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CRITICAL BEHAVIOUR OF THE PARTNER MODEL

BY ERIC FOXALL[1]

Arizona State University

We consider a stochastic model of infection spread incorporating monogamous partnership dynamics. In [*Ann. Appl. Probab.* **26** (2016) 1297–1328], a basic reproduction number R_0 is defined with the property that if $R_0 < 1$ the infection dies out within $O(\log N)$ units of time, while if $R_0 > 1$ the infection survives for at least $e^{\gamma N}$ units of time, for some $\gamma > 0$. Here, we consider the critical case $R_0 = 1$ and show that the infection dies out within $O(\sqrt{N})$ units of time, and moreover that this estimate is sharp.

1. Introduction. The contact process is a well-studied stochastic model of infection spread, in which an undirected graph $G = (V, E)$ determines a collection of sites V and edges E which we can think of as individuals and as links between individuals along which the infection can be transmitted. Each site is either healthy or infectious; infectious sites recover at a certain fixed rate which is usually normalized to 1, and transmit the infection to each of their neighbours at rate λ.

The contact process has been studied in a variety of different settings, including lattices [1, 3, 7, 8] (to cite just a few), infinite trees [10], power law graphs [2] and complete graphs [11]. In each case, there is a critical value λ_c below which the infection quickly vanishes from the graph, and above which the infection has a positive probability of surviving either for all time (if the graph is infinite), or for an amount of time that grows quickly (either exponentially or at least faster than polynomially) with the size of the graph; in the power law case $\lambda_c = 0$, so long-time survival is possible whenever $\lambda > 0$.

In [5], a version of the contact process on the complete graph, called the *partner model*, is introduced, in which the edges open and close dynamically, modelling the formation and breakup of monogamous partnerships. In this case, the edges E represent *possible* connections, and we have a process $\{E_t : t \geq 0\}$ with $E_t \subseteq E$ for each $t \geq 0$ that describes the set of open edges as a function of time. It is shown there exists a sharp phase transition with a critical value λ_c that depends on the edge opening and closing rates r_+ and r_-, with the property that for a population of N individuals, for large N:

- the infection dies off within $C \log N$ units of time when $\lambda < \lambda_c$ for some C that depends on λ, and

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- starting from a positive fraction of infectious individuals, the infection survives for at least $e^{\gamma N}$ units of time when $\lambda > \lambda_c$, for some $\gamma > 0$ that depends on λ.

However, the behaviour when $\lambda = \lambda_c$ is left open, and it is this regime that we consider here. The main result, which is stated more precisely in Theorem 2.1, is that the infection dies off within about $\sqrt{N}$ units of time.

We note that in the present setting our model is a one-sex model of infection spread. However, it can easily be generalized to a two-sex model, at the cost of introducing some additional equations, and a study of this model may be of interest in addressing questions of vaccination strategies for sexually transmitted infections, as in [12].

2. Description and main result. We begin by defining the partner model and identifying the location of the phase transition. There are N individuals, that we picture as vertices on the complete graph $K_N = (V, E)$, and we denote the set of infectious vertices at time t by V_t. Transmission and recovery are possible, as well as re-infection. At any moment in time, only a subset of the edges are open for transmission, and the open edges are denoted E_t, so the process is $\{(V_t, E_t) : t \geq 0\}$. The transitions are as follows:

- For each $x \in V_t$, $V_t \to V_t \setminus \{x\}$ at rate 1.
- For each $xy \in E_t$ such that $\{x, y\} \cap V_t = y$, $V_t \to V_t \cup \{x\}$ at rate λ.
- For each $xy \in E$ such that $xz \notin E_t$ and $yz \notin E_t$ for all $z \in V$, $E_t \to E_t \cup \{xy\}$ at rate r_+/N.
- For each $xy \in E_t$, $E_t \to E_t \setminus \{xy\}$ at rate r_-.

Each infectious individual becomes healthy at rate 1, and along each open edge, an infectious individual infects a healthy individual at rate λ. If x and y have no partners, they form a partnership at rate r_+/N, and if xy are partnered they break up at rate r_-. The normalization r_+/N is so that each individual finds a partner at a bounded rate. A graphical construction of the process in described in [5].

Letting S_t and I_t denote the total number of healthy and infectious *singletons* (i.e., unpartnered individuals), respectively, and SS_t, SI_t, II_t the number of partnered pairs of the three possible types, as noted in [5], $(S_t, I_t, SS_t, SI_t, II_t)$ is a continuous time Markov chain, whose transition rates can be easily written down. Defining $s_t = S_t/N$, $i_t = I_t/N$, $ss_t = SS_t/N$, $si_t = M_t/N$, $ii_t = II_t/N$, $(s_t, i_t, ss_t, si_t, ii_t)$ is a Markov chain as well, and sometimes more convenient to work with.

Defining $Y_t = S_t + I_t$ we have the transitions

$$Y_t \to \begin{cases} Y_t + 2, & \text{at rate } (N - Y_t)r_-/2, \\ Y_t - 2, & \text{at rate } Y_t(Y_t - 1)r_+/(2N), \end{cases}$$

so letting $y_t = Y_t/N$ denote the proportion of singletons, as shown in [5], y_t approaches and remains close to a stationary value y^* which is the unique equilibrium in (0, 1) for the ODE

$$y' = r_-(1 - y) - r_+y^2.$$

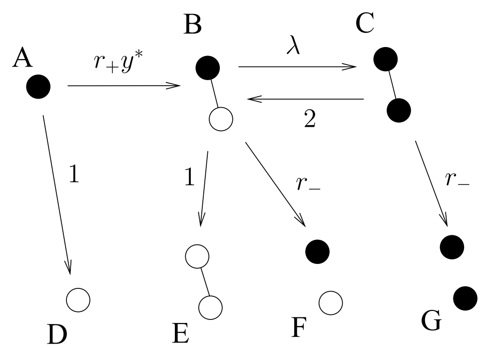

FIG. 1. *Markov chain used to compute R_0, with transition rates indicated; shaded sites are infectious.*

To decide whether the infection can spread, we use the following heuristic argument. We consider the effect of one infectious singleton in an otherwise healthy population, tracking the individual over one *partnership cycle*, which is the time interval that ends at the first moment when the individual either:

- recovers without finding a partner, or
- if it finds a partner before recovering, breaks up from that partnership.

Assuming $y \approx y^*$, this leads to the continuous time Markov chain $(Z_t)_{t\geq 0}$ whose transition rates are as shown in Figure 1. Let $\tau = \inf\{t : S_t \in \{D, E, F, G\}\} < \infty$ and define the *basic reproduction number*

$$R_0 = \mathbb{P}(Z_\tau = F | Z_0 = A) + 2\mathbb{P}(Z_\tau = G | Z_0 = A), \tag{2.1}$$

which is the expected number of infectious singletons upon absorption of the above Markov chain, starting from state A. As shown in [5], if $R_0 < 1$ the infection dies out by time $C \log N$, for some $C > 0$, with probability tending to 1 as $N \to \infty$, while if $R_0 > 1$ and $|V_0| \geq \varepsilon N$ the infection survives up to time $e^{\gamma N}$ with probability $\geq 1 - e^{-\gamma N}$ for some $\gamma > 0$.

If $R_0 = 1$, then in [5] it is shown for each $\varepsilon > 0$, after constant time depending on ε, $|V_t| \leq \varepsilon N$. However, the extinction time itself is not investigated. Here, we prove the extinction time is of order $\sqrt{N}$; the following is the main result.

THEOREM 2.1. *If $R_0 = 1$, then*:

- *there are $C, \gamma > 0$ so that for any (V_0, E_0), with probability $\geq 1 - e^{-\gamma m}$, $|V_{mC\sqrt{N}}| = 0$, and*
- *if $|V_0| \geq \sqrt{N}$ and $y_0 \geq y^* - \log N/\sqrt{N}$, there is $c > 0$ so that $|V_{c\sqrt{N}}| \neq 0$ with probability $\geq 1 - e^{-c(\log N)^2}$.*

To get some intuition for this result, we suppose $y \approx y^*$. There are three types of interactions involving infectious singletons—in the descriptions below, in order to track the change in I_t, we think of transitions of type 2 and 3 as accounting for the final state of the two partners, at the moment of breakup.

1. Recovery of infectious singletons, denoted $I \to S$, that occurs at rate I_t,
2. Partnering of infectious singletons with healthy singletons, denoted $S + I \to SI$, that occurs at rate $r_+ I_t(y^* N - I_t)(1 + o(1))/N$, and
3. Partnering of two infectious singletons, denoted $I + I \to II$, that occurs at rate $r_+ I_t^2(1 + o(1))/N$ and results in an average decrease in I_t by some constant $a > 0$ with each transition.

Since $R_0 = 1$, transitions 1 and 2 together result in zero average change in I_t. For simplicity, assume each transition causes a change of ± 1 in I_t—this is not quite true, but close enough, just to write something down for the variance in (2.2). If $I_t = O(\sqrt{N})$, then letting $\mathcal{I}_t = N^{-1/2} I_{\sqrt{N}t}$ and noting the change in timescale, in a time step $h = 1/N$,

$$
\begin{aligned}
E[\mathcal{I}_{t+h} - \mathcal{I}_t | \mathcal{I}_t] &= -\big(ar_+ + o(1)\big)\mathcal{I}_t^2/N, \\
\mathrm{Var}(\mathcal{I}_{t+h} - \mathcal{I}_t | \mathcal{I}_t) &= \big(1 + r_+\big(\mathcal{I}_t + y^*\big) + o(1)\big)\mathcal{I}_t/N.
\end{aligned} \tag{2.2}
$$

Thus, $\mathcal{I}_t$ behaves like an almost-critical branching process with a slight negative drift, given by the diffusion equation

$$
d\mathcal{I}_t = -ar_+ \mathcal{I}_t^2\, dt + \big(1 + r_+\big(\mathcal{I}_t + y^*\big)\big)^{1/2} \sqrt{\mathcal{I}_t}\, dB_t \tag{2.3}
$$

that hits zero within constant time, suggesting the correct time scale for extinction is $\sqrt{N}$.

Now, we have cut some obvious corners in arriving at (2.3). In particular, we have assumed that:

(i) $y_t \equiv y^*$ and that
(ii) partnerships have zero duration,

neither of which is true. Due to these complications, we do not actually try to prove convergence to a limiting diffusion. Instead, we first account for items (i) and (ii), then break up the problem into four zones, where $C > 0$ is a large constant and $c > 0$ is a small constant:

1. $N \ge I_t \ge C\sqrt{N}$,
2. $C\sqrt{N} \ge I_t \ge c\sqrt{N}$,
3. $c\sqrt{N} \ge I_t \ge N^{1/4-\varepsilon}$, and
4. $I_t \le N^{1/4-\varepsilon}$.

In each case, we will get just enough control on I_t (actually, on a related variable TI_t, the "total infections", described below, that plays the role of $|V_t|$) to show that it reaches the low end of the scale with positive probability, uniformly in the initial configuration, within $C'\sqrt{N}$ amount of time, for some $C' > 0$. Using the Markov property to iterate, then gives the exponential tail. The lower bound then follows more easily using the estimates we have gathered up to that point.

The proof is laid out as follows. In Section 4, we address item (i). The main result of that section is Lemma 4.8, in which we control the speed of decay, the maximum value and the integral over time, of $|y_t - y^*|$. Along the way, we note a diffusion limit for the process $\mathcal{Y}_t := N^{1/2}(y_t - y^*)$, that we then ignore in favour of a more bare-hands approach that allows us to account not only for the size of fluctuations of $y_t - y^*$ within the range $N^{-1/2}$, but also the rate of approach of $y_t - y^*$ to the range $N^{-1/2}$ from outside of that range. In addition, our approach yields fairly strong quantitative probability estimates on the events of interest, that are necessary later on, and that may be interesting in their own right in describing the rate of convergence of $\mathcal{Y}_t$ to its limiting diffusive behaviour.

In Section 5, we address item (ii) by changing the variables (I_t, SI_t, II_t) to a new set of variables (I_t, SIA_t, IIA_t) (the A is for anticipates) that take immediate account of the final state of partnerships, at the cost of losing the Markov property. Defining $TI_t := I_t + SIA_t + 2IIA_t$ that plays the role of $|V_t|$, and whose transition rate is directly a function of I_t, we then make up for this deficiency by producing a uniform lower bound on the ratio of I_t to TI_t, in Lemma 5.1, which helps us to ensure that what we want to happen, happens quickly enough.

In Section 6, we decompose transitions in TI_t into two parts, the *principal part* and the *auxiliary part*. The principal part assumes that $y_t \equiv y^*$ and the auxiliary part corrects for that assumption. As shown in Lemma 6.1, the principal part includes the slight negative drift observed in $\mathcal{I}_t$, while the auxiliary part includes the small, but inconvenient effects of fluctuation.

In Section 7, we prove the upper bound in four parts. In each case, we show the probability of the desired event is bounded away from 0 uniformly in N and in the particular choice of configuration, although it may depend on c, C. The greatest difficulty is showing that the estimated fluctuations in y_t do not interfere with things too much.

- In Proposition 7.1, we show that for some $C > 0$ the time it takes to reach $\leq C\sqrt{N}$ infectious is at most $C\sqrt{N}$.
- In Proposition 7.2, we show that for any $c, C > 0$ there is $C' > 0$ so that the time it takes to reach $\leq c\sqrt{N}$ from $C\sqrt{N}$ infectious is at most $C'\sqrt{N}$.
- In Proposition 7.3, we show that for some $c > 0$, the time it takes to reach $\leq N^\gamma$ from $\leq c\sqrt{N}$ infectious, for fixed $0 < \gamma < 1/4$, is $O(\sqrt{N})$.
- In Proposition 7.4, we show the time to go from N^γ to 0 infectious is $O(N^{2\gamma})$ for fixed $0 < \gamma < 1/4$. By comparison with a critical branching process, the correct order is probably $O(N^\gamma)$, since the drift is negligible at this point, but $N^{2\gamma}$ suffices and is easy enough to prove.

These are combined in Proposition 7.5 to prove the upper bound. The proof of the lower bound is simpler and is given in Proposition 7.6.

Note: After this paper was written, the referee has noted that it may be possible to prove Theorem 2.1 using the results of [6], that treat models with multiple timescales. In our case, the fluctuations in $y_t - y^*$ happen on an $O(1)$ timescale, while $\mathcal{I}_t$ changes on the longer timescale $\sqrt{N}$. Intuitively, the fluctuations in $y_t - y^*$, occurring on a shorter timescale, can be averaged out. However, we have not pursued this here.

