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INTERMITTENCY FOR THE WAVE AND HEAT EQUATIONS WITH FRACTIONAL NOISE IN TIME

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In this article, we consider the stochastic wave and heat equations driven by a Gaussian noise which is spatially homogeneous and behaves in time like a fractional Brownian motion with Hurst index $H > 1/2$. The solutions of these equations are interpreted in the Skorohod sense. Using Malliavin calculus techniques, we obtain an upper bound for the moments of order $p \geq 2$ of the solution. In the case of the wave equation, we derive a Feynman–Kac-type formula for the second moment of the solution, based on the points of a planar Poisson process. This is an extension of the formula given by Dalang, Mueller and Tribe [*Trans. Amer. Math. Soc.* **360** (2008) 4681–4703], in the case $H = 1/2$, and allows us to obtain a lower bound for the second moment of the solution. These results suggest that the moments of the solution grow much faster in the case of the fractional noise in time than in the case of the white noise in time.

1. Introduction. In this article, we consider the stochastic wave equation

$$\text{(SWE)}\qquad \begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,x) = \Delta u(t,x) + u(t,x)\dot{W}(t,x), \qquad (t>0, x \in \mathbb{R}^d), \\ u(0,x) = u_0, \\ \dfrac{\partial u}{\partial t}(0,x) = v_0, \end{cases}$$

and the stochastic heat equation

$$\text{(SHE)}\qquad \begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \dfrac{1}{2}\Delta u(t,x) + u(t,x)\dot{W}(t,x), \qquad (t>0, x \in \mathbb{R}^d), \\ u(0,x) = u_0, \end{cases}$$

where Δ stands for the Laplacian operator on $\mathbb{R}^d$, and $\dot{W}$ denotes the formal derivative of a Gaussian noise W (whose rigorous definition is given below). The definition of the solution to equations (SWE) and (SHE) is given in Section 3 below, using the Skorohod integral with respect to W. Intuitively, the noise $\dot{W}$ is homogeneous in space (with spatial covariance kernel f) and behaves in time like

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a fractional Brownian motion (fBm) with Hurst index $H > 1/2$. The initial conditions u_0 and v_0 are nonnegative constants. In the case of the wave equation (SWE), we assume that $d \leq 3$, while for the heat equation (SHE), $d \geq 1$ can be arbitrary.

There is a large amount of literature dedicated to the case $H = 1/2$, when the noise behaves in time like the Brownian motion. In this case, we say that the noise is white in time. We refer the reader to the lecture notes [23] for an introduction to the subject, as well as [16, 21, 22, 27, 29, 41, 44, 45, 48] for a sample of relevant references. The case $H \neq 1/2$ has to be treated by different methods, since the noise is not a semi-martingale in time. In recent years, there has been a growing interest in studying equations with general Gaussian noise, and in particular equations driven by a noise which behaves in time like a fBm with Hurst parameter $H \neq 1/2$; see [4, 6, 7, 13, 14, 32–36].

In the present article, the noise is introduced by a zero-mean Gaussian process $W = \{W(\varphi); \varphi \in \mathcal{H}\}$ with covariance

$$E\big(W(\varphi)W(\psi)\big) = \langle \varphi, \psi \rangle_{\mathcal{H}}. \tag{1}$$

Here $\mathcal{H}$ is a Hilbert space defined as the completion of the space $C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$ of infinitely differentiable functions with compact support on $\mathbb{R}_+ \times \mathbb{R}^d$, with respect to the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ defined by

$$\langle \varphi, \psi \rangle_{\mathcal{H}} = \alpha_H \int_{(\mathbb{R}_+ \times \mathbb{R}^d)^2} \varphi(t, x)\psi(s, y)|t - s|^{2H-2} f(x - y)\, dt\, dx\, ds\, dy, \tag{2}$$

where $\alpha_H = H(2H - 1)$. We assume that $H \in (\frac{1}{2}, 1)$, and f is the Fourier transform in $\mathcal{S}'(\mathbb{R}^d)$ of a tempered measure μ on $\mathbb{R}^d$, where $\mathcal{S}'(\mathbb{R}^d)$ is the dual of the space $\mathcal{S}(\mathbb{R}^d)$ of rapidly decreasing infinitely differentiable functions on $\mathbb{R}^d$.

Using the fact that

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(x)\psi(y) f(x - y)\, dx\, dy = \int_{\mathbb{R}^d} \mathcal{F}\varphi(\xi)\overline{\mathcal{F}\psi(\xi)}\mu(d\xi) \qquad \forall \varphi, \psi \in \mathcal{S}(\mathbb{R}^d),$$

we arrive at the following alternative expression for the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$:

$$\begin{aligned} &\langle \varphi, \psi \rangle_{\mathcal{H}} \\ &\quad = \alpha_H \int_0^\infty \int_0^\infty \int_{\mathbb{R}^d} |t - s|^{2H-2} \mathcal{F}\varphi(t, \cdot)(\xi)\overline{\mathcal{F}\psi(s, \cdot)(\xi)}\mu(d\xi)\, dt\, ds, \end{aligned} \tag{3}$$

where $\mathcal{F}$ denotes the Fourier transform in the x-variable.

In the present article, we consider the following four cases:

(i) $f(0) < \infty$ (i.e., μ is a finite measure);
(ii) $f(x) = |x|^{-\alpha}$ for some $0 < \alpha < d$ [i.e., $\mu(d\xi) = c_{\alpha,d}|\xi|^{-(d-\alpha)}\, d\xi$];
(iii) $f(x) = \prod_{j=1}^d |x_j|^{-\alpha_j}$ for some $\alpha_j \in (0, 1)$ [i.e., $\mu(d\xi) = c_{(\alpha_j)_j} \times \prod_{j=1}^d |\xi_j|^{\alpha_j - 1}\, d\xi$];
(iv) $d = 1$ and $f = \delta_0$ (i.e., μ is the Lebesgue measure).

Here we denote by $|x|$ the Euclidean norm of $x \in \mathbb{R}^d$.

Case (i) corresponds to a spatially smooth noise $\dot{W}$. In case (ii), f is called the *Riesz kernel* with exponent α. Case (iii) with the parametrization $\alpha_j = 2 - 2H_j$ for some $H_j \in (\frac{1}{2}, 1)$ leads to a noise $\dot{W}$ which is called a *fractional Brownian sheet* with indices $(H, H_1, \dots, H_d)$. Finally, case (iv) corresponds to a (rougher) noise $\dot{W}$ which is "white in space." This describes the spatial behavior of the noise in the four cases. On the other hand, in time, the noise is smoother than the white noise (the Brownian motion), since $H > 1/2$. We note in passing that the results of the present article can be extended to $H = 1/2$, recovering results which are already known for equations (SHE) and (SWE) with white noise in time. To ease the exposition, we discuss only the case $H > 1/2$.

The stochastic heat equation (SHE) driven by space–time white noise $\dot{W}$ arises in different contexts and has been studied by many authors. This equation is the continuous form of the parabolic Anderson model studied by Carmona and Molchanov in [11], and plays a major role in the study of the KPZ equation in physics; see [38]. The connection between the stochastic heat equation and the KPZ equation (via the Hopf–Cole transformation) was known informally by physicists for quite some time; see, for example, [9]. Recently, this connection has been made rigorous by Hairer in [31], using the theory of rough paths; see also [8]. Equation (SHE) with fractional noise in time has been studied in [6, 32, 35]. References [3, 10, 46] are dedicated to the wave equation with fractional noise.

In this article, we consider the Malliavin calculus approach for defining a solution to equations (SWE) and (SHE), as in [3], respectively [6]. In particular, we introduce the following assumption, known as *Dalang's condition*:

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1 + |\xi|^2} < \infty. \tag{DC}$$

This condition is necessary and sufficient for the existence of the solution to equations (SHE) and (SWE), when the noise is white in time; see [21]. It is also sufficient for the existence of the solution to these equations, when the noise is fractional in time and has spatial covariance given by the Riesz kernel; see [3, 6]. The necessity of (DC) in the case of (SHE) has been proved in [5].

Note that (DC) is satisfied in cases (i) and (iv). In cases (ii) and (iii), it holds if and only if $a < 2$, where a is defined by (9) below.

The purpose of this paper is to study intermittency properties for the solutions to equations (SHE) and (SWE). Intuitively, a space–time random field is called *physically intermittent* if it develops very high peaks concentrated on small spatial islands, as time becomes large. To give a formal mathematical definition of intermittency for a random-field $u = \{u(t, x); t \geq 0, x \in \mathbb{R}^d\}$, we consider the *upper Lyapunov exponent*

$$\gamma(p) := \limsup_{t \to \infty} \frac{1}{t} \log E|u(t, x)|^p \tag{4}$$

for any $p \geq 1$ [assuming that $\gamma(p)$ does not depend on x]. Traditionally, in the literature, the random-field u is called *weakly intermittent* if

$$\gamma(2) > 0 \quad \text{and} \quad \gamma(p) < \infty \qquad \text{for all } p \geq 2. \tag{5}$$

If $\gamma(1) = 0$, and $u(t,x) \geq 0$, then weak intermittency implies full intermittency. Recall that a random field u is *fully intermittent* if $p \mapsto \gamma(p)/p$ is strictly increasing; see [11]. Intuitively, full intermittency shows that for $p > q$,

$$\limsup_{t\to\infty} \frac{\|u(t,x)\|_p}{\|u(t,x)\|_q} = \infty, \tag{6}$$

where $\|\cdot\|_p$ denotes the norm in $L^p(\Omega)$. In other words, asymptotically, the pth moment of $u(t,x)$ is significantly larger than its qth moment. This suggests that the random variable $u(t,x)$ may take very large values with small (but significant) probabilities, and therefore it develops high peaks, when t is large. We refer to [9], Section 2.4, for a detailed explanation of this phenomenon.

Intermittency for the spatially-discrete heat equation was studied in [11]. In [28], Foondun and Khoshnevisan proved weak intermittency for the solution to equation (SHE) driven by space–time white noise, assuming that the initial condition u_0 is bounded away from 0. Similar investigations have been carried out in [9, 12, 20]. In the recent article [15], Chen, Hu, Song and Xing have given the exact asymptotics for the moments of the solution to equation (SHE) driven by a fractional noise in time, with spatial covariance kernel given by cases (ii)–(iv) above. Intermittency for the solution of the stochastic wave equation driven by a Gaussian noise which is white in time was studied in [18, 25].

The fractional aspect of the noise in time leads to a different notion of weak intermittency, which is obtained by a slight modification of the Lyapunov exponent. More precisely, for $\rho > 0$ and $p \geq 1$, we define the *modified upper Lyapunov exponent* (of index ρ) by

$$\gamma_\rho(p) := \limsup_{t\to\infty} \frac{1}{t^\rho} \log E\left|u(t,x)\right|^p. \tag{7}$$

By analogy with (5), we say that the random-field u is *weakly ρ-intermittent* if

$$\gamma_\rho(2) > 0 \quad \text{and} \quad \gamma_\rho(p) < \infty \qquad \text{for all } p \geq 2.$$

Also, we say that u is *fully ρ-intermittent* if $p \mapsto \gamma_\rho(p)/p$ is strictly increasing. These definitions guarantee that for a fully ρ-intermittent random-field u, relation (6) still holds, and so the intuitive (physical) notion of intermittency remains valid. A similar argument as the one developed in [9] still applies to explain the existence of the high peaks and the islands. Moreover, it remains true that weak ρ-intermittency of u implies its full ρ-intermittency, provided that $u(t,x) \geq 0$ and $\gamma_\rho(1) = 0$. (This can be proved by convexity arguments which do not depend on the exponent of t used in the definition of ρ-intermittency.)

This article is organized as follows. In Section 2, we describe our main results and introduce the exponents ρ for equations (SWE) and (SHE). Section 3 contains a review of some Malliavin calculus techniques which are needed for the definition of the solution. In Section 4, we prove the existence of the solution to equation (SWE) in *any* spatial dimension $d \geq 1$, and we give an upper bound for its second moment. An upper bound for its pth moment is given in Section 5. In Section 6, we obtain a Feynman–Kac-type representation for the second moment of the solution of (SWE) with $d \leq 3$, based on the points of a planar Poisson process. This result is used in Section 7 to obtain a lower bound for the second moment of the solution to (SWE). Section 8 is dedicated to the equation (SHE). An elementary estimate is given in Appendix A. Appendix B contains the proof of an inequality which is used in Section 4.

2. Main results.

In this section, we discuss the two main results of this article.

The following exponents are used for the weak ρ-intermittency of the solutions to equations (SWE), respectively (SHE):

$$\rho_{\mathrm{w}} = \frac{2H+2-a}{3-a}, \qquad \rho_{\mathrm{h}} = \frac{4H-a}{2-a}, \tag{8}$$

where

$$a = \begin{cases} 0, & \text{in case (i)}, \\ \alpha, & \text{in case (ii)}, \\ \displaystyle\sum_{j=1}^{d} \alpha_j, & \text{in case (iii)}, \\ 1, & \text{in case (iv)}. \end{cases} \tag{9}$$

We are now ready to state the first result about equation (SWE). We refer to (22) below for the definition of the solution.

THEOREM 2.1. *Let f be a kernel of cases* (i)–(iv). *Let ρ_{w} and a be the constants given by* (8), *respectively* (9). *Assume that condition* (DC) *holds.*

(a) *For any $d \geq 1$, equation* (SWE) *has a solution $\{u(t,x); t \geq 0, x \in \mathbb{R}^d\}$, given by relation* (28) *below. If $d \leq 2$, the solution is unique.*

(b) *For any $d \geq 1$, $p \geq 2$, $x \in \mathbb{R}^d$ and for any $t > 0$,*

$$E|u(t,x)|^p \leq c_1^p (u_0 + t v_0)^p \exp\bigl(c_2 p^{(4-a)/(3-a)} t^{\rho_{\mathrm{w}}}\bigr), \tag{10}$$

where $c_1 > 0$ is a constant depending on a, and $c_2 > 0$ is a constant depending on H and a.

(c) *Suppose that $d \leq 3$. Then for any $x \in \mathbb{R}^d$ and for any $t > 0$,*

$$E|u(t,x)|^2 \geq c_3 u_0^2 \exp\bigl(c_4 t^{\rho_{\mathrm{w}}}\bigr), \tag{11}$$

where $c_3 > 0$ and $c_4 > 0$ are constants depending on H and a.

A similar result holds for the parabolic equation (SHE).

THEOREM 2.2. *Let f be a kernel of cases* (i)–(iv). *Let ρ_{h} and a be the constants given by* (8), *respectively* (9). *Assume that condition* (DC) *holds. Let $d \geq 1$ be arbitrary.*

(a) *Equation* (SHE) *has a unique solution $\{u(t,x); t \geq 0, x \in \mathbb{R}^d\}$.*

(b) *For any $p \geq 2$, for any $x \in \mathbb{R}^d$ and for any $t > 0$,*

$$E|u(t,x)|^p \leq c_1^p u_0^p \exp\bigl(c_2 p^{(4-a)/(2-a)} t^{\rho_{\mathrm{h}}}\bigr), \tag{12}$$

where $c_1 > 0$ is a constant depending on a, and $c_2 > 0$ is a constant depending on H and a.

(c) *For any $x \in \mathbb{R}^d$ and for any $t > 0$,*

$$E|u(t,x)|^2 \geq c_3 u_0^2 \exp\bigl(c_4 t^{\rho_{\mathrm{h}}}\bigr), \tag{13}$$

where $c_3 > 0$ and $c_4 > 0$ are constants depending on H and a.

Most moment estimates for solutions to s.p.d.e.'s with white noise in time rely on martingale properties of stochastic integrals. Since the fBm is not a semimartingale, different techniques have to be used when the noise is fractional in time. In the case of equations (SWE) and (SHE), one can give explicitly the Wiener chaos representation of the solution. The upper bounds (10) and (12) are obtained directly using the equivalence of $L^2(\Omega)$- and $L^p(\Omega)$-norms on each Wiener chaos. The lower bounds require more work. For this, we follow the approach of Dalang and Mueller [25], which consists of using a Feynman–Kac (FK) type representation for the second moment of the solution, based on a Poisson process. Such a representation was originally developed in [26] for equations driven by a noise that is white in time. It was extended to the heat equation driven by fractional noise in time by the first author of this article in [2]. The extension to the wave equation with fractional noise in time is given in Section 6 below.

Article [26] contains also a FK representation for the nth moment of the solution of the wave (or heat) equation, for any integer $n \geq 2$ (Theorem 5.1 of [26]). The proof of this result uses the fact that the stochastic integral with respect to the noise W is a martingale in time, which allows the authors of [26] to apply Itô's formula. In the case of the fractional noise in time, the stochastic integral is not a semimartingale. There exists an Itô's formula for the Skorohod integral with respect to the classical fBm (Theorem 8 of [1]), which could probably be generalized to the case of the noise W. However, this formula contains an extra correction term involving the Malliavin derivative of the integrand process, which is difficult to handle. For this reason, we could not apply the method of Dalang, Mueller and Tribe [26] to obtain an FK representation (and an exponential lower bound) for the moment of order $n \geq 2$ of the solution to either wave of heat equation. We note that a lower bound for the nth moment of the solution to the heat equation has been recently obtained in [33], using an FK representation for the moments which is specific to the parabolic case (see Theorem 3.6 of [33]), and is different than the one used in the present paper. The lower bound for the nth moment of the solution to the wave equation remains an open problem.

As in [25], we focus mainly on the hyperbolic case (Theorem 2.1). The proof of Theorem 2.2 is very similar, and we only point out the differences in comparison to the hyperbolic case in Section 8. We made this choice since the results for the wave equation are completely new, in particular the second moment FK type representation.

In [15], Chen, Hu, Song and Xing obtained stronger results than our Theorem 2.2, by computing the exact Lyapunov exponent for the solution of equation (SHE), defined as the limit when $t \to \infty$, instead of the $\limsup$ in (7); see Theorem 6.1 of [15]. In [15], the solution is defined in the weak sense (i.e., using multiplication against test functions), and the stochastic integral is interpreted in the Stratonivich sense, according to Definition 4.2 of [36]. However, their method requires the additional assumptions $a < 4H - 2$ in cases (ii)–(iii), and $H > 3/4$ in case (iv), which are not needed in the present article. The proofs of [15] rely on a Feynman–Kac representation for the weak solution and its moments (due to [36]), which can only be proved under the above-mentioned additional assumptions. By Theorem 7.2 of [36], a similar Feynman–Kac representation exists for the mild solution (defined using the Skorohod integral, as in the present work), under the same assumptions mentioned above. Using this representation and under the same assumptions, it may be possible to compute the exact Lyapunov exponent for the mild solution, although this is not proved in [15]. We believe that in the absence of these assumptions, the methods of [36] and [15] cannot be applied for equation (SHE), even when it is interpreted in the Shorohod sense. These assumptions appear also in the recent preprint [33] for the Feynman–Kac representation for the solution of equation (SHE) interpreted in the Stratonovich sense (see Hypothesis 4.1 of [33]), but are not needed for obtaining exponential upper and lower bounds for the moments of the solution of (SHE), interpreted in the Stratonovich or Skorohod sense, as shown by Theorem 6.4 of [33]. In the case of the heat equation with noise as in case (iii) above, some exponential upper and lower bounds for the first moment of the solution (interpreted in the Stratonovich sense) have been obtained in [49].

