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Maximal displacement of critical branching symmetric stable processes

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Abstract. We consider a critical continuous-time branching process (a Yule process) in which the individuals independently execute symmetric α-stable random motions on the real line starting at their birth points. Because the branching process is critical, it will eventually die out, and so there is a well-defined maximal location M ever visited by an individual particle of the process. We prove that the distribution of M satisfies the asymptotic relation $P\{M \geq x\} \sim (2/\alpha)^{1/2} x^{-\alpha/2}$ as $x \to \infty$.

Résumé. Nous considérons un processus de branchement en temps continu critique (processus de Yule) dont les individus suivent indépendamment un processus α-stable symétrique réel issu de leur point de naissance. Le processus de branchement étant critique, il s'éteint presque surement et nous pouvons définir la valeur M qui représente la position maximale jamais atteinte par un individu. Nous montrons que la distribution de M satisfait l'estimée asymptotique suivante : $P\{M \geq x\} \sim (2/\alpha)^{1/2} x^{-\alpha/2}$.

1. Introduction and main result

1.1. *Critical branching symmetric stable process*

The subject of this paper is the *critical branching symmetric α-stable process* (henceforth abbreviated as "CBSS process"), a critical branching process in which each particle also moves in a one-dimensional space according to a symmetric stable process (specifically, the symmetric Lévy process with Lévy measure $|x|^{-1-\alpha} dx$). Formally, this is defined to be a continuous-time stochastic particle system initiated by a single particle at the origin $0 \in \mathbb{R}$ whose reproduction and dispersal mechanisms are as follows:

(A) Each particle, independently of all others and of the past of the process, waits an exponentially distributed time with parameter 1, and then either splits into two identical particles or dies with probability $1/2$;

(B) While not branching, each particle moves in $\mathbb{R}$ following a symmetric stable process of exponent α, independent of the reproduction process.

Since the branching mechanism is critical, the process will go extinct in finite time, with probability one. Consequently, there is a unique maximal real number $M \geq 0$, which we dub the *maximal displacement* of the process, such

that some particle of the process reaches the location M. Our interest is in the tail of the distribution of M. Let Y_t (Y for *Yule*, as this is a Yule process) be the total number of particles at time t, and let $\zeta_{t,i}$ ($i = 1, 2, \ldots, Y_t$) be the locations of the particles. Then the maximal displacement random variable is formally defined by

$$M = \max_{t \geq 0} M_t \quad \text{where } M_t = \max_{i=1,2,\ldots,Y_t} \zeta_{t,i},$$

with the convention that the max of the empty set is $-\infty$. The main result of the paper is the following theorem.

Theorem 1. *Let M be the maximal displacement of a critical branching symmetric stable process of exponent α ($0 < \alpha < 2$). Then*

$$P\{M \geq x\} \sim \sqrt{\frac{2}{\alpha}} \frac{1}{x^{\alpha/2}} \quad \text{as } x \to \infty. \tag{1}$$

1.2. *Discussion*

Theorem 1 is the natural analogue for the CBSS process of a theorem describing the maximal displacement of a critical driftless branching random walk, recently proved by the authors in [11]. The critical branching random walk is a discrete-time branching process in which particles alternately reproduce and move, as follows. The reproduction steps are governed by the law of a critical Galton–Watson process whose offspring distribution has finite variance η^2 and finite third moment; the movement steps are governed by the law of a finite-variance, mean-zero random walk on the integers $\mathbb{Z}$. See [11] or [10] for further details on the construction of the process. Since the branching mechanism is assumed to be critical (that is, the offspring distribution has mean 1), the process dies out after finitely many generations, and hence there is a well-defined maximal displacement random variable M, defined in the same manner as for the CBSS process discussed above. The main result of [11] states that if the step distribution of the random walk component of the branching random walk has mean 0, variance $\sigma^2 > 0$, and finite $4 + \varepsilon$ moment for some $\varepsilon > 0$, then as $x \to \infty$,

$$P\{M \geq x\} \sim \frac{6\eta^2}{\sigma^2 x^2}. \tag{2}$$

The result (2) is itself the natural extension to branching random walks of an earlier result of Sawyer and Fleischman [6] for critical branching Brownian motion.[2] For branching Brownian motion it is possible (and not difficult) to write a second-order ordinary differential equation for the distribution function of M; the tail asymptotics of solutions can then be obtained by relatively standard methods in ODE theory. See [6] for details. For branching random walk, it is also quite easy to write a nonlinear convolution equation for the distribution function (cf. [10,11]), but ODE methods cannot be used to determine tail asymptotics. (See the discussion on p. 924 of [10], in particular his eqn. 1.9, where the tail asymptotics is left as an open problem.) The primary technical contribution of [11] is a new method, based on Feynman–Kac formulas, for the analysis of such nonlinear convolution equations. The bulk of the current paper will be devoted to a parallel method for studying the asymptotic behavior of solutions to certain *pseudo-differential* equations that will be shown to govern the distribution function of the maximal displacement random variable for the CBSS process.

A careful reading of the proofs below will show that Theorem 1 can easily be generalized to a wider class of branching Markov processes on the line. The simple "double-or-nothing" branching mechanism (A) could be replaced by any critical (i.e., mean 1) distribution on the nonnegative integers with positive, finite variance and finite third moment; this would only change the asymptotic formula (1) by a multiplicative constant (see [11] for a detailed analysis of general branching laws in the case of branching random walk). Similarly, the assumption (B) that particles move according to the law of a symmetric, stable process could be relaxed to allow more general Lévy motions, as

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[2]See also [14]. The paper of Sawyer and Fleischman proposed the critical branching Brownian motion as a model for the dispersal of a mutant but neutral allele in a homogeneous environment. Branching random walks and branching diffusion processes are also used as models in combustion and reaction-diffusion processes, and they occur as low-density limits of certain spatial epidemics [12]. In all of these situations the maximal displacement is of natural interest.

long as the tails of the Lévy measure agree with those of the stable process. We have elected not to formulate and prove our result in the greatest possible generality in order to simplify the exposition and to emphasize what we consider the most interesting features of the model – specifically, (a) criticality of the branching mechanism and (b) stable scaling.

1.3. *Heuristics*

The following heuristic arguments suggest that $x^{-\alpha/2}$ is the correct order of magnitude for the tail probability (1). Fix a large time T and consider the event that the branching process survives to time T: by Kolmogorov's theorem for critical branching processes (cf. [2], ch. 1), the chance of this is on the order of $1/T$. Furthermore, by Yaglom's theorem, if the branching process survives to time T then at typical times $t \in [\varepsilon T, T]$ the number of particles alive will be on the order of T. Thus, the total "particle-time" will be on the order of T^2. Now in each small interval (Δt) of time, each particle has a chance $(\Delta t)T^{-2}$ of jumping a distance more than $T^{2/\alpha}$ to the right (by the Poisson point process representation of the symmetric α-stable process: see Section 2 below). Since the total particle-time is on the order of T^2, it follows that the conditional probability that some particle makes it past location $T^{2/\alpha}$ is on the order $O(1)$. Thus, the *unconditional* probability is on the order of $1/T$, since this is the probability that the process survives to time T. A similar argument shows that unless the process survives for significantly longer than time T then the chance that a particle moves much farther right than $T^{2/\alpha}$ is negligible.

These heuristics can, with some care (see Section 4 below), be made into rigorous arguments to prove that

$$P\{M \ge x\} \asymp x^{-\alpha/2}, \tag{3}$$

but there is little hope of obtaining the sharp asymptotic formula (1) by similar methods. (See [10] for a detailed analysis of such arguments in the case of a critical, driftless branching random walk.) The rough asymptotic formula (3) will be a necessary preliminary step in proving the sharper result (1) (see Proposition 7 in Section 4), but (1) will also require the use of different tools based on Feynman–Kac formulas.