3. Definitions and preliminaries. We begin with a couple of definitions that help us describe the likelihood of important events, and the intervals of time over which they hold.

DEFINITION 3.1. An event A holds:

- *with high probability* or whp in n if $\mathbb{P}(A) \geq 1 - e^{-\gamma n}$ for some $\gamma > 0$ and n large enough, and
- *for a very long time in n after $T(n)$* if it holds for $T(n) \leq t \leq e^{\gamma n}$ for some $\gamma > 0$ and n large enough.

Also, A holds:

- *with good probability* or wgp in n if $\mathbb{P}(A) \geq 1 - e^{-\gamma(\log n)^2}$ for some $\gamma > 0$ and n large enough, and
- *for a long time in n after $T(n)$* if it holds for $T(n) \leq t \leq e^{\gamma(\log n)^2}$ for some $\gamma > 0$ and n large enough.

In either case, if $T(n)$ is not mentioned then $T(n) \equiv 0$.

Note that both high probability and good probability are preserved under finite intersections. Also, since $e^{-c(\log n)^2} = n^{-c\log n}$ and $\log n$ is increasing and unbounded, if an event holds with good probability then for any $\alpha > 0$ and n large enough, it holds with probability $\geq 1 - n^{-\alpha}$. By the same token, for any $\alpha > 0$ and n large enough, if an event holds for a long time in n after $T(n)$ then in particular it holds for $T(n) \leq t \leq n^{\alpha}$.

REMARK 3.1. A useful trick lets us boost whp over a single time interval to whp for a very long time, and similar to boost wgp to wgp for a long time. Namely, E is an event measurable in terms of the state of the process, and has the property that

$$\mathbb{P}(E \text{ fails to hold for some } t \in [0, h]) \leq e^{-cn}$$

for some $c, h > 0$ uniformly in the initial values of the process, then by iterating on time intervals $[kh, (k+1)h]$ for $k = 0, 1, \dots, e^{cn/2}$ and taking a union bound we find that

$$\mathbb{P}(E \text{ fails to hold for some } t \in [0, e^{cn/2}h]) \leq e^{-cn/2}.$$

REMARK 3.2. In this paper, the ultimate goals are to show that:

1. extinction by time $C\sqrt{N}$ has positive probability uniformly in the initial configuration, and that
2. non-extinction by time $c\sqrt{N}$ has positive probability within a certain range of configurations.

Therefore:

1. We can restrict the process to the intersection of any finite collection of events $E_1, \dots, E_k$, if each E_i holds with probability $1 - o(1)$.
2. We only care about what happens to the process over the time horizon $\sqrt{N}$.

In particular, if $E_1, \dots, E_k$ each hold with good probability for a long time after $T_i(N)$, with each $T_i = o(\sqrt{N})$, then we can ignore what happens before $\max_i T_i$, and so can assume that $\bigcap_{i=1}^k E_i$ always holds, and we do not need to keep track of its probability.

The following basic large deviations estimate is proved in [5] and is useful throughout.

LEMMA 3.1. *Let X be Poisson distributed with mean μ, then*

$$\mathbb{P}\bigl(X > (1+\delta)\mu\bigr) \le e^{-\delta^2\mu/4} \qquad \text{for } 0 < \delta \le 1/2,$$
$$\mathbb{P}\bigl(X < (1-\delta)\mu\bigr) \le e^{-\delta^2\mu/2} \qquad \text{for } \delta > 0.$$

The following result, proved in [5], is the starting point for our investigations.

LEMMA 3.2. *For each $\varepsilon > 0$, there is $T > 0$ so that with high probability in N for a very long time after T, $|V_t| \le \varepsilon N$ and $|y_t - y^*| \le \varepsilon$.*

REMARK 3.3. As our first application of Remark 3.2, since $T > 0$ is constant, and thus $o(\sqrt{N})$ and since $\sqrt{N} = o$ (a very long time), we may always fix $\varepsilon > 0$ and assume that both $|V_t| \le \varepsilon N$ and $|y_t - y^*| \le \varepsilon$ holds for all time.

In [5], for $R_0 \neq 1$ we deal with the regime $|V_t| \le \varepsilon N$ using comparison to a branching process, but for $R_0 = 1$ that approach does not work. Since the value of y_t affects the rate of spread of the infection, our first step is to get better control on the proportion of singletons.

4. Proportion of singletons. In this section, we consider the proportion of singletons y_t, and control its distance from equilibrium, defined as $\delta y := y - y^*$.

First, we make an observation that guides our choice of spatial rescaling. Letting $\mathcal{Y}_t = \sqrt{N}(y_t - y^*)$, $\mathcal{Y}_t$ has transitions

$$\mathcal{Y}_t \to \begin{cases} \mathcal{Y}_t + 2/\sqrt{N}, & \text{at rate } r_-(1-y^*)N/2 - r_-\sqrt{N}\mathcal{Y}_t/2, \\ \mathcal{Y}_t - 2/\sqrt{N}, & \text{at rate } r_+(y^*)^2N/2 + r_+(\sqrt{N}y^*\mathcal{Y}_t + \mathcal{Y}_t^2/2). \end{cases}$$

Noting that $r_-(1-y^*) - r_+(y^*)^2 = 0$ by definition of y^*, in a time step $h = 1/N$,

$$E[\mathcal{Y}_{t+h} - \mathcal{Y}_t|\mathcal{Y}_t] = -(1/N)\big(r_-\mathcal{Y}_t + 2r_+y^*\mathcal{Y}_t + o(1)\big),$$

$$\mathrm{Var}(\mathcal{Y}_{t+h} - \mathcal{Y}_t|\mathcal{Y}_t) = (2/N)\big(r_-(1-y^*) + r_+(y^*)^2 + o(1)\big)$$

so letting

$$\mu(x) = -(r_- + 2r_+y^*)x \quad \text{and} \quad \sigma^2 = r_-(1-y^*) + r_+(y^*)^2$$

we find that $\mathcal{Y}_t$ approaches the solution to the diffusion equation

$$d\mathcal{Y}_t = \mu(\mathcal{Y}_t)\,dt + \sigma\,dB_t.$$

It may seem that making this rigorous would be a more direct way of showing fluctuations in $y_t - y^*$ are of order $N^{-1/2}$, as opposed to the somewhat bare hands approach that we take here. However, as seen in Lemma 4.8, our approach yields quantitative estimates on the probability that fluctuations in $y_t - y^*$ exceed the prescribed values. This will be useful later on when we need to apply these estimates a number of times that grows unboundedly with N.

We begin by showing that after a little while, δy_t has reached a fairly small value.

LEMMA 4.1. *Let $\tau = \inf\{t : |\delta y_t| \le \log N/\sqrt{N}\}$. Then there is $C > 0$ so that $\tau \le C\log N$ wgp in N.*

PROOF. In what follows, ε, c, C refer to positive constants such that $\varepsilon, c, 1/C$ may get smaller from step to step. Moreover, some inequalities hold only for N large enough. By Lemma 3.2, for any fixed $\varepsilon > 0$, there are $T, \gamma > 0$ so that

$$\text{whp } |\delta y_t| \le \varepsilon \qquad \text{for all } T \le t \le e^{\gamma N}$$

so we may assume $|\delta y| \le \varepsilon$. Let

$$r_u(y) = r_-(1-y)/2 \quad \text{and} \quad r_d(y) = r_+y^2/2 - r_+y/(2N)$$

so that $Nr_u(y_t), Nr_d(y_t)$ are the respective rates of upward and downward transitions in y_t. First, observe that

$$\max\big(r_u(y), r_d(y)\big) \le \max(r_+, r_-)/2 \qquad \text{for all } y \in [0, 1], \tag{4.1}$$

which is $\le C$ for any $C \ge \max(r_+, r_-)/2$, and since $y^* \in (0, 1)$, for any $0 < \varepsilon \le \min(y^*, 1-y^*)/2 > 0$ and $|y - y^*| \le \varepsilon$,

$$\min\big(r_u(y), r_d(y)\big) \ge \min(r_+, r_-)\varepsilon^2/2 - r_+/(2N),$$

which is $\geq \min(r_+, r_-)\varepsilon^2/4 > 0$ for N large enough. By definition of y^*, $r_u(y^*) - r_d(y^*) = r_+ y^*/(2N)$, so writing r_u and r_d as functions of δy,

$$r_u(\delta y) - r_d(\delta y) = \big(r_u'(y^*) - r_d'(y^*)\big)\delta y + o(\delta y) + r_+ y^*/(2N)$$

moreover $(r_u'(y^*) - r_d'(y^*)) < 0$. If $N \cdot |\delta y| \to \infty$ as $N \to \infty$ then $r_+ y^*/(2N) = o(\delta y)$—this is ok for this proof since we may assume $|\delta y| \geq 1/\sqrt{N}$. Thus, for $\varepsilon > 0$ small enough, N large enough and $|\delta y| \leq \varepsilon$,

$$\text{if } \delta y > 0 \text{ then } r_d(\delta y) - r_u(\delta y) \geq \varepsilon \cdot \delta y \tag{4.2}$$

and if $\delta y < 0$ then $r_u(\delta y) - r_d(\delta y) \geq \varepsilon \cdot |\delta y|$.

Suppose $\delta y_0 > 0$; a similar argument applies in the opposite case. Define

$$\tau = \inf\{t : \delta y_t \leq (1 - \varepsilon^2)\delta y_0\}.$$

Since $r_u(\delta y)$ decreases with δy, if $t < \tau$ then

$$r_u(\delta y_t) \leq r_u\big((1 - \varepsilon^2)\delta y_0\big) \tag{4.3}$$

and since $r_d(\delta y)$ increases with δy, and using (4.2), if $t < \tau$ then

$$r_d(\delta y_t) \geq r_d\big((1 - \varepsilon^2)\delta y_0\big) \geq r_u\big((1 - \varepsilon^2)\delta y_0\big) + \varepsilon(1 - \varepsilon^2)\delta y_0. \tag{4.4}$$

Using Lemma 3.1 with

$$\mu_u = r_u\big((1 - \varepsilon^2)\delta y_0\big) \cdot N \quad \text{and} \quad \delta_u \mu_u = \tfrac{1}{3}\varepsilon(1 - \varepsilon^2)\delta y_0 \cdot N$$

since $\mu_u \leq CN \cdot \delta y_0$ by (4.1), it follows that $\delta_u \geq c\delta y_0^2 > 0$ uniformly in N for some $c > 0$. Using (4.3), with probability $\geq 1 - \exp(-cN\delta y_0^2)$ for some $c > 0$, either $\tau < 1$ or there are at most $(1 + \delta_u)\mu_u$ upward transitions before time 1.

Similarly, using Lemma 3.1 with

$$\mu_d = r_d\big((1 - \varepsilon^2)\delta y_0\big) \cdot N \quad \text{and} \quad \delta_d \mu_d = \tfrac{1}{3}\varepsilon(1 - \varepsilon^2)\delta y_0 \cdot N$$

and using the first part of (4.4), with probability $\geq 1 - \exp(-cN\delta y_0^2)$, either $\tau < 1$ or there are at least $(1 - \delta_d)\mu_d$ downward transitions before time 1. Using (4.4),

$$(1 - \delta_d)\mu_d - (1 + \delta_u)\mu_u = \tfrac{1}{3}\varepsilon\big(1 - \varepsilon^2\big)\delta y_0$$

and since each transition moves δy by $2/N$, with probability $\geq 1 - 2\exp(-cN\delta y_0^2)$, either $\tau < 1$ or

$$\delta y_1 \leq \delta y_0 - \tfrac{1}{3}\varepsilon\big(1 - \varepsilon^2\big)\delta y_0$$

and for $\varepsilon > 0$ small enough this is at most $(1 - \varepsilon^2)\delta y_0$, which implies $\tau \leq 1$.

Summarizing so far, there are constants $\varepsilon, c > 0$ so that if $\delta y_0 \geq 1/\sqrt{N}$, then with probability $\geq 1 - 2\exp(-cN\delta y_0^2)$ there is $t \in (0, 1]$ so that $\delta y_t \leq (1 - \varepsilon^2)\delta y_0$. Also, since a similar argument applies when $\delta y_0 < 0$, the same is true if δy_t is replaced by $|\delta y_t|$.

If $|\delta y| \ge c \log N/\sqrt{N}$, then $1 - 2\exp(-cN\delta y_0^2)$ is at least

$$1 - \exp\bigl(-cN(c\log N/\sqrt{N})^2\bigr) = 1 - \exp\bigl(-c(\log N)^2\bigr)$$

for some possibly smaller $c > 0$. Since $|\delta y_0| \le 1$, choosing k so that $|\delta y_0|(1-c)^k \le \log N/\sqrt{N}$ gives $k \le C\log N$ with $C = 1/(2\log(1/(1-c)))$. Taking a union bound, with probability at least

$$1 - C\log(N)e^{-c(\log N)^2} \ge 1 - e^{-(c/2)(\log N)^2}$$

for N large enough, and using the strong Markov property at each stopping time, there is a time $t \le C\log N$ so that $|\delta y_t| \le \log N/\sqrt{N}$, as desired. $\square$

The next step is to examine what happens to δy_t on the scale $1/\sqrt{N}$. Before doing so, we collect some facts about random walk on an interval, absorbed at the boundary. The first is a restatement of Theorem 6.4.6 in [4].

LEMMA 4.2. *Let X_n be a* (*discrete time*) *random walk on* $\{0, \dots, M\}$, *absorbed at* $\{0, M\}$, *with*

$$X_{n+1} = \begin{cases} X_n + 1, & \textit{with probability } p_{x,x+1}, \\ X_n - 1, & \textit{with probability } p_{x,x-1} = 1 - p_{x,x+1}, \end{cases}$$

when $X_n \in \{1, \dots, M-1\}$. Let $T = \inf\{n \ge 0 : X_n \in \{0, M\}\}$ and let $b(x) = p_{x,x-1}/p_{x,x+1}$, then for $x = 1, \dots, M-1$,

$$\mathbb{P}(X_T = M | X_0 = x) = \frac{1 + \sum_{j=1}^{x-1}\prod_{i=1}^{j} b(i)}{1 + \sum_{j=1}^{M-1}\prod_{i=1}^{j} b(i)}$$

with the numerator equal to 1 *when* $x = 1$.