The appropriate exponents ρ are different in the hyperbolic and parabolic cases. Nevertheless, since $H > 1/2$, $\rho_{\mathrm{h}} > \rho_{\mathrm{w}} > 1$. Therefore, the lower bounds in Theorems 2.1 and 2.2 imply that $\gamma(2) = \infty$, which shows that the solutions to (SWE) and (SHE) are not weakly intermittent in the classical sense. However, these solutions are weakly ρ-intermittent (in the sense defined in Section 1) with $\rho = \rho_{\mathrm{w}}$ for the wave equation and $\rho = \rho_{\mathrm{h}}$ for the heat equation. The results of Theorems 2.1 and 2.2 do not provide full ρ-intermittency. When the noise is white in time, one typically obtains full ρ-intermittency by proving that $\gamma(1) = 0$. According to Song [49] (using the Stratonovitch integral), it appears that this may not be true in the case of the fractional noise in time. In this case, an alternative method is to obtain a sharp lower bound on the moments of order $p > 2$ of the solution, as in [25]. This is subject of ongoing research.

In the case when $H = 1/2$, $\rho_{\mathrm{w}} = \rho_{\mathrm{h}} = 1$, and we recover some of the known results of intermittency for the heat and wave equations with white noise in time. For instance, intermittency for the heat equation was studied in [28]. For the wave equation, full intermittency was obtained in [25] with the spatial covariance of case (i). Some upper bounds were obtained in [18].

As mentioned before, when $H > 1/2$, $\rho_{\mathrm{h}} > \rho_{\mathrm{w}} > 1$. A consequence of this is that the moments of the solution at some fixed time t are typically larger in the case of the fractional noise in time compared to the white-noise case. This would imply that the size of the peaks would be larger in the fractional case. Since $H > 1/2$, the noise is positively correlated in time, which explains why peaks build up larger values. Indeed, the fractional noise, when large, tends to remain large for a longer period of time, which then results in a higher build-up for the random-field u.

The upper and lower bounds given by Theorems 2.1 and 2.2 show that the exponents ρ_{w} and ρ_{h} are sharp. A lower bound result on the moments of order $p > 2$ would be needed in order to get the sharp behavior of the exponent $\gamma_\rho(p)$ as a function of p. This remains an open problem in the case of the wave equation. (In the case of the heat equation, this has been recently proved in the preprint [33].) We note that in our results, the behavior of $\gamma_\rho(p)$ as a function of p does not depend on H. For the wave equation, we obtain that $\gamma_{\rho_{\mathrm{w}}}(p) \le Cp^{4/3}$ in case (i) (spatially smooth noise), and $\gamma_{\rho_{\mathrm{w}}}(p) \le Cp^{3/2}$ in case (iv) (spatial white-noise), where $C > 0$ denotes a constant which does not depend on p. These confirm the behavior in the order p obtained in [25] for case (i) and in [18] for case (iv). For the heat equation, we obtain that $\gamma_{\rho_{\mathrm{h}}}(p) \le Cp^2$ in case (i) and $\gamma_{\rho_{\mathrm{h}}}(p) \le Cp^3$ in case (iv), for a constant $C > 0$ which does not depend on p, which correspond to the sharp order for white noise in time.

Finally, we would like to point out that Theorems 2.1 and 2.2 constitute a first step toward a more careful study of the intermittent behavior of the solution to the stochastic heat and wave equations. Indeed, following the program developed in [17, 19], sharp Lyapunov exponents for the moments of solutions to SPDEs (in particular their behavior as a function of p) are key ingredients for obtaining quantitative results regarding some physical properties of the solution, such as the height of the peaks, the size of the peak-islands and some space–time scaling results for the behavior of the peaks. These could lead to a careful understanding of the impact of the temporal (and spatial) correlation of the noise on the physical behavior of the solution. In particular, observing how the modified Lyapunov exponents impact the physical properties would be an important step in the understanding of mathematical intermittency. In the case of the white-noise in time, the existence of the sharp Lyapunov exponents has allowed the authors of [17, 19] to obtain KPZ-type scaling exponents for the solution to the stochastic heat equation.

3. Framework. In this section, we introduce the framework, and we give a brief summary of the results of Balan [3], which are needed in the present article.

We denote by G_{w}, respectively G_{h}, the fundamental solution of the wave equation, respectively the heat equation. In the case of the wave equation, recall that when $d \le 2$, $G_{\mathrm{w}}(t,\cdot)$ is a function given by

$$
\begin{aligned}
G_{\mathrm{w}}(t,x) &= \frac{1}{2} 1_{\{|x|\le t\}} \qquad \text{if } d=1 \quad \text{and} \\
G_{\mathrm{w}}(t,x) &= \frac{1}{2\pi} \frac{1}{\sqrt{t^2-|x|^2}} 1_{\{|x|<t\}} \qquad \text{if } d=2.
\end{aligned} \tag{14}
$$

In both cases, $\int_{\mathbb{R}^d} G_{\mathrm{w}}(t,x)\,dx = t$. When $d=3$, $G_{\mathrm{w}}(t,\cdot)$ is a finite measure on $\mathbb{R}^3$ given by

$$G_{\mathrm{w}}(t,\cdot) = \frac{1}{4\pi t}\sigma_t,$$

where σ_t is the surface measure on $\partial B(0,t)$, and $G_{\mathrm{w}}(t,\mathbb{R}^3)=t$. When $d \ge 4$, $G_{\mathrm{w}}(t,\cdot)$ is a distribution. For any $d \ge 1$, the Fourier transform of $G_{\mathrm{w}}(t,\cdot)$ is given by

$$\mathcal{F}G_{\mathrm{w}}(t,\cdot)(\xi) = \frac{\sin(t|\xi|)}{|\xi|}, \qquad \xi \in \mathbb{R}^d. \tag{15}$$

In the case of the heat equation, for any dimension $d \ge 1$, $G_{\mathrm{h}}(t,\cdot)$ is a function, known as the *heat kernel*. More precisely,

$$G_{\mathrm{h}}(t,x) = \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{|x|^2}{2t}\right) \quad \text{and} \quad \mathcal{F}G_{\mathrm{h}}(t,\cdot)(\xi) = \exp\left(-\frac{t|\xi|^2}{2}\right).$$

Below, we write G when the results apply for both G_{w} or G_{h}.

We denote by w_{w} (resp., w_{h}) the solution of the homogeneous wave (resp., heat) equation with the same initial condition as (SWE) [resp., (SHE)], that is,

$$w_{\mathrm{w}}(t,x) = u_0 + t v_0 \quad \text{and} \quad w_{\mathrm{h}}(t,x) = u_0. \tag{16}$$

We write w when the results apply for both w_{w} and w_{h}. Note that $w(t,x)$ does not depend on x in either case, and $w(t,x) \ge u_0 \ge 0$ for all $t>0$ and $x \in \mathbb{R}^d$.

We now discuss the concept of solution. Informally, a (mild) solution of (SWE) or (SHE) should be a process $\{u(t,x); t \ge 0, x \in \mathbb{R}^d\}$ which satisfies

$$u(t,x) = w(t,x) + \int_0^t \int_{\mathbb{R}^d} G(t-s,x-y) u(s,y) W(ds,dy), \tag{17}$$

provided the stochastic integral on the right-hand side is well defined (in a certain sense). Still informally, replacing $u(s,y)$ on the right-hand side of (17) by its definition and iterating this procedure, we conclude that the solution of (SWE) or (SHE) should be given by the following series of iterated integrals:

$$
\begin{aligned}
u(t,x) = {} & w(t,x) + \int_0^t \int_{\mathbb{R}^d} G(t-s,x-y) W(ds,dy) \\
& + \int_0^t \int_{\mathbb{R}^d} \int_0^s \int_{\mathbb{R}^d} G(t-s,x-y) \\
& \qquad \times G(s-r,y-z) W(dr,dz) W(ds,dy) + \cdots .
\end{aligned} \tag{18}
$$

To give a rigorous meaning to this procedure, we use an approach based on Malliavin calculus with respect to the isonormal Gaussian process $W = \{W(\varphi); \varphi \in \mathcal{H}\}$ with covariance specified by (1), where $\mathcal{H}$ is the Hilbert space defined as the completion of $C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$ with respect to the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ given by (2).

We recall the basic elements of Malliavin calculus; see [43] for more details. It is known that every square-integrable random variable F which is measurable with respect to W, has the Wiener chaos expansion

$$
F = E(F) + \sum_{n \ge 1} F_n \qquad \text{with } F_n \in \mathcal{H}_n,
$$

where $\mathcal{H}_n$ is the nth Wiener chaos space associated to W. Moreover, each F_n can be represented as $F_n = I_n(f_n)$ for some $f_n \in \mathcal{H}^{\otimes n}$, where $\mathcal{H}^{\otimes n}$ is the nth tensor product of $\mathcal{H}$, and $I_n : \mathcal{H}^{\otimes n} \to \mathcal{H}_n$ is the multiple Wiener integral with respect to W. By the orthogonality of the Wiener chaos spaces and an isometry-type property of I_n, we obtain that

$$
E|F|^2 = (EF)^2 + \sum_{n \ge 1} E\left|I_n(f_n)\right|^2 = (EF)^2 + \sum_{n \ge 1} n! \|\widetilde{f}_n\|^2_{\mathcal{H}^{\otimes n}},
$$

where $\widetilde{f}_n$ is the symmetrization of f_n in all n variables

$$
\widetilde{f}_n(t_1, x_1, \ldots, t_n, x_n) = \frac{1}{n!} \sum_{\rho \in S_n} f_n(t_{\rho(1)}, x_{\rho(1)}, \ldots, t_{\rho(n)}, x_{\rho(n)}),
$$

where S_n is the set of all permutations of $\{1, \ldots, n\}$.

Let $\mathcal{S}$ be the class of smooth random variables of the form

$$
F = f\bigl(W(\varphi_1), \ldots, W(\varphi_n)\bigr), \tag{19}
$$

where $f \in C_b^\infty(\mathbb{R}^n)$, $\varphi_i \in \mathcal{H}$, $n \ge 1$ and $C_b^\infty(\mathbb{R}^n)$ is the class of bounded C^∞-functions on $\mathbb{R}^n$, whose partial derivatives are bounded. The *Malliavin derivative* of F of the form (19) is an $\mathcal{H}$-valued random variable given by

$$
DF := \sum_{i=1}^n \frac{\partial f}{\partial x_i} \bigl(W(\varphi_1), \ldots, W(\varphi_n)\bigr) \varphi_i .
$$

We endow $\mathcal{S}$ with the norm $\|F\|^2_{\mathbb{D}^{1,2}} := E|F|^2 + E\|DF\|^2_{\mathcal{H}}$. The operator D can be extended to the space $\mathbb{D}^{1,2}$, the completion of $\mathcal{S}$ with respect to $\|\cdot\|_{\mathbb{D}^{1,2}}$.

The *divergence operator* δ is defined as the adjoint of the operator D. The domain of δ, denoted by $\operatorname{Dom}\delta$, is the set of $u \in L^2(\Omega;\mathcal{H})$ such that

$$\left|E\langle DF, u\rangle_{\mathcal{H}}\right| \le c\left(E|F|^2\right)^{1/2} \qquad \forall F \in \mathbb{D}^{1,2},$$

where c is a constant depending on u. If $u \in \operatorname{Dom}\delta$, then $\delta(u)$ is the element of $L^2(\Omega)$ characterized by the following duality relation:

$$E\bigl(F\delta(u)\bigr) = E\langle DF, u\rangle_{\mathcal{H}} \qquad \forall F \in \mathbb{D}^{1,2}. \tag{20}$$

In particular, $E[\delta(u)] = 0$. If $u \in \operatorname{Dom}\delta$, we use the notation

$$\delta(u) = \int_0^\infty \int_{\mathbb{R}^d} u(t,x) W(\delta t, \delta x),$$

and we say that $\delta(u)$ is the *Skorohod integral* of u with respect to W.

We recall the following criterion for Skorohod integrability; see also Proposition 1.3.7 of [43].

PROPOSITION 3.1 (Proposition 2.5 of [3]). *Assume that $u \in L^2(\Omega;\mathcal{H})$ has the Wiener chaos expansion*

$$u(t,x) = \sum_{n\ge 0} I_n\bigl(f_n(\cdot,t,x)\bigr), \tag{21}$$

where $f_0(t,x) = E(u(t,x))$, $I_0(x) = x$ and $f_n(\cdot,t,x) \in \mathcal{H}^{\otimes n}$ for any $n \ge 1$. Then $u \in \operatorname{Dom}\delta$ if and only if the series $\sum_{n\ge 0} I_{n+1}(f_n)$ converges in $L^2(\Omega)$, and in this case $\delta(u) = \sum_{n\ge 0} I_{n+1}(f_n)$.

We are now ready to give the rigorous definition of the solution to equations (SHE) and (SWE). Let $\mathcal{F}_t$ be the σ-field generated by $W(1_{[0,s]\times A})$ for $s \in [0,t]$, $A \in \mathcal{B}_b(\mathbb{R}^d)$, where $\mathcal{B}_b(\mathbb{R}^d)$ is the class of all bounded Borel sets in $\mathbb{R}^d$.

DEFINITION 3.2. An $(\mathcal{F}_t)_t$-adapted square-integrable process $u = \{u(t,x); t \ge 0, x \in \mathbb{R}^d\}$ is called a (*mild*) *solution* of (SWE) or (SHE) if it satisfies the following integral equation:

$$u(t,x) = w(t,x) + \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y) u(s,y) W(\delta s, \delta y), \tag{22}$$

that is, $v^{(t,x)} \in \operatorname{Dom}\delta$ and $u(t,x) = w(t,x) + \delta(v^{(t,x)})$ for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, where

$$v^{(t,x)}(s,\cdot) = 1_{[0,t]}(s) G(t-s, x-\cdot) u(s,\cdot), \qquad s \ge 0 \tag{23}$$

and $\cdot$ denotes the missing y-variable.

In the case of equation (SWE) in dimension $d \leq 2$ or equation (SHE) in any dimension d, $G(t, \cdot)$ is a function, and the existence and uniqueness of the solution can be proved similar to page 303 of [35]. We recall this argument here. Assume that a solution $u(t,x)$ exists and has the Wiener chaos expansion (21) for some functions $f_n(\cdot, t, x) \in \mathcal{H}^{\otimes n}$. Since G is a deterministic function, it follows that the process $v^{(t,x)}$ given by (23) has the Wiener chaos expansion $v^{(t,x)}(s,y) = \sum_{n\geq 0} I_n(g_n^{(t,x)}(\cdot, s, y))$, with kernels

$$g_n^{(t,x)}(\cdot, s, y) = 1_{[0,t]}(s) G(t-s, x-y) f_n(\cdot, s, y). \tag{24}$$

By Proposition 3.1, $v^{(t,x)} \in \operatorname{Dom} \delta$ if and only if $\sum_{n\geq 0} I_{n+1}(g_n^{(t,x)})$ converges in $L^2(\Omega)$. In this case, $\delta(v^{(t,x)}) = \sum_{n\geq 0} I_{n+1}(g_n^{(t,x)})$, and relation $u(t,x) = w(t,x) + \delta(v^{(t,x)})$ becomes

$$\sum_{n\geq 0} I_n\big(f_n(\cdot, t, x)\big) = w(t,x) + \sum_{n\geq 0} I_{n+1}\big(g_n^{(t,x)}\big).$$

By the uniqueness of the Wiener chaos expansion, we infer that $f_0(t,x) = w(t,x)$ and $f_{n+1}(\cdot, t, x) = g_n^{(t,x)}$ for any $n \geq 0$. This allows us to find f_n recursively:

$$\begin{aligned} &f_n(t_1, x_1, \dots, t_n, x_n, t, x) \\ &\quad = G(t-t_n, x-x_n) G(t_n - t_{n-1}, x_n - x_{n-1}) \cdots \\ &\qquad \times G(t_2 - t_1, x_2 - x_1) w(t_1, x_1) 1_{\{0<t_1<\cdots<t_n<t\}}. \end{aligned} \tag{25}$$

Therefore, if the series $\sum_{n\geq 0} I_{n+1}(g_n^{(t,x)}) = \sum_{n\geq 0} I_{n+1}(f_{n+1}(\cdot, t, x))$ converges in $L^2(\Omega)$, then the solution u exists and is unique, with the Wiener chaos expansion (21) with kernels $f_n(\cdot, t, x)$ given by (25). This coincides with the informal interpretation (18).

In the case of equation (SWE) with $d \geq 3$, the procedure for constructing a solution is more complicated, since $G_{\mathrm{w}}(t, \cdot)$ is a distribution in $\mathbb{R}^d$. We describe below the steps of this procedure, following [3].

Step 1. Define the kernel $f_n(\cdot, t, x)$ as a distribution in $\mathcal{S}'(\mathbb{R}^{nd})$, identifying its action on a test function, as in Section 2.1 of [3]. By Proposition 2.1 of [3], for any $0 < t_1 < \cdots < t_n < t$, $f_n(t_1, \cdot, \dots, t_n, \cdot, t, x)$ is a distribution in $\mathbb{R}^{nd}$ whose Fourier transform [in $\mathcal{S}'(\mathbb{R}^{nd})$] is the function

$$\begin{aligned} &\mathcal{F} f_n(t_1, \cdot, \dots, t_n, \cdot, t, x)(\xi_1, \dots, \xi_n) \\ &\quad = (u_0 + t_1 v_0) e^{-i(\xi_1+\cdots+\xi_n)\cdot x} \overline{\mathcal{F} G_{\mathrm{w}}(t_2 - t_1, \cdot)(\xi_1)} \\ &\qquad \times \overline{\mathcal{F} G_{\mathrm{w}}(t_3 - t_2, \cdot)(\xi_1 + \xi_2)} \cdots \overline{\mathcal{F} G_{\mathrm{w}}(t - t_n, \cdot)(\xi_1 + \cdots + \xi_n)}. \end{aligned} \tag{26}$$

$f_n(t_1, \cdot, \dots, t_n, \cdot, t, x)$ is defined to be 0 for $(t_1, \dots, t_n) \in [0,t]^n \setminus T_n(t)$ where $T_n(t) = \{0 < t_1 < \cdots < t_n < t\}$. Note that in Proposition 2.1 of [3], it is assumed that $u_0 = 1$ and $v_0 = 0$, so that $w_{\mathrm{w}} = 1$. This result continues to hold when the function w_{w} is given by (16), since w_{w} does not depend on x.