1.4. *Superprocess limits*

The asymptotic relation (2) is shown in [11] to be closely related to the Dawson–Watanabe scaling limit (super-Brownian motion) for critical branching random walks. In brief, if n independent copies of a critical branching random walk (satisfying the hypotheses of [11]) are all started at time 0 at the origin, if time and mass are scaled by a factor n and space by $n^{1/2}$ then the resulting measure-valued processes converge as $n \to \infty$ to super-Brownian motion. This by itself does not imply that the distribution of the rightmost particle converges to that of the rightmost support point of the limiting super-Brownian motion, because the rightmost support point is not a continuous functional of the path, but the asymptotic relation (2) implies that the convergence does indeed hold. See [11] for an extended discussion.

There is a similar scaling for the measure-valued processes connected with the CBSS. In particular, if n independent copies of the CBSS are all started at time 0 at the origin, if time and mass are scaled by n and space is scaled by $n^{-1/\alpha}$ then the resulting measure-valued process becomes the symmetric α-stable superprocess in the $n \to \infty$ limit. (See, for instance, [13], ch. 2 for details of the construction.) But it is known (cf. [16], Theorem 1.5 and the preceding discussion) that the symmetric α-stable superprocess $\{X_t\}_{t \ge 0}$ has the property that with probability 1, for every $t > 0$ the support of the random measure X_t is either $\varnothing$ or $\mathbb{R}$, so unlike super-Brownian motion the symmetric α-stable superprocess has no "rightmost support point." Thus, the scaling relation (1) is not related to any obvious feature of the limiting superprocess.

1.5. *Plan of the paper*

Theorem 1 will be proved by first showing that the distribution function of M satisfies a pseudo-differential equation (21) involving the fractional Laplacian operator. This will be done in Section 3. A comparison principle for solutions to the pseudo-differential equation will be proved in Section 4, and this will be used to prove the a priori estimates (3). Finally, in Section 5, a Feynman–Kac representation of solutions to the pseudo-differential equation (21) will be used to obtain sharp asymptotics. The Feynman–Kac representation will involve path integrals of the symmetric α-stable process, and in analyzing these it will be necessary to call on some structural features of these processes: the relevant facts are collected in Section 2.

2. Preliminaries on symmetric stable processes

Recall [3] that a *symmetric α-stable process* in $\mathbb{R}$ is a real-valued Lévy process $\{X_t\}_{t \geq 0}$ whose distribution X_t is symmetric (i.e. X_t has the same distribution as $-X_t$) for any $t \geq 0$, and satisfies the scaling property

$$\frac{X_t}{t^{1/\alpha}} =^{\mathcal{D}} X_1 \quad \forall t > 0. \tag{4}$$

Henceforth, we shall reserve the symbol X for a symmetric α-stable process, and we shall use the usual convention of attaching a superscript x to the probability and expectation operators E^x, P^x to denote that under P^x the process X_t has initial value $X_0 = x$. When the superscript x is omitted, it should be understood that $x = 0$.

The characteristic function of a symmetric α-stable process has the form $Ee^{i\theta X(t)} = \exp(-\gamma t |\theta|^\alpha)$ for some constant $\gamma > 0$ (whose exact value will not play a role in this paper). This is clearly integrable, and so it follows by the Fourier inversion theorem that the distribution of X_t has a density $f_t(x)$ with respect to Lebesgue measure dx. We shall only consider the case $\alpha < 2$. For such α, the symmetric stable process is a pure jump process and has a Poisson point process representation

$$X(t) = \iint_{\mathcal{X}} y \mathbb{1}_{(0,t]}(s) N(ds, dy), \tag{5}$$

where $N(ds, dy)$ is a Poisson random measure with intensity μ given by

$$\mu(dt, dy) = dt \cdot \lambda(dy) \quad \text{with Lévy measure,} \tag{6}$$
$$\lambda(dy) = |y|^{-1-\alpha} dy \quad \text{on } \mathbb{R}.$$

The infinitesimal generator of a symmetric α-stable process is (see [3], p. 24) the *fractional Laplacian pseudo-differential operator* $-(-\Delta)^{\alpha/2}$. Thus, if X_t is symmetric α-stable and $f : \mathbb{R} \to \mathbb{R}$ is a suitable function (for example, a compactly supported smooth function), then

$$\lim_{t \to 0+} \frac{E^x[f(X_t)] - f(x)}{t} = -(-\Delta)^{\alpha/2} f(x), \tag{7}$$

where $(-\Delta)^{\alpha/2}$ is the nonlocal linear operator defined by the singular integral

$$(-\Delta)^{\alpha/2} f(x) := \int_{-\infty}^{\infty} \frac{f(x) - f(y)}{|x - y|^{1+\alpha}} \, dy, \tag{8}$$

where the integral is interpreted as a Cauchy principal value. The domain of this operator is understood to be the set of all bounded, continuous functions such that the limit (7) exists.

Several elementary first-passage properties of the symmetric α-stable process $\{X_t\}_{t \geq 0}$ will be used repeatedly in the analysis that follows. First, by the Hewitt–Savage 0–1 Law, the limsup and liminf of any sample path are $\pm \infty$, and so for any $A > 0 > B$ the first-passage times

$$\tau_A^+ = \tau^+(A) = \inf\{t > 0 : X_t \geq A\} \quad \text{and}$$
$$\tau_B^- = \tau^-(B) = \inf\{t > 0 : X_t \leq B\} \tag{9}$$

are almost surely finite. Note that by symmetry the random variables $\tau^+(A)$ and $\tau^-(-A)$ have the same distribution, and similarly so do $X_{\tau^+(A)}$ and $-X_{\tau^-(-A)}$. Second, by the scaling law, for any $A > 0$ the joint distribution of $(\tau^+(A)/A^\alpha, X_{\tau^+(A)}/A)$ is identical to that of $(\tau^+(1), X_{\tau^+(1)})$. Third, there is the following analogue of the "reflection principle" for Brownian motion.

Lemma 2. *Let X_t be a symmetric α-stable process with $X_0 = 0$, and define*

$$X_t^* := \max_{s \in [0,t]} X_s. \tag{10}$$

Then for any $y > 0$,

$$P\{X_t^* \geq y\} \leq 2P\{X_t \geq y\}. \tag{11}$$

Proof. Fix $y > 0$, and abbreviate $\tau = \tau_y$. Then $X_\tau \geq y$, and by the strong Markov property,

$$\tilde{X}_t := \begin{cases} X_t, & \text{when } t < \tau, \\ 2X_\tau - X_t, & \text{when } t \geq \tau \end{cases}$$

is a symmetric α-stable process as well. Hence

$$\begin{aligned} P\{X_t^* \geq y\} &= P\{X_t^* \geq y, X_t \geq y\} + P\{X_t^* \geq y, X_t < y\} \\ &= P\{X_t \geq y\} + P\{\tau < t, \tilde{X}_t > 2X_\tau - y\} \\ &\leq P\{X_t \geq y\} + P\{\tilde{X}_t > y\} \\ &= 2P\{X_t \geq y\}. \qquad \Box \end{aligned}$$

Some of the arguments in Sections 3 and 5 will require estimates on first-passage probabilities on very short time intervals. The following (elementary) lemma asserts that a first passage to an interval $[A, \infty)$ with $A > 0$ can occur within a very short time $\varepsilon > 0$ can occur only if there is a single jump of size nearly A or greater during this time interval. For any interval $J \subset \mathbb{R}$, define

$$\nu_J = \inf\{t \geq 0 : X_t - X_{t-} \in J\} \quad \text{and} \tag{12}$$

$$\sigma_J = \inf\{t \geq 0 : X_t \notin J\}. \tag{13}$$

Lemma 3. *For any $A > 0$ and $0 < \delta < A/2$,*

$$P\{\tau_A^+ < \varepsilon\} \sim P\{\nu_{[A,\infty)} < \varepsilon\} = \varepsilon\lambda[A, \infty) \quad \text{as } \varepsilon \to 0 \quad \text{and} \tag{14}$$

$$\lim_{\varepsilon \to 0} P\left(\tau_A^+ \neq \sigma_{(-\delta,\delta)} \mid \tau_A^+ < \varepsilon\right) = 0. \tag{15}$$

