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PERMANENTAL VECTORS WITH NONSYMMETRIC KERNELS

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A permanental vector with a symmetric kernel and index 2 is a squared Gaussian vector. The definition of permanental vectors is a natural extension of the definition of squared Gaussian vectors to nonsymmetric kernels and to positive indexes. The only known permanental vectors either have a positive definite kernel or are infinitely divisible. Are there some others? We present a partial answer to this question.

1. Introduction. A real-valued positive vector $\psi = (\psi_i, 1 \le i \le n)$ is a permanental vector if its Laplace transform satisfies for every $(\alpha_1, \alpha_2, \dots, \alpha_n)$ in $\mathbb{R}_+^n$

$$\mathbb{E}\left[\exp\left\{-\frac{1}{2}\sum_{i=1}^{n}\alpha_i\psi_i\right\}\right] = |I + \alpha G|^{-\beta}, \tag{1.1}$$

where I is the $n \times n$-identity matrix, α is the diagonal matrix $\mathrm{Diag}((\alpha_i)_{1\le i\le n})$, $G = (G(i,j))_{1\le i,j\le n}$ and β is a fixed positive number.

Such a vector $(\psi_i, 1 \le i \le n)$ is a permanental vector with kernel $(G(i,j), 1 \le i, j \le n)$ and index β. Note that the kernel of ψ is not uniquely determined. Indeed any matrix DGD^{-1} with D $n \times n$-diagonal matrix with nonzero entries is a kernel for ψ. The matrices G and DGD^{-1} are said to be *diagonally equivalent*. But remark that ψ also admits G^t for kernel. More generally, the kernels of ψ are said to be *effectively equivalent*.

Vere-Jones has established a necessary and sufficient condition on the couple (G, β) for the existence of such a vector. His criterion is reminded at the beginning of Section 3.

For G $n \times n$-symmetric positive definite matrix and $\beta = 2$, (1.1) is the Laplace transform of the vector $(\eta_1^2, \eta_2^2, \dots, \eta_n^2)$ where $(\eta_1, \eta_2, \dots, \eta_n)$ is a centered Gaussian vector with covariance G. The definition of permanental vectors hence represents an extension of the definition of squared Gaussian vectors. The question is: to which point? More precisely, one already knows two classes of matrices that satisfy Vere-Jones criterion: the symmetric positive definite matrices and the inverse M-matrices (a nonsingular matrix A is a M-matrix if its off-diagonal entries are nonpositive and the entries of A^{-1} are nonnegative). Up to effective equivalence, these are the only known examples of permanental kernels. The question becomes:

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Is there an irreducible permanental kernel that would not belong to any of this two classes?
This two classes correspond respectively to vectors with the Laplace transform of a squared Gaussian vector to a positive power and to infinitely divisible permanental vectors. Infinitely divisible permanental processes are connected to local times of Markov processes thanks to Dynkin's isomorphism theorem and its extensions (see [3]). Besides, we have shown in [1] that for permanental vectors, infinite divisibility and positive correlation are equivalent properties.
In dimension one, obviously, the above two classes are identical and the answer is negative. One easily checks that a 2-dimensional permanental vector with index 2 is a squared Gaussian couple. Moreover, Vere-Jones [7], solving a question raised by Lévy [6], proved that a squared Gaussian couple is always infinitely divisible. Hence, in this case also the two classes are identical and the answer is negative. In dimension 3, the situation is different. Indeed, Kogan and Marcus [5] have shown that if the kernel of a 3-dimensional permanental vector is not effectively equivalent to a symmetric matrix (in short, is not symmetrizable), then it is diagonally equivalent to an inverse M-matrix. Since there exist inverse M-matrices that are not symmetrizable, the two classes are not identical and have a nonempty intersection. But the answer to the above question remains negative.
The case of dimension d strictly greater than 3 is still an open question. We show here that if the kernel of a d-dimensional permanental vector is strongly not symmetrizable, meaning that none of its principal submatrices of dimension 3 is symmetrizable, then it is diagonally equivalent to the inverse of an M-matrix. The result presented below is actually a little stronger and suggests that the answer should still be negative in the general case. In other words, one might think that the permanental vectors with a kernel not effectively equivalent to a symmetric matrix, are always infinitely divisible.
For a set of indexes I, we adopt the notation: $G_{I\times I} = (G(i,j))_{(i,j)\in I\times I}$.

THEOREM 1.1. *For $d > 3$, let ψ be a d-dimensional permanental vector with kernel G. Assume that there exists at most one subset I of three indexes such that $G_{I\times I}$ is symmetrizable. Then ψ is infinitely divisible.*

Theorem 1.1 can also be stated as follows:

For $\psi = (\psi_i)_{1\le i\le d}$ permanental vector of dimension $d > 3$, assume that there exists at most three integers $1 \le i_1, i_2, i_3 \le d$ such that $(\psi_{i_1}, \psi_{i_2}, \psi_{i_3})$ has the Laplace transform of a squared Gaussian vector to some power. Then ψ is infinitely divisible.

The proof of Theorem 1.1 is given in Section 3. Section 2 introduces the needed preliminaries and definitions. Section 4 presents some examples and remarks.

2. Preliminaries. We remind first the necessary and sufficient condition established by Vere-Jones [8] for a given matrix K to be the kernel of a permanental vector.

