

# The Liouville property for groups acting on rooted trees

Gideon Amir<sup>a</sup>, Omer Angel<sup>b</sup>, Nicolás Matte Bon<sup>c</sup> and Bálint Virág<sup>d</sup>

<sup>a</sup>*Department of Mathematics, Bar-Ilan University. Ramat Gan, 52900 Israel. E-mail: gidi.amir@gmail.com*

<sup>b</sup>*Department of Mathematics, University of British Columbia. 121 - 1984 Mathematics Rd, Vancouver, BC, V6T 1Z2 Canada.  
E-mail: angel@math.ubc.ca*

<sup>c</sup>*Laboratoire de Mathématiques d'Orsay, Faculté des Sciences d'Orsay, Université Paris-Sud, Orsay, F-91405, France.  
E-mail: nicolas.matte.bon@ens.fr*

<sup>d</sup>*Department of Mathematics University of Toronto 40 St George St. Toronto, ON, M5S 2E4, Canada. E-mail: balint@math.utoronto.ca*

Received 16 October 2013; revised 2 November 2014; accepted 7 July 2015

---

**Abstract.** We show that on groups generated by bounded activity automata, every symmetric, finitely supported probability measure has the Liouville property. More generally we show this for every group of automorphisms of bounded type of a rooted tree. For automaton groups, we also give a uniform upper bound for the entropy of convolutions of every symmetric, finitely supported measure.

**Résumé.** Nous démontrons que les groupes engendrés par les automates d'activité bornée ont la propriété de Liouville pour tout choix d'une mesure de probabilité symétrique, de support fini. Plus généralement, nous montrons ce résultat pour tous les groupes agissants sur un arbre enraciné par automorphismes de type borné. Dans le cas des groupes d'automate nous obtenons aussi une borne supérieure uniforme pour l'entropie, qui ne dépend pas du choix de la mesure symétrique, de support fini.

MSC: 20F69; 05C81; 20E08

Keywords: Groups acting on rooted trees; Liouville property; Random walk entropy; Recurrent Schreier graphs

---

## 1. Introduction

Groups acting on rooted trees are a source of finitely generated groups with a number of interesting properties concerning amenability, growth and random walks. An important special case are automata groups. These include the Grigorchuk group of intermediate growth [15], Gupta and Sidki's examples of finitely generated torsion  $p$ -groups [17], the Hanoi Tower groups [14], the Basilica group [16] and iterated monodromy groups arising from holomorphic dynamics: see [23,25] for a survey of the topic and literature.

One aspect of groups acting on rooted trees that has attracted some attention is the behaviour of random walks on them, especially random walks with the *Liouville property*. (For the definition and preliminaries on the Liouville property see Section 1.2.) On one hand, construction based on these groups have provided new examples of asymptotic behaviours for the rate of escape and entropy of random walks in the sublinear range [2,9] and for the relationship between the Liouville property and growth of groups [4,12]. On the other hand the Liouville property has turned out to be a useful tool to prove *amenability* of several classes of groups acting on rooted trees [1,5,6,8,19]. The motivation of this paper has its roots in these results, that we now outline.

The first result of this kind is due to Bartholdi and Virág [6], who used random walks to prove amenability of the *Basilica group*. A key observation was that the Basilica group admits a special, so-called *self-similar*, non-degenerate symmetric, finitely supported measure. From this, they deduce amenability of the group. Their method was later generalized and simplified by Kaimanovich [19] who gave a general definition of a self-similar measure on a group

acting on a rooted tree, and showed that any such measure has the Liouville property. In particular a group supporting a non-degenerate self-similar measure is amenable. As noted in [19], existence of a finitely supported self-similar measure is rare as it relies on strong combinatorial assumptions on the action on the tree.

These ideas were further developed and used to show amenability of a large class of automata groups, namely groups generated by *finite automata of bounded activity* by Bartholdi, Kaimanovich and Nekrashevych [5] and groups generated by *finite automata of linear activity* in [1] (see Section 2.4 for definitions regarding automata groups and their activity degree). The idea of the proofs in [1,5] is to embed all such groups in a special family of groups called the *mother groups*, and show that these admit a special generating measure with the Liouville property. Thus the mother groups are amenable. Since amenability is inherited by subgroups, so are all groups generated by finite automata of bounded and linear activity. The measure considered on the mother groups also has a certain self-similarity, weaker than the self-similarity required in [19]. These results have been unified and are now part of a more general amenability criterion due to Juschenko, Nekrashevych and de la Salle [18], which applies to a wider class of groups (and is not concerned with the Liouville property).

The results and methods from [1,5,6,19] raise the following question: does the Liouville property hold for *all* symmetric, finitely supported measures on the groups considered there? Note that the results above do not even imply that every bounded or linear automaton group *admits* a symmetric, finitely supported, generating probability measure with the Liouville property (see Section 1.2 for an account on open questions concerning the stability of the Liouville property). A positive answer was conjectured in [1] for groups generated by finite automata of bounded, linear and quadratic activity. It is shown in [3] that this does not hold in general for automata groups of polynomial activity of degree at least 3, and it is not yet known if these groups are amenable as asked by Sidki [27].

### 1.1. Statement of results

The aim of this paper is to establish the Liouville property for a large class of random walks on groups acting on rooted trees, that may lack self-similar properties. Our method combines ideas from papers cited above, together with an analysis of the orbital Schreier graphs for the group action on the rooted tree. Our first result gives a partial answer to the conjecture in [1], covering the case of bounded automata groups. In fact we do not need the assumption that the groups are generated by a *finite state* automaton: our result applies to general groups of automorphisms of *bounded type* (see Definition 2.4) of a spherically homogeneous rooted tree.

**Theorem 1.** *Let  $G$  be a group of automorphisms of bounded type of a spherically homogeneous rooted tree of bounded valencies. Then every symmetric, finitely supported probability measure  $\mu$  on  $G$  has the Liouville property.*

Amenability of general groups of automorphisms of bounded type of a rooted tree is a particular case of the result of Juschenko, Nekrashevych and de la Salle [18] which answers a question of Nekrashevych [24]. Since the Liouville property implies amenability, Theorem 1 also implies this result.

A key ingredient of the proof of Theorem 1 is *recurrence* of the orbital Schreier graphs for the action on the boundary of the rooted tree [7,18]. More can be said in cases where the Schreier graphs have explicit descriptions. In such cases a closer analysis of the Schreier graphs yields explicit upper bounds for the entropy of the convolutions  $H(\mu^{*k})$  (see Section 3.2 for preliminaries regarding entropy and its relationship to the Liouville property).

We illustrate this with the *principal group of directed automorphism*  $M(A, B)$  (see Section 5.1 for the definition). These groups were first defined and studied by Briussel, who proved their amenability in [8] using random walks and in [10] by exhibiting Følner sets. These are generalizations of the mother groups from [5]. In particular they contain as subgroups all groups generated by finite-state automata with bounded activity (see Theorem 5.2 below). A particular case of the group  $M(A, B)$  was also used in [2,9].

**Theorem 2.** *Let  $M(A, B)$  be a group as in Definition 5.1 acting on a spherically homogeneous rooted tree  $\mathbb{T}_{\bar{m}}$  with bounded valencies  $\bar{m} = (m_n)$ . Then every symmetric, finitely supported measure  $\mu$  on  $M(A, B)$  has the Liouville property. Moreover there exists a constant  $C$  depending on  $\text{supp}(\mu)$  only such that*

$$H(\mu^{*k}) \leq Ck^\alpha,$$

where  $\alpha = \log m_*/\log \frac{m_*^2}{m_*-1}$  and  $m_* = \max(\bar{m})$ .

In [5] this bound was obtained in the case when  $M(A, B)$  is the mother group and  $\mu$  is in a special class of measures defined there. Note the support of  $\mu$  need not generate all of  $M(A, B)$ . Since every group generated by a finite automaton of bounded activity is a subgroup of some group of the form  $M(A, B)$ , we get the following corollary.

**Corollary 1.1.** *Let  $G$  be a group generated by a finite automaton of bounded activity, and  $\mu$  a symmetric, finitely supported probability measure on  $G$ . Then  $H(\mu^{*k}) \leq Ck^\alpha$ , where  $\alpha < 1$  depends only on the group  $G$  and the constant  $C$  depends only on the support of  $\mu$ .*

The exponent  $\alpha$  can be explicitly determined from the structure of the automaton by following the argument in [5, Theorem 3.3] to embed  $G$  in the mother group.

A comment on the linear and quadratic activity case in the conjecture in [1] seems in order. Can these cases be attacked using the method of this paper? The ascension diagrams (see Definition 3.5) become more complicated. To analyse them effectively, a more precise understanding of simple random walk on the Schreier graphs seems needed, beyond the fact that the infinite graphs are recurrent. This task becomes harder together with the level of precision required, as the graphs also become more complicated. We believe that this can be done to prove the conjecture in the linear case. However this would require a considerably more complicated analysis relying on quantitative resistance estimates. We do not know if there is any hope to apply our method to the quadratic case.

### 1.2. Preliminaries on the Liouville property

Given a probability measure  $\mu$  on a countable group  $G$ , a function  $f : G \rightarrow \mathbb{R}$  is said to be  $\mu$ -harmonic if  $f(g) = \sum_{h \in G} f(gh)\mu(h)$  for every  $g \in G$ . The measure  $\mu$  is said to have the *Liouville property* if every bounded  $\mu$ -harmonic function on  $G$  is constant on the subgroup  $H = \langle \text{supp}(\mu) \rangle$ . An equivalent formulation of the Liouville property is triviality of the *Poisson boundary* of  $(H, \mu)$  [20]. If moreover the measure  $\mu$  is symmetric and has finite first moment with respect to a word metric, the Liouville property is equivalent to the random walk with step measure  $\mu$  having 0 asymptotic speed [21, Corollary 3]. Under the weaker assumption that  $\mu$  has finite entropy, the Liouville property is equivalent to vanishing of the asymptotic entropy  $h(\mu)$  ([11,20]). The latter will be the characterisation of the Liouville property that we use in this paper. See Section 3.2 for preliminaries regarding entropy.

Amenability of a countable group  $G$  is equivalent to the existence of a Liouville symmetric measure supported on a generating set of  $G$  [20,26]. This measure may have infinite support, as in the well-known case of the Lamplighter group over  $\mathbb{Z}^3$ , namely  $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}^3$ , see [20]. In some amenable groups it must even have infinite entropy, see [13]. Thus existence of a *finitely supported* Liouville symmetric measure whose support generates the group is strictly stronger than amenability. The Liouville property depends on the choice of  $\mu$ ; however it is an important open question whether it is a group property when one restricts to symmetric measures with finite generating support, and whether it is inherited by subgroups for the same class of measures.

### 1.3. Structure of the paper and overview of the proofs

Section 2 contains preliminaries on groups acting on rooted trees.

Section 3 contains general facts on random walks on groups acting on rooted trees. Most of this section is based on the connection between groups acting on rooted trees and *random walks with internal degrees of freedom*, introduced by Kaimanovich [19]. A random walk with internal degrees of freedom on a group  $G$  with space of degrees  $Y$  is a Markov chain on  $G \times Y$  that can be described in terms of a random walk on a *diagram*: a finite graph with vertex set  $Y$  where edges are labelled by probability measures on  $G$ . We revisit and slightly generalise ideas in [19] by considering the *ascension diagram* (see Definition 3.5), a smaller diagram obtained by stopping the walker on the Schreier graph when it visits a fixed subset of  $Y$  (rather than a single vertex as in previous works). We then prove an inequality linking the asymptotic entropy of the random walk on the group with the ascension diagrams. In [1,5,19], random walk with internal degrees of freedom arising from self-similar random walks were used, via explicit calculations using matrices with entries in the group algebra. For random walks lacking of self-similarity properties, these calculations become more complicated. To avoid these we take advantage of recurrence of the Schreier graphs through a simple fact proven at the end of the section.

In Section 4 we prove Theorem 1. The proof is based on the tools introduced in Section 3. A key observation is that *sections* of elements in the support of  $\mu$  at high enough levels of the tree belong either to a finite group of *finitary automorphisms* or to a finite *groupoid of directed automorphisms* (a notion introduced in Section 2). Combined with recurrence of the orbital Schreier graphs, this yields bounds on the asymptotic speed of random walks with internal degrees of freedom determined by the ascension diagrams, which are used to bound the entropy of the original random walk.

Finally, in Section 5 we prove Theorem 2. The additional ingredient needed is a analysis of the orbital Schreier graphs for the action on the finite level of the tree using electric network theory. We give lower bounds on effective resistances between certain points in the graph, and use them to get explicit entropy estimates through arguments similar to Section 4.

## 2. Rooted trees and their automorphisms

### 2.1. Spherically homogeneous rooted trees and their automorphisms

Let  $\bar{m} = (m_i)_{i \geq 1}$  be a bounded sequence of positive integers. The *spherically homogeneous rooted tree*  $\mathbb{T}_{\bar{m}}$  is the tree where each vertex at level  $k$  has  $m_{k+1}$  children in level  $k + 1$ . The tree  $\mathbb{T}_{\bar{m}}$  has a root in level 0, which is denoted  $\emptyset$ . A vertex at level  $k$  is naturally encoded by a word  $x_k x_{k-1} \cdots x_1$ , where  $x_i \in X_{m_i} = \{0, \dots, m_i - 1\}$ . The children of  $v$  are words of the form  $xv$  where  $x$  is a single letter. We denote by  $\mathbb{T}_{\bar{m}}^n \subset \mathbb{T}_{\bar{m}}$  the set of words of length  $n$ , i.e. the  $n$ th level of the tree. Note that words are read from right to left.