Proof. By Lemma 2 and the scaling law, for any fixed $\delta > 0$,

$$\begin{aligned} P\{\tau^+(\delta) < \varepsilon\} &\leq 2P\{X_\varepsilon \geq \delta\} \\ &= 2P\{X_1 \geq \delta\varepsilon^{-1/\alpha}\} \\ &\sim \kappa\delta^\alpha\varepsilon, \end{aligned}$$

where $\kappa > 0$ is a constant that depends on the exponent α. (See, e.g., [18], p. 95, or [5], sec. XVII.6 for the fact that the tail of the α-stable law is regularly varying with exponent α.) It follows by symmetry that

$$P\{\sigma_{(-\delta,\delta)} < \varepsilon\} \sim 2\kappa\delta^\alpha\varepsilon. \tag{16}$$

Next, we will show that for any $A > 0$ and $0 < \delta < A/2$,

$$\lim_{\varepsilon \to 0} P\left(|X_{\nu_{[A,\infty)}-}| > \delta \mid \nu_{[A,\infty)} \leq \varepsilon\right) = 0. \tag{17}$$

For this we use the Poisson point process representation of the symmetric stable process. Let $\hat{X}_t$ be a second version of the stable process whose Poisson point process agrees with that of X_t on $[0, \infty) \times (-\infty, A)$ but is independent on $[0, \infty) \times [A, \infty)$. Then clearly the process $\hat{X}_t$ is independent of the random variable $\nu_{[A,\infty)}$, and $|X_{\nu_{[A,\infty)}}| \leq |\hat{X}_\varepsilon^*|$ on the event that $\nu_{[A,\infty)} \leq \varepsilon$ and $\hat{\nu}_{[A,\infty)} > \varepsilon$, because on this event the processes X_t and $\hat{X}_t$ will coincide up to time $\nu_{[A,\infty)}$. Thus,

$$P\left(|X_{\nu_{[A,\infty)}-}| > \delta \mid \nu_{[A,\infty)} \leq \varepsilon\right) \leq P\left(|\hat{X}_\varepsilon| > \delta\right) + P(\nu_{[A,\infty)} \leq \varepsilon).$$

The assertion (17) now follows from the inequality (16) and the fact that $\hat{X}_t \to 0$ in probability as $t \to 0$.

The results (14)–(15) follow routinely from (16) and (17). First, if $X_{\sigma_{(-\delta,\delta)}} \le A - \delta$ then in order to exit $(-\infty, A]$ by time ε the post-$\sigma_{(-\delta,\delta)}$ process would have to make a second exit of the interval $(-\delta, \delta)$, and so by the strong Markov property,

$$P\left(\tau_A^+ \le \varepsilon \text{ and } X_{\sigma_{(-\delta,\delta)}} \le A - \delta > \varepsilon\right) \le P(\sigma_{(-\delta,\delta)} \le \varepsilon)^2.$$

Since $P\{\tau_A^+ \le \epsilon\}$ is of order at least ε, it follows that as $\varepsilon \to 0$, the conditional probability that $\sigma_{(-\delta,\delta)} = v_{[A-2\delta,\infty)}$ given that $\tau_A^+ \le \varepsilon$ converges to 1. Hence,

$$\lim_{\varepsilon \to 0} P\left(\sigma_{(-\delta,\delta)} = v_{[A-2\delta,\infty)} \mid \tau_A^+ \le \varepsilon\right) = 1.$$

Since $\sigma_{(-\delta,\delta)} \le \tau_A^+$, it follows that

$$P\left\{\tau_A^+ < \varepsilon\right\} \le P\{v_{[A-2\delta,\infty)} \le \varepsilon\}\left(1 + o(1)\right).$$

On the other hand, if $v_{[A+2\delta,\infty)} \le \varepsilon$ and $|X_{v_{[A+2\delta,\infty)}}| < \delta$ then it must be the case that $\tau_A^+ = v_{[A+2\delta,\infty)}$, and hence, by (17),

$$\begin{aligned} P\left\{\tau_A^+ < \varepsilon\right\} &\ge P\left\{v_{[A+2\delta,\infty)} \le \varepsilon \text{ and } |X_{v_{[A+2\delta,\infty)}}| < \delta\right\} \\ &\sim P\{v_{[A+2\delta,\infty)} \le \varepsilon\}. \end{aligned}$$

Since $\delta > 0$ can be made arbitrarily small, this proves (14). Finally, if $\sigma_{(-\delta,\delta)} = v_{[A+\delta,\infty)}$ then both are equal to τ_A^+. Therefore,

$$P\left(\sigma_{(-\delta,\delta)} \ne \tau_A^+ \mid \tau_A^+ \le \varepsilon\right) \le P\left(v_{[A-2\delta,A+\delta]} \le \varepsilon \mid \tau_A^+ \le \varepsilon\right) = O(\delta).$$

Since $\delta > 0$ is arbitrary, (15) follows. $\qquad\square$

Lemma 3 indicates that when the first-passage time $\tau^+(A)$ is very small it is because the path makes a single jump of size $\ge A$ at time $\tau^+(A)$. The following lemma – a consequence of the Poisson point process representation of the stable process – asserts that the size of this jump is independent of the path up to the jump time. For any interval $J \subset \mathbb{R}$, define $v_J = v(J)$ to be the first time that $X_{v(J)} - X_{v(J)-} \in J$, equivalently,

$$v_J = \inf\left\{t > 0 : N(t, J) = 1\right\}. \tag{18}$$

Lemma 4. *Let $J \subset \mathbb{R}$ be a nonempty, open interval such that 0 is in the interior of J^c. Then the jump size $X_{v(J)} - X_{v(J)-}$ is independent of $\mathcal{F}_{v(J)-}$.*

Proof. This is an elementary consequence of the Poisson point process representation, using the fact that any event in the σ-algebra $\mathcal{F}_{v(J)-}$ is determined by the restriction of the Poisson point process to $[0, \infty) \times J^c$. $\qquad\square$

Corollary 5. *For all $x > A > 0$, as $\varepsilon \to 0$,*

$$\lim_{\varepsilon \to 0} P\left(X_{\tau_A^+} > x \mid \tau_A^+ < \varepsilon\right) = \frac{\lambda[x, \infty)}{\lambda[A, \infty)} = (x/A)^{-\alpha}. \tag{19}$$

Moreover, this relation holds uniformly in the region $x > A \ge 1$.

3. A pseudo-differential equation for the CDF

Let M be the maximal displacement of a CBSS process. The tail distribution function of M will be denoted by

$$u(x) = P\{M \geq x\}. \tag{20}$$

Clearly, $u(x) = 1$ for all $x \leq 0$, and it is easily seen that $0 < u(x) < 1$ for all $x > 0$. Since $M < \infty$ with probability one, $\lim_{x \to \infty} u(x) = 0$. Furthermore, the strong Markov property for the CBSS implies that u is continuous, as the following argument shows. Fix $x \geq 0$, and denote by T_x the first time that a particle of the CBSS process reaches x; this is a stopping time. By the strong Markov property, the post-T_x process initiated by the particle at x is itself a CBSS process; this process will, with (conditional) probability 1, place a particle in (x, ∞) at some time after T_x, because (i) the initiating particle will not immediately die, and (ii) a symmetric stable process started at 0 must immediately enter both the positive and negative halflines.

The key to our analysis of the tail behavior of u is that u satisfies the pseudo-differential equation in the following proposition. The analogous differential equation in the case of branching Brownian motion was exploited by [6] (see also [8], who were apparently unaware of [6]).