Step 2. Let $\widetilde{f}_n(\cdot, t, x)$ be the symmetrization of $f_n(\cdot, t, x)$. By Remark 2.3 of [3], if $\|\widetilde{f}_n(\cdot, t, x)\|^2_{\mathcal{H}^{\otimes n}} < \infty$, then $\widetilde{f}_n(\cdot, t, x) \in \mathcal{H}^{\otimes n}$ and the multiple Wiener integral $I_n(f_n(\cdot, t, x)) = I_n(\widetilde{f}_n(\cdot, t, x))$ is a well-defined element of $\mathcal{H}_n$.

Step 3. Suppose that the series $\sum_{n\geq 1} I_n(f_n(\cdot, t, x))$ converges in $L^2(\Omega)$, that is,

$$\sum_{n\geq 1} n! \|\widetilde{f}_n(\cdot, t, x)\|^2_{\mathcal{H}^{\otimes n}} < \infty. \tag{27}$$

Let

$$u(t, x) := w(t, x) + \sum_{n\geq 1} I_n\big(f_n(\cdot, t, x)\big). \tag{28}$$

Step 4. Define $v^{(t,x)}(s, \cdot)$ by relation (23). This is a product between the distribution $G_{\mathrm{w}}(t - s, x - \cdot)$ and the function $u(s, \cdot)$. The process $v^{(t,x)}$ has the Wiener chaos expansion $v^{(t,x)}(\bullet) = \sum_{n\geq 0} I_n(f_{n+1}(\cdot, \bullet, t, x))$, where $\bullet$ denotes the missing (s, y) variable; see the proof of Theorem 2.8 in [3]. By Proposition 3.1, $v^{(t,x)} \in \operatorname{Dom}\delta$ and $\delta(v^{(t,x)}) = \sum_{n\geq 0} I_{n+1}(f_{n+1}(\cdot, t, x)) = u(t, x) - w(t, x)$. Hence the process $u = \{u(t, x); t \geq 0, x \in \mathbb{R}^d\}$ with the Wiener chaos expansion (28) is a solution of (SWE). Moreover,

$$E\big|u(t, x)\big|^2 = w(t, x)^2 + \sum_{n\geq 1} \frac{1}{n!}\alpha_n(t), \tag{29}$$

where $\alpha_n(t) = n! E|I_n(f_n(\cdot, t, x))|^2 = (n!)^2 \|\widetilde{f}_n(\cdot, t, x)\|^2_{\mathcal{H}^{\otimes n}}$.

Step 5. It remains to prove (27). When the spatial covariance function f is given by case (ii) above, this follows by Proposition 3.4 of [3]. A similar argument can be used for cases (i), (iii) and (iv); see Proposition 4.2 below.

Summarizing, to prove that a solution of (SWE) exists in the case $d \geq 3$, we only need to show that the series $\sum_{n\geq 1} I_n(f_n(\cdot, t, x))$ converges in $L^2(\Omega)$; that is, (27) holds. In this case, one such solution is given by (28).

REMARK 3.3. The uniqueness of the solution of (SHE) for $d \geq 3$ was not treated in [3]. It may be possible to show that the solution is unique in this case too. This would require significant modifications to the method described above for the case $d \leq 2$, since both terms $G(t - s, x - y)$ and $f_n(\cdot, s, y)$ encountered in definition (24) of $g_n^{(t,x)}(\cdot, s, y)$ are distributions in y. We do not investigate this problem here. We note in passing that the classical method for proving uniqueness does not seem to work for equations (SWE) or (SHE) when the solution is interpreted in the sense of Definition 3.2. To see this, assume that there are two solutions u and v, and let $d = u - v$. Then

$$d(t, x) = \int_0^t \int_{\mathbb{R}^d} G(t - s, x - y)\, d(s, y) W(\delta s, \delta y).$$

The $L^2(\Omega)$-norm of the Skorohod integral above is a sum of two terms, the second one involving the Malliavin derivative of d; see [42], relation (1.11). This second term vanishes when the noise is white in time, but when the noise is fractional in time, it is not clear how to treat this term.

REMARK 3.4. After examining (26), we infer that in the case of equation (SWE) with $d = 3$, $f_n(t_1, \cdot, \dots, t_n, \cdot, t, x)$ is a finite measure on $\mathbb{R}^{3n}$ given by

$$\begin{aligned} &f_n(t_1, \cdot, \dots, t_n, \cdot, t, x) \\ &\quad = G(t - t_n, x - dx_n)G(t_n - t_{n-1}, x_n - dx_{n-1}) \cdots \\ &\quad\quad \times G(t_2 - t_1, x_2 - dx_1)w(t_1, x_1)1_{\{0<t_1<\cdots<t_n<t\}}, \end{aligned}$$

where for fixed $a \in \mathbb{R}^3$, we denote by $G(t, a - \cdot)$ the measure defined by $G(t, a - \cdot)(A) = G(t, a - A)$ for all $A \in \mathcal{B}(\mathbb{R}^3)$.

REMARK 3.5. Notice that in both the hyperbolic and parabolic cases, the function (or distribution) f_n is stationary in the sense that, for all $t_1, \dots, t_n \in [0, t]$ and for any $x_1, \dots, x_n, x \in \mathbb{R}^d$,

$$f_n(t_1, x_1, \dots, t_n, x_n, t, x) = f_n(t_1, x_1 - x, \dots, t_n, x_n - x, t, 0).$$

This remains valid for $\widetilde{f}_n$. A direct consequence is that $\|\widetilde{f}_n(\cdot, t, x)\|^2_{\mathcal{H}^{\otimes n}}$, and hence $\alpha_n(t)$, do not depend on x. Since the initial conditions are constant, w does not depend on x either and the moments of u are independent of x. This justifies the definition of Lyapunov exponent independent of x. Also, notice that it is possible to show that the law of $u(t, x)$ is independent of x; see, for instance, [21] in the white noise case. These remarks are not true if the initial conditions are not constant.

We return now to series (27), which is also related to the second moment of the solution $u(t, x)$; see (29). An important role in the present paper is played by the nth term of this series, which depends on $\alpha_n(t)$. First, note that an expression similar to (3) exists for the n-fold inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}^{\otimes n}}$. Using this expression, we have

$$\alpha_n(t) = \alpha_H^n \int_{[0,t]^{2n}} \prod_{j=1}^n |t_j - s_j|^{2H-2} \psi_n(\mathbf{t}, \mathbf{s}) \, d\mathbf{t} \, d\mathbf{s}, \tag{30}$$

where we denote $\mathbf{t} = (t_1, \dots, t_n)$ and $\mathbf{s} = (s_1, \dots, s_n)$, and we define

$$\begin{aligned} \psi_n(\mathbf{t}, \mathbf{s}) = \int_{\mathbb{R}^{nd}} &\mathcal{F} g_{\mathbf{t}}^{(n)}(\cdot, t, x)(\xi_1, \dots, \xi_n) \\ &\times \overline{\mathcal{F} g_{\mathbf{s}}^{(n)}(\cdot, t, x)(\xi_1, \dots, \xi_n)} \mu(d\xi_1) \cdots \mu(d\xi_n) \end{aligned} \tag{31}$$

with $g_{\mathbf{t}}^{(n)}(\cdot, t, x) = n! \widetilde{f}_n(t_1, \cdot, \dots, t_n, \cdot, t, x)$. Note that $\psi_n(\mathbf{t}, \mathbf{s})$ depends also on t, so that the correct notation should be $\psi_n(\mathbf{t}, \mathbf{s}, t)$. To simplify the notation, we omit writing t in $\psi_n(\mathbf{t}, \mathbf{s}, t)$.

An alternative calculation of the function $\psi_n(\mathbf{t},\mathbf{s})$ is needed in Section 6 below, for equation (SWE) with $d \leq 3$. For this, let $\rho, \sigma \in S_n$ be such that

$$0 < t_{\rho(1)} < \cdots < t_{\rho(n)} < t \quad \text{and} \quad 0 < s_{\sigma(1)} < \cdots < s_{\sigma(n)} < t,$$

and denote $t_{\rho(n+1)} = s_{\sigma(n+1)} = t$. Then if $d \leq 2$, we have

$$\begin{aligned}\psi_n(\mathbf{t},\mathbf{s}) = \int_{\mathbb{R}^{2nd}} &\prod_{j=1}^{n} G(t_{\rho(j+1)} - t_{\rho(j)}, x_{\rho(j+1)} - x_{\rho(j)}) w(t_{\rho(1)}, x_{\rho(1)}) \\ &\times \prod_{j=1}^{n} G(s_{\sigma(j+1)} - s_{\sigma(j)}, y_{\sigma(j+1)} - y_{\sigma(j)}) w(s_{\sigma(1)}, y_{\sigma(1)}) \\ &\times \prod_{j=1}^{n} f(x_j - y_j)\, d\mathbf{x}\, d\mathbf{y},\end{aligned} \tag{32}$$

with the notation $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^{nd}$, whereas if $d = 3$,

$$\begin{aligned}\psi_n(\mathbf{t},\mathbf{s}) = \int_{\mathbb{R}^{2nd}} &\prod_{j=1}^{n} G(t_{\rho(j+1)} - t_{\rho(j)}, x_{\rho(j+1)} - dx_{\rho(j)}) w(t_{\rho(1)}, x_{\rho(1)}) \\ &\times \prod_{j=1}^{n} G(s_{\sigma(j+1)} - s_{\sigma(j)}, y_{\sigma(j+1)} - dy_{\sigma(j)}) w(s_{\sigma(1)}, y_{\sigma(1)}) \\ &\times \prod_{j=1}^{n} f(x_j - y_j).\end{aligned} \tag{33}$$

(In both integrals above, we use the notation $x_{\rho(n+1)} = y_{\sigma(n+1)} = x$.)

This concludes the summary of the results of [3] which are needed here.

4. Hyperbolic case: Existence of the solution. In this section, we prove the existence of a solution of equation (SWE) [given by (28)] in any space dimension $d \geq 1$, when f is a kernel of cases (i)–(iv). This yields the conclusion of Theorem 2.1(a) and (b) (with $p = 2$).

We let $G = G_{\mathrm{w}}$ and $w = w_{\mathrm{w}}$. We introduce the following constant:

$$K(\mu) := \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{1 + |\xi - \eta|^2} \mu(d\xi). \tag{34}$$

Note that $K(\mu) < \infty$ if and only if (DC) holds; see the proof of Lemma 8 in [24].

Note that (DC) is satisfied in cases (i) and (iv). In cases (ii) and (iii), (DC) holds if and only if

$$a < 2. \tag{35}$$

We define a constant K_{w} by

$$K_{\mathrm{w}} = \begin{cases} \mu(\mathbb{R}^d), & \text{in case (i)}, \\ 4K(\mu), & \text{in cases (ii) and (iii)}, \\ \pi, & \text{in case (iv)}. \end{cases} \tag{36}$$

We have the following preliminary result.

LEMMA 4.1. *Let f be a kernels of cases* (i)–(iv). *Assume that* (DC) *holds. For any $t > 0$ and for any $\mathbf{t} = (t_1, \dots, t_n)$ in $[0, t]^n$,*

$$\psi_n(\mathbf{t}, \mathbf{t}) \le (u_0 + t v_0)^2 K_{\mathrm{w}}^n (u_1, \dots, u_n)^{2-a},$$

where a is given by (9), *$u_j = t_{\rho(j+1)} - t_{\rho(j)}$ for $j = 1, \dots, n$, $t_{\rho(1)} < \cdots < t_{\rho(n)}$ for some $\rho \in S_n$, $t_{\rho(n+1)} = t$, and K_{w} is the constant defined in* (36).

PROOF. As in the proof of Lemma 3.2 of [3], by (31), (26) and (15), we obtain

$$\begin{aligned} &\psi_n(\mathbf{t}, \mathbf{t}) \\ &\quad = (u_0 + t_{\rho(1)} v_0)^2 \int_{\mathbb{R}^{nd}} \frac{\sin^2(u_1|\xi_1|)}{|\xi_1|^2} \cdots \\ &\qquad\qquad \times \frac{\sin^2(u_n|\xi_1 + \cdots + \xi_n|)}{|\xi_1 + \cdots + \xi_n|^2} \mu(d\xi_1) \cdots \mu(d\xi_n). \end{aligned}$$

We consider separately the four cases:

- *Case* (i). Using the fact that $|x^{-1} \sin x| \le 1$, we have

$$\psi_n(\mathbf{t}, \mathbf{t}) \le (u_0 + t v_0)^2 [\mu(\mathbb{R}^d)]^n (u_1, \dots, u_n)^2.$$

- *Case* (ii). This case was treated in Lemma 3.2 of [3].
- *Case* (iii). Let $c = c_{(\alpha_j)_j}$. Using the change of variables $\eta_j = \xi_1 + \cdots + \xi_j$,

$$\begin{aligned} \psi_n(\mathbf{t}, \mathbf{t}) &= c^n (u_0 + t_{\rho(1)} v_0)^2 \int_{\mathbb{R}^d} d\eta_1 \frac{\sin^2(u_1|\eta_1|)}{|\eta_1|^2} \prod_{j=1}^d |\eta_{1,j}|^{\alpha_j - 1} \\ &\quad \times \int_{\mathbb{R}^d} d\eta_2 \frac{\sin^2(u_2|\eta_2|)}{|\eta_2|^2} \prod_{j=1}^d |\eta_{2,j} - \eta_{1,j}|^{\alpha_j - 1} \\ &\quad \vdots \\ &\quad \times \int_{\mathbb{R}^d} d\eta_n \frac{\sin^2(u_n|\eta_n|)}{|\eta_n|^2} \prod_{j=1}^d |\eta_{n,j} - \eta_{n-1,j}|^{\alpha_j - 1}, \end{aligned}$$

where $\eta_i = (\eta_{i,j})_{j=1,\dots,d}$ with $\eta_{i,j} \in \mathbb{R}$. Note that for any $t > 0$ and $\eta \in \mathbb{R}^d$,

$$
\begin{aligned}
c \int_{\mathbb{R}^d} & \frac{\sin^2(t|\xi|)}{|\xi|^2} \prod_{j=1}^d |\xi_j - \eta_j|^{\alpha_j - 1} \, d\xi \\
&= c t^{2-a} \int_{\mathbb{R}^d} \frac{\sin^2(|\xi|)}{|\xi|^2} \prod_{j=1}^d |\xi_j - t\eta_j|^{\alpha_j - 1} \, d\xi \\
&= t^{2-a} \int_{\mathbb{R}^d} \frac{\sin^2(|\xi + t\eta|)}{|\xi + t\eta|^2} \mu(d\xi) \\
&\le 4 t^{2-a} \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1 + |\xi + t\eta|^2} \le 4 t^{2-a} K(\mu),
\end{aligned}
$$

since $(\sin(x)/x)^2 \le 4/(1 + x^2)$ for all $x > 0$. Hence

$$
\psi_n(\mathbf{t}, \mathbf{t}) \le (u_0 + t v_0)^2 \big(4K(\mu)\big)^n (u_1, \dots, u_n)^{2-a}.
$$

- *Case* (iv). Using the change of variables $\eta_j = \xi_1 + \cdots + \xi_j$, we have

$$
\psi_n(\mathbf{t}, \mathbf{t}) = (u_0 + t_{\rho(1)} v_0)^2 \int_{\mathbb{R}^n} \frac{\sin^2(u_1|\eta_1|)}{|\eta_1|^2} \cdots \frac{\sin^2(u_n|\eta_n|)}{|\eta_n|^2} \, d\eta_1 \cdots d\eta_n.
$$

Using (14), (15) and Plancherel's theorem, we obtain that for any $t > 0$,

$$
\int_{\mathbb{R}} \frac{\sin^2(t|\xi|)}{|\xi|^2} \, d\xi = \pi t.
$$

Hence

$$
\psi_n(\mathbf{t}, \mathbf{t}) = (u_0 + t v_0)^2 \pi^n u_1, \dots, u_n. \qquad \square
$$

The following result is an extension of Proposition 3.1 of [25] to the case of the fractional noise in time.

PROPOSITION 4.2. *Let f be a kernel of cases* (i)–(iv), *and $\rho_{\mathrm{w}}, a, K_{\mathrm{w}}$ be the constants given by* (8), (9), *respectively* (36). *Assume that* (DC) *holds*. *Then*:

(a) *for any $t > 0$ and for any integer $n \ge 1$*,

$$
\alpha_n(t) \le (u_0 + t v_0)^2 c^n K_{\mathrm{w}}^n \frac{t^{(2H+2-a)n}}{(n!)^{2-a}}, \tag{37}
$$

where $\alpha_n(t)$ is given by (30) *and c is a constant depending on H and a*;

(b) *for any $d \ge 1$*, *equation* (SWE) *has a solution $u(t,x)$* [*given by* (28)] *which has the following property*: *for any $x \in \mathbb{R}^d$ and for any $t > 0$*,

$$
E|u(t,x)|^2 \le c_1 (u_0 + t v_0)^2 \exp\big(c_2 K_{\mathrm{w}}^{1/(3-a)} t^{\rho_{\mathrm{w}}}\big),
$$

where $c_1 > 0$ is a constant depending on a, *and $c_2 > 0$ is a constant depending on H and a*.

PROOF. (a) We proceed as in the proof of Proposition 3.3 of [3].

For any $\mathbf{t} = (t_1, \dots, t_n) \in [0,t]^n$, we define $\beta(\mathbf{t}) = \prod_{j=1}^n u_j$, where $u_j = t_{\rho(j+1)} - t_{\rho(j)}$, and $\rho \in S_n$ is chosen such that $t_{\rho(1)} < \cdots < t_{\rho(n)}$, and $t_{\rho(n+1)} = t$.

