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Slowdown in branching Brownian motion with inhomogeneous variance

Pascal Maillard[a,1] and Ofer Zeitouni[b,2]

[a] *Département de Mathématiques, Université Paris-Sud, 91405 Orsay Cedex, France.*
[b] *Department of Mathematics, The Weizmann Institute of Science, POB 26, Rehovot 76100, Israel and Courant Institute, New-York University.*
E-mail: fer.zeitouni@weizmann.ac.il

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Abstract. We consider the distribution of the maximum M_T of branching Brownian motion with time-inhomogeneous variance of the form $\sigma^2(t/T)$, where $\sigma(\cdot)$ is a strictly decreasing function. This corresponds to the study of the time-inhomogeneous Fisher–Kolmogorov–Petrovskii–Piskunov (F-KPP) equation $F_t(x,t) = \sigma^2(1-t/T)F_{xx}(x,t)/2 + g(F(x,t))$, for appropriate nonlinearities $g(\cdot)$. Fang and Zeitouni (*J. Stat. Phys.* **149** (2012) 1–9) showed that $M_T - v_\sigma T$ is negative of order $T^{1/3}$, where $v_\sigma = \int_0^1 \sigma(s)\,\mathrm{d}s$. In this paper, we show the existence of a function m'_T, such that $M_T - m'_T$ converges in law, as $T \to \infty$. Furthermore, $m'_T = v_\sigma T - w_\sigma T^{1/3} - \sigma(1)\log T + O(1)$ with $w_\sigma = 2^{-1/3}\alpha_1 \int_0^1 \sigma(s)^{1/3}|\sigma'(s)|^{2/3}\,\mathrm{d}s$. Here, $-\alpha_1 = -2.33811\ldots$ is the largest zero of the Airy function Ai. The proof uses a mixture of probabilistic and analytic arguments.

Résumé. Nous étudions la loi du maximum M_T d'un mouvement brownien branchant avec une variance inhomogène en temps de la form $\sigma^2(t/T)$, où $\sigma(\cdot)$ est une fonction strictement décroissante. Ceci correspond à étudier l'équation Fisher–Kolmogorov–Petrovskii–Piskunov (F-KPP) inhomogène en temps, $F_t(x,t) = \sigma^2(1-t/T)F_{xx}(x,t)/2 + g(F(x,t))$, pour des nonlinéarités $g(\cdot)$ appropriées. Fang et Zeitouni (*J. Stat. Phys.* **149** (2012) 1–9) ont montré que $M_T - v_\sigma T$ est negatif de l'ordre $T^{1/3}$, où $v_\sigma = \int_0^1 \sigma(s)\,\mathrm{d}s$. Dans cet article, nous montrons l'existence d'une fonction m'_T telle que $M_T - m'_T$ converge en loi quand $T \to \infty$. De plus, $m'_T = v_\sigma T - w_\sigma T^{1/3} - \sigma(1)\log T + O(1)$ avec $w_\sigma = 2^{-1/3}\alpha_1 \int_0^1 \sigma(s)^{1/3}|\sigma'(s)|^{2/3}\,\mathrm{d}s$. Ici, $-\alpha_1 = -2.33811\ldots$ est la plus grande racine de la fonction d'Airy Ai. La démonstration repose sur un mélange d'arguments probabilistes et analytiques.

1. Introduction

The classical branching Brownian motion (BBM) model in $\mathbb{R}$ can be described probabilistically as follows. Fix a law μ of finite variance on $[2,\infty) \cap \mathbb{Z}$. At time $t = 0$, one particle exists and is located at the origin. This particle starts performing standard Brownian motion on the real line, up to an exponentially distributed random time, with parameter $\beta_0 = (2(\mathbf{E}_\mu[L]-1))^{-1}$ (i.e., branching occurs at rate β_0). At that time, the particle instantaneously splits into a random number $L \geq 2$ of independent particles, and those start afresh performing Brownian motion until their (independent) exponential clocks ring. There is an extensive literature on this model and its discrete analog, the branching random walk, in particular concerning the position of the right-most particle (see, e.g., [1,4,5,8,18,22]). In order to state the main result, introduce the F–KPP travelling wave equation

$$\phi : \mathbb{R} \to (0,1) \text{ increasing}, \qquad \frac{1}{2}\phi'' + \phi' + \beta_0\bigl(\mathbf{E}_\mu\bigl[\phi^L\bigr] - \phi\bigr) = 0, \qquad \phi(-\infty) = 0, \qquad \phi(+\infty) = 1. \tag{1.1}$$

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One has the following theorem:

Theorem (Bramson [5]). *Let M_t denote the position of the right-most particle at time t in branching Brownian motion as defined above. Then there exists a solution ϕ to (1.1), such that for all $x \in \mathbb{R}$,*

$$\mathbf{P}\left(M_t \le t - \frac{3}{2}\log t + x\right) \to \phi(x), \quad \text{as } t \to \infty.$$

We discuss in this paper a variant of the BBM model, first introduced in [8], where the motion of the particle(s) is controlled by a time-inhomogeneous variance. More precisely, let $\sigma \in C^2([0,1])$ be a strictly decreasing function with $\sigma(1) > 0$ and $\inf_{t\in[0,1]} |\sigma'(t)| > 0$. We assume that the variance of the Brownian motions at time $t \in [0,T]$ is given by $\sigma^2(t/T)$.

Let $N(t)$, $t \in [0,T]$ denote the collection of particles alive at time t and for any particle $v \in N(t)$, let $X_v(s)$, $s \in [0,1]$ denote the trajectory performed by the particle and its ancestors. Then $M_t = \max_{u\in N(t)} X_u(t)$ denotes the location of the rightmost particle at time t. The cumulative distribution function of M_T is $F(\cdot, T)$, where $F(x,t)$ is the solution of the time-inhomogeneous Fisher–Kolmogorov–Petrovskii–Piskunov (F-KPP) equation

$$\begin{aligned} &\frac{\partial F}{\partial t}(x,t) = \frac{\sigma^2(1-t/T)}{2}\frac{\partial^2 F}{\partial^2 x}(x,t) + \beta_0\left(\mathbf{E}_\mu\left[F(x,t)^L\right] - F(x,t)\right), \quad t \in [0,T], x \in \mathbb{R}, \\ &F(x,0) = \mathbf{1}_{x\ge 0}. \end{aligned} \tag{1.2}$$

See [18] for this probabilistic interpretation of the F-KPP equation in the time homogeneous case.

In [10], the authors prove the following.

Theorem (Fang and Zeitouni [10]). *There exist constants $C, C' > 0$ so that*

$$-C \le \liminf_{T\to\infty} \frac{M_T - v_\sigma T}{T^{1/3}} \le \limsup_{T\to\infty} \frac{M_T - v_\sigma T}{T^{1/3}} \le -C' < 0, \tag{1.3}$$

where $v_\sigma = \int_0^1 \sigma(s)\,\mathrm{d}s$.

(The derivation in [10] is for the case that $P(L=2)=1$, but applies with no changes to the current setup. The linear in T asymptotics, i.e., the speed v_σ, can be read off with some effort from the results in [8] and [3].)

Our goal in this paper is to significantly refine Theorem 1. To state our results, introduce the functions $v, w : [0,1] \to \mathbb{R}_+$ by

$$v(t) = \int_0^t \sigma(s)\,\mathrm{d}s, \tag{1.4}$$

and

$$w(t) = 2^{-1/3}\alpha_1 \int_0^t \sigma(s)^{1/3}\left|\sigma'(s)\right|^{2/3}\mathrm{d}s, \tag{1.5}$$

where $-\alpha_1 = -2.33811\ldots$ is the largest zero of the *Airy function of the first kind*

$$\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{t^3}{3} + xt\right)\mathrm{d}t, \tag{1.6}$$

see [2], Section 10.4, for definitions; note that Ai satisfies the Airy differential equation $\mathrm{Ai}''(x) - x\,\mathrm{Ai}(x) = 0$. Note also that $v_\sigma = v(1)$. Set

$$m_T = v(1)T - w(1)T^{1/3} - \sigma(1)\log T.$$

Our main result is the following.

Theorem 1.1. *The family of random variables* $(M_T - m_T)_{T\geq 0}$ *is tight. Further, there exists a solution* $\phi(x)$ *to* (1.1) *and a function* m'_T *with* $C_\sigma = \limsup_{T\geq 0} |m'_T - m_T| < \infty$, *such that for all* $x \in \mathbb{R}$,

$$\lim_{T\to\infty} \mathbf{P}\bigl(M_T \leq m'_T + x\bigr) = \phi\bigl(x/\sigma(0)\bigr).$$

Furthermore, for a fixed travelling wave ϕ, *the constant* C_σ *above is uniformly bounded for*

$$\sigma \in \Bigl\{\sigma \in C^2\bigl([0,1]\bigr) : \sigma(0) + 1/\sigma(1) < c_0,\ \sup_{t\in[0,1]} \bigl|\sigma''(t)\bigr| < c_0,\ \inf_{t\in[0,1]} \bigl|\sigma'(t)\bigr| > 1/c_0\Bigr\} =: \Xi_{c_0}.$$

Parallel to our work, and an inspiration to it, was the study [20], by PDE techniques, of a class of time-inhomogeneous F-KPP equations that includes (1.2). Compared with [20], we deal with a slightly restricted class of equations, but are able to obtain finer (up to order 1) asymptotics and convergence to a travelling wave. We hope that our techniques can be pushed to yield convergence in distribution of the family $(M_T - m_T)_{T\geq 0}$ (instead of $(M_T - m'_T)_{T\geq 0}$), in parallel with the recent results in [6], but this requires significant changes in the approach of [6] (mainly, because unlike in the time-homogeneous case, extremal particles at time T will, with positive probability, be extremal at some random intermediate time between ϵT and $(1-\epsilon)T$). We therefore leave the adaptation for possible future work.

