Their theorem states that if  $G$  is a bounded degree one-ended triangulation, then it can be circle packed so that the carrier is either the entire plane or the open unit disc  $U$  according to whether  $G$  is recurrent or transient, respectively. Since we are interested in nonconstant bounded harmonic functions, we consider here only the latter case.

Consider a transient, bounded degree, one-ended triangulation  $G$  and its circle packing  $P = \{C_v\}_{v \in V}$  with  $\text{carr}(P) = U$ . We identify each vertex  $v$  with the center of  $C_v$ —it will always be clear from the context if the letter  $v$  represents a vertex or a point in  $\mathbb{R}^2$ . Let  $\{X_n\}$  be the simple random walk on  $G$ . A principal result of Benjamini and Schramm [7] is that  $\lim_{n \rightarrow \infty} X_n$  exists and is a point  $X_\infty \in \partial U$  almost surely, and furthermore its distribution is nonatomic. This immediately implies that any bounded measurable function  $g : \partial U \rightarrow \mathbb{R}$  can be extended to a bounded harmonic function  $h : V \rightarrow \mathbb{R}$  by setting  $h(v) = \mathbb{E}_v[g(X_\infty)]$ , where for a vertex  $x$  we write  $\mathbb{E}_x$  for the expectation w.r.t. the random walk started at  $x$ . Since the distribution of  $X_\infty$  is nonatomic they deduce that a nonconstant bounded harmonic function exists. Note that since each vertex  $v$  is in  $\text{carr}(P)$ , we cannot have a vertex on the boundary of  $U$ .

The main result of this paper is that there are no other bounded harmonic functions, that is, any bounded harmonic function can be represented this way. Recall that a graph is one-ended if removal of any finite set of vertices leaves only one infinite connected component.

**THEOREM 1.1.** *Let  $G = (V, E)$  be a transient bounded degree, one-ended triangulation and let  $P$  be a circle packing of  $G$  with  $\text{carr}(P) = U$ . Then for any bounded harmonic function  $h : V \rightarrow \mathbb{R}$  there exists a bounded measurable function  $g : \partial U \rightarrow \mathbb{R}$  such that  $h(v) = \mathbb{E}_v[g(X_\infty)]$ .*

For a vertex  $x$ , we write  $\mathbb{P}_x$  for the probability measure on  $G^\mathbb{N}$  of the Markov chain started at  $x$ . The measure space  $(G^\mathbb{N}, \mathcal{I}, \mathbb{P}_x)$  is often called the *Poisson boundary* of the chain. The choice of  $x$  does not matter much because the measures  $\mathbb{P}_x$  are absolutely continuous with respect to each other. As mentioned before, there is a correspondence between bounded harmonic functions and  $L^\infty(G^\mathbb{N}, \mathcal{I}, \mathbb{P}_x)$  and for that reason the space of bounded harmonic functions is sometimes also referred to as the Poisson boundary. Theorem 1.1 shows that if we circle pack  $G$  in  $U$ , then  $\partial U$  is a representation of the Poisson boundary. More precisely, let  $f : G^\mathbb{N} \rightarrow \partial U$  be the measurable function (defined  $\mathbb{P}_x$ -almost everywhere)  $f(\{x_n\}) = \lim x_n$  and let  $\mathcal{B} \subset \mathcal{I}$  be the pull back  $\sigma$ -algebra on  $G^\mathbb{N}$ . Then  $\mathcal{B}$  and  $\mathcal{I}$  are in fact equivalent, that is, for any  $A \in \mathcal{I}$  there exists  $B \in \mathcal{B}$  such that the measure of  $A \Delta B$  is zero.

The Martin boundary [12, 23, 31] is another concept of a boundary of a Markov chain, associated with the space of positive harmonic functions. While the Poisson boundary is naturally defined as a measure space, the Martin boundary is a topological space. It is well known (see Chapter 24 of [31]) that the Poisson boundary may be obtained by endowing the Martin boundary with a suitable measure. Hence, in addition to its intrinsic interest, the Martin boundary studied here will provide more information and will yield Theorem 1.1 rather abstractly.

An illustrative example of the difference between the boundaries is the following. Let  $G$  be the graph obtained from  $\mathbb{Z}^3$  by connecting its root to a disjoint one-sided infinite path. It is possible for a positive harmonic function to diverge only along the path. Thus, the Martin boundary will consist of two points (corresponding to the two “infinities” of  $G$ ), however, since the simple random walk has probability 0 of staying in the infinite path forever, and  $\mathbb{Z}^3$  has no nonconstant bounded harmonic functions, the Poisson boundary will have all its mass on one of the points.

Let us formally define the Martin boundary. Let  $x_0$  be an arbitrary fixed root of  $G$  and  $M(x, y)$  be the *Martin kernel*

$$M(x, y) = \frac{G(x, y)}{G(x_0, y)},$$

where  $G(x, y) = \mathbb{E}_x[\# \text{ visits to } y]$  is the Green function. For any fixed  $y$ , the function  $M(\cdot, y)$  is a positive function that is harmonic everywhere except at  $y$ . Hence, if for some sequence  $y_n$ , such that the graph distance between  $x_0$  and  $y_n$  tends to infinity, the functions  $M(\cdot, y_n)$  converges pointwise, then the limit is a positive harmonic function on  $G$ . The *Martin boundary* is defined to be the set  $\mathcal{M}$  of all such limit points, endowed with the pointwise convergence topology.

A positive harmonic function  $h : V \rightarrow \mathbb{R}$  such that  $h(x_0) = 1$  is called *minimal* if for any positive harmonic function  $g$  such that  $g(x) \leq h(x)$  for all  $x$ , then  $g = ch$  for some constant  $c > 0$ . The minimal functions are the extremal points of the convex set of positive harmonic functions, normalized to have  $h(x_0) = 1$ . By Choquet’s theorem and [31], Theorem 24.8, it follows that any positive harmonic function  $h$  can be written as  $h = \int g d\mu(g)$  for some measure  $\mu$  depending on  $h$ , and supported on the set of minimal harmonic functions. If we normalize so that  $h(x_0) = 1$ , then  $\mu$  is a probability measure.

**THEOREM 1.2.** *Let  $G = (V, E)$  be a transient bounded degree, one-ended triangulation and let  $P$  be a circle packing of  $G$  with  $\text{carr}(P) = U$ . Then:*

- (1) *For a sequence  $y_n \in V$  we have that  $M(\cdot, y_n)$  converges pointwise if and only if  $y_n$  converges in  $\mathbb{R}^2$  (in particular, the limit only depends on  $\lim y_n$ ).*
- (2) *If  $y_n \rightarrow \xi \in \partial U$ , then  $\lim M(\cdot, y_n)$  is a minimal harmonic function.*
- (3) *The map  $\xi \mapsto \lim M(\cdot, y_n)$ , where  $y_n \rightarrow \xi$ , is a homeomorphism.*

*In particular, the Martin boundary is homeomorphic to  $\partial U$ .*

The limit  $\lim_{y \rightarrow \xi} M(\cdot, y)$  is denoted by  $M_\xi$ . Thus, for any positive harmonic function  $h$  there is some measure  $\mu$  on  $\partial U$ , so that  $h = \int_{\partial U} M_\xi d\mu(\xi)$ .

A similar characterization of the Poisson boundary of planar graphs in terms of their square tiling was recently obtained by Georgakopoulos [15]. His results allow him to characterize the Poisson boundary for a somewhat more general set of graphs, namely, of bounded degree *uniquely absorbing* planar graphs. The analysis in this paper of random walk via circle packings and other embeddings requires a completely different set of tools and in return allows us to characterize the Martin boundary with no additional cost.

**1.1. Good embeddings of planar graphs.** Recall that a *proper embedding* of a planar graph is a map sending the vertices to points in the plane and edges to continuous curves connecting the corresponding vertices such that no two edges cross. If each edge is mapped to a straight line we call it an *embedding with straight lines*. Given a circle packing of a graph  $G$ , we may obtain such an embedding by mapping vertices to the corresponding circle’s center and edges to straight lines between the corresponding vertices.

We will prove our results for more general embeddings than the one obtained from circle packing. The setting below has risen in the study of critical 2D lattice models and was formalized by Chelkak [8]. Let  $G = (V, E)$  be an infinite, connected, simple planar graph together with an embedding with straight lines. As before, we identify a vertex  $v$  with its image in the embedding. We write  $|u - v|$  for the Euclidean distance between points in the plane. For constants  $D \in (1, \infty)$  and  $\eta > 0$ , we say that the embedding is  $(D, \eta)$ -good if it satisfies:

(a) *No flat angles.* For any face, all the inner angles are at most  $\pi - \eta$ . In particular, all faces are convex, there is no outer face and the number of edges in a face is at most  $2\pi/\eta$ .

(b) *Adjacent edges have comparable lengths.* For any two adjacent edges  $e_1 = (u, v)$  and  $e_2 = (u, w)$ , we have that  $|u - w|/|u - v| \in [D^{-1}, D]$ .

We say that an embedding is *good* if it has straight lines and it is  $(D, \eta)$ -good for some  $D, \eta$ . A classical lemma of Rodin and Sullivan [24] (known as the *Ring lemma*) asserts that the ratio between radii of tangent circles in a circle packing of a bounded degree triangulation is bounded above and away from 0. We immediately get the following.

PROPOSITION 1.3. *Any circle packing of a bounded degree triangulation is  $(D, \eta)$ -good for some  $D$  and  $\eta$  that only depend on the maximum degree.*

In a similar fashion to the circle packing setting, we define the carrier of the embedding of  $G$ , denoted by  $\text{carr}(G)$ , to be the union of all the (closed) faces of the embedding. Note that if  $G$  is a one-ended triangulation, then  $\text{carr}(G)$  is always an open simply connected set in the plane. Lastly, suppose that the edges of the graph are equipped with positive weights  $\{w_e\}_{e \in E}$  and consider the weighted random walk  $\{X_n\}$  defined by  $P(X_1 = u | X_0 = v) = w_{(v,u)}/w_v$  for any edge  $(u, v)$ , where  $w_v = \sum_{u:u \sim v} w_{(u,v)}$ . A function  $h : V \rightarrow \mathbb{R}$  is harmonic with respect to the weighted graph when

$$(1.1) \quad h(v) = \sum_{u:u \sim v} \frac{w_{(u,v)}}{w_v} h(u),$$

or in other words, when  $h(X_n)$  is a martingale. The general version of Theorems 1.1 and 1.2 is now stated in a straightforward manner.

THEOREMS 1.1' AND 1.2'. *Let  $G = (V, E)$  be a bounded degree planar graph with a good embedding with straight lines such that  $\text{carr}(G) = U$ . Assume that  $G$  is equipped with positive edge weights bounded above and away from 0. Then the conclusions of Theorems 1.1 and 1.2 hold verbatim.*

Theorems 1.1 and 1.2 are immediate corollaries of this statement together with Proposition 1.3.