A $n \times n$-matrix K is the kernel of a permanental vector with index $\beta > 0$ if and only if:

(I) *All the real eigenvalues of K are nonnegative.*
(II) *For every $\gamma > 0$, set $K_\gamma = (I + \gamma K)^{-1} K$, then K_γ is β-positive definite.*

A $n \times n$-matrix $M = (M(i, j))_{1 \leq i, j \leq n}$ is said to be β-positive definite if for every integer m, every (not necessarily distinct) $k_1, k_2, \dots, k_m$ in $\{1, 2, \dots, n\}$

$$\mathrm{per}_\beta\big((M(k_i, k_j))_{1 \leq i, j \leq m}\big) \geq 0,$$

where for any $m \times m$-matrix $A = (A(i, j))_{1 \leq i, j \leq m}$, the quantity $\mathrm{per}_\beta(A)$ is defined as follows: $\mathrm{per}_\beta(A) = \sum_{\tau \in \mathcal{S}_m} \beta^{\nu(\tau)} \prod_{i=1}^m A_{i,\tau(i)}$, with $\mathcal{S}_m$ the set of the permutations on $\{1, 2, \dots, m\}$, and $\nu(\tau)$ the signature of τ.

Note that the property of β-positive definiteness for a matrix M is supported by an infinite family of matrices derived from M.

The proposition below is just the regrouping of results of Kogan and Marcus on the three dimensional permanental kernels. For the sake of clarity, we explain where to find this results in [5]. Adopting their convention, 0 is both positive and negative.

DEFINITION 2.1. Two $d \times d$-matrices A and B are said to be effectively equivalent if for every x in $\mathbb{R}^d : |I + xA| = |I + xB|$.

DEFINITION 2.2. A squared matrix is symmetrizable if it is effectively equivalent to a symmetric matrix.

PROPOSITION 2.3. *Let ψ be a 3-dimensional permanental vector with kernel $G = (G_{ij})_{1 \leq i, j \leq 3}$. Then we have*:

(i) *G is diagonally equivalent either to a matrix with all positive entries or to a matrix with all negative off-diagonal entries.*
(ii) *If G has all its off-diagonal entries strictly negative, then G is diagonally equivalent to a symmetric matrix.*
(iii) *If G has all its off-diagonal entries strictly positive, then G is either an inverse M-matrix or it is diagonally equivalent to a symmetric matrix.*
(iv) *If G has one or more zero off-diagonal entries, it is effectively equivalent to a symmetric matrix $\tilde{G}$ such that $\tilde{G}_{ij} = 0$ when $G_{ij} G_{ji} = 0$.*

Up to some misprints, (i) is Remark 2.1 in [5], which is a consequence of the fact that $G_{ij} G_{ji} \geq 0$ for every $1 \leq i, j \leq 3$. Indeed, for example, for G with only positive entries

$$\begin{pmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} G_{11} & -G_{12} & -G_{13} \\ -G_{21} & G_{22} & G_{23} \\ -G_{31} & G_{32} & G_{33} \end{pmatrix}$$

are diagonally equivalent.

(ii) is established in the first part of the proof of Lemma 4.1 in [5].

(iii) is established in the first part of the proof of Lemma 5.1 in [5].

(iv) When G is diagonally equivalent to a matrix with all negative off-diagonal entries, this is a consequence of the second part of the proof of Lemma 4.1 in [5]. When G is diagonally equivalent to a matrix with all positive off-diagonal entries, (iv) is a consequence of the last paragraph of the proof of Lemma 5.1 in [5] together with its Lemma 2.3 cleaned from a misprint. For the last sentence of Lemma 2.3 to be correct, the word “diagonally” should be replaced by “effectively.” Indeed Lemma 2.3 in [5], assuming that the two matrices

$$\begin{pmatrix} 1 & 0 & c_2 \\ a_2 & 1 & b_1 \\ c_1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & \sqrt{c_1 c_2} \\ 0 & 1 & 0 \\ \sqrt{c_1 c_2} & 0 & 1 \end{pmatrix}$$

are permanental kernels, states that for $a_2 b_1 c_1 c_2 \neq 0$, they are diagonally equivalent. But they cannot be diagonally equivalent. They are effectively equivalent.

We will use repeatedly the following lemma which is an elementary remark.

LEMMA 2.4. *For A $n \times n$-matrix, the following points are equivalent*:

(i) *A is diagonally equivalent to a symmetric matrix.*

(ii) *For every couple (D_1, D_2) of diagonal $n \times n$-matrices with strictly positive diagonal entries, $D_1 A D_2$ is diagonally equivalent to a symmetric matrix.*

(iii) *There exist two diagonal $n \times n$-matrices with strictly positive diagonal entries D_1 and D_2 such that $D_1 A D_2$ is diagonally equivalent to a symmetric matrix.*

REMARK 2.5. In view of Proposition 2.3, a permanental kernel of dimension 3, $G = (G_{ij})_{1 \le i,j \le 3}$, is symmetrizable iff:

- either G has an off-diagonal entry equal to zero,
- either G has no zero off-diagonal entry and it is diagonally equivalent to a symmetric matrix with strictly positive entries.

To check whether a 3×3-matrix K without zero off-diagonal entry, is symmetrizable, one has first to check the existence of a signature matrix σ (a diagonal matrix with $|\sigma_{ii}| = 1$, $1 \le i \le 3$) such that: $\sigma K \sigma = (|K_{ij}|)_{1 \le i,j \le 3}$, and then check that

$$\big|K(1,2)K(2,3)K(3,1)\big| = \big|K(2,1)K(1,3)K(3,1)\big|.$$

3. Proof of Theorem 1.1. *Step 1*: Assume that $d = 4$ and that G has no symmetrizable 3×3-principal submatrices, we show then that ψ is infinitely divisible.