By the Cauchy–Schwarz inequality, $\psi_n(\mathbf{t},\mathbf{s}) \le \psi_n(\mathbf{t},\mathbf{t})^{1/2}\psi_n(\mathbf{s},\mathbf{s})^{1/2}$. By Lemma 4.1, it follows that

$$\psi_n(\mathbf{t},\mathbf{s}) \le (u_0 + tv_0)^2 K_{\mathrm{w}}^n \big[\beta(\mathbf{t})\beta(\mathbf{s})\big]^{(2-a)/2}. \tag{38}$$

Using definition (30) of $\alpha_n(t)$ and (38), we obtain

$$\alpha_n(t) \le (u_0 + tv_0)^2 K_{\mathrm{w}}^n \alpha_H^n \int_{[0,t]^{2n}} \prod_{j=1}^n |t_j - s_j|^{2H-2} \big[\beta(\mathbf{t})\beta(\mathbf{s})\big]^{(2-a)/2}\, d\mathbf{t}\, d\mathbf{s}.$$

We now use the fact that for any $\varphi \in L^{1/H}(\mathbb{R}^n)$,

$$\alpha_H^n \int_{\mathbb{R}^{2n}} \prod_{j=1}^n |t_j - s_j|^{2H-2} |\varphi(\mathbf{t})||\varphi(\mathbf{s})|\, d\mathbf{t}\, d\mathbf{s} \le b_H^n \left(\int_{\mathbb{R}^n} |\varphi(\mathbf{t})|^{1/H}\, d\mathbf{t}\right)^{2H} \tag{39}$$

for some constant $b_H > 0$; see Lemma B.3, Appendix B. We obtain

$$\begin{aligned}\alpha_n(t) &\le (u_0 + tv_0)^2 K_{\mathrm{w}}^n b_H^n \left(\int_{[0,t]^n} \prod_{j=1}^n \beta(\mathbf{t})^{(2-a)/(2H)}\, d\mathbf{t}\right)^{2H} \\ &= (u_0 + tv_0)^2 K_{\mathrm{w}}^n b_H^n \left(n! \int_{T_n(t)} \big[(t-t_n)\cdots(t_2-t_1)\big]^{(2-a)/(2H)}\, d\mathbf{t}\right)^{2H},\end{aligned}$$

where $T_n(t) = \{0 < t_1 < \cdots < t_n < t\}$. By Lemma 3.5 of [6], for any $h > -1$,

$$\int_{T_n(t)} \big[(t-t_n)(t_n - t_{n-1})\cdots(t_2-t_1)\big]^h d\mathbf{t} = \frac{\Gamma(1+h)^{n+1}}{\Gamma((1+h)n+1)} t^{(1+h)n}.$$

By Stirling's formula, $\Gamma((1+h)n+1) \sim C_n (n!)^{1+h}$, where C_n is such that $\lambda^{-n} \le C_n \le \lambda^n$ for some constant $\lambda > 1$ depending on h; see the proof of Lemma A.1, Appendix A. Hence

$$\int_{T_n(t)} \big[(t-t_n)(t_n - t_{n-1})\cdots(t_2-t_1)\big]^h\, d\mathbf{t} \le \frac{\Gamma(1+h)^n c_0^n}{(n!)^{1+h}} t^{(1+h)n}$$

for some $c_0 > 0$. In our case, $h = (2-a)/(2H)$. We obtain:

$$\begin{aligned}\alpha_n(t) &\le (u_0 + tv_0)^2 K_{\mathrm{w}}^n b_H^n \left(n! \frac{\Gamma(1+h)^n c_1^n}{(n!)^{1+h}} t^{(1+h)n}\right)^{2H} \\ &= (u_0 + tv_0)^2 K_{\mathrm{w}}^n c^n \frac{1}{(n!)^{2-a}} t^{(2H+2-a)n},\end{aligned}$$

where $c = b_H \Gamma(1+h)^{2H} c_0^{2H}$ depends on H and a.

(b) We use (29) and the result from part (a). We obtain that for any $t > 0$,

$$E|u(t,x)|^2 \leq (u_0 + tv_0)^2 \sum_{n\geq 0} \frac{c^n K_{\mathrm{w}}^n t^{(2H+2-a)n}}{(n!)^{3-a}}.$$

Since this series is convergent for any fixed $t > 0$, this proves the existence result. Now, using Lemma A.1 (Appendix A), we have that for all $t > 0$

$$E|u(t,x)|^2 \leq c_1 (u_0 + tv_0)^2 \exp\bigl(c_2' \bigl(c K_{\mathrm{w}} t^{2H+2-a}\bigr)^{1/(3-a)}\bigr),$$

where $c_1 > 0$ and $c_2' > 0$ are some constants depending on a. The conclusion follows taking $c_2 = c_2' c^{1/(3-a)}$. $\square$

5. Hyperbolic case: Upper bound on the moments. In this section, we give an upper bound for the moments of order $p > 2$ of a solution of equation (SWE) [given by (28)]. This yields the conclusion of Theorem 2.1(b).

Recall that this solution of (SWE) has the Wiener chaos expansion given by (28). This means that $u(t,x) = \sum_{n\geq 0} J_n(t,x)$ where $J_n(t,x)$ is in the nth Wiener chaos $\mathcal{H}_n$ associated to the noise W, and

$$E|u(t,x)|^2 = \sum_{n\geq 0} E|J_n(t,x)|^2 = \sum_{n\geq 0} \frac{1}{n!}\alpha_n(t),$$

where $\alpha_n(t)$ is defined in (29) and is estimated by (37).

The following result is an extension of Theorem 3.2 of [25] to the case of the fractional noise in time.

PROPOSITION 5.1. *Let f be one of the kernels* (i)–(iv), *and $\rho_{\mathrm{w}}, a, K_{\mathrm{w}}$ be the constants given by* (8), (9), *respectively* (36). *Assume that* (DC) *holds. Let $u(t,x)$ be a solution of* (SWE), *given by* (28). *Then for any $p \geq 2$, for any $x \in \mathbb{R}^d$ and for any $t > 0$,*

$$E|u(t,x)|^p \leq c_1^p (u_0 + tv_0)^p \exp\bigl(c_2 K_{\mathrm{w}}^{1/(3-a)} p^{(4-a)/(3-a)} t^{\rho_{\mathrm{w}}}\bigr),$$

where $c_1 > 0$ is a constant depending on a, and $c_2 > 0$ is a constant depending on H and a.

PROOF. When $p = 2$, the result is given by Propostion 4.2.

When $p > 2$, we use the same idea as in the proof of Theorem 4.1 of [3]. We denote by $\|\cdot\|_p$ the $L^p(\Omega)$-norm. We use the fact that for elements in a *fixed* Wiener chaos $\mathcal{H}_n$, the $\|\cdot\|_p$-norms are equivalent; see the last line of page 62 of [43] with $q = p$ and $p = 2$. More precisely,

$$\|J_n(t,x)\|_p \leq (p-1)^{n/2} \|J_n(t,x)\|_2 = (p-1)^{n/2} \biggl(\frac{1}{n!}\alpha_n(t)\biggr)^{1/2}.$$

Using (37), we obtain

$$\|J_n(t,x)\|_p \le (u_0+tv_0)C^n_{p,K_{\mathrm{w}}} t^{n(2H+2-a)/2}\frac{1}{(n!)^{(3-a)/2}},$$

where $C_{p,K_{\mathrm{w}}}=(p-1)^{1/2}c^{1/2}K_{\mathrm{w}}^{1/2}$ and c depends on H and a.

Recall Minkowski's inequality for integrals (see Appendix A.1 of [50]),

$$\left[\int_Y\left(\int_X|F(x,y)|\mu(dx)\right)^p\nu(dy)\right]^{1/p} \le \int_X\left(\int_Y|F(x,y)|^p\nu(dy)\right)^{1/p}\mu(dx).$$

We use this inequality for $(X,\mathcal{X})=(\mathbb{N},2^{\mathbb{N}})$ with μ the counting measure, $(Y,\mathcal{Y},\nu)=(\Omega,\mathcal{F},P)$ and $F(n,\omega)=J_n(\omega,t,x)$. We have

$$\begin{aligned}\|u(t,x)\|_p &= \left\|\sum_{n\ge 0}J_n(t,x)\right\|_p \le \sum_{n\ge 0}\|J_n(t,x)\|_p\\ &\le (u_0+tv_0)\sum_{n\ge 0}\frac{C^n_{p,K_{\mathrm{w}}}t^{n(2H+2-a)/2}}{(n!)^{(3-a)/2}}.\end{aligned}$$

Using Lemma A.1 (Appendix A), we infer that for any $t>0$,

$$\|u(t,x)\|_p \le c_1(u_0+tv_0)\exp\{c_2'(C_{p,K_{\mathrm{w}}}t^{(2H+2-a)/2})^{2/(3-a)}\},$$

where $c_1>0$ and $c_2'>0$ are some constants depending on a. The conclusion follows taking $c_2=c_2'c^{1/(3-a)}$, since $\frac{2H+2-a}{2}\cdot\frac{2}{3-a}=\rho_{\mathrm{w}}$ and

$$pC^{2/(3-a)}_{p,K_{\mathrm{w}}}=p(p-1)^{1/(3-a)}c^{1/(3-a)}K_{\mathrm{w}}^{1/(3-a)}. \qquad \square$$

6. Hyperbolic case: FK representation for the second moment. In this section, we develop a Feynman–Kac (FK) representation for the second moment of a solution $u(t,x)$ of the wave equation (SWE) [given by (28)], similar to the one obtained in [26] in the case of white noise in time. Due to the fractional component of the noise, our representation is based on a Poisson random measure on $\mathbb{R}^2_+$, rather than a simple Poisson process. This extension follows the approach of [2] for the parabolic case.

The following theorem is the main result of this section. This theorem is valid for any function f for which covariance (2) of the noise W is well defined, but may not be valid in case (iv) (since in this case, f is a distribution). Theorem 6.1 will be used in Section 7 to obtain a lower bound for the second moment of a solution to (SWE) in cases (i)–(iii). Case (iv) will be treated differently using an approximation based on case (ii).

THEOREM 6.1. *Suppose that equation* (SWE) *with* $d\le 3$ *has a solution* $u(t,x)$ [*given by* (28)], *where* $W=\{W(\varphi);\varphi\in\mathcal{H}\}$ *is a zero-mean Gaussian process with covariance specified by* (1) *and* (2), *and* f *is a nonnegative function on*

$\mathbb{R}^d$, which is the Fourier transform of a tempered measure μ on $\mathbb{R}^d$. Then for any $t > 0, x \in \mathbb{R}^d$,

$$\begin{aligned} E\lvert u(t,x)\rvert^2 &= e^{t^2} \sum_{n\geq 0} \sum_{\substack{i_1,\dots,i_n \\ \text{distinct}}} E_x\Biggl[w_{\mathrm{w}}(t-\tau_n, X^1_{\tau_n}) w_{\mathrm{w}}(t-\tau'_n, X^2_{\tau'_n}) \prod_{j=1}^n (\tau_j - \tau_{j-1}) \\ &\qquad \times \prod_{j=1}^n (\tau'_j - \tau'_{j-1}) \prod_{j=1}^n f(X^1_{T_{i_j}} - X^2_{S_{i_j}}) \alpha_H^n \\ &\qquad \times \prod_{j=1}^n |T_{i_j} - S_{i_j}|^{2H-2} 1_{B_{i_1,\dots,i_n}(t)} \Biggr], \end{aligned}$$

where, by convention, the term for $n = 0$ is taken to be $w_{\mathrm{w}}(t,x)^2$, and w_{w} is defined by (16). Here:

- *$N = \sum_{i\geq 1} \delta_{P_i}$ is a Poisson random measure on $\mathbb{R}^2_+$ of intensity the Lebesgue measure, with $P_i = (T_i, S_i)$;*
- *$B_{i_1,\dots,i_n}(t)$ is the event that N has points $P_{i_1},\dots,P_{i_n}$ in $[0,t]^2$;*
- *$\tau_1 \leq \cdots \leq \tau_n$ and $\tau'_1 \leq \cdots \leq \tau'_n$ are the points $T_{i_1},\dots,T_{i_n}$, respectively $S_{i_1},\dots,S_{i_n}$ arranged in increasing order;*
- *the processes $X^1 = (X^1_s)_{s\in[0,t]}$ and $X^2 = (X^2_s)_{s\in[0,t]}$ are defined by (41) and (42) below, and we denote by P_x a probability measure under which $X^1_0 = X^2_0 = x$. (E_x stands for the expectation with respect to P_x.)*

The processes X^1 and X^2 are constructed as in [26], using the coordinates of the points of N on the two axes. We explain this construction below. On the event $B_{i_1,\dots,i_n}(t)$, we arrange the two sets of points $\{T_{i_1},\dots,T_{i_n}\}$ and $\{S_{i_1},\dots,S_{i_n}\}$ in increasing order as $\tau_1 \leq \cdots \leq \tau_n$, respectively $\tau'_1 \leq \cdots \leq \tau'_n$. More precisely, if we denote $U_j = T_{i_j}$ and $V_j = S_{i_j}$ for $j = 1,\dots,n$, then there exist some permutations ρ and σ of $\{1,\dots,n\}$ such that

$$U_{\rho(n)} \leq U_{\rho(n-1)} \leq \cdots \leq U_{\rho(1)} \qquad \text{and} \qquad V_{\sigma(n)} \leq V_{\sigma(n-1)} \leq \cdots \leq V_{\sigma(1)}.$$

We let $\tau_j = U_{\rho(n+1-j)}$ and $\tau'_j = V_{\sigma(n+1-j)}$ for any $j = 1,\dots,n$.

We let $(\Theta^1_i)_{i\geq 1}$ and $(\Theta^2_i)_{i\geq 1}$ be two independent i.i.d. collections of random variables with the same law as Θ_0, where Θ_0 is a random variable with values in $\mathbb{R}^d$ such that if $d \leq 2$, Θ_0 has density function $G_{\mathrm{w}}(1,\cdot)$, and if $d = 3$, Θ_0 has distribution $G_{\mathrm{w}}(1,\cdot)$. The importance of the variable Θ_0 stems from the fact that for any $t > 0$,

$$\frac{G_{\mathrm{w}}(t,\cdot)}{t} \quad \text{is the density/distribution of } t\Theta_0. \tag{40}$$

Using the points $\tau_1 \le \cdots \le \tau_n$ and the variables $(\Theta_i^1)_{i\ge 1}$, we construct the process $X^1 = (X_s^1)_{s\in[0,t]}$ by setting

$$X_s^1 = X_{\tau_i}^1 + (s-\tau_i)\Theta_{i+1}^1 \qquad \text{if } \tau_i \le s \le \tau_{i+1} \tag{41}$$

for any $1 \le i \le n$, where $\tau_0 = 0$, $\tau_{n+1} = t$ and $X_0^1 = 0$. We use a similar construction for the process $X^2 = (X_s^2)_{s\in[0,t]}$ using the points $\tau_1' \le \cdots \le \tau_n'$ and the variables $(\Theta_i^2)_{i\ge 1}$, that is, $\tau_0' = 0$, $\tau_{n+1}' = t$, $X_0^2 = 0$ and for any $1 \le i \le n$,

$$X_s^2 = X_{\tau_i'}^2 + (s-\tau_i')\Theta_{i+1}^2 \qquad \text{if } \tau_i' \le s \le \tau_{i+1}'. \tag{42}$$

We now give some remarks about the statement of Theorem 6.1.

REMARK 6.2. A similar formula can be obtained for $E[u(t,x)u(s,y)]$ using the points of N in $[0,t]\times[0,s]$ and assuming that $X_0^1 = x$ and $X_0^2 = y$.

REMARK 6.3. Note that $|T_{i_j} - S_{i_j}|^{2H-2} = \infty$ if the point (T_{i_j}, S_{i_j}) falls on the diagonal $D = \{(s,s); 0 \le s \le t\}$ of the square $[0,t]^2$. This is not a problem since with probability 1, N has no points in D: $P(N(D)=0) = e^{-\mathrm{Leb}(D)} = 1$.

REMARK 6.4. Without loss of generality we may assume that $\tau_1 < \cdots < \tau_n$ and $\tau_1' < \cdots < \tau_n'$ since the event for which $\tau_j = \tau_{j-1}$ (or $\tau_j' = \tau_{j-1}'$) for some $j = 1,\dots,n$ has probability zero: with probability 1, no vertical (or horizontal) line contains two distinct points of N; see page 223 of [47].

REMARK 6.5. Theorem 6.1 is valid for any function f, not necessarily as in one of the cases (i)–(iii). In fact, this representation remains valid if we replace $\alpha_H|t-s|^{2H-2}$ in (2) by a function $\eta(t,s)$, provided that $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ defines an inner product. We only need to assume that a solution of (SWE) [given by (28)] exists. In the new representation, $\alpha_H|T_{i_j} - S_{i_j}|^{2H-2}$ is replaced by $\eta(t-T_{i_j}, t-S_{i_j})$.

We now introduce the necessary ingredients for the proof of Theorem 6.1.

Recall first that if $(N_t)_{t\ge 0}$ is a Poisson process on $\mathbb{R}_+$ of rate 1 with jump times $\tau_1 < \tau_2 < \cdots$, then the conditional distribution of $(\tau_1,\dots,\tau_n)$ given $N_t = n$ coincides with the distribution of the order statistics of a sample of size n from the uniform distribution on $[0,t]$. This property lies at the core of the FK formula obtained in [26] and can be seen very easily as follows. For any $t>0$ fixed, the process $(N_s)_{s\in[0,t]}$ can be constructed as $N_s = \sum_{i=1}^{Y} 1_{\{X_i \le s\}}$, where $(X_i)_{i\ge 1}$ are i.i.d. random variables with a uniform distribution on $[0,t]$, and Y is an independent Poisson random variable with mean t. If $N_t = n$, the jump times of N in $[0,t]$ coincide with the order statistics $X_{(1)} < \cdots < X_{(n)}$.

A similar property holds for the planar Poisson process. This basic observation has enabled the first author to obtain in [2] an FK formula similar to that of [26] in the case of the heat equation with fractional noise in time. In this section, we develop a similar formula for the wave equation with $d \leq 3$.

More precisely, let N be a Poisson random measure as in Theorem 6.1. Since the Lebesgue measure does not have any atoms, N is a.s. simple, that is, $N(\{\mathbf{t}\}) \leq 1$ for all $\mathbf{t} = (t_1, t_2) \in \mathbb{R}_+^2$ a.s. (see Exercise 2.4 of [37]). This means that with probability 1, the points $(P_i)_{i \geq 1}$ are distinct. For any $t > 0$ fixed, we consider the event $B_{i_1,\dots,i_n}(t)$ for distinct indices $i_1, \dots, i_n \geq 1$.

The following result plays an important role in the present paper; see also Problem 5.2, page 162 of [47].

LEMMA 6.6. *Let $N = \sum_{i \geq 1} \delta_{P_i}$ be a Poisson random measure on $\mathbb{R}_+^2$ of intensity the Lebesque measure, with $P_i = (S_i, T_i)$. For $t > 0$ and distinct indices $i_1, \dots, i_n$, let $B_{i_1,\dots,i_n}(t)$ be the event that N has points $P_{i_1}, \dots, P_{i_n}$ in $[0, t]^2$. Given $B_{i_1,\dots,i_n}(t)$, both vectors $(P_{i_1}, \dots, P_{i_n})$ and $(\mathbf{t} - P_{i_1}, \dots, \mathbf{t} - P_{i_n})$ have a uniform distribution on $[0, t]^{2n}$, where $\mathbf{t} = (t, t) \in \mathbb{R}_+^2$.*

PROOF. The restriction of N to $[0, t]^2$ can be constructed as $N = \sum_{i=1}^{Y} \delta_{X_i}$, where $(X_i)_{i \geq 1}$ are i.i.d. random variables with a uniform distribution on $[0, t]^2$, and Y is an independent Poisson random variable with mean t^2. If N has points $P_{i_1}, \dots, P_{i_n}$ in $[0, t]^2$, the vector $(P_{i_1}, \dots, P_{i_n})$ of the n points coincides with a vector $(X_{j_1}, \dots, X_{j_n})$ for some distinct indices $j_1, \dots, j_n$, which clearly has a uniform distribution on $[0, t]^{2n}$. The argument for the vector $(\mathbf{t} - P_{i_1}, \dots, \mathbf{t} - P_{i_n})$ is similar; see Lemma 2.1 of [2] for an alternative proof. $\square$

As a consequence of the previous lemma, any n-fold integral over $([0, t]^2)^n$ of a deterministic function F has a stochastic representation based on the points of N; see page 257 of [2] for the proof.