1.2. *Harmonic measure and exit time of discrete balls.* In the following, let  $G = (V, E)$  be a planar graph with a good embedding. A discrete domain  $S$  is a subset of  $V$  along with the induced edges  $E(S) = (S \times S) \cap E$ . The boundary of  $S$ , denoted  $\partial S$  is the external vertex boundary, that is, all vertices not in  $S$  with a neighbor in  $S$ . For  $u \in \mathbb{R}^2$  we denote by  $B_{\text{euc}}(u, r)$  the Euclidean ball  $\{y \in \mathbb{R}^2 :$

$|u - y| \leq r$  of radius  $r$  centred at  $u$ , and the discrete Euclidean ball  $V_{\text{euc}}(u, r)$  is the vertex set

$$V_{\text{euc}}(u, r) = V \cap B_{\text{euc}}(u, r).$$

As before, assume that the edges are equipped with positive weights and consider the weighted random walk  $\{X_n\}$ . For  $A \subset V$  let  $\tau_A$  be the first hitting time of  $A$ , that is,  $\tau_A = \min\{n : X_n \in A\}$  or  $\infty$  if  $A$  is never hit. The following two theorems answer a question recently posed by Chelkak ([8], page 9).

**THEOREM 1.4.** *For any positive constants  $D, \eta$  there exists  $c = c(D, \eta) > 0$  with the following. Assume that  $G$  is a graph with a  $(D, \eta)$ -good embedding, and all edges weights in  $[D^{-1}, D]$ . Then for any vertex  $u$ , any  $r \geq 0$  such that  $B_{\text{euc}}(u, r) \subset \text{carr}(G)$  and any closed interval  $I \subset \mathbb{R}/(2\pi\mathbb{Z})$  of length  $\pi - \eta$  we have*

$$\mathbb{P}_u(\arg(X_{T_r} - u) \in I) \geq c,$$

where  $T_r = \tau_{\partial V_{\text{euc}}(u, r)}$  is the first exit time from  $V_{\text{euc}}(u, r)$ .

Note that for smaller intervals of arguments the statement above may be false; for example, the left-hand side is 0 if  $\partial V_{\text{euc}}(u, r)$  contains no vertex in these directions.

For a vertex  $u \in V$ , we denote its *radius of isolation* by  $r_u = \min_{V \setminus \{u\}}\{|u - v|\}$ . We use  $f \asymp g$  when there is some  $C = C(D, \eta)$  so that  $C^{-1}g \leq f \leq Cg$ .

**THEOREM 1.5.** *For any positive constants  $D, \eta$  there exists  $C = C(D, \eta) \geq 1$  with the following. Assume that  $G$  is a graph with a  $(D, \eta)$ -good embedding and all edges weights are in  $[D^{-1}, D]$ . Then for any vertex  $u$  and any  $r \geq r_u$  with  $B_{\text{euc}}(u, Cr) \subset \text{carr}(G)$  we have*

$$\mathbb{E}_u \sum_{t=0}^{T_r} r_{X_t}^2 \asymp r^2,$$

where  $T_r = \tau_{\partial V_{\text{euc}}(u, r)}$  is the first exit time from  $V_{\text{euc}}(u, r)$ .

The reader may wonder why we require  $B_{\text{euc}}(x_0, Cr) \subset \text{carr}(G)$ , while the theorem only talks about the time to exit the smaller  $B_{\text{euc}}(x_0, r)$ . This is an artifact of our proof, and the stronger requirement can indeed be removed. This requires showing that it is possible to “extend” the embedding to a good embedding of a larger graph with carrier  $\mathbb{R}^2$ . This is indeed possible, and we plan to address this in a future paper.

1.3. *About the proofs and the organization of the paper.* We would like to compare the random walk on a well-embedded graph to Brownian motion, and certainly our results above justify such a comparison. However, the simple random walk on a good embedding can behave rather irregularly. For example, its Euclidean trajectory is not a martingale and can have a local drift. The random walk is also much slower when traversing areas of short edges compared to areas of longer edges. To fix the second problem, we could study the variable speed random walk which waits at each vertex an amount of time comparable to  $r_x^2$ , or to the area of one of the faces containing the vertex (a good embedding guarantees that all faces sharing a vertex have comparable area). Instead, we use the *cable process* on the graph, which can be thought of as Brownian motion on the embedding (see Section 3). The vertex trajectory of this process has the same distribution as the simple random walk, so the harmonic measures do not change.

A central step in this work is showing that well-embedded graphs satisfy volume doubling and a Poincaré inequality with respect to the Euclidean metric (rather than the graph metric). This is done in Section 3. The work of Sturm [28] (which applies in the very general setting of local Dirichlet spaces) then enables us to obtain various corollaries: an elliptic Harnack inequality (Theorem 5.4) and heat kernel estimates (see Theorem 3.6). These already give us enough control to prove Theorems 1.4 and 1.5 in Section 4.

To prove Theorems 1.1 and 1.2, we require a boundary Harnack inequality (see Theorem 5.5). Roughly speaking, this states that two positive harmonic functions that vanish on most of the boundary of the domain do so in a uniform way. In our setting, the boundary Harnack inequality is a consequence of the volume doubling and Poincaré inequality, as shown in [21], following an argument of Aikawa [1] that originates in the work of Bass and Burdzy [6]. Given the boundary Harnack inequality, it is possible to prove Theorem 1.1 by constructing an explicit coupling between two random walks starting at two different points conditioned to converge to some  $\xi \in \partial U$  (by conditioning that the random walk is swallowed in a small neighborhood and taking a weak limit) so that with probability 1 their traces coincide after a finite number of steps. This coupling is constructed by showing that for any annulus of constant aspect ratio around  $\xi$  the conditioned random walks have a positive chance to meet.

We do not use this proof approach and instead use the more succinct approach of Aikawa [1]. His argument (following Jerison and Kenig [18]) shows how the characterization of the Martin boundary of Brownian motion on a uniform domain follows from the boundary Harnack principle. Our argument in Section 5 is very similar to [1] except for the complication that our process is not a martingale. Thus, a separate argument is necessary to show the convergence of the random walk to the boundary and that the distribution of the limit is nonatomic.

**2. Preliminaries.** We begin with some geometric consequences of having a good embedding. In this section, we assume that we are given a  $(D, \eta)$ -good embedding of a graph  $G$ .

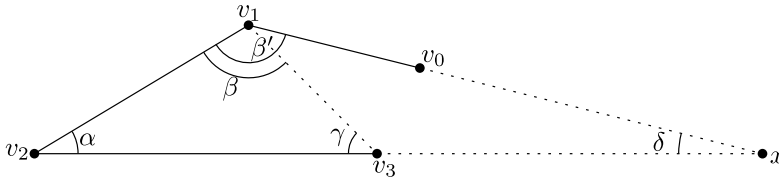


FIG. 1. *No thin acute angles: In a good embedding, the angle  $\alpha$  must be at least  $D^{-1} \sin(\eta/2)$ .*

LEMMA 2.1 (No thin acute angles). *The angle between any two adjacent edges is at least  $D^{-1} \sin(\eta/2)$ .*

PROOF. Let  $\alpha$  be the angle between three consecutive vertices on a face  $v_1, v_2, v_3$  such that the edge  $[v_1, v_2]$  is not longer than the edge  $[v_2, v_3]$ . By convexity, the triangle  $v_1, v_2, v_3$  is contained in the face. Let  $\beta, \gamma$  be the angles  $\angle v_2 v_1 v_3$  and  $\angle v_2 v_3 v_1$ , respectively. By our assumption, we learn that  $\beta \geq \gamma$ , hence  $\gamma \leq \pi/2$ . See Figure 1.

If  $\alpha \geq \eta/2$ , then we are done since  $\eta/2 \geq D^{-1} \sin(\eta/2)$ . Otherwise, let  $v_0$  be the vertex before  $v_1$  on the face (if the face is a triangle, then  $v_0 = v_3$ ). Let  $\beta'$  be the angle  $\angle v_0 v_1 v_2$  so that  $\beta' \geq \beta$ , and by (b) we have  $\beta' \leq \pi - \eta$ . Let  $x \in \mathbb{R}^2$  be the meeting point of the ray emanating from  $v_1$  toward  $v_0$  and the ray emanating from  $v_2$  toward  $v_3$  (since  $\alpha + \beta' < \pi$  these rays must intersect and the intersection point  $x$  must be on the same side of the infinite line through  $v_1, v_2$  as  $v_0$  and  $v_3$ ). Let  $\delta$  be the angle  $\angle v_1 x v_2$ . We have that  $\delta = \pi - \beta' - \alpha$  hence  $\eta/2 \leq \delta \leq \gamma \leq \pi/2$ . Hence, by the law of sines

$$\alpha \geq \sin(\alpha) = \frac{|v_1 x| \sin(\delta)}{|v_1 v_2|} \geq \frac{|v_1 v_0| \sin(\delta)}{|v_1 v_2|} \geq D^{-1} \sin(\eta/2),$$

where  $|v_1 x| \geq |v_1 v_0|$  since by convexity  $v_0$  and  $v_1$  are on the same side of the infinite line passing through  $v_2$  and  $v_3$ .  $\square$

LEMMA 2.2 (Sausage lemma). *There exists  $c = c(D, \eta) > 0$  such that if  $e, f$  are nonadjacent edges then  $d(e, f) \geq c|e|$ , where  $|e|$  and  $d(\cdot, \cdot)$  are Euclidean length and distance. In particular, any vertex  $u \in V \setminus e$  is of distance at least  $c|e|$  from  $e$ .*

PROOF. Write  $e = \{v, w\}$  and see Figure 2. Let  $v_1, v_2$  be two consecutive neighbors of  $v$ . Because the angle  $\angle v_1 v v_2$  is at most  $\pi - \eta$ , one of the angles  $\angle v_1 v_2 v$  or  $\angle v_2 v_1 v$  is at least  $\eta/2$  and by (c) both  $|v v_1|$  and  $|v v_2|$  are at least  $D^{-1}|e|$ . Hence, the distance between  $v$  and the line through  $v_1$  and  $v_2$  is at least  $D^{-1} \sin(\eta/2)|e|$ . We conclude that there are no points of  $X$  inside a ball around  $v$  of radius  $D^{-1} \sin(\eta/2)|e|$  except for the edges emanating from  $v$  and the same holds for  $w$ .