Thanks to Remark 2.5, we know that G has no entry equal to 0. Moreover, in view of (i) and (ii) of Proposition 2.3, every 3×3-principal submatrix of G has to be diagonally equivalent to a matrix with all positive entries. This means that for every subset of three indexes I, there exists S_I from I into $\{-1, +1\}$ such that $S_I(i)G(i, j)S_I(j) \geq 0$, for $i, j \in I$. This leads to

$$G(i, j)G(j, k)G(k, i) > 0 \qquad \forall i, j, k \in \{1, 2, 3, 4\}.$$

Since G has no zero entry, this property implies the existence of S from $\{1, 2, 3, 4\}$ into $\{-1, +1\}$ such that: $S(i)G(i, j)S(j) > 0, \forall i, j \in \{1, 2, 3, 4\}$. Since ψ also admits for kernel SGS, we will assume from now that the entries of G are all strictly positive.

For $\sigma > 0$, consider the 3-dimensional vector ϕ_σ with Laplace transform

$$\frac{\mathbb{E}[\exp\{-(1/2)\sum_{j=1}^{3} \lambda_j \psi_j\} \exp\{-(\sigma/2)\psi_4\}]}{\mathbb{E}[\exp\{-(\sigma/2)\psi_4\}]}. \tag{3.1}$$

This vector is a permanental vector with the same index as ψ and admits for kernel $H(\sigma, G)$ (see [5])

$$H(\sigma, G) = \left(G(i, j) - \frac{\sigma}{1 + \sigma G(4, 4)} G(i, 4)G(4, j)\right)_{1 \leq i, j \leq 3}.$$

For which values of σ, is $H(\sigma, G)$ symmetrizable? For $\sigma > 0$, we have $\frac{\sigma}{1+\sigma G(4,4)} < \frac{1}{G(4,4)}$. We set: $\Gamma = (\frac{G(i,j)}{G(i,4)G(4,j)})_{1 \leq i,j \leq 4}$. Making use of Lemma 2.4, we are looking for the values of c in $(0, \frac{1}{G(4,4)})$ such that $(\Gamma(i, j) - c)_{1 \leq i, j \leq 3}$ is symmetrizable. In view of Remark 2.5, this can occur in two ways:

- either $(\Gamma(i, j) - c)_{1 \leq i, j \leq 3}$ has an off-diagonal entry equal to zero.
- either $(\Gamma(i, j) - c)_{1 \leq i, j \leq 3}$ has no zero off-diagonal entry and it is diagonally equivalent to a symmetric matrix.

The first possibility is excluded because it would imply the existence of i and j distinct from 4, such that: $\frac{G(i,j)}{G(i,4)G(4,j)} < \frac{1}{G(4,4)}$. But since the entries of G are all strictly positive, we know by assumption that $G_{\{i,j,4\} \times \{i,j,4\}}$ is an inverse M-matrix. This last property implies in particular that: $G(i, j)G(4, 4) \geq G(i, 4)G(4, j)$. This can bee seen by computing the inverse of $G_{\{i,j,4\} \times \{i,j,4\}}$ or by using Willoughby's paper [9].

We now study the second possibility. Since $(\Gamma(i, j) - c)_{1 \leq i, j \leq 3}$ has only strictly positive entries, we know, thanks to Lemma 2.4(iii), that $(\Gamma(i, j) - c)_{1 \leq i, j \leq 3}$ is diagonally equivalent to a symmetric matrix if and only if $(\frac{\Gamma(i,j)-c}{(\Gamma(i,3)-c)(\Gamma(3,j)-c)})_{1 \leq i,j \leq 3}$ is. Denote this last matrix by A_c. Since $A_c(i, 3) =$

$A_c(3,j)=\frac{1}{\Gamma(3,3)-c}$ for every $1\le i,j\le 3$, A_c is diagonally equivalent to a symmetric matrix if and only if $(A_c(i,j))_{1\le i,j\le 2}$ is symmetric. This translates into

$$\frac{\Gamma(1,2)-c}{(\Gamma(1,3)-c)(\Gamma(3,2)-c)}=\frac{\Gamma(2,1)-c}{(\Gamma(2,3)-c)(\Gamma(3,1)-c)},$$

which means that c must solve a polynomial equation with degree 3. Hence, only the two following cases might occur:

- either there are at most three distinct values for c such that $(\Gamma(i,j)-c)_{1\le i,j\le 3}$ is diagonally equivalent to a symmetric matrix,
- either for every value of c, $(\Gamma(i,j)-c)_{1\le i,j\le 3}$ is diagonally equivalent to a symmetric matrix.

In the later case, one obtains in particular $(\Gamma(i,j))_{1\le i,j\le 3}$ is diagonally equivalent to a symmetric matrix. Thanks to Lemma 2.4, this implies that $(G(i,j))_{1\le i,j\le 3}$ is diagonally equivalent to a symmetric matrix. But this is excluded by assumption. Consequently, except for at most three distinct values of σ, $H(\sigma,G)$ is not symmetrizable.