We remark that Mallein [17] has recently published results similar to ours which are less precise but hold for a rather general class of (not necessarily Gaussian) time-inhomogeneous branching random walks.

The core of the proof of Theorem 1.1 is based on a constrained first and second moment analysis of the number of particles that reach a target value but remain below a barrier for the duration of their lifetime. Due to the time-inhomogeneity of $\sigma(\cdot)$, the choice of barrier is not straight-forward, and in particular it is not a straight line; "rectifying" it introduces a killing potential. The analysis of the survival of Brownian motion in this potential eventually leads to a time-inhomogeneous Airy-type differential equation which we study by analytic means, exploiting the anti-symmetry of the differential operator. (As pointed out to us by Dima Ioffe, a similar phenomenon with related $T^{1/3}$ scaling was already observed in [11,13].) These methods together lead to estimates of the right tail of M_T which are sharp up to a multiplicative factor (Proposition 3.1). By a bootstrapping procedure that may be of independent interest, these estimates are then turned into convergence in law by using a convergence result for the derivative Gibbs measure of (time-homogeneous) branching Brownian motion.

The structure of the paper is as follows. In the next section, we introduce a barrier $\gamma_T(\cdot)$, and show that with high probability, no particle crosses (a shifted version of) the barrier, see Lemma 2.1. Using the barrier, we then control the distribution of extremal particles at all times large enough (Lemma 2.2). In these lemmas, results concerning time-inhomogeneous Airy-type PDE's are needed, and the proof of those is given in Appendix A. Section 3 combines the results of Section 2 (taken at time $T - T^{2/3}$) together with an analysis of the last segment of time of length $T^{2/3}$, and provides the first-and-second moment results needed to obtain lower and upper bound on the right tail of M_T. The proof of Theorem 1.1 is then completed in Section 4, using a result about the convergence of the derivative Gibbs measure of (time-homogeneous) branching Brownian motion, which is given in Appendix B.

Notation. *In the rest of this article* (*except in the appendix*), *the symbols* C, C', C_1, C_2 *etc. stand for positive constants, possibly depending on* c_0 (*see Theorem* 1.1), *whose values may change from line to line. The phrase "*X *holds for large* T*" means that there exists* T_0, *possibly depending on* c_0, *such that* X *holds for* $T \geq T_0$ *for all* $\sigma \in \Xi_{c_0}$. *We further use the Landau symbols* $O(\cdot)$ *and* $o(\cdot)$, *which are always to be interpreted with respect to* $T \to \infty$, *and which may depend on* c_0 *as well. Finally, the symbols* $\mathbf{P}$ *and* $\mathbf{E}$ (*possibly with sub-/superscripts*) *always stand for the law of a branching Markov process* (*branching Brownian motion with time-varying or constant variance and with or without absorption of particles*) *and the expectation with respect to this law. On this other hand, the symbols* P *and* E *are used for probability and expectation with respect to a single particle* (*i.e., a Markov process, usually a Brownian motion with time-varying or constant variance or a three-dimensional Bessel process*). *The location of the initial particle is denoted by a subscript, e.g.,* $\mathbf{P}_x$, *without a subscript the initial particle is implicitly located at the origin.*

2. Crossing estimates

Fix T. Define the curve $\gamma_T : [0,T] \to \mathbb{R}$ by

$$\gamma_T(t) = Tv(t/T) - T^{1/3}w(t/T).$$

Introduce the constant

$$\kappa := 8/\sigma^2(1). \tag{2.1}$$

In this section we prove two lemmas. The first lemma bounds, for any fixed $K \ge 1$, the probability that there exists a particle that reaches the curve $\gamma_T(t) + K$. The second lemma estimates the expected number of particles that have stayed below the curve up to time t, and reach a given terminal value at time t.

Lemma 2.1. *There exists a constant $C = C(c_0)$, such that for large T, for any $\sigma \in \Xi_{c_0}$ and every $K \in [1, T^{1/3}]$,*

$$\mathbf{P}\Big(\exists t \in [0,T] : \max_{u \in N(t)} X_u(t) \ge \gamma_T(t) + K\Big) \le CKe^{-K/\sigma(0)}.$$

Proof. The proof goes by a first moment estimate of the number of particles hitting the curve $\gamma_T + K$. For an interval $I \subset [0,T]$, let R_I be the number of particles hitting the curve $\gamma_T + K$ for the first time during the interval I. Let B_t be a Brownian motion with variance $\sigma^2(t/T)$ started from the point x under P_x (see the remarks on notation in the introduction). For a path $(X_t)_{t \ge 0}$, define $H_0(X) = \inf\{t \ge 0 : X_t = 0\}$. By the first moment formula[1] for branching Markov processes ([14], Theorem 4.1) (also known as "Many-to-one lemma") we then have (taking $x = K$)

$$\mathbf{E}[R_I] = E_0\big[e^{H_0(\gamma_T + K - B)/2} \mathbb{1}_{H_0(\gamma_T + K - B) \in I}\big] = E_K\big[e^{H_0(B+\gamma_T)/2} \mathbb{1}_{H_0(B+\gamma_T)\in I}\big],$$

where the second equality follows from the fact that the law of $K - B_t$ under P_0 is equal to the law of B_t under P_K by symmetry. Applying Girsanov's theorem we get that

$$\begin{aligned}\mathbf{E}[R_I] &= E_K\Bigg[\exp\bigg(\int_0^{H_0(B)} \frac{\gamma_T'(t)}{\sigma^2(t/T)}\,\mathrm{d}B_t + \frac{H_0(B)}{2} - \int_0^{H_0(B)} \frac{(\gamma_T'(t))^2}{2\sigma^2(t/T)}\,\mathrm{d}t\bigg)\mathbb{1}_{H_0(B)\in I}\Bigg] \\ &= e^{-K\gamma_T'(0)/\sigma^2(0)+o(1)} E_K\Bigg[\exp\bigg(\frac{1}{T}\int_0^{H_0(B)}\bigg(-q_T(t/T)B_t + T^{1/3}\frac{w'(t/T)}{\sigma(t/T)}\bigg)\,\mathrm{d}t\bigg)\mathbb{1}_{H_0(B)\in I}\Bigg],\end{aligned}$$

where the last equation follows by integration by parts and the function $q_T : [0,1] \to \mathbb{R}$ is defined by

$$q_T(t) = \frac{|\sigma'(t)|}{\sigma^2(t)} + T^{-2/3}\big(w'/\sigma^2\big)'(t).$$

For large T, this yields by (1.5) and the assumptions on σ and K,

$$\begin{aligned}\mathbf{E}[R_I] = e^{-K/\sigma(0)+o(1)} E_K\Bigg[\exp\bigg(\frac{1}{T}\int_0^{H_0(B)}\bigg\{&-q_T(t/T)B_t \\ &+ \alpha_1 q_T(t)^{2/3}\bigg(\frac{1}{2}\sigma^2(t/T)\bigg)^{1/3} T^{1/3}\bigg\}\,\mathrm{d}t\bigg)\mathbb{1}_{H_0(B)\in I}\Bigg].\end{aligned} \tag{2.2}$$

Set

$$J(t) = \int_0^t \frac{1}{2}\sigma(s)^2\,\mathrm{d}s. \tag{2.3}$$

[1] Note that due to our choice of the branching rate β_0, the expected number of particles in the system at the time t is $\mathbf{E}[N(t)] = e^{t/2}$, which is the reason for the exponential term arising in the formula.

Recall the constant κ from (2.1). We will bound separately $\mathbf{E}[R_{[0,\kappa T^{2/3}]}]$ and $\mathbf{E}[R_{[\kappa T^{2/3},T]}]$. For the first term, (2.2) immediately gives

$$\mathbf{E}[R_{[0,\kappa T^{2/3}]}] \le Ce^{-K/\sigma(0)}, \tag{2.4}$$

because under P_K, B_t is positive until the time $H_0(B)$ and the factor in front of $T^{1/3}$ in the integral in (2.2) is bounded by a constant C. In order to bound $\mathbf{E}[R_{\kappa T^{2/3},T]}]$, we note that the expectation on the right-hand side of (2.2) equals

$$\int_I \frac{1}{2}\sigma^2(t/T)\frac{\mathrm{d}G(K,y;t)}{\mathrm{d}y}\bigg|_{y=0}\mathrm{d}t, \tag{2.5}$$

where $G(x,y;t)$ is the fundamental solution to the PDE (A.4), with $Q(t) = q_T(J(t/T))$ (see [12], Sections 5.2.1 and 5.2.8, for an elementary, but somewhat non-rigorous proof of this fact, and [7], Sections I.XI.7 and 2.IX.13, for the formal definition of parabolic measure and its relation to hitting time distributions for Brownian motion). Now, by (A.7) of Corollary A.4,

$$\frac{\mathrm{d}G(K,y;t)}{\mathrm{d}y}\bigg|_{y=0} \le CT^{-1}K,$$

for any $t \in [\kappa T^{2/3}, T]$. Together with (2.2) and (2.5), this yields for large T,

$$\mathbf{E}[R_{[\kappa T^{2/3},T]}] \le CKe^{-K/\sigma(0)}. \tag{2.6}$$

The lemma now follows from (2.4) and (2.6) and Markov's inequality. □

We next control the expected number of particles that stay below the curve $\gamma_T(\cdot) + K$ up to time $t \le T$ and reach a prescribed value at time t. In what follows, for measures μ, ν we use the notation $\mu(\cdot \in dy) \le \nu(\cdot \in dy)$, $y \ge 0$, to mean that for any interval $I \subset \mathbb{R}_+$, $\mu(\cdot \in I) \le \nu(\cdot \in I)$.