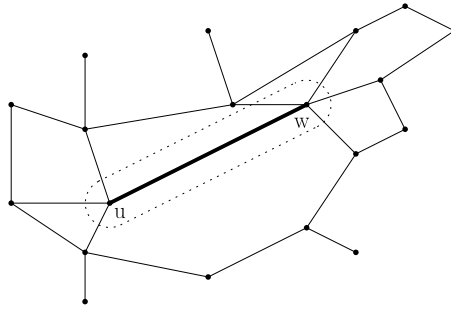


FIG. 2. *The sausage lemma: no edge can intersect the marked “sausage” with width  $c|e|$ .*

Next, consider one of the two faces containing  $e$  and let  $v_1$  be the neighbor of  $v$  in the face that is not  $w$  and similarly  $w_1$  be the neighbor of  $w$  in the face that is not  $v$  (if the face is a triangle, then  $w_1 = v_1$ ). By Lemma 2.1 the angles  $\angle v_1 v w$  and  $\angle v w w_1$  are at least  $D^{-1} \sin(\eta/2)$  and by condition (b) these angles are at most  $\pi - \eta$ . Hence, by condition (c), the face contains a trapezoid in which  $e$  is a base and the two sides are sub-intervals containing  $v$  and  $w$  of the edges  $(v, v_1)$  and  $(w, w_1)$ , respectively, and of height at least  $D^{-1}|e| \sin(D^{-1} \sin(\eta/2))$ .  $\square$

**COROLLARY 2.3.** *There exists  $c = c(D, \eta) > 0$  such that for any edge  $e$  and vertex  $u \notin e$  and  $r > 0$  we have that if  $e$  intersects  $B_{\text{euc}}(u, cr)$ , then  $e$  is contained in  $B_{\text{euc}}(u, r)$ .*

**LEMMA 2.4.** *There exists a constant  $C = C(D, \eta) < \infty$  such that if an edge  $e$  is contained in a face  $f$ , then  $\text{diam}(f) \leq C|e|$ , and  $C^{-1} \mathfrak{L}eb(f) \leq |e|^2 \leq C \mathfrak{L}eb(f)$ , where  $\mathfrak{L}eb(f)$  is the usual Lebesgue area measure in  $\mathbb{R}^2$ .*

**PROOF.** Since external angles in a polygon add up to  $2\pi$ , the number of sides of a face is at most  $2\pi/\eta$ , and since consecutive sides have length ratio at most  $D$ , any two sides of a face have ratio at most  $C$ , and the diameter of the face is at most some constant times the shortest edge.

The relation to the area of  $f$  follows from Lemma 2.1 and that edges adjacent to  $e$  have comparable lengths.  $\square$

Most of our arguments will take place in the metric space  $(X, d_0)$  defined as follows. For an edge  $(u, v) \in E$ , write  $[u, v]$  for the closed line segment in the plane from  $u$  to  $v$ . We put

$$X = \bigcup_{(u,v) \in E} [u, v],$$

and let  $d_0$  be the shortest path distance in  $X$ . For  $x \in X$  and  $r > 0$  we write  $B_{d_0}(x, r)$  for the ball  $\{y \in X : d_0(x, y) \leq r\}$ . An idea that we will use frequently

is to take a curve in  $\mathbb{R}^2$  with some useful properties and modify it slightly to get a curve in  $X$  with similar properties.

PROPOSITION 2.5. *Assume that  $\text{carr}(G) = U$ . There exists a constant  $C_1 = C_1(D, \eta)$  such that for any  $x, y \in X$  we have*

$$(2.1) \quad |x - y| \leq d_0(x, y) \leq C_1|x - y|.$$

PROOF. Since  $d_0(x, y)$  is the Euclidean length of the shortest path in  $X$  between  $x$  and  $y$  the inequality  $|x - y| \leq d_0(x, y)$  is obvious.

To prove the other inequality, we first prove the assertion for  $x$  and  $y$  that are on the same face. If  $x$  and  $y$  are on the same edge then  $d_0(x, y) = |x - y|$ . If  $x$  and  $y$  are on two different edges that share a vertex  $v$ , then since the angle at  $v$  is bounded away from 0 (by Lemma 2.1) we deduce by the law of sines on the triangle  $x, v, y$  that  $d_0(x, y) \leq |x - v| + |y - v| \leq C|x - y|$ . Lastly, when  $x$  and  $y$  are on two edges of the same face not sharing a vertex, Lemma 2.2 immediately gives that  $|x - y|$  is at least  $c|e|$ , where  $e$  is some edge that contains  $x$ . Lemma 2.4 gives that  $|e|$  is at least a constant multiple times the diameter of the face and the assertion follows.

Finally, when  $x$  and  $y$  are not on the same face let  $[x, y]$  be the straight segment connecting  $x$  and  $y$  and let  $x = x_0, x_1, \dots, x_k = y$  be the points on  $[x, y]$  where the segment intersects  $X$ , so that  $x_i$  and  $x_{i+1}$  are on the boundary of some face for all  $i = 0, \dots, k - 1$ . Then  $d_0(x_i, x_{i+1}) \leq C|x_i - x_{i+1}|$  and summing over  $i$  finishes the proof of the lemma.  $\square$

Consequently, the completion of  $(X, d_0)$  is  $\overline{X} = X \cup \partial U$  with the topology induced from  $\mathbb{R}^2$ , and (2.1) extends to the space  $\overline{X}$ .

LEMMA 2.6 ( $X$  is inner uniform). *Assume that  $G$  has a  $(D, \eta)$ -good embedding and that  $\text{carr}(G) = U$ . There exist constants  $C = C(D, \eta) < \infty$  and  $c = c(D, \eta) > 0$  such that for any  $\xi_1, \xi_2 \in \partial U$  with  $\xi_1 \neq \xi_2$  there exists a continuous curve  $\Gamma : [0, L] \rightarrow \overline{X}$  such that the following holds:*

- (1)  $\Gamma$  is parametrized by length, that is,  $\text{length}(\Gamma[0, t]) = t$  for all  $t \in [0, L]$ .
- (2)  $\Gamma(0) = \xi_1$  and  $\Gamma(L) = \xi_2$ .
- (3)  $L \leq C d_0(\xi_1, \xi_2)$ .
- (4) For any  $t \in (0, L)$  we have

$$d_0(\Gamma(t), \partial U) \geq c \min(t, L - t).$$

PROOF. Consider a circle orthogonal to  $U$  through  $\xi_1, \xi_2$ , and the continuous curve  $\gamma$  which is the arc from  $\xi_1$  to  $\xi_2$  in that circle. Let  $(\dots, x_{-2}, x_{-1}, x_0, x_1, x_2, \dots)$  be the set  $\gamma \cap X$  with the order induced by  $\gamma$ . Two consecutive points  $x_i$  and  $x_{i+1}$  are on the same face, so write  $\Gamma_i$  for a piecewise linear curve on the

boundary of that face connecting  $x_i$  to  $x_{i+1}$  in the shorter way according to  $d_0$ . Let  $\Gamma$  be the concatenation of  $(\Gamma_i)_{\{-\infty < i < \infty\}}$ , parametrized by arc length (which we shall see below is finite). Note that  $\Gamma$  might not be a simple curve, which does not cause any difficulty. Thus, (1) holds. Let us show that  $\Gamma$  satisfies requirements (2)–(4).

We first note that by Lemma 2.2 we have that there exists  $C > 0$  such that for any face  $f$

$$(2.2) \quad \max_{z \in \partial f} d_0(z, \partial U) \leq C \min_{z \in \partial f} d_0(z, \partial U),$$

where by  $z \in \partial f$  we mean that  $z \in X$  is on one of the edges encompassing  $f$ . Now, it is clear that  $x_k \rightarrow \xi_1$  when  $k \rightarrow -\infty$  and  $x_k \rightarrow \xi_2$  when  $k \rightarrow \infty$ , so by (2.2) we get that  $\Gamma$  satisfies requirement (2).

Next, we have that  $\text{length}(\Gamma_i) = d_0(x_i, x_{i+1})$  since  $x_i$  and  $x_{i+1}$  are on the same face  $f$ , and since the shortest curve between two points on the boundary of a convex face  $f$  that does not enter the face is along its boundary. Thus, we have  $\text{length}(\Gamma) = \sum_i d_0(x_i, x_{i+1}) \leq C|\xi_1 - \xi_2|$  by Proposition 2.5, so requirement (3) holds. Lastly, (4) holds immediately for the points  $x_k$ , and by (2.2) we obtain this for any point on  $\Gamma$ .  $\square$

**3. The cable process.** It will be convenient for us to obtain useful estimates using results of Sturm [28]. For that, we need to introduce the *cable process* which can be thought of as Brownian motion on the embedding of  $G$ . (See, e.g., [29].) Recall that  $(w_e)$  are edge weights on  $G$ , which are bounded above and away from 0. An intuitive description of the process is as follows: Let  $x$  be a vertex and  $e_1, \dots, e_k$  the edges emanating from it, and let  $\{W_t\}_{t \geq 0}$  be standard Brownian motion. It is well known that  $W_t$  can be decomposed into countably many excursions in which  $W_t \neq 0$ . For each such excursion, we choose the edge  $e_i$  with probability proportional to  $w_{e_i}|e_i|$  for  $i = 1, \dots, k$  and embed the excursion on the edge  $e_i$ . We stop when we hit one of the neighbors  $x_1, \dots, x_k$  of  $x$ , and continue from this neighbor using the strong Markov property. Thus, this process is a standard Brownian motion on the edges (or “cables”), and behaves like a Walsh Brownian motion (see [30]) at the vertices.

Note that it is possible for this process to “explode”, or visit infinitely many vertices in finite time, and indeed this does happen almost surely in the transient setting.

Before defining the process formally, let us state two useful properties that will make the connection to the discrete time weighted random walk evident. Denote by  $Z_t$  the process and let  $T$  be the hitting time of  $\{x_1, \dots, x_k\}$ . Then for  $1 \leq i \leq k$ , we have (see [13], Theorem 2.1)

$$(3.1) \quad \mathbb{P}_x(Z_T = x_i) = \frac{w_{e_i}}{\sum_{i=1}^k w_{e_i}},$$

that is, the process  $Z_t$  observed on vertices has the same trace as the simple random walk (we ignore the uncountably many times it visits each  $x$  before proceeding to one of its neighbors). Also, in our setting, there exists a constant  $C = C(D, \eta) > 0$  such that (see [13], Theorem 2.2)

$$(3.2) \quad C^{-1} \leq \frac{\mathbb{E}_x T}{r_x^2} \leq C,$$

where  $r_x$  is the length of the shortest edge touching  $x$  (and so is comparable to the length of any edge touching  $x$ ). Intuitively, the process  $Z_t$  behaves like the variable speed random walk that waits roughly  $r_x^2$  time at vertex  $x$  before proceeding.