Set now $G_\sigma=(I+\sigma G)^{-1}G$. We have shown (Proposition 3.2 in [1]) that there exists a permanental vector ψ_σ with the same index as ψ, admitting G_σ for kernel and such that its Laplace transform satisfies

$$\begin{aligned}&\mathbb{E}\left[\exp\left\{-\frac{1}{2}\sum_{j=1}^{4}\lambda_j\psi_\sigma(j)\right\}\right]\\&\quad=\mathbb{E}\left[\frac{\exp\{-(\sigma/2)\sum_{i=1}^{4}\psi(i)\}}{\mathbb{E}[\exp\{-(\sigma/2)\sum_{i=1}^{4}\psi(i)\}]}\exp\left\{-\frac{1}{2}\sum_{j=1}^{4}\lambda_j\psi(j)\right\}\right].\end{aligned}$$

Hence, thanks to (3.1), it also satisfies

$$\mathbb{E}\left[\exp\left\{-\frac{1}{2}\sum_{j=1}^{3}\lambda_j\psi_\sigma(j)\right\}\right]=c(\sigma)\mathbb{E}\left[\exp\left\{-\frac{1}{2}\sum_{j=1}^{3}(\lambda_j+\sigma)\phi_\sigma(j)\right\}\right], \tag{3.2}$$

where $c(\sigma)=\frac{\mathbb{E}[\exp\{-(\sigma/2)\psi(4)\}]}{\mathbb{E}[\exp\{-(\sigma/2)\sum_{i=1}^{4}\psi(i)\}]}$.

We show now that $(\psi_\sigma(i))_{1\le i\le 3}$ admits a kernel without zero entry. To do so, we first adopt the notation: $\underline{G}_\sigma=(G_\sigma(i,j))_{1\le i,j\le 3}$. Note that (3.2) can be written as follows for every $x\in\mathbb{R}^3$:

$$|I+x\underline{G}_\sigma|=\frac{|I+(x+\sigma I)H_\sigma|}{|I+\sigma H_\sigma|}.$$

Besides denote by R_α the α-resolvent of H_σ : $R_\alpha=(I+\alpha H_\sigma)^{-1}H_\sigma$, then we have

$$\begin{aligned}|I+xR_\sigma|&=|I+x(I+\sigma H_\sigma)^{-1}H_\sigma|=|I+\sigma H_\sigma|^{-1}|I+(x+\sigma I)H_\sigma|\\&=|I+x\underline{G}_\sigma|.\end{aligned}$$

Consequently: $\underline{G}_\sigma$ and R_σ are effectively equivalent. This implies that $(\underline{G}_\sigma)^{-1}$ and R_σ^{-1} are effectively equivalent.

Assume that $\underline{G}_\sigma$ has an off-diagonal entry equal to zero. Then thanks to Proposition 2.3(iv), $\underline{G}_\sigma$ is effectively equivalent to a symmetric matrix. This implies that $(\underline{G}_\sigma)^{-1}$, and consequently R_σ^{-1}, is effectively equivalent to a symmetric matrix. Since $R_\sigma^{-1} = H_\sigma^{-1} + \sigma I$, we obtain that $(H_\sigma^{-1} + \sigma I)$ is effectively equivalent to a symmetric matrix, which easily implies that H_σ^{-1}, and then H_σ must be effectively equivalent to a symmetric matrix. Except for at most three values of σ, this is not true.

Consequently, we have obtained, except for at most three values of σ, that $(G_\sigma(i,j))_{1\le i,j\le 3}$ has no zero entry and is not effectively equivalent to a symmetric matrix. In view of Proposition 2.3, the only possibility for $(G_\sigma(i,j))_{1\le i,j\le 3}$ is to be diagonally equivalent to an inverse M-matrix.

Hence, there exists a function S from $\{1,2,3\}$ into $\{-1,+1\}$ such that for every i, j in $\{1,2,3\}$:

$$S(i)G_\sigma(i,j)S(j) > 0,$$

which leads to

$$G_\sigma(i,j)G_\sigma(j,k)G_\sigma(k,i) > 0 \tag{3.3}$$

for every i, j, k in $\{1,2,3\}$.

The choice of the three indexes 1, 2 and 3, being arbitrary, we conclude that excepted for at most a finite number of values of σ, G_σ has no zero entry and satisfies

$$G_\sigma(i,j)G_\sigma(j,k)G_\sigma(k,i) > 0$$

for every i, j, k in $\{1,2,3,4\}$.

This last property implies that there exists a function S_σ from $\{1,2,3,4\}$ into $\{-1,+1\}$ such that for every i, j in $\{1,2,3,4\}$

$$S_\sigma(i)G_\sigma(i,j)S_\sigma(j) > 0. \tag{3.4}$$

For the three values of σ that we have excluded, we still have (3.4). Indeed assume that there exists such a value and that it is strictly positive. Denote it by α. We know now that there exists $\varepsilon > 0$, such that for every σ in $(\alpha, \alpha+\varepsilon]$, G_σ satisfies (3.4). Note that G_σ is the $(\sigma-\alpha)$-resolvent of G_α:

$$G_\alpha = G_\sigma\big(I - (\sigma-\alpha)G_\sigma\big)^{-1}.$$

For $(\sigma-\alpha)$ small enough, we have $G_\alpha = \sum_{k=1}^{\infty}(\sigma-\alpha)^k G_\sigma^k$. Making use of (3.4), one obtains for every $1 \le i, j \le 4$:

$$S_\sigma G_\alpha S_\sigma(i,j) = \sum_{k=1}^{\infty}(\sigma-\alpha)^k S_\sigma G_\sigma^k S_\sigma(i,j) = \sum_{k=1}^{\infty}(\sigma-\alpha)^k (S_\sigma G_\sigma S_\sigma)^k(i,j) > 0.$$

Obviously, (3.4) implies that G_σ is β-positive definite for every $\beta > 0$. Vere-Jones criteria allows then to conclude that ψ is infinitely divisible.
Conclusion of Step 1: We have actually established that if G is a permanental kernel of dimension 4, with no symmetrizable 3×3-principal submatrices, then it is the kernel of an infinitely divisible permanental vector, and moreover, for every $\sigma > 0$, its σ-resolvent G_σ has a no zero entry.

