

# Le Her and Other Problems in Probability Discussed by Bernoulli, Montmort and Waldegrave

David R. Bellhouse and Nicolas Fillion

*Abstract.* Part V of the second edition of Pierre Rémond de Montmort’s *Essay d’analyse sur les jeux de hazard* published in 1713 contains correspondence on probability problems between Montmort and Nicolaus Bernoulli. This correspondence begins in 1710. The last published letter, dated November 15, 1713, is from Montmort to Nicolaus Bernoulli. There is some discussion of the strategy of play in the card game Le Her and a bit of news that Montmort’s friend Waldegrave in Paris was going to take care of the printing of the book. From earlier correspondence between Bernoulli and Montmort, it is apparent that Waldegrave had also analyzed Le Her and had come up with a mixed strategy as a solution. He had also suggested working on the “problem of the pool,” or what is often called Waldegrave’s problem. The Universitätsbibliothek Basel contains an additional forty-two letters between Bernoulli and Montmort written after 1713, as well as two letters between Bernoulli and Waldegrave. The letters are all in French, and here we provide translations of key passages. The trio continued to discuss probability problems, particularly Le Her which was still under discussion when the *Essay d’analyse* went to print. We describe the probability content of this body of correspondence and put it in its historical context. We also provide a proper identification of Waldegrave based on manuscripts in the Archives nationales de France in Paris.

*Key words and phrases:* History of probability, history of game theory, strategy of play.

## 1. INTRODUCTION

The earliest extant correspondence between Pierre Rémond de Montmort and a member of the Bernoulli family is a letter from Montmort to Johann Bernoulli dated February 27, 1703, concerning a paper on calculus that the latter had written for the Académie royale des sciences in Paris (Bernoulli, 1702). They corresponded sporadically over the next few years. On April

29, 1709, Montmort sent Bernoulli a copy of his book on probability, *Essay d’analyse sur les jeux de hazard*, that he recently had published (Montmort, 1708). The book is the first in a series of books in probability published by several others over the years 1708 to 1718 in what Hald [(1990), page 191] calls the “Great Leap Forward” in probability. Bernoulli replied with a gift of a copy of his nephew’s doctoral dissertation (Bernoulli, 1709), the second book in Hald’s “Great Leap Forward”; Nicolaus Bernoulli’s book dealt with applications of probability. Once Johann Bernoulli received his copy of *Essay d’analyse*, he sent, on March 17, 1710, a detailed set of comments on the book. In the letter Bernoulli included another set of comments on *Essay d’analyse*, this one by his nephew Nicolaus (Montmort, 1713, pages 283–303). Thus began a series of correspondence between Montmort and Nico-

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laus Bernoulli on problems in probability. Montmort included much of this correspondence in Part V of the second edition of *Essay d'analyse* (Montmort, 1713). The correspondence between Montmort and Nicolaus Bernoulli after 1713, left unpublished and largely ignored by historians, contains scientific news and further discussion of problems in probability. The major topic is a continuing discussion of issues related to the card game Le Her. Next, in terms of ink spilt on probability, are discussions of the “problem of the pool,” or Waldegrave’s problem, generalized to more than three players, and of the game Les Étrennes (which may be translated as “the gifts”). The correspondence also contains discussions of various problems in algebra, geometry, differential equations and infinite series.

As an aristocrat, Montmort’s network included both political and scientific connections. His letters to Bernoulli contain some references to his political activities that sometimes kept him from replying promptly. His brother, Nicolas Rémond, was Chef de conseil for Philippe duc d’Orléans, who became regent of France after his uncle Louis XIV died in 1715 (Leibniz, 1887, page 599). Among the mathematicians of the era, Montmort corresponded with Isaac Newton, Gottfried Leibniz, Brook Taylor and Abraham De Moivre, in addition to the Bernoullis as well as several others. As a talented amateur mathematician, his work was well regarded by the mathematicians of his day. He was generous to his scientific friends. He received as guests to the Château de Montmort Nicolaus Bernoulli, Brook Taylor and one of the sons of Johann Bernoulli. He also sent gifts of cases of wine and champagne to both Newton and Taylor.