The construction based on excursions can be made precise. However, it is easier to define the cable process via the methods of Dirichlet forms (see [14]). Let  $(\bar{X}, d_0)$  be the compact metric space defined in Section 2. For an edge  $(u, v)$ , write  $dx$  for Lebesgue measure on  $[u, v]$ , and define a measure  $m$  on  $\bar{X}$  by taking

$$m(dx) = \sum_{(u,v) \in E} |u - v| w_{uv} dx.$$

LEMMA 3.1. *We have  $m(\bar{X}) < \infty$ .*

PROOF. By Lemma 2.4, the measure of any edge  $e$  is less than  $C \mathfrak{L}eb(f_1 \cup f_2)$ , where  $f_1, f_2$  are the two faces containing  $e$ . Since  $\text{carr}(G) = U$ , it follows that  $m(\bar{X}) \leq 2C \mathfrak{L}eb(U)$ .  $\square$

We say that a function  $f$  on  $\bar{X}$  is piecewise differentiable if it is continuous at each vertex, and is differentiable w.r.t. the length measure on every edge. (We require that the one sided derivatives exist at the end of each edge.) The derivative  $f'$  depends on the direction, and only makes sense if we fix a direction for every edge. However,  $f'g'$  is well defined for differentiable  $f, g$  and does not depend on choosing a direction on the edges. Let  $\mathcal{D}_0$  be the space of differentiable functions  $f$  on  $\bar{X}$  such that  $f'$  is continuous on each edge, and  $|f'|$  is bounded. For  $f, g \in \mathcal{D}_0$ , let

$$d\Gamma(f, g)(dx) = \sum_{(u,v) \in E} 1_{[u,v]}(x) f'(x) g'(x) |u - v| w_{uv} dx,$$

$$\mathcal{E}(f, g) = \sum_{(u,v)} \int_{[u,v]} f'(x) g'(x) |u - v| w_{uv} dx = \int_{\bar{X}} d\Gamma(f, g).$$

Let  $\mathcal{D}$  be the completion of  $\mathcal{D}_0$  with respect to the norm

$$\|f\|_{\mathcal{E}_1} = \left( \mathcal{E}(f, f) + \int_{\bar{X}} f^2 dm \right)^{1/2}.$$

It is straightforward to verify that the bilinear form  $(\mathcal{E}, \mathcal{D}_0)$  is closed and Markov (see [14], page 4 and [9], Section 2.2), so that  $(\mathcal{E}, \mathcal{D})$  is a Dirichlet form. Since

$\mathcal{D}_0$  is dense in  $C(\overline{X})$ , the continuous functions on  $\overline{X}$ ,  $(\mathcal{E}, \mathcal{D})$  has a core and is thus a regular Dirichlet form. The associated strong Markov process  $Z$  is the cable process on  $\overline{X}$ . We remark that with this construction the functions  $f$  in the domain  $\mathcal{D}$  do not vanish on the boundary  $\partial U$ , so that the process  $Z$  is conservative and will reflect from the boundary after its first hit.

The space  $(\overline{X}, m, \mathcal{E})$  has an intrinsic metric associated with it (see [28]), where the distance between  $x, y$  is given by

$$\sup\{f(x) - f(y) : f \in \mathcal{D} \text{ and } |f'| \leq 1\}.$$

In our case, it is clear that this metric coincides with  $d_0$  defined above. We will show that this space is doubling and has a weak Poincaré inequality.

### 3.1. Doubling.

LEMMA 3.2. *Let  $X$  be a good embedding of some graph. There exists a integer  $M = M(D, \eta) > 0$  such that for any  $x \in X$  and any  $r > 0$  there exists  $x_1, \dots, x_M \in X$  such that*

$$B_{d_0}(x, 2r) \subset \bigcup_{i=1}^M B_{d_0}(x_i, r).$$

PROOF. This is an easy consequence of the equivalence between  $d_0$  and the Euclidean metric. We have that  $B_{d_0}(x, 2r) \subset B_{\text{euc}}(x, 2r)$ . Let  $C_1$  be the constant from Proposition 2.5. The Euclidean ball can be covered by  $M$  Euclidean balls  $B_{\text{euc}}(y_i, r/2C_1)$  of radius  $r/2C_1$  for some  $M \asymp C_1^2$ . For each  $i$ , if  $B_{\text{euc}}(y_i, r/2C_1)$  intersects  $X$  and  $x_i$  is an arbitrary point in the intersection then  $B_{\text{euc}}(y_i, r/2C_1) \cap X \subset B_{d_0}(x_i, r)$ . Otherwise, we ignore this ball. So the collection of balls  $B_{d_0}(x_i, r)$  covers  $B_{d_0}(x, 2r)$ .  $\square$

Recall the radius of isolation  $r_u$  defined for  $u \in V$  as the distance to the nearest vertex  $v \neq u$ . We extend this to  $x \in X$  by letting  $r_x$  be the length of the edge containing  $x$  when  $x \in X \setminus V$ , and setting  $r_x = 0$  if  $x \in \partial U$ . Note that if  $r \in (0, 1)$  and  $x \in \overline{X}$  then we have  $\mathcal{L}eb(B_{\text{euc}}(x, r)) \geq cr^2$ .

LEMMA 3.3. *For any  $x \in \overline{X}$  and  $r \in (0, 1)$ , we have*

$$(3.3) \quad m(B_{d_0}(x, r)) \asymp r \cdot (r \vee r_x).$$

*In particular, there exists a constant  $C > 0$  such that for  $r > 0$*

$$m(B_{d_0}(x, 2r)) \leq Cm(B_{d_0}(x, r)).$$

PROOF. First, we assume that  $x$  is a vertex. When  $r \leq r_x$ , we have that  $m(B_{d_0}(x, r)) \asymp r_x r$ , because the degrees of  $G$  are bounded and adjacent edges have comparable length. So it suffices to prove that  $m(B_{d_0}(x, r)) \asymp r^2$  when  $r \geq r_x$ .

By Lemma 2.4, for an edge  $e$  we have  $m(e) \asymp |e|^2 \leq C \mathfrak{L}eb(f)$  where  $f$  is a face containing  $e$ . Since each face has a bounded degree, and since faces intersecting  $B_{d_0}(x, r)$  are fully contained in  $B_{\text{euc}}(x, Cr)$  we find  $m(B_{d_0}(x, r)) \leq Cr^2$ , where  $C = C(D, \eta) < \infty$ . For the lower bound, if  $r_x \leq r \leq Cr_x$  then  $m(B_{d_0}(x, r)) \geq r_x^2 \geq C^{-2}r^2$ . If  $r > Cr_x$  by Corollary 2.3 and Proposition 2.5, the union of the set of faces  $f$  adjacent to edges  $e$  contained in  $B_{d_0}(x, cr)$  contains  $U \cap B_{\text{euc}}(x, c^2r)$ , giving the required lower bound.

If  $x \in X$  is not a vertex, then let  $y$  be a closest vertex to  $x$ , and let  $s = d_0(x, y)$ . If  $r \leq s$  then  $m(B_{d_0}(x, r)) = 2rr_x$ , which equals  $2r(r \vee r_x)$  since  $r_x \geq s$ . Now suppose that  $r > s$ . Then  $B_{d_0}(x, r) \subset B_{d_0}(y, 2r)$ , and since  $r_y \asymp r_x$ , this gives the upper bound in (3.3). If  $r \in [s, 2s]$ , then  $m(B_{d_0}(x, r)) \geq \frac{1}{2}rr_x$ , while if  $r > 2s$  then  $B_{d_0}(y, r/2) \subset B_{d_0}(x, r)$ , so in either case we have the lower bound.

Finally, if  $x \in \partial U$  and  $r > 0$  then we can find a vertex  $y$  such that  $d_0(x, y) < r/2$  and  $r_y < r/2$ , and use the bounds for  $m(B_{d_0}(y, \cdot))$ .  $\square$

3.2. *Poincaré inequality.* The strong Poincaré inequality states that for any differentiable  $f$  on  $B = B_{d_0}(x, r)$  we have

$$(3.4) \quad \int_B |f(x) - \bar{f}|^2 m(dx) \leq Cr^2 \int_B |f'(x)|^2 m(dx),$$

where  $\bar{f} = \frac{1}{m(B)} \int_B f m(dx)$  is the mean of  $f$ . A well-known technique due to Jerison (see [17], Section 5 and also [26], Section 5.3, for a simpler proof) shows that for spaces satisfying the doubling property, this follows from the weak Poincaré inequality which we now prove.

THEOREM 3.4 (Weak Poincaré inequality). *There exist positive constants  $C = C(D, \eta)$  and  $C' = C'(D, \eta)$  such that for any  $x_0 \in X$  and  $r \in (0, 1)$  with  $B_{\text{euc}}(x_0, Cr) \subset \text{carr}(G)$ , and all  $f$  piecewise differentiable on  $B_{d_0}(x_0, Cr)$  we have*

$$\int_{B_{d_0}(x_0, r)} |f(x) - \bar{f}|^2 m(dx) \leq C'r^2 \int_{B_{d_0}(x_0, Cr)} |f'(x)|^2 m(dx).$$

We begin with the following lemma.

LEMMA 3.5. *Let  $x$  have law  $\frac{m(dx)}{m(B)}$  on some set  $B \subset X$ , and conditioned on  $x$  let  $\hat{x}$  be uniform in the union of the two faces incident to the edge containing  $x$ . Then there is some constant  $C = C(D, \eta)$  so that the law of  $\hat{x}$  is bounded by  $\frac{C \mathfrak{L}eb}{m(B)}$ , where  $\mathfrak{L}eb$  is the usual Lebesgue measure on  $\mathbb{R}^2$ .*

PROOF. For an edge  $e$  of  $X$ , we have that  $\mathbb{P}(x \in e) = \frac{m(e \cap B)}{m(B)} \leq \frac{|e|^2}{m(B)}$  (with equality holding when  $e \subset B$ ). If  $e$  is incident to some face  $f$  then the conditional contribution to the density of  $\hat{x}$  in  $f$  is at most  $1/\mathcal{L}\epsilon\mathfrak{b}(f)$ , and so the density on a face  $f$  surrounded by edges  $e_1, \dots, e_k$  is at most  $\frac{1}{m(B)} \sum \frac{|e_i|^2}{\mathcal{L}\epsilon\mathfrak{b}(f)}$ . The number of edges surrounding a face is at most  $2\pi/\eta$ , and the square of each is comparable to the area of  $f$ , giving the claim.  $\square$

PROOF OF THEOREM 3.4. We let  $B = B_{d_0}(x_0, r)$ . Let  $x, y$  be independent points chosen in  $B$  with law  $\frac{m(dx)}{m(B)}$ . We start with the simple identity

$$\int_B |f(x) - \bar{f}|^2 m(dx) = \frac{m(B)}{2} \mathbb{E} |f(x) - f(y)|^2,$$

that follows from expanding. Let  $\gamma = \gamma_{xy}$  be some (possibly random) path in  $X$  between  $x$  and  $y$ , then  $f(y) - f(x) = \int_\gamma f'(z) dz$ , where  $dz$  is the length element along  $\gamma$ . Applying Cauchy–Schwarz gives

$$|f(x) - f(y)|^2 \leq |\gamma| \int_\gamma |f'(z)|^2 dz,$$

where  $|\gamma|$  denotes the length of  $\gamma$ .

To construct the path, we need to consider two cases. If  $B_{d_0}(x_0, r)$  contains a single vertex of the graph, then the graph in  $B_{d_0}(x_0, r)$  is a star, and there is an obvious choice of path  $\gamma_{xy}$ . If there are at least two vertices, we proceed as follows. Let  $\hat{x}$  (resp.,  $\hat{y}$ ) be uniformly chosen in the union of the two faces of  $X$  incident to  $x$  (resp., to  $y$ ). The straight line segment  $\hat{x}\hat{y}$  begins at a face containing  $x$ , ends at a face containing  $y$ , and possibly passes through some other faces in between. As in the proof of Proposition 2.5, we can approximate this line segment by a path  $\gamma_{xy}$  in  $X$ , which stays in the boundaries of faces crossed by the line segment.