Step 2: Define the claim (R_n) as follows.

(R_n): If G is a $n \times n$-square matrix without symmetrizable 3×3-principal submatrix, then ψ is infinitely divisible and for every $\sigma > 0$, its σ-resolvent G_σ has no zero entry.

We have just established (R_4). Assume that (R_n) is satisfied. We now establish (R_{n+1}). First note that G is diagonally equivalent to a matrix with only strictly positive entries. Indeed, using exactly the same argument as at the beginning of Step 1, one obtains

$$G(i,j)G(j,k)G(k,i) > 0 \qquad \forall i,j,k \in \{1,2,\dots,n\}.$$

Similarly, as in Step 1, one concludes that there exists S from $\{1,2,\dots,n\}$ into $\{-1,+1\}$ such that $S(i)G(i,j)S(j) > 0, \forall i,j \in \{1,2,\dots,n\}$.

Hence, we can assume that all the entries of G are strictly positive. Then consider the vector $(\phi_\sigma(i))_{1 \le i \le n}$ with Laplace transform

$$\frac{\mathbb{E}[\exp\{-(1/2)\sum_{j=1}^n \lambda_j \psi_j\}\exp\{-(\sigma/2)\psi_{n+1}\}]}{\mathbb{E}[\exp\{-(\sigma/2)\psi_{n+1}\}]}. \tag{3.5}$$

The vector $(\phi_\sigma(i))_{1 \le i \le n}$ is a permanental vector admitting for kernel $H(\sigma, G)$ defined by

$$H(\sigma,G) = \left(G(i,j) - \frac{\sigma}{1+\sigma G(n+1,n+1)} G(i,n+1)G(n+1,j)\right)_{1 \le i,j \le n}.$$

We look for the values of σ such that $H(\sigma, G)$ would have a 3×3-principal symmetrizable matrix. We set $\Gamma = (\frac{G(i,j)}{G(i,n+1)G(n+1,j)})_{1 \le i,j \le n+1}$. We hence look for the values of c in $(0, \frac{1}{G(n+1,n+1)})$ such that $(\Gamma(i,j) - c)_{1 \le i,j \le n}$ would have a symmetrizable 3×3-principal submatrix. We fix I, a subset of three elements of $\{1,2,\dots,n\}$. Similarly, as in the case $d = 4$, we know that $(\Gamma - c)_{I \times I}$ has no zero entry. The only way for $(\Gamma - c)_{I \times I}$ to be symmetrizable is to be diagonally equivalent to a symmetric matrix. Again as in Step 1, we have:

- either $(\Gamma - c)_{I \times I}$ is symmetrizable for at most three distinct values of c,
- either for every real c, $(\Gamma - c)_{I \times I}$ is diagonally equivalent to a symmetric matrix.

In the second case, one obtains that $\Gamma_{I \times I}$, and consequently $G_{I \times I}$ thanks to Lemma 2.4, is diagonally equivalent to a symmetric matrix, which is excluded by assumption. Consequently, for σ outside of a finite set, $H(\sigma, G)$ does not contain any 3×3-principal symmetrizable matrix. Thanks to our assumption on the case $d = n$, we know that ϕ_σ is infinitely divisible and that for every $\alpha > 0$, R_α, the α-resolvant of H_σ has no zero entry. Besides there exists a permanental vector with the same index as ϕ_σ, admitting R_α for kernel (see Proposition 3.2 in [1]). Making use of Vere-Jones criteria, one easily shows that this permanental vector is infinitely divisible too.

Setting $G_\sigma = (I + \sigma G)^{-1} G$, we know (Proposition 3.2 in [1]) that there exists a permanental vector ψ_σ with the same index as ψ, admitting G_σ for kernel and such that its Laplace transform satisfies

$$\mathbb{E}\left[\exp\left\{-\frac{1}{2}\sum_{j=1}^{n+1}\lambda_j \psi_\sigma(j)\right\}\right]$$
$$= \mathbb{E}\left[\frac{\exp\{-(\sigma/2)\sum_{i=1}^{n+1}\psi(i)\}}{\mathbb{E}[\exp\{-(\sigma/2)\sum_{i=1}^{n+1}\psi(i)\}]}\exp\left\{-\frac{1}{2}\sum_{j=1}^{n+1}\lambda_j \psi(j)\right\}\right].$$

Hence, thanks to (3.5), it also satisfies

$$\mathbb{E}\left[\exp\left\{-\frac{1}{2}\sum_{j=1}^{n}\lambda_j \psi_\sigma(j)\right\}\right] = c(\sigma)\mathbb{E}\left[\exp\left\{-\frac{1}{2}\sum_{j=1}^{n}(\lambda_j + \sigma)\phi_\sigma(j)\right\}\right], \tag{3.6}$$

where $c(\sigma) = \frac{\mathbb{E}[\exp\{-(\sigma/2)\psi(n+1)\}]}{\mathbb{E}[\exp\{-(\sigma/2)\sum_{i=1}^{n+1}\psi(i)\}]}$.

Similarly, as in Step 1, one shows that the two $n \times n$-matrices R_σ and $(G_\sigma(i, j))_{1\le i,j\le n}$ are effectively equivalent. Note that for every $1 \le i, j \le n$: $G_\sigma(i, j)G_\sigma(j, i) = R_\sigma(i, j)R_\sigma(j, i)$.