COROLLARY 6.7. *For any measurable function $F : [0, t]^{2n} \to \mathbb{R}$ which is either bounded or nonnegative, we have*

$$\int_{[0,t]^{2n}} F(t_1, s_1, \dots, t_n, s_n)\, d\mathbf{t}\, d\mathbf{s} = n! e^{t^2} \sum_{\substack{i_1,\dots,i_n \\ \text{distinct}}} E\big[F(t - T_{i_1}, t - S_{i_1}, \dots, t - T_{i_1}, t - S_{i_n}) 1_{B_{i_1,\dots,i_n}(t)}\big],$$

where $\mathbf{t} = (t_1, \dots, t_n)$ and $\mathbf{s} = (s_1, \dots, s_n)$ with $t_i \in [0, t]$ and $s_i \in [0, t]$.

The next result gives a stochastic representation for the nth term of series (29).

LEMMA 6.8. *For any $t > 0$ and for any integer $n \geq 1$, we have*

$$\alpha_n(t) = n!\alpha_H^n e^{t^2} \times \sum_{\substack{i_1,\dots,i_n \\ \text{distinct}}} E\Bigg[\prod_{j=1}^n |T_{i_j} - S_{i_j}|^{2H-2} \times \psi_n(t - T_{i_1}, \dots, t - T_{i_n}, t - S_{i_1}, \dots, t - S_{i_n}) 1_{B_{i_1,\dots,i_n}(t)}\Bigg],$$

where $\psi_n(\mathbf{t}, \mathbf{s})$ is given by (31).

PROOF. The integral on the right-hand side of (30) can be represented in the desired form by applying Corollary 6.7 to the function

$$F(t_1, s_1, \dots, t_n, s_n) = \alpha_H^n \prod_{j=1}^n |t_j - s_j|^{2H-2} \psi_n(\mathbf{t}, \mathbf{s}).$$

□

The next result will be used to evaluate the term $\psi_n(t - T_{i_1}, \dots, t - T_{i_n}, t - S_{i_1}, \dots, t - S_{i_n})$ which appears in Lemma 6.8. For simplicity, we work first with some nonrandom points $(t_1, s_1), \dots, (t_n, s_n)$ in $[0, t]^2$. These points will be replaced later by $(T_{i_1}, S_{i_1}), \dots, (T_{i_n}, S_{i_n})$.

LEMMA 6.9. *Let $(t_1, s_1), \dots, (t_n, s_n) \in [0, t]^2$. Let $\rho, \sigma \in S_n$ be such that*

$$0 < t_{\rho(n)} < \cdots < t_{\rho(1)} < t \quad \text{and} \quad 0 < s_{\sigma(n)} < \cdots < s_{\sigma(1)} < t.$$

If $d \leq 2$, then

$$\begin{aligned}
&\psi_n(t - t_1, \dots, t - t_n, t - s_1, \dots, t - s_n) \\
&\quad = \int_{\mathbb{R}^{2nd}} d\mathbf{z}\, d\mathbf{w} \prod_{j=1}^n f\left(\sum_{k=1}^{n+1-\rho^{-1}(j)} z_k - \sum_{k=1}^{n+1-\sigma^{-1}(j)} w_k\right) \\
&\qquad \times G_{\mathrm{w}}(t_{\rho(n)}, z_1) G_{\mathrm{w}}(t_{\rho(n-1)} - t_{\rho(n)}, z_2) \cdots G_{\mathrm{w}}(t_{\rho(1)} - t_{\rho(2)}, z_n) \\
&\qquad \times G_{\mathrm{w}}(s_{\sigma(n)}, w_1) G_{\mathrm{w}}(s_{\sigma(n-1)} - s_{\sigma(n)}, w_2) \cdots G_{\mathrm{w}}(s_{\sigma(1)} - s_{\sigma(2)}, w_n) \\
&\qquad \times w\left(t - t_{\rho(1)}, x + \sum_{k=1}^n z_k\right) w\left(t - s_{\sigma(1)}, x + \sum_{k=1}^n w_k\right),
\end{aligned}$$

where $\mathbf{z} = (z_1, \dots, z_n)$ and $\mathbf{w} = (w_1, \dots, w_n)$ with $z_i \in \mathbb{R}^d$, $w_i \in \mathbb{R}^d$. A similar relation holds for $d = 3$, replacing $G_{\mathrm{w}}(t_{\rho(n)}, z_1)\, dz_1$ by $G_{\mathrm{w}}(t_{\rho(n)}, dz_1)$, etc.

PROOF. Assume first that $d \leq 2$. We use the alternative definition (32) of $\psi_n(\mathbf{t},\mathbf{s})$. We proceed as in the first part of the proof of Lemma 2.2 of [2]. Denote $t_{\rho(n+1)} = s_{\sigma(n+1)} = 0$ and $x_{\rho(n+1)} = y_{\sigma(n+1)} = x$. Note that

$$0 < t - t_{\rho(1)} < \cdots < t - t_{\rho(n)} < t \quad \text{and} \quad 0 < t - s_{\sigma(1)} < \cdots < t - s_{\sigma(n)} < t$$

and $G_{\mathrm{w}}(t,x) = G_{\mathrm{w}}(t,-x)$. By definition, $\psi_n(t-t_1,\ldots,t-t_n,t-s_1,\ldots,t-s_n)$ is equal to

$$\begin{aligned}\int_{\mathbb{R}^{2nd}} d\mathbf{x}\,d\mathbf{y} \prod_{j=1}^n G_{\mathrm{w}}(t_{\rho(j)} - t_{\rho(j+1)}, x_{\rho(j)} - x_{\rho(j+1)}) w(t - t_{\rho(1)}, x_{\rho(1)}) \\ \times \prod_{j=1}^n G_{\mathrm{w}}(s_{\sigma(j)} - s_{\sigma(j+1)}, y_{\sigma(j)} - y_{\sigma(j+1)}) w(t - s_{\sigma(1)}, y_{\sigma(1)}) \\ \times \prod_{j=1}^n f(x_j - y_j).\end{aligned}$$

The result follows by the change of variables $x_{\rho(j)} - x_{\rho(j+1)} = z_{n+1-j}$ and $y_{\sigma(j)} - y_{\sigma(j+1)} = w_{n+1-j}$ for $j = 1,\ldots,n$.

The same argument works also for $d = 3$, using the alternative definition (33) of $\psi_n(\mathbf{t},\mathbf{s})$. To see this, assume for simplicity that $n = 2$, $0 < t_1 < t_2 < t$ and $0 < s_2 < s_1 < t$. (The same argument applies in the general case.) Then $\psi_2(t - t_1, t - t_2, t - s_1, t - s_2)$ is equal to

$$\begin{aligned}\int_{\mathbb{R}^{4d}} h(x_1,x_2,y_1,y_2) G_{\mathrm{w}}(t_1, dx_1 - x) G_{\mathrm{w}}(t_2 - t_1, dx_2 - x_1) \\ \times G_{\mathrm{w}}(s_2, dy_2 - x) G_{\mathrm{w}}(s_1 - s_2, dy_1 - y_2),\end{aligned}$$

where $h(x_1,x_2,y_1,y_2) = f(x_1 - y_1) f(x_2 - y_2) w(t - t_2, x_2) w(t - s_1, y_1)$ and we used the fact that $G_{\mathrm{w}}(t, a - dx) = G_{\mathrm{w}}(t, dx - a)$. We claim that for any nonnegative measurable function $\varphi : \mathbb{R}^{4d} \to \mathbb{R}$,

$$\begin{aligned}&\int_{\mathbb{R}^{4d}} \varphi(x_1,x_2,y_1,y_2) G_{\mathrm{w}}(t_1, dx_1 - x) G_{\mathrm{w}}(t_2 - t_1, dx_2 - x_1) \\ &\quad \times G_{\mathrm{w}}(s_2, dy_2 - x) G_{\mathrm{w}}(s_1 - s_2, dy_1 - y_2) \\ &\quad = \int_{\mathbb{R}^{4d}} \varphi(x + z_1, x + z_1 + z_2, x + w_1 + w_2, x + w_1) \\ &\qquad \times G_{\mathrm{w}}(t_1, dz_1) G_{\mathrm{w}}(t_2 - t_1, dz_2) G_{\mathrm{w}}(s_2, dw_1) G_{\mathrm{w}}(s_1 - s_2, dw_2).\end{aligned} \tag{43}$$

(This means that we can apply informally the change of variables $x_1 - x = z_1$, $x_2 - x_1 = z_2$ and $y_2 - x = w_1$, $y_1 - y_2 = w_2$.) Assuming that $\varphi(x_1,x_2,y_1,y_2) = \phi_1(x_1)\phi_2(x_2)\psi_1(y_1)\psi_2(y_2)$, relation (43) follows using the fact that for any nonnegative measurable function $\phi : \mathbb{R}^d \to \mathbb{R}$,

$$\int_{\mathbb{R}^d} \phi(x) G_{\mathrm{w}}(t, dx - a) = \int_{\mathbb{R}^d} \phi(a + y) G_{\mathrm{w}}(t, dy).$$

The case of an arbitrary function φ follows by approximation. The conclusion follows applying (43) to the function $\varphi = h$. $\square$

REMARK 6.10. In the case of the heat equation, $G_{\mathrm{h}}(t-s,\cdot)$ is the density of $B^1_t - B^1_s$, where $(B^1_t)_{t\geq 0}$ is a d-dimensional Brownian motion, and the product

$$G_{\mathrm{h}}(t_{\rho(n)}, z_1)G_{\mathrm{h}}(t_{\rho(n-1)} - t_{\rho(n)}, z_2)\cdots G_{\mathrm{h}}(t_{\rho(1)} - t_{\rho(2)}, z_n),$$

which appears in Lemma 6.9 is the density of the random vector

$$(B^1_{t_{\rho(n)}}, B^1_{t_{\rho(n-1)}} - B^1_{t_{\rho(n)}}, \dots, B^1_{t_{\rho(1)}} - B^1_{t_{\rho(2)}}).$$

Applying a similar argument for the other n-term product (depending on s) and using an independent Brownian motion $(B^2_t)_{t\geq 0}$, we infer that

$$\begin{aligned}&\psi_n(t\mathbf{e} - \mathbf{t}, t\mathbf{e} - \mathbf{s})\\ &\quad = E\left[w\big(t - t_{\rho(1)}, x + B^1_{t_{\rho(1)}}\big)w\big(t - s_{\sigma(1)}, x + B^2_{s_{\sigma(1)}}\big)\prod_{j=1}^n f\big(B^1_{t_j} - B^2_{s_j}\big)\right],\end{aligned}$$

where $\mathbf{e} = (1, \dots, 1) \in \mathbb{R}^n$, $\mathbf{t} = (t_1, \dots, t_n)$ and $\mathbf{s} = (s_1, \dots, s_n)$. Something similar will happen in the case of the wave equation, conditionally on N.

REMARK 6.11. Due to (40), when $d \leq 2$, the product

$$\frac{G_{\mathrm{w}}(\tau_1, z_1)}{\tau_1}\cdot\frac{G_{\mathrm{w}}(\tau_2 - \tau_1, z_2)}{\tau_2 - \tau_1}\cdots\frac{G_{\mathrm{w}}(\tau_n - \tau_{n-1}, z_n)}{\tau_n - \tau_{n-1}}$$

is the conditional density of $\mathbf{Y}^1 = (X^1_{\tau_1}, X^1_{\tau_2} - X^1_{\tau_1}, \dots, X^1_{\tau_n} - X^1_{\tau_{n-1}})$ given N. Let $\mathbf{Y}^2 = (X^2_{\tau'_1}, X^2_{\tau'_2} - X^2_{\tau'_1}, \dots, X^2_{\tau'_n} - X^2_{\tau'_{n-1}})$. Since X^1 and X^2 are conditionally independent given N,

$$\prod_{j=1}^n \frac{G_{\mathrm{w}}(\tau_j - \tau_{j-1}, z_j)}{\tau_j - \tau_{j-1}}\prod_{j=1}^n \frac{G_{\mathrm{w}}(\tau'_j - \tau'_{j-1}, w_j)}{\tau'_j - \tau'_{j-1}}$$

is the conditional density of $(\mathbf{Y}^1, \mathbf{Y}^2)$ given N. A similar thing happens when $d = 3$. Therefore, for the wave equation, the processes X^1, X^2 play the same role (conditionally on N), as the Brownian motions B^1, B^2 for the heat equation; see Remark 6.10.

PROOF OF THEOREM 6.1. By applying Lemma 6.9 to the points $(t_j, s_j) = (T_{i_j}, S_{i_j})$ we obtain that on the event $B_{i_1,\dots,i_n}(t)$,

$$\begin{aligned}&\psi_n(t - T_{i_1}, \dots, t - T_{i_n}, t - S_{i_1}, \dots, t - S_{i_n})\\ &\quad = \int_{\mathbb{R}^{2nd}} d\mathbf{z}\, d\mathbf{w}\prod_{j=1}^{nd} f\left(\sum_{k=1}^{n+1-\rho^{-1}(j)} z_k - \sum_{k=1}^{n+1-\sigma^{-1}(j)} w_k\right)\end{aligned}$$

$$\begin{aligned}&\times G_{\mathrm{w}}(\tau_1, z_1)G_{\mathrm{w}}(\tau_2-\tau_1, z_2)\cdots G_{\mathrm{w}}(\tau_n-\tau_{n-1}, z_n)\\ &\times G_{\mathrm{w}}(\tau_1', w_1)G_{\mathrm{w}}(\tau_2'-\tau_1', w_2)\cdots G_{\mathrm{w}}(\tau_n'-\tau_{n-1}', w_n)\\ &\times w_{\mathrm{w}}\left(t-\tau_n, x+\sum_{k=1}^n z_k\right)w_{\mathrm{w}}\left(t-\tau_n', x+\sum_{k=1}^n w_k\right),\end{aligned}$$

assuming that $d\le 2$. A similar identity holds for $d=3$ replacing $G_{\mathrm{w}}(\tau_1,z_1)\,dz_1$ by $G_{\mathrm{w}}(\tau_1,dz_1)$, and so on. Inside this integral, we multiply and divide by $\prod_{j=1}^n(\tau_j-\tau_{j-1})\prod_{j=1}^n(\tau_j'-\tau_{j-1}')$.

We assume that $X_0^1=X_0^2=0$. Using Remark 6.11, we infer that on the event $B_{i_1,\dots,i_n}(t)$, $\psi_n(t-T_{i_1},\dots,t-T_{i_n},t-S_{i_1},\dots,t-S_{i_n})$ is equal to the conditional expectation of

$$\begin{aligned}&\prod_{j=1}^n f\left(\sum_{k=1}^{n+1-\rho^{-1}(j)}(X_{\tau_k}^1-X_{\tau_{k-1}}^1)-\sum_{k=1}^{n+1-\rho^{-1}(j)}(X_{\tau_k'}^2-X_{\tau_{k-1}'}^2)\right)\\ &\qquad\times w_{\mathrm{w}}\left(t-\tau_n, x+\sum_{k=1}^n(X_{\tau_k}^1-X_{\tau_{k-1}}^1)\right)w_{\mathrm{w}}\left(t-\tau_n', x+\sum_{k=1}^n(X_{\tau_k'}^2-X_{\tau_{k-1}'}^2)\right)\\ &\qquad\times\prod_{j=1}^n(\tau_j-\tau_{j-1})\prod_{j=1}^n(\tau_j'-\tau_{j-1}')\end{aligned}$$

given N. Note that

$$\sum_{k=1}^{n+1-\rho^{-1}(j)}(X_{\tau_k}^1-X_{\tau_{k-1}}^1)=X_{\tau_{n+1-\rho^{-1}(j)}}^1\quad\text{and}\quad\sum_{k=1}^n(X_{\tau_k}^1-X_{\tau_{k-1}}^1)=X_{\tau_n}^1$$

(these are telescopic sums whose first term is $X_{\tau_0}^1=0$). Recall that $\tau_k=U_{\rho(n+1-k)}$ for any $k=1,\dots,n$ (where $U_j=T_{i_j}$). Hence

$$\tau_{n+1-\rho^{-1}(j)}=U_{\rho(n+1-n-1+\rho^{-1}(j))}=U_{\rho(\rho^{-1}(j))}=U_j=T_{i_j}.$$

A similar argument applies to the terms depending on X^2. We obtain that on the event $B_{i_1,\dots,i_n}(t)$,

$$\begin{aligned}&\psi_n(t-T_{i_1},\dots,t-T_{i_n},t-S_{i_1},\dots,t-S_{i_n})\\ &\quad=E\left[\prod_{j=1}^n f(X_{T_{i_j}}^1-X_{S_{i_j}}^2)w_{\mathrm{w}}(t-\tau_n,x+X_{\tau_n}^1)w_{\mathrm{w}}(t-\tau_n',x+X_{\tau_n'}^2)\right.\\ &\qquad\qquad\left.\times\prod_{j=1}^n(\tau_j-\tau_{j-1})\prod_{j=1}^n(\tau_j'-\tau_{j-1}')\middle|N\right].\end{aligned}$$

Looking now back at the representation of $\alpha_n(t)$ (Lemma 6.8), we obtain

$$
\begin{aligned}
&\frac{1}{n!}\alpha_n(t) \\
&\quad = e^{t^2} \sum_{\substack{i_1,\dots,i_n \\ \text{distinct}}} E\Bigg[1_{B_{i_1,\dots,i_n}(t)} \prod_{j=1}^n |T_{i_j} - S_{i_j}|^{2H-2} \\
&\qquad\qquad \times E\Bigg[\prod_{j=1}^n f(X^1_{T_{i_j}} - X^2_{S_{i_j}}) w_{\mathrm{w}}(t-\tau_n, x + X^1_{\tau_n}) \\
&\qquad\qquad\qquad \times w_{\mathrm{w}}(t-\tau'_n, x + X^2_{\tau'_n}) \\
&\qquad\qquad\qquad \times \prod_{j=1}^n (\tau_j - \tau_{j-1}) \prod_{j=1}^n (\tau'_j - \tau'_{j-1}) \Big| N \Bigg]\Bigg].
\end{aligned}
$$

Note that $1_{B_{i_1,\dots,i_n}(t)} \prod_{j=1}^n |T_{i_j} - S_{i_j}|^{2H-2}$ is measurable with respect to N, and so, this term goes inside the conditional expectation with respect to N. The result follows using the fact that $E[E[\cdot|N]] = E[\cdot]$ and taking the sum over $n \geq 1$. In the final step, the values $x + X^1_{\tau_n}$ and $x + X^2_{\tau'_n}$ are replaced by $X^1_{\tau_n}$, respectively $X^2_{\tau'_n}$, under the probability measure P_x. $\square$

7. Hyperbolic case: Lower bound on the moment of order 2. In this section, we give a lower bound for the second moment of a solution u to (SWE) [given by (28)], when f is a kernel of cases (i)–(iv). This yields the conclusion of Theorem 2.1(c).