Le Her is a game of strategy and chance played with a standard deck of fifty-two playing cards. The simplest situation is when two players play the game, and the solution is not simply determined even in that situation. Montmort calls the two players Pierre and Paul. Pierre deals a card from the deck to Paul and then one to himself. Paul has the option of switching his card for Pierre’s card. Pierre can only refuse the switch if he holds a king (the highest valued card). After Paul makes his decision to hold or switch, Pierre now has the option to hold whatever card he now has or to switch it with a card drawn from the deck. However, if he draws a king, he must retain his original card. The player with the highest card wins the pot, with ties going to the dealer Pierre. The game can be expanded to more than two players. Montmort [(1708), pages 186–187] originally described the problem for four players and posed the question: What are the chances of each

player relative to the order in which they make their play?

Because of the winning conditions, it is obvious that one would want to switch low cards and keep high ones. The key is to find what to do with the middle cards, such as seven and eight, when two players are playing the game. In other cases, cards are clearly too low to keep or too high to switch, being much below or above the average in a random draw. Naturally, the threshold would be lower with more than two players.

In Part V of *Essay d'analyse*, only the game with two players is considered. Initially, Montmort and Nicolaus Bernoulli wrote back and forth about the problem and came to the same solution. However, two of Montmort’s friends contended that this solution was incorrect. These were an English gentleman named Waldegrave and an abbot whose abbey was only a league and a half (about 5.8 kilometers) from Château de Montmort (Montmort, 1713, page 338). Montmort identified Waldegrave only as the brother of the Lord Waldegrave who married the natural daughter of King James II of England. Lord Waldegrave is Henry Waldegrave, 1st Baron Waldegrave, and his wife is Henrietta Fitz-James, daughter of James II, and his mistress Arabella Churchill. The abbot is the Abbé d’Orbais; Montmort also refers to him as the Abbé de Monsoury. The reason for the two appellations for the abbot is that his full name is Pierre Cuvier de Montsoury, Abbé d’Orbais. He has been described as “un prodige de bon coeur, d’urbanité et de science” (Bout, 1887). For the spelling choice between Montsoury and Monsoury, it should be noted that Montmort often spelled his name “Montmort.”

Two other problems were discussed extensively in the correspondence. The first is the problem of the pool, a problem that Waldegrave suggested to Montmort and solved himself (Montmort, 1713, page 318). In *Essay d'analyse* the problem is solved for three players. It is often called Waldegrave’s problem (Bellhouse, 2007). The “pool” is a way of getting three or more players to gamble against one another, when the game put into play is for two players only. In the situation for three players (Montmort uses the names Pierre, Paul and Jacques), all three begin by putting an ante into the pot. Then Pierre and Paul play a game against each other. The winner plays against Jacques and the loser puts money into the pot. The game continues until one player has beaten the other two in a row. That player takes the pot. The game can be expanded to more than three players, but that situation was not fully treated in

*Essay d'analyse*. The second is the problem of solving the game Les Étrennes (or “estrene,” an alternative old French spelling). As described by Montmort [(1713), pages 406–407], this is a strategic game between a father and his son. The father holds an odd or even number of tokens in his hand, which his son cannot see. When the son guesses even, he receives a gift of four écus (silver coins) if he is correct and nothing if wrong. When the son guesses odd, he receives one écu if he is correct and nothing if wrong. The discussion of this game in the correspondence is only brought in to enlighten Le Her whose strategic nature is in some important respects essentially similar.

Montmort concludes the last letter (to Bernoulli) that appears in *Essay d'analyse* with a remark that Waldegrave had volunteered to take care of getting the book printed in Paris. Montmort's letter was dated November 15, 1713, and was written from Paris. What is also of concern to us are the letters after this date and how these letters relate to earlier discussions. The unpublished correspondence begins with a letter from Montmort to Bernoulli dated January 25, 1714, in which he says that he has sent Bernoulli two copies of the second edition of *Essay d'analyse*. Montmort was still in Paris, where he claimed to have been for three months. He was staying at a hotel in Rue des Bernardins, which in modern Paris is only a walk of 350 meters to the printer, Jacques Quillau in Rue Galande. Presumably, Waldegrave's help consisted mainly in dealing with the printer and the proof sheets as they came off the press, thus relieving Montmort of some tedious work.