We first observe that due to Lemma 2.2 we have that  $\hat{x}, \hat{y} \in B_0 := \{u \in \mathbb{R}^2 : |u - x_0| < C_0 r\}$  for some  $C_0 = C_0(D, \eta) \geq 1$ , and  $B_0 \subset \text{carr}(G)$  by our assumptions. By increasing  $C_0$ , we can guarantee that  $\gamma_{xy}$  also does not leave  $B_0$ , and as in Proposition 2.5, we have  $|\gamma_{xy}| \leq Cr$ . We shall see below that  $\mathbb{P}(z \in \gamma_{xy}) \leq \frac{C\rho_z}{r}$  where  $\rho_z$  is the length of the edge containing  $z$  (we neglect the measure 0 set of vertices). Given that we conclude the proof as follows:

$$\begin{aligned} \int_B |f(x) - \bar{f}|^2 m(dx) &\leq Cm(B) \mathbb{E} \left[ r \int_\gamma |f'(z)|^2 dz \right] \\ &\leq Crm(B) \int_{z \in B_0} \frac{C\rho_z}{r} |f'(z)|^2 dz \\ &\leq Cr^2 \int_{B_0} |f'(z)|^2 m(dz), \end{aligned}$$

since  $\rho_z dz = m(dz)$ , and  $m(B) \leq Cr^2$ .

To bound the probability that  $z \in \gamma_{xy}$ , note that the faces incident to  $z$  are contained in  $\{u \in \mathbb{R}^2 : |u - z| \leq C_1 \rho_z\}$ . Let  $A$  be the event that the segment  $\hat{x}\hat{y}$  intersects a face incident to  $z$ , and  $A'$  the event that the segment passes within distance  $C_1 \rho_z$  of  $z$ , so that  $A \subset A'$ . Let  $\hat{m}$  be the law of  $\hat{x}$  and  $\hat{y}$ . By Lemma 3.5, we have that

$$\hat{m} \leq \frac{C}{m(B)} \mathcal{L}eb \leq \frac{C_2}{r^2} \mathcal{L}eb,$$

and  $\text{supp } \hat{m} \subset B_0 \subset \{|\hat{x} - z| \leq 2C_0 r\}$ . We have now

$$\begin{aligned} \mathbb{P}(z \in \gamma_{xy}) &\leq \mathbb{E}1_A \leq \mathbb{E}1_{A'} = \iint 1_{A'} d\hat{m} \times d\hat{m} \\ &\leq \int_{|\hat{x}-z| \leq 2C_0 r} \int_{|\hat{y}-z| \leq 2C_0 r} 1_{A'} \left(\frac{C_2}{r^2}\right)^2 d\hat{x} d\hat{y}. \end{aligned}$$

By scaling and translating this is  $(C_2/2C_0)^2$  times the probability that the segment between two uniform points  $u, v \in U$  passes within  $C_1 \rho_z/r$  of the origin. For such  $u, v$ , the distance between the segment and the origin is a continuous random variable with finite density, so the distance is at most  $C_1 \rho_z/r$  with probability at most  $C \rho_z/r$ .  $\square$

3.3. *Heat kernel estimates.* Finally, we are able to deduce estimates for the heat kernel of the cable process on  $X$ . Let  $q_t(x, y)$  denote the heat kernel for the Markov process  $\{Z_t\}_{t \geq 0}$  associated with  $(\bar{X}, m, \mathcal{E})$ , that is,  $q_t(x, \cdot)$  is the density (with respect to  $m$ ) of  $Z_t$  conditioned on  $Z_0 = x$ . For a set  $A \subset \bar{X}$ , we let  $q_t^A(x, y)$  denote the heat kernel for the process killed when it exits  $A$ . (If  $A = \bar{X}$ , then  $q^A$  is just the unkilled heat kernel.)

**THEOREM 3.6.** *There exists constants  $c, C$  depending only on  $D, \eta$  such that for any  $x_0 \in X$  and  $r > 0$  we have that for any  $t \leq r^2$  and  $x, y \in X \cap B_{\text{euc}}(x_0, \sqrt{t})$ ,*

$$q_t^A(x, y) \geq \frac{c}{m(B_{\text{euc}}(x_0, \sqrt{t}))},$$

where  $A = \bar{X} \cap B_{\text{euc}}(x_0, Cr)$ .

**PROOF.** This is obtained by combining Theorem 3.5 of [28] with (3.4) and Lemma 3.3 (giving parabolic Harnack inequality), and then appealing to Theorem 3.2 in [5] [the assertion that (c) implies (b) is what we use with the function  $\tau(t) = t^2$ ]. Finally, using Proposition 2.5 to move from balls in  $d_0$  to Euclidean balls.  $\square$



**4. Harmonic measure and exit time of discrete discs.** In this section, we prove Theorems 1.4 and 1.5. Let  $G = (V, E)$  be a planar graph with a  $(D, \eta)$ -good embedding and let  $(\bar{X}, d_0)$  be the associated metric space. We consider the cable process  $Z$  on  $G$  defined in Section 3. We slightly abuse notation, and use  $\tau_A$  to denote the hitting time of  $A$  by the cable process, that is,  $\tau_A = \inf\{t : Z_t \in A\}$ . Recall that the restriction of  $Z_t$  to  $V$  is the simple random walk, and so when  $A \subset V$ , the law of  $X_{\tau_A}$  is the same for the cable process and for the simple random walk.

For  $u \in \mathbb{R}^2$ , radius  $r > 0$  and an interval of angles  $I \subset \mathbb{R}/(2\pi\mathbb{Z})$  let  $\text{Cone}(u, r, I)$  denote the intersection of  $X$  and the cone of radius  $r$  centered at  $u$  with opening angles  $I$ , that is,

$$\text{Cone}(u, r, I) = \{v \in X : |v - u| \leq r \text{ and } \arg(v - u) \in I\}.$$

A *wide cone* is a cone where  $|I| \geq \pi - \eta$ . By definition, if  $u$  is a vertex in a good embedding then there is an edge containing  $u$  entering every wide cone with tip at  $u$ .

LEMMA 4.1. *For any vertex  $u \in V$  and any wide cone  $A = \text{Cone}(u, r, I)$  such that  $B_{\text{euc}}(u, r) \subset \text{carr}(G)$ , we have*

$$m(A) \asymp r(r \vee r_u).$$

PROOF. The case  $r \leq r_u$  is easy and we omit the details. Assume  $r \geq r_u$  and write  $A'$  for the Euclidean cone  $A' = \{x \in \mathbb{R}^2 : |x - u| \leq r \text{ and } \arg(x - u) \in I\}$ . For any face  $f$ , we will show that  $m(A \cap \partial f) \geq c \mathfrak{Leb}(A' \cap f)$  for some  $c(D, \eta)$ . This implies the lower bound, since summing over all faces gives  $m(A) \geq cr^2$ .

Let  $\ell = \text{diam}(A' \cap f) \leq \text{diam}(f)$ , and note that every edge of  $f$  has length at least  $c\ell$  for some  $c$ . Consider the circle  $C_s = \{z : |z - u| = s\}$ . We have that the length  $|C_s \cap f \cap A'|$  is at most  $C\ell$ , and is nonzero for  $s$  in some interval  $J$ . Integrating over  $s$  gives

$$\mathfrak{Leb}(A' \cap f) = \int_J |C_s \cap f \cap A'| \leq C\ell|J|.$$

We next argue that  $\partial f$  must cross inside  $A$  any circle  $C_s$  that intersects  $f \cap A'$ . To see this, note that we can construct a path from  $u$  taking only edges with directions in  $I$  until we exit  $A$  after finitely many steps [since  $A \subset \text{carr}(G)$ ]. The face  $f$  is restricted to one side of the path, and so  $\partial f$  intersects  $A \cap C_s$ . It now follows that the length of  $A \cap \partial f$  is at least  $|J|$ , and since the length of edges of  $f$  is at least  $c\ell$  we get  $m(A \cap \partial f) \geq c\ell|J|$ , and the lower bound follows.

For the upper bound, we prove only the case  $r \geq r_u$ , as the other is immediate. By Lemma 2.2 every edge intersecting  $A$  has length at most  $Cr$ , and all incident faces are contained in  $B_{\text{euc}}(u, C'r)$ . For any such edge  $e$ , taking all of  $m(e)$  still gives at most the area of the faces containing  $e$ , and since each face is counted a bounded number of times, the claim follows by summing over the edges.  $\square$

COROLLARY 4.2. *There exists a constant  $c = c(D, \eta) > 0$  such that for any vertex  $u \in V$  and any  $r \geq r_u$  with  $B_{\text{euc}}(u, r) \subset \text{carr}(G)$ , and any wide cone  $A = \text{Cone}(u, r, I)$  we have*

$$m(A \setminus B_{\text{euc}}(u, cr)) \geq cr^2.$$

LEMMA 4.3. *There exist constants  $c = c(D, \eta) > 0$  such that for any interval  $I$  with  $|I| = \pi - \eta$ , any vertex  $u \in V$  and any  $r \geq r_u$  satisfying  $B_{\text{euc}}(u, r) \subset \text{carr}(G)$  we have*

$$\mathbb{P}_u(\tau_S < \tau_{V \setminus V_{\text{euc}}(u, 2r)}) \geq c,$$

where

$$S = V \cap \text{Cone}(u, r, I) \setminus B_{\text{euc}}(u, cr).$$

PROOF. Write  $C$  for the unbounded Euclidean cone  $\{x \in \mathbb{R}^2 : \arg(x - u) \in I\}$  and construct an infinite simple path  $P$  from  $u$  that remains in  $C$ , as we did in the previous lemma. The existence of  $P$  implies that any edge  $e$  that intersects  $C$  must have at least one endpoint in  $C$ .

Write  $c < 1$  for the smaller of the constants in Corollaries 2.3 and 4.2. Let  $B \subset X \cap C$  be constructed as follows: consider an edge  $e$  that intersects  $C \cap B_{\text{euc}}(u, cr) \setminus B_{\text{euc}}(u, c^2r)$  and does not contain  $u$ ; if  $e$  is entirely contained in  $C$ , then we add  $e$  to  $B$ , otherwise  $e = (v_1, v_2)$  where only  $v_1$  is in  $C$  and we add to  $B$  the straight line segment between  $v_1$  and  $(v_1 + v_2)/2$  (i.e., half the edge  $e$ , starting at  $v_1$ ). We have that  $m(B) \geq \frac{1}{2}m(X \cap C \cap B_{\text{euc}}(u, cr) \setminus B_{\text{euc}}(u, c^2r))$  since for any edge  $e$  that intersects  $C \cap B_{\text{euc}}(u, cr)$  we added to  $B$  at least half of  $e \cap C$ . Hence, by Corollary 4.2 we get that  $m(B) \geq c^3r^2/2$ .