For cases (i)–(iii), we follow the approach of Dalang and Mueller [25]. This means that for any $x, y \in \mathbb{R}^d$ with $x \neq y$, we consider the solid (infinite) cone $C(x, y)$ in $\mathbb{R}^d$, with vertex y, axis oriented in the direction of the vector $x - y$ and an angle of $\pi/4$ between the axis and any lateral side. This cone has the following properties:

(i) if $|z - y| \leq \delta$, $|y - x| \leq \delta$ and $z \in C(x, y)$, then $|z - x| \leq \delta$;
(ii) $y + z \in C(x, y)$ if and only if $y + rz \in C(x, y)$ for any $r > 0$;
(iii) $C(x, y) + z = C(x + z, y + z)$.

7.1. *Case* (i): *Spatially smooth noise.* In this case, since f is continuous at 0, $\lim_{x\to 0} f(x) = f(0) = \mu(\mathbb{R}^d) = K_{\mathrm{w}}$. Letting $\alpha_0 = K_{\mathrm{w}}/2$, we infer that there exists $\delta > 0$ such that

$$f(x) \geq \alpha_0 \qquad \text{for all } x \in \mathbb{R}^d, |x| \leq 2\delta; \tag{44}$$

that is, f satisfies Assumption C of [25]. We assume that δ is a rational number.

The next result corresponds to Theorem 2.1(c), in case (i). Its proof relies on the Feynman–Kac formula developed in Section 6.

THEOREM 7.1. *Let f be a kernel of case* (i). *Then*, *for any $x \in \mathbb{R}^d$ and for any $t > 0$*,

$$E\left|u(t,x)\right|^2 \geq c_3 u_0^2 \exp\left(c_5 K_{\mathrm{w}}^{1/3} t^{\rho_{\mathrm{w}}}\right),$$

where $c_3 > 0$ and $c_5 > 0$ are some constants depending on H, and the constants ρ_{w} and K_{w} are given by (8), *respectively* (36).

PROOF. We proceed as in the proof of Theorem 4.1 of [25]. To facilitate the comparison with the proof of these authors, we use the same notation; that is, we denote n by k in the statement of Theorem 6.1 above. We let $N_t = N([0,t]^2)$.

Step 1. *First step for the lower bound of $E|u(t,x)|^2$.*

Let $k \in \mathbb{Z}_+$ be a large enough value (depending on t) such that

$$m := k\delta \in \mathbb{Z}_+, \tag{45}$$

where δ is given by (44). (The precise value of k will be given in step 8 below.) Notice that for all $t > 0$ and $x \in \mathbb{R}^d$, $w_{\mathrm{w}}(t,x) = u_0 + v_0 t \geq u_0$, since $v_0 \geq 0$. Hence, by Theorem 6.1,

$$\begin{aligned} E\left|u(t,x)\right|^2 \geq{}& u_0^2 e^{t^2} \alpha_H^k \\ &\times \sum_{\substack{i_1,\dots,i_k \\ \text{distinct}}} E_x\Bigg[\prod_{j=1}^k (\tau_j - \tau_{j-1}) \prod_{j=1}^k (\tau'_j - \tau'_{j-1}) \\ &\times \prod_{j=1}^k f\left(X^1_{T_{i_j}} - X^2_{S_{i_j}}\right) \prod_{j=1}^k |T_{i_j} - S_{i_j}|^{2H-2} 1_{B_{i_1,\dots,i_k}(t)}\Bigg]. \end{aligned} \tag{46}$$

Step 2. *The event $D(t)$.*

We consider the event $D(t) = D^1(t) \cap D^2(t)$, where

$$D^1(t) = \left\{X^1_{\tau_j} + \Theta^1_{j+1} \in C\left(x, X^1_{\tau_j}\right) \text{ for all } j = 1,\dots,k-1\right\} \cap B_{i_1,\dots,i_k}(t),$$

$$D^2(t) = \left\{X^2_{\tau'_j} + \Theta^2_{j+1} \in C\left(x, X^2_{\tau'_j}\right) \text{ for all } j = 1,\dots,k-1\right\} \cap B_{i_1,\dots,i_k}(t).$$

On the event $D^1(t)$, if we assume that $\tau_j - \tau_{j-1} \leq \delta$ for all $j = 1,\dots,k$, then

$$\left|X^1_{\tau_j} - x\right| \leq \delta \qquad \text{for all } j = 1,\dots,k. \tag{47}$$

We first prove (47) by induction on j, using the properties of the cone. The argument is the same as in [25]. We include it for the sake of completeness. If $j = 1$, then $X^1_{\tau_1} = x + \tau_1 \Theta^1_1$ and $|X^1_{\tau_1} - x| = \tau_1 |\Theta^1_1| \leq \delta$ since $|\Theta^1_1| \leq 1$. Assume now that $|X^1_{\tau_j} - x| \leq \delta$. We use property (i) of the cone for points $x' = 0$, $y' = X^1_{\tau_j} - x$ and $z' = X^1_{\tau_{j+1}} - x$. We note that $|z' - y'| = |X^1_{\tau_{j+1}} - X^1_{\tau_j}| = (\tau_{j+1} - \tau_j)|\Theta^1_{j+1}| \leq \delta$, and $|y' - x'| = |X^1_{\tau_j} - x| \leq \delta$ by the induction hypothesis. We also have $z' \in C(x', y')$, that is, $X^1_{\tau_{j+1}} - x \in C(0, X^1_{\tau_j} - x)$. [This is equivalent to $X^1_{\tau_{j+1}} \in C(0, X^1_{\tau_j} - x) + x = C(x, X^1_{\tau_j})$, using property (iii) of the cone for the last equality, which is in turn equivalent to $X^1_{\tau_j} + (\tau_{j+1} - \tau_j)\Theta^1_{j+1} \in C(x, X^1_{\tau_j})$, using the definition of $X^1_{\tau_{j+1}}$. By property (ii) of the cone, this last property is equivalent to $X^1_{\tau_j} + \Theta^1_{j+1} \in C(x, X^1_{\tau_j})$, which holds true on the event $D^1(t)$.] By property (i) of the cone, it follows that $|z' - x'| \le \delta$, that is, $|X^1_{\tau_{j+1}} - x| \le \delta$. This completes the proof of (47).

Recall that $\tau_j = U_{\rho(k+1-j)}$ for some permutation ρ of $\{1, \dots, k\}$, where $U_j = T_{i_j}$. As j runs through the set $\{1, \dots, k\}$, so does the value $\rho(k+1-j)$. Therefore, on the event $D^1(t)$, if we assume that $\tau_j - \tau_{j-1} \le \delta$ for all $j = 1, \dots, k$, then $|X^1_{T_{i_j}} - x| \le \delta$ for all $j = 1, \dots, k$, by (47). A similar property holds for X^2 on the event $D^2(t)$. Hence, on the event $D(t)$, if we assume that $\tau_j - \tau_{j-1} \le \delta$ and $\tau'_j - \tau'_{j-1} \le \delta$ for all $j = 1, \dots, k$, then

$$\left|X^1_{T_{i_j}} - X^2_{S_{i_j}}\right| \le 2\delta \qquad \text{for all } j = 1, \dots, k$$

and so

$$f\left(X^1_{T_{i_j}} - X^2_{S_{i_j}}\right) \ge \alpha_0 \qquad \text{for all } j = 1, \dots, k. \tag{48}$$

Step 3. *The islands* $(I_{j,l})_{1 \le j,l \le k}$.

The idea of the proof is to build some small islands around the k points of the process N in the region $[0, t]^2$. Figure 1 shows these islands for $k = 4$. To define

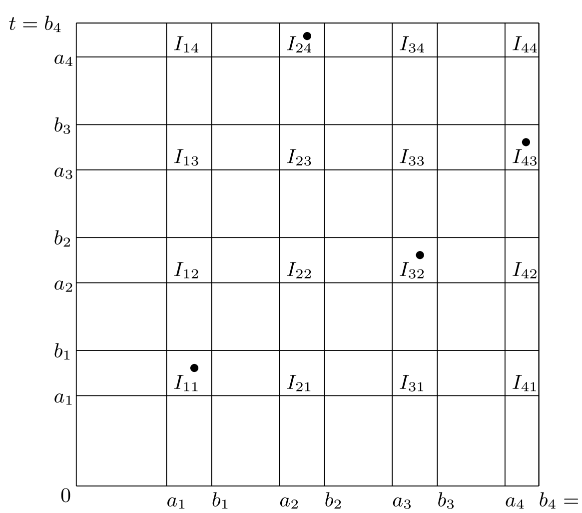

FIG. 1. *The islands $I_{j,l}$ (for $k = 4$) with points situated on the islands $I_{11}, I_{24}, I_{32}, I_{43}$ corresponding to the permutation $(l_1, l_2, l_3, l_4) = (1, 4, 2, 3)$.*

these islands, we let $\varepsilon = \frac{\delta t}{m+1}$ and $t_j = j\varepsilon$ for any $j = 1, \dots, k$. Due to (45), we have

$$t_k = k\varepsilon = \frac{m}{m+1} t \approx t \qquad \text{if } m \text{ is large.}$$

We consider the intervals $I_j = [a_j, b_j]$ with $j = 1, \dots, k$, where $a_j = t_j - \varepsilon/4$ for $j = 1, \dots, k$, $b_j = t_j + \varepsilon/4$ if $j \le k-1$, and $b_k = t$. For any $j, l = 1, \dots, k$, we define

$$I_{j,l} = I_j \times I_l.$$

The area of each square island $I_{j,l}$ is greater than $(\varepsilon/4)^2$. In both the horizontal and vertical directions, the islands are separated by intervals of length $\varepsilon/2$.

Step 4. *The event* $C_{i_1,\dots,i_k}(t)$.

Let $C_{i_1,\dots,i_k}(t)$ be the event that N has points $P_{i_1}, \dots, P_{i_k}$ in $[0,t]^2$ located on the islands $I_{1,l_1}, \dots, I_{k,l_k}$, for some permutation $(l_1, \dots, l_k)$ of $\{1, \dots, k\}$.

Clearly, $C_{i_1,\dots,i_k}(t)$ is included in $B_{i_1,\dots,i_k}(t)$. Notice that on the event $C_{i_1,\dots,i_k}(t)$, it is not possible to have two points (T_{i_p}, S_{i_p}) and (T_{i_q}, S_{i_q}) of N in $[0,t]^2$ such that T_{i_p}, T_{i_q} are in the same interval I_j or S_{i_p}, S_{i_q} are in the same interval I_l. Therefore, on the event $C_{i_1,\dots,i_k}(t)$, for any $j = 1, \dots, k$, we have $\tau_j \in I_j, \tau'_j \in I_j$, and hence

$$\frac{\varepsilon}{2} \le \tau_j - \tau_{j-1} \le 2\varepsilon \quad \text{and} \quad \frac{\varepsilon}{2} \le \tau'_j - \tau'_{j-1} \le 2\varepsilon. \tag{49}$$

In particular, if

$$m \ge m_0(t) := [2t - 1], \tag{50}$$

then $\tau_j - \tau_{j-1} \le \delta$ and $\tau'_j - \tau'_{j-1} \le \delta$ for all $j = 1, \dots, k$. It follows that

$$\text{inequality (48) holds on the event } D(t) \cap C_{i_1,\dots,i_k}(t), \tag{51}$$

provided that $m \ge m_0 = m_0(t)$.

Step 5. *Second step for the lower bound of* $E|u(t,x)|^2$.

On the event $B_{i_1,\dots,i_k}(t)$, we define $\widetilde{Z}_t = \prod_{j=1}^k (\tau_j - \tau_{j-1}) \prod_{j=1}^k (\tau'_j - \tau'_{j-1})$. Using (46) and (51), we obtain

$$\begin{aligned} &E\left|u(t,x)\right|^2 \\ &\quad \ge u_0^2 e^{t^2} \alpha_H^k \sum_{\substack{i_1,\dots,i_k \\ \text{distinct}}} E_x \Bigg[\widetilde{Z}_t \prod_{j=1}^k f\big(X^1_{T_{i_j}} - X^2_{S_{i_j}}\big) \\ &\qquad\qquad \times \prod_{j=1}^k |T_{i_j} - S_{i_j}|^{2H-2} 1_{D(t)} 1_{C_{i_1,\dots,i_k}(t)} \Bigg] \end{aligned}$$

$$\geq u_0^2 e^{t^2} \alpha_H^k \alpha_0^k \sum_{\substack{i_1,\dots,i_k \\ \text{distinct}}} E_x\left[\widetilde{Z}_t \prod_{j=1}^k |T_{i_j} - S_{i_j}|^{2H-2} 1_{D(t)} 1_{C_{i_1,\dots,i_k}(t)}\right]$$

$$= u_0^2 e^{t^2} \alpha_H^k \alpha_0^k \sum_{\substack{i_1,\dots,i_k \\ \text{distinct}}} E_x\left[\widetilde{Z}_t \prod_{j=1}^k |T_{i_j} - S_{i_j}|^{2H-2} 1_{C_{i_1,\dots,i_k}(t)} P_x\bigl[D(t)|N\bigr]\right].$$

Since the events $D^1(t)$ and $D^2(t)$ are conditionally independent given N,

$$P_x\bigl[D(t)|N\bigr] = P_x\bigl[D^1(t)|N\bigr] P_x\bigl[D^2(t)|N\bigr].$$

Using the properties of the cone and the independence of $(\Theta_i^1)_{i\geq 1}$, it can be shown that $P_x[D^1(t)|N] = \gamma^{N_t - 1}$, where $\gamma = P(y + \Theta_0 \in C(0, y)) \in (0, 1)$ does not depend on $y \in \mathbb{R}^d$. Note that γ depends on d. A similar property holds for $D^2(t)$. Hence,

$$P_x\bigl[D(t)|N\bigr] = \gamma^{2(N_t - 1)} > \gamma^{2N_t}.$$

Combining this with the previous lower bound for $E|u(t,x)|^2$, we obtain

$$E\bigl|u(t,x)\bigr|^2 \geq u_0^2 e^{t^2} \alpha_H^k \alpha_0^k \gamma^{2k} \times \sum_{\substack{i_1,\dots,i_k \\ \text{distinct}}} E_x\left[\widetilde{Z}_t \prod_{j=1}^k |T_{i_j} - S_{i_j}|^{2H-2} 1_{C_{i_1,\dots,i_k}(t)}\right].$$

We define the conditional expectation of a random variable X with respect to an event B by $E[X|B] = E[X 1_B]/P(B)$. (This is not the same as $E[X|\mathcal{G}]$, where $\mathcal{G} = \sigma(\{B\}) = \{\varnothing, B, B^c, \Omega\}$ since $E[X|\mathcal{G}] = E[X|B]1_B + E[X|B^c]1_{B^c}$.) In our case, X is the random variable appearing in the expectation above and $B = B_{i_1,\dots,i_k}(t)$. We obtain

$$E\bigl|u(t,x)\bigr|^2 \geq u_0^2 e^{t^2} \alpha_H^k \alpha_0^k \gamma^{2k} \times \sum_{\substack{i_1,\dots,i_k \\ \text{distinct}}} E_x\left[\widetilde{Z}_t \prod_{j=1}^k |T_{i_j} - S_{i_j}|^{2H-2} 1_{C_{i_1,\dots,i_k}(t)} \middle| B_{i_1,\dots,i_k}(t)\right] \times P_x\bigl(B_{i_1,\dots,i_k}(t)\bigr).$$

Note that by (49), on the event $C_{i_1,\dots,i_k}(t)$, we have $\widetilde{Z}_t \geq (\varepsilon/2)^{2k}$. Using the fact that $\delta = m/k$ [by the definition (45) of m], we see that

$$\frac{\varepsilon}{2} = \frac{\delta t}{2(m+1)} = \frac{m}{m+1} \cdot \frac{t}{2k} \geq \frac{ct}{k} \tag{52}$$

with $c = 1/8$. Hence $\widetilde{Z}_t \geq (ct/k)^{2k}$ and

$$E|u(t,x)|^2 \geq e^{t^2} \alpha_H^k \alpha_0^k \gamma^{2k} \left(\frac{ct}{k}\right)^{2k} \times \sum_{\substack{i_1,\dots,i_k \\ \text{distinct}}} E_x \left[\prod_{j=1}^k |T_{i_j} - S_{i_j}|^{2H-2} 1_{C_{i_1,\dots,i_k}(t)} \Big| B_{i_1,\dots,i_k}(t) \right] \times P_x(B_{i_1,\dots,i_k}(t)).$$

Since both T_{i_j} and S_{i_j} are in $[0,t]$, we obviously have $|T_{i_j} - S_{i_j}| < t$. Thus since $2H - 2 < 0$,

$$\prod_{j=1}^k |T_{i_j} - S_{i_j}|^{2H-2} > t^{(2H-2)k}.$$

This turns out to be enough for our purposes. With this bound, we have

$$E|u(t,x)|^2 \geq u_0^2 e^{t^2} \alpha_H^k \alpha_0^k \gamma^{2k} \left(\frac{ct}{k}\right)^{2k} t^{(2H-2)k} \times \sum_{\substack{i_1,\dots,i_k \\ \text{distinct}}} P_x(C_{i_1,\dots,i_k}(t) | B_{i_1,\dots,i_k}(t)) P_x(B_{i_1,\dots,i_k}(t)). \tag{53}$$

Step 6. *The conditional probability* $P_x(C_{i_1,\dots,i_k}(t)|B_{i_1,\dots,i_k}(t))$.

Let S_k be the set of all permutations $(l_1,\dots,l_k)$ of $\{1,\dots,k\}$. By the definition of the event $C_{i_1,\dots,i_k}(t)$,

$$P_x(C_{i_1,\dots,i_k}(t)|B_{i_1,\dots,i_k}(t)) = \sum_{(l_1,\dots,l_k)\in S_k} P_x(A_{i_1,\dots,i_k}(t,(l_1,\dots,l_k))|B_{i_1,\dots,i_k}(t)),$$

where $A_{i_1,\dots,i_k}(t,(l_1,\dots,l_k))$ is the event that N has points $P_{i_1},\dots,P_{i_k}$ in $[0,t]^2$ located on the islands $I_{1,l_1},\dots,I_{k,l_k}$. Note that

$$A_{i_1,\dots,i_k}(t,(l_1,\dots,l_k)) = \bigcup_{(j_1,\dots,j_k)\in S_k} \{P_{i_1} \in I_{j_1,l_1},\dots,P_{i_k} \in I_{j_k,l_k}\}.$$

Given $B_{i_1,\dots,i_k}(t)$, $(P_{i_1},\dots,P_{i_k})$ has a uniform distribution on $[0,t]^{2k}$. Hence

$$\begin{aligned} P_x(P_{i_1} \in I_{j_1,l_1},\dots,P_{i_k} \in I_{j_k,l_k}|B_{i_1,\dots,i_k}(t)) &= \frac{\mathrm{Leb}(I_{j_1,l_1}\times\cdots\times I_{j_k,l_k})}{\mathrm{Leb}([0,t]^{2k})} \\ &\geq \frac{1}{t^{2k}}\left(\frac{\varepsilon}{4}\right)^{2k}. \end{aligned}$$

Since the last quantity does not depend on the permutations $(j_1, \dots, j_k)$ and $(l_1, \dots, l_k)$, we obtain that

$$P_x\big(C_{i_1,\dots,i_k}(t)\,|\,B_{i_1,\dots,i_k}(t)\big) = (k!)^2 \frac{1}{t^{2k}}\left(\frac{\varepsilon}{4}\right)^{2k} \ge (k!)^2\left(\frac{c}{k}\right)^{2k}, \tag{54}$$

using (52) for the inequality. Relation (54) is the analogue of (4.7) of [25] (with $n = 2$) for the fractional noise.