LEMMA 4.3. *Let M, X_n, T, b be as in Lemma 4.2. There is $C > 0$ so that if $b(x) \equiv b$ is constant in x* (*although it may depend on M*) *and M is large enough, then for integer $m \ge 1$,*

$$\mathbb{P}\bigl(T \ge mCM^2\bigr) \le 2^{-m}.$$

If $b = 1$, then for each $c > 0$, there is $p(c) > 0$ with $p(c) \to 1^-$ as $c \to 0^+$, such that

$$\mathbb{P}\bigl(T \ge cM^2\bigr) \ge p(c).$$

The above two statements remain true if X_t is a continuous time random walk moving at rate 1.

PROOF. If $b(x) \equiv b$ is constant, by symmetry it is enough to consider $b \geq 1$. In this case a simple coupling argument shows that X_n is stochastically dominated by symmetric simple random walk Y_n on $\mathbb{Z}$, provided $X_0 \leq Y_0$. Letting $Y_0 = X_0$ and $T' = \inf\{n : Y_n = 0\}$, T is stochastically dominated by T'. Letting $T_\pm$ denote $\inf\{n : Y_n = M/2 \pm M/2\}$, the reflection principle plus the local central limit theorem gives, for each $a > 0$,

$$\lim_{M\to\infty} \mathbb{P}(T_+ \leq aM^2/4) = 2\int_{1/\sqrt{a}}^{\infty} e^{-x^2/2}\,dx$$

and the same holds, by symmetry, for T_-. Taking a large enough that the right-hand side is $\geq 3/4$, letting $C = a/4$ and noting that $T' = \min(T_+, T_-)$, we find $\mathbb{P}(T' > CM^2) \leq 1/2$ for large enough M, and the same holds for T by comparison. Using the Markov property to iterate gives $\mathbb{P}(T > mCM^2) \leq (1/2)^m$ as desired.

To get the second statement, given $c > 0$ let $r(c) = 1 - 2\int_{1/\sqrt{4c}}^{\infty} e^{-x^2/2}\,dx > 0$ and let $p(c) = r(c)e^{-c}$. Then, $\lim_{M\to\infty} \mathbb{P}(T' > cM^2) = r(c) > p(c)$, for large enough M the value is $\geq p(c)$, moreover $p(c) \to 1^-$ as $c \to 0^+$.

The result in continuous time follows in the same way after controlling the number of transitions up to time t with the help of Lemma 3.1; details are omitted. □

Next, we look at a specific Markov chain that as shown later roughly corresponds to the sequence of visits of δy_t to the points $\{k/\sqrt{N} : k = 1, \dots\}$. The choice $p_{k,k-1}/p_{k,k+1} = e^{ck}$ is motivated by the upcoming (4.6).

LEMMA 4.4. *Let K_n be the Markov chain on $\{1, 2, \dots\}$ with $p_{1,2} = 1$ and $p_{k,k-1} + p_{k,k+1} = 1$ with $p_{k,k-1}/p_{k,k+1} = e^{ck}$ for $k > 1$, for some fixed $c > 0$. Let $T = \inf\{n > 0 : K_n = 1\}$ and for $j, k > 1$ let $\rho_{j,k} = \mathbb{P}(K_n = k$ for some $0 < n < T | K_0 = j)$ and let*

$$G(j,k) = \sum_{i=0}^{T-1} \mathbf{1}(K_i = k | K_0 = j).$$

Then:

1. $\rho_{j,k} = 1$ *if* $j > k$,
2. $\rho_{k,k} = p_{kk+1} + p_{k,k-1}\rho_{k-1,k}$ *and*
3. $\rho_{j,k} \leq j e^{-c(\binom{k}{2} - \binom{j}{2})}$ *if* $j < k$

and for $d > 0$, $\mathbb{P}(G(j,k) > d) = \rho_{j,k}\rho_{k,k}^d$.

PROOF. Since K_n moves only step at a time, it visits all of $\{2, \dots, K_0\}$ before hitting 1, so if $j > k$ then $\rho_{j,k} = 1$. The formula for $\rho_{k,k}$ follows from this fact after conditioning on the value of K_1. If $j < k$ then letting $T_k = \inf\{n : K_n \in \{1,k\}\}$, $\rho_{j,k} = \mathbb{P}(K_{T_k} = k | K_0 = j)$ so using Lemma 4.2,

$$\rho_{j,k} = \frac{1 + \sum_{\ell=1}^{j-1} e^{c(\ell+1)\ell/2}}{1 + \sum_{\ell=1}^{k-1} e^{c(\ell+1)\ell/2}} \leq j e^{(c/2)(j(j-1)-k(k-1))}$$

as desired. The last statement follows from the definition of $\rho_{j,k}$ and the Markov property. $\square$

Using this result, we can control the sum of K_n from the time it starts at level k until it hits level 1.

LEMMA 4.5. *For K_n as in Lemma 4.4 and $j > 1$, let $K_0 = j$ and let $G(j) = \sum_{n=0}^{T-1} K_n = \sum_{k>1} kG(j,k)$, then for d large enough and some $C > 0$ that depends on c,*

$$\mathbb{P}\big(G(j) \leq 9j^2 d + jd^2/2 + d^3/4\big) \geq 1 - Ce^{-cd/4}.$$

PROOF. We use the results of Lemma 4.4 without mention. Let d_k be a sequence of integers, then

$$\mathbb{P}\bigg(G(j) > \sum_{k>1} kd_k\bigg) \leq \sum_{k>1} \mathbb{P}\big(G(j,k) > d_k\big)$$

so given d, we want a sequence d_k such that

$$\sum_{k>1} kd_k \leq j^2 d + jd^2 + d^3 \quad \text{and} \quad \sum_{k>1} \mathbb{P}\big(G(j,k) > d_k\big) \leq 3e^{-2cd/3}.$$

By definition $p_{k,k+1} = 1/(1+e^{ck}) \leq e^{-ck}$ and since $\binom{k}{2} - \binom{k-1}{2} = k-1$, $\rho_{k-1,k} \leq (k-1)e^{-c(k-1)}$, and so

$$\rho_{k,k} \leq ke^{-ck/3} \qquad \text{for all } k > 1.$$

For any k, it follows that $\mathbb{P}(G(j,k) > d_k) \leq ke^{-ckd_k/3}$. If $k > j$ then for $i = k - j \geq 1$, since $\binom{i+j}{2} - \binom{j}{2} = (i^2 + 2ji - 1)/2 \geq i^2/2$,

$$\rho_{j,k} \leq je^{-ci^2/2}$$

and so

$$\mathbb{P}\big(G(j,k) > d_k\big) \leq jke^{-c(i^2/2 + kd_k/3)}.$$

If $i \geq 2k/3$, that is, $k \geq 3j$, then use $d_k = d - i$ so that $i^2 + kd_k/3 \geq kd/3$. If $k < 3j$ use $d_k = d$. Then

$$\sum_{k>1} \mathbb{P}\big(G(j,k) > d_k\big) \leq \sum_{2\leq k\leq 3j} ke^{-ckd/3} + \sum_{k>3j} jke^{-cdk/3},$$

which is at most $Ce^{-cd/4}$ when d is large enough, and

$$\sum_{k>1} kd_k = \sum_{k=2}^{3j} kd + \sum_{i=1}^{\lfloor d \rfloor}(j+i)(d-i) \le 9j^2 d + jd^2/2 + d^3/4. \qquad \square$$

Before continuing, we collect some large deviations estimates. Since these are standard facts, their proof is given in the Appendix.

LEMMA 4.6. *If $X \sim \text{binomial}(n, p)$ then for $x > 0$ and letting $r = x/np$,*

$$\mathbb{P}(X > x) \le e^{-x(1/r + \log(r/e))}$$

and for $0 < \delta < 1$,

$$\mathbb{P}\big(X < (1-\delta)np\big) \le e^{-np\delta^2/2}.$$

If $X_i, i = 1, \dots, m$ are independent and $X_i \sim \text{geometric}(p)$, that is, $\mathbb{P}(X_i > d) = p^d$ for $i = 1, \dots, m$ then letting $S_m = X_1 + \cdots + X_m$,

$$\mathbb{P}\big(S_m > (1+\delta)m/(1-p)\big) \le e^{-m(\delta - \log(1+\delta))}.$$

Next, we control the sum of K_n over repeated excursions away from level 1. The proof consists of repeated application of large deviations estimates, as well as a truncation/estimation step.

LEMMA 4.7. *Let $T_0 = 0$ and for $n = 1, 2, \dots$ let $T_n = \inf\{m > T_{n-1} : K_m = 1\}$. If $K_0 = 1$ then there is $C > 0$ so that with good probability in n,*

$$\sum_{i=0}^{T_n - 1} K_i \le Cn.$$

PROOF. For $k > 1$, let $G_j(2,k) = \sum_{i=T_{j-1}+1}^{T_j - 1} \mathbf{1}(K_i = k)$, then

$$\sum_{i=0}^{T_n - 1} K_i = n + \sum_{1 \le j \le n, k > 1} kG_j(2,k)$$

and for fixed k, $\{G_j(2,k) : j \in \mathbb{N}\}$ are independent and distributed like $G(2,k)$ from Lemma 4.4. Letting $J_n(k) = \{j \in \{1, \dots, n\} : G_j(2,k) > 0\}$ be the excursions that reach level k, and letting $V_n(k) = |J_n(k)|$, since $\mathbb{P}(G_j(2,k) > 0) = \rho_{2,k}$ and by independence of successive excursions, $V_n(k) \sim \text{binomial}(n, \rho_{2,k})$ and

$$\sum_{i=0}^{T_n - 1} K_i = n + \sum_{k>1} k \sum_{j \in J_n(k)} G_j(2,k)$$

and for fixed k, $\{G_j(2,k) : j \in J_n(k)\}$ are independent and distributed like $\{G(2,k)|G(2,k) > 0\}$, which has $\mathbb{P}(G(2,k) > d|G(2,k) > 0) = \rho_{k,k}^d$, that is, is geometric with parameter $\rho_{k,k}$. Since by Lemma 4.4,

$$\rho_{2,k} \le e^{-(c/2)(k(k-1)-2)} \le e^{-(c/2)((k/2)^2)} = e^{-ck^2/8}$$

it follows that $V_n(k)$ is at most binomial(n,p) with $p = e^{-ck^2/8}$. Recalling Lemma 4.6, to find the values k such that $V_n(k) = 0$ with good probability set $x = 1/2$ so that $r = x/(np) \ge e^{ck^2/8}/(2n)$ and then

$$\mathbb{P}\big(V_n(k) > 0\big) \le e^{-(1/2)[ck^2/8 - \log(2ne)]} = (2ne)^{1/2} e^{-ck^2/16},$$

which if $k = \lceil \log n \rceil + k'$ is at most

$$(2ne)^{1/2} e^{-c(\log n + k')^2/16} \le (2e)^{1/2} n^{1/2-(c/16)\log n} e^{-ck'^2/16}.$$

Since $(2e)^{1/2} \sum_{k' \ge 1} e^{-ck'^2/16} < \infty$, for $C > 0$ large enough we find that

$$\mathbb{P}\big(V_n(k) = 0 \text{ for all } k > \lceil \log n \rceil\big) \ge 1 - Cn^{1/2-(c/16)\log n},$$

which is at most $Cn^{-(c/32)\log n}$, for some possibly larger $C > 0$. For $k = 2, \dots, \lceil \log n \rceil$ and $C > 0$ let $x = Cnk^{-3}$, then $r = x/(np) \ge Ck^{-3}e^{ck^2/8}$ and

$$\mathbb{P}\big(V_n(k) > Cnk^{-3}\big) \le e^{-nk^{-3}(ck^2/8 - 3\log k + \log(C/e))},$$

which is at most $e^{-(c/16)n/k}$ for every $k \ge 2$ provided C is taken large enough that $3\log k - \log(C/e) \le ck^2/16$ uniformly for $k \ge 2$. Since $k \le \log n + 1 = \log(ne)$, $e^{-(c/16)n/k} \le e^{-(c/16)n/(\log(ne))}$ and after summing and taking the complement we find

$$\mathbb{P}\big(V_n(k) \le Cnk^{-3} \text{ for } k = 2, \dots, \lceil \log n \rceil\big) \ge 1 - \log(n) e^{-(c/16)n/(\log(ne))}.$$

It remains to control the sums

$$\sum_{j \in J_n(k)} G_j(2,k).$$

If $j \in J_n(k)$ then $G_j(2,k) > 0$ and as noted above, $\{G_j(2,k)|G_j(2,k) > 0\}$ is geometric with parameter $p = \rho_{k,k}$, that is, $\mathbb{P}(G_j(2,k) = d|G_j(2,k) > 0) = p^{d-1}(1-p)$. With the notation of Lemma 4.6, and setting $\delta = 1/2$ while noting $1/2 - \log(3/2) \ge 1/16$,

$$\mathbb{P}\big(S_m > (3/2)m/(1-p)\big) \le e^{-m/16}.$$

Thus, on the event that $V(k) \le Cnk^{-3}$ for each $k \in \{2, \dots, \lceil \log n \rceil\}$, using the above estimate for each k with $m = Cnk^{-3} \ge Cn/(3\log ne)$ and recalling that $p = \rho_{k,k}$, we find

$$\mathbb{P}\bigg(\sum_{j \in J(k)} G_j(2,k) \le (3/2)Cnk^{-3}/(1-\rho_{k,k}) \text{ for } k = 2, \dots, \lceil \log n \rceil\bigg)$$
$$\ge 1 - \log(n) e^{-Cn/(48 \log ne)}.$$

If $k > 1$ then $\rho_{k,k} < 1$, and from the proof of Lemma 4.5 we have $\rho_{k,k} \le 2e^{-ck/3}$, so $\sum_{k>1} k^{-2}/(1-\rho_{k,k}) < \infty$. Summarizing, for some $C > 0$ the events considered occur together with probability $\ge 1 - Cn^{-(c/32)\log n}$, and when they occur,

$$\sum_{i=0}^{T_n-1} K_i = n + \sum_{j \in J, k>1} kG_j(2,k) \le n\left(1 + (3/2)C \sum_{k=2}^{\lceil \log n \rceil} k^{-2} \Big/ (1-\rho_{k,k})\right) \le Cn$$

for some possibly larger $C > 0$ that does not depend on n. $\square$

Next, we come to the main event of this section. Since the proof is somewhat lengthy, it is broken up into a few pieces.