Proposition 6. $u(x)$ *solves the following nonlinear boundary value problem*

$$\begin{cases} (-\Delta)^{\alpha/2} u(x) + \frac{1}{2}(u(x))^2 = 0, & \text{for } x > 0, \\ u(x) = 1, & \text{for } x \leq 0. \end{cases} \tag{21}$$

Proof. Fix any $x > 0$. Let us calculate $P\{M < x\} = 1 - u(x)$ by conditioning on the evolution of the CBSS process up to time $\varepsilon > 0$. Up until the time T that it first fissions or dies, the initiating particle follows a symmetric α-stable trajectory. Let $\{X_s\}_{s \geq 0}$ be a generic symmetric α-stable process, and write $X_t^* := \max_{s \in [0,t]} X_s$. By Lemma 2 (the "reflection principle"), for any fixed $x > 0$, as $\varepsilon \to 0$,

$$P\{X_\varepsilon^* \geq x\} \leq 2P\{X_\varepsilon \geq x\} = O(\varepsilon). \tag{22}$$

The distribution of T is exponential with mean one, and the event that the initiating particle fissions rather than dies at time T is Bernoulli-1/2, independent of T. If the initiating particle fissions then the event $M < x$ requires that *both* of the CBSS processes engendered by the fission have maximal displacements $< x$ *and that the path of the initial particle up to the time of fission stays below the level x*. Hence,

$$P\{M < x, \text{ initial particle fissions before time } \varepsilon\}$$

$$= \frac{1}{2} \int_0^\varepsilon e^{-t} \left(\int_{-\infty}^x \left(P\{M < x - y\} \right)^2 dF_t(y) - P\{X_t^* \geq x\} \right) dt$$

$$= \frac{1}{2} \int_0^\varepsilon e^{-t} \int_{-\infty}^x \left(1 - u(x - y) \right)^2 dF_t(y) \, dt + o(\varepsilon)$$

$$= \frac{1}{2} \varepsilon \left(1 - u(x) \right)^2 + o(\varepsilon), \tag{23}$$

where F_t is the distribution of the random variable X_t. The last equality holds because u is continuous and bounded and $F_t \Rightarrow F_0 = \delta_0$ as $t \to 0$. The second equality follows from the estimate (22). A similar argument shows that

$$P\{M < x, \text{ initial particle dies before time } \varepsilon\}$$

$$= \frac{1}{2} \int_0^\varepsilon e^{-t} \cdot P^0\{X_t^* < x\} \, dt$$

$$= \frac{1}{2} \varepsilon + o(\varepsilon). \tag{24}$$

Next, by the Markov property,

$$
\begin{aligned}
&P\{M < x \text{ and } T > \varepsilon\} \\
&= e^{-\varepsilon}\left(\int_{-\infty}^{x} P\{M < x - y\}\,\mathrm{d}F_\varepsilon(y) - P\{X_\varepsilon^* \geq x, X_\varepsilon < x\}\right).
\end{aligned}
\tag{25}
$$

The first term in (25) is equal to

$$
\begin{aligned}
e^{-\varepsilon}&\int_{-\infty}^{x}\left(1 - u(x - y)\right)\mathrm{d}F_\varepsilon(y) \\
&= e^{-\varepsilon}\left(\int_{-\infty}^{\infty}\left(1 - u(x - y)\right)\mathrm{d}F_\varepsilon(y) - \left(1 - u(x)\right)\right) + e^{-\varepsilon}\left(1 - u(x)\right) \\
&= e^{-\varepsilon}\int_{-\infty}^{\infty}\left(u(x) - u(x - y)\right)\mathrm{d}F_\varepsilon(y) + e^{-\varepsilon}\left(1 - u(x)\right) \\
&= \varepsilon(-\Delta)^{\alpha/2}u(x) + (1 - \varepsilon)\left(1 - u(x)\right) + o(\varepsilon).
\end{aligned}
$$

In the third equality we exploited the boundary condition $u(x - y) = 1$ when $x - y \leq 0$. The last equality follows from (7).

The three terms (23), (24), and (25) account for all of the terms in the pseudo-differential equation (21). Thus, to complete the proof it remains only to show that the second term in (25) satisfies

$$
P\{X_\varepsilon^* \geq x, X_\varepsilon < x\} = o(\varepsilon).
$$

For this we appeal to Lemmas 2–3. Relation (14) of Lemma 3 implies that $P(\tau(x) \leq \varepsilon) = P(X_\varepsilon^*) = O(\varepsilon)$, so it is enough to show that as $\varepsilon \to 0$,

$$
P\left(X_\varepsilon < x \mid \tau(x) \leq \varepsilon\right) = o(1).
$$

Now if $X_{\tau(A)} > x + \beta$ then in order that $X_\varepsilon < x$ the process must traverse an interval of size β in time $< \varepsilon$, and by Lemma 2 the chance of this is no more than $2P(X_\varepsilon > \beta)$. Furthermore, by the scaling law (4), if $\beta = \varepsilon^\varrho$ for some $\varrho < 1/\alpha$ then $P(X_\varepsilon > \beta) = o(1)$ as $\varepsilon \to 0$. Consequently,

$$
\begin{aligned}
P&\left(X_\varepsilon < x \mid \tau(x) \leq \varepsilon\right) \\
&= P\left(X_\varepsilon < x \text{ and } X_{\tau(x)} > x + \varepsilon^\varrho \mid \tau(x) \leq \varepsilon\right) \\
&\quad + P\left(X_\varepsilon < x \text{ and } X_{\tau(x)} \leq x + \varepsilon^\varrho \mid \tau(x) \leq \varepsilon\right) \\
&= o(1) + o(1),
\end{aligned}
$$

the last by relation (19), which implies that $P(X_{\tau(x)} \leq x + \varepsilon^\varrho \mid \tau(x) \leq \varepsilon) \to 0$ as $\varepsilon \to 0$.

Finally, recall that $P\{M < x\} = 1 - u(x)$ is equal to the sum of the three probabilities (23), (24), and (25) above. Therefore,

$$
\varepsilon\left(1 - u(x)\right) = \frac{1}{2}\varepsilon\left(1 - u(x)\right)^2 + \frac{1}{2}\varepsilon + \varepsilon\int_{-\infty}^{\infty}\left(u(x) - u(x - y)\right)\mathrm{d}\nu(y) + o(\varepsilon).
$$

Dividing both sides by ε, then letting $\varepsilon \to 0$, we conclude that

$$
\int_{-\infty}^{\infty}\left(u(x) - u(x - y)\right)\mathrm{d}\nu(y) + \frac{1}{2}\left(u(x)\right)^2 = 0. \qquad \square
$$

4. A priori bounds for $u(x)$

The first step toward establishing the sharp asymptotic formula (1) will be to show that the function u satisfies the rough asymptotic formula (3). We will give two different arguments, one probabilistic, the other analytic, the first showing that the *particular* function u defined by (20) satisfies the inequalities (3), the second proving the following (superficially) more general result. (It will follow from the Feynman–Kac formula (33) below that the solution to the boundary value problem (21) is unique, hence must coincide with (20).)

Proposition 7. *Let $u(x)$ be a continuous positive solution to the boundary value problem* (21), *and suppose that $u(x) \to 0$ as $x \to \infty$. Then there exist positive constants C_1 and C_2 such that*

$$\frac{C_1}{x^{\alpha/2}} \le u(x) \le \frac{C_2}{x^{\alpha/2}} \tag{26}$$

for all $x \ge 1$.

4.1. *Probabilistic approach*

These arguments apply specifically to the tail distribution function $u(x)$ of the maximal displacement M of a CBSS process. Recall that in a CBSS process, the number of particles alive at time t is a standard Yule (binary fission) process. The CBSS can be constructed by first running a Yule process Y_t, then running independent symmetric α-stable processes along the edges of the resulting genealogical tree. The Yule process itself can be built by first constructing a discrete-time double-or-nothing Galton–Watson process (i.e., a Galton–Watson process whose offspring distribution is $p_0 = p_2 = 1/2$) and then attaching independent unit exponential random variables to the edges of the resulting Galton–Watson tree.