## 2. THE TREATMENT OF THE GAME OF LE HER IN *ESSAY D'ANALYSE*

To understand the discussion of Le Her after 1713, it is necessary to describe the treatment of the game as it appears in the second edition of *Essay d'analyse*. Hald [(1990), pages 314–322] provides a detailed description of the mathematical calculations involved in assessing the game. Yet he devotes little space to elucidating the discussions among Bernoulli, Montmort and Waldegrave, as well as the Abbé d'Orbais, concerning the issues surrounding these mathematical calculations. It is the substance of these discussions that are of interest to us.

Hald's only comment on the discussion over Le Her concerns a comment made by Waldegrave and Abbé d'Orbais to the effect that Bernoulli's reasoning in obtaining his mathematical solution is faulty. After pointing out their observation that Bernoulli's solution fails

to account for a player's probability of playing in a certain way, Hald [(1990), page 315] claims:

It is no wonder that Bernoulli does not understand the implications of this remark, since the writers themselves have not grasped the full implication of their point of view.

It is indeed true that there was some confusion on Bernoulli's side which he deftly tried to hide.

Henny [(1975), page 502] comments that he is amazed to find expressed in the letters many concepts and ideas that appear in modern game theory. At the same time, he is surprised to find Waldegrave defending his position so strongly against Bernoulli who was the superior mathematician. Henny states further that Waldegrave did not have the necessary mathematical skills to provide a mathematical proof of his results.

As we will show in a review of the treatment of Le Her in *Essay d'analyse* and the subsequent unpublished correspondence, both Hald's and Henny's insights fall short of the mark.<sup>1</sup> One reason they fall short is that they do not consider the full range of the various events that were under discussion and their associated probabilities. Two events are natural for a probabilist to consider. The first is the distribution of the cards to Pierre and Paul. The second is the randomizing device used to come up with the mixed strategy prescribing when the players should hold and when they should switch. The randomizing device considered in *Essay d'analyse* is a bag containing black and white counters or tokens (the old French word used is “jetton”). The third event that Montmort, Waldegrave and Bernoulli consider (but not Hald or Henny) is difficult, or perhaps impossible, to quantify. This is the possibility that Paul, say, is a poor player and does not follow a strategy that is mathematically optimal, or the possibility that Paul, say, is a very good player who tries to trick Pierre into making a poor choice. This kind of event unfolds regularly in modern poker games.

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<sup>1</sup>The same could be said of others who have passed harsh judgments on Montmort and Bernoulli. For instance, Fisher (1934) argues that “Montmort's conclusion [that no absolute rule could be given], though obviously correct for the limited aspect in which he viewed the problem, is unsatisfactory to common sense, which suggests that in all circumstances there must be, according to the degree of our knowledge, at least one rule of conduct which shall be not less satisfactory than any other; and this his discussion fails to provide.” Our discussion below will show that this assessment is misinformed.

Another reason for which Hald and Henny see some confusion in the discussions among Bernoulli, Montmort and Waldegrave is that what we are seeing in the correspondence is the complete unfolding of a problem from its initial statement, and discussions around it, to a complete solution. This is different from a “textbook” statement of a problem followed by a solution. In the latter case, the problem and solution are both well laid out. In the former case, there is some grappling with the problem until it becomes clear how to proceed.

We begin with the correspondence in *Essay d’analyse* where Le Her is first mentioned. In Johann Bernoulli’s 1710 letter to Montmort in *Essay d’analyse*, he suggests more efficient methods to reach Montmort’s conclusions for a variety of problems and in some cases generalizes Montmort’s results. There is only one reference to the problem of Le Her, which is the second of four problems proposed in Montmort [(1708), pages 185–187]:

The second and the third [problem] seem to me amenable, but not without much difficulty and work, that I prefer to defer to you and learn the solution, than to work long at the expense of my ordinary occupations that leave me scarcely any time to apply myself to other things.