We denote by  $\text{Aut}(\mathbb{T}_{\bar{m}})$  the group of automorphisms of  $\mathbb{T}_{\bar{m}}$  that fix the root. Note that for some sequences  $\bar{m}$  (in particular the constant sequences) all automorphisms of  $\mathbb{T}_{\bar{m}}$  fix the root. However, there are sequences for which the tree has additional automorphisms which do not fix the root and so do not belong to  $\text{Aut}(\mathbb{T}_{\bar{m}})$  in our notations. We write actions of automorphisms on the right and use the notation

$$w \mapsto w \cdot g$$

for  $w \in \mathbb{T}_{\bar{m}}$  and  $g \in \text{Aut}(\mathbb{T}_{\bar{m}})$ . For  $w \in \mathbb{T}_{\bar{m}}$ , consider the sub-tree rooted at  $w$ . If  $w$  is at level  $n$ , then this sub-tree is isomorphic to the spherically homogeneous rooted tree  $\mathbb{T}_{\sigma^n \bar{m}}$ , where  $\sigma$  denotes the shift operator

$$\sigma(m_1, m_2, \dots) = (m_2, m_3, \dots).$$

Automorphisms  $g \in \text{Aut}(\mathbb{T}_{\bar{m}})$  preserve the levels of the tree, so that every word  $w \in \mathbb{T}_{\bar{m}}$  is mapped by  $g$  to a word  $w \cdot g$  of the same length, say  $n$ . Since the sub-trees above  $w$  and  $w \cdot g$  are canonically isomorphic,  $g$  induces a bijection of the sub-trees rooted at  $w$  and  $w \cdot g$ , which can be identified with a unique element of  $\text{Aut}(\mathbb{T}_{\sigma^n \bar{m}})$ . This element is called the *section* of  $g$  at  $w$  and it is denoted  $g|_w$ . Formally, the section is the unique element  $g|_w \in \text{Aut}(\mathbb{T}_{\sigma^n \bar{m}})$  such that for every word  $v \in \mathbb{T}_{\bar{m}}^n$ ,

$$vw \cdot g = (v \cdot g|_w)(w \cdot g),$$

where the parenthesis juxtaposition denotes concatenation of words. It immediately follows from the definition that sections are multiplied and inverted according to the following rules

$$(gh)|_w = g|_w h|_{w \cdot g}; \quad g^{-1}|_w = (g|_{w \cdot g^{-1}})^{-1}. \tag{1}$$

Using an equivalent terminology, there is an isomorphism (a *wreath recursion*)

$$\begin{aligned} \text{Aut}(\mathbb{T}_{\bar{m}}) &\rightarrow \text{Aut}(\mathbb{T}_{\sigma \bar{m}}) \wr_{X_{m_1}} S_{m_1} = \text{Aut}(\mathbb{T}_{\sigma \bar{m}})^{X_{m_1}} \rtimes S_{m_1}, \\ g &\mapsto (g|_0, \dots, g|_{m_1-1})\sigma, \end{aligned}$$

where  $g|_0, \dots, g|_{m_1-1}$  are the first level sections of  $g$  and the permutation  $\sigma$  gives its action on the first level  $\mathbb{T}_{\bar{m}}^1 = X_{m_1}$ .

**Definition 2.1.** Let  $G < \text{Aut}(\mathbb{T}_{\bar{m}})$ . For  $n \in \mathbb{N}$  we denote by  $G^{(n)}$  the group of  $n$ th level sections of  $G$ , i.e. the subgroup of  $\text{Aut}(\mathbb{T}_{\sigma^n \bar{m}})$  generated by  $\{g|_w : g \in G, w \in \mathbb{T}_{\bar{m}}^n\}$ .

**Remark 2.2.** If the group  $G$  is generated by the set  $S$ , the groups of sections  $G^{(n)}$  are generated by the  $n$ th level sections of elements in  $S$ , see (1).

The action of  $\text{Aut}(\mathbb{T}_{\bar{m}})$  naturally extends to an action by homeomorphism on the *boundary at infinity* of the tree  $\partial\mathbb{T}_{\bar{m}}$ . The boundary  $\partial\mathbb{T}_{\bar{m}}$  is the set of infinite geodesic rays starting from the root. In our notations it identifies with the set of left-infinite sequences  $\gamma = \cdots x_3 x_2 x_1$  where  $x_i \in X_{m_i}$ . The set  $\partial\mathbb{T}_{\bar{m}}$  is endowed with the natural product topology, which makes it homeomorphic to the Cantor set.

The *Schreier graph* associated with a group action is defined as follows. If a group  $G$  generated by a finite symmetric set  $S$  acts on a set  $Y$ , the Schreier graph has vertex set  $Y$  and edges  $(y, y \cdot s)$  for  $y \in Y, s \in S$ . We admit that the action of  $G$  on  $Y$  can be non-transitive and then the Schreier graph is disconnected. A connected component of the Schreier graph is called an *orbital Schreier graph*.

In our setting, a finitely generated subgroup  $G < \text{Aut}(\mathbb{T}^d)$  naturally defines a sequence of finite Schreier graphs arising from the action on the finite levels of the tree. It also defines a family of infinite graphs given by the orbital Schreier graphs for the action of  $G$  on  $\partial\mathbb{T}_{\bar{m}}$ . The Schreier graph for level  $n + 1$  covers the graph for level  $n$ .

### 2.2. Activity and automorphisms of bounded type

The *activity function* of an automorphism  $g \in \text{Aut}(\mathbb{T}_{\bar{m}})$  is the function  $\Gamma_g : \mathbb{N} \rightarrow \mathbb{N}$  that counts the number of level  $n$  vertices  $v$  so that  $g|_v \neq e$ . By (1) the activity satisfies

$$\Gamma_{gh}(n) \leq \Gamma_g(n) + \Gamma_h(n); \quad \Gamma_{g^{-1}}(n) = \Gamma_g(n).$$

This allows to define several subgroups of  $\text{Aut}(\mathbb{T}_{\bar{m}})$  in terms of the activity function. For instance elements whose activity function is bounded (respectively grows at most polynomially, respectively grows subexponentially) form a subgroup of  $\text{Aut}(\mathbb{T}_{\bar{m}})$ .

**Definition 2.3.** An element  $g \in \text{Aut}(\mathbb{T}_{\bar{m}})$  is called *finitary* if the sections  $g|_v$  are non-trivial only for finitely many vertices  $v \in \mathbb{T}_{\bar{m}}$ . We define the *depth of  $g$*  to be the smallest level  $n$  so that all sections at level  $n$  are trivial.

Finitary automorphisms of  $\mathbb{T}_{\bar{m}}$  form a locally finite subgroup of  $\text{Aut}(\mathbb{T}_{\bar{m}})$ .

Automorphisms of *bounded type* are automorphisms that have bounded activity in a strong sense, that we now define.

**Definition 2.4 (Automorphism of bounded type).** An automorphism  $g \in \text{Aut}(\mathbb{T}_{\bar{m}})$  is said to be of *bounded type* if there exists a finite set of rays in  $\partial\mathbb{T}_{\bar{m}}$ , called the *singular rays of  $g$* , and a  $K > 0$  so that  $g|_v$  is finitary with depth at most  $K$  whenever  $v$  does not belong to a singular ray. The minimal such  $K$  is called the *depth of  $g$* .

In other word automorphisms of bounded type are those that have non-trivial sections only in a bounded neighbourhood of a finite set of rays. Obviously automorphisms of bounded type have bounded activity. In some special cases the two notions coincide (for instance for automorphisms defined by a finite-state automaton, see Section 2.4).

**Remark 2.5.** It is easy to see from (1) that automorphisms of bounded type form a subgroup of  $\text{Aut}(\mathbb{T}_{\bar{m}})$ .

### 2.3. The groupoid of directed automorphisms of a rooted tree

A special case of automorphism of bounded type are the *directed automorphisms*.

**Definition 2.6.** An automorphism of bounded type  $g \in \text{Aut}(\mathbb{T}_{\bar{m}})$  is said to be *directed* if it has at most one singular ray  $\gamma \in \partial\mathbb{T}_{\bar{m}}$ . If there is such a ray, we say  $g$  is directed along  $\gamma$ . By convention we say that a finitary  $g$  is directed along every ray.

Unlike automorphisms of bounded type, directed automorphisms do not form a subgroup of  $\text{Aut}(\mathbb{T}_{\bar{m}})$ . However we have the following properties (cf. the section multiplication rule (1)):

1. if  $g, h$  are directed along  $\gamma, \eta$  respectively and  $\gamma \cdot g = \eta$  then  $gh$  is directed along  $\gamma$ ;
2. if  $g$  is directed along  $\gamma$  then  $g^{-1}$  is directed along  $\gamma \cdot g$ .

These properties suggest that the set of directed automorphisms of  $\mathbb{T}_{\bar{m}}$  form essentially a *groupoid* (up to some ambiguity originated by finitary automorphisms). We shall now make this intuition precise.

Recall that any right action of a group  $G$  on a set  $X$  defines a groupoid, called the *action groupoid* and denoted  $\mathcal{G} = \mathcal{G}(X, G)$ . By definition  $\mathcal{G} = X \times G$  as a set; the product of two elements  $(x, g)$  and  $(y, h)$  in  $\mathcal{G}$  is defined whenever  $x \cdot g = y$  and in this case  $(x, g)(y, h) = (x, gh)$ ; the inverse of  $(x, g)$  is  $(x \cdot g, g^{-1})$ . Elements of the form  $(x, e) \in \mathcal{G}$  are called *units*. A *subgroupoid* of  $\mathcal{G}$  is a subset  $\mathcal{H} \subset \mathcal{G}$  which is closed under taking inverses and products (i.e. whenever the product of two elements in  $\mathcal{H}$  is defined in  $\mathcal{G}$ , it belongs to  $\mathcal{H}$ ) and that contains all units  $(y, e)$  for  $y \in Y$ , where  $Y \subset X$  is the projection of  $\mathcal{H}$  to  $X$  (note that this is allowed to be a proper subset of  $X$ ). The subgroupoid generated by a family  $\mathcal{S} \subset \mathcal{G}$  is the smallest subgroupoid containing  $\mathcal{S}$ .

The *groupoid of directed automorphism* of the rooted tree  $\mathbb{T}_{\bar{m}}$  is the subgroupoid  $\mathcal{D}_{\bar{m}}$  of the action groupoid  $\mathcal{G}(\partial\mathbb{T}_{\bar{m}}, \text{Aut}(\mathbb{T}_{\bar{m}}))$  which consists of couples  $(\gamma, g) \in \partial\mathbb{T}_{\bar{m}} \times \text{Aut}(\mathbb{T}_{\bar{m}})$  such that  $g$  is directed along  $\gamma$ . We define the *depth* of  $(\gamma, g) \in \mathcal{D}_{\bar{m}}$  to be the depth of  $g$ . There is a natural projection

$$\begin{aligned} \mathcal{D}_{\bar{m}} &\rightarrow \text{Aut}(\mathbb{T}_{\bar{m}}), \\ (\gamma, g) &\mapsto g, \end{aligned}$$

which maps the groupoid product, whenever defined, to the usual group product.

Any *non-finitary* directed automorphism  $g \in \text{Aut}(\mathbb{T}_{\bar{m}})$  has a unique pre-image in  $\mathcal{D}_{\bar{m}}$ . This allows to think of  $g$  either as an element of  $\text{Aut}(\mathbb{T}_{\bar{m}})$  or as an element of  $\mathcal{D}_{\bar{m}}$ . Finitary automorphisms however have several pre-images and thus the groupoid  $\mathcal{D}_{\bar{m}}$  cannot be properly identified with a subset of  $\text{Aut}(\mathbb{T}_{\bar{m}})$ .

**Lemma 2.7.** *The groupoid  $\mathcal{D}_{\bar{m}}$  is locally finite: every finite family  $\mathcal{S} \subset \mathcal{D}_{\bar{m}}$  generates a finite subgroupoid. Moreover the cardinality of this finite subgroupoid has an upper bound which depends only on the cardinality of  $\mathcal{S}$ , on the maximal depth of elements in  $\mathcal{S}$ , and on  $m_*$ .*

**Proof.** It is a classical and elementary fact that the unrestricted infinite direct product of finite groups of bounded size is locally finite, and the cardinality of the subgroup generated by a finite subset has an upper bound that only depends on the size of the subset.

Given a ray  $\gamma = \dots x_3x_2x_1 \in \partial\mathbb{T}_{\bar{m}}$  define the element  $h_\gamma \in \text{Aut}(\mathbb{T}_{\bar{m}})$  by the wreath recursion

$$h_\gamma = (h_{\sigma\gamma}, e, \dots, e)(0x_1),$$

where  $\sigma\gamma = \dots x_3x_2$ . This element is directed along the zero ray  $\rho = \dots 000$ , moreover  $\rho \cdot h_\gamma = \gamma$ . It is straightforward to check that if  $g$  is directed along  $\gamma$  with depth at most  $K$ , then  $h_\gamma g h_{\gamma \cdot g}^{-1}$  is directed along  $\rho$ , fixes  $\rho$  and has depth at most  $K$ . The set of automorphisms of  $\text{Aut}(\mathbb{T}_{\bar{m}})$  with these properties is isomorphic to an infinite direct product of finite groups with bounded cardinalities.