LEMMA 4.8. *Let $\delta y_t = y_t - y^*$ and let $T = \inf\{t : \delta y_t \le 1/\sqrt{N}\}$, then*:

1. *there is $C > 0$ so that if N large enough, then with good probability in t, if $|\delta y_0| \le 1/\sqrt{N}$ then*

$$\int_0^t |\delta y_s| \, ds \le Ct/\sqrt{N}$$

2. *for each $c > 0$, with good probability in N, if $|\delta y_0| \le c \log N/\sqrt{N}$ then $|\delta y_t| \le 2c \log N/\sqrt{N}$ for a long time, and*
3. *there is $C' > 0$ so that for each $C > 0$, if $|\delta y_0| \le C \log N/\sqrt{N}$ then with good probability in N*,

$$\int_0^T |\delta y_t| \, dt \le C'(\log N)^6/\sqrt{N}.$$

PROOF. As in the proof of Lemma 4.1, we use c and C that may change from step to step, and we may assume that $|\delta y_t| \le \varepsilon$ for small $\varepsilon > 0$. Since it is nicer to work with integer-valued things, we will work with $Y_t = Ny_t$ instead of y_t. Recall from Section 2 that

$$Y_t \to \begin{cases} Y_t + 2, & \text{at rate } (N - Y_t)r_-/2, \\ Y_t - 2, & \text{at rate } Y_t(Y_t - 1)r_+/(2N). \end{cases}$$

Defining $\delta Y_t := Y_t - 2\lfloor Ny^*/2 \rfloor$, if N is even the range of δY_t is then a subset of $\{x \in 2\mathbb{Z} : |x| \le N\}$. We will stick with this convention, but nothing is really different if N is odd.

Step 1: *Upper bound on* $|\delta Y_t|$. Our first task is to define a nicer looking chain X_t that dominates $|\delta Y_t|$. Notice that $\delta y - \delta Y/N = \lfloor Ny^* \rfloor/N - y^* = O(1/N)$, and so

$$\sqrt{N}\delta y - \delta Y/\sqrt{N} = O(1/\sqrt{N}).$$

In other words, δy, on the scale $1/\sqrt{N}$, is for large N very nearly equal to δY, on the scale $\sqrt{N}$, so it is enough to prove our results for δY on the scale $\sqrt{N}$.

Recall r_u and r_d from the proof of Lemma 4.1, that can be viewed as functions of δY. From that proof, it follows that for some $c > 0$ and N large enough, if $|\delta Y| \geq \sqrt{N}$ then

$$r_d(\delta Y) - r_u(\delta Y) \geq c \cdot \delta Y/N \qquad \text{for } \delta Y > 0$$

and similarly $r_u - r_d \geq c \cdot |\delta Y|/N$ for $\delta Y < 0$. Since $\max(r_u, r_d) \leq C$, if $\delta Y > 0$ and suppressing the argument,

$$r_d/r_u = (r_d - r_u)/r_u + 1 \geq 1 + (c/C) \cdot \delta Y/N$$

and similarly $r_u/r_d \geq 1 + (c/C) \cdot |\delta Y|/N$ when $\delta Y < 0$. Let

$$q_+(\delta Y) := N r_u(\delta Y) \quad \text{and} \quad q_-(\delta Y) := N r_d(\delta Y)$$

denote the respective rates of upward and downward transitions in δY. It follows that

$$q_-/q_+ \geq 1 + c \cdot \delta Y/N \qquad \text{when } \delta Y > 0$$

and $q_+/q_- \geq 1 + c \cdot |\delta Y|/N$ when $\delta Y < 0$. Letting $M = 2\lceil \sqrt{N}/2 \rceil$, a simple coupling argument then shows that $|\delta Y_t|$ is stochastically dominated by the continuous time Markov chain X_t on $2\mathbb{N} \cap [M, \infty)$ with $X_0 = \max(\delta Y_0, M)$ and

$$X_t \to \begin{cases} X_t + 2, & \text{at rate } q_+(X_t), \\ X_t - 2, & \text{at rate } q_+(X_t)(1 + cX_t/N), \end{cases}$$

with the proviso that if $X_t = M$ then the rate of decrease is zero.

Step 2: *Upper bound on* X_t. In order to apply earlier lemmas, we make one more upper bound. Let $S = \{jM : j \in \mathbb{Z}\}$ and define a process Z_t on $2\mathbb{N} \cap [M, \infty)$ as follows. Let $t_0 = \inf\{t > 0 : Z_t \in S\}$ and define recursively

$$t_i = \inf\bigl\{i > t_{i-1} : Z_t \in \{Z_{t_{i-1}} \pm M\}\bigr\}$$

and let $K_i = Z_{t_i}/M$. Then let

$$r_-(x, k) = \begin{cases} 0, & \text{if } x = M, \\ q_+(x), & \text{if } x > M, k = 1, \\ q_+(x)\bigl(1 + ck/(2\sqrt{N})\bigr), & \text{if } x > M, k > 1, \end{cases}$$

and for $t_{i-1} \leq t < t_i$ define Z_t by the rates

$$Z_t \to \begin{cases} Z_t + 2, & \text{at rate } q_+(Z_t), \\ Z_t - 2, & \text{at rate } r_-(Z_t, K_i). \end{cases}$$

Comparing the rates one verifies that X_t is stochastically dominated by Z_t, which in turn means that δY_t is dominated by Z_t. Notice that K_i is a discrete time Markov chain on the state space $\{1, 2, \ldots\}$ and satisfies $p_{12} = 1$ and $p_{k,k-1} + p_{k,k+1} = 1$ for $k > 1$. Since its transition rates depend on past values several steps back in time, Z_t is not, globally in time, a Markov chain. However, if we define

$$\tilde{Z}_t^{(i)} = Z_{t_{i-1}+t} \qquad \text{for } t \in [0, t_i - t_{i-1})$$

then for each i, $\tilde{Z}_t^{(i)}$ is a continuous time Markov chain on its interval of definition.

Step 3: *Item* 1. Since Z_t dominates δY_t, it is enough to show $\int_0^t Z_s\,ds \le Ct\sqrt{N}$ with good probability in t. To tackle this integral, we break it up as follows: if $t_{n-1} \le t < t_n$ then

$$\frac{1}{t}\int_0^t Z_s\,ds \le \frac{1}{t_{n-1}}\sum_{i=1}^n (t_i - t_{i-1})(K_{i-1}+1)M. \tag{4.5}$$

Transition probabilities of K_i. We examine what happens on time intervals $[t_{i-1}, t_i)$, where K_{i-1} is fixed. For $i > 0$, letting $s_0 = 0$ and

$$s_j = \inf\{t > s_{j-1} : \tilde{Z}_t^{(i)} \neq \tilde{Z}_{s_{j-1}}^{(i)}\}$$

be the jump times of $\tilde{Z}_t^{(i)}$ and defining $\tilde{Z}_j^{(i)} = \tilde{Z}_{s_j}^{(i)}$, as shown in [9], $\tilde{Z}_j^{(i)}$ is a discrete time Markov chain with

$$\tilde{Z}_j^{(i)} = \begin{cases} \tilde{Z}_j^{(i)} + 2, & \text{w.p. } p_+ = q_+/(q_- + q_+), \\ \tilde{Z}_j^{(i)} - 2, & \text{w.p. } p_- = q_-/(q_- + q_+), \end{cases}$$

and the random variables $\{s_j - s_{j-1} : n \ge 0\}$ are independent and exponentially distributed with exponential rate $q_+ + q_-$. Note that $p_-/p_+ = q_-/q_+$. In our case, $cN \le q_+ + q_- \le CN$ and $p_-/p_+ = 1$ if $K_{i-1} = 1$ and $= 1 + cK_{i-1}/(2\sqrt{N})$ if $K_{i-1} > 1$, and in particular is constant. If $k > 1$, using the transition probabilities for $\tilde{Z}_j^{(i)}$ and Lemma 4.2, the transition probability $p_{k,k+1}$ for K_i satisfies

$$\begin{aligned} p_{k,k+1} &= \frac{1 + \sum_{n=1}^{M-1}(1 + ck/(2\sqrt{N}))^n}{1 + \sum_{n=1}^{2M-1}(1 + ck/(2\sqrt{N}))^n} \\ &= \frac{(1 + ck/(2\sqrt{N}))^M - 1}{(1 + ck/(2\sqrt{N}))^{2M} - 1} \\ &= \bigl((1 + ck/(2\sqrt{N}))^M + 1\bigr)^{-1} \end{aligned} \tag{4.6}$$

since the denominator is a difference of squares. Letting $x = (1 + ck/(2\sqrt{N}))^M$, $p_{k,k+1} = 1/(x+1)$ so $p_{k,k-1} = x/(x+1)$ and $p_{k,k-1}/p_{k,k+1} = x$, and since $M = \lceil\sqrt{N}\rceil$, $x \ge e^{ck}$ for some $c > 0$ uniformly for $k \in \{2, \dots, \sqrt{N}\}$. Therefore, K_i is stochastically dominated by the Markov chain with $p_{k,k-1}/p_{k,k+1} = e^{ck}$ defined in Lemma 4.4, and so are the quantities $T_j - T_{j-1}$ and

$$G_j(2,k) = \sum_{i=T_{j-1}}^{T_j - 1} \mathbf{1}(K_i = k)$$

from the proof of Lemma 4.7. Rewriting (4.5) in terms of the times T_j and emphasizing the different levels k, for $t_{T_{n-1}} \le t < t_{T_n}$ we have

$$\frac{1}{t}\int_0^t Z_s\,ds \le \frac{1}{t_{T_{n-1}}}\sum_{k\ge 1}(k+1)\sum_{i=1}^{T_n}(t_i - t_{i-1})\mathbf{1}(K_{i-1} = k)M. \tag{4.7}$$

Controlling $t_i - t_{i-1}$. Let j_0 be such that $s_{j_0} = t_i - t_{i-1}$.

Case 1: $K_{i-1} = 1$. In this case, the process $\tilde{Z}_t^{(i)}$ is a symmetric simple random walk on $\{0, 2, \dots, 2M\}$ reflected to remain above M. Using Lemma 4.3 with $b = 1$, for some $c > 0$ and M large enough we have

$$\mathbb{P}\big(j_0 \ge cM^2\big) \ge 1/2.$$

Since the rate of transitions in $\tilde{Z}_t^{(i)}$ is at most CN, using Lemma 3.1 with $\mu = cM^2/2$ and $\delta = 1/2$, with probability $\ge 1 - e^{-cM^2/32}$, at most $3cM^2/4$ transitions have occurred by time $cM^2/(2CN) \ge c/(2C)$, so combining with the above estimate on j_0,

$$\mathbb{P}\big(t_i - t_{i-1} \ge c/(2C)\big) \ge 1/2 - e^{-cM^2/32},$$

which is at least $1/4$ for M large enough. Since $K_{T_i} = 1$ for each i, for some $c > 0$, the cardinality of the set $\{i \le n : t_{T_i+1} - t_{T_i} \ge c\}$ is at least binomial$(n, 1/4)$. Then, using Lemma 4.6 with $\delta = 1/2$ and the fact that $t_{T_n} = \sum_{i=0}^{n-1} t_{T_i+1} - t_{T_i}$,

$$\mathbb{P}(t_{T_n} \ge cn/8) \ge 1 - e^{-n/32}.$$

Case 2: K_{i-1} *is any value*. In this case, $\tilde{Z}_t^{(i)}$ has the form described in Lemma 4.3 for some1 b, so for integer $m \ge 1$ and any value of K_{i-1},

$$\mathbb{P}\big(j_0 \ge CmM^2\big) \le 2^{-m}.$$

Since the rate of transitions in $\tilde{Z}_t^{(i)}$ is at least cN, using Lemma 3.1 with $\mu = mCM^2$ and $\delta = 1/2$, with probability $\ge 1 - e^{-mCM^2/16}$, at least $mCM^2/2$ transitions have occurred by time $mCM^2/(cN) \le mC$ for some possibly larger $C > 0$, so for some $c > 0$,

$$\mathbb{P}(t_i - t_{i-1} \ge mC) \le 2^{-m} + e^{-mCM^2/16} \le e^{-cm},$$

which implies that $t_i - t_{i-1}$ is at most $C(1 + \text{geometric}(e^{-c}))$.

Controlling (4.7). As shown in the proof of Lemma 4.7, for $i = 1, \dots, T_n$, with good probability in n there are no visits to levels $k > \log n + 1$, and there are at most Cnk^{-3} visits to levels $k \in \{2, \dots, \lceil \log n \rceil\}$. Using Lemma 4.6 and the above estimate on $t_i - t_{i-1}$, for $k \in \{2, \dots, \lceil \log n \rceil\}$ and noting $Cnk^{-3} \ge Cn/(3 \log n)$, with good probability in n,

$$\sum_{i=1}^{T_n} (t_i - t_{i-1}) \mathbf{1}(K_{i-1} = k) \le C^2\big(1 + (3/2)/\big(1 - e^{-c}\big)\big) nk^{-3},$$

which is at most Cnk^{-3} for some possibly larger $C > 0$, and since $K_i = 1$ exactly when $i = T_j$ for some j, with good probability in n, $\sum_{i=1}^{T_n} (t_i - t_{i-1}) \mathbf{1}(K_{i-1} = 1) \le Cn$. Summing on k, with good probability in n,

$$\sum_{k \ge 1} (k+1) \sum_{i=1}^{T_n} (t_i - t_{i-1}) \mathbf{1}(K_{i-1} = k) \le Cn$$

for some possibly larger $C > 0$ not depending on n. Combining this with the above estimate on t_{T_n} and using (4.7) shows that with good probability in n, for $t_{T_{n-1}} \le t < t_{T_n}$,

$$\frac{1}{t}\int_0^t Z_s\,ds \le \frac{8CnM}{c(n-1)} \le C\sqrt{N} \tag{4.8}$$

for some possibly larger $C > 0$; this is nearly enough to prove item 1 since we still need to show (4.8) holds with good probability in t. To do so, first note the random variables $\{T_n - T_{n-1} : n \ge 1\}$ are independent and identically distributed, and that K_n is stochastically dominated by the Markov chain L_n on $\{1, 2, \dots\}$ with $p_{12} = 1$, $p_{k,k-1} + p_{k,k+1} = 1$ for $k > 1$ and $p_{k,k-1}/p_{k,k+1} = e^{2c}$, $k \ge 2$, so defining

$$T = \inf\{i > 0 : L_i = 1 | L_0 = 1\},$$

$T_n - T_{n-1}$ is dominated by T. If $T > i$ then $L_i > 1$ and $L_i = 1 + (2X - i)$ where $X \sim \text{binomial}(i, p)$ with $p = 1/(1 + e^{2c}) < 1/2$, which implies $X > i/2$. Using Lemma 4.6 with $x = i/2$ gives $r = x/(ip) = 1/(2p) > 1$, so letting $c = 1/r + \log(r/e)$, since $1/r + \log(r/e)$ is positive when $r > 1$, $c_0 > 0$ so for $d \ge 1$, if i_0 is taken larger than 1,

$$\mathbb{P}(T > i) \le e^{-ci/2}$$

so T is at most $1 + Y$ where $Y \sim \text{geometric}(e^{-c/2})$. Writing $T_n = \sum_{j=1}^n T_j - T_{j-1}$, taking $C \ge n[1 + (1 + \delta)/(1 - e^{-ci/2})]$ and using again Lemma 4.6 with $\delta = 1/2$ while noting $1/2 - \log(3/2) \ge 1/16$,

$$\mathbb{P}(T_n \le Cn) \ge 1 - e^{-n/16}.$$

Using the fact that $t_n - t_{n-1}$ is at most $C(1 + \text{geometric}(e^{-c}))$, a similar estimate as for T_n shows that for N large enough, with high probability in n, $t_n \le Cn$, and combining these, with good probability in n, $t_{T_n} \le t_{Cn} \le Cn$, so if $t_{T_{n-1}} \le t < t_{T_n}$ then $n \ge t_{T_n}/C > t/C$ and since $x \mapsto e^{-c(\log x)^2}$ is decreasing, $e^{-c(\log n)^2} \le e^{-c(\log t - \log C)^2}$ which for t large enough is at most $e^{-(c/2)(\log t)^2}$, so that if an event holds with good probability in n, then it holds with good probability in t.