Lemma 2.2. *For large T, we have for all $t \in [0,T]$, $K \in [0, T^{1/3}]$ and $y > 0$,*

$$\begin{aligned}&\mathbf{E}\big[\#\big\{u \in N(t) : \gamma_T(t) + K - X_u(t) \in \mathrm{d}y \text{ and } X_u(s) \le \gamma_T(s) + K, \forall s \le t\big\}\big]\\ &\quad\le 2e^{y/\sigma(t/T) - K/\sigma(0)}G(K,y;t)\,\mathrm{d}y,\end{aligned}$$

where $G(x,y;t)$ is the fundamental solution to the PDE (A.4), with $Q(t) = q_T(J(t/T))$.

Proof. By a similar argument as the one leading to (2.2), the expectation in the statement of the lemma equals

$$e^{y\gamma_T'(t)/\sigma^2(t/T) - K\gamma_T'(0)/\sigma^2(0)+o(1)}G(K,y;t)\,\mathrm{d}y.$$

By the assumption on K, we have $K\gamma_T'(0)/\sigma^2(0) = K/\sigma(0) + o(1)$ and by definition of γ_T, we have $\gamma_T'(t) \le \sigma(t/T)$. The claim follows. □

3. Tail estimates

We derive in this section tail estimates on the distribution of M_T summarized in the following proposition.

Proposition 3.1. *There exists a constant $C = C(c_0)$, such that for large T, for any $\sigma \in \Xi_{c_0}$ and every $K \in [1, T^{1/3}]$,*

$$C^{-1}Ke^{-K/\sigma(0)} \le \mathbf{P}\big(M_T \ge m_T + K\big) \le CKe^{-K/\sigma(0)}.$$

The proof of Proposition 3.1 goes by a suitably truncated first-second moment method, inspired by analogous results in the time-homogeneous case [1,4,6,22]. The key ingredients are estimates on a single Brownian particle with time-inhomogeneous variance staying below a curve and reaching a certain point at a given time t. These results, which have already been used in the previous section, are obtained in the appendix by analytic methods. However, as in the time-homogeneous case, the first-second moment method applied directly to the particles staying under the curve γ_T would not yield the $O(1)$ precision on the maximum at time T that we are aiming at, but would rather induce an error of magnitude $O(\log\log T)$. This can be rectified in our case by slightly changing the curve in the time interval $[T - T^{2/3}, T]$ in a way similar to the time-homogeneous case (namely, by having it end at the point $\gamma_T(T) - \sigma(1)\log T$). Luckily, for the upper bound it is possible to shortcut this approach, as Slepian's inequality allows us here to directly use existing results in the time-homogeneous case for the system during the time interval $[T - T^{2/3}, T]$ (see Section 3.1 for details).

3.1. *Proof of Proposition* 3.1: *Upper bound*

Set $t_0 = T - T^{2/3}$ and let $K \geq 2$. Let $(\mathcal{F}_t)_{t\geq 0}$ be the natural filtration of the BBM. A union bound gives,

$$\mathbf{P}\big(M_T \geq m_T + K \mid \mathcal{F}_{t_0}\big) \leq \sum_{u\in N(t_0)} \mathbf{P}_{(X_u(t_0),t_0)}\big(M_T \geq m_T + K\big),$$

where $\mathbf{P}_{(x,t)}$ denotes the law of BBM with variance $\sigma^2(\cdot/T)$ starting with one particle at the point x at time t. We will estimate the summands on the right-hand side by comparison with a BBM with constant variance. Set $\sigma_c^2 = T^{1/3}\int_{1-T^{-1/3}}^1 \sigma^2(t)\,\mathrm{d}t$. By the assumption on σ, we have

$$m_T - \gamma_T(t_0) \geq T\int_{1-T^{-1/3}}^1 \sigma(t)\,\mathrm{d}t - \sigma(1)\log T - C \geq \sigma_c\left(T^{2/3} - \frac{3}{2}\log T^{2/3}\right) - C_1,$$

for some constant C_1 that we fix for the remainder of this proof. Now, let $(Y_u(T^{2/3}))_u$ and $(Y_u^c(T^{2/3}))_u$ be the positions of the particles at time $T^{2/3}$ in branching Brownian motions with branching rate β_0 and variances $\sigma^2((\cdot + t_0)/T)$ and σ_c^2, respectively. Conditioned on the genealogy, we have $\mathbf{E}[Y_u(T^{2/3})^2] = \mathbf{E}[Y_u^c(T^{2/3})^2]$ and $\mathbf{E}[Y_u(T^{2/3})Y_v(T^{2/3})] \geq \mathbf{E}[Y_u^c(T^{2/3})Y_v^c(T^{2/3})]$ for every u and v, by the definition of σ_c^2 and the fact that σ^2 is decreasing. Hence, setting $M^c = \max_u Y_u^c(T^{2/3})$, we have by Slepian's inequality [23] for every $x \geq 1$,

$$\mathbf{P}_{(\gamma_T(t_0)+K-C_1-x,t_0)}\big(M_T \geq m_T + K\big) \leq \mathbf{P}\left(M^c \geq \sigma_c\left(T^{2/3} - \frac{3}{2}\log T^{2/3}\right) + x\right).$$

The tail estimates for the maximum of time-homogeneous BBM are available, e.g., in [5], and we obtain that

$$\mathbf{P}_{(\gamma_T(t_0)+K-C_1-x,t_0)}\big(M_T \geq m_T + K\big) \leq Cxe^{-x/\sigma_c} \leq Cxe^{-x/\sigma(t_0/T)}, \tag{3.1}$$

for large T, uniformly in $x \geq 1$.

Let A denote the event that no particle reaches the curve $\gamma_T(t) + K - C_1 - 1$ until time t_0. Integrating the upper bound in Lemma 2.2 (taken at time $t = t_0$) against the distribution in (3.1) and using Corollary A.5 now yields for $K \geq 2(C_1 + 1)$ and large T,

$$\mathbf{P}\big(\{M_T \geq m_T + K\} \cap A\big) \leq CKe^{-K/\sigma(0)}.$$

The upper bound in the statement of Proposition 3.1 now follows from this inequality, together with the fact that $\mathbf{P}(A^c) \leq CKe^{-K/\sigma(0)}$ for large T by Lemma 2.1.

3.2. *Proof of Proposition* 3.1: *Lower bound*

As discussed above, the proof involves a second moment ("Many-to-two") argument. In order to carry it out, we need to modify the curve $\gamma_T(\cdot)$ at the last interval $[T - T^{2/3}, T]$. Toward this end, fix $K > 1$ and let $\phi_T(t)$ be an increasing, twice differentiable function[2] such that $\phi_T(t) \equiv 0$ on $[0, T - T^{2/3}]$, $\phi_T(T) = \sigma(1)\log T$, $\phi_T'(t) \le 2\sigma(1)\log T/T^{2/3}$ and $\phi_T''(t) \le 4\sigma(1)\log T/T^{4/3}$. Define the curve

$$\zeta_T(t) = \gamma_T(t) + K - \phi_T(t).$$

From the definitions, one obtains after some algebraic manipulations

$$\frac{(\zeta_T'(t))^2}{2\sigma^2(t/T)} = \frac{1}{2} - T^{-2/3}\frac{w'(t/T)}{\sigma(t/T)} - \frac{\phi_T'(t)}{\sigma(1)} + o(1/T). \tag{3.2}$$

For $s, t \in [0, T]$, let

$$G_\zeta(x, y; s, t)\,\mathrm{d}y = \mathbf{E}_{(K-x,s)}\bigl[\#\bigl\{u \in N(t) : X_u(r) \le \zeta_T(r)\ \forall s \le r \le t, \zeta_T(t) - X_u(t) \in \mathrm{d}y\bigr\}\bigr]$$

denote the expected number of descendants at time t of a particle present at time s at location $K - x$, so that the path of the descendant stayed below the curve $\zeta_T(\cdot)$ until time t, and reached, at time t, an infinitesimal neighborhood of the value $\zeta_T(t) - y$. Similarly to the proof of (2.2), we have, using (3.2), that

$$\begin{aligned} G_\zeta(x, y; s, t)\,\mathrm{d}y &= E_{(x,s)}\biggl[\exp\biggl(\int_s^t \frac{\zeta_T'(r)}{\sigma^2(r/T)}\,\mathrm{d}B_r + \frac{t-s}{2} - \int_s^t \frac{(\zeta_T'(r))^2}{2\sigma^2(r/T)}\,\mathrm{d}r\biggr)\mathbb{1}_{B_t \in \mathrm{d}y, H_0(B_{s+\cdot}) > t-s}\biggr] \\ &= \exp\biggl(\frac{\zeta_T'(t)}{\sigma^2(t/T)}y - \frac{\zeta_T'(s)}{\sigma^2(s/T)}x + \frac{\phi_T(t) - \phi_T(s)}{\sigma(1)} + o(1)\biggr)G(x, y; s, t)\,\mathrm{d}y, \end{aligned} \tag{3.3}$$

where under $P_{(x,s)}$, $(B_t)_{t \ge s}$ is the time-inhomogeneous Brownian motion starting at time s at x and with instantaneous variance $\sigma^2(\cdot/T)$ and $G(x, y; s, t)$ is the fundamental solution to (A.4), with $Q(t) = |\sigma'(J(t/T))|/\sigma^2(J(t/T)) + O(\log T/T^{2/3})$. In particular, if N_T denotes the number of particles, at time T, whose trajectory stayed under the curve $\zeta_T(\cdot)$ and reached the interval $[\zeta_T(T) - 2, \zeta_T(T) - 1]$ at time T, then, for large T,

$$\mathbf{E}[N_T] = \int_1^2 G_\zeta(0, y; 0, T)\,\mathrm{d}y \ge CTe^{-K/\sigma(0)}\int_1^2 G(K, y; 0, T)\,\mathrm{d}y \ge CKe^{-K/\sigma(0)}, \tag{3.4}$$

where the last inequality follows from (A.6) of Corollary A.4.