It follows that if  $\mathcal{S} \subset \mathcal{D}_{\bar{m}}$  is finite and consist of elements with depth at most  $K$ , elements of the form  $(\rho, h_\gamma)(\gamma, s)(\gamma \cdot s, h_{\gamma \cdot s}^{-1})$  where  $(\gamma, s) \in \mathcal{S}$  belong to an infinite direct product of finite groups of bounded cardinality. Let  $\mathcal{H} \subset \mathcal{D}_{\bar{m}}$  be the subgroupoid that they generate, which is in fact a group. The cardinality of  $\mathcal{H}$  has an upper bound that depends only on  $|\mathcal{S}|$  and  $K$ . Now observe that the subgroupoid generated by  $\mathcal{S}$  is contained in  $\cup_{\gamma, \eta} (\gamma, h_\gamma^{-1})\mathcal{H}(\rho, h_\eta^{-1})$  where  $\gamma, \eta$  run along singular rays of elements in  $\mathcal{S} \cup \mathcal{S}^{-1}$ . The conclusion follows.  $\square$

From now on we adopt the following notation: calligraphic letters (e.g.  $\mathcal{S}$ ) will always denote subsets (or subgroupoids) of the groupoid  $\mathcal{D}_{\bar{m}}$ , and we will sometimes denote with the corresponding capital letter (e.g.  $S$ ) the projection to the group  $\text{Aut}(\mathbb{T}_{\bar{m}})$ .

2.4. Automata groups and their activity degree

We now recall some basic notions in the relevant particular case of *automata groups* acting on regular rooted trees. These notions do not play an active role in the proofs, but this is the most relevant source of examples.

If the sequence  $\bar{m}$  is constant, equal to some positive integer  $m$ , the tree  $\mathbb{T}_m$  is called the *regular rooted tree* of degree  $m$ . It is indexed by the set of words in the alphabet  $X = \{0, \dots, m - 1\}$ .

An important class of finitely generated groups acting on the tree  $\mathbb{T}_m$  are groups generated by *finite automata*. An *invertible automaton* over the alphabet  $X$  is a set  $A$  (the automaton state space) together with a pair of maps

$$\begin{aligned} A &\rightarrow S_m, & A \times X &\rightarrow A, \\ a &\mapsto \sigma_a, & (a, x) &\mapsto a_x. \end{aligned}$$

Such an automaton acts on words in the alphabet  $X$  as follows: if the current state is  $a \in A$ , and the automaton receives as input a letter  $x$  it outputs the letter  $x \cdot \sigma_a$ , and switches to state  $a_x$ . Given an initial state  $a \in A$ , any word input into the automaton yields an output of equal length, and it is readily seen that for any initial state this action defines an automorphism of  $\mathbb{T}_m$ . This automorphism is as follows:  $a$  acts on the first level by the permutation  $\sigma_a$ ; its first level section at vertex  $x$  is the automorphism defined by the state  $a_x \in A$ . If a state defines the identity automorphism  $e$  of  $\mathbb{T}_m$ , it is said to be *trivial*.

Every automaton  $A$  generates a subgroup of  $\text{Aut}(\mathbb{T}_m)$ , generated by the automorphisms corresponding to all states. An equivalent description is that we have a finite set  $A \subset \text{Aut}(\mathbb{T}_m)$ , so that for any  $g \in A$  and any  $v \in \mathbb{T}_m$  we have  $g|_v \in A$ . Such a set naturally defines an automaton.

We shall always suppose that automata are *reduced*, i.e. two distinct states of  $A$  define distinct automorphisms of the tree. Any automaton can be brought to a reduced form by identifying states with the same action on the tree.

If  $a$  is a state of an automaton  $A$ , the *activity function*  $\Gamma_a(n)$  (see Section 2.2) is determined in a simple manner by the structure of the automaton as we shall now explain. First note that in the automaton case  $\Gamma_a(n)$  grows either polynomially with some integer exponent  $d_a$  or exponentially (in which case  $d_a$  is set to be  $+\infty$ ). (This is since these functions satisfy a linear recursion among themselves, and since  $A$  is finite.) The *activity degree* of  $A$  is defined to be  $d = \max_{a \in A} d_a$ . This invariant was introduced by Sidki in [27]. When  $d = 0$  the automaton is said to be of *bounded activity*. Some well-studied examples of groups acting on rooted trees belong to the class of bounded activity automata groups, including the Grigorchuk group, the Basilica group and iterated monodromy groups of postcritically finite polynomials (see [23]).

An automaton gives rise to a directed graph, possibly with loops and multiple edges, called the *Moore diagram* of the automaton. The vertex set is  $A$ , and there is an oriented edge from  $a$  to  $a_x$  for every  $x \in X$ . This directed edge is labelled by  $(x, x \cdot \sigma_a)$ . The trivial state is a sink. For clarity, the loops based at the trivial state are usually omitted from the Moore diagram. See Figure 1 in Section 3.4 for an example.

The activity degree can easily be computed by looking at the structure of the Moore diagram. A *non-trivial simple cycle* (henceforth, just *cycle*) in the diagram is a closed oriented path visiting each vertex at most once which visits states other than the trivial state. Note that a path is its set of edges, so that it is possible for two distinct cycles to visit the same vertices, and even in the same order. The activity is exponential ( $d = +\infty$ ) if and only if some strongly connected component of the Moore diagram contains more than one cycle (in particular  $d = +\infty$  whenever two distinct cycles intersect). If this is not the case, then there is a partial order on the set of cycles: say that  $c \rightarrow c'$  if there is an oriented path from some state in  $c$  to some state in  $c'$ . The activity degree is then equal to the largest  $d \geq 0$  for which there are distinct cycles with  $c_d \rightarrow c_{d-1} \rightarrow \dots \rightarrow c_0$ .

It easily follows from this description that an automaton of bounded activity generates a group of automorphisms of bounded type in the sense of Definition 2.4.

Another diagram associated to an automaton is the *dual Moore Diagram*, which is a special case of a Schreier graph. This is the oriented graph that has the alphabet  $X$  as vertex set, and for every  $x \in X$  and  $a \in A$  there is an oriented edge going from  $x$  to  $x \cdot \sigma_a$ . Such an edge is labelled by  $(a, a_x)$ .

The  $n$ th iteration of the dual Moore diagram is defined to be the oriented graph that has as a vertex set the  $n$ th level of the tree  $\mathbb{T}_m^n$  and for every word  $w \in \mathbb{T}_m^n$  and every state  $a \in A$  there is an edge going from  $w$  to  $w \cdot a$ . Such an edge is labelled by  $(a, a|_w)$ . The  $n$ th iteration of the dual Moore diagram is thus isomorphic as a graph to the Schreier graphs of  $G = \langle A \rangle$  acting on  $\mathbb{T}_m^n$  with generating set  $A$ .

### 3. Tools for random walks on groups acting on rooted trees

#### 3.1. Random walk with internal degrees of freedom

Let  $G$  be a group and  $Y$  be a finite set. Consider a Markov chain with state space  $Y$  and transition probabilities given by a stochastic matrix

$$P = (p_{xy})_{x,y \in Y}.$$

We shall always suppose that this Markov chain is irreducible. Consider also a collection of probability measures on  $G$ , denoted  $\mu_{xy}$  for  $x, y \in Y$ . These are called *edge measures*, and we denote the collection by  $M = (\mu_{xy})_{x,y \in Y}$ . Only measures  $\mu_{xy}$  for pairs with  $p_{xy} \neq 0$  are used.

Note that our notation are different from those in [19] in that there  $M = (\mu_{xy})$  denotes a matrix of *sub-probability* measures with total mass  $p_{xy}$  and that equal our  $\mu_{xy}$  only after renormalization.

Given such a pair  $(M, P)$  we draw the following *diagram*: take the (oriented) graph with vertex set  $Y$  induced by stochastic matrix  $P$  (with an edge  $(x, y)$  whenever  $p_{xy} \neq 0$ ). Label the edge  $(x, y)$  by the pair  $(\mu_{xy}, p_{xy})$ . We will hereinafter make no distinction between  $(M, P)$  and the associated diagram.

**Definition 3.1.** *The random walk with internal degrees of freedom corresponding to  $(M, P)$  is the Markov chain  $(\mathbf{g}_k, \mathbf{y}_k)$  on  $G \times Y$ , defined as follows:  $\mathbf{y}_k$  performs a random walk on  $Y$  with transition probabilities given by  $P$ . When  $\mathbf{y}_k$  crosses a given edge, the group element  $\mathbf{g}_k$  is multiplied on the right by a sample of the corresponding edge measure. Formally, the transition probability from  $(h, x)$  to  $(g, y)$  is  $p_{xy}\mu_{xy}(h^{-1}g)$ .*

Recall that for a Markov chain with state space  $S$  and  $T \subset S$ , the induced Markov chain on  $T$  has transition probabilities  $p_{xy} = \mathbb{P}_x(X_\tau = y)$  where  $\tau = \inf\{n > 0 : X_n \in T\}$ .

**Definition 3.2 (Trace).** *Let  $(\mathbf{g}_k, \mathbf{y}_k)$  be a random walk with internal degrees of freedom on  $G \times Y$  with diagram given by  $(M, P)$ . For a non-empty  $W \subset Y$ , the induced Markov chain on  $G \times W$  is called the trace over  $W$  of the original random walk with internal degrees of freedom.*

It is easy to see that the trace of a random walk with internal degrees of freedom is also a random walk with internal degrees of freedom  $(M_W, P_W)$ . In general, the measures making up  $M_W$  can be much more complex than the measures in  $M$ . For example, they may have infinite support even if measures of  $M$  have finite support. However, the walks we study below are such that we retain some control over the support of the new edge measures.

Note that the diagram  $(M_W, P_W)$  of the trace does not depend on the initial distribution of  $(\mathbf{g}_0, \mathbf{y}_0)$ . Hence taking the trace might be seen as an operation on diagrams. The diagram  $(M_W, P_W)$  can be explicitly computed from the diagram  $(M, P)$  and formulae can be given in term of matrices with entries in the group algebra  $\ell^1(G)$ , as shown in [19].

#### 3.2. Entropy and speed of random walks with internal degrees of freedom

Let  $\nu$  be a probability measure on a countable space  $E$ . Recall that its *entropy* is the quantity

$$H(\nu) = - \sum_{\nu(e) > 0} \nu(e) \log \nu(e).$$

For a random variable  $\mathbf{X}$  taking values in a countable space, the entropy  $H(\mathbf{X})$  is defined as the entropy of its distribution. Let us recall some basic properties of entropy.

**Proposition 3.3.**

1. *If  $\mathbf{X}$  has finite support, then  $H(\mathbf{X}) \leq \log |\text{supp}(\mathbf{X})|$ , and equality holds if and only if  $\mathbf{X}$  is uniformly distributed on  $\text{supp}(\mathbf{X})$ .*

2. Let  $\mathbf{Y}, \mathbf{X}_1, \dots, \mathbf{X}_n$  be discrete random variables defined on the same probability space, and suppose that  $\mathbf{Y}$  is a function of  $\mathbf{X}_1, \dots, \mathbf{X}_n$ . Then

$$H(\mathbf{Y}) \leq H(\mathbf{X}_1, \dots, \mathbf{X}_n) \leq H(\mathbf{X}_1) + \dots + H(\mathbf{X}_n),$$

where the middle term denotes the entropy of the joint distribution of  $(\mathbf{X}_1, \dots, \mathbf{X}_n)$ .

3. Let  $G$  be a group generated by a finite set  $S$  with the shortest word metric  $|\cdot|$ . There exists a constant  $C$ , depending only on  $|S|$ , such that if  $\mathbf{g}$  is a random variable taking values in  $G$ , then

$$H(\mathbf{g}) \leq C\mathbb{E}|\mathbf{g}| + C.$$

Let  $\mu$  be a probability measure on a group  $G$ , and  $(\mathbf{g}_k)_k$  be the corresponding random walk. By (2) above and sub-additivity, the following limit exists:

$$h(\mu) = \lim_{k \rightarrow \infty} \frac{1}{k} H(\mu^{*k}) = \lim_{k \rightarrow \infty} \frac{1}{k} H(\mathbf{g}_k).$$

The limit is called the *asymptotic entropy* of  $\mu$ . The asymptotic entropy is related to the Liouville property by the following fundamental result:

**Theorem 3.4 ([11,20]).** *Let  $\mu$  have  $H(\mu) < \infty$ . Then  $h(\mu) = 0$  if and only if  $(G, \mu)$  has the Liouville property.*

Another fundamental quantity associated to  $\mu$  is the *asymptotic speed*. Let  $G$  be generated by a finite  $S$  and let  $|\cdot|$  be the associated word metric. The asymptotic speed with respect to  $S$  is the limit

$$\ell_S(\mu) = \lim_{k \rightarrow \infty} \frac{1}{k} \mathbb{E}|\mathbf{g}_k|,$$

which exists by sub-additivity, provided  $\mu$  has finite first moment (i.e.  $\sum |g|\mu(g) < \infty$ ).

The definitions of asymptotic entropy and speed extend to the setting of random walks with internal degrees of freedom. Namely let  $(\mathbf{g}_k, \mathbf{y}_k)$  be random walk with internal degrees of freedom on  $G \times Y$  with diagram  $(M, P)$ . Suppose that the initial distribution of  $(\mathbf{g}_0, \mathbf{y}_0)$  and all edge measures  $\mu_{xy}$  have finite entropy. Then the asymptotic entropy of the random walk with internal degrees of freedom is well defined and does not depend on the initial distribution of  $(\mathbf{g}_0, \mathbf{y}_0)$  (hence it is a numerical invariant of the diagram):

$$h(M, P) = \lim_{k \rightarrow \infty} \frac{1}{k} H(\mathbf{g}_k, \mathbf{y}_k) = \lim_{k \rightarrow \infty} \frac{1}{k} H(\mathbf{g}_k). \tag{2}$$

Similarly the asymptotic speed  $\ell_S(M, P)$  is well-defined whenever all edge measures and the starting point  $\mathbf{g}_0$  have finite first moment. Asymptotic speed and entropy are related by the inequality

$$h(M, P) \leq v_S \ell_S(M, P) \leq \log |S| \ell_S(M, P), \tag{3}$$

where  $v_S = \lim_n \frac{1}{n} \log(|S^n|) \leq \log |S|$  is the *exponential growth rate* of the group  $G$  with generating set  $S$ . We will only use that the asymptotic entropy has a linear upper bound in terms of the speed, with constant depending only on the number of generators.