Step 4: *Item* 2. By Lemma 4.4, the probability of visiting level $2c \log N$ on any excursion starting from level $c \log N$, is at most $e^{-c'(\log N)^2}$ for some $c' > 0$, so with good probability in N, after $e^{c'(\log N)^2/2}$ excursions, level $2c \log N$ has still not been visited. By the above estimate on t_{T_n}, with high probability in $e^{c'(\log N)^2/2} \ge N$ for N large enough, this many excursions requires time at least $ce^{c'(\log N)^2/2}/8$, that is, a long time with respect to N.

Step 5: *Item* 3. If $Z_0 = \lceil C \log N \rceil M$, we have

$$\int_0^{t_{T_1}} Z_s\,ds \le \sum_{i=0}^{T_1 - 1} (t_i - t_{i-1})(K_{i-1} + 1)M.$$

From the proof of Lemma 4.5, it follows that with probability $\geq 1 - e^{-cd}$ for some $c > 0$ there are at most d visits each to levels $2, \dots, C \log N + d/3$, and no visits to higher levels before hitting level 1, in which case $T_1 \leq d(C \log N + d/3)$. Letting $d = (\log N)^2$, $C \log N + d/3$ is at most $(\log N)^2$ for N large enough which gives $T_1 \leq (\log N)^4$, and $e^{-cd} = e^{-c(\log N)^2}$. Using a similar estimate as above for $\sum_{i=0}^{T_1-1} t_i - t_{i-1}$ and using $K_i + 1 \leq \log N + d/3 + 1 \leq (\log N)^2$, with good probability in N, and recalling $M = \lceil \sqrt{N} \rceil$,

$$\sum_{i=0}^{T_1-1} (t_i - t_{i-1})(K_{i-1} + 1)M \leq C' M (\log N)^6 \leq C'(\log N)^6(\sqrt{N} + 1). \qquad \square$$

5. Change of variables. We return to the partner model and make a slight modification to the way we record its progress. First, define the three types SSA_t, SIA_t and IIA_t, where the A stands for anticipated, as follows. If a partnership xy is formed at time t, let $s > t$ be the first time of breakup of xy after t, and record, in advance, the state of x and y at time s, and the duration $s - t$ of the partnership. Then SSA_t is the number of partnerships at time t that upon breakup will consist of two healthy individuals, and similarly for SIA_t and IIA_t.

The choice of variables SSA_t, SIA_t, IIA_t may seem unusual as they are not adapted to the natural filtration of the Poisson point processes that determine the transitions in the model. However, consider the following modification. Attach an independent uniform random variable at each partnering event, then use this variable to sample the joint distribution of the final state and duration of the partnership conditioned on its initial state. This modification preserves the sample path distribution of $(S_t, I_t, SSA_t, SIA_t, IIA_t)$, and the modified process is adapted. Although we do not make use of this fact, we note the process can be made Markov by introducing "countdown" variables for the time of breakup of each partnership, that are incremented at formation, and decrease linearly in time with slope one until breakup occurs.

Let $TI_t = I_t + SIA_t + 2IIA_t$ (the TI stands for "total infectious"), which is the analogue of $|V_t|$ for these new variables. Since the $SIA \to S + I$ and $IIA \to I + I$ transitions leave TI_t unchanged, the only transitions affecting TI_t are the ones affecting infectious singletons. Also, since the final state of a partnership is decided at the moment of partnership formation and recorded in the variables SSA_t, SIA_t, IIA_t, the corresponding change in TI_t is felt immediately.

There are three types of transition affecting infectious singletons:

(i) $I \to S$ at rate I,
(ii) $S + I \to SI$ at rate $r_+(y - i)I$ and
(iii) $I + I \to II$ at rate $(r_+ i/2)I$.

The second and third type of transition are followed by an immediate transition $SI \to SSA$, SIA or IIA, and $II \to SSA$, SIA or IIA, with a probability determined by the Markov chain from Figure 1. At each transition, TI_t can increase by 1, stay the same, or decrease by 1 or 2.

Since the rate of transitions in TI is a multiple of I, to get TI to decrease quickly enough it would help to know the ratio I/TI is not too small. As shown in the next lemma, this can be achieved with good probability in N provided $TI \geq (\log N)^2$.

LEMMA 5.1. *There are $c, h > 0$ so that, so long as $TI_t \geq (\log N)^2$, with good probability in N, $I_t \geq c \cdot TI_t$ for a long time after h.*

PROOF. For any $t \geq 0$, since S_t and I_t are identical in the partner model and the modified partner model, it follows that $SS_t + SI_t + II_t = SSA_t + SIA_t + IIA_t$. Moreover, $SSA_t \geq SS_t$ since an SS partnership cannot become an SI or an II partnership, which implies that $SIA_t + IIA_t \leq SI_t + II_t$. Define

$$IP_t = SI_t + II_t,$$

which is so named because it counts infectious partnerships. Then

$$TI_t = I_t + SIA_t + 2IIA_t \leq I_t + 2(SIA_t + IIA_t) \leq I_t + 2IP_t. \tag{5.1}$$

If $I_t \geq c \cdot IP_t$, then $I_t + 2IP_t \leq I_t + (2/c)I_t$ and so $I_t \geq TI_t/(1 + (2/c))$, so it is enough to show $I_t \geq c \cdot IP_t$ for some $c > 0$. Along the way, we will show also that $IP_t \geq c \cdot I_t$.

Since $TI_t \geq (\log N)^2$ by assumption, then using (5.1), $\max(I_t, IP_t) \geq (1/3) \times (\log N)^2$, and by definition of whp and wgp,

$$\text{whp in } (\log N)^2 \quad \Leftrightarrow \quad \text{wgp in } N.$$

There are two main steps. Step 1 is to show that for some $c > 0$, small enough $h > 0$ and all $t \geq 0$,

$$\text{whp in } \max(I_t, IP_t), \qquad \min(I_{t+h}, IP_{t+h}) \geq ch \max(I_t, IP_t). \tag{5.2}$$

Step 2 is to show that for some $c' > 0$, small enough $h > 0$ and all $t \geq 0$, if $\min(I_t, IP_t) \geq ch \max(I_t, IP_t)$ then

$$\text{whp in } \min(I_t, IP_t), \qquad \inf_{s \in [0,h]} I_{t+s} \geq c'h \sup_{s \in [0,h]} IP_{t+s} \tag{5.3}$$

and to show also that the same holds after switching the roles of I and IP. Applying step 1 at $t = 0$, then applying both steps at times $t = nh$, $n = 1, \ldots,$ (long time) and making use of the trick mentioned in Remark 3.2 establishes the result.

Step 1. If $I_t \geq IP_t$, then to establish step 1 it is enough to show that whp in I_t,

$$I_{t+h} \geq chI_t \quad \text{and} \quad IP_{t+h} \geq chI_t.$$

We will show this much is true. A similar argument works, switching the roles of I and IP, and together this establishes step 1. By Remark 3.3, we may assume $y \geq y^*/2$ so that $S + I \geq y^*N/2$. By including the partnering of only half of the infectious singletons with other (infectious or healthy) singletons, we find the rate of increase of IP_t is at least $r_+(I/2)(y^*/4)$ and the rate of transition of each infectious partnership to a healthy partnership or a breakup is at most $1+r_-$, then by counting the number of new (over the interval $[t, t+h]$) infectious partnerships still present at time $t+h$ and using a pair of large deviations estimates we find that whp in I_t,

$$IP_{t+h} \geq \big(r_+y^*h/16\big)\big(1-(3/2)h(1+r_-)\big)I_t \geq chI_t \tag{5.4}$$

for some $c > 0$ not depending on h, if $h > 0$ is small enough. On the other hand, the rate of transition of each infectious single is at most $1+r_+$. Using Lemma 3.1 with $\mu = (1+r_+)hI_t$ and $\delta = 1/2$, from among the I_t infectious singletons present at time t, whp in I_t at least $(1-(3/2)h(1+r_+))I_t$ of them remain infectious singletons over the time interval $[t, t+h]$, and in particular,

$$\inf_{0\leq s\leq h} I_{t+s} \geq \big(1-(3/2)h(1+r_+)\big)I_t \geq (1-Ch)I_t \tag{5.5}$$

for some $C > 0$ not depending on h. Taking $h > 0$ small enough, $1-Ch \geq ch$.

Step 2. The rate of increase of IP_t is at most r_+I_t. Using again Lemma 3.1 with $\mu = r_+hI_t$ and $\delta = 1/2$, whp in I_t,

$$\sup_{0\leq s\leq h} IP_{t+s} \leq IP_t + (3/2)r_+h \sup_{0\leq s\leq h} I_{t+s}. \tag{5.6}$$

Since the rate of increase of I_t is at most $2r_-IP_t$, whp in IP_t,

$$\sup_{0\leq s\leq h} I_{t+s} \leq I_t + 3r_-h \sup_{0\leq s\leq h} IP_{t+s}. \tag{5.7}$$

Combining (5.6) and (5.7), whp in $\min(IP_t, I_t)$,

$$\sup_{0\leq s\leq h} IP_{t+s} \leq \frac{IP_t + (3/2)r_+hI_t}{1-(9/2)r_+r_-h^2} \leq IP_t(1+Ch) + ChI_t \tag{5.8}$$

for some $C > 0$ not depending on h, if $h > 0$ is small enough. Choosing $h > 0$ small enough that $Ch \leq 1/2$, if $I_t \geq chIP_t$ then using (5.8) and (5.5), whp in $\min(I_t, IP_t)$,

$$IP_t(1+Ch) \geq \sup_{0\leq s\leq h} IP_{t+s} - I_t/2 \geq \sup_{0\leq s\leq h} IP_{t+s} - \inf_{0\leq s\leq h} I_{t+s}$$

moreover

$$\inf_{0\leq s\leq h} I_{t+s} \geq I_t/2 \geq chIP_t/2 \geq (ch/3) \sup_{0\leq s\leq h} IP_{t+s} - (ch/2) \inf_{0\leq s\leq h} I_{t+s}$$

so combining these two expressions,

$$I_{t+s} \geq (ch/3)/(1+ch/2)IP_{t+s} \qquad \text{for all } s \in [0, h].$$

A similar argument shows that for some $c, c' > 0$ if $IP_t \geq chI_t$ then whp in $\min(I_t, IP_t)$, $IP_{t+s} \geq c'hI_{t+s}$ for all $s \in [0, h]$. $\square$

6. Principal and auxiliary parts. Recall the three types of transitions affecting TI_t described above in Section 5. Notice that only the $S+I \to SI$ transition depends on y. We split $TI_t - TI_0$ into two pieces:

1. *the principal part* X_t with
   (a) transitions of type $I \to S$ at rate I_t,
   (b) transitions of type $S+I \to SI$ at rate $I_t r_+(y^* - i_t)$ and
   (c) transitions of type $I+I \to II$ at rate $I_t r_+(i_t/2)$, and
2. *the auxiliary part* A_t with transitions of type $S+I \to SI$ at rate $r_+ I|\delta y_t|$.

When TI_t, X_t and A_t are coupled in the natural way we have

$$TI_t = TI_0 + X_t + \operatorname{sgn}(\delta y_t)A_t \qquad \text{for all } t \geq 0,$$

where $\operatorname{sgn}(x)$ is equal to $1, 0$ or -1 respectively as x is $> 0, = 0$ or < 0. When defined in this way, X_t has a slight negative drift, as shown in the upcoming Lemma 6.1. We now consider the principal part. Let

$$z = 1 + r_+(y^* - i) + r_+ i/2 \tag{6.1}$$

so that the principal part is Iz and define

$$p_S = 1/z, \qquad p_{SI} = r_+(y^* - i)/z, \qquad p_{II} = r_+ i/(2z).$$

As in the Introduction, let $(Z_t)_{t\geq 0}$ denote the Markov chain whose transition rates are depicted in Figure 1. Recall $\tau = \inf\{t | Z_t \in \{D, E, F, G\}\}$, and use $\{B \to E\}$ to denote the event $\{Z_\tau = F | Z_0 = A\}$ and similarly for other states. Then X_t has the following transitions:

$X_t \to \cdot$	at rate
$X_t + 1$	$q_{x,x+1} = Iz \cdot p_{SI}\mathbb{P}(B \to G)$
$X_t - 1$	$q_{x,x-1} = Iz \cdot (p_S + p_{SI}\mathbb{P}(B \to E) + p_{II}\mathbb{P}(C \to F) + O(1/N))$
$X_t - 2$	$q_{x,x-2} = Iz \cdot (p_{II}\mathbb{P}(C \to E) + O(1/N))$

Similarly, A_t has the following transitions:

$$A_t \to \begin{cases} A_t + 1, & \text{at rate } r_{x,x+1} = r_+ I_t |\delta y_t| \mathbb{P}(B \to G), \\ A_t - 1, & \text{at rate } r_{x,x-1} = r_+ I_t |\delta y_t| \mathbb{P}(B \to E). \end{cases}$$

We now show why the principal part is useful. Equation (6.2) shows it has a negative drift, while equation (6.3) is a consequence of (6.2) that will allow us to define a useful supermartingale.