We now estimate the second moment of N_T. For the rest of the proof, fix the constant $C_1 = 1/(2\sigma(0))$, which satisfies $C_1 \le \inf_{t \in [0,T]} \zeta_T'(t)/\sigma^2(t/T)$ for large T. The second moment formula[3] (“Many-to-two lemma”) for branching Markov processes ([14], Theorem 4.15), then yields for large T,

$$\begin{aligned} \mathbf{E}\bigl[N_T^2\bigr] &= \mathbf{E}[N_T] + \beta_0\mathbf{E}_\mu\bigl[L^2 - L\bigr]\int_0^T \mathrm{d}t \int_0^\infty \mathrm{d}y G_\zeta(K, y; 0, t)\biggl(\int_1^2 G_\zeta(y, z; t, T)\,\mathrm{d}z\biggr)^2 \\ &\le \mathbf{E}[N_T] + Ce^{-K/\sigma(0)}\int_0^T T\,\mathrm{d}t \\ &\quad\times \int_0^\infty \mathrm{d}y G(K, y; 0, t)\biggl(\int_1^2 G(y, z; t, T)\,\mathrm{d}z\biggr)^2 e^{-C_1 y + ((\phi_T(T) - \phi_T(t))/\sigma(1))}. \end{aligned} \tag{3.5}$$

We split the integral into three parts, according to intervals of time $[0, \kappa T^{2/3}]$, $[\kappa T^{2/3}, T - \kappa T^{2/3}]$ and $[T - \kappa T^{2/3}, T]$ and denote the three parts by I_1, I_2 and I_3. In order to estimate the first and third part, we bound the Green kernel

[2]The construction of such a function is possible for large enough T, for example by gluing together a parabola on $[T - T^{2/3}, T - T^{2/3}/2]$ and a line on $[T - T^{2/3}/2, T]$.
[3]It can be derived by conditioning on the splitting time of pairs of particles.

$G(x, y; s, t)$ for $t - s \le \kappa T^{2/3}$ by the Green kernel of Brownian motion killed at the origin. Namely, writing $V(t) = \int_0^t \sigma^2(s/T)\,\mathrm{d}s$, we have for $t - s \le \kappa T^{2/3}$ and $x, y \ge 0$,

$$G(x, y; s, t) \le \frac{C}{\sqrt{t-s}} \exp\biggl(-\frac{(x-y)^2}{2(V(t) - V(s))}\biggr)\biggl(\frac{xy}{t-s} \wedge 1\biggr). \tag{3.6}$$

For $t \ge \kappa T^{2/3}$, we use Corollary A.4 in order to bound $G(K, y; 0, t)$ and $G(y, z; T - t, T)$ (for the latter, we consider the time-reversal of (A.4), and $Q(\cdot)$ as above). This yields $G(K, y; 0, t) \le CT^{-1}Ky$ and $G(y, z; T - t, T) \le CT^{-1}y$ for every $t \ge \kappa T^{2/3}$ and $z \in [1, 2]$.

For the first part, we now get by exchanging integrals,

$$I_1 \le T^2 \int_0^\infty \bigl(T^{-1}y\bigr)^2 e^{-C_1 y}\biggl(\int_0^{\kappa T^{2/3}} G(K, y; 0, t)\,\mathrm{d}t\biggr)\mathrm{d}y \le C\int_0^\infty y^2 e^{-C_1 y}(1 + Ky)\,\mathrm{d}y \le CK,$$

for $K \ge 1$ and large T. Here, we used the fact that by (3.6),

$$\int_0^{\kappa T^{2/3}} G(K, y; 0, t)\,\mathrm{d}t \le C\int_0^1 \frac{1}{\sqrt{t}}\,\mathrm{d}t + C\int_1^\infty \frac{Ky}{t^{3/2}}\,\mathrm{d}t \le C(1 + Ky).$$

For the second part, we have

$$I_2 \le CT^2 \int_{\kappa T^{2/3}}^{T - \kappa T^{2/3}} \mathrm{d}t \int_0^\infty T^{-3}Ky^3 e^{-C_1 y}\,\mathrm{d}y \le CK.$$

For the third part, we note that by (3.6) and the assumptions on ϕ_T, we have for every $y \ge 0$, for large T,

$$\begin{aligned}&\int_1^{\kappa T^{2/3}} \biggl(\int_1^2 G(y, z; T - t, T)\,\mathrm{d}z\biggr)^2 e^{(\phi_T(T) - \phi_T(T-t))/\sigma(1)}\,\mathrm{d}t \\ &\quad\le Cy^2\biggl(\int_1^{T^{2/3}/\log T} t^{-3}\,\mathrm{d}t + T^{2/3}T\biggl(\frac{T^{2/3}}{\log T}\biggr)^{-3}\biggr) \le Cy^2.\end{aligned}$$

Furthermore, for $t \le 1$, we have $(\int_1^2 G(y, z; T - t, T)\,\mathrm{d}z)^2 \exp((\phi_T(T) - \phi_T(T - t))/\sigma(1)) \le C$ for every y. This gives,

$$I_3 \le \int_0^\infty Ky^3 e^{-C_1 y}\,\mathrm{d}y + \int_0^1 CK\,\mathrm{d}t \le CK.$$

In total, we have

$$\mathbf{E}\bigl[N_T^2\bigr] \le \mathbf{E}[N_T] + Ce^{-K/\sigma(0)}(I_1 + I_2 + I_3) \le C\mathbf{E}[N_T],$$

by (3.4). This now yields,

$$\mathbf{P}(N_T \ge 1) \ge \frac{\mathbf{E}[N_T]^2}{\mathbf{E}[N_T^2]} \ge C^{-1}\mathbf{E}[N_T],$$

which, together with (3.4), finishes the proof of the lower bound in Proposition 3.1.

4. Proof of Theorem 1.1

Armed with the tail estimates provided by Proposition 3.1, the proof of Theorem 1.1 follows by considering the descendants of the particles living at a large (but fixed) time t. Here are the details.

We assume without loss of generality that $\sigma(0) = 1$ (otherwise we can rescale space). Write $\mathbf{P}^T$ and $\mathbf{E}^T$ in place of $\mathbf{P}$ and $\mathbf{E}$, similarly, we write $\mathbf{P}^T_{(x,t)}$ in place of $\mathbf{P}_{(x,t)}$ (see Section 3.1). Furthermore, we will denote by $\mathbf{P}_{\mathrm{hom}}$ and $\mathbf{E}_{\mathrm{hom}}$ the law of (time-homogeneous) branching Brownian motion with variance 1 and branching rate β_0, starting with one particle at the origin. In what follows, we fix $y \in \mathbb{R}$ and let $t \geq 0$ large enough, such that $|y| < \log t - 2$. We will later let first T, then t go to infinity, i.e., we will choose t as a function of T, such that $t(T)$ goes to infinity slowly enough as $T \to \infty$.

As in Section 3.1, let $(\mathcal{F}_{t'})_{t' \geq 0}$ be the natural filtration of the BBM. Define the $\mathcal{F}_t$-measurable random variable $W_{t,T}$ by

$$W_{t,T} = \mathbf{P}^T\big(M_T \leq m_T + y \mid \mathcal{F}_t\big) = \prod_{u \in N(t)} \big(1 - \mathbf{P}^T_{(X_u(t),t)}\big(M_T \geq m_T + y\big)\big).$$

Furthermore, define

$$D_t = \sum_{u \in N(t)} \big(t - X_u(t)\big) e^{X_u(t) - t}.$$

By Proposition 3.1 applied with the function $\bar{\sigma}(t') = \sigma((t'(T-t)+t)/T)$, there exists a constant C and for each large T a function $g_{t,T} : \mathbb{R}_+ \to [C^{-1}, C]$, such that for each $x \in [-t, t - \log t]$,

$$1 - \mathbf{P}^T_{(x,t)}\big(M_T \geq m_T + y\big) = \exp\big(-g_{t,T}\big((y - x + t)/\sqrt{t}\big)(y - x + t)e^{-(y-x+t)}\big). \tag{4.1}$$

By the continuity of $\mathbf{P}^T_{(x,t)}$ in x, the functions $g_{t,T}$ are actually continuous, in particular, they are Lebesgue-measurable.