Proof of the lower bound $u(x) \ge C_1/x^{\alpha/2}$. Denote by ξ the total progeny of the Yule process, that is, the number of distinct particles born in the course of the branching process. Equivalently, ξ is $1 +$ the number of edges in the genealogical tree. A well known (but somewhat difficult to trace[3]) result from the elementary theory of Galton–Watson processes has it that for a critical Galton–Watson process whose offspring distribution has positive, finite variance,

$$P\{\xi \ge m\} \asymp \frac{1}{\sqrt{m}}. \tag{27}$$

(This can also be deduced from the results of [1].) Hence, there exists $C > 0$ such that with probability at least $C/\sqrt{m}$ the Yule tree has at least m branches.

The branch lengths of the Yule tree are independent unit exponentials, independent of both the Galton–Watson skeleton and the stable random motions of the particles, and so the spatial displacements D_e of particles along these edges e are unit exponential mixtures of symmetric α-stable random variables X_t. Consequently, since the tail of a symmetric α-stable random variable is regularly varying with exponent α, there is a constant $C' > 0$ such that, conditional on the Galton–Watson tree, for each edge e

$$P\{|D_e| \ge 3m^{1/\alpha}\} \ge C'/m. \tag{28}$$

Therefore, since the random variables D_e are conditionally independent given the Galton–Watson tree, it follows from (27) and (28) that with probability at least $C''/\sqrt{m}$ there will be some edge e of the tree for which $|D_e| \ge 3m^{1/\alpha}$. But

---

[3]The probability generating function of ξ was derived by I. J. Good [7] in 1949, and related results were later obtained by Dwass [4] and Pakes [15]. It was known to T. Harris [9] that in the special case where the offspring distribution is the geometric distribution with mean 1 the genealogical tree of the associated Galton–Watson process is the excursion tree of a simple random walk excursion, from which it follows directly that ξ is $1/2$ the number of steps in the simple random walk excursion.

on this event there must be at least one particle of the CBSS process that finds its way out of the interval $[-m^{1/\alpha}, m^{1/\alpha}]$. Since the CBSS process is invariant under reflection of the space axis, it follows that

$$u(m^{1/\alpha}) = P\{M \geq m^{1/\alpha}\} \geq C''/\sqrt{m}. \qquad \square$$

Proof of the upper bound $u(x) \leq C_2/x^{\alpha/2}$. This relies on the following elementary property of the CBSS process: the mean particle density at location dx at time t is $f_t(x)\, dx$, where $f_t(x)$ is the density of the symmetric α-stable random variable X_t. Consequently, for any $x \geq 0$ and $t > 0$, the conditional expectation of the number of particles to the right of x at time t given that some particle of the CBSS reaches the halfline $[x, \infty)$ before time t is at least $1/2$. It follows that

$$u(x) = P\{M \geq x\} \leq 2P\{X_t \geq x\} + P\{Y_t \geq 1\},$$

where Y_t is the skeletal Yule process, M is the maximal displacement of the CBSS process, and X_t is a generic symmetric α-stable process. By setting $t = x^{-\alpha/2}$ and using the fact that the distribution of X_1 has regularly varying tail with exponent α and the fact (essentially Kolmogorov's theorem for critical branching processes) that $P\{Y_t \geq 1\} \sim C/t$, we obtain the desired estimate

$$u(x) \leq C'/x^{\alpha/2}. \qquad \square$$

4.2. *Analytic approach*

We shall prove Proposition 7 in general by first establishing a comparison principle for the boundary value problem (21), then comparing our $u(x)$ to explicit supersolutions and subsolutions of (21), both of which decay to zero as a constant times $x^{-\alpha/2}$. (Thanks to Professor Luis Silvestre for suggesting this.)

Proposition 8 (Comparison Principle). *Let $u(x)$ be a continuous positive solution to the boundary value problem (21), and suppose that $u(x) \to 0$ as $x \to \infty$.*
   (A) Suppose that $U(x)$ is a continuous positive super-solution to (21), meaning that

$$\begin{cases} (-\Delta)^{\alpha/2}U(x) + \frac{1}{2}(U(x))^2 \geq 0, & \text{for } x > 0, \\ U(x) \geq 1, & \text{for } x \leq 0. \end{cases}$$

Furthermore, assume that $U(x) \to 0$ as $x \to \infty$. Then,

$$u(x) \leq U(x) \quad \text{for all } x \in \mathbb{R}.$$

   (B) Suppose that $V(x)$ is a continuous positive sub-solution to (21), meaning that

$$\begin{cases} (-\Delta)^{\alpha/2}V(x) + \frac{1}{2}(V(x))^2 \leq 0, & \text{for } x > 0, \\ V(x) \leq 1, & \text{for } x \leq 0. \end{cases}$$

Furthermore, assume that $V(x) \to 0$ as $x \to \infty$. Then,

$$u(x) \geq V(x) \quad \text{for all } x \in \mathbb{R}.$$

Proof. We will only prove part (A). The proof of part (B) can be done in an analogous manner.
   We proceed by contradiction. Suppose that $u(x_0) > U(x_0)$ at some point $x_0 \in \mathbb{R}$. Then $(u - U)(x)$, a continuous function that is nonpositive for $x \leq 0$ and goes to zero as $x \to \infty$, would attain a strictly positive global maximum value at a certain point $x_1 > 0$:

$$(u - U)(x_1) = \max_{x \in \mathbb{R}}(u - U)(x) > 0.$$

Now consider the quantity $(-\Delta)^{\alpha/2}(u - U)(x_1)$. On one hand,

$$(-\Delta)^{\alpha/2}(u - U)(x_1) = \int_{-\infty}^{\infty} \frac{(u - U)(x_1) - (u - U)(y)}{|x_1 - y|^{1+\alpha}}\, dy \geq 0$$

because $(u - U)(x_1) \geq (u - U)(y)$ for all y. On the other hand,

$$\begin{aligned}
(-\Delta)^{\alpha/2}(u - U)(x_1) &= (-\Delta)^{\alpha/2}u(x_1) - (-\Delta)^{\alpha/2}U(x_1) \\
&\leq -\frac{1}{2}\big(u(x_1)\big)^2 + \frac{1}{2}\big(U(x_1)\big)^2 \\
&< 0.
\end{aligned}$$

This is a contradiction. Thus, $u(x) \leq U(x)$ for all $x \in \mathbb{R}$. $\qquad\square$

Proof of Proposition 7. Consider the function

$$w(x) = \begin{cases} (1 + x)^{-\alpha/2}, & \text{for } x > 0, \\ 1, & \text{for } x \leq 0. \end{cases}$$

We will show that, for a large enough constant C, $Cw(x + 1)$ is a supersolution to (21), and for a small enough positive constant C, $Cw(x + 1)$ is a subsolution to (21). Notice that $(-\Delta)^{\alpha/2}(Cw)(x + 1) + \frac{1}{2}(Cw(x + 1))^2 = C((-\Delta)^{\alpha/2}w(x + 1) + \frac{1}{2}C(w(x + 1))^2)$. Hence it suffices to show

$$\sup_{x>0} \frac{-(-\Delta)^{\alpha/2}w(x + 1)}{\frac{1}{2}(w(x + 1))^2} < \infty \quad \text{and} \quad \inf_{x>0} \frac{-(-\Delta)^{\alpha/2}w(x + 1)}{\frac{1}{2}(w(x + 1))^2} > 0.$$

Because $-(-\Delta)^{\alpha/2}w(x + 1)$ is obviously continuous for $x \in [0, \infty)$, it eventually boils down to proving

$$\limsup_{x \to \infty} \frac{-(-\Delta)^{\alpha/2}w(x)}{x^{-\alpha}} < \infty \quad \text{and} \quad \liminf_{x \to \infty} \frac{-(-\Delta)^{\alpha/2}w(x)}{x^{-\alpha}} > 0. \tag{29}$$