Step 7. *Third step for the lower bound of* $E|u(t,x)|^2$.

Combining (53) and (54), we get

$$E\big|u(t,x)\big|^2 \ge u_0^2 e^{t^2} \alpha_H^k \alpha_0^k \gamma^{2k}\left(\frac{ct}{k}\right)^{2k} t^{(2H-2)k}(k!)^2\left(\frac{c}{k}\right)^{2k} \times \sum_{\substack{i_1,\dots,i_k \\ \text{distinct}}} P_x\big(B_{i_1,\dots,i_k}(t)\big).$$

We now use the fact that $\{N_t = k\}$ is the disjoint union of all events $B_{i_1,\dots,i_k}(t)$ for all sets $\{i_1, \dots, i_k\}$ of cardinality k. Moreover, N_t has a Poisson distribution with mean t^2. Hence $P(N_t = k) = e^{-t^2} t^{2k}/k!$ and

$$\begin{aligned} E\big|u(t,x)\big|^2 &\ge u_0^2 e^{t^2} \alpha_H^k \alpha_0^k \gamma^{2k}\left(\frac{ct}{k}\right)^{2k} t^{(2H-2)k}(k!)^2\left(\frac{c}{k}\right)^{2k} e^{-t^2}\frac{t^{2k}}{k!} \\ &= u_0^2\big(\alpha_0\alpha_H\gamma^2c^4\big)^k t^{(2H+2)k}\frac{1}{k^{4k}}k!. \end{aligned}$$

By Stirling's formula, there exists some $k_0 \ge 1$ such that $k! \ge e^{-k}k^k$ for all $k \ge k_0$. It follows that if $k \ge k_0$, then

$$E\big|u(t,x)\big|^2 \ge u_0^2\left(\alpha_0 c_H \frac{t^{2H+2}}{k^3}\right)^k, \tag{55}$$

where $c_H = \alpha_H\gamma^2c^4e^{-1}$ depends on H. (c_H depends also on d, through γ.)

Step 8. *The choice of* k.

Let

$$k = \big[e^{-1/3}\alpha_0^{1/3}c_H^{1/3}t^{(2H+2)/3}\big],$$

where $[x] = k \in \mathbb{Z}$ if $k \le x < k+1$. Since $k \le e^{-1/3}\alpha_0^{1/3}c_H^{1/3}t^{(2H+2)/3}$, it follows that $e \le \alpha_0 c_H t^{2H+2}/k^3$. On the other hand, letting

$$k_1 = \tfrac{1}{2}\big(e^{-1}\alpha_0 c_H\big)^{1/3} = \alpha_0^{1/3}\cdot\tfrac{1}{2}\big(\alpha_H\gamma^2c^4e^{-2}\big)^{1/3} =: \alpha_0^{1/3}c_1^*,$$

we have $k > 2k_1t^{(2H+2)/3} - 1 \ge k_1t^{(2H+2)/3}$ if $k_1t^{(2H+2)/3} \ge 1$. Using (55), we infer that

$$E\big|u(t,x)\big|^2 \ge u_0^2e^k \ge u_0^2\exp\big(k_1t^{(2H+2)/3}\big) \qquad \text{if } \alpha_0t^{2H+2} \ge t_1' := \big(c_1^*\big)^{-3}.$$

Note that $k \geq k_0$ if $\alpha_0 t^{2H+2} \geq t_1'' := k_0^3 e c_H$. We take $t_1 = t_1' \vee t_1''$.

Let $c_4 = c_1^* \alpha_0^{1/3} = c_1^* 2^{-1/3} K_{\mathrm{w}}^{1/3}$. This proves that for any $t > 0$ such that $\alpha_0 t^{2H+2} \geq t_1$ (i.e., for all $t \geq t_0$ for some $t_0 > 0$),

$$E|u(t,x)|^2 \geq u_0^2 \exp(c_4 t^{\rho_{\mathrm{w}}}).$$

Step 9. *Extension to all $t > 0$.*

Using (29) and the fact that $u_0 > 0$ and $v_0 > 0$, we infer that for any $0 < t < t_0$,

$$E|u(t,x)|^2 \geq w(t,x)^2 = (u_0 + t v_0)^2 \geq c_3^* u_0^2 \exp(c_4 t_0^{\rho_{\mathrm{w}}}) \geq c_3^* u_0^2 \exp(c_4 t^{\rho_{\mathrm{w}}}),$$

where $c_3^* = \exp(-c_4 t_0^{\rho_{\mathrm{w}}})$. Finally, we let $c_3 = \min(1, c_3^*)$ and $c_5 = c_1^* 2^{-1/3}$. $\square$

7.2. *Cases* (ii) *and* (iii): *Fractional noise in space.* These cases are treated similar to case (i), using Theorem 6.1. The difference is that instead of (44), we use the fact that for any $\delta > 0$,

$$f(x) \geq \alpha_0(\delta) := (2\delta)^{-a} \qquad \text{for all } x \in \mathbb{R}^d, |x| \leq 2\delta, \tag{56}$$

where a is given by (9).

The next result corresponds to Theorem 2.1(c), in cases (ii)–(iii).

THEOREM 7.2. *Let f be a kernel of either case* (ii) *or* (iii). *If* (35) *holds, then for any $x \in \mathbb{R}^d$ and for any $t > 0$,*

$$E|u(t,x)|^2 \geq c_3 u_0^2 \exp(c_4 t^{\rho_{\mathrm{w}}}),$$

where $c_3 > 0$ and $c_4 > 0$ are some constants depending on H and a, and the constants ρ_{w} and a are given by (8), *respectively* (9).

PROOF. We use the same argument as in the proof of Theorem 7.1, but with a different method of specifying the parameters.

More precisely, we let $k \in \mathbb{Z}_+$ be a large enough value (depending on t) which will be chosen later. We choose $\delta = m/k$ where $m = [2t]$. This ensures that (45) and (50) are satisfied. Note that δ depends on t/k.

Let $c_H = \alpha_H \gamma^2 c^4 e^{-1}$. Relation (55) says that if $k \geq k_0$, then

$$\begin{aligned} E|u(t,x)|^2 &\geq u_0^2 \left(c_H (2\delta)^{-a} \frac{t^{2H+2}}{k^3} \right)^k = u_0^2 \left(c_H 2^{-a} \left(\frac{m}{k} \right)^{-a} \frac{t^{2H+2}}{k^3} \right)^k \\ &\geq u_0^2 \left(c_H 2^{-a} \left(\frac{2t}{k} \right)^{-a} \frac{t^{2H+2}}{k^3} \right)^k = u_0^2 \left(c_H^* \frac{t^{2H+2-a}}{k^{3-a}} \right)^k, \end{aligned}$$

where $c_H^* = c_H 4^{-a}$. We let

$$k = \left[\left(e^{-1} c_H^* t^{2H+2-a} \right)^{1/(3-a)} \right].$$

(This choice will ensure that δ is small since $\delta \approx 2t/k \approx Ct^{1-\rho_w}$ and $\rho_w > 1$.) Then $e \leq c_H^* t^{2H+2-a}/k^{3-a}$. On the other hand, letting

$$c_4 = \frac{1}{2}\left(e^{-1}c_H^*\right)^{1/(3-a)}, \tag{57}$$

we have $k > 2c_4 t^{\rho_w} - 1 \geq c_4 t^{\rho_w}$ if $c_4 t^{\rho_w} \geq 1$. Hence

$$E\left|u(t,x)\right|^2 \geq u_0^2 e^k \geq u_0^2 \exp\left(c_4 t^{\rho_w}\right) \qquad \text{for all } t \geq t_0,$$

where

$$t_0 = \left(ec_H^{-1} 2^{3+a}\right)^{1/(2H+2-a)}. \tag{58}$$

For $0 < t < t_0$, we argue as in step 9 of the proof of Theorem 7.1. $\square$

7.3. *Case* (iv): *White noise in space.* In this case, we cannot apply directly Theorem 6.1 since f is not a function. Instead of this, we will use an approximation technique based on case (ii).

The fact that we use this approximation may be surprising, since in many instances, it is easier to deal with the white noise than a correlated noise. This is due to the fact that our method for proving the lower bound relies on the representation given by Theorem 6.1. Obtaining a similar representation in the case $f = \delta_0$ is more delicate. (The Dirac distribution would have to be approximated in some sense, so that the representation make sense.) Instead of this, we decided to use an approximation directly for obtaining the lower bound.

Our procedure can be viewed as another method to smoothen the noise, paralleling the method used in [35] and [33]. The fact that we approximate δ_0 by a Riesz kernel allows us to use the result that we proved for case (ii). A more standard procedure in the literature is to approximate δ_0 by the heat kernel $p_\varepsilon(x) = (2\pi\varepsilon)^{-1/2}\exp(-|x|^2/(2\varepsilon))$ as $\varepsilon \downarrow 0$. This is a kernel of case (i), with $\lim_{x\to 0} p_\varepsilon(x) = (2\pi\varepsilon)^{-1/2}$. Denoting by $u_\varepsilon(t,x)$ the solution of (SWE) driven by a noise W_ε with spatial covariance p_ε, one infers by Theorem 7.1 that $E|u_\varepsilon(t,x)|^2 \geq u_0^2 \exp(c_1\alpha_0(\varepsilon)t^{(2H+2)/3})$, with $\alpha_0(\varepsilon) = (2\pi\varepsilon)^{-1/2}/2$. However, this approximation is not suitable for our purposes, since $\lim_{\varepsilon\downarrow 0}\alpha_0(\varepsilon) = \infty$.

We begin to explain this approximation technique. For any $a \in (0,1)$, let $W_a = \{W_a(\varphi); \varphi \in \mathcal{H}_a\}$ be an isonormal Gaussian noise with covariance $E[W_a(\varphi) \times W_a(\psi)] = \langle \varphi, \psi \rangle_{\mathcal{H}_a}$ where $\langle \cdot, \cdot \rangle_{\mathcal{H}_a}$ is given by (2) with $f(x)$ replaced by $f_a(x) = |x|^{-a}$. Note that $f = \mathcal{F}\mu_a$ where $\mu_a(\xi) = (2\pi)^{-1}|\xi|^{a-1}\,d\xi$. Let $u_a(t,x)$ be the solution of the equation

$$\frac{\partial^2 u}{\partial t^2} = \Delta u + u\dot{W}_a \qquad (t > 0, x \in \mathbb{R})$$

with initial conditions $u(0,x) = u_0$ and $\frac{\partial u}{\partial t}(0,x) = v_0$. This solution has the Wiener chaos expansion $u_a(t,x) = \sum_{n\geq 0} I_{n,a}(f_n(\cdot,t,x))$ where $I_{n,a}$ denotes the multiple Wiener integral with respect to W_a. Hence

$$E\left|u_a(t,x)\right|^2 = \sum_{n\geq 0} \frac{1}{n!}\alpha_{n,a}(t),$$

where

$$\alpha_{n,a}(t) = \alpha_H^n \int_{[0,t]^{2n}} \prod_{j=1}^n |t_j - s_j|^{2H-2} \psi_{n,a}(\mathbf{t},\mathbf{s})\, d\mathbf{t}\, d\mathbf{s} \tag{59}$$

and

$$\begin{aligned} &\psi_{n,a}(\mathbf{t},\mathbf{s}) \\ &\quad = \int_{\mathbb{R}^{2n}} g_{\mathbf{t}}^{(n)}(x_1,\ldots,x_n,t,x) g_{\mathbf{s}}^{(n)}(y_1,\ldots,y_n,t,x) \prod_{j=1}^n f_a(x_j - y_j)\, d\mathbf{x}\, d\mathbf{y} \end{aligned} \tag{60}$$

and $g_{\mathbf{t}}^{(n)}(x_1,\ldots,x_n,t,x) = \prod_{j=1}^n G_{\mathrm{w}}(t_{\rho(j+1)} - t_{\rho(j)}, x_{\rho(j+1)} - x_{\rho(j)}) w(t_{\rho(1)}, x_{\rho(1)})$ if $t_{\rho(1)} < \cdots < t_{\rho(n)}$.

LEMMA 7.3. *For any integer $n \geq 1$ and for any $\mathbf{t},\mathbf{s} \in [0,t]^n$,*

$$\lim_{a\uparrow 1} \psi_{n,a}(\mathbf{t},\mathbf{s}) = \psi(\mathbf{t},\mathbf{s}),$$

where $\psi_n(\mathbf{t},\mathbf{s})$ is given by (32) with $d=1$ and $f = \delta_0$, that is,

$$\begin{aligned} \psi_n(\mathbf{t},\mathbf{s}) = \int_{\mathbb{R}^n} \prod_{j=1}^n & G_{\mathrm{w}}(t_{\rho(j+1)} - t_{\rho(j)}, x_{\rho(j+1)} - x_{\rho(j)}) w_{\mathrm{w}}(t_{\rho(1)}, x_{\rho(1)}) \\ & \times \prod_{j=1}^n G_{\mathrm{w}}(s_{\sigma(j+1)} - s_{\sigma(j)}, x_{\sigma(j+1)} - x_{\sigma(j)}) w_{\mathrm{w}}(s_{\sigma(1)}, x_{\sigma(1)})\, d\mathbf{x}, \end{aligned}$$

where the permutations $\rho,\sigma \in S_n$ are chosen such that $t_{\rho(1)} < \cdots < t_{\rho(n)}$ and $s_{\sigma(1)} < \cdots < s_{\sigma(n)}$, $t_{\rho(n+1)} = s_{\sigma(n+1)} = t$ and $x_{\rho(n+1)} = x_{\sigma(n+1)} = x$.

PROOF. Note that for any $g,h \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$,

$$\begin{aligned} &\lim_{a\uparrow 1} \int_{\mathbb{R}}\int_{\mathbb{R}} g(x)h(y) f_a(x-y)\, dx\, dy \\ &\qquad = \lim_{a\uparrow 1} \frac{1}{2\pi} \int_{\mathbb{R}} \mathcal{F}g(\xi)\overline{\mathcal{F}h(\xi)} |\xi|^{a-1}\, d\xi \\ &\qquad = \frac{1}{2\pi}\int_{\mathbb{R}} \mathcal{F}g(\xi)\overline{\mathcal{F}h(\xi)}\, d\xi = \int_{\mathbb{R}} g(x)h(x)\, dx, \end{aligned}$$

by the dominated convergence theorem. To justify the application of this theorem, we note that for $a > 1/2$, the integrand $|\mathcal{F}g(\xi)||\mathcal{F}h(\xi)||\xi|^{a-1}$ is bounded by the integrable function

$$\|g\|_1\|h\|_1|\xi|^{-1/2}1_{\{|\xi|\le 1\}} + \big|\mathcal{F}g(\xi)\big|\big|\mathcal{F}h(\xi)\big|1_{\{|\xi|\ge 1\}}.$$

From here we infer that for any $g, h \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$,

$$\lim_{a\uparrow 1}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} g(\mathbf{x})h(\mathbf{y})\prod_{j=1}^{n} f_a(x_j - y_j)\,d\mathbf{x}\,d\mathbf{y} = \int_{\mathbb{R}^n} g(\mathbf{x})h(\mathbf{x})\,d\mathbf{x},$$

with $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$. We apply this to $g = g_{\mathbf{t}}^{(n)}(\cdot, t, x)$ and $h = g_{\mathbf{s}}^{(n)}(\cdot, t, x)$, using the fact that

$$\psi_n(\mathbf{t}, \mathbf{s}) = \int_{\mathbb{R}^n} g_{\mathbf{t}}^{(n)}(\mathbf{x}, t, x) g_{\mathbf{s}}^{(n)}(\mathbf{x}, t, x)\,d\mathbf{x}. \qquad \square$$

LEMMA 7.4. *For any $t > 0$ and for any integer $n \ge 1$,*

$$\lim_{a\uparrow 1}\alpha_{n,a}(t) = \alpha_n(t).$$

PROOF. This follows by Lemma 7.3 and the dominated convergence theorem. It remains to justify the application of this theorem. For this, we note that $\psi_{n,a}(\mathbf{t}, \mathbf{s}) \le \psi_{n,a}(\mathbf{t}, \mathbf{t})^{1/2}\psi_{n,a}(\mathbf{s}, \mathbf{s})^{1/2}$. Let $u_j = t_{\rho(j+1)} - t_{\rho(j)}$. As in the proof of Lemma 4.1, it follows that for any $t \ge 1$,

$$\begin{aligned}
\psi_{n,a}(\mathbf{t}, \mathbf{t}) &\le (u_0 + t v_0)^2 \frac{1}{(2\pi)^n}(4K_a)^n (u_1, \dots, u_n)^{2-a} \\
&\le (u_0 + t v_0)^2 \frac{1}{(2\pi)^n}(4K_a)^n t^{n(1-a)} u_1, \dots, u_n \\
&\le (u_0 + t v_0)^2 \frac{t^n}{(2\pi)^n}(4K_a)^n u_1, \dots, u_n,
\end{aligned}$$

where $K_a := K(\mu_a)$ is given by (34). We now prove that

$$K_a = L_a := \int_{\mathbb{R}} \frac{1}{1 + |\xi|^2}\mu_a(d\xi). \tag{61}$$

To see this, note first that $L_a \le K_a$. On the other hand, for any $\eta \in \mathbb{R}$,

$$\int_{\mathbb{R}} \frac{1}{1 + |\xi - \eta|^2}\mu_a(d\xi) = \int_{\mathbb{R}} e^{i\eta x} p(x)|x|^{-a}\,dx,$$

where $p(x) = (4\pi)^{-1/2}\int_0^\infty e^{-u}u^{-1/2}e^{-|x|^2/(4u)}\,du$; see (3.4) of [24]. Taking the modulus on both sides and using (3.5) of [24], we obtain that for any $\eta \in \mathbb{R}$,

$$\int_{\mathbb{R}} \frac{1}{1 + |\xi - \eta|^2}\mu_a(d\xi) \le \int_{\mathbb{R}} p(x)|x|^{-a}\,dx = L_a.$$

Taking the supremum over $\eta \in \mathbb{R}$, we obtain that $K_a \le L_a$. This proves (61).

By considering separately the regions $\{|\xi| \le 1\}$ and $\{|\xi| \ge 1\}$, we see that $L_a \le 2(a^{-1} + (2-a)^{-1})$. Hence $L_a \le 6$ if $a > 1/2$.