If  $(M, P)$  is a random walk with internal degrees of freedom on  $G \times Y$ , let  $\nu$  be the stationary distribution of  $P$ , which is unique since we assume  $P$  is irreducible. If  $(M_W, P_W)$  is the trace over  $W$ , then the asymptotic entropies satisfy the relation (see [19, Proof of Theorem 3.3]):

$$h(M_W, P_W) = \frac{1}{\nu(W)} h(M, P). \tag{4}$$

Note that the fraction of time spent in a subset  $W$  converges a.s. to  $\nu(W)$ .

### 3.3. Random walks with internal degrees of freedom and groups acting on rooted trees

Let  $\mu$  be a probability measure on a group  $G < \text{Aut}(\mathbb{T}_{\bar{m}})$  whose support generates  $G$ , and consider the associated random walk  $(\mathbf{g}_k)$ . Fix a level  $n \geq 0$ , and recall that  $G^{(n)}$  denotes the subgroup of  $\text{Aut}(\mathbb{T}_{\sigma^n \bar{m}})$  generated by  $n$ th level sections of elements in  $G$  (Definition 2.1).

Pick a vertex  $v \in \mathbb{T}_{\bar{m}}^n$ , and let  $\mathbb{O} \subset \mathbb{T}_{\bar{m}}^n$  be its orbit under the action of  $G$ . Then  $(v \cdot \mathbf{g}_k)$  is a Markov chain on  $\mathbb{O}$ , and a key observation made in [19] is that  $(\mathbf{g}_k|_v, v \cdot \mathbf{g}_k)$  is a random walk with internal degrees of freedom on  $G^{(n)} \times \mathbb{O}$  (restricting to an orbit assures the irreducibility condition for the marginal Markov chain). Let  $(M, P)$  be its diagram.

It easily follows from the section multiplication rule (1) that  $(M, P)$  has transition probabilities and edge measures given for every  $v, w \in \mathbb{O}$  by

$$\begin{aligned}
 p_{vw} &= \mu\{g : v \cdot g = w\}, \\
 \mu_{vw}(h) &= \mu\{g : v \cdot g = w, g|_v = h\} / p_{vw} \quad \text{whenever } p_{vw} \neq 0.
 \end{aligned}
 \tag{5}$$

Note moreover, that if  $\mu$  is symmetric one has  $\mu_{vw} = \hat{\mu}_{wv}$ , where  $\hat{\nu}(g) = \nu(g^{-1})$  denotes the reflected measure with respect to group inversion. If  $\mu$  is symmetric and finitely supported, the diagram  $(M, P)$  is isomorphic as a graph to the Schreier graph of  $G$  acting on  $\mathbb{O}$  with generating set  $\text{supp}(\mu)$ . When  $G$  is an automaton group, this diagram might also be seen as a weighted version of the *dual Moore diagram* of the  $n$ th iteration of the automaton.

**Definition 3.5 (Ascension diagram).** Let  $G < \text{Aut}(\mathbb{T}_{\bar{m}})$ , and  $\mu$  be a probability measure on  $G$  supported on a generating set.

1. Let  $\mathbb{O} \subset \mathbb{T}_{\bar{m}}^n$  be a  $G$ -orbit. We denote by  $T_{\mathbb{O}}(\mu) = (M, P)$  the random walk with internal degrees of freedom on  $G^{(n)} \times \mathbb{O}$ , whose transition probabilities and edge measures are given by (5).
2. More generally, let  $\mathbb{W} \subset \mathbb{O}$  be non-empty. We denote by  $T_{\mathbb{W}}(\mu) = (M_{\mathbb{W}}, P_{\mathbb{W}})$  the trace over  $\mathbb{W}$  of  $T_{\mathbb{O}}(\mu)$ .

We call  $T_{\mathbb{W}}(\mu)$  the ascension diagram of measure  $\mu$  with respect to vertex set  $\mathbb{W}$ . The case when  $\mathbb{W}$  coincides with the whole orbit is seen as a particular case of the same definition.

The simplest case of the above construction is when  $\mathbb{W} = \{w\}$  is a single point. In this case  $T_w(\mu)$  is just a new probability measure on  $G^{(n)}$ , that admits a clear interpretation: it is the step measure of the random walk on  $G^{(n)}$  that one sees by looking to the action on the subtree rooted at  $w$  at the times when  $w$  is stabilized (see [1,19]). In this case  $T_w$  is an operator acting on measures and was called the *ascension operator* in [1]. The next theorem was stated and proved in [19], in the above simpler situation and when the action of  $G$  on levels is transitive.

**Theorem 3.6.** Let  $G < \text{Aut}(\mathbb{T}_{\bar{m}})$ , and  $\mu$  a measure on  $G$  with finite entropy. Let  $\mathbb{T}_{\bar{m}}^n = \mathbb{O}_1 \sqcup \dots \sqcup \mathbb{O}_r$  be the partition of the  $n$ th level of the tree into  $G$ -orbits. Consider a collection of non-empty subsets  $\mathbb{W}_i \subset \mathbb{O}_i$ . Then

$$h(\mu) \leq \sum |\mathbb{W}_i| \cdot h(T_{\mathbb{W}_i}(\mu)).$$

**Proof.** Consider first the case that  $\mathbb{W}_i = \mathbb{O}_i$  for every  $i$ . The element  $\mathbf{g}_k$  is completely determined by its action on the  $n$ th level and its sections at vertices of that level, hence by the data of  $(\mathbf{g}_k|_v, v \cdot \mathbf{g}_k)$  for every  $v \in \mathbb{T}_{\bar{m}}^n$ . By Proposition 3.3(2)

$$H(\mathbf{g}_k) \leq \sum_{v \in X^n} H(\mathbf{g}_k|_v, v \cdot \mathbf{g}_k) = \sum_{v \in \mathbb{O}_1} H(\mathbf{g}_k|_v, v \cdot \mathbf{g}_k) + \dots + \sum_{v \in \mathbb{O}_r} H(\mathbf{g}_k|_v, v \cdot \mathbf{g}_k).$$

The latter are random walks with internal degrees of freedom with diagrams  $T_{\mathbb{O}_i}(\mu)$ . Dividing by  $k$  and letting  $k \rightarrow \infty$

$$h(\mu) \leq \sum_{i=1}^r |\mathbb{O}_i| \cdot h(T_{\mathbb{O}_i}(\mu)).$$

For general  $\mathbb{W}_i \subset \mathbb{O}_i$  the theorem follows from relation (4) and the observation that the stationary measure on each orbit is the uniform measure on it. □

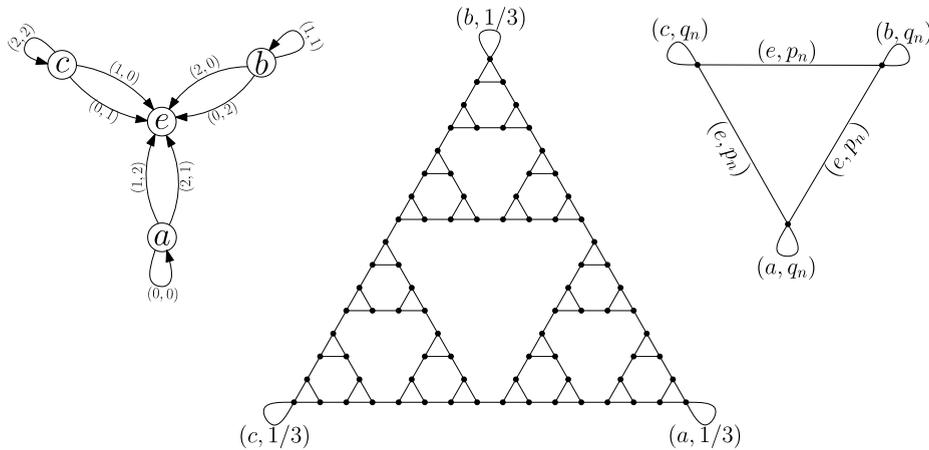


Fig. 1. Left: the Moore diagram for the Hanoi automaton. Centre: The diagram for  $T_{\mathbb{O}_n}(\mu)$  for  $n = 4$ . Here  $a, b, c, e$  denote delta measures at those elements of the group. Every edge except the self-loops at vertices  $\mathbb{W}_n$  is labelled  $(e, 1/3)$ . Right: the diagram for  $T_{\mathbb{W}_n}(\mu)$ .

### 3.4. An illustrative example: The Hanoi Tower group

Before turning to the proof of Theorem 1 in full generality, let us illustrate how the notions from the previous paragraph are used in one particularly simple example – the *Hanoi Tower group*. This group is generated by a 4-state automaton over the 3-elements alphabet, and it is related to the classical Hanoi Tower game on 3 pegs. Its Schreier graphs on the levels of the tree are discrete approximation of the Sierpinski gasket (see for instance [14]).

The Hanoi group is the automaton group  $G < \text{Aut}(\mathbb{T}_3)$  generated by the three automorphisms of finite type  $a, b, c$  defined by the wreath recursions

$$a = (a, e, e)(12), \quad b = (e, b, e)(02), \quad c = (e, e, c)(01).$$

Note that  $a^2 = b^2 = c^2 = e$ . The Moore diagram of the automaton is shown in Figure 1.

One can prove that for every symmetric measure  $\mu$  supported on any generating set of  $G$ , and for every single vertex  $w \in \mathbb{T}_3$ , the ascension operator  $T_w(\mu)$  is infinitely supported. In particular,  $G$  admits no finitely supported *self-similar measure* in the sense of [19]. However the Liouville property can be shown as follows.

Consider the uniform measure  $\mu$  on the standard generators  $\{a, b, c\}$ . The group  $G$  acts transitively on the levels of the tree, so there is a single orbit. For every level  $n$  set  $\mathbb{W}_n = \{0^n, 1^n, 2^n\} \subset \mathbb{T}_3^n$ . The diagram of  $T_{\mathbb{W}_n}(\mu)$  is a triangle with self-loops. The self-similarity of the generators  $a, b, c$  (their sections are either themselves or trivial) yields that  $T_{\mathbb{W}_n}(\mu)$  has the same measures  $\mu_{xy}$  on the edges for every  $n$ . Figure 1 also shows the ascension diagrams  $T_{\mathbb{O}_n}(\mu)$  with respect to the whole orbit  $\mathbb{O}_n = \mathbb{T}_3^n$  and  $T_{\mathbb{W}_n}(\mu)$  with respect to set  $\mathbb{W}_n$ .

The diagrams  $T_{\mathbb{W}_n}(\mu)$  differ only in the transition probabilities  $p_n$  and  $q_n$  (which satisfy  $2p_n + q_n = 1$ ). These can be determined in turn by analysing simple random walk on the Schreier graphs of the group  $G$  acting on the levels of the tree (shown in the left). It is easy to see that these Schreier graphs converge to an infinite recurrent graph (in the local topology, rooted at a vertex of  $\mathbb{W}_n$ ). This implies that  $p_n \rightarrow 0$ . Since the generators are involutions, this roughly tells us that the random walks with internal degrees of freedom  $T_{\mathbb{W}_n}(\mu)$  get “lazier” as  $n$  grows. More precisely, using (3) one can find a sequence of real numbers  $\alpha_n$  decreasing to zero and prove an *a-priori* upper bound  $h(T_{\mathbb{W}_n}(\mu)) \leq \alpha_n$  (we can have  $\alpha_n = Cp_n$ ). Theorem 3.6 then yields

$$h(\mu) \leq 3h(T_{\mathbb{W}_n}(\mu)) \leq 3\alpha_n \rightarrow 0,$$

which implies *a-fortiori* that  $h(\mu) = 0$  (and also  $h(T_{\mathbb{W}_n}(\mu)) = 0$  for every  $n$ ).

A similar argument actually applies to every symmetric and finitely supported measure on the Hanoi group  $G$ , with a different choice of  $\mathbb{W}_n$ . We omit further details, as this is a special case of Theorem 1.

### 3.5. Groups acting on a rooted tree with recurrent Schreier graphs

We say that a finitely generated subgroup  $G$  of  $\text{Aut}(\mathbb{T}_{\bar{m}})$  acts on  $\partial\mathbb{T}_{\bar{m}}$  with recurrent Schreier graphs if every orbital Schreier graph for the action of  $G$  on  $\partial\mathbb{T}_{\bar{m}}$  is recurrent. Since recurrence is stable under rough isometries, this property does not depend on the choice of the finite symmetric generating set of  $G$ , and more generally of a symmetric and finitely supported probability measure  $\mu$  on  $G$ , see [22, Theorem 2.17]. Recurrence of the Schreier graphs is related to groups of automorphisms of bounded type by the following result.

**Proposition 3.7 ([7,18]).** *Let  $G$  be a finitely generated group of automorphisms of bounded type of a spherically homogeneous rooted tree  $\mathbb{T}_{\bar{m}}$ . Then  $G$  acts on  $\partial\mathbb{T}_{\bar{m}}$  with recurrent Schreier graphs.*

This fact was first proved by Bondarenko [7] for groups generated by bounded automata; see [18, Lemma 4.3] for a more general version which includes groups of automorphisms of bounded type of a rooted tree.

Let  $G < \text{Aut}(\mathbb{T}_{\bar{m}})$  be a finitely generated group acting on  $\partial\mathbb{T}_{\bar{m}}$  with recurrent Schreier graphs, and endow it with a symmetric finitely supported probability measure  $\mu$ . Fix a starting ray  $\gamma = (v_0, v_1, \dots) \in \partial\mathbb{T}_{\bar{m}}$ .