LEMMA 6.1. *There are $\varepsilon, c > 0$ and $\alpha > 1$ so that if $\log N/N \leq i \leq \varepsilon$ then for N large enough,*

$$q_{x,x+1} - q_{x,x-1} - 2q_{x,x-2} \leq -cNi^2 \tag{6.2}$$

and

$$(\alpha - 1)q_{x,x+1} + (\alpha^{-1} - 1)q_{x,x-1} + (\alpha^{-2} - 1)q_{x,x-2} \leq -cNi^2. \tag{6.3}$$

PROOF. Define

$$\Delta(i) = p_{SI}\mathbb{P}(B \to G) \tag{6.4}$$
$$- \big(p_S + p_{SI}\mathbb{P}(B \to E) + p_{II}\mathbb{P}(C \to F)\big)$$
$$- 2p_{II}\mathbb{P}(C \to E).$$

Then, recalling the definition of z in (6.1), it follows that

$$q_{x,x+1} - q_{x,x-1} - 2q_{x,x-2} = Iz\big(\Delta(i) + O(1/N)\big). \tag{6.5}$$

If $i = 0$, then $p_{II} = 0$ so $p_S + p_{SI} = 1$, and $p_{SI} = r_+ y^*/(1 + r_+ y^*) = \mathbb{P}(A \to B)$, and since $1 = p_S + p_{SI} = p_S + p_{SI}(\mathbb{P}(B \to E \cup F \cup G)$,

$$\Delta(0) + 1 = 2p_{SI}\mathbb{P}(B \to G) + p_{SI}\mathbb{P}(B \to F)$$
$$= 2\mathbb{P}(A \to G) + \mathbb{P}(A \to F) = R_0$$

so if $R_0 = 1$ then $\Delta(0) = 0$. Moreover, it is not hard to check that $\partial_i \Delta$ is well-defined.

In the proof of Lemma 4.2 in [5], it is shown for $R_0 \geq 1$ that $\Delta(i)$ decreases with i and $\partial_i \Delta(0) < 0$, so $\Delta(i) = -ci + o(i)$ for some $c > 0$ and small $i > 0$. If $i \geq (\log N)/N$, then $O(1/N) = o(i)$, also, $z \geq 1$, so for some $c > 0$ and N large enough, using (6.5) and the fact that $I = Ni$, it follows that

$$q_{x,x+1} - q_{x,x-1} - 2q_{x,x-2} \leq -cNi^2,$$

which proves the first part. If $\alpha = 1 + \beta$ with $0 < \beta < 1$, then $\alpha - 1 = \beta$ and using a series expansion, one easily finds that $\alpha^{-1} - 1 \leq -\beta$ and $\alpha^{-2} - 1 \leq -2\beta$. For such α, we then have

$$(\alpha - 1)q_{x,x+1} + (\alpha^{-1} - 1)q_{x,x-1} + (\alpha^{-2} - 1)q_{x,x-2}$$
$$\leq \beta(q_{x,x+1} - q_{x,x-1} - 2q_{x,x-2})$$

and the second part follows after combining the last two expressions, and taking $c > 0$ smaller if necessary. □

7. Extinction time. We begin with a couple of simple lemmas that will eventually be used to control the principal part.

LEMMA 7.1. *Let X_t be a continuous-time Markov chain on $\{-1, 0, \ldots, M\}$ absorbed at $\{-1, 0, M\}$ with adapted and bounded jump rates*

$$q_{x,x+1}(t), \qquad q_{x,x-1}(t), \qquad q_{x,x-2}(t)$$

and let $\tau = \inf\{t : X_t \in \{-1, 0, M\}\}$. Suppose that for some $\alpha > 1$, almost surely for all $t \geq 0$,

$$(\alpha - 1)q_{x,x+1} + (\alpha^{-1} - 1)q_{x,x-1} + (\alpha^{-2} - 1)q_{x,x-2} \leq 0. \tag{7.1}$$

Then $\mathbb{P}(X_\tau = M | X_0 = x) \leq \alpha^{x-M}$.

PROOF. First let

$$Q_t = (\alpha - 1)q_{x,x+1}(t) + (\alpha^{-1} - 1)q_{x,x-1}(t) + (\alpha^{-2} - 1)q_{x,x-2}(t).$$

Consider the random variable α^{X_t}. From the definition of the rates, and using (7.1), we have

$$E[\alpha^{X_{t+h}}|X_t] = \alpha^{X_t}\left(1 + E\left[\int_t^{t+h} Q_s\,ds\Big|X_t\right] + O(h^2)\right) \le \alpha^{X_t}(1 + O(h^2)).$$

Since the rates are bounded, $O(h^2)$ is uniform in t and α^{X_t}, so is at most Ch^2 for some $C > 0$. Taking expectations on both sides and using the inequality $1 + x \le e^x$ then gives

$$E[\alpha^{X_{t+h}}|X_t] \le \alpha^{X_t} e^{Ch^2}.$$

Fixing s, t and n and letting $h = s/n$, then iterating the above, gives

$$E[\alpha^{X_{t+s}}|X_t] \le e^{Ch^2 n}\alpha^{X_t} = e^{Ch}\alpha^{X_t}.$$

Letting $h \to 0^+$ shows that α^{X_t} is a supermartingale. Using the optional stopping theorem,

$$E[\alpha^{X_\tau}|X_0] \le \alpha^{X_0}$$

so the result follows from the fact that

$$\alpha^x \ge E[\alpha^{X_\tau}|X_0 = x] \ge \alpha^M P(X_\tau = M|X_0 = x). \qquad \square$$

LEMMA 7.2. *Let X_t be as in Lemma 7.2 except on $\{-1, 0, \dots\}$ with $X_0 \in \{0, \dots, M-1\}$, and let $\tau = \inf\{t : X_t \in \{-1, 0\}\}$. Suppose that for some $\alpha > 1$, $b > 0$ that almost surely for all $t \ge 0$,*

$$(\alpha - 1)q_{x,x+1} + (\alpha^{-1} - 1)q_{x,x-1} + (\alpha^{-2} - 1)q_{x,x-2} \le -b.$$

Then

$$\mathbb{P}(\tau > t) \le \alpha^{M-1}e^{-bt}. \tag{7.2}$$

PROOF. Proceeding as in the proof of Lemma 7.1 and using (7.2), for some $C > 0$ we find that

$$E[\alpha^{X_{t+h}}|X_t] \le \alpha^{X_t}(1 - bh + Ch^2).$$

As before, fixing s, t and n and letting $h = s/n$, then iterating, gives

$$E[\alpha^{X_{t+s}}|X_t] \le \alpha^{X_t}(1 + (-bs + Cs^2/n)/n)^n \le \alpha^{X_t}e^{-bs+Cs^2/n}$$

and since n is arbitrary,

$$E[\alpha^{X_{t+s}}|X_t] \le \alpha^{X_t}e^{-bs}.$$

Let $\theta > 0$ be such that $\alpha = e^{\theta}$. Multiplying both sides of the above equation by $e^{b(t+s)}$ then gives

$$E\big[e^{\theta X_{t+s}+b(t+s)}|X_t\big] \le e^{\theta X_t + bt}.$$

In other words, $e^{\theta X_t + bt}$ is a supermartingale. Using this fact together with Markov's inequality and the fact that X_t is integer-valued,

$$\begin{aligned} P(X_t > 0|X_0) &= P\big(e^{\theta X_t} > 1|x_0\big) \\ &\le E\big[e^{\theta X_t}|X_0\big] = e^{-bt}E\big[e^{\theta X_t + bt}|X_0\big] \le e^{-bt}e^{\theta X_0} \end{aligned}$$

and since $X_0 \le M-1$ by assumption, this is at most $\alpha^{M-1}e^{-bt}$. $\square$

First, we show that TI_t can be brought down to $C\sqrt{N}$ by time $C\sqrt{N}$, and with good control on δy_t, with positive probability. The approach is to use the drift in the principal part while controlling the contribution from the auxiliary part. To facilitate this, we break up the movement of TI_t into *levels*—a new level begins when TI_t either dips below half, or rises above twice, of its previous value. This is also used in Proposition 7.3.

PROPOSITION 7.1. *There is* $C > 0$ *so that with probability* $\ge 1/2$, *for some* $t \le C\sqrt{N}$, $TI_t \le C\sqrt{N}$ *and* $|\delta y_t| \le 2\log N/\sqrt{N}$.

PROOF. Using Lemma 4.1, with good probability in N, for some $t \le C\log N$, $|\delta y_t| \le \log N/\sqrt{N}$. By Lemma 4.8, with good probability in N for a long time in N, $|\delta y_t| \le 2\log N/\sqrt{N}$, which establishes the easy part of the above statement. Note that since $C\log N = o(\sqrt{N})$, by Remark (3.2) we may assume that $|\delta y_t| \le 2\log N/\sqrt{N}$ holds for all time, and in particular at $t = 0$. Let $t_0 = 0$ and $L_0 = TI_0/\sqrt{N}$ then define recursively

$$t_j = \inf\{t > t_{j-1} : TI_t/TI_{t_{j-1}} \notin [1/2, 2]\} \quad \text{and} \quad L_j = TI_{t_j}/\sqrt{N}.$$

Since it only remains to bring TI_t below $C\sqrt{N}$ for some $C > 0$, we may assume that $L_j \ge C$ for all j under consideration, and for some fixed $C > 0$ that can be chosen as large as needed.

Step 1: *Fixed* j. Given $j \ge 1$ let X_t and A_t denote the contribution from the principal and auxiliary parts, not from $t = 0$ but starting from t_{j-1}. That is, $X_0 = A_0 = 0$ and

$$TI_t = TI_{t_{j-1}} + X_{t-t_{j-1}} + \operatorname{sgn}(\delta y_t)A_{t-t_{j-1}} \qquad \text{for } t \ge t_{j-1}.$$

We use the following fact to estimate t_j as well as the value of TI_{t_j}. Let w be a fixed small number and let $t^* > 0$. Then

$$\sup_{t\le t*}|A_t| \le w\sqrt{N}L_{j-1}, \qquad \inf_{t\le t^*} X_t \le -(1/2+w)L_{j-1}\sqrt{N} \tag{7.3}$$
$$\text{implies } t^* > t_j - t_{j-1}.$$

Also, if the two conditions in (7.3) hold then

$$\sup_{t\le t^*} X_t < (1-w)L_{j-1}\sqrt{N} \qquad \text{implies } L_j \le L_{j-1}/2. \tag{7.4}$$

Controlling X_t. For integer m to be chosen, let $t^* = m\sqrt{N}/L_{j-1}$, and for now assume the first condition in (7.3) holds. By Lemma 5.1, we may assume $I_t \ge c\sqrt{N}L_{j-1}/2$ for all $t \in [t_{j-1}, t_j]$, so using Lemma 6.1, X_t satisfies the conditions of Lemma 7.2 with $b = cL_{j-1}^2$, for some $c > 0$. Using (7.3) and Lemma 7.2 with $M = ((3/2)+2w)\sqrt{N}L_{j-1}$ which is at most $2\sqrt{N}L_{j-1}$ if $w > 0$ is small enough, and letting $\alpha = e^{\theta}$,

$$\begin{aligned}\mathbb{P}(t_j - t_{j-1} \le m\sqrt{N}/L_{j-1}) &\ge 1 - \alpha^{2\sqrt{N}L_{j-1}} e^{-(cL_{j-1}^2)(m\sqrt{N}/L_{j-1})}\\ &\ge 1 - e^{\sqrt{N}L_{j-1}(2\theta - cm)}.\end{aligned}$$

If m is taken large enough that $2\theta - cm < 0$ the above is at least $1 - e^{-c\sqrt{N}L_{j-1}}$ for some possibly smaller $c > 0$. To estimate the probability that $L_j \le L_{j-1}/2$, use Lemma 7.1 with $x = (1/2+w)\sqrt{N}L_{j-1}$ and $M = (3/2)\sqrt{N}L_{j-1}$ to find that for some $\alpha > 1$,

$$\mathbb{P}(L_j = L_{j-1}/2) \ge 1 - \alpha^{-(1+w)\sqrt{N}L_{j-1}},$$

which is at least $1 - \alpha^{-(1/2)\sqrt{N}L_{j-1}}$, if $w \le 1/2$. Before moving onto the next step, fix w small enough and m large enough that the above is valid.

Controlling A_t. We now show the first condition in (7.3) holds. Recall that A_t has transitions at rate $r_+ I_t|\delta y_{t_{j-1}+t}|)$. Let t^* be as above and note that $|\delta y_{t_{j-1}}| \le 2\log N/\sqrt{N}$. Using items 3 and 1 of Lemma 4.8, we find that with good probability in $\min(N, m\sqrt{N}/L_{j-1}) \ge m\sqrt{N}/L_{j-1}$, for some $C' > 0$,

$$\begin{aligned}\sup_{t\le t^*} |A_t| &\le C'\sqrt{N}L_{j-1}\big((\log N)^6 + m\sqrt{N}/L_{j-1}\big)/\sqrt{N}\\ &\le C'\sqrt{N}L_{j-1}\big((\log N)^6/\sqrt{N} + m/L_{j-1}\big).\end{aligned}$$

Recall that $L_{j-1} \ge C$, by assumption, for some fixed $C > 0$ that can be chosen, if C and N are taken large enough that $(\log N)^6/\sqrt{N} + m/L_{j-1} \le w/C'$, then

$$\sup_{t\le t^*} |A_t| \le w\sqrt{N}L_{j-1}.$$

Step 2: *Several* j. We now estimate the probability that starting from $TI_0 \le \varepsilon N$, $TI_{t_j} \le TI_{j-1}/2$ repeatedly until $TI_t \le C\sqrt{N}$, as well as the amount of time for this to happen. Looking to the estimates we have made up to this point, for fixed j the desired event has good probability in $\sqrt{N}/L_j$, so the complementary event has probability at most $e^{-c(\log(\sqrt{N}/L_j))^2}$ for some $c > 0$ that does not depend on j.