For σ outside a finite set, R_σ has no zero entry, and hence neither $(G_\sigma(i, j))_{1\le i,j\le n}$. Moreover, we know also that $(\psi_\sigma(i))_{1\le i\le n}$ is infinitely divisible. In particular for every triplet of indexes i, j and k in $\{1, 2, \dots, n\}$, $(\psi_\sigma(i), \psi_\sigma(j), \psi_\sigma(k))$ is infinitely divisible. Consequently, $(G_\sigma)_{\{i,j,k\}\times\{i,j,k\}}$ is diagonally equivalent to an inverse M-matrix. The choice of the index $(n + 1)$ being arbitrary, we actually obtain that for σ outside of a finite set there exists a function S_σ from $\{1, 2, \dots, n, n + 1\}$ into $\{-1, +1\}$ such that for every i, j in $\{1, 2, \dots, n, n + 1\}$

$$S_\sigma(i)G_\sigma(i, j)S_\sigma(j) > 0. \tag{3.7}$$

For σ element of the finite set of excluded values, one shows that (3.7) is still true exactly as we did it for (3.4) in Step 1.

We conclude that for every $\sigma > 0$, G_σ has no zero entry and is β-positive definite for every $\beta > 0$. Thanks to Vere-Jones criteria, ψ is infinitely divisible and (R_{n+1}) is established.

Step 3: Assume that $d = 4$ and that G is such that the matrices $(G(i, j))_{i,j\in\{1,2,4\}}$, $(G(i, j))_{i,j\in\{1,3,4\}}$ and $(G(i, j))_{i,j\in\{2,3,4\}}$ are not symmetrizable. We show that ψ is infinitely divisible and that for every $\sigma > 0$, its σ-resolvent G_σ has no zero entry.

First, note that according Remark 2.5, the three matrices $(G(i,j))_{i,j\in\{1,2,4\}}$, $(G(i,j))_{i,j\in\{1,3,4\}}$ and $(G(i,j))_{i,j\in\{2,3,4\}}$ have no zero entry. Hence, G has no zero entry. Since these three matrices are all diagonally equivalent to inverse of M-matrices, we can then easily establish the existence of S from $\{1,2,3,4\}$ into $\{-1,+1\}$ such that: $S(i)G(i,j)S(j) > 0, \forall i,j \in \{1,2,3,4\}$. We can hence assume that the entries of G are all strictly positive.

We now make the notation for $H(\sigma, G)$ more precise, by writing

$$H(\sigma, G, 4) = \left(G(i,j) - \frac{\sigma}{1+\sigma G(4,4)} G(i,4)G(4,j)\right)_{1\le i,j\le 3}.$$

Similarly, for any k in $\{1,2,3,4\}$, $H(\sigma, G, k)$ is defined by

$$H(\sigma, G, k) = \left(G(i,j) - \frac{\sigma}{1+\sigma G(k,k)} G(i,k)G(k,j)\right)_{i,j\in\{1,2,3,4\}\setminus\{k\}}.$$

Making use of the argument developed in Step 1, we know that for each of the three matrices $H(\sigma, G, 3)$, $H(\sigma, G, 2)$ and $H(\sigma, G, 1)$, there are at most three distinct values of σ for which they are not symmetrizable. Consequently, for σ outside of a finite set, the three matrices $(G_\sigma(i,j))_{i,j\in\{1,2,4\}}$, $(G_\sigma(i,j))_{i,j\in\{1,3,4\}}$ and $(G_\sigma(i,j))_{i,j\in\{2,3,4\}}$ have no zero entry and are diagonally equivalent to inverse M-matrices. Setting $I_3 = \{1,2,4\}$, $I_2 = \{1,3,4\}$ and $I_1 = \{2,3,4\}$, we hence know that there exist three functions S_3, S_2 and S_1 from respectively I_3, I_2 and I_1 into $\{-1,+1\}$ such that for every $p = 1, 2$ or 3, and every couple (i,j) of I_p, we have

$$S_p(i)G_\sigma(i,j)S_p(j) > 0.$$

To determine the sign of $G_\sigma(1,2)G_\sigma(2,3)G_\sigma(3,1)$, note that it has the same sign as $S_3(1)S_3(2) \cdot S_1(2)S_1(3) \cdot S_2(3)S_2(1)$. But $S_3(1)S_2(1)$ has the same sign as $S_3(4)S_2(4)G_\sigma(4,1)^2$; $S_3(2)S_1(2)$ has the same sign as $S_3(4)S_1(4)G_\sigma(2,4)^2$ and $S_1(3)S_2(3)$ has the same sign as $S_1(4)S_2(4)G_\sigma(4,3)^2$. One obtains

$$G_\sigma(1,2)G_\sigma(2,3)G_\sigma(3,1) > 0.$$

Consequently, for every i, j, k in $\{1,2,3,4\}$ we have

$$G_\sigma(i,j)G_\sigma(j,k)G_\sigma(k,i) > 0,$$

which leads to the existence of a function S from $\{1,2,3,4\}$ to $\{-1,+1\}$ such that for every i,j in $\{1,2,3,4\}$:

$$S(i)G_\sigma(i,j)S(j) > 0. \tag{3.8}$$

For σ element of the finite set of excluded values, one shows that (3.8) is still true exactly as we did it for (3.4) in Step 1.

We conclude that for every $\sigma > 0$, G_σ has no zero entry and that ψ is infinitely divisible.

Step 4: We assume that G has exactly one symmetrizable principal 3×3-submatrix. Denote by I the subset of the corresponding three distinct indexes. We show that ψ is infinitely divisible and that for every $\sigma > 0$, its σ-resolvent G_σ has no zero entry. Define the claim $(\tilde{R}_n)$ as follows.