We now appeal to Theorem 3.6 with  $x_0 = u$  and  $t = r^2$  and integrate over  $y \in B$  to get that

$$\mathbb{P}_u(Z_t \in B \text{ and } t < \tau_{\partial B_{\text{euc}}(u, 2r)}) \geq c' > 0,$$

for some constant  $c' = c'(D, \eta) > 0$ . By Corollary 2.3, we have that  $B \cap V \subset X \cap C \cap V_{\text{euc}}(u, r) \setminus V_{\text{euc}}(u, c^3r)$  and since we added either full edges or half edges, it is clear that starting from any point in  $B$ , the probability that the first vertex that we visit is in  $B \cap V$  is at least  $1/2$ . This completes our proof.  $\square$

LEMMA 4.4. *For any  $\varepsilon > 0$ , there exists  $c = c(\varepsilon, D, \eta) > 0$  such that for any vertex  $u \in V$  and any  $r \geq r_u$  satisfying  $B_{\text{euc}}(u, r) \subset \text{carr}(G)$ , and any interval  $I$  with  $|I| = \pi - \eta$  we have*

$$\mathbb{P}_u(\tau_S < \tau_O) > c,$$

where

$$S = V \cap \text{Cone}(u, \infty, I) \setminus V_{\text{euc}}(u, r)$$

and

$$O = \{v \in V : d(v, \text{Cone}(u, \infty, I)) \geq \varepsilon r\}.$$

PROOF. We iterate  $2(c\varepsilon)^{-1}$  times Lemma 4.3 with a cone of radius  $r' = \varepsilon r/2$  and opening  $I$ .  $\square$

LEMMA 4.5. *For any  $\varepsilon > 0$ , there exists  $c = c(\varepsilon, D, \eta) > 0$  such that for any vertex  $u \in V$ , any  $r \geq r_u$  satisfying  $B_{\text{euc}}(u, r) \subset \text{carr}(G)$ , any interval  $I$  with  $|I| = \pi - \eta$ , and any vertex  $v$  such that  $\varepsilon r \leq |u - v| \leq (1 - \varepsilon)r$ , and  $\arg(v - u) \in I$  we have the following. Let*

$$S = V \cap \text{Cone}(u, \infty, I) \setminus V_{\text{euc}}(u, r)$$

and

$$Q = V \setminus (\text{Cone}(u, \infty, I) \cup V_{\text{euc}}(u, r)),$$

then  $\mathbb{P}_v(\tau_S < \tau_Q) \geq c$ .

PROOF. Write  $C$  for the Euclidean cone  $\{x \in \mathbb{R}^2 : \arg(x - u) \in I\}$ . If the distance of  $v$  from  $\mathbb{R}^2 \setminus C$  is at least  $\varepsilon r$ , then we apply Lemma 4.4 on the cone parallel to  $C$  emanating from  $v$  and the assertion follows. Assume now the opposite, and write  $R_1, R_2$  for the two rays of the cone  $C$  so that  $R_1$  is before  $R_2$  clockwise and assume without loss of generality that  $v$  is closer to  $R_1$ . Let  $C'$  be the cone

$$C' = \{x \in \mathbb{R}^2 : \arg(x - v) \in I - \alpha\},$$

where  $\alpha = \alpha(\varepsilon) > 0$  is the largest number so that  $d(u, C' \setminus C) \geq (2c^{-1} + 2)r$  where  $c > 0$  is the constant from Corollary 2.3 (see Figure 3). Define the set  $O'$  by

$$O' = \{v \in V : d(v, V \cap C' \cap V_{\text{euc}}(v, 2r)) \geq \varepsilon' r\},$$

where  $\varepsilon' = \varepsilon'(\varepsilon, \alpha) > 0$  is chosen so that  $(V \setminus O') \setminus V_{\text{euc}}(u, r) \subset C$ .

We now apply Lemma 4.4 with  $\varepsilon', v$  and  $C'$  to obtain that with probability uniformly bounded below we visit  $(V \cap C') \setminus V_{\text{euc}}(v, 2r)$  before visiting  $O'$ . By Corollary 2.3, when this event occurs the length of the last edge traversed has length at most  $2c^{-1}r$ . Hence, by our choice of  $\alpha$  in this last step we find ourselves in  $S$  and by our choice of  $\varepsilon'$  we have not stepped outside of  $C \cup V_{\text{euc}}(u, r)$ , as required.  $\square$

PROOF OF THEOREM 1.4. Let  $\varepsilon = \varepsilon(D, \eta) > 0$  be a fixed small number to be chosen later. Let us consider several cases. Recall that by condition (a) there always exist an edge  $(u, v)$  such that  $\arg(v - u) \in I$ . First, if there exists such an edge  $(u, v)$  with  $|u - v| > r$ , then with probability at least  $D^{-3}$  we take this edge in the first step and we are done. Second, if there is such an edge so that  $(1 - \varepsilon)r \leq |u - v| \leq r$ , then as long as  $\varepsilon$  is small with respect to  $D$ , then by condition (b)  $v$  has a neighbor  $w$  such that  $\arg(w - u) \in I$  and  $|w - u| > r$ , so with probability at least  $D^{-6}$  we take two steps from  $u$  to  $w$  and we are done. Third, if there exists such an edge so that  $\varepsilon r \leq |u - v| \leq (1 - \varepsilon)r$  then  $\mathbb{P}_u(X_{T_r} \in S) \geq D^{-3}\mathbb{P}_v(X_{T_r} \in S)$

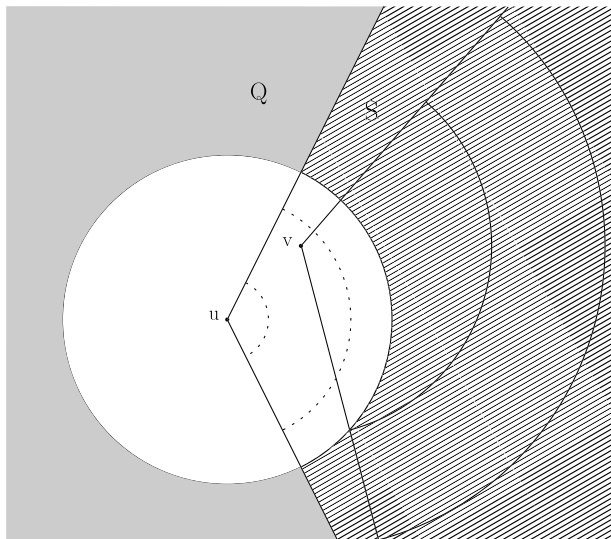


FIG. 3. Illustration of Lemma 4.5. The probability from  $v$  of hitting  $S$  before  $Q$  cannot be too small. Also shown: the possible locations for  $v$ , the rotated cone  $C'$  and the set likely to be hit from  $v$  by Lemma 4.4.

where  $S$  is defined in Lemma 4.5, and by that lemma the last quantity is uniformly bounded below and we are done. Lastly, if all neighbors  $v$  of  $u$  satisfy  $|u - v| \leq \varepsilon r$ , then we apply Lemma 4.4 with radius  $\varepsilon r$  and obtain that with probability uniformly bounded from below we visit  $V \cap \text{Cone}(u, \varepsilon r, I) \setminus V_{\text{euc}}(u, \varepsilon r)$  before visiting  $O = \{v \in V : d(v, \text{Cone}(u, \varepsilon r, I)) \geq \varepsilon^2 r\}$ . When this occurs, the last edge taken by the random walk has length at most  $c^{-1} \varepsilon r$  by Corollary 2.3, where  $c > 0$  is the constant of that lemma. Hence, if  $\varepsilon$  is chosen so that  $\varepsilon \leq (c^{-1} + 2)^{-1}$  we get that at that hitting time we are at a vertex  $v$  such that  $\varepsilon r \leq |u - v| \leq (1 - \varepsilon)r$  and  $\arg(v - u) \in I$ . The assertion of the theorem now follows by another application of Lemma 4.5, completing the proof.  $\square$

PROOF OF THEOREM 1.5. Let  $T_r^Z$  denote the exit time from  $B_{\text{euc}}(u, r)$  of the cable process and  $T_r$  the exit time from  $V_{\text{euc}}(u, r)$  for the simple random walk. Our first goal is to prove that  $\mathbb{E}_u T_r^Z \asymp r^2$ . We begin with the lower bound. To that aim, let  $C_{3.6}$  be the constant from Theorem 3.6 and let  $A = B_{\text{euc}}(u, C_{3.6}r)$  and assume that  $A \subset \text{carr}(G)$ . We apply Theorem 3.6 with  $t = r^2$  and integrate over  $y \in B_{\text{euc}}(u, r)$  to get that  $\mathbb{P}(T_{C_{3.6}r}^Z \geq r^2) \geq c$ , hence  $\mathbb{E}_u T_{C_{3.6}r}^Z \geq cr^2$  and so  $\mathbb{E}_u T_r^Z \geq c'r^2$  for some constant  $c' > 0$ .

To show the upper bound, Lemma 3.3 immediately implies that there exists some constant  $C_{3.3} > 0$  such that for any  $r \geq r_u$  and any  $x \in X \cap B_{\text{euc}}(u, r)$  we have

$$m(X \cap B_{\text{euc}}(x, C_{3.3}r) \setminus B_{\text{euc}}(u, r)) \geq r^2.$$

We prove the theorem with  $C = C_{3.3}C_{3.6} + 1$ . We apply Theorem 3.6 with  $A = B_{\text{euc}}(x, C_{3.3}C_{3.6}r)$  [so that  $A \subset \text{carr}(G)$ ] and  $t = r^2$  and integrate over  $y \in X \cap B_{\text{euc}}(x, C_{3.3}r) \setminus B_{\text{euc}}(u, r)$  to get that for any  $x \in X \cap B_{\text{euc}}(u, r)$  we have  $\mathbb{P}_x(T_r \geq r^2) \leq 1 - c$ , for some constant  $c > 0$ . Hence,  $\mathbb{E}_u T_r^Z \leq C'r^2$  for some  $C' > 0$ .

We got that  $\mathbb{E}_u(T_r^Z) \asymp r^2$ . Recall that the trace of the cable process along the vertices is distributed as the discrete weighted random walk. Hence, by writing  $T_r^Z$  as a the sum of possible random walks paths and the time it takes the cable process to traverse between vertices we obtain using (3.2) that

$$\mathbb{E}_u(T_r^Z) \asymp \mathbb{E} \sum_{t=0}^{T_r} r^2_{X_t},$$

where  $\{X_t\}$  is the discrete weighted random walk.  $\square$

**5. The Martin boundary.** It will be convenient to approximate the graph  $G$ , embedded in the plane with carrier  $U$  by finite subgraphs  $G_\varepsilon$ . For  $\varepsilon > 0$ , consider the subgraph  $G_\varepsilon$  induced by the vertices  $V_\varepsilon$  where

$$V_\varepsilon = \{v \in V : |v| \leq 1 - \varepsilon\}.$$

For two vertices  $a, z$  in a finite weighted graph we write  $R_{\text{eff}}(a, z)$  for the effective electrical resistance between  $a$  and  $z$  (for a definition and introduction to electrical resistance, see [22]). For disjoint sets  $A, Z$  of vertices, we write  $R_{\text{eff}}(A, Z)$  for the electrical resistance between  $A$  and  $Z$  in the graph obtained by contracting  $A$  and  $Z$  to two vertices.