In his reply to this letter, which is dated November 15, 1710 (Montmort, 1713, pages 303–307), Montmort makes no reference to this passage.

Nicolaus Bernoulli’s first letter to Montmort, dated February 26, 1711, makes no reference to the game Le Her. It is a note in Montmort’s reply to Nicolaus Bernoulli, dated April 10, 1711 (Montmort, 1713, pages 315–323) that initiates the discussion of Montmort’s second problem:

I started some time ago to work on the solution of problems that I propose at the end of my book; I find that in Le Her, when there are only two players left, Pierre and Paul, Paul’s advantage is greater than 1 in 85, and less than 1 in 84. This problem has difficulties of a singular nature.

In a postscript to this letter, Montmort makes an additional remark:

As there are few copies of my book left, there will soon be a new edition. When I have decided, I will ask you permission, and your uncle, to insert your beautiful letters which will make the principal embellishment.

It is this announcement that may have motivated Nicolaus Bernoulli to continue his correspondence with Montmort and to send him much interesting material. Publishing mathematical material outside a scientific society or without a patron to cover the costs was an expensive proposition, one that Montmort could afford. Because of the specialized type that was used and the accompanying necessary skill of the typesetter, the cost of a mathematical publication was well above the norm for less technical books. Bernoulli could get his results in print at no cost to himself.

Bernoulli responded with a long letter, dated November 10, 1711 (Montmort, 1713, pages 323–337). In this letter, he announces, among many other things, that he has also solved the two-person case for Le Her (Montmort, 1713, page 334):

I also solved the problem on Le Her in the simplest case; here is what I found. If we suppose that each player observes the conduct that is most advantageous to him, Paul must only hold to a card that is higher than a seven and Pierre to one that is higher than an eight, and we find under this supposition that the lot of Pierre will be to that of Paul as 2697 is to 2828. Supposing that Paul also holds to a seven, then Pierre must hold to an eight, and their lots will still be as 2697 to 2828. Nevertheless it is more advantageous for him not to hold to a seven than to hold to it, which is a puzzle that I leave you to develop.

This passage is carefully worded, yet it will be misinterpreted by Montmort and Waldegrave. As we will see, a key aspect that is neglected by Montmort and Waldegrave is the antecedent of Bernoulli’s conditional statement starting with “If we suppose that each player observes the conduct. . .”

Montmort’s reply, dated March 1, 1712 (Montmort, 1713, pages 337–347), highly praises Bernoulli’s prior letter. He complains that, being in Paris, he has had no time and peace to think on his own and, as a consequence, the main object of his letter is to report progress made by his two friends, the Abbé d’Orbais and Waldegrave, on a problem proposed by Bernoulli, and on the problem of Le Her. On the latter, Montmort reports that “they dare however not submit to your decisions” (Montmort, 1713, page 338). However, as he says in a passage that is key to understanding the forthcoming controversy, the Abbé d’Orbais also previously disagreed with Montmort:

When I worked on Le Her a few years ago, I told M. l'Abbé de Monsoury what I had found, but neither my calculations nor my arguments could convince him. He always maintained that it was impossible to determine the lot of Pierre and Paul, because we could not determine which card Pierre must hold to, and vice versa, which results in a circle, and makes in his opinion the solution impossible. He added a quantity of subtle reasonings which made me doubt a little that I had caught the truth. That is where I was when I proposed that you examine this problem; my goal was to make sure from you of the goodness of my solution, without having the trouble of recalling my ideas on this which were completely erased.

Montmort then claims that Bernoulli's solution confirms what he had found, a decision that prompts a reply from Waldegrave objecting to Bernoulli's solution, quoted at length in Montmort (1713), pages 339–340.