$(\tilde{R}_n)$: If G is a $n \times n$-square matrix with exactly one symmetrizable 3×3-principal submatrix, then ψ is infinitely divisible and for every $\sigma > 0$, its σ-resolvent G_σ has no zero entry.

We just established $(\tilde{R}_4)$. Assume that $(\tilde{R}_n)$ is satisfied we show now that $(\tilde{R}_{n+1})$ is satisfied.

As in Step 3, one shows that we can assume that the entries of G are strictly positive. Note that for every index p in $\{1, 2, \dots, n+1\}$, $H(\sigma, G, p)$ is the kernel of a n-dimensional permanental vector. We still set $\Gamma = (\frac{G(i,j)}{G(i,p)G(p,j)})_{1 \le i,j \le n+1}$. Fix J subset of three elements of $\{1, 2, \dots, n+1\} \setminus \{p\}$. We look for the values c in $(0, \frac{1}{G(p,p)})$ for which $(\Gamma(i,j) - c)_{i,j \in J \times J}$ is symmetrizable. Unless $J = I$, we know, similarly as in Step 2, that $(\Gamma(i,j) - c)_{i,j \in J \times J}$ has no off-diagonal zero entry. Hence, for $J \ne I$, the only way for $(\Gamma(i,j) - c)_{i,j \in J \times J}$ to be symmetrizable is to be diagonally equivalent to a symmetric matrix. We know that:

- either $(\Gamma(i,j) - c)_{i,j \in J \times J}$ is diagonally equivalent to a symmetric matrix for at most three distinct values of c,
- either $(\Gamma(i,j) - c)_{i,j \in J \times J}$ is diagonally equivalent for every value of c.

In the later case, one obtains $G_{J \times J}$ is symmetrizable, which implies that $J = I$. Consequently for every p, and every σ outside of a finite set, $H(\sigma, G, p)$ contains at most one 3×3-symmetrizable principal submatrix. If there is none, then Step 2 tells us that the corresponding permanental vector is infinitely divisible and that for every $\alpha > 0$, its α-resolvent has no zero entry. If $H(\sigma, G, p)$ has exactly one 3×3-symmetrizable principal submatrix, we obtain the same property thanks to $(\tilde{R}_n)$. Making use of the argument developed in Step 2, one shows that ψ is infinitely divisible and for every $\sigma > 0$, its σ-resolvent has no zero. We have hence obtained $\tilde{R}_{n+1}$. This completes the proof of Theorem 1.1.

4. Remarks and examples.

REMARK 4.1. Theorem 1.1 can be reformulated in terms of linear algebra as follows.

For $d > 3$, let G be a $d \times d$-matrix with no zero entry such that at most one of its 3×3-principal submatrices is diagonally equivalent to a symmetric matrix. Assume that:

(I) *all the real eigenvalues of G are nonnegative*,

(II) *there exists $\beta > 0$, such that for every $\gamma > 0$, setting $G_\gamma = (I + \gamma G)^{-1} G$, G_γ is β-positive definite*,

then G is diagonally equivalent to an inverse M-matrix.

Assumptions (I) and (II) are necessary to obtain the conclusion. Indeed, consider the following nonsingular matrix borrowed from [4]:

$$A = \begin{pmatrix} 1 & 0,10 & 0,40 & 0,30 \\ 0,40 & 1 & 0,40 & 0,65 \\ 0,10 & 0,20 & 1 & 0,60 \\ 0,15 & 0,30 & 0,60 & 1 \end{pmatrix}.$$

It is not an inverse M-matrix, since $A^{-1}(2,3)$ is positive. But note that every 3×3-principal submatrix is an inverse M-matrix and is not symmetrizable. Consequently, A is not the kernel of a permanental vector.

REMARK 4.2. The condition required by Theorem 1.1 to obtain infinite divisibility, is sufficient and not necessary. Indeed, there exist nonsymmetrizable inverse M-matrices with more than one symmetrizable 3×3-principal submatrix. Here is a family of such matrices with dimension 4:

$$\Gamma = \begin{pmatrix} \Gamma(1,1) & a & a & \Gamma(4,4) \\ b & \Gamma(2,2) & e & \Gamma(4,4) \\ b & e & \Gamma(3,3) & \Gamma(4,4) \\ \Gamma(4,4) & \Gamma(4,4) & \Gamma(4,4) & \Gamma(4,4) \end{pmatrix},$$

with $\Gamma(i,i) > e$ for $i = 1, 2, 3$; $a, b, e > \Gamma(4,4)$ and $e > a, b$.
For $a \neq b$, Γ is not symmetrizable and has exactly two symmetrizable 3×3-principal submatrices, $\Gamma_{|\{1,2,3\}\times\{1,2,3\}}$ and $\Gamma_{|\{2,3,4\}\times\{2,3,4\}}$, and two nonsymmetrizable 3×3-principal submatrices.

Here are now examples of matrices illustrating Theorem 1.1.

$$\text{Set: } K = \begin{pmatrix} K(1,1) & e & a & K(4,4) \\ b & K(2,2) & a & K(4,4) \\ b & e & K(3,3) & K(4,4) \\ K(4,4) & K(4,4) & K(4,4) & K(4,4) \end{pmatrix},$$

with a, b and e positive: $K(i,i) > K(4,4)$ for $i = 1, 2, 3$; $K(i,i) > \sup\{a, b, e\}$ for $i = 1, 2, 3$; $\inf\{a, b, e\} > K(4,4)$.
For a, b, e distinct, K is not symmetrizable and $K_{|\{1,2,3\}\times\{1,2,3\}}$ is its unique symmetrizable principal submatrix of order 3. Moreover, the matrix K is an inverse M-matrix.