Now for any $x > 0$, let us compute

$$\begin{aligned}
-(-\Delta)^{\alpha/2}w(x) &= -\int_{-\infty}^{\infty} \frac{w(x) - w(y)}{|x - y|^{1+\alpha}}\, dy \\
&= -\int_{-\infty}^{-1} \frac{(1 + x)^{-\alpha/2} - 1}{(x - y)^{1+\alpha}}\, dy - \int_{-1}^{\infty} \frac{(1 + x)^{-\alpha/2} - (1 + y)^{-\alpha/2}}{|x - y|^{1+\alpha}}\, dy \\
&:= -A - B.
\end{aligned}$$

The first integral can be easily evaluated:

$$-A = \frac{1}{\alpha} \cdot \big(1 - (1 + x)^{-\alpha/2}\big) \cdot (x + 1)^{-\alpha} \sim \frac{1}{\alpha} \cdot x^{-\alpha} \quad \text{as } x \to \infty.$$

To deal with the second integral B, consider an auxiliary function

$$\begin{aligned}
F(x) &:= \int_0^{\infty} \frac{x^{-\alpha/2} - y^{-\alpha/2}}{|x - y|^{1+\alpha}}\, dy \\
&= \int_0^{\infty} \frac{x^{-\alpha/2} - \lambda^{-\alpha/2}z^{-\alpha/2}}{|x - \lambda z|^{1+\alpha}} \lambda\, dz \quad (y = \lambda z),
\end{aligned}$$

where $\lambda > 0$ is an arbitrarily chosen constant. Then, observe that

$$
\begin{aligned}
F(\lambda x) &= \int_0^\infty \frac{\lambda^{-\alpha/2} x^{-\alpha/2} - \lambda^{-\alpha/2} z^{-\alpha/2}}{|\lambda x - \lambda z|^{1+\alpha}} \lambda \, dz \\
&= \lambda^{-3\alpha/2} \int_0^\infty \frac{x^{-\alpha/2} - z^{-\alpha/2}}{|x - z|^{1+\alpha}} \, dz \\
&= \lambda^{-3\alpha/2} F(x)
\end{aligned}
$$

for all $\lambda > 0$ and all $x > 0$. This scaling property of F immediately implies that there exists constant C such that

$$
F(x) = C \cdot x^{-3\alpha/2}.
$$

To relate $F(x)$ to our integral B, we notice that

$$
\begin{aligned}
F(1+x) &= \int_0^\infty \frac{(1+x)^{-\alpha/2} - y^{-\alpha/2}}{|(1+x) - y|^{1+\alpha}} \, dy \quad \text{(by definition of } F) \\
&= \int_{-1}^\infty \frac{(1+x)^{-\alpha/2} - (1+z)^{-\alpha/2}}{|x - z|^{1+\alpha}} \, dz \quad (y = 1+z) \\
&= B.
\end{aligned}
$$

Hence

$$
B = F(1+x) \sim C \cdot x^{-3\alpha/2} = o(x^{-\alpha}).
$$

Thus,

$$
-(-\Delta)^{\alpha/2} w(x) = -A - B = \frac{1}{\alpha} \cdot x^{-\alpha} + o(x^{-\alpha}),
$$

verifying (29).

Therefore, there exist positive constants C_1' and C_2' such that $C_1' w(x)$ is a subsolution to (21) and $C_2' w(x)$ is a supersolution to (21). By Proposition 8, $C_1' w(x) \le u(x) \le C_2' w(x)$ for all sufficiently large x. Since $w(x) \sim x^{-\alpha/2}$ as $x \to \infty$, there are positive constants C_1 and C_2 such that $C_1 x^{-\alpha/2} \le u(x) \le C_2 x^{-\alpha/2}$ for all sufficiently large x, proving (26). $\qquad\square$

5. Proof of Theorem 1

5.1. *Feynman–Kac representation of solutions*

Our approach to Theorem 1 will rely on an analogue of the Feynman–Kac formula for solutions to pseudo-differential equations of the form $-(-\Delta)^{\alpha/2} v(x) = q(x) v(x)$. The operator $-(-\Delta)^{\alpha/2}$ is the infinitesimal generator of the symmetric α-stable process, and hence the Feynman–Kac representations will be functional integrals with respect to paths X_t of the symmetric α-stable process. Denote by P^x and E^x the probability and expectation operators under which the initial point of the process is $X_0 = x$, and recall that τ_0^- is the first-passage time to the half-line $(-\infty, 0)$.

Theorem 9 (Feynman–Kac Formula). *Let $v : \mathbb{R} \to \mathbb{R}$ be a bounded, continuous solution of*

$$
-(-\Delta)^{\alpha/2} v(x) = q(x) v(x) \quad \text{for all } x > 0, \tag{30}
$$

where $q(x)$ is a nonnegative and continuous. Then

$$
Z_t = \exp\left(-\int_0^{t \wedge \tau_0^-} q(X_s) \, ds \right) \cdot v(X_{t \wedge \tau_0})
$$

is a bounded martingale with respect to the filtration $\{\mathcal{F}^x_{t \wedge \tau_0^-}\}_{t \geq 0}$. *Consequently, by the Optional Stopping Theorem, for any stopping time* $\tau \leq \tau_0^-$,

$$v(x) = E^x \left[\exp\left(-\int_0^\tau q(X_s)\, ds \right) \cdot v(X_\tau) \right]. \tag{31}$$

The Feynman–Kac formula has been proved to hold for arbitrary Markov processes satisfying the Feller property – see, for instance, [17], sec. III.19 for a formal development and [10] for some of the history of the formula. Theorem 9 is a special case of the results of [17], sec. III.19, as the symmetric α-stable process is Feller.

Corollary 10. *If* $u(x)$ *is a solution to the boundary value problem* (21), *then for any stopping time* $\tau \leq \tau_0^-$,

$$u(x) = E^x \left[\exp\left(-\frac{1}{2} \int_0^\tau u(X_s)\, ds \right) \cdot u(X_\tau) \right]. \tag{32}$$

In particular,

$$u(x) = E^x \left[\exp\left(-\frac{1}{2} \int_0^{\tau_0^-} u(X_s)\, ds \right) \right]. \tag{33}$$

Proof. The representation (32) directly follows from Theorem 9 by setting $q(x) = \frac{1}{2}u(x)$. The result (33) follows from setting $\tau = \tau^-(0)$, since $u(X_{\tau_0}) = 1$. $\qquad \square$

5.2. *Consequences of the Feynman–Kac formula*

Formula (33) restricts the decay of $u(x)$ both above and below, because the function u appears on both sides of (33) but with opposite signs. When combined with the a priori estimates of Proposition 7, the integral representation (33) will lead to sharp asymptotic estimates, as we will show in Section 5.3. In this section we collect some preliminary consequences of the representation (33). Henceforth, we will use the notational shorthand

$$\Psi_t = \int_0^t u(X_s)\, ds \tag{34}$$

for the path integrals that occur in the Feynman–Kac formulas. Recall that for any interval J the random variable σ_J is the time of first exit from J, and ν_J is the time of the first jump of size $X_t - X_{t-} \in J$.