Denote $\beta(\mathbf{t}) = \prod_{j=1}^{n}(t_{\rho(j+1)} - t_{\rho(j)})$. It follows that for any $a \in (1/2, 1)$,

$$\psi_{n,a}(\mathbf{t},\mathbf{s}) \le (u_0 + tv_0)^2 \frac{t^n}{(2\pi)^n} 24^n \big[\beta(\mathbf{t})\beta(\mathbf{s})\big]^{1/2}. \tag{62}$$

The claim is justified since $\int_{[0,t]^{2n}} \prod_{j=1}^{n} |t_j - s_j|^{2H-2} [\beta(\mathbf{t})\beta(\mathbf{s})]^{1/2}\, d\mathbf{t}\, d\mathbf{s} < \infty$; see the proof of Theorem 4.2. □

LEMMA 7.5. *For any $t > 0$ and for any $x \in \mathbb{R}^d$,*

$$\lim_{a\uparrow 1} E\big|u_a(t,x)\big|^2 = E\big|u(t,x)\big|^2.$$

PROOF. The result follows by Lemma 7.4 and the dominated convergence theorem. We justify the application of this theorem. By (59) and (62),

$$\begin{aligned}\alpha_{n,a}(t) &\le (u_0 + tv_0)^2 \frac{t^n}{(2\pi)^n} 24^n \int_{[0,t]^{2n}} \prod_{j=1}^{n} |t_j - s_j|^{2H-2} \big[\beta(\mathbf{t})\beta(\mathbf{s})\big]^{1/2}\, d\mathbf{t}\, d\mathbf{s}\, d\mathbf{t}\, d\mathbf{s} \\ &\le (u_0 + tv_0)^2 c^n \frac{1}{n!} t^{(2H+1)n},\end{aligned}$$

for any $a \in (1/2, 1)$, where the last inequality follows as in the proof of Theorem 4.2. Since $\sum_n c^n t^{(2H+1)n}/(n!)^2 < \infty$, the proof is complete. □

The next result corresponds to Theorem 2.1(c), in case (iv).

THEOREM 7.6. *Let f be the kernel of case* (iv). *Then, for any $x \in \mathbb{R}$ and for any $t > 0$, we have*

$$E\big|u(t,x)\big|^2 \ge c_3 u_0^2 \exp\big(c_4 t^{\rho_{\mathrm{w}}}\big),$$

where $c_3 > 0$ and $c_4 > 0$ are some constants depending on H, and ρ_{w} is given by (8).

PROOF. By Theorem 7.2, for any $x \in \mathbb{R}^d$ and for any $t \ge t_a$,

$$E\big|u_a(t,x)\big|^2 \ge u_0^2 \exp\big(c_a t^{(2H+2-a)/(3-a)}\big), \tag{63}$$

where the constants $c_a > 0$ and $t_a > 0$ are given by (57) and (58), that is,

$$c_a = \tfrac{1}{2}\big(e^{-1} c_H 4^{-a}\big)^{1/(3-a)} \quad \text{and} \quad t_a = \big(e c_H^{-1} 2^{3+a}\big)^{1/(2H+2-a)}.$$

Let $c_4 = \lim_{a\uparrow 1} c_a$ and $t_0' = \lim_{a\uparrow 1} t_a$. Then $t_a \le 2t_0' =: t_0$ for all $a \in (a_0, 1)$. Fix $t \ge t_0$. We let $a \uparrow 1$ in (63). Using Lemma 7.5, we infer that

$$E|u(t,x)|^2 \ge u_0^2 \exp(c_4 t^{\rho_{\mathrm{w}}}) \qquad \text{for all } t \ge t_0.$$

For $0 < t < t_0$, we argue as in step 9 of the proof of Theorem 7.1. $\square$

Summarizing the results of this section, we can say that Theorems 7.1 and 7.2 generalize Theorem 4.1 of [25] (with $p = 2$) to the case of the fractional noise in time. However, reference [25] does not contain a result analogous to Theorem 7.6 for the case $H = 1/2$, that is, when W is a space–time white noise.

8. Parabolic case: Proof of Theorem 2.2. In this section, we examine equation (SHE). We state and sketch the proof of two results, which together give the conclusion of Theorem 2.2. The proofs are similar to those presented above in the hyperbolic case, taking $G = G_{\mathrm{h}}$ and $w = w_{\mathrm{h}}$. For the lower bound, we use a FK representation similar to the one given in [2], except that here we work with processes X^1, X^2 defined by (65) and (66) below, instead of Brownian motions B^1, B^2.

We define a different constant K_{h} than in the hyperbolic case, namely

$$K_{\mathrm{h}} = \begin{cases} \mu(\mathbb{R}^d), & \text{in case (i)}, \\ K(\mu), & \text{in cases (ii) and (iii)}, \\ \sqrt{\pi}, & \text{in case (iv)}. \end{cases} \tag{64}$$

We recall that in the parabolic case, the spatial dimension $d \ge 1$ is arbitrary. The first result gives the existence of the solution and an upper bound for its moments of order $p \ge 2$.

PROPOSITION 8.1. *Let f be a kernel of cases* (i)–(iv), *and $\rho_{\mathrm{h}}, a, K_{\mathrm{h}}$ be the constants given by* (8), (9), *respectively* (64). *Assume that* (DC) *holds. Then*:

(a) *for any $t > 0$ and for any integer $n \ge 1$*,

$$\alpha_n(t) \le u_0^2 K_{\mathrm{h}}^n c^n (n!)^{a/2} t^{(4H-a)n/2},$$

where c is a constant depending on H and a;

(b) *equation* (SHE) *has a unique solution $u(t,x)$ which has the following property*: *for any $p \ge 2$, for any $x \in \mathbb{R}^d$ and for any $t > 0$*,

$$E|u(t,x)|^p \le c_1^p u_0^p \exp\bigl(c_2 K_{\mathrm{h}}^{2/(2-a)} p^{(4-a)/(2-a)} t^{\rho_{\mathrm{h}}}\bigr),$$

where $c_1 > 0$ is a constant depending on a, and $c_2 > 0$ is a constant depending on H and a.

PROOF. (a) Similar to Lemma 4.1, it can be shown that

$$\begin{aligned}\psi_n(\mathbf{t},\mathbf{t}) &= u_0^2\int_{\mathbb{R}^{nd}}\exp(-u_1|\xi_1|^2)\cdots\exp(-u_n|\xi_1+\cdots+\xi_n|^2)\mu(d\xi_1)\cdots\mu(d\xi_n)\\ &\leq u_0^2K_{\mathrm{h}}^n(u_1,\ldots,u_n)^{-a/2}.\end{aligned}$$

To prove this in cases (ii) and (iii), one uses the following inequality:

$$\int_{\mathbb{R}^d}\exp(-t|\xi-\eta|^2)\mu(d\xi)\leq K(\mu)t^{-a/2}.$$

The conclusion follows as in the proof of Proposition 4.2(a). Note that

$$\psi_n(\mathbf{t},\mathbf{s})\leq u_0^2K_{\mathrm{h}}^n[\beta(\mathbf{t})\beta(\mathbf{s})]^{-a/4}.$$

(b) The conclusion follows as in the proof of Proposition 4.2(b) (case $p=2$), respectively Proposition 5.1 (case $p>2$). □

For the lower bound, we use the following representation for the second moment of the solution to (SHE), which can be obtained as in Section 6, assuming that f is a function:

$$E|u(t,x)|^2 = e^{t^2}u_0^2\sum_{n\geq 0}\sum_{\substack{i_1,\ldots,i_n\\ \text{distinct}}}E_x\left[\prod_{j=1}^n f(X^1_{T_{i_j}}-X^2_{S_{i_j}})\alpha_H^n\prod_{j=1}^n|T_{i_j}-S_{i_j}|^{2H-2}1_{B_{i_1,\ldots,i_n}(t)}\right].$$

Here, the event $B_{i_1,\ldots,i_n}(t)$ and the points (T_{i_j},S_{i_j}) are defined as in Section 6, but the processes X^1 and X^2 are given by

$$X^1_s = X^1_{\tau_i}+\sqrt{s-\tau_i}\,\Theta^1_{i+1}\qquad \text{if } \tau_i\leq s\leq\tau_{i+1}, \tag{65}$$

$$X^2_s = X^2_{\tau'_i}+\sqrt{s-\tau'_i}\,\Theta^2_{i+1}\qquad \text{if } \tau'_i\leq s\leq\tau'_{i+1}, \tag{66}$$

where $(\Theta^1_i)_{i\geq 1}$ and $(\Theta^2_i)_{i\geq 1}$ are two independent collections of i.i.d. random variables with values in $\mathbb{R}^d$ with the same law as Θ_0, and Θ_0 has a d-dimensional standard normal distribution. Note that in this case,

$$G_{\mathrm{h}}(t,\cdot)\qquad \text{is the density of } \sqrt{t}\,\Theta_0. \tag{67}$$

(Alternatively, X^1, X^2 can be two independent d-dimensional standard Brownian motions; see Remark 6.11 and [2].)

PROPOSITION 8.2. *Let f be a kernel of cases* (i)–(iv), *and ρ_{h} be the constant given by* (8). *Then for any $x\in\mathbb{R}^d$ and for any $t>0$,*

$$E|u(t,x)|^2\geq c_3u_0^2\exp(c_4t^{\rho}),$$

where $c_3>0$ and $c_4>0$ are some constants depending on H and a.

PROOF. In case (i), the argument is similar to the one used in Theorem 7.1. One difference is that we replace δ by δ^2. This is essentially due to the use of a parabolic rather than hyperbolic scaling; compare (67) with (40). In addition, in the events $D^1(t)$, $D^2(t)$, we add the condition $|\Theta^1_{j+1}| \le 1$, respectively $|\Theta^2_{j+1}| \le 1$, for all $j = 1, \dots, k-1$. Note that the variable $\widetilde{Z}_t$ (in step 5) is replaced by 1. Instead of (55), we obtain that for all $k \ge k_0$,

$$E|u(t,x)|^2 \ge u_0^2 \left(\alpha_0 c_H \frac{t^{2H}}{k}\right)^k.$$

The argument for cases (ii)–(iii) is similar to the proof of Theorem 7.2, leading to the following lower bound: there exists $k_0 > 0$ such that for all $k \ge k_0$,

$$E|u(t,x)|^2 \ge u_0^2 \left(c_H^* \frac{t^{2H-a/2}}{k^{1-a/2}}\right)^k.$$

Choosing k appropriately completes the proof. The argument for case (iv) is similar to the proof of Theorem 7.6. In all cases, the argument is extended to all $t > 0$, as in step 9 of the proof of Theorem 7.1. □

APPENDIX A: AN ELEMENTARY RESULT

LEMMA A.1. *For any $a > 0$, we have*

$$\sum_{n \ge 0} \frac{x^n}{(n!)^a} \le c_1 \exp(c_2 x^{1/a}) \qquad \textit{for all } x > 0,$$

where $c_1 > 0$ and $c_2 > 0$ are some constants depending on a.

PROOF. Note that for any $a > 0$, we have

$$\lim_{n\to\infty} \frac{\Gamma(an+1)}{(n!)^a a^{an+1/2}(2\pi n)^{(1-a)/2}} = 1; \tag{68}$$

see also (3.19) of [35]. To see this, we use Stirling's formula in the following format:

$$\lim_{x\to\infty} \frac{\Gamma(x+1)}{x^x e^{-x}\sqrt{2\pi x}} = 1$$

(see, e.g., Corollary 3 of [39]), from which we infer that

$$\Gamma(an+1) \sim (an)^{an} e^{-an} (2\pi an)^{1/2} \quad \text{and} \quad n! = \Gamma(n+1) \sim n^n e^{-n} (2\pi n)^{1/2}.$$

Here we use the notation $a_n \sim b_n$ to indicate that $a_n/b_n \to 1$ as $n \to \infty$.

From (68), it follows that there exists a constant $C_1 > 0$ depending on a, such that

$$\frac{\Gamma(an+1)}{C_n (n!)^a} \le C_1 \qquad \text{for all } n \ge 0,$$

where $C_n = a^{an} n^{(1-a)/2}$. Clearly, we can choose a constant $\lambda > 1$ (depending on a) such that $\lambda^{-n} \le C_n \le \lambda^n$ for all n. Therefore,

$$\sum_{n\ge 0} \frac{x^n}{(n!)^a} \le C_1 E_a(\lambda x), \tag{69}$$

where $E_a(x) = \sum_{n\ge 0} x^n / \Gamma(an+1)$ denotes the Mittag–Leffler function.

We now use the asymptotic behavior of $E_a(x)$ for $x > 0$:

$$\lim_{x\to\infty} \frac{E_a(x)}{\exp(x^{1/a})} = \frac{1}{a} \qquad \text{for all } a > 0$$

(see Theorem 1 of [30]). Hence there exist some constants $C_2 > 0$ and $x_0 > 0$ depending on a such that

$$E_a(x) \le C_2 \exp(x^{1/a}) \qquad \text{for all } x \ge x_0.$$

If $0 < x < x_0$, then $x^n \le x_0^n$ and $E_a(x) \le C_3 \le C_3 \exp(x^{1/a})$, where $C_3 = E_a(x_0)$ depends only on a. Taking $C_4 = \max(C_2, C_3)$, it follows that

$$E_a(x) \le C_4 \exp(x^{1/a}) \qquad \text{for all } x > 0. \tag{70}$$

The conclusion follows from (69) and (70). $\square$

APPENDIX B: A FUNDAMENTAL INEQUALITY

In this section, we prove inequality (39) which is used in the proof of Proposition 4.2. Note that this inequality is a simplified form of (2.5) of [35].

We first recall the Hardy–Littlewood–Sobolev theorem.

THEOREM B.1 (Theorem 1, page 119 of [50]). *Let $0 < \alpha < n$ and $1 < p < \infty$. Let $q > p$ be such that $1/p - 1/q = \alpha/n$. For any $\varphi \in L^p(\mathbb{R}^n)$, the integral*

$$(I_\alpha \varphi)(x) := \int_{\mathbb{R}^n} \varphi(y)|x-y|^{-n+\alpha}\, dy$$

converges absolutely for almost all $x \in \mathbb{R}^n$, and

$$\|I_\alpha \varphi\|_{L^q(\mathbb{R}^n)} \le C_{n,\alpha,p} \|\varphi\|_{L^p(\mathbb{R}^n)}, \tag{71}$$

where $C_{n,\alpha,p} > 0$ is a constant depending on n, α and p.

The following inequality is due to [40]. We include its proof for the sake of completeness.

LEMMA B.2. *Let $H \in (1/2, 1)$ and $\alpha_H = H(2H-1)$. For any $f, g \in L^{1/H}(\mathbb{R})$,*

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |f(t)||g(s)||t-s|^{2H-2}\,dt\,ds \tag{72}$$
$$\leq C_H\left(\int_{\mathbb{R}} |f(t)|^{1/H}\,dt\right)^H \left(\int_{\mathbb{R}} |g(t)|^{1/H}\,dt\right)^H,$$

where $C_H > 0$ is the constant from (71) with $n = 1, \alpha = 2H-1$ and $p = 1/H$.

PROOF. Using Hölder's inequality with $p = 1/H$ and $q = 1/(1-H)$, we infer that the left-hand side of (72) is smaller than

$$\left(\int_{\mathbb{R}} |f(t)|^{1/H}\,dt\right)^H \left[\int_{\mathbb{R}}\left(\int_{\mathbb{R}} |g(s)||t-s|^{2H-2}\,ds\right)^{1/(1-H)} dt\right]^{1-H}$$
$$= \|f\|_{L^{1/H}(\mathbb{R})} \cdot \left\{\int_{\mathbb{R}} \left[\left(I_{2H-1}|g|\right)(t)\right]^{1/(1-H)} dt\right\}^{1-H}$$
$$= \|f\|_{L^{1/H}(\mathbb{R})} \|I_{2H-1}|g|\|_{L^{1/(1-H)}(\mathbb{R})}.$$

The conclusion now follows by (71) with $n = 1, \alpha = 2H-1$, $p = 1/H$ and $q = 1/(1-H)$. $\square$

LEMMA B.3. *For any $\varphi \in L^{1/H}(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \varphi(\mathbf{t})\varphi(\mathbf{s}) \prod_{i=1}^n |t_i - s_i|^{2H-2}\,d\mathbf{t}\,d\mathbf{s} \leq C_H^n \left(\int_{\mathbb{R}^n} |\varphi(\mathbf{t})|^{1/H}\,d\mathbf{t}\right)^{2H}, \tag{73}$$

where $C_H > 0$ is the constant from Lemma B.2, and we denote $\mathbf{t} = (t_1, \dots, t_n)$ and $\mathbf{s} = (s_1, \dots, s_n)$.

PROOF. We proceed by induction on n. For $n = 1$, the result holds by Lemma B.2. Suppose that (73) holds for $n-1$. By applying Lemma B.2 to the functions $f(\cdot) = \varphi(t_1, \dots, t_{n-1}, \cdot)$ and $g(\cdot) = \varphi(s_1, \dots, s_{n-1}, \cdot)$ for fixed $(t_1, \dots, t_{n-1}) \in \mathbb{R}^{n-1}$ and $(s_1, \dots, s_{n-1}) \in \mathbb{R}^{n-1}$, we obtain that

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |\varphi(t_1, \dots, t_{n-1}, t_n)||\varphi(s_1, \dots, s_{n-1}, s_n)||t_n - s_n|^{2H-2}\,dt_n\,ds_n$$
$$\leq C_H \|\varphi(t_1, \dots, t_{n-1}, \cdot)\|_{1/H} \|\varphi(s_1, \dots, s_{n-1}, \cdot)\|_{1/H},$$

where $\|\cdot\|_{1/H}$ denotes the $L^{1/H}(\mathbb{R})$-norm. [By Fubini's theorem, the functions f and g are in $L^{1/H}(\mathbb{R})$ for almost all $(t_1, \dots, t_{n-1}) \in \mathbb{R}^{n-1}$ and $(s_1, \dots, s_{n-1}) \in \mathbb{R}^{n-1}$.] Hence, the left-hand side of (73) is less that

$$C_H \int_{\mathbb{R}}\int_{\mathbb{R}} \|\varphi(t_1, \dots, t_{n-1}, \cdot)\|_{1/H} \|\varphi(s_1, \dots, s_{n-1}, \cdot)\|_{1/H} \tag{74}$$
$$\times \prod_{i=1}^{n-1} |t_i - s_i|^{2H-2}\,d\mathbf{t}_{n-1}\,d\mathbf{s}_{n-1},$$

where $\mathbf{t}_{n-1} = (t_1, \dots, t_{n-1})$ and $\mathbf{s}_{n-1} = (s_1, \dots, s_{n-1})$.

By the induction hypothesis, (74) is less than

$$C_H^n \left(\int_{\mathbb{R}^{n-1}} \| \varphi(t_1, \dots, t_{n-1}, \cdot) \|_{1/H}^{1/H} \, dt_1, \dots, dt_{n-1} \right)^{2H}$$
$$= C_H^n \left(\int_{\mathbb{R}} \int_{\mathbb{R}} |\varphi(\mathbf{t})|^{1/H} \, d\mathbf{t} \right)^{2H},$$

where $\mathbf{t} = (t_1, \dots, t_n)$. This proves (73). $\square$

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