If  $(\mathbf{g}_k)_{k \in \mathbb{N}}$  is the random walk on  $G$  driven by  $\mu$ , then  $(v_n \cdot \mathbf{g}_k)$  is a Markov chain on level  $n$  of the tree. These chains are naturally coupled, and the  $n$ th level Markov chain projects to the previous ones. The Markov chain on the boundary of the tree  $(\gamma \cdot \mathbf{g}_k)_{k \in \mathbb{N}}$  projects onto all of these. Consider a family of rays  $\mathbb{U}_\infty \subset \partial\mathbb{T}_{\bar{m}}$  containing  $\gamma$  and denote  $\mathbb{U}_n \subset \mathbb{T}_{\bar{m}}^n$  the set of projections of rays in  $\mathbb{U}_\infty$  to the  $n$ th level of the tree. Let  $\mathbf{T}_\infty$  be the first positive return time of  $(\gamma \cdot \mathbf{g}_k)$  to  $\mathbb{U}_\infty$  and let  $\mathbf{T}_n$  be the first positive return time of  $(v_n \cdot \mathbf{g}_k)$  to  $\mathbb{U}_n$ . Since the chains project onto each other, we have

$$\mathbf{T}_1 \leq \mathbf{T}_2 \leq \mathbf{T}_3 \leq \dots \leq \mathbf{T}_\infty.$$

By recurrence of  $(\gamma \cdot \mathbf{g}_k)$ , the sequence  $(\mathbf{T}_n)$  is bounded, and so is constant for  $n \geq R$  for some random  $R$ . We therefore have  $\mathbb{P}(\mathbf{T}_n \neq \mathbf{T}_m) \leq \mathbb{P}(R > m \wedge n)$  which yields the following proposition.

**Proposition 3.8.** *With the notations above, the return times satisfy  $\mathbb{P}(\mathbf{T}_n \neq \mathbf{T}_m) \rightarrow 0$  as  $n, m \rightarrow \infty$  with  $n, m \in \mathbb{N} \cup \{\infty\}$ .*

## 4. Proof of Theorem 1

Throughout this section we fix a finitely generated group  $G$  acting faithfully on  $\mathbb{T}_{\bar{m}}$  by automorphisms of bounded type, equipped with a finite, symmetric generating set  $S$ . We also let  $K > 0$  be the maximal depth of a generator  $s \in S$  (see Definition 2.4).

### 4.1. Deep level sections in groups of automorphisms of bounded type

The aim of this subsection is to construct generating sets for the groups of level  $n$  sections  $G^{(n)}$  that have a special form adapted to our purpose. For every level  $n$ , denote by  $\mathbb{W}_n \subset \mathbb{T}_{\bar{m}}^n$  the set of vertices  $w \in \mathbb{T}_{\bar{m}}^n$  such that the section  $s|_w$  is non-trivial for some  $s \in S$ . Note that since each generator of  $s$  has finitely many singular rays, and non-trivial sections are all within distance  $K$  of one of these rays, the size of  $\mathbb{W}_n$  is uniformly bounded in  $n$ .

Denote by  $\mathbb{A}_\infty \subset \partial\mathbb{T}_{\bar{m}}$  the finite set of rays of the tree which are singular for some generator  $s \in S$  (see Definition 2.4), and  $\mathbb{A}_n$  its projection to level  $n$  of the tree. The other non-trivial sections are at vertices of  $\mathbb{B}_n = \mathbb{W}_n \setminus \mathbb{A}_n$ .

**Remark 4.1.** *Observe that if  $n$  is large enough, rays in  $\mathbb{A}_\infty$  have distinct projections to level  $n$ . We assume henceforth that  $n$  is large enough for this to hold. For any such  $n$  and every  $v \in \mathbb{A}_n$  we denote by  $\gamma_v \in \partial\mathbb{T}_{\sigma^n \bar{m}}$  the continuation of this ray above  $v$ , i.e. the unique ray of the shifted tree  $\mathbb{T}_{\sigma^n \bar{m}}$  such that  $\gamma_v v \in \mathbb{A}_\infty$ .*

For every  $v \in \mathbb{A}_n$  and every  $s \in S$  the section  $s|_v$  is either finitary or directed along  $\gamma_v$ . Therefore  $(\gamma_v, s|_v)$  belongs to the groupoid of directed automorphisms  $\mathcal{D}_{\sigma^n \bar{m}}$ . Let  $\mathcal{A}_n < \mathcal{D}_{\sigma^n \bar{m}}$  be the subgroupoid generated by  $(\gamma_v, s|_v)$  when  $v$  runs in  $\mathbb{A}_n$  and  $s$  runs in  $S$ . By Lemma 2.7 the groupoid  $\mathcal{A}_n$  is finite, moreover its cardinality is uniformly bounded in

$n$  since the cardinality of  $\mathbb{A}_n$  is bounded (in fact, it is constant and equal to the cardinality of  $\mathbb{A}_\infty$  if  $n$  is large enough) and all elements  $(\gamma_v, s|_v)$  have depth at most  $K$ . Let  $A_n$  be the projection of  $\mathcal{A}_n$  to the group  $\text{Aut}(\mathbb{T}_{\sigma^n \bar{m}})$ , i.e.

$$A_n = \{g : \exists \gamma \text{ such that } (\gamma, g) \in \mathcal{A}_n\}.$$

Consider now the case  $v \in \mathbb{B}_n$ . Then for every  $s \in S$  the section  $s|_v$  is finitary with depth at most  $K$ . All such sections generate a finite group; let us denote it  $B_n$ . This finite group also has uniformly bounded cardinality.

Hence for every generator  $s$  and every  $v \in \mathbb{W}_n$  the section  $s|_v$  belongs to  $A_n \cup B_n$ . It follows that the set  $S_n = A_n \cup B_n$  generates the group of sections  $G^{(n)}$  (see Definition 2.1 and Remark 2.2).

We summarize the discussion above as a proposition:

**Proposition 4.2.** *For every large enough  $n$ , the group of sections  $G^{(n)}$  admits a finite, symmetric generating set  $S_n$  whose cardinality is bounded uniformly in  $n$  and which can be written as a union  $S_n = A_n \cup B_n$ , where:*

- $A_n$  is the projection to  $\text{Aut}(\mathbb{T}_{\sigma^n \bar{m}})$  of a finite subgroupoid  $\mathcal{A}_n$  of the groupoid  $\mathcal{D}_{\sigma^n \bar{m}}$  of directed automorphisms of  $\mathbb{T}_{\sigma^n \bar{m}}$ ;
- $B_n$  is a finite group of finitary automorphisms of  $\mathbb{T}_{\sigma^n \bar{m}}$ .

Moreover, for every  $s \in S$  we have that  $s|_v \in A_n$  for  $v \in \mathbb{A}_n$ ,  $s|_v \in B_n$  for  $v \in \mathbb{B}_n$ , and  $s|_v = e$  otherwise.

#### 4.2. Vanishing of asymptotic entropy

We keep all notations introduced in the previous section: the vertex sets  $\mathbb{W}_n, \mathbb{A}_n, \mathbb{B}_n$ , the set of rays  $\mathbb{A}_\infty$ , and the generating sets  $S_n = A_n \cup B_n$  for the groups of sections. We consider a symmetric, finitely supported probability measure  $\mu$  on  $G$  with support  $S$ .

Fix a level  $n$  large enough so that Remark 4.1 applies. Take any orbit  $\mathbb{O} \subset \mathbb{T}_m^n$  for the action of  $G$  and consider at first the ascension diagram  $T_{\mathbb{O}}(\mu)$  with respect to the orbit. Let  $(\mu_{vw})_{v,w \in \mathbb{O}}$  be the edge measures of this ascension diagram. Proposition 4.2, with the definition of the edge measures (5) and their symmetry property  $\mu_{vw} = \hat{\mu}_{vw}$  imply the following facts, which we summarize for later reference.

#### Claim 4.3.

1. If  $v, w \in \mathbb{A}_n$ , then the measure  $\mu_{vw}$  is supported in  $A_n$ . Moreover  $(\gamma_v, h)$  belongs to the groupoid  $\mathcal{A}_n$  for any  $h \in \text{supp}(\mu_{vw})$ .
2. If  $v, w \in \mathbb{B}_n$ , then the measure  $\mu_{vw}$  is supported in the finite group  $B_n$ .
3. Otherwise,  $\mu_{vw}$  is concentrated on the identity.

As a first consequence, observe that whenever the orbit  $\mathbb{O}$  does not intersect  $\mathbb{W}_n = \mathbb{A}_n \sqcup \mathbb{B}_n$  one has immediately  $h(T_{\mathbb{O}}\mu) = 0$ , since all edge measures of  $T_{\mathbb{O}}$  are trivial and the corresponding random walk with internal degrees of freedom is just a finite Markov chain.

Suppose now that there are  $r = r(n)$  orbits in level  $n$  that have non-trivial intersection with  $\mathbb{W}_n$ , and denote them  $\mathbb{O}_{1,n}, \dots, \mathbb{O}_{r,n}$ . Set  $\mathbb{W}_{i,n} = \mathbb{W}_n \cap \mathbb{O}_{i,n}$ , and consider the ascension diagram  $T_{\mathbb{W}_{i,n}}(\mu)$ . Note that the edge measure of this diagram also satisfy Claim 4.3 (part 3 is vacuous here). Theorem 3.6 and the above observation that  $h(T_{\mathbb{O}}(\mu)) = 0$  whenever  $\mathbb{O} \cap \mathbb{W}_n = \emptyset$  give

$$h(\mu) \leq \sum_{i=1}^r |\mathbb{W}_{i,n}| \cdot h(T_{\mathbb{W}_{i,n}}(\mu)). \quad (6)$$

To prove that  $h(\mu) = 0$  we estimate the asymptotic speed of the diagrams  $T_{\mathbb{W}_{i,n}}(\mu)$ :

**Proposition 4.4.** *With the above notations, there exists a sequence  $a_n \rightarrow 0$  so that the speed of the diagrams  $T_{\mathbb{W}_{i,n}}(\mu)$  with respect to the generating set  $S_n$  satisfies*

$$\ell_{S_n}(T_{\mathbb{W}_{i,n}}(\mu)) \leq a_n,$$

for every  $i = 1, \dots, r(n)$ .

Let us explain how this concludes the proof of Theorem 1. Since the generating sets  $S_n$  have bounded cardinalities, we deduce from (3) that  $h(T_{\mathbb{W}_{i,n}}(\mu)) \leq C a_n$  for a constant  $C$  that does not depend on  $n$ . Since the cardinalities of  $\mathbb{W}_{i,n}$  and  $r(n)$  are uniformly bounded, (6) implies that there exists  $C' > 0$  so that  $h(\mu) \leq C' a_n \rightarrow 0$ . Hence  $h(\mu) = 0$ . As noted, this is equivalent to the Liouville property for  $(G, \mu)$  (see [11,20]). The rest of this section contains the proof of this speed estimate.

**Proof of Proposition 4.4.** Let  $(\mathbf{g}_k, \mathbf{v}_k)$  be a random walk with internal degrees of freedom with diagram  $T_{\mathbb{W}_{i,n}}(\mu)$  starting from  $(e, v)$ , where  $v \in \mathbb{W}_{i,n}$  is arbitrary. Let  $|\cdot|$  be the word metric on  $G^{(n)}$  with respect to the generating set  $S_n = A_n \cup B_n$ . We shall prove that there exists a sequence  $a_n \rightarrow 0$  such that for every  $k \geq 0$  we have

$$\mathbb{E}|\mathbf{g}_k| \leq 1 + a_n k,$$

uniformly in  $i$  and the starting point  $v \in \mathbb{W}_{i,n}$ . To simplify the notations we will henceforth omit the index  $i = 1, \dots, r(n)$ , writing  $\mathbb{W}_n, \mathbb{A}_n, \mathbb{B}_n$  for  $\mathbb{W}_{i,n}, \mathbb{A}_{i,n}, \mathbb{B}_{i,n}$ .

Let  $\mathbf{h}_1, \dots, \mathbf{h}_k$  be the increments  $\mathbf{h}_j = \mathbf{g}_{j-1}^{-1} \mathbf{g}_j$ . Recall that, conditionally to the positions of  $\mathbf{v}_j, \mathbf{v}_{j+1}$  the distribution of the increment  $\mathbf{h}_{j+1}$  is given by the edge measure  $\mu_{\mathbf{v}_j, \mathbf{v}_{j+1}}$  of the diagram  $T_{\mathbb{W}_n}(\mu)$ , which satisfies Claim 4.3. We consider two types of “bad events” that may happen at some times  $j \geq 1$ .

**Traverse:** One of  $\mathbf{v}_{j-1}, \mathbf{v}_j$  belongs to  $\mathbb{A}_n$  and the other to  $\mathbb{B}_n$ .

**Bad alignment:** Both  $\mathbf{v}_{j-1}, \mathbf{v}_j$  belong to  $\mathbb{A}_n$  and  $\gamma_{\mathbf{v}_{j-1}} \cdot \mathbf{h}_j \neq \gamma_{\mathbf{v}_j}$ . (Recall the notation  $\gamma_v$  from Remark 4.1.)

Let  $N_k$  be the total number of bad events of either type up to time  $k$ . We divide the proof of Proposition 4.4 into three steps, given by Lemmas 4.5–4.7 below, stating that the word length is bounded by the number of bad events, and that the probability of the two types of bad events at each step is small.

Note that the probability of a bad event happening at the  $j$ th step conditionally to  $\mathbf{v}_j = v$  only depends on  $v$  and on the diagram  $T_{\mathbb{W}_n}(\mu)$ . We prove that there exists a sequence  $a_n \rightarrow 0$  so that this conditional probability is bounded above by  $a_n$ , uniformly in  $v \in \mathbb{W}_n$ . This implies  $\mathbb{E}N_k \leq c_n k$ , concluding the proof of Proposition 4.4 by Lemma 4.5. For both types of bad events, the proof of this bound is based on Proposition 3.8, which applies since  $G$  acts on  $\partial \mathbb{T}_{\bar{m}}$  with recurrent Schreier graphs (see Proposition 3.7).