Since $TI_t \le 2|V_t|$, we may assume $TI_t \le \varepsilon$ for $t \ge 0$, which gives $L_0 \le \varepsilon\sqrt{N}$. Let $\varepsilon = 1/C$ where C is as above. Given j, if the desired event holds for every $i \le j$ then $L_j \le 2^{-j} L_0$. Using a union bound, the probability the good event holds until $TI \le C\sqrt{N}$ is at least

$$1 - \sum_{j=0}^{\infty} e^{-c(\log(2^j C))^2} \ge 1 - \sum_{j=0}^{\infty} e^{-c(j\log 2 + \log C)^2},$$

which is at least $1/2$ if C is taken large enough. At each step, the time required is at most $m\sqrt{N}/L_{j-1}$, where m is fixed, so the total amount of time is at most

$$(mc\sqrt{N}/C)\sum_{j=0}^{\infty} 2^{-j} \le 2mc\sqrt{N}/C.$$

□

Next, we show that for fixed $C > c > 0$, TI_t can be brought down from $C\sqrt{N}$ to $c\sqrt{N}$ with positive probability within time $C'\sqrt{N}$ for some $C' > 0$.

PROPOSITION 7.2. *For any $C > c > 0$, there are $C', p > 0$ so that with probability $\ge p > 0$ uniformly in N, if $TI_0 \le C\sqrt{N}$ and $|\delta y_0| \le 2\log N/\sqrt{N}$ then for some $t \le C'\sqrt{N}$, $TI_t \le c\sqrt{N}$ and $|\delta y_t| \le 4\log N/\sqrt{N}$.*

PROOF. The estimate on $|\delta y_t|$ is the same as in Proposition 7.1. Here, let X_t and A_t denote the contributions due to the principal and auxiliary parts starting from $t = 0$, so that $TI_t = TI_0 + X_t + \operatorname{sgn}(\delta y_t) A_t$ for all $t \ge 0$.

From (6.2), it follows that X_t is dominated by a symmetric simple random walk $\tilde{X}_t$ moving at rate Iz. Since $TI_t \ge c\sqrt{N}$ on the region of interest, using Lemma 5.1 we may assume $I_t \ge cc'\sqrt{N}$ for some $c' > 0$, which, since $z \ge 1$, implies $Iz \ge c'\sqrt{N}$ for some possibly smaller $c' > 0$.

Let $T = \inf\{n : \tilde{X}_n = \pm M/2\}$. Using Lemma 4.3, for any $a > 0$ and large enough (even) M,

$$\mathbb{P}\bigl(T \le aM^2/(c'\sqrt{N})\bigr) \ge p(a) > 0$$

and by symmetry,

$$\mathbb{P}\bigl(T \le aM^2/(c'\sqrt{N}), \tilde{X}_T = -M/2\bigr) \ge p(a)/2.$$

Setting $M/2 = \lceil C\sqrt{N}\rceil$ and comparing to $\tilde{X}_n$, with probability at least $p(a)/2$, there exists $t \le aC^2N/(c'\sqrt{N}) = aC'\sqrt{N}$ such that either $TI_t \le c\sqrt{N}$ or $X_t \le -C\sqrt{N}$.

Following the proof of Proposition 7.1, with good probability in $\sqrt{N}$, for $t^* = aC'\sqrt{N}$ and some $C'' > 0$ not depending on a,

$$\sup_{t \le t^*} |A_t| \le C''\sqrt{N}\bigl((\log N)^6/\sqrt{N} + aC'\bigr),$$

which if a is taken small enough, is at most $c\sqrt{N}$ when N is large. Since

$$X_t \le -C\sqrt{N} \quad \text{and} \quad |A_t| \le c\sqrt{N} \qquad \text{implies } TI_t \le c\sqrt{N}$$

the proof is complete. $\square$

Now, we start from $TI_0 \le c\sqrt{N}$ for small $c > 0$ and show that TI_t can be brought down to N^γ for some $\gamma < 1/4$ within time $C\sqrt{N}$. The proof is similar to that of Proposition 7.1 except that here, the drift is not strong enough to jump straight down a whole bunch of levels all at once. Instead, there is a constant amount of drift from level to level, so we treat it like a biased random walk. In the proof, we will make use of the following fact, that we record now.

LEMMA 7.3. *If $x_0, \dots, x_{2k}$ are positive integers such that*

$$x_{i+1} = \inf\{x : x > 2x_i\}$$

for exactly k of the numbers $\{0, \dots, 2k-1\}$ and

$$x_{i+1} = \sup\{x : x < x_i/2\}$$

for the remaining k numbers in $\{0, \dots, 2k-1\}$, then $x_{2k} \le x_0$.

PROOF. Define

$$f_+(x) = \inf\{x : x > 2x_i\} \quad \text{and} \quad f_-(x) = \sup\{x : x < x_i/2\}$$

so that $x_{2k} = f_{a_{2k-1}} \circ f_{a_{2k-2}} \circ \cdots \circ f_{a_0}(x_0)$ for some string $a_0 \cdots a_{2k-1}$ satisfying $|\{i : a_i = +\}| = |\{i : a_i = -\}| = k$. Since f_+ and f_- are non-decreasing functions it suffices to show that if $y = f_+(f_-(x))$ or $y = f_-(f_+(x))$, then $y \le x$. Clearly, $f_+(x) = 2x + 1$, and

$$f_-(x) = \begin{cases} x/2 - 1, & \text{if } x \text{is even,} \\ x/2 - 1/2, & \text{if } x \text{is odd.} \end{cases}$$

Since $2x + 1$ is odd, $f_-(f_+(x)) = (2x+1)/2 - 1/2 = x$, and we compute

$$f_+\bigl(f_-(x)\bigr) = \begin{cases} 2(x/2 - 1) + 1 = x - 1, & \text{if } x \text{is even,} \\ 2(x/2 - 1/2) + 1 = x, & \text{if } x \text{is odd.} \end{cases} \qquad \square$$

PROPOSITION 7.3. *There are $c, C > 0$ so that if $TI_0 \le c\sqrt{N}$ and $|\delta y_0| \le 4 \log N/\sqrt{N}$ then with probability $\ge 1/4$ there is $t \le C\sqrt{N}$ so that $TI_t \le N^\gamma$ for some $\gamma < 1/4$.*

PROOF. Define t_j, L_j, and for fixed j define X_t and A_t, all as in the proof of Proposition 7.1. As noted in the proof of Proposition 7.3, X_t is dominated by a symmetric simple random walk $\tilde{X}_t$ moving at rate $I_t z \ge I_t$. Since $TI_t \ge N^\gamma$ on the region of interest, using Lemma 5.1 we may assume $I_t \ge c' TI_t$ for some $c' > 0$.

Since $TI_t \geq \sqrt{N}L_{j-1}/2$ for all $t \in [t_{j-1}, t_j)$, it follows that $\tilde{X}_t$ has transition rate at least $c'\sqrt{N}L_{j-1}/2$, for $t \in [0, t_j - t_{j-1})$.

Step 1: *Fixed* j. Recall from (7.3) that $|A_t| \leq w\sqrt{N}L_{j-1}$ and $X_t \leq -(1/2 + w)\sqrt{N}L_{j-1}$ implies $t \geq t_j - t_{j-1}$. Later we will need to force exit from the low end of $[TI_{t_{j-1}}/2, 2TI_{t_{j-1}}]$, starting from any value in that interval, so we will require $X_t \leq -(3/2 + w)\sqrt{N}L_{j-1}$ instead. Using Lemma 4.3 and dividing by the transition rate, for N large enough and $w \leq 1/2$, with probability $\geq 1 - 2^{-m}$,

$$\begin{aligned} &\inf\{t : t = t_j - t_{j-1} \text{ or } X_t \leq -(3/2 + w)\sqrt{N}L_{j-1}\} \\ &\quad \leq 2mCNL_{j-1}^2/(c'\sqrt{N}L_{j-1}/2) = mC\sqrt{N}L_{j-1} \end{aligned} \tag{7.5}$$

for some possibly larger $C > 0$. Letting $t^* = mC\sqrt{N}L_{j-1}$, with good probability in $\min(N, mC\sqrt{N}L_{j-1})$, which, since $TI_t \geq N^\gamma$ is at least good probability in N,

$$\begin{aligned} \sup_{t \leq t^*} |A_t| &\leq (\sqrt{N}L_{j-1})C'((\log N)^6 + m\sqrt{N}L_{j-1})/\sqrt{N} \\ &\leq 2mC'\sqrt{N}L_{j-1}^2 \end{aligned} \tag{7.6}$$

for N large enough, which is at most $w\sqrt{N}L_{j-1}$ provided $L_{j-1} \leq (w/2mC')$. We will force this inequality in the next step by choosing a small enough $c > 0$ and by restricting to $L_j \leq c$. On the event in (7.6), and on the event that

$$\begin{aligned} &\inf\{t : X_t \leq -(1/2 + w)\sqrt{N}L_{j-1}\} \\ &\quad < \min(\inf\{t : X_t \geq (1 - w)\sqrt{N}L_{j-1}\}, mC\sqrt{N}L_{j-1}) \end{aligned} \tag{7.7}$$

it holds that $TI_{t_j} \leq TI_{t_{j-1}}/2$. Gathering estimates and noting X_t is dominated by symmetric simple random walk, the intersection of (7.5), (7.6), (7.7) has, for some $c' > 0$, probability at least

$$\begin{aligned} &(1 - w)/((1/2 + w) + (1 - w)) - 2^{-m} - e^{-c'(\log N)^2} \\ &\quad = 2(1 - w)/3 - 2^{-m} - e^{-c'(\log N)^2}. \end{aligned}$$

For $w > 0$ small enough and m large enough, for large N the above is at least $3/5$. Moreover, on the intersection of (7.5), (7.6), (7.7),

$$t_j - t_{j-1} \leq mC\sqrt{N}L_{j-1} \quad \text{and} \quad TI_{t_j} \leq TI_{t_{j-1}}/2. \tag{7.8}$$

Step 2: *Several* j. Note that $L_{j-1} \leq w/(2mC')$ for all j if both $TI_0 \leq c\sqrt{N}$ for small enough $c > 0$, and $L_j \leq L_0$ for all j under consideration. In this case, the values k_j defined by $k_0 = 0$ and recursively by $k_j = k_{j-1} \pm 1$, according as $L_j \geq 2L_{j-1}$ or $L_j \leq L_{j-1}/2$, are dominated by a random walk with $p_{x,x-1} = 3/5$ and $p_{x,x+1} = 2/5$, which we now consider.

Define the absorbing states $K_+ = 1$ and $K_- = \sup\{k : 2^k \le N^\gamma / c\sqrt{N}\}$, then let $J = \min\{j : k_j \in \{K_-, K_+\}\}$ and let $\tau = \inf\{t : TI_t \le N^\gamma\}$. On the event that $k_J = K_-$, it follows that $\tau \le \sum_{j=1}^{J}(t_j - t_{j-1})$. Using Lemma 4.2 with $M = K_+ - K_-$, $b = 3/2$ and $x = M - 1$,

$$\begin{aligned}\mathbb{P}(k_J = K_-) &= 1 - \frac{1 + (3/2) + \cdots + (3/2)^{x-1}}{1 + (3/2) + \cdots + (3/2)^x} \\ &= 1 - \frac{2}{3} \cdot \frac{1 - (2/3)^x}{1 - (2/3)^{x+1}} \ge 1/3.\end{aligned} \tag{7.9}$$

Fix $k \in \{K_- + 1, \dots, K_+ - 1\}$ and let $G(k) = \sum_{j=0}^{J-1} \mathbf{1}(k_j = k | k_0 = K_+ - 1)$. Using Lemma 4.4, $P(G(k) > d) \le \rho_{k,k}^d$. Repeating the above calculation with $x = k - 1$ and $M = k$ shows that $\rho(k-1, k) \le 2/3$ and so

$$\rho(k,k) \le p(k,k+1) + p(k,k-1)\rho(k-1,k) \le 2/5 + (3/5)(2/3) = 4/5$$

for any value of $k \in \{K_- + 1, \dots, K_+ - 1\}$, so it follows that $\mathbb{P}(G(k) > d) \le (4/5)^d$. Summing, we find

$$\begin{aligned}&\mathbb{P}\bigl(G(K_+ - \ell) \le \ell d \text{ for } \ell \in \{1, \dots, K_+ - K_- - 1\}\bigr) \\ &\quad \ge 1 - \sum_{\ell=1}^{\infty} (4/5)^{\ell d} = 1 - \frac{(4/5)^d}{1 - (4/5)^d}.\end{aligned} \tag{7.10}$$

On the event in (7.10), and on the event the time spent at each visit to level $K_+ - \ell$ is at most $\ell d m C \sqrt{N} c 2^{-\ell+1}$,

$$\tau \le mcC\sqrt{N} \sum_{\ell=1}^{K_+ - K_- - 1} (\ell d)^2 2^{-\ell+1} \le mC' d^2 \sqrt{N} \tag{7.11}$$

for some $C' > 0$ not depending on N or d. On each visit to level $K_+ - \ell$, using Lemma 7.3 and the assumption $TI_0 \le c\sqrt{N}$, we have

$$L_j \le 2^{-\ell+1} L_0 \le c 2^{-\ell+1}.$$

Using (7.5) and the Markov property, with probability $\ge 1 - 2^{-m\ell d}$ the time spent at that level is at most $\ell d m C \sqrt{N} c 2^{-\ell+1}$. The probability of the intersection of the events in (7.9), (7.10), (7.11) is, for large N, at least

$$1/3 - \frac{(4/5)^d}{1 - (4/5)^d} - \sum_{\ell=0}^{\log_2 N} (\ell d) 2^{-m\ell d},$$

which, for some d is at least $1/4$ when N is large enough. □

If TI_0 and δy_0 are small enough, then we can send the process to extinction with positive probability within $o(\sqrt{N})$ amount of time. Note that we first need to send TI_t to zero, *then* send $|V_t|$ to zero to kill the infection.