REMARK 4.3. For permanental vectors with symmetrizable kernel, one might think that assuming the infinite divisibility of all its triplets would lead to the infinite divisibility of the vector itself. As it has been noticed in [2], the Brownian sheet provides a counter-example. Here is another one found in [4]. Indeed the following matrix B is a 4×4-covariance matrix of a centered Gaussian vector $(\eta_1, \eta_2, \eta_3, \eta_4)$ such that for every triplet of distinct indexes i, j, k, $(\eta_i^2, \eta_j^2, \eta_k^2)$ is infinitely divisible, but $(\eta_1^2, \eta_2^2, \eta_3^3, \eta_4^3)$ is not infinitely divisible:

$$B = \begin{pmatrix} 1 & 0,50 & 0,35 & 0,40 \\ 0,50 & 1 & 0,50 & 0,26 \\ 0,35 & 0,50 & 1 & 0,50 \\ 0,40 & 0,26 & 0,50 & 1 \end{pmatrix}$$

and $B^{-1}(2,4)$ is positive.

REMARK 4.4. Let $(G(i,j))_{1\le i,j\le n}$ be the kernel of a permanental vector ψ. We assume that G is nonsingular. For α in $[0,1]$, consider now the $2n \times 2n$-matrix $H(\alpha)$ defined by

$$H(\alpha) = \begin{bmatrix} G & \alpha G \\ \alpha G & G \end{bmatrix}.$$

The matrix $H(1)$ is the kernel of the vector (ψ, ψ). The matrix $H(0)$ is the kernel of the permanental vector $(\psi, \tilde{\psi})$, where $\tilde{\psi}$ is an independent copy of ψ.

PROPOSITION 4.5. *If G does not contain any symmetrizable 3×3-principal submatrix, then for any α in $(0,1)$, $H(\alpha)$ is not the kernel of a permanental vector.*

PROOF. For x complex number, we have

$$\begin{aligned} \det\bigl(H(\alpha) - xI\bigr) &= \det\begin{bmatrix} G - xI & \alpha G \\ \alpha G & G - xI \end{bmatrix} \\ &= \bigl|(G - xI)^2 - \alpha^2 G^2\bigr| = \bigl|(1+\alpha)G - xI\bigr|\bigl|(1-\alpha)G - xI\bigr|, \end{aligned}$$

since αG and $(G - xI)$ commute. Hence, $H(\alpha)$ satisfies the first condition of Vere-Jones criterion of existence of a permanental vector. Moreover, for $\alpha < 1$, $H(\alpha)$ is not singular.

By assumption, G has no zero entry (if not it would contain a symmetrizable 3×3-principal submatrix) and thanks to Theorem 1.1, it is hence diagonally equivalent to an inverse M-matrix. In particular, there exists a signature matrix σ such that the entries of $\sigma G \sigma$ are all strictly positive. Consequently, we can assume that the entries of $H(\alpha)$ are all strictly positive.

For α in $(0,1)$, $H(\alpha)$ is not an inverse M-matrix. Indeed, write

$$H(\alpha) = H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix},$$

with $H_{11} = H_{22} = G$ and $H_{12} = H_{21} = \alpha G$, then

$$H^{-1} = \begin{bmatrix} (H/H_{22})^{-1} & -(H/H_{22})^{-1} H_{12} (H_{22})^{-1} \\ -H_{22}^{-1} H_{21} (H/H_{22})^{-1} & (H/H_{11})^{-1} \end{bmatrix},$$

where H/H_{11} is the Schur complement of H_{11} in H defined by

$$H/H_{11} = H_{22} - H_{21} H_{22}^{-1} H_{12},$$

and similarly H/H_{22} is the Schur complement of H_{22} in H:

$$H/H_{22} = H_{11} - H_{12}H_{22}^{-1}H_{21}.$$

Then, as it has been noticed by Johnson and Smith [4], H is an inverse M-matrix iff:

(i) H/H_{11} is an inverse M-matrix,
(ii) H/H_{22} is an inverse M-matrix,
(iii) $(H_{22})^{-1}H_{21}(H/H_{22})^{-1}$ has nonnegative entries only,
(iv) $(H/H_{22})^{-1}H_{12}(H_{22})^{-1}$ has nonnegative entries only.

For $H = H(\alpha)$ with α in [0, 1), this criterion gives the following:
(i) and (ii): $(1-\alpha^2)G$ is an inverse M-matrix.
(iii) and (iv): $\frac{\alpha}{1-\alpha^2}G^{-1}$ has only nonnegative entries.
Hence unless $\alpha = 0$, $H(\alpha)$ is never an inverse M-matrix.

Now assume that there exists a permanental vector admitting $H(\alpha)$ for kernel. Then we know that this permanental vector is not infinitely divisible. But as soon as G does not contain any 3×3-covariance matrix, neither does $H(\alpha)$. Thanks to Theorem 1.1, a permanental vector that would admit $H(\alpha)$ for kernel should be infinitely divisible. Hence, $H(\alpha)$ can not be the kernel of a permanental vector. □

Note that for G symmetric positive definite matrix, we know that $H(\alpha)$ is still a covariance matrix. But the corresponding $2n$-dimensional permanental vector is never infinitely divisible, because Conditions (i) and (iii) above are always antagonistic.

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