According to Waldegrave and the Abbé d'Orbais, it is not true that Paul must hold only to an eight and Pierre to a nine. Rather, that Paul should be indifferent to hold to a seven or to switch, and that Pierre should be indifferent to hold to an eight or to switch. Waldegrave wrote the following to Montmort (Montmort, 1713, page 339):

We argue that it is indifferent to Paul to switch or hold with a seven, and to Pierre to switch or hold with an eight. To prove this, I must first explain their lot in all cases. That of Paul having a seven, is  $\frac{780}{50 \times 51}$  when he switches, and when he holds on to it his lot is  $\frac{720}{50 \times 51}$  if Pierre holds on to an eight, and  $\frac{816}{50 \times 51}$  if Pierre switches with an eight. The lot of Pierre having an eight is  $\frac{150}{23 \times 50}$  if he holds on to it, and  $\frac{210}{23 \times 50}$  if he switches in the case that Paul only holds on to a seven; and  $\frac{350}{27 \times 50}$  by holding on to it, and  $\frac{314}{27 \times 50}$  by switching in the case that Paul holds on to a seven, so here they are. The lots of Paul  $\frac{780 \text{ or } 720 \text{ or } 816}{50 \times 51}$ , those of Pierre  $\frac{150 \text{ or } 210}{23 \times 50}$  or  $\frac{350 \text{ or } 314}{27 \times 50}$ .

Based on the numbers he obtains, Waldegrave observes that “720 being more below 780 than 816 is above, it appears that Paul must have a reason to switch with 7” (Montmort, 1713, page 339). The differences, 780 –

720 and 816 – 780, are in the ratio 60:36, or 5:3, a ratio which later enters the discussion.

In the rest of his argument, Waldegrave talks of a weight instead of a reason. He first lets the weight that leads Paul to switch be  $A$ , and the weight that leads Pierre to switch be  $B$ . And he argues that the same weights lead Paul and Pierre to both strategies.  $A$  leads Paul to switch with 7 and, as a consequence, it also leads Pierre to switch his 8; but what leads Pierre to switch his 8 must lead Paul to hold with 7. So,  $A$  leads Paul to both switch with a 7 and hold on to it. The same goes for Pierre. Therefore, “it is false that Paul must only hold on to an 8, and Pierre to a 9,” which was Bernoulli's claimed solution. The word “probability” comes up only once in this discussion, in the conclusion of the excerpt from Waldegrave's letter to Montmort. Waldegrave writes (Montmort, 1713, page 340):

Apparently Mr. Bernoulli was simply looking at the fractions that express the different lots of Pierre and Paul, without paying attention to the probability of what the other will do.

Montmort leaves the discussion there without further comment.

Upon receiving Montmort's letter, Bernoulli agrees with these figures, saying that “the lots they found for Pierre and Paul are very right” (Montmort, 1713, page 348). And yet, when Bernoulli proposes his solution, and when Montmort eventually publishes a table of probabilities as an appendix to *Essay d'analyse* (Montmort, 1713, page 413), the numbers are different. The Bernoulli–Montmort probabilities are shown in Table 1, which appears in Hald (1990), page 318. None of the parties in this debate actually explain their calculations. Waldegrave's probabilities are justified in Todhunter (1865), pages 107–110; the Bernoulli–Montmort probabilities are in Hald (1990), pages 315–318. The difference in the probabilities is that Walde-

TABLE 1  
*Probabilities that Paul wins depending on the strategies of play*

	Pierre	Switch the 8 (and under)	Hold the 8 (and over)
Paul			
Switch the 7 (and under)		$\frac{2828}{5525}$	$\frac{2838}{5525}$
Hold the 7 (and over)		$\frac{2834}{5525}$	$\frac{2828}{5525}$

grave's probabilities are conditional on Paul having a seven in his hand and the Bernoulli–Montmort probabilities are the marginal probabilities for all cards that Paul may hold.

In a letter dated June 2, 1712, Bernoulli replies to Waldegrave's argument by accusing him of committing a fallacy. He argues that if we suppose that *A* leads Paul to switch with a seven, and so leads Pierre to switch with an eight (if Pierre knows Paul switches with seven), then it also leads Paul to hold on to a seven. Therefore, *A* both leads Paul to switch with a seven and to hold on to a seven. His conclusion is that (Montmort, 1713, page 348):

we are supposing two contradictory things at the same time; that is, that Paul knows and ignores at the same time what Pierre will do, and Pierre what Paul will do.