PROPOSITION 7.4. *If* $TI_0 \le N^\gamma$ *for* $\gamma < 1/4$ *and* $\delta y_0 \le \log N/\sqrt{N}$ *then for some* $C, p > 0$,

$$\mathbb{P}(|V_{CN^{2\gamma}}| = 0) \ge p.$$

PROOF. Let $T = \inf\{t : TI_t = 0 \text{ or } TI_t \ge 2N^\gamma\}$. If $t < T$ then since $I_t \le TI_t \le 2N^\gamma$ and $\delta y_0 \le \log N/\sqrt{N}$, using items 1 and 3 of Lemma 4.8, with good probability in $\min(N, t)$ the number of transitions due to the auxiliary part up to time t is at most Poisson with rate

$$r_+(2N^\gamma)C\big((\log N)^6/\sqrt{N} + t/\sqrt{N}\big),$$

which is at most $CtN^{\gamma-1/2}$ for large enough N, for some $C > 0$. Since $\mathbb{P}(\text{Poisson}(\alpha) = 0) = e^{-\alpha}$, there are no auxiliary transitions with probability $\ge e^{-CtN^{\gamma-1/2}} \ge 1 - CtN^{\gamma-1/2}$, which for $t \le CN^{1/2-\gamma-\varepsilon}$ is at least $1 - CN^{-\varepsilon}$.

We now examine the contribution from the principal part, starting from $t = 0$, that we denote as usual by X_t. As noted in the proof of Proposition 7.3, X_t is dominated by a symmetric simple random walk $\tilde{X}_t$, although in this case we will need to take greater care when estimating the rate.

Using Lemma 4.3, starting from $\le N^\gamma$, with probability $\ge 1/2$, $\tilde{X}_t$ hits either 0 or $2N^\gamma$ after at most $CN^{2\gamma}$ transitions, so using symmetry, $\tilde{X}_t$, and by comparison X_t, hits 0 before hitting $2N^\gamma$ after at most $CN^{2\gamma}$ transitions with probability at least $1/4$. We now need to estimate the rate of transitions.

Given the initial state of a partnership, for a final state F and duration τ of the partnership, using Bayes' rule for the density functions we have

$$d\mathbb{P}(\tau|F) = \frac{d\mathbb{P}(F|\tau)d\mathbb{P}(\tau)}{\mathbb{P}(F)}$$

and since for a given duration each possible final state occurs with probability ≤ 1, $d\mathbb{P}(F|\tau) \le 1$, and in particular $d\mathbb{P}(\tau|F) \le Cd\mathbb{P}(\tau)$ for some $C > 0$ not depending on F. This implies that

$$\mathbb{P}(\tau > t|F) \le C\mathbb{P}(\tau > t) = Ce^{-r_- t} = e^{-r_-(t-\log C)} \tag{7.12}$$

and so $(\tau|F)$ is at most $\log C + \text{exponential}(r_-)$, uniformly in the final state F.

If $TI_t > 0$, then either $I_t > 0$ or $IPA_t > 0$, where IPA_t is defined by $IPA_t = SIA_t + IIA_t$. If $I_t > 0$, then by definition of TI_t, TI_t has transition rate at least 1. If $I_t = 0$ but $IPA_t > 0$, then since breakup of an *SIA* or an *IIA* partnership gives at least one infectious singleton, from (7.12), the amount of time until $I_t > 0$ is at most $\log C + \text{exponential}(r_-)$. Thus, for $CN^{2\gamma}$ transitions in TI_t to occur requires at most $2CN^{2\gamma}$ transitions in total from either I_t or IPA_t which, with high probability in $N^{2\gamma}$, requires time at most

$$CN^{2\gamma}\big(2\log C + 3\max(1, 1/r_-)\big).$$

Summarizing so far, for some $C > 0$, with probability $\ge 1/4 - O(N^{-\varepsilon})$ which is at least $1/8$ for N large enough, $TI_t = 0$ for some $t \le CN^{2\gamma}$.

It remains to control the time until $|V_t| = 0$. Using (7.12) and noting that the time for n particles, each decaying at rate r, to all decay is of order $\log n$, with probability $\geq 1/2$, if $TI_t = 0$ then after at most an additional $\log C + C \log N^\gamma$ amount of time, $V_t = 0$. Taking $p = (1/8)(1/2) = 1/16$, the result follows. $\square$

It is now easy to show that when $R_0 = 1$, the infection dies out by time $C\sqrt{N}$ with positive probability.

PROPOSITION 7.5. *There are $C, \gamma > 0$ so that from any initial distribution of (V_0, E_0), for N large enough and integer m,*

$$\mathbb{P}(V_{mC\sqrt{N}} = 0) \geq 1 - e^{-\gamma m}.$$

PROOF. Let:

- $t_1 = \inf\{t : TI_t \leq C\sqrt{N}, |\delta y_t| \leq \log N/\sqrt{N}\}$,
- $t_2 = \inf\{t > t_1 : TI_t \leq c\sqrt{N}, |\delta y_t| \leq 2\log N/\sqrt{N}\}$,
- $t_3 = \inf\{t > t_2 : TI_t \leq N^\gamma, |\delta y_t| \leq 4\log N\sqrt{N}\}$ and
- $t_4 = \inf\{t > t_3 : |V_t| = 0\}$.

Apply Propositions 7.1, 7.2, 7.3 and 7.4 in that order, and use the Markov property at each step, to deduce that $t_4 \leq C'\sqrt{N}$ with probability $\geq p > 0$ for some p, C' uniformly in N. To get the above statement, let $1 - p = e^{-\gamma}$ and apply the Markov property repeatedly. $\square$

We conclude with a matching lower bound that works when $|V_0| \geq \sqrt{N}$ and $\delta y_0 \leq -\log N/\sqrt{N}$.

PROPOSITION 7.6. *If $|V_0| \geq \sqrt{N}$ and $\delta y_0 \geq -\log N/\sqrt{N}$ there is $c > 0$ so that with good probability in N,*

$$V_{c\sqrt{N}} \neq 0.$$

PROOF. First, we show that if $|V_0| \geq \sqrt{N}$ then for some $c > 0$, $TI_0 \geq c\sqrt{N}$ whp in $\sqrt{N}$, as follows. First of all,

$$|V_0| \geq \sqrt{N} \qquad \text{implies } \max(I_0, IP_0) \geq \sqrt{N}/3.$$

Since $TI_t \geq I_t$, if $I_0 \geq \sqrt{N}/3$ then $TI_0 \geq \sqrt{N}/3$. If $IP_0 \geq \sqrt{N}/3$, note that:

1. with positive probability, an infectious partnership breaks up while at least one partner is still infectious, and
2. the final state of partnerships existing at $t = 0$ are independent.

From this and a standard large deviations argument, it follows that for some $c > 0$, $TI_0 \ge c\sqrt{N}$ whp in $\sqrt{N}$.

Define $\tau = \inf\{t : |TI_t - TI_0| \ge TI_0/2\}$. Since $I_t \le TI_t$, it follows that $I_t = O(\sqrt{N})$ for $t < \tau$, and from Lemma 5.1, for $t < \tau$ we have $I_t \ge c\sqrt{N}$ for some $c > 0$. Following the proof of Lemma 6.1, and since $1/N = o(i)$ for $t < \tau$, from (6.5) we have

$$q_{x,x+1} - q_{x,x-1} - 2q_{x,x-2} = Iz\big(\Delta(i) + o(i)\big). \tag{7.13}$$

As before, let

$$Q_t = (\alpha - 1)q_{x,x+1}(t) + (\alpha^{-1} - 1)q_{x,x-1}(t) + (\alpha^{-2} - 1)q_{x,x-2}(t).$$

Taking now $\alpha = 1 - \beta$ for small $\beta > 0$, so that $\alpha < 1$, $\alpha - 1 = -\beta$, $\alpha^{-1} - 1 = \beta + O(\beta^2)$ and $\alpha^{-2} - 1 = 2\beta + O(\beta^2)$ then using (7.13), we have

$$\begin{aligned} Q_t &= \big(-\beta + O(\beta^2)\big)\big(q_{x,x+1}(t) - q_{x,x-1}(t) - 2q_{x,x-2}\big)(t) \\ &= \big(-\beta + O(\beta^2)\big)\big(\Delta(i_t) + o(i_t)\big)I_t z. \end{aligned}$$

If $\beta > 0$ is small enough, then $-2\beta \le -\beta + O(\beta^2) \le 0$. Recall that $\Delta(i) = -bi + o(i)$ for some $b > 0$. Since $i_t = O(1/\sqrt{N}) = o(1)$ for $t < \tau$, for N large enough and some $C > 0$ it follows that

$$-C/\sqrt{N} \le \Delta(i_t) + o(i_t) \le 0.$$

Since z is bounded and $I_t = O(\sqrt{N})$, it follows that $0 \le Q_t \le b$ for some $b > 0$ and $t < \tau$. In order to make it true for all t, for $t \ge \tau$ simply "freeze" the rates $q_{x,x+\cdot}$ to their values at time τ^-. Since the desired event falls in $\{t < \tau\}$, this does not affect the conclusion.

Let $\theta > 0$ be such that $e^{-\theta} = \alpha$. Proceeding as in the proof of Lemmas 7.1 and 7.2, we find that $e^{-\theta X_t}$ is a submartingale and $e^{-\theta X_t - bt}$ is a supermartingale. Applying Doob's inequality to the submartingale $e^{-\theta X_t}$, for any $t, C > 0$ we have

$$P\Big(\inf_{s \le t} X_s < -Ct\Big) = P\Big(\sup_{s \le t} e^{-\theta X_s} > e^{\theta Ct}\Big) \le e^{-\theta Ct} E\big[e^{-\theta X_t}\big]$$

then using the supermartingale $e^{-\theta X_t - bt}$ and the fact that $X_0 = 0$,

$$E\big[e^{-\theta X_t}\big] = e^{bt} E\big[e^{-\theta X_t - bt}\big] \le e^{bt} E\big[e^{-\theta X_0}\big] = e^{bt}$$

so combining the two,

$$P\Big(\inf_{s \le t} X_s < -Ct\Big) \le e^{bt - \theta Ct}$$

so for $C > 0$ large enough the event holds whp in t. It is easy to check, as in the proof of Propositions 7.1 and 7.3, that for $t = c'\sqrt{N}$, with good probability in $c'\sqrt{N}$, $\sup_{s \le t} |A_s|$ is at most $c'C\sqrt{N}$ for some $C > 0$ that does not depend on

c'. By first taking $C > 0$ large enough, then taking c' small enough, with good probability in $\sqrt{N}$,

$$\inf_{t \le c'\sqrt{N}} X_t - |A_t| \le -TI_0/2,$$

which implies $TI_{c'\sqrt{N}} \ge TI_0/2$. In particular, $TI_{c'\sqrt{N}} \ne 0$ which implies that $V_{c'\sqrt{N}} \ne 0$. □

APPENDIX

PROOF OF LEMMA 4.6. If $X \sim \text{binomial}(n, p)$ then $\mathbb{E}e^{\theta X} = ((1-p) + pe^{\theta})^n = (1 + p(e^{\theta} - 1))^n \le e^{np(e^{\theta}-1)}$ so using Markov's inequality, for $\theta \ge 0$, $\mathbb{P}(X > x) \le e^{-\theta x}\mathbb{E}e^{\theta X} \le e^{-\theta x + np(e^{\theta}-1)}$, and setting $x = npr$ with $r = e^{\theta}$ gives

$$\mathbb{P}(X > x) \le e^{-x\log r + x - x/r}$$

from which the first estimate follows. For the lower bound, note for $\theta \ge 0$, $\mathbb{E}e^{-\theta X} \le e^{np(e^{-\theta}-1)}$ so $\mathbb{P}(X < x) = \mathbb{P}(e^{-\theta X} > e^{-\theta x}) \le e^{\theta x + np(e^{-\theta}-1)}$. Setting $x = np(1-\delta)$ with $1-\delta = e^{-\theta}$, the exponent is $-np(1-\delta)\log(1-\delta) - np\delta$, and since $\log(1-\delta) \ge -\delta - \delta^2/2$ for $\delta \le 1$, $(1-\delta)\log(1-\delta) + \delta \ge \delta^2/2 + \delta^3/2 \ge \delta^2/2$ and the second estimate follows.

If $X \sim \text{geometric}(p)$ then $\mathbb{E}e^{\theta X} = (1-p)e^{\theta}/(1-pe^{\theta})$, so if $X_i \sim \text{geometric}(p)$ are independent, $i = 1, \dots, m$ and $S_m = X_1 + \cdots + X_m$ then $\mathbb{E}e^{\theta S_m} = [(1-p)e^{\theta}/(1-pe^{\theta})]^m$, and

$$\mathbb{P}(S_m > x) \le e^{-\theta(x-m) + m(\log(1-p) - \log(1-pe^{\theta}))}$$

and optimizing in θ gives $(1 - pe^{\theta}) = m/x$ and $\theta = \log((1/p)(1 - m/x))$. Setting $x = (1+\delta)m/(1-p)$ gives $\theta = \log(1+\delta/p) - \log(1+\delta)$ and $\log(1-pe^{\theta}) = \log(1-p) - \log(1+\delta)$ and the exponent $-\log(1+\delta/p)(x-m) + \log(1+\delta)x$ which since $x - m = (\delta + p)m/(1-p)$ is equal to $(-m/(1-p))(p(1+\delta/p)\log(1+\delta/p) - (1+\delta)\log(1+\delta))$. Now, the function $f(x) := x\log x$ has $f'(x) = 1 + \log x$ which increases with x, so $f(1+\delta/p) - f(1+\delta) \ge (1/p - 1)\delta f'(1+\delta) = (1/p-1)\delta(1 + \log(1+\delta))$ and so

$$\begin{aligned}
&p(1+\delta/p)\log(1+\delta/p) - (1+\delta)\log(1+\delta)\\
&\quad = pf(1+\delta/p) - f(1+\delta)\\
&\quad = p\big(f(1+\delta/p) - f(1+\delta)\big) + (p-1)f(1+\delta)\\
&\quad \ge (1-p)\big[\delta\big(1 + \log(1+\delta)\big) - (1+\delta)\log(1+\delta)\big]\\
&\quad = (1-p)\big[\delta - \log(1+\delta)\big]
\end{aligned}$$

and the desired estimate follows. □

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SCHOOL OF MATHEMATICAL AND STATISTICAL SCIENCES
ARIZONA STATE UNIVERSITY
P.O. BOX 871804
TEMPE, ARIZONA 85287-1804
USA
E-MAIL: e.t.foxall@gmail.com