Proposition 11. *Fix* $\delta \in (0, \frac{1}{2})$ *and abbreviate* $\sigma = \sigma_{(x-\delta x, x+\delta x)}$. *For all sufficiently small* δ, *as* $x \to \infty$,

$$E^x \exp\{-\Psi_\sigma/2\} u(X_\sigma) \mathbb{1}\{X_\sigma \geq \delta x\} = o\big(u(x)\big), \tag{35}$$

and consequently,

$$E^x \exp\{-\Psi_\sigma/2\} \mathbb{1}\{X_\sigma \geq \delta x\} = o(1). \tag{36}$$

Proof. The monotonicity of u and the a priori bounds (26) imply that the ratio $u(X_\sigma)/u(x)$ remains bounded above on the event $X_\sigma \geq \delta x$ by a constant $C = C_\delta < \infty$ depending on $\delta > 0$ but not on x. Hence, the second relation (36) will follow from the first relation (35). Now consider the exponential $\exp\{-\Psi_\sigma/2\}$. The integrand $u(X_s)$ is bounded below by $u(x + \delta x)$ up to time σ, by the monotonicity of u, and so by the a priori bounds (26), with $C = C_\delta$ as above,

$$\Psi_\sigma \geq C\sigma x^{-\alpha/2}.$$

Hence, on the event $\sigma > x^{\alpha/2+\eta}$ the exponential $e^{-\Psi_\sigma/2}$ will be bounded above by $\exp\{-Cx^\eta\} = o(u(x))$. On the other hand, the scaling law (4) implies that the distribution of σ/x^α under P^x is the same as that of σ under P^1, so as

$x \to \infty$ the probability that $\sigma \le x^{\alpha/2+\eta}$ converges to zero, for any $\eta < \alpha/2$. Thus,

$$
\begin{aligned}
E^x e^{-\Psi_\sigma/2} u(X_\sigma) \mathbb{1}\{X_\sigma \ge \delta x\} &\le C u(x) E^x e^{-\Psi_\sigma/2} \\
&\le C u(x) E^x e^{-\Psi_\sigma/2} \big(\mathbb{1}\{\sigma \ge x^{\alpha/2+\eta}\} + \mathbb{1}\{\sigma < x^{\alpha/2+\eta}\}\big) \\
&= C u(x) \big(o(1) + o(1)\big). \qquad\qquad\qquad\qquad\qquad \square
\end{aligned}
$$

Proposition 11 implies that for large x the expectation in the Feynman–Kac formula (32), with $\tau = \sigma$, is dominated by those sample paths that exit the interval $(x - \delta x, x + \delta x)$ by jumping to the interval $(-\infty, \delta x)$. The next result asserts that the relative contribution from those paths for which the jump lands somewhere in $(-2\delta x, 2\delta x)$ is small.

Proposition 12. *For each $\varepsilon > 0$ there exists $\delta > 0$ such that for all sufficiently large x*

$$
E^x \exp\{-\Psi_\sigma/2\}\mathbb{1}\big\{X_\sigma \in [-2\delta x, 2\delta x]\big\} \le \varepsilon u(x), \tag{37}
$$

where $\sigma = \sigma_{(x-\delta x, x+\delta x)}$.

Proof. If $\delta < 1/4$ then the event $X_\sigma \in [-2\delta x, 2\delta x]$ can only occur if there is a jump of size $\Delta = X_\sigma - X_{\sigma-} < -x + 3\delta x$ at time σ. Moreover, because $X_{\sigma-} \in [x - \delta x, x + \delta x]$, the jump must be the *first* jump of magnitude more than $2\delta x$, and so $\sigma = \nu$, where $\nu = \nu_{(-\infty, -x+3\delta x)}$. In order that $X_\sigma \in [-2\delta x, 2\delta x]$, the size of the jump must satisfy

$$
\Delta \in [-x - 4\delta x, -x + 4\delta x].
$$

Similarly, if at time σ the process X_s makes a jump of size $\Delta < -x - \delta x$ then $X_\sigma < 0$ and so $\sigma = \tau^-(0)$.

By Lemma 4, the random variable Δ is independent of the σ-algebra $\mathcal{F}_{\nu-}$ under P^x, and furthermore the distribution of Δ is

$$
P^x\{\Delta \le -x - tx\} = \frac{\lambda(-\infty, -x - tx)}{\lambda(-\infty, -x + 2\delta x)} = \left(\frac{1 - 3\delta}{1 + t}\right)^\alpha.
$$

Hence, since $\tau^-(0) = \nu$ on the event $\{\sigma = \nu\} \cap \{\Delta \le -x - \delta x\}$, the Feynman–Kac formula (33) implies that

$$
\begin{aligned}
u(x) &\ge E^x \big(\exp\{-\Psi_\sigma/2\}\mathbb{1}\{\sigma = \nu\}\mathbb{1}\{\Delta \le -x - \delta x\}\big) \\
&= E^x \exp\{-\Psi_\sigma/2\}\mathbb{1}\{\sigma = \nu\}) \times P^x\{\Delta \le -x - \delta x\}.
\end{aligned}
$$

But the independence of Δ and $\mathcal{F}_{\nu-}$ also implies that

$$
\begin{aligned}
&E^x \big(\exp\{-\Psi_\sigma/2\}\mathbb{1}\{\sigma = \nu\}\mathbb{1}\big\{\Delta \in [-x - 4\delta x, -x + 4\delta x]\big\}\big) \\
&= E^x \exp\{-\Psi_\sigma/2\}\mathbb{1}\{\sigma = \nu\} \times P^x\big\{\Delta \in [-x - 4\delta x, -x + 4\delta x]\big\},
\end{aligned}
$$

since Ψ_ν is measurable with respect to $\mathcal{F}_{\nu-}$, so it now follows that

$$
\begin{aligned}
&E^x (\exp\{-\Psi_\sigma/2\}\mathbb{1}\big\{X_\sigma \in [-2\delta x, 2\delta x] \text{ and } \sigma = \nu\big\} \\
&\le u(x) \cdot \frac{P^x\{\Delta \in [-x - 4\delta x, -x + 4\delta x]\}}{P^x\{\Delta \le -x - \delta x\}}.
\end{aligned}
$$

The ratio of the two probabilities on the right is $O(\delta)$ (uniformly in x, by scaling), so this proves (37). $\qquad \square$

Proposition 13. *For any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \ge 1$,*

$$
1 - \varepsilon \le \frac{u((1 + \delta)x)}{u(x)} < 1. \tag{38}
$$

Proof. The law of the symmetric α-stable process X_s under $P^{x+\delta x}$ is the same as that of $X_s + \delta x$ under P^x, and the first passage time $\tau^-(\delta x)$ under $P^{x+\delta x}$ is the same as that of $\tau^-(0)$ under P^x. Hence, by the Feynman–Kac formula (32),

$$\frac{u((1+\delta)x)}{u(x)} = \frac{E^{(1+\delta)x}[\exp\{-\Psi_{\tau^-(\delta x)}/2\} \cdot u(X_{\tau^-(\delta x)})]}{u(x)}$$

$$= \frac{E^x[\exp(-(1/2)\int_0^{\tau^-(0)} u(X_s + \delta x)\,ds) \cdot u(X_{\tau^-(0)} + \delta x)]}{u(x)}$$

$$\geq \frac{E^x[\exp(-(1/2)\int_0^{\tau^-(0)} u(X_s)\,ds) \cdot \mathbb{1}\{X_{\tau^-(0)} \leq -\delta x\}]}{E^x[\exp(-(1/2)\int_0^{\tau^-(0)} u(X_s)\,ds)]}.$$

Thus, to prove Proposition 13 it suffices to show that for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$E^x\big[\exp\{-\Psi_{\tau^-(0)}/2\} \cdot \mathbb{1}\big\{X_{\tau^-(0)} \in (-\delta x, 0]\big\}\big] \leq \varepsilon u(x)$$

for all sufficiently large x. But this follows directly from Proposition 12. $\qquad\square$

Propositions 11–12 imply that the expectation in the Feynman–Kac formula (33) is dominated by paths X_s for which the first escape from $[x - \delta x, x + \delta x]$ coincides with the first jump of size $< -x + 2\delta x$. On this event, the stopping time τ_0^- coincides with the time σ of first exit from $[x - \delta x, x + \delta x]$ and with the time ν of the first jump of size $< -x + 2\delta x$. The importance of Proposition 13 is that it guarantees that on this event the value of $\Psi_{\tau^-(0)}$ is nearly the same as $\nu u(x)$. Thus, it is not unreasonable to hope that the Feynman–Kac expectation (33) should be well-approximated by $E^x \exp\{-\nu u(x)/2\}$. Since ν is the first occurrence time in a Poisson process of rate $\lambda[x - 2\delta x, \infty) = Cx^{-\alpha}$, where $C = (1 - 2\delta)^{-\alpha}/\alpha$, it is exponentially distributed and so the latter expectation can be evaluated exactly:

$$E^x e^{-\nu u(x)/2} = C/\big(C + x^\alpha u(x)/2\big). \tag{39}$$

Given this, Theorem 1 will follow, because together with the Feynman–Kac formula it leads to the limiting relation

$$u(x) \sim C/\big(C + x^\alpha u(x)/2\big),$$

from which (1) can be easily deduced, using the fact that $\delta > 0$ can be chosen arbitrarily small.