Bernoulli explains that if we do not commit this fallacy regarding what Paul and Pierre know about the other's intent, we are led to reasoning in a circle, which shows that Waldegrave's argument cannot show anything. This argument is peculiar, and seems to suggest that Bernoulli does not understand Waldegrave's point. It might, however, be simply a misinterpretation of Waldegrave's argument, for it is expressed in terms of weight rather than in terms of probability. The word "weight" or "poids" in French offers more opportunity for misinterpretation. Moreover, Bernoulli admits having written his letter hastily, as he was preparing for a long trip through the Netherlands and England. As a result of this travel, some subsequent letters are delayed, and the arguments they contain do not follow the chronological order of when the letters were written.

A letter to Bernoulli, dated September 5, 1712 (Montmort, 1713, pages 361–370), announces that Waldegrave and the Abbé d'Orbais have seen Bernoulli's reply in which he accuses them of committing a fallacy. Montmort includes a note from the Abbé d'Orbais in which he claims that Waldegrave has written a beautiful and precise reply to Bernoulli's objection; the rebuttal, however, is not included. In this note, the Abbé d'Orbais also enjoins Montmort to take a side in this dispute between them. This suggests that, even if Montmort thanked Bernoulli for his solution, which he claimed agreed with his own, Montmort has not yet made up his mind as to whether Bernoulli really solved the problem.

The next letter concerning Le Her is from Bernoulli to Montmort, dated December 30, 1712 (Montmort, 1713, pages 375–394). Adding important pieces to the

puzzle, it contains a three-page discussion of Le Her (Bernoulli mentions having just received the June 2 letter, since it was sent from Switzerland to Holland, then to England, and finally back to Switzerland). Bernoulli insists that, despite Waldegrave's arguments, Paul does not do as well by abiding to the maxim of holding to a seven, than that of switching with a seven. Bernoulli then says (Montmort, 1713, page 376):

If it were impossible to decide this problem, Paul having a seven would not know what to do; and to rid himself [from deciding], he would subject himself to chance, for example, he would put in a bag an equal number of white tokens and black tokens, with the intent of holding to a seven if he draws a white one, & to switch with a seven if he draws a black one; because if he put an unequal number he would be lead more to one party than to the other, which is against the assumption. Pierre with an eight would do the same thing to see whether he must switch or not.

This comment introduces with clarity the idea of chance by "the way of tokens" (as they will say later). What Bernoulli says here seems to confirm that, at first, when he accused Waldegrave of committing a fallacy, he did not interpret Waldegrave's weights as probabilities. Nonetheless, he suggests that the only probability allocation compatible with the supposed state of ignorance of the players is that each player chooses a strategy with probability  $\frac{1}{2}$ . Under these choices, he computes the lot of Paul (which is then  $\frac{774}{51 \times 50}$ ) and concludes that it would be a bad thing for Paul to randomize in this way, since he could guarantee himself a lot of  $\frac{780}{51 \times 50}$ . Therefore, Bernoulli concludes Paul must always switch with a seven. As Bernoulli says (Montmort, 1713, page 376), "it is better to make the choice where we risk less." He then explains the reasoning that he had left out of his hastily written letter from June 2. In contemporary terms, he calculated the unconditional probability of winning under each pure strategy profile (without assuming that any card has been dealt yet). He displays a refined version of the reasoning that led to accusing Waldegrave of a fallacy, yet it does not do full justice to Waldegrave's idea.