**Lemma 4.5.** *The word metric  $|\mathbf{g}_k|$  is bounded above by  $1 + N_k$ .*

**Proof.** Let  $s < t$  be such that no bad event happens for  $j \in [s, t]$ . Then either  $v_j \in \mathbb{A}_n$  for all  $j \in [s, t]$ , or else  $v_j \in \mathbb{B}_n$  for all  $j \in [s, t]$ . In the second case, the increment  $\mathbf{h}_j$  belongs to the finite group  $B_n$ , hence their product has length 1 with respect to the generating set  $S_n = A_n \cup B_n$ . In the first case, since there is no bad alignment, we have  $\gamma_{\mathbf{v}_{j-1}} \cdot \mathbf{h}_j = \gamma_{\mathbf{v}_j}$  for every  $j \in [s, t]$ . Assume that the second case holds. By Claim 4.3, for every  $j$  the couple  $(\gamma_{\mathbf{v}_{j-1}}, \mathbf{h}_j)$  belongs to the finite groupoid  $\mathcal{A}_n$ . The condition that  $\gamma_{\mathbf{v}_{j-1}} \cdot \mathbf{h}_j = \gamma_{\mathbf{v}_j}$  guarantees that the product of two consecutive such couples is defined in the groupoid, hence belongs to  $\mathcal{A}_n$ . It follows that  $(\gamma_{\mathbf{v}_{s-1}}, \mathbf{h}_s) \cdots (\gamma_{\mathbf{v}_{t-1}}, \mathbf{h}_t) = (\gamma_{\mathbf{v}_t}, \mathbf{h}_s \cdots \mathbf{h}_t) \in \mathcal{A}_n$  and thus  $\mathbf{h}_{j_1+1} \cdots \mathbf{h}_{j_2} \in A_n$  has length 1.

We conclude that the word length of  $\mathbf{g}_k = \mathbf{h}_1 \cdots \mathbf{h}_k$  is bounded by one more than the total number of bad events.  $\square$

Recall that  $(\mu_{vw})$  and  $(p_{vw})$  denote the edge measures and the marginal transition probabilities of the ascension diagram  $T_{\mathbb{W}_n}(\mu) = (M, P)$ .

**Lemma 4.6.** *There exists a sequence  $a'_n \rightarrow 0$  so that*

$$P(\mathbb{B}_n, \mathbb{A}_n) = P(\mathbb{A}_n, \mathbb{B}_n) := \sum_{v \in \mathbb{A}_n, w \in \mathbb{B}_n} p_{vw} \leq a'_n.$$

*In particular the probability that a traverse happens at any time is bounded above by  $a'_n$ .*

**Proof.** First, observe that the matrix  $P$  is symmetric, as it is the trace of a symmetric Markov chain on a recurrent subset. Hence  $P(\mathbb{B}_n, \mathbb{A}_n) = P(\mathbb{A}_n, \mathbb{B}_n)$ .

We now argue this quantity is small. Fix some  $\gamma \in \mathbb{A}_\infty$ , and let  $v \in \mathbb{A}_n$  be the unique vertex that belongs to  $\gamma$ . We shall prove that  $P(v, \mathbb{B}_n) := \sum_{w \in \mathbb{B}_n} p_{vw}$  tends to zero as  $n \rightarrow \infty$ . This is sufficient, since there are finitely many choices for  $\gamma$ .

Let  $(\tilde{\mathbf{g}}_j)$  the random walk on the group  $G$  with step measure  $\mu$ . Recall from the definition of the ascension diagram (see Section 3.3) that  $P(v, \mathbb{B}_n)$  is the probability that  $v \cdot \tilde{\mathbf{g}}_{\mathbf{t}} \in \mathbb{B}_n$ , where  $\mathbf{t}$  is the first return time of  $v \cdot \tilde{\mathbf{g}}_j$  to  $\mathbb{W}_n$ .

Consider simultaneously level  $n - K$ , and recall that (from the definition of  $K$ ) the projection of  $\mathbb{W}_n$  to this level is contained in  $\mathbb{A}_{n-K}$ . Let  $w \in \mathbb{A}_{n-K}$  be the projection of  $v$ . Let  $\mathbf{T}_{n-K}$  be the first return time of  $w \cdot \tilde{\mathbf{g}}_j$  to  $\mathbb{A}_{n-K}$  and  $\mathbf{T}_n$  be the first return time of  $v \cdot \tilde{\mathbf{g}}_j$  to  $\mathbb{A}_n$ . We now apply Proposition 3.8 to the family of rays  $\mathbb{A}_\infty$  with starting point  $\gamma$ . Since  $w \cdot \tilde{\mathbf{g}}_{\mathbf{t}}$  is the projection of  $v \cdot \tilde{\mathbf{g}}_{\mathbf{t}}$  and the latter is in  $\mathbb{W}_n$  it follows that  $w \cdot \tilde{\mathbf{g}}_{\mathbf{t}} \in \mathbb{A}_{n-K}$ , and in particular  $\mathbf{T}_{n-k} \leq \mathbf{t}$ .

Suppose  $v \cdot \tilde{\mathbf{g}}_{\mathbf{t}} \in \mathbb{B}_n$ , then  $(v \cdot \tilde{\mathbf{g}}_j)$  returns to  $\mathbb{A}_n$  strictly after time  $\mathbf{t}$ . Hence  $\mathbf{T}_{n-K} \leq \mathbf{t} < \mathbf{T}_n$ . By Proposition 3.8 the probability of this event tends to 0 as  $n \rightarrow \infty$ , concluding the proof.  $\square$

**Lemma 4.7.** *There exists a sequence  $a''_n \rightarrow 0$ , so that for every  $v \in \mathbb{A}_n$  we have  $q_v \leq a''_n$ , where*

$$q_v := \sum_{w \in \mathbb{A}_n} p_{vw} \tilde{\mu}_{vw} \{h : \gamma_v \cdot h \neq \gamma_w\}$$

is the probability that a bad alignment event happens at the  $j$ th step conditioned on  $\mathbf{v}_{j-1} = v$ .

**Proof.** This proof too is based on Proposition 3.8. Fix again  $\gamma \in \mathbb{A}_\infty$  and let  $v \in \mathbb{A}_n$  be the unique vertex belonging to  $\gamma$ . We assume that  $n$  is large enough so that Remark 4.1 applies. As before, it is enough to prove that  $q_v$  tends to zero as  $n \rightarrow \infty$  and  $v$  belongs to  $\gamma$ . Let  $(\tilde{\mathbf{g}}_j)$  be the random walk on  $G$  with step measure  $\mu$  and let  $\mathbf{t}$  be the first return time of  $v \cdot \tilde{\mathbf{g}}_j$  to  $\mathbb{W}_n$ . Let  $\mathbf{T}_n$  (resp.  $\mathbf{T}_\infty$ ) be the return time of  $v \cdot \tilde{\mathbf{g}}_j$  (resp.  $\gamma \cdot \tilde{\mathbf{g}}_j$ ) to  $\mathbb{A}_n$  (resp.  $\mathbb{A}_\infty$ ). Setting  $\mathbf{w} := v \cdot \tilde{\mathbf{g}}_{\mathbf{t}}$  the probability  $q_v$  equals the probability that  $\mathbf{w} \in \mathbb{A}_n$  and  $\gamma_v \cdot \tilde{\mathbf{g}}_{\mathbf{t}|v} \neq \gamma_{\mathbf{w}}$ . If this event happens we have  $\mathbf{T}_n = \mathbf{t}$ , while  $\mathbf{T}_\infty > \mathbf{t}$ . Indeed,  $\gamma \cdot \tilde{\mathbf{g}}_{\mathbf{t}} = (\gamma_v \cdot \tilde{\mathbf{g}}_{\mathbf{t}|v})\mathbf{w} \neq \gamma_{\mathbf{w}}\mathbf{w}$  and hence  $\gamma \cdot \tilde{\mathbf{g}}_{\mathbf{t}} \notin \mathbb{A}_\infty$  (since  $\gamma \cdot \tilde{\mathbf{g}}_{\mathbf{t}}$  contains  $\mathbf{w}$ , but the unique ray in  $\mathbb{A}_\infty$  that contains  $\mathbf{w}$  is  $\gamma_{\mathbf{w}}\mathbf{w}$ ). This implies that  $\mathbf{T}_n < \mathbf{T}_\infty$ . The probability of this event tends to zero as  $n \rightarrow \infty$  by Proposition 3.8.  $\square$

Setting  $a_n = a'_n + a''_n$  we have  $\mathbb{E}N_k \leq a_n k$ . This concludes the proof of Proposition 4.4, and thus the proof of Theorem 1 as noted.  $\square$

## 5. Proof of Theorem 2

To make the proof of Theorem 1 quantitative, the key idea is to let the level  $n$  tend to infinity together with the time  $k$ , at a carefully chosen rate. One needs an estimate on the rate of convergence to 0 of the probabilities from Lemmas 4.6 and 4.7. Such estimates can be obtained from a closer analysis of the Schreier graphs of the action of  $G$  on the finite levels of the tree using electric network theory.

This section is organized as follows: In Section 5.1 we define principal groups of directed automorphisms, and in Section 5.2 we study sections in such groups. Some simplifications occur in this setting, in particular the groupoid  $\mathcal{A}_n$  can be chosen to be a group, and the bad alignment events cannot occur. Then in Section 5.3 we calculate lower bounds on the resistance in the relevant Schreier graphs, and finally in Section 5.4 we combine all ingredients to prove Theorem 2.

### 5.1. Principal groups of directed automorphisms and the mother group

Let  $\bar{m}$  be a bounded sequence as before, and set  $m_* = \max_i m_i$ . The 0-ray in  $\mathbb{T}_{\bar{m}}$ , consists of all vertices of the form  $0^n = 0 \cdots 0$ . The neighbours of the zero ray in the tree are vertices of the form  $v = x0^n$ , where  $x \in X_{m_i}$  is the only non-zero letter in  $v$ . Let

$H_{\bar{m}} < \text{Aut}(\mathbb{T}_{\bar{m}})$  be the subgroup consisting of elements that are directed along the zero ray, fix the zero ray, and have depth at most one (recall that this means that their sections can be non-trivial only on the zero ray or its neighbours). Equivalently, an element  $h \in H_{\bar{m}}$  has a wreath recursion of the form

$$h = (h', \tau_1, \dots, \tau_{m_1-1})\rho,$$

where  $\tau_1, \dots, \tau_{m_1-1} \in S_{m_2}$  are the sections at first-level vertices other than 0, the permutation  $\rho \in S_{m_1}$  is such that  $0 \cdot \rho = 0$  and  $h' \in H_{\sigma \bar{m}}$ . The group  $H_{\bar{m}}$  is locally finite.

We identify the symmetric group  $S_{m_1}$  with the subgroup of  $\text{Aut}(\mathbb{T}_{\bar{m}})$  consisting of automorphisms that permute vertices on the first level and have trivial sections on them. In the same way,  $S_{m_n}$  identifies with a subgroup of  $\text{Aut}(\mathbb{T}_{\sigma^n \bar{m}})$  for every  $n \geq 0$ .

The following groups were defined and studied by Brioussell (see [8–10]). They are a generalization of the mother group from [5].

**Definition 5.1.** Let  $A < H_{\bar{m}}$ ,  $B < S_{m_1}$  be finite subgroups. The principal group of directed automorphisms generated by  $A$  and  $B$  is the group  $\langle A \cup B \rangle < \text{Aut}(\mathbb{T}_{\bar{m}})$ . We denote it by  $M(A, B)$ .

Note that the term “principal group of directed automorphisms” should be taken as whole, in fact the group  $M(A, B)$  is generated by directed automorphisms but also contains automorphisms that are not directed.

Many groups acting on  $\mathbb{T}_m$  embed in a group of the form  $M(A, B)$ , see Theorem 5.2 below and also [8, Section 9] for a slight generalization. We shall omit  $A, B$  from the notation when there is no ambiguity, and write simply  $M$  for  $M(A, B)$ .

There is an important particular case of Definition 5.1. Take  $\bar{m}$  a constant sequence, and set  $B = S_m$ . For  $A$  we take all elements  $h \in H_m$  for which the section at 0 is  $h$  itself:  $h|_0 = h$ . Equivalently,  $A$  is the group of automorphisms that admit a wreath recursion of the form

$$h = (h, \tau_1, \dots, \tau_{m-1})\rho,$$

where  $\tau_1, \dots, \tau_{m-1} \in S_m$  and  $\rho \in S_m$  is such that  $0 \cdot \rho = 0$ . It is easy to see that  $h$  is determined by  $\rho$  and the  $\tau_i$ , and that  $A$  is a finite group, isomorphic to  $S_m \wr_{X \setminus \{0\}} S_{m-1}$ . With these choices of  $A$  and  $B$ , the group  $M = M(A, B)$  is generated by a bounded automaton, and is called the *mother group of bounded activity* over the  $m$ -element alphabet.

The mother group was first defined in [5] in the bounded activity case. An analogous generalization to higher activity degrees was provided in [1]. Its significance relies on the fact that every polynomial activity automaton group embeds in a mother group of the same activity degree, possibly acting on a bigger alphabet. We only use this result in the bounded activity case:

**Theorem 5.2 ([1,5]).** Let  $G < \text{Aut}(\mathbb{T}_m)$  be a group generated by a bounded activity automaton. Then there exists  $m'$  such that  $G$  embeds isomorphically in the mother group of bounded activity over  $m'$  elements.

Henceforth, we shall fix a sequence  $\bar{m} = (m_n)_n$  of natural numbers bounded by  $m_* = \max_n m_n$ , as well as two finite groups  $A < H_{\bar{m}}$  and  $B \subset S_{m_1}$  generating a principal group of directed automorphisms  $M = M(A, B)$ . We also fix a subgroup  $G < M$ , generated by a finite symmetric set  $S \subset M$ .