To justify the replacement of the Feynman–Kac expectation (33) by the expectation $E^x \exp\{-\nu u(x)/2\}$, we must verify that the contribution to this last expectation from paths for which $\tau^-(0) = \sigma = \nu$ does not hold is small.

Proposition 14. *Fix $\delta > 0$ and let σ be the time of first exit from $(x - \delta x, x + \delta x)$ and ν the time of the first jump of size $< -x + 2\delta x$. For any $\varepsilon > 0$, if $\delta > 0$ is sufficiently small then as $x \to \infty$,*

$$E^x \exp\big\{-(1+\varepsilon)\nu u(x)/2\big\}\mathbb{1}\{\sigma \neq \nu\} \leq o\big(u(x)\big). \tag{40}$$

Proof. The event $\{\sigma \neq \nu\}$ can occur only if $\sigma < \nu$ and $X_\sigma \geq \delta x$. Moreover, by the strong Markov property for the underlying Poisson point process, the conditional distribution of the residual waiting time $\nu - \sigma$ given $\mathcal{F}_\sigma$ on the event $\{\sigma \neq \nu\}$ is the same as the *unconditional* distribution of ν, and so

$$E^x e^{-(1+\varepsilon)\nu u(x)/2}\mathbb{1}\{\sigma \neq \nu\} \leq E^x e^{-(1+\varepsilon)(\nu-\sigma)u(x)/2}e^{-(1+\varepsilon)\sigma u(x)/2}\mathbb{1}\{\sigma \neq \nu\}$$

$$= E^x e^{-(1+\varepsilon)\nu u(x)/2} E^x e^{-(1+\varepsilon)\sigma u(x)/2}\mathbb{1}\{\sigma \neq \nu\}$$

$$\leq E^x e^{-(1+\varepsilon)\nu u(x)/2} E^x e^{-\Psi_\sigma/2}\mathbb{1}\{\sigma \neq \nu\}$$

$$\leq E^x e^{-(1+\varepsilon)\nu u(x)/2} E^x e^{-\Psi_\sigma/2}\mathbb{1}\{X_\sigma \geq \delta x\}.$$

(The third inequality holds by Proposition 13, provided $\delta > 0$ is sufficiently small and $x \geq 1$.) Hence, by Proposition 11, as $x \to \infty$,

$$E^x e^{-(1+\varepsilon)vu(x)/2} \mathbb{1}\{\sigma \neq v\} \leq o\big(E^x e^{-(1+\varepsilon)vu(x)/2}\big).$$

Thus, to complete the proof we need only show that

$$E^x e^{-(1+\varepsilon)vu(x)/2} = O\big(u(x)\big).$$

But this follows routinely from the fact that v is exponentially distributed with rate $\lambda[x - 2\delta x, \infty) = Cx^{-\alpha}$, where $C = (1 - 2\delta)^{-\alpha}/\alpha$:

$$E^x e^{-(1+\varepsilon)vu(x)/2} = C/\big(C + (1+\varepsilon)x^\alpha u(x)\big).$$

The a priori estimates (26) now yield the desired conclusion.   □

5.3. Proof of Theorem 1

Fix $\delta > 0$ small and write $\sigma = \sigma_{(x-\delta x, x+\delta x)}$ for the first exit time from the interval $(x - \delta x, x + \delta x)$ and $v = v_{(-\infty, -x+2\delta x)}$ for the time of the first discontinuity of size $< -x + 2\delta x$. By Propositions 11–12, for any $\varepsilon > 0$ there exists $\delta > 0$ so small that

$$u(x) = E^x \exp\{-\Psi_{\tau^-(0)}/2\}$$
$$\leq (1-\varepsilon)^{-1} E^x \exp\{-\Psi_\sigma/2\}\mathbb{1}\big\{\tau^-(0) = \sigma = v\big\}.$$

On the event $\{\sigma = v = \tau^-(0)\}$, the path X_s remains in the interval $(x - \delta x, x + \delta x)$ up to time $\tau^-(0)$, so by Proposition 13 the path integral in the exponential is approximately $vu(x)$: more precisely, for any $\varepsilon > 0$ there exists $\delta > 0$ so small that

$$u(x) \leq (1+\varepsilon) E^x \exp\big\{-(1-\varepsilon)vu(x)/2\big\}\mathbb{1}\big\{\tau^-(0) = \sigma = v\big\}$$
$$\leq (1+\varepsilon) E^x \exp\big\{-(1-\varepsilon)vu(x)/2\big\} \quad \text{and} \tag{41}$$
$$u(x) \geq (1-\varepsilon) E^x \exp\big\{-(1+\varepsilon)vu(x)/2\big\}\mathbb{1}\big\{\tau^-(0) = \sigma = v\big\}.$$

The expectation in the upper bound can be evaluated exactly, as in the proof of Proposition 14, using the fact that v is exponentially distributed. This yields the inequality

$$u(x) \leq \frac{(1+\varepsilon)x^{-\alpha}(1-2\delta)^{-\alpha}/\alpha}{(1-\varepsilon)u(x)/2 + x^{-\alpha}(1-2\delta)^{-\alpha}/\alpha}. \tag{42}$$

To obtain a usable lower bound from the last inequality in (41) we use Proposition 14. The complement of the event $\{\tau^-(0) = \sigma = v\}$ is contained in the union of $\{\sigma \neq v\}$ with $\{X_\sigma \in (-2\delta x, 2\delta x)\}$. Proposition 14 implies that as $x \to \infty$,

$$E^x \exp\big\{-(1+\varepsilon)vu(x)/2\big\}\mathbb{1}\{\sigma \neq v\} = o\big(u(x)\big),$$

while Propositions 13 and 12 imply that for sufficiently small $\delta > 0$, if x is large then

$$E^x \exp\big\{-(1+\varepsilon)vu(x)/2\big\}\mathbb{1}\big\{X_\sigma \in (-2\delta x, 2\delta x)\big\}$$
$$\leq E^x \exp\{-\Psi_\sigma/2\}\mathbb{1}\big\{X_\sigma \in (-2\delta x, 2\delta x)\big\}$$
$$\leq \varepsilon u(x).$$

Consequently, for sufficiently small $\delta > 0$ and large x,

$$E^x \exp\{-(1+\varepsilon)\nu u(x)/2\}\mathbb{1}\{\tau^-(0) = \sigma = \nu\}$$
$$\geq E^x \exp\{-(1+\varepsilon)\nu u(x)/2\} - 2\varepsilon u(x).$$

The last expectation can now be evaluated, using once again the fact that ν is exponentially distributed; this gives the lower bound

$$u(x) \geq \left(\frac{1}{1-2\varepsilon}\right) \frac{x^{-\alpha}(1-2\delta)^{-\alpha}/\alpha}{(1+\varepsilon)u(x)/2 + x^{-\alpha}(1-2\delta)^{-\alpha}/\alpha}. \tag{43}$$

Since $\varepsilon > 0$ and $\delta > 0$ can be made arbitrarily small, it now follows from the inequalities (42)–(43) that

$$\lim_{x \to \infty} x^{\alpha}\big(u(x)\big)^2 = \frac{2}{\alpha}.$$

Therefore,

$$u(x) \sim \sqrt{\frac{2}{\alpha}} \frac{1}{x^{\alpha/2}}. \qquad \square$$

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