Eight months later, on August 20, 1713, Montmort [(1713), pages 395–400] finally replies to Bernoulli, complaining that he has, despite his philosophical inclinations, been involved in political activities, and so he did not have the leisure for intellectual work. Thus,

his letter only contains scientific news. There is only one brief mention of Le Her; he tells Bernoulli that, despite his last effort to provide a thorough and precise argument, Waldegrave and the Abbé d’Orbais are still unconvinced by his claimed solution. Shortly after, in a letter dated September 9, 1713, Bernoulli also asks Montmort to explain his own views on the dispute. Montmort obliges him in his letter dated November 15, 1713. This is the last letter published in the second edition of *Essay d’analyse* (Montmort, 1713, pages 403–413). The letter also contains an excerpt of a letter from Waldegrave and a table of the lots of Paul and Pierre for the four crucial combinations of strategies, which are summarized in Table 1.

Here, then, is Montmort’s understanding of the situation. To begin with, he agrees with Bernoulli that it is not indifferent to Paul to switch or hold with a seven, and to Pierre to switch or hold with an eight, because of Bernoulli’s calculations of the unequal chances for each strategy. (This shows that Bernoulli and Montmort use “indifferent” in the sense of having the same probability of winning. For Waldegrave and d’Orbais, however, “indifferent” seems to mean, perhaps more awkwardly, that no strategy dominates the other in probability.) This being said, Montmort nonetheless disagrees with Bernoulli that this establishes the strategy as a maxim, that is, as a rule of conduct that must be obeyed invariably to obtain the best results. Rather, he thinks that it is impossible to establish such a maxim (Montmort, 1713, page 403):

[T]he solution of the problem is impossible, that is, we cannot prescribe to Paul the conduct that he must adopt when he has a seven, and to Pierre the conduct he must adopt when he has an eight.

He grants that, if one is to choose a fixed and determined maxim, then switching on seven, for Paul, will be better than any other, yet Paul can hope to make his lot better.

Why, then, would a solution be impossible? Would the solution not be the optimum that one can reach in Paul’s hope of making his lot better? Montmort claims that, whereas he used to think that the use of black and white tokens to randomize strategies could avoid the “circle,” he does not think that anymore. He gives a general formula to find the probability of winning with a certain probability allocation for what we call a mixed strategy:

$$\frac{2828ac + 2834bc + 2838ad + 2828bd}{13 \cdot 17 \cdot 25(a + b + c + d)},$$

where  $a$  is Paul’s probability of switching with seven,  $b$  is Paul’s probability of holding the seven,  $c$  is Pierre’s probability of switching with an eight, and  $d$  is Pierre’s probability of holding on to an eight. But how should the probabilities be chosen? Montmort claims that any argument will only inform us of what Paul must do conditionally to what Pierre does and vice versa, which leads us into a circle once again. He concludes that Bernoulli’s arguments to show that a circle does not occur are wrong, and instead formulates this thesis (Montmort, 1713, page 404):

[W]e must suppose that both players are equally subtle, and that they will choose their conduct only based on their knowledge of the conduct of the other player. However, since there is here no fixed point, the maxim of a player depends on the yet unknown maxim of the other, so that if we establish one, we draw from this supposition a contradiction that shows that we must not have established it.

Montmort also disagrees with Bernoulli that, under pain of contradiction, if we are to use white and black tokens to randomize, we must use an equal number of tokens. Instead, he thinks that the probability of winning calculated for the fixed and determined maxims shows that Paul must switch more often with a seven than hold on to it. Yet, he maintains (Montmort, 1713, page 405):

But how much more often must he switch rather than hold, and in particular what he must do (here and now) is the principal question: the calculation does not teach us anything about that, and I take this decision to be impossible.

Thus, Montmort believes, it seems, that there is no optimal probability allocation.

But he has another reason for believing that the solution of the game is impossible. He has in mind the game *Les Étrennes* (Montmort, 1713, pages 406–407). Montmort also believes that it is impossible to prescribe any strategy of play in *Les Étrennes* because the players might always try, and indeed good players will try, to deceive other players into thinking that they will play something they are not playing, thus trying to outsmart each other (“jouer au plus fin” is the phrase used in French).