As in Appendix B (note that if $(B_t)_{t\geq 0}$ is a Brownian motion started at the origin, then $(t - B_t)_{t \geq 0}$ is a Brownian motion with drift $+1$ started at the origin), define the derivative Gibbs measure

$$\mu_t = \sum_{u \in \mathcal{N}(t)} \big(t - X_u(t)\big) e^{-(t - X_u(t))} \delta_{(t - X_u(t))/\sqrt{t}}.$$

Then, on the event $A_t = \{\forall u \in N(t) : -t \leq X_u(t) \leq t - \log t\}$, we get by (4.1)

$$W_{t,T} \mathbb{1}_{A_t} = \exp\biggl(-e^{-y} \int_0^\infty g_{t,T}(y/\sqrt{t} + x)\mu_t(\mathrm{d}x)\biggr)\mathbb{1}_{A_t} \tag{4.2}$$

and $\mathbf{P}_{\mathrm{hom}}(A_t) \to 1$ as t goes to infinity [5]. Now, note that as $T \to \infty$, the law of the process until time t converges to its law under $\mathbf{P}_{\mathrm{hom}}$, because conditioned on the genealogical structure and the branching times, the particle motion until time t on each of the finitely many branches of the genealogical tree converges to Brownian motion with variance 1. Moreover, thanks to the continuity and positivity of the Gaussian density, we can construct a probability space with probability measure $\widetilde{\mathbf{P}}$ which supports random variables $(\widetilde{\mu}_T)_{T\geq 0}$ and $\widetilde{\mu}$, such that, under $\widetilde{\mathbf{P}}$, $\widetilde{\mu}_T$ follows the law of μ_t under $\mathbf{P}^T$, $\widetilde{\mu}$ follows the law of μ_t under $\mathbf{P}_{\mathrm{hom}}$ and $\widetilde{\mu}_T = \widetilde{\mu}$ on an event $\widetilde{G}_T$ with $\widetilde{\mathbf{P}}(\widetilde{G}_T) \to 1$ as $T \to \infty$. In particular,

$$\int_0^\infty g_{t,T}(y/\sqrt{t} + x)\widetilde{\mu}_T(\mathrm{d}x) = \int_0^\infty g_{t,T}(y/\sqrt{t} + x)\widetilde{\mu}(\mathrm{d}x) \quad \text{on } \widetilde{G}_T \text{, for every } y. \tag{4.3}$$

By a diagonalization argument, we can now choose $t = t(T)$ growing slowly with T, so that (4.3) continues to hold with this choice of $t(T)$. By Theorem B.1, we have that for every bounded continuous function f,

$$\begin{aligned}&\mathbf{E}_{\mathrm{hom}}\biggl[f\biggl(\int_0^\infty g_{t(T),T}\big(y/\sqrt{t(T)} + x\big)\mu_{t(T)}(\mathrm{d}x)\biggr)\biggr]\\ &\quad - \mathbf{E}_{\mathrm{hom}}\biggl[f\biggl(D_\infty \int g_{t(T),T}\big(y/\sqrt{t(T)} + x\big)\rho(\mathrm{d}x)\biggr)\biggr] \to_{T\to\infty} 0,\end{aligned}$$

where ρ is the law of a BES(3) process at time 1, started at 0, and the variable D_∞ is the derivative martingale limit from Appendix B. Using the above coupling we conclude that

$$\mathbf{E}^T\left[f\left(\int_0^\infty g_{t(T),T}\big(y/\sqrt{t(T)}+x\big)\mu_{t(T)}(\mathrm{d}x)\right)\right] - \mathbf{E}_{\mathrm{hom}}\left[f\left(D_\infty\int g_{t(T),T}\big(y/\sqrt{t(T)}+x\big)\rho(\mathrm{d}x)\right)\right]\to 0. \tag{4.4}$$

On the other hand, since ρ has a continuous density with respect to Lebesgue measure, we have,

$$\limsup_{T\to\infty}\left|\int g_{t(T),T}\big(y/\sqrt{t(T)}+x\big)\rho(\mathrm{d}x) - \int g_{t(T),T}(x)\rho(\mathrm{d}x)\right| = 0. \tag{4.5}$$

Setting $C_T = \int g_{t(T),T}(x)\rho(\mathrm{d}x)$, we get by (4.2), (4.4), (4.5) and dominated convergence,

$$\lim_{T\to\infty}\mathbf{P}^T\big(M_T\le m_T+y-\log C_T\big) = \mathbf{E}_{\mathrm{hom}}\big[e^{-e^{-y}D_\infty}\big] = \phi(x),$$

where ϕ is a solution to (1.1), see Appendix B. This yields Theorem 1.1.

Remark. *While a-priori, the constant C_T depends on the particular choice of sequence $t(T)$, it is clear that the conclusion of Theorem 1.1 implies that a-posteriori, it is independent of this choice.*

Appendix A: An Airy-type PDE with time-varying parameters

We are interested in the following parabolic PDE:

$$w_t = \varepsilon^{-1}\big\{w_{xx} - q(t)xw\big\},\qquad w(t,0)=0\quad \forall t\ge 0, \tag{A.1}$$

for $q\in C^1[0,1]$, $q>0$. We want to study its behaviour as $\varepsilon\to 0$.

Before solving this equation, we recall some facts about the *Airy differential operator* $L\psi = \psi'' - x\psi$. Let $L^2([0,\infty))$ be the space of square-integrable functions on $[0,\infty)$ and let $\langle\cdot,\cdot\rangle$ be the associated scalar product[4] with norm $\|\cdot\|_2$. Recall the definition (1.6) of the *Airy function of the first kind* $\mathrm{Ai}(x)$. We denote by $-\alpha_1 > -\alpha_2 > \cdots$ its discrete set of zeros, with $\alpha_1 = 2.33811\ldots$. The functions ψ_n defined by

$$\psi_n(x) = \frac{\mathrm{Ai}(x-\alpha_n)}{\|\mathrm{Ai}(\cdot-\alpha_n)\|_2},\quad n=1,2,\ldots$$

then form an ONB of $L^2([0,\infty))$ and ψ_n is an eigenfunction of L with eigenvalue $-\alpha_n$ [24, Section 4.4].

The following lemma collects some other facts about the functions $\psi_n(x)$, which are probably well-known, although we could not find a reference to some of them.

Lemma A.1.

1. $\|\mathrm{Ai}(\cdot-\alpha_n)\|_2 = |\mathrm{Ai}'(-\alpha_n)|$ *for all* n. *In particular,* $\psi_n'(0)=1$ *for all* n.
2. $\alpha_n n^{-2/3}\to 3\pi/2$ *as* $n\to\infty$.
3. $|\psi_n(x)|\le x$ *for all* $n\ge 1$ *and* $x\ge 0$.
4. *For some numerical constant* C, $\langle|\psi_n|,x\rangle\le Cn^{4/3}$ *for all* $n\ge 1$.

Proof. The first and second points are [24, (4.52) and (2.52)], respectively. For the third point, we first note that since $\mathrm{Ai}''(x) = x\,\mathrm{Ai}(x)$, the local extrema of Ai' on $\mathbb{R}$ are exactly the zeros of Ai and the origin. Furthermore, by the first point of the lemma, $|\mathrm{Ai}'(-\alpha_n)|$ is increasing in n and by [24, (3.50)], $|\mathrm{Ai}'(0)| < |\mathrm{Ai}'(-\alpha_1)|$. This yields $|\mathrm{Ai}'(x)|\le|\mathrm{Ai}'(-\alpha_n)|$ for all $x\ge-\alpha_n$, from which the third point of the lemma follows.

[4] We will also use the notation $\langle f,g\rangle = \int_0^\infty f(x)g(x)\,\mathrm{d}x$ for $g\notin L^2([0,\infty))$, as long as the integral is well defined.

The third point of the lemma in particular implies $\langle|\psi_n|\mathbb{1}_{x\le\alpha_n}, x\rangle \le \alpha_n^2/2$ for all n. Now,

$$\big\langle|\psi_n|\mathbb{1}_{x\ge\alpha_n}, x\big\rangle = \big\|\mathrm{Ai}(\cdot-\alpha_n)\big\|_2^{-1}\big\langle|\mathrm{Ai}|, x+\alpha_n\big\rangle \le \|\mathrm{Ai}\|_2^{-1}\big\langle|\mathrm{Ai}|, x+\alpha_n\big\rangle.$$

By the tail bound $\mathrm{Ai}(x) \le \exp(-(2/3)x^{3/2})$ for large x [2, 10.4.59], the expression on the right-hand side of the last inequality is finite, whence $\langle|\psi_n|\mathbb{1}_{x\ge\alpha_n}, x\rangle \le C\alpha_n$, for some numerical constant C. Applying the second point of the lemma shows the fourth point. □

We get back to the equation (A.1). Define for a constant q the operator $L_q u = u_{xx} - qxu$. One easily checks that the function $\psi_n^q(x) = q^{1/6}\psi_n(q^{1/3}x)$ is an eigenfunction of L_q with eigenvalue $-\alpha_n q^{2/3}$ and the functions ψ_n^q form an ONB of $L^2([0,\infty))$. We further denote by $g(x,y;t) := g(x,y;0,t)$ the fundamental solution of (A.1).