LEMMA 5.1. *There exists  $c = c(D, \eta) > 0$  such that for any  $r > 0$  and  $\varepsilon \leq r/10$  and any  $\xi \in \partial U$  we have the resistance bound*

$$R_{\text{eff}}^{(\varepsilon)}(V_{\text{euc}}(\xi, r), V_{\text{euc}}(\xi, 2r)^c) \geq c,$$

where  $R_{\text{eff}}^{(\varepsilon)}$  denote the resistance is in the graph  $G_\varepsilon$ .

PROOF. We use the discrete Dirichlet principle for effective resistance; see Exercise 2.13 of [22]. Define a function  $f : V_\varepsilon \rightarrow \mathbb{R}$  by

$$f(x) = \begin{cases} 0, & \text{if } |x - \xi| \leq r, \\ \frac{|x - \xi| - r}{r}, & \text{if } |x - \xi| \in [r, 2r], \\ 1, & \text{if } |x - \xi| \geq 2r, \end{cases}$$

and let us estimate the Dirichlet energy of the function. Note that  $f$  is  $r^{-1}$ -Lipschitz, so that for any edge  $(x, y)$  we have  $|f(x) - f(y)| \leq |x - y|/r$ . All edges  $(x, y)$  such that  $x, y \in V_{\text{euc}}(\xi, r)$  or  $x, y \notin V_{\text{euc}}(\xi, 2r)$  contribute 0 to the energy. Any other edge  $(x, y)$  contributes at most  $|x - y|^2/r^2$  to the energy. Since  $|x - y|^2$  is proportional to the area of the faces adjacent to the edge  $(x, y)$ , all these

faces are contained in  $V_{\text{euc}}(\xi, Cr)$  for some  $C = C(D, \eta) < \infty$ , and each face has degree at most  $C$ , we get that the energy is bounded by some constant and the result follows.  $\square$

**COROLLARY 5.2.** *There exist constants  $K = K(D, \eta) < \infty$  and  $c = c(D, \eta) > 0$  such that for any  $R, r$  satisfying  $0 < Kr \leq R$ , any  $\varepsilon \leq r/10$  and any  $\xi \in \partial U$  we have the resistance bound*

$$R_{\text{eff}}^{(\varepsilon)}(V_{\text{euc}}(\xi, r), V_\varepsilon \setminus V_{\text{euc}}(\xi, R)) \geq c \log \frac{R}{r},$$

where  $R_{\text{eff}}^{(\varepsilon)}$  denotes the resistance in the graph  $G_\varepsilon$ .

**PROOF.** Let  $K \geq 2$  be such that there are no edges  $(x, y)$  such that  $|x - \xi| \leq r$  and  $|y - \xi| \geq Kr$ . Such a choice is possible by Lemma 2.2 and the fact that  $\xi$  is an accumulation point of vertices.

Suppose first that  $R = K^{2m-1}r$  for some integer  $m \geq 1$ . Define sets  $A_0 = \{v : |v - \xi| < r\}$ , and  $A_i = \{v : |v - \xi| \in [K^{i-1}r, K^i r]\}$ . By Lemma 2.2, there are no edges connecting  $A_i$  to  $A_j$  for  $|i - j| > 1$ . By Lemma 5.1, we have that  $R_{\text{eff}}^{(\varepsilon)}(A_i, A_{i+2}) \geq c$ . Contracting all edges in  $A_{2i}$  for each  $i$  (recall that by Thompson’s principle [22], Chapter 2, this operation can only decrease the effective resistance) and using the series law for resistance we find

$$R_{\text{eff}}^{(\varepsilon)}(V_{\text{euc}}(\xi, r), V_\varepsilon \setminus V_{\text{euc}}(\xi, R)) \geq cm \geq c' \log \frac{R}{r}.$$

For general  $R > Kr$ , the claim follows by monotonicity in  $R$ .  $\square$

**PROPOSITION 5.3 (Random walk convergence).** *Let  $X_n$  be the simple random walk on  $G$ , then  $X_n$  converges a.s. to a limit  $X_\infty \in \partial U$ . Furthermore, the law of  $X_\infty$  has no atoms.*

Consequently, for any starting point  $X_0$ , we may define the harmonic measure  $\omega$  on  $\partial U$  to be the law of  $X_\infty$ .

**PROOF OF PROPOSITION 5.3.** By Lemma 2.4, for each edge  $e = (u, v)$  we have that  $|v - u|^2$  is bounded by a constant times the area of either faces adjacent to  $e$ . Since each face has degree at most  $2\pi/\eta$ , this immediately gives that the Dirichlet energy of the Euclidean location function, that is,  $\sum_{e=(u,v)} |u - v|^2$ , is bounded by some constant. By [4], Theorem 1.1, this implies that  $X_n$  converges almost surely. (The theorem is stated for real valued functions, so we apply it to each coordinate separately.) It is trivial that the limit cannot be a vertex of  $G$ , so must be in  $\partial U$ .

Let us now fix  $X_0$ , and show that for any  $\xi \in \partial U$  we have  $\mathbb{P}(X_\infty = \xi) = 0$ . We have that  $R_{\text{eff}}(X_0, \partial V_\varepsilon) \leq C$  for some  $C$ , since  $G$  is transient. By Corollary 5.2 with  $R = |X_0 - \xi|$ , for  $r$  small enough and any  $\varepsilon < r/10$  we have  $R_{\text{eff}}^{(\varepsilon)}(X_0, V_{\text{euc}}(\xi, r)) \geq c|\log r|$  and, therefore,

$$\mathbb{P}(\{X_n\} \text{ visits } V_{\text{euc}}(\xi, r) \text{ before } \partial V_\varepsilon) \leq \frac{C}{|\log r|}.$$

Since this estimate is uniform in  $\varepsilon$ , we learn that the probability that  $\{X_n\}$  ever visits  $V_{\text{euc}}(\xi, r)$  is at most  $C|\log r|^{-1}$ . This bound is uniform in  $\varepsilon$ , and so holds also for the random walk on  $G$ . Finally,  $X_\infty = \xi$  implies that  $V_{\text{euc}}(\xi, r)$  is visited for all  $r$ , and so  $\mathbb{P}(X_\infty = \xi) \leq \inf_r C|\log r|^{-1} = 0$ .  $\square$

We now state two variations of the Harnack principle that apply to well embedded graphs.

**THEOREM 5.4 (Elliptic Harnack inequality).** *For any  $A > 1$ , there exists  $C = C(D, \eta, A) > 0$  such that for any  $x \in X$  and  $r > 0$  such that  $d_0(x, \partial U) > Ar$ , and any positive, harmonic function  $h$  on  $B_{d_0}(x, Ar)$  we have*

$$\max_{y \in B_{d_0}(x, r)} h(y) \leq C \min_{y \in B_{d_0}(x, r)} h(y).$$

**THEOREM 5.5 (Boundary Harnack principle).** *There exists positive constants  $A_0, A_1$  and  $R$ , depending only on  $D$  and  $\eta$ , such that for any  $\xi \in \partial U$ , any  $r \in (0, R)$  and any two functions  $h_1, h_2 : X \rightarrow \mathbb{R}$  that are positive, harmonic, bounded on  $B_{d_0}(\xi, A_0r)$ , and almost surely  $h_i(X_n) \rightarrow 0$  as  $n \rightarrow \infty$  for  $i = 1, 2$ , we have*

$$A_1^{-1} \leq \frac{h_1(x)/h_2(x)}{h_1(y)/h_2(y)} \leq A_1 \quad \forall x, y \in B_{d_0}(\xi, r) \cap X.$$

Theorem 5.4 follows from Theorem 3.5 of [28]. To obtain Theorem 5.5, we use [21], Theorem 4.2. We take their  $\hat{\mathcal{E}}$  and  $\mathcal{E}$  to be our  $\mathcal{E}$ , their spaces  $X$  and  $Y$  to be  $\bar{X}$ , and their  $\Omega$  to be  $X$ . Since in this case  $\hat{\mathcal{E}} = \mathcal{E}$  Assumptions 1 and 2 of [21] hold, and the conditions of volume doubling and the Poincaré inequality needed in [21] are provided by Theorem 3.4 and Lemma 3.3. Finally, Lemma 2.6 shows that  $\Omega$  is inner uniform.

We now proceed to the proof of Theorem 1.2. It has been known since Ancona [3] that a boundary Harnack principle such as Theorem 5.5 implies that the Martin boundary is homeomorphic to the Euclidean boundary. The papers [1, 2, 21] all contain results of this kind. In particular, the proof in [1] is quite robust, and translates to our setting with only minor changes. However, since the argument is both reasonably short and illuminating, we include it for the sake of completeness.

For convenience, we consider the Martin kernels as a function of the first coordinate, that is, we denote  $M_y(\cdot) = G(\cdot, y)/G(x_0, y)$ . Let  $\mathcal{H}^+$  denote the set of

positive harmonic functions  $h$  on  $X$ , normalized to have  $h(x_0) = 1$ . Note that on any locally finite connected graph,  $\mathcal{H}^+$  is compact w.r.t. the product (pointwise) topology. Of those, we let  $\mathcal{H}_0^+$  denote the set of functions so that  $h(X_n) \xrightarrow[n \rightarrow \infty]{\text{a.s.}} 0$  for any starting point  $X_0$ . By the martingale convergence theorem [11], a.s. convergence holds for any positive harmonic function; it is of course enough to assume the limit is a.s. 0 for a single starting point. Finally, for  $\xi \in \partial U$  let us denote by  $\mathcal{H}_\xi$  those functions  $h \in \mathcal{H}_0^+$  which are bounded on  $X \setminus B_{d_0}(\xi, r)$  for any  $r > 0$ . Our immediate goal is the following.

PROPOSITION 5.6. *For any  $\xi \in \partial U$ , the set  $\mathcal{H}_\xi$  is a singleton.*

We first prove that  $\mathcal{H}_\xi$  is not empty.

LEMMA 5.7. *Let  $y_n$  be a sequence of vertices and suppose  $y_n \rightarrow \xi \in \partial U$ . Then there exists a subsequence  $y_{n_k}$  such that  $M_{y_{n_k}}$  converges pointwise to some  $h \in \mathcal{H}_\xi$ .*

PROOF. Since  $\mathcal{H}^+$  is compact, there exist a subsequence  $y_{n_k}$  such that  $M_{y_{n_k}}$  converges pointwise. For clarity, we pass to the subsequence. Let  $M_\xi$  be the limit. Let us now prove that  $M_\xi \in \mathcal{H}_\xi$ . It is clear that  $M_\xi$  is harmonic and  $M_\xi(x_0) = 1$  since these are local constraints and are immediately satisfied by the limiting procedure, so we need to show that  $M_\xi(X_n) \xrightarrow[n \rightarrow \infty]{\text{a.s.}} 0$  and that  $M_\xi$  is bounded outside any neighborhood of  $\xi$ .