Furthermore, it will be useful to suppose that  $A$  contains the following elements. Let  $\bar{\sigma} = (\sigma_1, \dots, \sigma_{m_*})$  with  $\sigma_i \in S_i$  be a collection of permutations in the symmetric groups up to  $m_*$  elements. Define  $h_{\bar{\sigma}} \in \text{Aut}(\mathbb{T}_{\bar{m}})$  to act on words as follows. If the first non-zero letter of word  $w$  is at position  $i$ , then  $w \cdot h_{\bar{\sigma}}$  is equal to  $w$  except for the  $i + 1$ st letter which is permuted by  $\sigma_{m_{i+1}}$ . It is easy to see that elements of the form  $h_{\bar{\sigma}}$  are in  $H_{\bar{m}}$  and form a finite group. We shall suppose that  $A$  contains this finite group. Adding any finite set of elements to  $A$  does not cause any loss of generality, since the group  $H_{\bar{m}}$  is locally finite.

## 5.2. Sections in the principal groups of directed automorphisms

We now describe the sections of the generators  $s \in S$  of the group  $G$ . We will use notations analogous to those in Section 4.1.

**Definition 5.3.**

- Let  $A_n = \langle a|_{0 \dots 0} \rangle_{a \in A}$  the finite subgroup of  $H_{\sigma^n \bar{m}}$  consisting of sections of elements of  $A$  at  $n$ th level along the zero ray.

- Let  $B_n = \langle a|_{x0\dots 0} \rangle_{a \in A, x \in X_{m_n} \setminus \{0\}} = \langle a'|_x \rangle_{a' \in A_{n-1}, x \in X_{m_n} \setminus \{0\}}$  the subgroup of  $S_{m_n}$  generated by the  $n$ th level sections of  $A$  at neighbors of the zero ray.

Note that  $A_n \cup B_n$  generate the group of  $n$  level sections  $M^{(n)}$ . In particular,  $M^{(n)} = M(A_n, B_n)$  is a principal group of directed automorphisms of  $\mathbb{T}_{\sigma^n \bar{m}}$ .

As in Section 4.1 we denote by  $\mathbb{W}_n \subset \mathbb{T}_{\bar{m}}^n$  the set of  $n$ th level vertices  $w \in \mathbb{T}_{\bar{m}}^n$  such that the section  $s|_w$  is non-trivial for some generator  $s \in S$ . We also keep the same definitions of the set of singular rays  $\mathbb{A}_\infty$  and the sets  $\mathbb{A}_n, \mathbb{B}_n$ . The following lemma is a more explicit version of Proposition 4.2 in this setting.

**Lemma 5.4.** *The set  $\mathbb{A}_\infty$  consists of rays ending with an infinite sequence of zeros. In particular there is an  $n_0$  and set  $\{w_1, \dots, w_k\} \subset \mathbb{T}_{\bar{m}}^n$  independent of  $n$ , so that for  $n > n_0$ , the sets  $\mathbb{A}_n$  and  $\mathbb{B}_n$  have the form*

- $\mathbb{A}_n = \{00\dots 0w_j\}$ ,
- $\mathbb{B}_n = \{x00\dots 0w_j : x \in X_{m_n} \setminus \{0\}\}$ .

Moreover for every generator  $s$ , we have  $s|_w \in A_n$  (resp.  $s|_w \in B_n$ ) if  $w \in \mathbb{A}_n$  (resp.  $w \in \mathbb{B}_n$ ) and  $s|_w = e$  otherwise.

**Proof.** We first show that for every  $h \in M$  there exist a  $n_h$  such that for  $n \geq n_h$  all of its  $n$ th level sections are in the generating set  $A_n \cup B_n$ . From the definition of  $A_n$  and  $B_n$ , it suffices to prove this for  $n = n_h$ . We do this by induction on the word metric  $|h|$  associated to the generating set  $A \cup B$ . For  $h \in A \cup B$  the claim holds with  $n_h = 0$ .

First of all, observe that if  $h$  is a product of two generators then its first level sections are in  $A_1 \cup B_1$ , so that one can take  $n_h = 1$ . Indeed, if  $h = s_1 s_2$ , with  $s_1, s_2 \in A$  then  $h \in A$  and its first level sections are in  $A_1 \cup B_1$  by definition. If  $s_1$  is in  $B$  then its sections are trivial, and from (1) we see that first level sections of  $h$  are those of  $s_2$  possibly in a different order, and are in  $A_1 \cup B_1$ . Similarly, this is the case if  $s_2 \in B$ .

Generally, suppose that the conclusion holds for  $h$ , and consider  $g = hs$ . Then sections of  $g$  at level  $n_h + 1$  are first level sections of products from  $A_{n_h} \cup B_{n_h}$ . The case of a product of two generators applies, and the sections are in  $A_{n_h+1} \cup B_{n_h+1}$ , so  $n_g = n_h + 1$  will do.

We deduce that for every large enough level  $n$ , the sections of every generator  $s \in S$  are in  $A_n \cup B_n$ . To conclude observe that elements of  $B_n$  are finitary and elements of  $A_n$  are directed along the zero ray. It follows that the singular set  $\mathbb{A}_\infty$  of generators consists of rays ending with an infinite sequence of zeros, and that  $\mathbb{A}_n$  and  $\mathbb{B}_n$  have the claimed form. □

### 5.3. Resistances in Schreier graphs

In this subsection we analyze effective resistances in the Schreier graphs of the group  $G$  acting on the levels of the tree, with respect to the fixed generating set  $S$ . See [22, Chapter 2] for a general background on electric network theory.

It is convenient to first consider the Schreier graph  $\Lambda_n$  for the whole group  $M$  acting on the  $n$ th level  $\mathbb{T}_{\bar{m}}^n$ , equipped with the standard generating set  $A \cup B$ . Call the vertex  $0^n \in \mathbb{T}_{\bar{m}}^n$  the root. Vertices of the form  $x0^{n-1} \in \mathbb{T}_{\bar{m}}^n$  with  $x \neq 0$  are called the anti-roots. The following proposition determines a lower bound for the asymptotics of resistance in  $\Lambda_n$  between the root and any anti-root, as  $n \rightarrow \infty$ . See also [2,3] for more on resistances in these graphs.

**Lemma 5.5.** *There exists a constant  $c$ , not depending on  $n$ , such that for every  $x \neq 0$  we have the resistance bound*

$$\text{Res}_{\Lambda_n}(0^n, x0^{n-1}) \geq c \prod_{i=1}^n \frac{m_i}{m_i - 1} \geq c \left( \frac{m_*}{m_* - 1} \right)^n.$$

**Proof.** A word in  $\mathbb{T}_{\bar{m}}^n$  can be mapped to a word in  $\{0, *\}^n$ , by substituting every non-zero letter with the symbol  $*$ . The set of anti-roots is exactly the pre-image of  $*0^{n-1}$ . The graph  $\Lambda_n$  projects to a graph  $\hat{\Lambda}_n$  with vertex set  $\{0, *\}^n$  and multiple edges. By Rayleigh monotonicity, resistances in the projected graph are no larger than resistances in the original graph.

The key observation is that  $\hat{\Lambda}_n$  is just a path with multiple edges, and some self loops. The root  $0^n$  and anti-root  $*0^{n-1}$  are the ends of the path. To see this, observe by looking to the action of the generators that there are only two

kind of non-trivial moves on elements of  $\{0, *\}^n$ : changing the rightmost (first) letter (generators in  $B$ ) or changing the letter after the first appearance of  $*$  from the right (generators in  $A$ ). Moves of the second kind give loops for the root  $0^n$  and the anti-root  $*0^{n-1}$ , so these vertices have only one other neighbour. A connected multi-graph having all vertices of degree two, except two vertices of degree 1, is a path with multiple edges and loops.

To get a bound on the resistances in  $\Lambda_n$  we need to find the edge multiplicities in  $\hat{\Lambda}_n$ . The degree of vertices in  $\Lambda_n$  is bounded by some  $C$ , so the degree of a vertex  $x \in \{0, *\}^n$  is at most  $C$  times the number of vertices in  $\mathbb{T}_m^n$  that project to  $x$ , i.e. it is bounded above by  $C \prod_{i|x_i=*} (m_i - 1)$ . Hence the total resistance is bounded below by

$$\text{Res}_{\hat{\Lambda}_n}(0^n, *0^{n-1}) \geq \sum_{\{0,*\}^n} C^{-1} \prod_{i|x_i=*} \frac{1}{m_i - 1} = C^{-1} \prod_{i=1}^n \frac{m_i}{m_i - 1}. \quad \square$$

The Rayleigh monotonicity principle and rough invariance of resistances under quasi-isometries [22, Chapter 2] allow us to deduce a similar consequence for the group  $G$  equipped with any symmetric generating set  $S$ . Fix a level  $\mathbb{T}_m^n$  deep enough, and recall the definition of the vertex sets  $\mathbb{A}_n$  and  $\mathbb{B}_n$  from Lemma 5.4.

Recall that the resistance between two vertex sets  $\mathbb{A}, \mathbb{B}$  in a graph is defined as the resistance from  $\bar{a}$  to  $\bar{b}$  in the graph where  $\mathbb{A}$  and  $\mathbb{B}$  have been collapsed to points  $\bar{a}, \bar{b}$  (this definition makes sense also for disconnected graphs).

**Lemma 5.6.** *Consider a fixed generating set  $S$  of  $G$ , and let  $\Gamma_n$  be the (possibly disconnected) corresponding Schreier graph of the action of  $G$  on  $\mathbb{T}_m^n$ . There exists a constant  $c$  depending only on the generating set  $S$  such that for any large enough  $n$*

$$\text{Res}_{\Gamma_n}(\mathbb{A}_n, \mathbb{B}_n) \geq c \prod_{i=1}^n \frac{m_i}{m_i - 1} \geq c \left( \frac{m_*}{m_* - 1} \right)^n.$$

Moreover, the same holds if each  $s \in S$  has some conductance which applies to the corresponding edges in  $\Gamma_n$ .

**Proof.** Since  $\mathbb{A}_n, \mathbb{B}_n$  have uniformly bounded cardinalities, it is sufficient to prove that for any  $v \in \mathbb{A}_n$  and  $w \in \mathbb{B}_n$ , the effective resistance between  $v$  and  $w$  in  $\Gamma_n$  satisfies the same bound (with a possibly larger constant). Consider the larger generating set  $\tilde{S} = S \cup A \cup B$  of  $M$ , and let  $\tilde{\Gamma}_n$  be the corresponding Schreier graph of the action of  $M$  on  $\mathbb{T}_m^n$ . Then  $\Gamma_n$  and  $\Lambda_n$  are both subgraph of  $\tilde{\Gamma}_n$ , and by Rayleigh monotonicity

$$\text{Res}_{\Gamma_n}(v, w) \geq \text{Res}_{\tilde{\Gamma}_n}(v, w).$$

Next, note that the graph  $\tilde{\Gamma}_n$  is roughly equivalent [22, p. 51] to the standard Schreier graph  $\Lambda_n$  with constants not depending on  $n$ . Thus effective resistances of the graphs  $\tilde{\Gamma}_n$  and  $\Lambda_n$  are equivalent up to multiplicative constants. With Lemma 5.5, this implies that for every  $x \neq 0$

$$\text{Res}_{\tilde{\Gamma}_n}(0^n, x0^{n-1}) \geq c' \prod_{i=1}^n \frac{m_i}{m_i - 1}$$

for some constant  $c'$ .

Finally, we shall show that there is some constant  $K$ , so that for  $v \in \mathbb{A}_n$  and  $w \in \mathbb{B}_n$ , the distance in  $\tilde{\Gamma}_n$  from  $0^n$  to  $v$  (and similarly from  $x0^{n-1}$  to  $w$ ) are at most  $K$ . Since resistance is bounded by distance and by the triangle inequality for resistances we get

$$\text{Res}_{\Gamma_n}(v, w) \geq c' \prod_{i=1}^n \frac{m_i}{m_i - 1} - 2K.$$

Taking  $n$  large enough, this completes the proof. Since  $\Lambda_n$  is a subgraph of  $\tilde{\Gamma}_n$  it suffices to bound distances in  $\Lambda_n$ .

By Lemma 5.4 we have  $v = 0 \cdots 0w_i$  and  $w = x0 \cdots 0w_j$  where  $w_i$  and  $w_j$  are words in some fixed and finite set. Note that there is an element  $g$  of length at most  $2^l - 1$  such that  $0^l \cdot g = w_i$ , and  $g|_{0^l} = e$ , this is easy to see by

induction on  $l$  using only the generators of  $B$  and  $h_{\bar{v}}$  (which we assumed to be in  $A$ ). It follows that the distance from  $0^n$  to  $v$  is at most  $2^l - 1$  for any  $n$ , and similarly for  $w$  and  $x0^{n-1}$ .

If the elements of  $S$  have associated conductances, then these are bounded by some constant. By monotonicity, the resistance is decreased by at most that constant.  $\square$

### 5.4. Entropy estimate

We keep notations from the previous section:  $\mu$  is supported on the set  $S$ , and we denote by  $\Gamma_n$  the Schreier graph of the action of  $G = \langle S \rangle$  on the level set  $\mathbb{T}_m^n$  with generating set  $S$ . If the action is not transitive  $\Gamma_n$  is not connected, but this is irrelevant in what follows. Let  $V_n = m_1 \cdots m_n$  be the volume of the level sets of the tree. With  $\mathbb{A}_n$  and  $\mathbb{B}_n$  as above (Lemma 5.4), we shall consider two resistances:

$$R_n = \text{Res}_{\Gamma_n}(\mathbb{A}_n, \mathbb{B}_n), \quad R_n^\mu = \text{Res}_{(\Gamma_n, \mu)}(\mathbb{A}_n, \mathbb{B}_n).$$

Here  $R_n$  is computed in the graph with all edge weights equal to 1, and  $R_n^\mu$  is the resistance with edge weights given by  $\mu$ . Since  $\mu(g) \leq 1$  for any  $g$  we have  $R_n^\mu \geq R_n$ .