Proposition A.2. *Set $Q_1 = \inf_{t\in[0,1]} q(t)^{2/3}$ and $Q_2 = \sup_{t\in[0,1]} |(\log q)'(t)|$. Suppose $Q_1 > 0$. Then there exists $C_0 = C_0(Q_1^{-1}, Q_2) > 0$ depending continuously on its parameters, such that for all $\varepsilon > 0$, $t \in [4\varepsilon, 1]$ and $\delta \in [\varepsilon, \sqrt{\varepsilon}]$ there exist*

- *$(c_{*n})_{n\ge2}$ and $(c_n^*)_{n\ge2}$ with $|c_{*n}| \vee |c_n^*| \le C_0 \exp(-C_0^{-1}(t\wedge\delta)\varepsilon^{-1}n^{2/3})$,*
- *$q_*(t) \le q(t) \le q^*(t)$ with $q^*(t) - q_*(t) \le 2(t\wedge\delta)^2\varepsilon^{-1}\sup_{t\in[0,1]}|q'(t)|$,*
- *$C_1 = C_1(\varepsilon,\delta,C_0) > 1$ depending continuously on its parameters and satisfying $C_1 \to 1$ as $\varepsilon \to 0$ and $\delta/\sqrt{\varepsilon} \to 0$,*

such that for all $x \in [0,1]$,

$$C_1^{-1}\left(\psi_1^{q^*(t)} + \varepsilon\sum_{n=2}^\infty c_n^*\psi_n^{q^*(t)}\right) \le \frac{g(x,\cdot;t)}{\psi_1^{q(0)}(x)}\exp\left(\varepsilon^{-1}\alpha_1\int_0^t q(s)^{2/3}\,\mathrm{d}s\right) \le C_1\left(\psi_1^{q_*(t)} + \varepsilon\sum_{n=2}^\infty c_{*n}\psi_n^{q_*(t)}\right).$$

Before providing the proof of Proposition A.2, we derive some a-priori estimates on solutions of (A.1).

Lemma A.3. *Define Q_1 and Q_2 as in Proposition A.2 and assume $Q_1 > 0$. Let $w(t,x)$ be the solution to (A.1) with initial condition satisfying $\|w(0,\cdot)\|_2 \le 1$. Define for each $t \ge 0$ the function $W_t(x) = \exp(\int_0^t \varepsilon^{-1}\alpha_1 q(s)^{2/3}\,\mathrm{d}s) \times w(t,x)$. Then there exist numerical constants C, C_1, such that for all $t \in [0,1]$,*

1. $\|W_t\|_2 \le 1$,
2. $|\langle W_t, \psi_1^{q(t)}\rangle - \langle W_0, \psi_1^{q(0)}\rangle| \le C\frac{Q_2+1}{Q_1}\varepsilon$ *and*
3. $(\sum_{n\ge2}\langle W_t, \psi_n^{q(t)}\rangle^2)^{1/2} \le C\frac{Q_2+1}{Q_1}\varepsilon(\frac{Q_2+1}{Q_1}\varepsilon + |\langle W_0, \psi_1^{q(0)}\rangle|) + \exp(-C_1\varepsilon^{-1}Q_1 t)$.

Proof. After decomposing the solution of (A.1) in the eigen-basis determined by the Airy functions, the proof proceeds by analyzing a coupled system of linear, time inhomogeneous, ordinary differential equations.

Throughout the proof, C, C_1 and C_2 are some numerical constants which may change from line to line. Define the vector $\mathbf{c}(t) = (c_1(t), c_2(t), \dots)^T$, where $c_n(t) = \langle W_t, \psi_n^{q(t)}\rangle$. From (A.1), one gets

$$\dot{c}_n(t) = -\varepsilon^{-1}(\alpha_n - \alpha_1)q(t)^{2/3}c_n(t) + \sum_{k\ge1} c_k(t)q'(t)\left\langle \psi_k^{q(t)}, \frac{\mathrm{d}}{\mathrm{d}\tilde{q}}\psi_n^{\tilde{q}}(t)\bigg|_{\tilde{q}=q(t)}\right\rangle,$$

whence

$$\dot{\mathbf{c}}(t) = \big(D(t) + A(t)\big)\mathbf{c}(t), \qquad D(t) = -\varepsilon^{-1}q(t)^{2/3}\operatorname{diag}(\alpha_i - \alpha_1)_{i\ge1}, \qquad A(t) = (\log q)'(t)A. \tag{A.2}$$

Here, A is the antisymmetric matrix

$$A = \frac{1}{6}\big(I + 2\big(\langle x\psi_i', \psi_j\rangle\big)_{i,j\ge1}\big) = \frac{1}{6}\big(\langle x\psi_i', \psi_j\rangle - \langle x\psi_j', \psi_i\rangle\big)_{i,j\ge1},$$

where the equality is easily verified by integration by parts[5].

[5]In fact, $A_{ij} = 2(-1)^{i+j}(\alpha_i - \alpha_j)^{-3}$ for $i \ne j$, which can be easily verified by the equation (3.54) in [24], however, we will not use this fact.

Since $D(t) + A(t)$ and $D(t') + A(t')$ do not commute unless $q(t)^{2/3}(\log q)'(t') = q(t')^{2/3}(\log q)'(t)$, there is no obvious explicit expression for the solution to (A.2). However, since D is diagonal and A antisymmetric, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{c}(t)\|_2^2 = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{c}^T(t)\mathbf{c}(t) = \mathbf{c}^T(t)\big(D^T(t) + A^T(t) + D(t) + A(t)\big)\mathbf{c}(t) = 2\mathbf{c}^T(t)D(t)\mathbf{c}(t) \le 0,$$

by the positivity of $q(t)$. This implies the first claim. In particular, $|c_1(t)| \le 1$ for all $t \ge 0$. Setting $\bar{\mathbf{c}}(t) = (0, c_2(t), c_3(t), \dots)^T$, the previous equation yields,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\bar{\mathbf{c}}^T(t)\bar{\mathbf{c}}(t) &= 2\bar{\mathbf{c}}^T(t)D(t)\bar{\mathbf{c}}(t) - 2c_1(t)\sum_{j=2}^{\infty} A_{1j}(t)c_j(t) \\ &\le -2\varepsilon^{-1}q(t)^{2/3}(\alpha_2 - \alpha_1)\bar{\mathbf{c}}^T(t)\bar{\mathbf{c}}(t) + 2\big|c_1(t)\big|\big\|\big(A_{1j}(t)\big)_{j\ge 2}\big\|_2\big\|\bar{\mathbf{c}}(t)\big\|_2,\end{aligned}$$

by the Cauchy–Schwarz inequality. By Parseval's formula, $\|(A_{1j})_{j\ge2}\|_2 \le \|x\psi_1'\|_2/3 < \infty$. This yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\|\bar{\mathbf{c}}(t)\big\|_2 \le -C_1\varepsilon^{-1}Q_1\big\|\bar{\mathbf{c}}(t)\big\|_2 + C_2Q_2\big|c_1(t)\big|.$$

Note that the general solution to the equation $f'(t) = -af(t) + b$ is $f(t) = (b/a) + \widetilde{C}e^{-at}$, for arbitrary $\widetilde{C} \in \mathbb{R}$. Since $\bar{\mathbf{c}}(0) \le 1$, Grönwall's inequality now yields that

$$\big\|\bar{\mathbf{c}}(t)\big\|_2 \le C(Q_2/Q_1)\varepsilon \sup_{s\in[0,1]}\big|c_1(s)\big| + \exp\big(-C_1\varepsilon^{-1}Q_1 t\big), \tag{A.3}$$

In order to show the second claim, we note that by (A.2), for every $t \in [0, 1]$,

$$\big|c_1(t) - c_1(0)\big| \le \int_0^t \left|\sum_{j=2}^{\infty} A_{1j}(t)c_j(t)\right| \mathrm{d}t \le C\int_0^t \big\|\bar{\mathbf{c}}(t)\big\|_2\,\mathrm{d}t,$$

where the last inequality follows from the Cauchy–Schwarz inequality as above. Together with (A.3) and the fact that $\sup_{t\in[0,1]}|c_1(t)| \le 1$, this implies the second claim. The third claim follows from this, together with (A.3). □

Proof of Proposition A.2. Fix $t \in [4\varepsilon, 1]$ and $\delta \in [\varepsilon, t - 3\varepsilon]$. We can construct $q^*, q_* \in C^1([0, 1])$ such that the following holds:

- $Q_1 \le q_* \le q \le q^*$
- $q_* \equiv q \equiv q^*$ on $[2\varepsilon, t - \varepsilon - \delta]$,
- q^* and q_* are constant on $[0, \varepsilon] \cup [t - \delta, t]$,
- $\sup_{s\in[0,1]} \max\{|(\log q_*)'(s)|, |(\log q^*)'(s)|\} \le Q_2$ and
- $q^* - q_* \le 2(t \wedge \delta)^2\varepsilon^{-1}\sup_{t\in[0,1]}|q'(t)|$.

Now let $x \in [0, 1]$. Let w^* and w_* denote the solutions to (A.1) with q replaced by q^* or q_*, respectively, and with initial condition $w^*(0, \cdot) = w_*(0, \cdot) = \delta(\cdot - x)$. By the parabolic maximum principle ([9], Theorem 7.1.9), we then have $w^*(t', y) \le g(x, y; t') \le w_*(t', y)$ for all $y \ge 0$ and $t' \in [0, 1]$.