Recall that by the reversibility of the random walk we have

$$(5.1) \quad \text{deg}(x) \cdot G(x, y) = \text{deg}(y) \cdot G(y, x),$$

and since degrees are bounded,  $G(x, y)$  and  $G(y, x)$  are equivalent up to constants. We therefore have that

$$M_{y_k}(x) \asymp \frac{G(y_k, x)}{G(y_k, x_0)}.$$

Let  $A_0$  be the constant from Theorem 5.5, let  $r > 0$  be arbitrary small such that  $x_0 \notin B_{d_0}(\xi, A_0 r)$  and let  $x$  be an arbitrary vertex satisfying  $x \notin B_{d_0}(\xi, A_0 r)$ . Define the functions  $h_0 = G(\cdot, x_0)$  and  $h_1 = G(\cdot, x)$ . The functions  $h_0, h_1$  are positive, harmonic on  $X \cap B_{d_0}(\xi, r)$  and bounded above by  $G(x, x)$  and  $G(x_0, x_0)$ , respectively. Furthermore, both tend to 0 almost surely over the random walk since  $G$  is a transient graph. Hence, we may apply Theorem 5.5 to them and deduce that

$$\frac{G(z, x)}{G(z, x_0)} \asymp \frac{G(v_r, x)}{G(v_r, x_0)},$$

where  $v_r, z$  are any two vertices in  $B_{d_0}(\xi, r)$  and the constants in the  $\asymp$  do not depend on the choice of  $x$ . Let  $k_0$  be a number so that for all  $k \geq k_0$  we have



$y_k \in B_{d_0}(\xi, r)$  so by the previous approximate equality we get that for any  $k \geq k_0$  we have

$$(5.2) \quad M_{y_k}(x) \asymp \frac{G(v_r, x)}{G(v_r, x_0)} \asymp \frac{G(x, v_r)}{G(x_0, v_r)},$$

for all  $x \notin B_{d_0}(\xi, A_0r)$  and  $v_r$  a fixed vertex (its choice may depend on  $r$ ). Since  $G(x, v_r) \leq G(v_r, v_r)$ , we learn that  $M_{y_k}$  is bounded outside of  $B_{d_0}(\xi, r)$  for any  $r > 0$  and  $k > k_0$ , and we deduce the same for  $M_\xi$  immediately.

Next, by Proposition 5.3 the probability that  $\lim X_t = \xi$  is 0. We learn that almost surely there exists  $r > 0$  such that  $X_t \notin B_{d_0}(\xi, A_0r)$  for all  $t \geq 0$ . Let  $k_0$  be as above. By (5.2), we get that almost surely for any  $t \geq 0$

$$M_{y_k}(X_t) \asymp \frac{G(v_r, X_t)}{G(v_r, x_0)},$$

and by taking a limit  $k \rightarrow \infty$  we have that almost surely

$$M_\xi(X_t) \leq A \frac{G(v_r, X_t)}{G(v_r, x_0)},$$

for all  $t \geq 0$  where  $A = A(D, \eta) < \infty$ . Since  $G(v_r, X_t) \rightarrow 0$  as  $t \rightarrow \infty$  almost surely, we deduce that  $\lim M_\xi(X_n) = 0$  almost surely, concluding the proof.  $\square$

**PROOF OF PROPOSITION 5.6.** We first prove that there exists  $A = A(D, \eta) < \infty$  such that for any  $h_1, h_2 \in \mathcal{H}_\xi$  we have

$$(5.3) \quad A^{-1} \leq \frac{h_1(x)}{h_2(x)} \leq A \quad \text{for all } x \in X.$$

Let  $r > 0$  be an arbitrary small number and let  $\xi_1, \xi_2 \in \partial U$  be the two boundary points so that  $|\xi - \xi_1| = |\xi - \xi_2| = r$ . We appeal to Lemma 2.6 and get a curve  $\Gamma : (0, L) \rightarrow X$  satisfying the conditions of the lemma. We now use the curve to construct balls  $B_0, \dots, B_N$  for some  $N = N(D, \eta) < \infty$  such that for some small  $c \in (0, 1/2)$  the following holds:

- (1)  $B_0 = B_{d_0}(\xi_1, r/(2A_0))$  and  $B_N = B_{d_0}(\xi_2, r/(2A_0))$ ,
- (2) for  $i = 1, \dots, N - 1$  we have  $B_i = B_{d_0}(x_i, cr)$  where  $x_i \in \gamma$  and  $d_0(x_i, \partial U) > 2cr$ ,
- (3)  $B_i \cap B_{i+1} \neq \emptyset$  for  $i = 0, \dots, N - 1$ .

We apply Theorems 5.5 and 5.4 to obtain that there exists  $A = A(D, \eta) < \infty$  such that

$$A^{-1} \leq \frac{h_1(x)/h_2(x)}{h_1(x')/h_2(x')} \leq A \quad \forall x, x' \in \bigcup_{i=1}^N B_i.$$

Indeed, the assertion for  $x, x' \in B_0$  and  $x, x' \in B_N$  is precisely Theorem 5.5. Moreover, Theorem 5.4 gives that the values of  $h_1, h_2$  within  $B_1 \cup \dots \cup B_{N-1}$  change by at most a multiplicative constant.

Fix  $x' \in \gamma$  and note that  $h_1(x)/h_2(x) \leq q$  for all  $x \in \gamma$  where  $q = Ah_1(x')/h_2(x')$ . Then  $g(x) = h_1(x) - qh_2(x)$  is harmonic and nonpositive on  $\gamma$ . The martingale  $g(X_n)$  stopped when hitting  $\gamma$  is bounded, converges to 0 if  $\gamma$  is not hit, and is stopped at a negative value if  $\gamma$  is hit. By  $L^1$ -convergence for bounded martingales  $g(x) \leq 0$  everywhere, and so

$$A^{-1} \leq \frac{h_1(x)/h_2(x)}{h_1(x')/h_2(x')} \leq A \quad \forall x \in X \setminus B_{d_0}(\xi, r).$$

In particular,  $h_1(x')/h_2(x') \leq Ah_1(x_0)/h_2(x_0) = A$  and similarly  $h_1(x')/h_2(x') \geq A^{-1}$ . Hence,  $A^{-2} \leq h_1(x)/h_2(x) \leq A^2$  for all  $x \in X \setminus V_{\text{euc}}(\xi, r)$ . Since  $r > 0$  was arbitrary, this gives (5.3).

Next, we show that in fact  $A = 1$ ; the following argument is due to Ancona [3]. Indeed, write

$$c = \sup_{h_1, h_2 \in H_\xi, x \in X} \frac{h_1(x)}{h_2(x)},$$

so that  $c \in [1, \infty)$ . Assume by contradiction that  $c > 1$  and let  $h_1, h_2 \in H_\xi$ . Then  $h_3 = (ch_1 - h_2)/(c - 1)$  is a function in  $H_\xi$  so  $h_2 \leq ch_3$  which simplifies to  $(2c - 1)h_2 \leq c^2h_1$ . Since  $c^2/(2c - 1) < c$ , we have reached a contradiction.  $\square$

PROOF OF THEOREM 1.2'. Minimality of  $M_\xi$  follows easily from Proposition 5.6, since if  $0 \leq h \leq M_\xi$  then  $h(\cdot)/h(x_0)$  is easily seen to be in  $\mathcal{H}_\xi$ , and so it must equal  $M_\xi$ .

Suppose  $y_n \rightarrow \xi \in \partial U$  then Proposition 5.6 and Lemma 5.7 together show that  $\lim_{y_n \rightarrow \xi} M_{y_n}(\cdot)$  exists and is the unique function in  $\mathcal{H}_\xi$ . Thus, convergence of  $y_n$  implies convergence of  $M_{y_n}$ .

Next, note that if  $\xi \neq \xi'$  are two points on  $\partial U$ , then  $M_\xi \neq M_{\xi'}$ . Indeed,  $M_\xi$  is an unbounded function, since otherwise, by the bounded martingale convergence theorem we would get that  $\mathbb{E} \lim M_\xi(X_n) = M_\xi(x_0) = 1$ , contradicting the fact that  $M_\xi(X_n) \rightarrow 0$  almost surely. However,  $M_\xi$  is bounded away from  $\xi$  and so must be unbounded in any neighborhood of  $\xi$ . It follows that  $M_\xi \neq M_{\xi'}$ .

Now, suppose we have a convergent sequence  $M_{y_n} \rightarrow M_\infty$  for some sequence  $y_n$ . Since  $U$  is compact, there is a convergent subsequence  $y_{n_k} \rightarrow \xi$ . If  $\xi$  is not in  $\partial U$  then eventually  $y_{n_k} = \xi$ . Otherwise,  $M_{y_{n_k}} \rightarrow M_\xi$ , and in either case  $M_\infty = M_\xi$ . Since  $\xi$  is determined by  $M_\xi$ , we have that  $y_n \rightarrow \xi$ , completing the proof of (1).

Finally, we show that the map  $\xi \mapsto M_\xi(\cdot)$  is a homeomorphism. It is invertible, so we need continuity of the map and its inverse. Suppose  $\xi_n \rightarrow \xi$  are points in  $\partial U$ . For an arbitrary  $x$ , we may find  $y_n$  so that  $d(y_n, \xi_n) < \frac{1}{n}$ , and also  $|M_{y_n}(x) - M_{\xi_n}(x)| \leq \frac{1}{n}$ . We have that  $y_n \rightarrow \xi$  and, therefore,  $M_{y_n}(x) \rightarrow M_\xi(x)$ , and so also

$M_{\xi_n}(x) \rightarrow M_\xi(x)$ . Similarly, if  $M_{\xi_n} \rightarrow M_\xi$  we can diagonalize to find  $y_n$  with  $d(y_n, \xi_n) < \frac{1}{n}$  so that  $M_{y_n} \rightarrow M_\xi$ . By (1), we have  $y_n \rightarrow \xi$  and, therefore,  $\xi_n \rightarrow \xi$ .  $\square$

PROOF OF THEOREM 1.1'. We appeal to general properties of the Martin boundary; see Chapter 24 of [31] for a concise introduction. This theory implies that any positive harmonic function  $h$  can be represented as an integral on the Martin boundary  $\mathcal{M}$  with respect to some measure. When  $h$  is bounded, this measure is absolutely continuous with respect to the exit measure on the Martin boundary, hence it can be written as

$$(5.4) \quad h(x) = \int_{\mathcal{M}} M(x) f(M) d\nu_{x_0}(M),$$

where  $\nu_{x_0}$  is the law of  $\lim_n M_{X_n}(\cdot)$  starting from  $x_0$  and  $f : \mathcal{M} \rightarrow \mathbb{R}$  is some bounded measurable function; see Theorem 24.12 in [31]. Theorem 24.10 in [31] states that the Radon–Nikodym derivative of  $\nu_x$  with respect to  $\nu_{x_0}$  is the function from  $\mathcal{M}$  to  $\mathbb{R}$  mapping each  $M \in \mathcal{M}$  to  $M(x)$ . Hence, we may rewrite (5.4) as

$$h(x) = \int_{\mathcal{M}} f(M) d\nu_x(M).$$

Now, apply Theorem 1.2 and let  $\iota : \partial U \rightarrow \mathcal{M}$  be the homeomorphism  $\xi \mapsto M_\xi$ . Theorem 1.2 implies that the image under  $\iota$  of the random walk's exit measure on  $\partial U$  coincides with  $\nu_x$ , completing our proof.  $\square$

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