We will use the following slight generalization of the classical formula for random walk hitting probabilities (in the case when  $A = \{a\}$  is a singleton). We have not located a reference for this, but this is a reformulation of Exercise 2.45 in [22].

**Lemma 5.7.** *Let  $\Gamma$  be a finite weighted graph (possibly disconnected) where each edge  $e$  has weight  $w_e$ . Set  $Q = \sum_e w_e$ , the total weight of the graph. Consider the random walk  $(X_n)_{n \geq 0}$  started at its stationary measure  $\nu(x) = Q^{-1} \sum_{e \ni x} w_e$ . For a vertex set  $W \subset \Gamma$  denote the hitting time by  $T_W = \min\{n \geq 1 : X_n \in W\}$ . Then for disjoint vertex sets  $A, B$*

$$\mathbb{P}(X_0 \in A, T_B < T_A) = \frac{1}{2Q \text{Res}(A, B)}.$$

The following proposition is a quantitative version of our previous proof.

**Proposition 5.8.** *With the above notations, there exists a constant  $C$  depending only on  $\bar{m}$ , on the ambient group  $M(A, B)$  and on  $\text{supp}(\mu)$  such that for every  $n$  and  $k$  we have*

$$H(\mu^{*k}) \leq C \left( V_n + \frac{k}{R_n} \right).$$

Before proving Proposition 5.8, let us see how this implies Theorem 2 by an appropriate choice of the level  $n$ . Let  $n = n(k)$  be the smallest integer such that  $k \leq V_n R_n$ . Proposition 5.8 applied for this choice of  $n$  gives

$$H(\mu^{*k}) \leq 2C V_n = 2C \prod_{i=1}^n m_i.$$

Recall that  $\alpha = \log m_* / \log \frac{m_*^2}{m_* - 1}$  is as in the theorem and note that for  $m \leq m_*$  we have  $m \leq (\frac{m^2}{m-1})^\alpha$ . Recall also from Lemma 5.6 that  $R_n \geq c \prod_{i=1}^n (\frac{m_i}{m_i - 1})$ . Using these inequalities, we get

$$\begin{aligned} H(\mu^{*k}) &\leq 2C m_* \prod_{i=1}^{n-1} m_i \leq 2C m_* \left( \prod_{i=1}^{n-1} \frac{m_i^2}{m_i - 1} \right)^\alpha \\ &= 2C m_* \left( V_{n-1} \prod_{i=1}^{n-1} \frac{m_i}{m_i - 1} \right)^\alpha \leq 2C m_* (V_{n-1} R_{n-1})^\alpha \leq 2C m_* k^\alpha, \end{aligned}$$

where we used that  $V_{n-1} R_{n-1} \leq k$  by the choice of  $n$ .

**Proof of Proposition 5.8.** Fix  $k$  and  $n$  big enough, and let  $\mathbf{g}_k$  be the  $k$ th step of a random walk on  $G$  with law  $\mu^{*k}$ . We have that  $\mathbf{g}_k$  is determined by its action  $\sigma_k$  on  $\mathbb{T}_m^n$  together with its sections at the vertices there, and we shall estimate the entropy coming from each of these parts.

Consider the finite tree  $\tilde{\mathbb{T}}_m^n$  consisting of all vertices up to and including level  $n$ , and let  $\tilde{V}_n \leq 2V_n$  be its cardinality. An automorphism of  $\tilde{\mathbb{T}}_m^n$  is determined by the permutation associated to each of its vertices, that give the action on the children (this is called the *portrait* of the automorphism). Hence the set of automorphisms of the finite tree has cardinality at most  $(m_*)^{\tilde{V}_n} \leq C^{V_n}$ , where  $C$  depends only on  $m$ . It follows that  $\sigma_k$  has at most  $C^{V_n}$  possible values, and by Proposition 3.3(1) we have

$$H(\sigma_k) \leq C V_n$$

for some  $C$  depending only on  $m$ .

To make the next estimates cleaner it is convenient to add randomness in the form of an independent uniform automorphism  $\varepsilon$  of  $\tilde{\mathbb{T}}_m^n$ . As for  $\sigma_k$ , we have  $H(\varepsilon) \leq C V_n$ . We then have

$$\begin{aligned} H(\mathbf{g}_k) &\leq H(\mathbf{g}_k, \varepsilon) = H(\sigma_k, \varepsilon, (\mathbf{g}_k|_v)_{v \in \mathbb{T}_m^n}) = H(\sigma_k, \varepsilon, (\mathbf{g}_k|_{\varepsilon(v)})_{v \in \mathbb{T}_m^n}) \\ &\leq H(\sigma_k) + H(\varepsilon) + \sum_{v \in \mathbb{T}_m^n} H(\mathbf{g}_k|_{\varepsilon(v)}). \end{aligned}$$

The first two terms here are at most  $C V_n$ . The advantage of using  $\varepsilon$  independent of  $\mathbf{g}_k$  is that  $\varepsilon(v)$  is uniform in  $\mathbb{T}_m^n$ , and in particular all terms in the last sum are now equal and the sum equals  $V_n H(\mathbf{g}_k|_v)$ , where  $v$  is a uniform random vertex of  $\mathbb{T}_m^n$ . (Alternatively, this could be achieved with an  $\varepsilon$  with smaller entropy  $\log V_n$ , by taking a random power of some fixed cyclic permutation of  $\mathbb{T}_m^n$ .)

To estimate  $H(\mathbf{g}_k|_v)$  we note that  $(\mathbf{g}_i|_v, \mathbf{v} \cdot \mathbf{g}_i)$  is a random walk with internal degrees of freedom with state space  $\mathbb{T}_m^n$ , and that the edge measures are supported on the finite groups  $A_n$  and  $B_n$  when  $\mathbf{v} \cdot \mathbf{g}_i$  is at  $\mathbb{A}_n$  and  $\mathbb{B}_n$  respectively, and on the identity otherwise (Claim 4.3). It follows from Proposition 3.3(3) that  $H(\mathbf{g}_k|_v) \leq C \mathbb{E}|\mathbf{g}_k|_v| + C$ , where the length  $|\mathbf{g}_k|_v|$  is measured w.r.t. the generating set  $A_n \cup B_n$ , and the constant  $C$  depends only on the cardinalities of these groups (hence on  $M(A, B)$  only).

Let  $\ell$  be the number of times the walk  $(\mathbf{v} \cdot \mathbf{g}_i)$  moves from  $\mathbb{A}_n$  to  $\mathbb{B}_n$  and back to  $\mathbb{A}_n$  up to time  $k$ . Then we have  $|\mathbf{g}_k|_v| \leq 2\ell + 2$ , and so we need to estimate  $\mathbb{E}\ell$ . Say that a traverse begins at time  $i$  if  $\mathbf{v} \cdot \mathbf{g}_i \in \mathbb{A}_n$  and if the walk then visits  $\mathbb{B}_n$  before returning to  $\mathbb{A}_n$ . Note that we do not care whether the visit to  $\mathbb{B}_n$  or the return to  $\mathbb{A}_n$  occur before or after time  $k$ . We now use Lemma 5.7, which applies since  $\mathbf{v} \cdot \mathbf{g}_i$  is stationary. The total weight of edges leaving each vertex is 1, and so (recalling that  $R_n^\mu \geq R_n$ )

$$\mathbb{P}(\text{a traverse begins at time } i) = \frac{1}{2V_n R_n^\mu} \leq \frac{1}{2V_n R_n}.$$

Thus  $\mathbb{E}\ell \leq \frac{k}{2V_n R_n}$ , and so

$$H(\mathbf{g}_k|_v) \leq C \mathbb{E}(2\ell + 2) \leq \frac{Ck}{V_n R_n} + 2C,$$

and

$$H(\mathbf{g}_k) \leq C' \left( V_n + \frac{k}{R_n} \right).$$

□

### Acknowledgements

We thank Mikael de la Salle for pointing out an imprecision in the statement of Proposition 3.3 in a previous version. GA’s research was supported by the Israel Science Foundation (Grant 1471) and by a grant from the GIF, the German–Israeli Foundation for Scientific Research and Development. OA was partially supported by NSERC and ENS in Paris. NMB was introduced to this subject by Anna Erschler, and thanks her for several conversations. The work of NMB was partially supported by the ERC staring Grant GA 257110 “RaWG.”

## References

- [1] G. Amir, O. Angel and B. Virág. Amenability of linear-activity automaton groups. *J. Eur. Math. Soc. (JEMS)* **15** (3) (2013) 705–730. [MR3085088](#)
- [2] G. Amir and B. Virág. Speed exponents for random walks on groups. Preprint, 2012. Available at [arXiv:1203.6226](#).
- [3] G. Amir and B. Virág. Positive speed for high-degree automaton groups. *Groups Geom. Dyn.* **8** (1) (2014) 23–38. [MR3209701](#)
- [4] L. Bartholdi and A. Erschler. Poisson–Furstenberg boundary and growth of groups. Preprint, 2011. Available at [arXiv:1107.5499](#).
- [5] L. Bartholdi, V. A. Kaimanovich and V. V. Nekrashevych. On amenability of automata groups. *Duke Math. J.* **154** (3) (2010) 575–598. [MR2730578](#)
- [6] L. Bartholdi and B. Virág. Amenability via random walks. *Duke Math. J.* **130** (1) (2005) 39–56. [MR2176547](#)
- [7] I. Bondarenko. *Groups Generated by Bounded Automata and Their Schreier Graphs*. Thesis (Ph.D.)—Texas, A&M University. ProQuest LLC, Ann Arbor, MI, 2007. [MR2711289](#)
- [8] J. Brioussel. Amenability and non-uniform growth of some directed automorphism groups of a rooted tree. *Math. Z.* **263** (2) (2009) 265–293. [MR2534118](#)
- [9] J. Brioussel. Behaviors of entropy on finitely generated groups. *Ann. Probab.* **41** (6) (2013) 4116–4161. [MR3161471](#)
- [10] J. Brioussel. Følner sets of alternate directed groups. *Ann. Inst. Fourier (Grenoble)* **64** (3) (2014) 1109–1130. [MR3330165](#)
- [11] Y. Derriennic. Quelques applications du théorème ergodique sous-additif. In *Conference on Random Walks (Kleebach, 1979) (French)* 183–201, 4. *Astérisque* **74**. Soc. Math. France, Paris, 1980. [MR0588163](#)
- [12] A. Erschler. Boundary behavior for groups of subexponential growth. *Ann. of Math. (2)* **160** (3) (2004) 1183–1210. [MR2144977](#)
- [13] A. Erschler. Liouville property for groups and manifolds. *Invent. Math.* **155** (1) (2004) 55–80. [MR2025301](#)
- [14] R. Grigorchuk and Z. Šunić. Asymptotic aspects of Schreier graphs and Hanoi Towers groups. *C. R. Math. Acad. Sci. Paris* **342** (8) (2006) 545–550. [MR2217913](#)
- [15] R. I. Grigorchuk. On the Milnor problem of group growth. *Dokl. Akad. Nauk SSSR* **271** (1) (1983) 30–33. [MR0712546](#)
- [16] R. I. Grigorchuk and A. Žuk. On a torsion-free weakly branch group defined by a three state automaton. *Internat. J. Algebra Comput.* **12** (1–2) (2002) 223–246. International Conference on Geometric and Combinatorial Methods in Group Theory and Semigroup Theory (Lincoln, NE, 2000). [MR1902367](#)
- [17] N. Gupta and S. Sidki. On the Burnside problem for periodic groups. *Math. Z.* **182** (3) (1983) 385–388. [MR0696534](#)
- [18] K. Juschenko, V. Nekrashevych and M. de la Salle. Extensions of amenable groups by recurrent groupoids. Preprint, 2013. Available at [arXiv:1305.2637v1](#).
- [19] V. A. Kaimanovich. “Münchhausen trick” and amenability of self-similar groups. *Internat. J. Algebra Comput.* **15** (5–6) (2005) 907–937. [MR2197814](#)
- [20] V. A. Kaimanovich and A. M. Vershik. Random walks on discrete groups: Boundary and entropy. *Ann. Probab.* **11** (3) (1983) 457–490. [MR0704539](#)
- [21] A. Karlsson and F. Ledrappier. Linear drift and Poisson boundary for random walks. *Pure Appl. Math. Q.* **3** (4, Special Issue: In honor of Grigory Margulis. Part 1) (2007) 1027–1036. [MR2402595](#)
- [22] R. Lyons and Y. Peres. *Probability on Trees and Networks*. Cambridge Univ. Press, Cambridge, 2015. Available at <http://pages.iu.edu/~rdlyons/>.
- [23] V. Nekrashevych. *Self-Similar Groups. Mathematical Surveys and Monographs* **117**. American Mathematical Society, Providence, RI, 2005. [MR2162164](#)
- [24] V. Nekrashevych. Free subgroups in groups acting on rooted trees. *Groups Geom. Dyn.* **4** (4) (2010) 847–862. [MR2727668](#)
- [25] V. Nekrashevych. Iterated monodromy groups. In *Groups St Andrews 2009 in Bath. Vol. 1* 41–93. *London Math. Soc. Lecture Note Ser.* **387**. Cambridge Univ. Press, Cambridge, 2011. [MR2858850](#)
- [26] J. Rosenblatt. Ergodic and mixing random walks on locally compact groups. *Math. Ann.* **257** (1) (1981) 31–42. [MR0630645](#)
- [27] S. Sidki. Finite automata of polynomial growth do not generate a free group. *Geom. Dedicata* **108** (2004) 193–204. [MR2112674](#)