Write $W^*_{t'}(y) = w^*(t', y)\exp(\varepsilon^{-1}\alpha_1\int_0^{t'} q^*(s)^{2/3}\,\mathrm{d}s)$ for all t', y. For every n, we have by the first point of Lemma A.1 and the fact that $\psi_1(x) > 0$ for all $x > 0$,

$$\big|\big\langle W_0^*, \psi_n^{q^*(0)}\big\rangle\big| = \big|\psi_n^{q^*(0)}(x)\big| \le C\psi_1^{q(0)}(x),$$

for some constant C depending on Q_1. By (A.2), we then have

$$\big|\big\langle W_\varepsilon^*, \psi_n^{q^*(0)}\big\rangle\big| \le C\exp\big(-(\alpha_n - \alpha_1)q^*(0)^{2/3}\big)\psi_1^{q(0)}(x),$$

for every n, since the off-diagonal terms cancel by the fact that q^* is constant on $[0,\varepsilon]$. Together with the second point of Lemma A.1, this yields $\|W_\varepsilon^*\|_2 \le C_1$ for some constant C_1 as ε is small enough. Furthermore,

$$\langle W_\varepsilon^*, \psi_1^{q^*(0)}\rangle = \langle W_0^*, \psi_1^{q^*(0)}\rangle = \big(1+o(1)\big)\psi_1^{q(0)}(x),$$

where $o(1)$ is a term depending on Q_1 and Q_2 which vanishes as $\varepsilon \to 0$. Applying Lemma A.3 with initial condition $w(0,\cdot) = W_\varepsilon^*/C_1$, we get that $\langle W_t^*, \psi_1^{q^*(t)}\rangle = (1+o(1))\psi_1^{q(0)}(x)$ and $(\sum_{n\ge 2}\langle W_{t-\delta}^*, \psi_n^{q^*(t)}\rangle^2)^{1/2} \le C_2\varepsilon\psi_1^{q(0)}(x)$ for small ε, where C_2 depends on Q_1 and Q_2. As above, this now implies that for every $n \ge 2$, for small ε,

$$\big|\langle W_t^*, \psi_n^{q^*(0)}\rangle\big| \le \exp\big(-CQ_1(t\wedge\delta)\varepsilon^{-1}n^{2/3}\big)C_2\varepsilon\psi_1^{q(0)}(x).$$

Together with the previous estimates, this finally yields the existence of a sequence of constants $(c_n^*)_{n\ge 2}$ with $|c_n^*| \le \exp(-CQ_1(t\wedge\delta)\varepsilon^{-1}n^{2/3})C_2$, such that as $\varepsilon \to 0$,

$$w^*(t,\cdot) \ge \big(1+o(1)\big)\exp\left(-\varepsilon^{-1}\alpha_1\int_0^t q^*(s)^{2/3}\,\mathrm{d}s\right)\psi_1^{q(0)}(x)\left(\psi_1^{q^*(t)} + \varepsilon\sum_{n=2}^\infty c_n^*\psi_n^{q^*(t)}\right).$$

An analogous formula holds for w_*. The statement now follows from the fact that $\int_0^1 q^*(s)^{2/3} - q_*(s)^{2/3}\,\mathrm{d}s = O((t\wedge\delta)^2)$ by construction. □

Fix $T > 0$. The results obtained in the current section can be easily transported to the following PDE on $[0,T]\times\mathbb{R}_+$, encountered in Sections 2 and 3.

$$u_t(t,x) = \frac{1}{2}\sigma^2(t/T)u_{xx}(t,x) + \left\{-T^{-1}Q(t)x + T^{-2/3}\alpha_1 Q(t)^{2/3}\left(\frac{1}{2}\sigma^2(t/T)\right)^{1/3}\right\}u(t,x), \tag{A.4}$$

with Dirichlet boundary condition at 0 and where $Q \in C^1([0,T])$ with $Q(t) > 0$ for all $t \in [0,T]$. Setting $J(t) = \int_0^t \frac{1}{2}\sigma(s)^2\,\mathrm{d}s$ as in (2.3), defining $q(t)$ by $q(J(t)/J(1)) = 2Q(t)/\sigma^2(t)$, and changing variables by

$$u(t,x) = w\big(J(t/T)/J(1), T^{-1/3}x\big)\exp\left(J(1)T^{1/3}\alpha_1\int_0^{J(t/T)/J(1)} q(s)^{2/3}\,\mathrm{d}s\right),$$

we see that the function $w(t,x)$ solves (A.1) on $[0,1]\times\mathbb{R}_+$ with $\varepsilon^{-1} = J(1)T^{1/3}$ and the $q(t)$ defined here. In particular, if $G(x,y;t) := G(x,y;0,t)$ and $g(x,y;t) = g(x,y;0,t)$ denote the fundamental solutions of (A.4) and (A.1), respectively, then we have the relation

$$\begin{aligned} G(x,y;t) &= T^{-1/3}g\big(T^{-1/3}x, T^{-1/3}y; J(t/T)/J(1)\big) \\ &\quad\times\exp\left(J(1)T^{1/3}\alpha_1\int_0^{J(t/T)/J(1)} q(s)^{2/3}\,\mathrm{d}s\right). \end{aligned} \tag{A.5}$$

The following estimates on $G(x,y;t)$ are used in the main text. Both are corollaries of Proposition A.2. Recall the constant κ from (2.1).

Corollary A.4. *For large T, we have for all $x,y\ge 0$ and $t\in[\kappa T^{2/3}, T]$,*

$$C_0^{-1}T^{-1}xy\mathbb{1}_{x,y\le T^{1/3}} \le G(x,y;t) \le C_0T^{-1}xy, \tag{A.6}$$

where $C_0 > 0$ depends continuously on $\sigma(0)$, $\sigma(1)^{-1}$, $(\inf_{t\in[0,1]} q(t))^{-1}$, $\sup_{t\in[0,1]}|q'(t)|$ and $q(0)$. In particular, with the same assumptions,

$$\frac{\mathrm{d}}{\mathrm{d}y}G(x,y;t)\bigg|_{y=0} \le C_0T^{-1}x. \tag{A.7}$$

Corollary A.5. *For large T, we have for all $x \ge 0$, $t \in [\kappa T^{2/3}, T]$,*

$$\int_0^\infty G(x, y; t) y \,\mathrm{d}y \le C_0 T^{-1} x, \tag{A.8}$$

where C_0 is as in the previous corollary.

Proof of Corollary A.4. Throughout the proof, we will use the fact that $|\psi_n^q(x)| \le \sqrt{q} x$ for every $q \ge 0$, $x \ge 0$ and $n \in \mathbb{N}^*$, by the third part of Lemma A.1. Note that for $t \ge \kappa T^{2/3}$, we have with $\varepsilon^{-1} = J(1) T^{1/3}$,

$$J(t/T)/J(1) = J(1)^{-1} \int_0^{t/T} \frac{1}{2}\sigma^2(s)\,\mathrm{d}s \ge J(1)^{-1} \frac{1}{2}\sigma^2(1) t/T \ge \frac{1}{2}\sigma^2(1)\kappa\varepsilon = 4\varepsilon,$$

where the first inequality follows from the fact that σ^2 is a decreasing function and the last equality follows from the definition of κ in (2.1). By (A.5) and Proposition A.2, with the notation introduced there, we then have for every $x, y \ge 0$,

$$\begin{aligned} G(x, y; t) &\lesssim T^{-1/3} \psi_1^{q(0)}\big(T^{-1/3} x\big) \left(\psi_1^{q_*(t)}\big(T^{-1/3} y\big) + \varepsilon \sum_{n=2}^\infty c_{*n} \big|\psi_n^{q_*(t)}\big(T^{-1/3} y\big)\big| \right) \\ &\le C_0 T^{-1} x y \sqrt{q(0)} \sup_{t \in [0,1]} \sqrt{q(t)}. \end{aligned} \tag{A.9}$$

Here, C_0 is a constant as in the statement of the corollary; this follows from the fact that the quantities Q_1^{-1} and Q_2 in Proposition A.2 can be expressed as

$$Q_1^{-1} = \Big(\inf_{t \in [0,1]} q(t) \Big)^{-2/3}, \qquad Q_2 = \sup_{t \in [0,1]} \big| q'(t)/q(t) \big| \le \frac{\sup_{t \in [0,1]} |q'(t)|}{\inf_{t \in [0,1]} q(t)}.$$

Together with the fact that $\sup_{t \in [0,1]} q(t) \le q(0) + \sup_{t \in [0,1]} |q'(t)|$, Equation (A.9) now implies the right-hand inequality of (A.6).

As for the left-hand inequality in (A.7), we have by Proposition A.2, again with the notation introduced there, for every $x, y \ge 0$,

$$G(x, y; t) \gtrsim T^{-1/3} \psi_1^{q(0)}\big(T^{-1/3} x\big) \left(\psi_1^{q^*(t)}\big(T^{-1/3} y\big) - \varepsilon \sum_{n=2}^\infty c_n^* \big|\psi_n^{q^*(t)}\big(T^{-1/3} y\big)\big| \right). \tag{A.10}$$

Now note that the function ψ_1 is by definition (strictly) positive on $(0, \infty)$ and continuous on $[0, \infty)$. Furthermore, $\psi_1'(0) = 1$ by the first part of Lemma A.1. This implies that the function $x \mapsto \psi_1(x)/x$ can be extended to a continuous and strictly positive function on $[0, \infty)$. In particular, for every $q > 0$, $\psi_1(x) \ge C^{-1} x$ for all $x \in [0, q]$, where $C = \inf_{x \in [0,q]} \psi_1(x)/x > 0$ depends continuously on q.

Letting $\varepsilon \to 0$ (i.e., $T \to \infty$), the left-hand inequality of (A.6) now readily follows from (A.10) by a reasoning similar to the one used above for the right-hand inequality of (A.6), taking into the account the above lower bound on ψ_1.

Equation (A.7) immediately follows from (A.6). □

Proof of Corollary A.5. Similar to the proof of the last corollary, using in addition the fourth part of Lemma A.1. We omit the details. □

Appendix B: Convergence of the derivative Gibbs measure of (time–homogeneous) branching Brownian motion

In this section, we consider branching Brownian motion with (time-homogeneous) variance $\sigma^2 = 1$, drift $+1$ and reproduction law and branching rate as before. In particular, the left-most particle drifts off to $+\infty$ with zero speed, i.e., if $M_t = \min_{u\in\mathcal{N}(t)} X_u(t)$, then almost surely, as $t\to\infty$, $M_t/t\to 0$ and $M_t\to+\infty$ [4]. Define the *derivative Gibbs measure* at time t:

$$\mu_t = \sum_{u\in\mathcal{N}(t)} X_u(t)e^{-X_u(t)}\delta_{X_u(t)/\sqrt{t}}.$$

The quantity $D_t = \int 1\,\mathrm{d}\mu_t$ is then known as the *derivative martingale*, and it is known ([15,19,25]) that D_t converges almost surely as $t\to\infty$ to a (strictly) positive limit D_∞ whose Laplace transform is given by $\mathbf{E}[\exp(-e^{-x}D_\infty)] = \phi(x)$, where ϕ is a solution to (1.1).

Let ρ denote the law of a BES(3) process at time 1, started at 0, i.e.,

$$\rho(\mathrm{d}x) = \sqrt{\frac{2}{\pi}}x^2 e^{-x^2/2}\mathbb{1}_{x\ge 0}\,\mathrm{d}x.$$

Theorem B.1. *In probability, μ_t converges weakly to $D_\infty\rho$. Moreover, for every family $(f_t)_{t\ge 0}$ of uniformly bounded measurable functions (i.e., $\sup_{t,x}|f_t(x)|<\infty$), we have*

$$\int f_t\,\mathrm{d}\mu_t - D_\infty\int f_t\,\mathrm{d}\rho \to 0,\quad \textit{in probability.}$$

Remark B.2. *Convergence in probability of the Gibbs measure*

$$\mu_t^* = \sqrt{t}\times\sum_{u\in\mathcal{N}(t)} e^{-X_u(t)}\delta_{X_u(t)/\sqrt{t}},$$

has recently been shown by Madaule [16] for general branching random walks. While Theorem B.1 (at least the first statement) could be in principle recovered from the results in [16] (see in particular Proposition 3.4 of that paper), we present below for completeness a fairly simple proof.

Proof. Note that we can (and will) assume w.l.o.g. that $f_t\ge 0$ for each $t\ge 0$. For every $s\le t$, define the measure

$$\mu_t^s = \sum_{u\in\mathcal{N}(t)} X_u(t)e^{-X_u(t)}\mathbb{1}_{(X_u(r)\ge 0\ \forall s\le r\le t)}\delta_{X_u(t)/\sqrt{t}}.$$

Since $\min_{u\in\mathcal{N}(t)} X_u(t)\to+\infty$ almost surely [18], there exists a random time S, such that we have $\mu_t^s = \mu_t$ for all $S\le s\le t$. Since moreover $D_s\to D_\infty$ almost surely, as $s\to\infty$, it is enough to show that almost surely, for any family of nonnegative functions $(f_t)_{t\ge 0}$ as in the statement of the theorem,

$$\lim_{s\to\infty}\limsup_{t\to\infty}\bigl|\mathbf{E}\bigl[e^{-\int f_t\,\mathrm{d}\mu_t^s}\mid\mathcal{F}_s\bigr] - e^{-D_s\int f_t\,\mathrm{d}\rho}\bigr| = 0,\quad \text{a.s.}\tag{B.1}$$

Let $s\le t$. Define $f_{s,t}(x) = f_t(x\sqrt{(t-s)/t})$. By the branching property and Jensen's inequality,

$$\mathbf{E}\bigl[e^{-\int f_t\,\mathrm{d}\mu_t^s}\mid\mathcal{F}_s\bigr] = \prod_{u\in\mathcal{N}(s)}\mathbf{E}_{X_u(s)}\bigl[e^{-\int f_{s,t}\,\mathrm{d}\mu_{t-s}^0}\bigr]\ge\exp\biggl(-\sum_{u\in\mathcal{N}(s)}\mathbf{E}_{X_u(s)}\biggl[\int f_{s,t}\,\mathrm{d}\mu_{t-s}^0\biggr]\biggr).\tag{B.2}$$

We now have for every $x\ge 0$, by the first moment formula for branching Markov processes [14, Theorem 4.1] and Girsanov's theorem, for every bounded measurable function f,

$$\begin{aligned}\mathbf{E}_x\biggl[\int f\,\mathrm{d}\mu_t^0\biggr] &= e^{t/2}E_x\bigl[(B_t+t)e^{-(B_t+t)}f\bigl((B_t+t)/\sqrt{t}\bigr)\mathbb{1}_{(B_r\ge 0\ \forall r\le t)}\bigr]\\ &= e^{-x}E_x\bigl[B_t f(B_t/\sqrt{t})\mathbb{1}_{(B_r\ge 0\ \forall r\le t)}\bigr] = xe^{-x}E_x\bigl[f(R_t/\sqrt{t})\bigr] = xe^{-x}E_{x/\sqrt{t}}\bigl[f(R_1)\bigr],\end{aligned}$$

where under P_x, $(R_t)_{t\geq 0}$ is a three-dimensional Bessel process started at x [21, Section XI.1]. The law of R_1 under P_x has a continuous density with respect to Lebesgue measure for every x which converges uniformly to the density of ρ as $x\to 0$. It follows easily from this that for every $x\geq 0$,

$$\mathbf{E}_x\left[\int f_{s,t}\,\mathrm{d}\mu^0_{t-s}\right]-xe^{-x}\int f_t\,\mathrm{d}\rho\to 0,\quad \text{as } t\to\infty. \tag{B.3}$$

Equations (B.2) and (B.3) now yield the inequality "$\geq$" in (B.1). In order to obtain the other inequality, we have by Lemma B.3 below, for some constants C, C',

$$\mathbf{E}_x\left[\left(\int f_{s,t}\,\mathrm{d}\mu^0_{t-s}\right)^2\right]\leq C'\mathbf{E}^0_x\left[D^2_{t-s}\right]\leq Ce^{-x}, \tag{B.4}$$

where the superscript in $\mathbf{E}^0_x$ indicates that the particles are killed upon hitting the origin. By the branching property and the inequalities $e^{-x}\leq 1-x+x^2\leq e^{-x+x^2}$ for $x\geq 0$, we then get by (B.4),

$$\begin{aligned}\mathbf{E}\left[e^{-\int f_t\,\mathrm{d}\mu^s_t}\mid\mathcal{F}_s\right]&\leq\exp\left(\sum_{u\in\mathcal{N}(s)}-\mathbf{E}_{X_u(s)}\left[\int f_{s,t}\,\mathrm{d}\mu^0_{t-s}\right]+\mathbf{E}_{X_u(s)}\left[\left(\int f_{s,t}\,\mathrm{d}\mu^0_{t-s}\right)^2\right]\right)\\&\leq\exp\left(-D_s\int f_t\,\mathrm{d}\rho+CW_s+E_{s,t}\right),\end{aligned} \tag{B.5}$$

where $W_s=\sum_{u\in\mathcal{N}(s)}e^{-X_u(s)}$ and

$$E_{s,t}=\sum_{u\in\mathcal{N}(s)}-\mathbf{E}_{X_u(s)}\left[\int f_{s,t}\,\mathrm{d}\mu^0_{t-s}\right]+D_s\int f_t\,\mathrm{d}\rho$$

is an $\mathcal{F}_s$-measurable term. By (B.4) and the fact that ρ has a continuous density with respect to Lebesgue's measure, $E_{s,t}$ tends to zero almost surely, as $t\to\infty$, for each fixed s. Since $W_s\to 0$ almost surely, as $s\to\infty$ (see, e.g., [15, 19]), the inequality (B.5) yields the inequality "$\leq$" in (B.1). This finishes the proof of the theorem. □

Lemma B.3. *Let $\mathbf{E}^0_x$ be the law of BBM as in the beginning of this section but where in addition particles are killed upon hitting the origin. For some constant C, $\mathbf{E}^0_x[D^2_t]\leq Ce^{-x}$ for every $x\geq 0$ and $t\geq 0$.*

Proof. We first note that $(D_t)_{t\geq 0}$ is a martingale as well under $\mathbf{E}^0_x$. In particular, $\mathbf{E}^0_x[D_t]=xe^{-x}$ for every $x\geq 0$ and $t\geq 0$. By the second moment formula for branching Markov processes [14, Theorem 4.15], this gives for some constant C,

$$\mathbf{E}^0_x\left[D^2_t\right]=\mathbf{E}^0_x\left[\sum_{u\in\mathcal{N}(t)}X_u(t)^2e^{-2X_u(t)}\right]+C\mathbf{E}^0_x\left[\int_0^t\sum_{u\in\mathcal{N}(s)}X_u(s)^2e^{-2X_u(s)}\,\mathrm{d}s\right].$$

By the first moment formula for branching Markov processes and Girsanov's theorem we get as in the proof of Theorem B.1,

$$\mathbf{E}^0_x\left[D^2_t\right]=e^{-x}\left(E_x\left[B^2_te^{-B_t}\mathbb{1}_{B_s\geq 0\ \forall s\leq t}\right]+CE_x\left[\int_0^{t\wedge T_0}B^2_se^{-B_s}\,\mathrm{d}s\right]\right),$$

where T_0 is the first hitting time of the origin. The term in the first expectation is bounded by a constant. As for the second expectation, by the inequality $x^2e^{-x}\leq C'e^{-x/2}$ and Ito's formula, we have

$$E_x\left[\int_0^{t\wedge T_0}B^2_se^{-B_s}\,\mathrm{d}s\right]\leq 4C'E_x\left[e^{-B_{t\wedge T_0}/2}-e^{-x/2}\right]\leq 4C'.$$

This yields the lemma. □

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