As he was finishing his letter, Montmort received one from Waldegrave and quoted extensively from it to

Bernoulli. *Essay d'analyse* essentially concludes with Waldegrave's letter. Waldegrave refers to a formula, which is not included by Montmort; it presumably is the formula displayed above. He explains that, if  $a = 3$  and  $b = 5$  (so that the probability of Paul switching with a seven is 0.625), then the lot of Pierre is going to be  $\frac{2831}{5525} + \frac{3}{4.5525}$  no matter what  $c$  and  $d$  are. This shows that  $\frac{2831}{5525} + \frac{3}{4.5525}$  is Paul's minimum lot. He can only adopt another conduct in the hope of making his lot better. This shows, he claims, that both Bernoulli and (formerly) himself were wrong to claim that the lots of Paul was to that of Pierre as 2828 is to 2697; if both players play in the most advantageous way, Paul's lot is  $\frac{2831}{5525} + \frac{3}{4.5525}$ . Waldegrave is convinced that this is something that both Bernoulli and Montmort will agree to, now that it is agreed that one can use a randomized strategy. He also explains that, if Pierre uses  $c = 5$  and  $d = 3$ , then  $\frac{2831}{5525} + \frac{3}{4.5525}$  will also be Paul's maximum lot.

Waldegrave also asserts that it is impossible to establish a maxim; he grants, however, that it is impossible for him to show this with the same level of evidence. This is often taken incorrectly as evidence of a lack of Waldegrave's mathematical abilities. Waldegrave is instead referring to the situation in which players may try to outsmart each other. Waldegrave agrees that if Paul does not use  $a = 3$  and  $b = 5$ , then it is possible for Paul to do better than  $\frac{2831}{5525} + \frac{3}{4.5525}$  provided that Pierre does not play in the best way. On the other hand, it would be worse if Pierre plays correctly. Furthermore, Waldegrave remarks (Montmort, 1713, page 411):

What means are there to discover the ratio of the probability that Pierre will play correctly to the probability that he will not? This appears to me to be absolutely impossible, and thus leads us into a circle.

As with Montmort, his main concern is that it is always possible for the players to try to outsmart each other (“jouer au plus fin”).

### 3. ISSUES ARISING FROM THE PUBLISHED CORRESPONDENCE

Examining the detailed arguments provided by Montmort, Bernoulli and Waldegrave reveals a picture that contrasts with the judgment that they were essentially confused on the fundamental concepts and methods required to solve a strategic game such as Le Her. In fact, we maintain that they understood most

of the aspects of the problem with clarity. There are, however, a number of important outstanding issues left unresolved in the correspondence on Le Her as it appears in *Essay d'analyse*. Let us review them briefly.

It is true that the letters reveal a certain type of misunderstanding; however, it is not conceptual confusion, but rather mutual misinterpretation due to using terms differently. An instance of this is whether it is indifferent to Paul to switch or hold to a seven. On the one hand, both Montmort and Bernoulli claim that it is not indifferent to Paul because the chances of winning are not identical. On the other hand, Waldegrave claims that it is indifferent, and the reason for that seems to be that neither pure strategy dominates the other in probability.

Another instance of this is the disagreement they appear to have on the existence of a circularity in the analysis of the game. Montmort and Waldegrave assert that there is a vicious circle that prevents one from establishing a maxim; the circle they discuss, however, is really a regression *ad infinitum*, that is, to establish a maxim, we always need to go one step further in the “A must know what B does” loop (Bernoulli agrees with this point). However, Bernoulli claims that there is a circle in Waldegrave's argument, in the sense that either his argument is contradictory or a *petitio principii* (but Bernoulli is not considering randomizing strategies at this point). Again, they are only contradicting each other in the wording, not in the idea.

Finally, a third instance is that Montmort and Waldegrave claim that the solution of the game is impossible, whereas Bernoulli does not. Here again, they disagree on what it means to “solve” the game Le Her. Bernoulli claims that the solution is the strategy that guarantees the best minimal gain—what we would call a minimax solution—and that as such there is a solution. However, despite understanding this “solution concept,” Montmort and Waldegrave refuse to affirm that it “solves” the game, since there are situations in which it might not be the best rule to follow, namely, if a player is weak and can be taken advantage of. Clearly, the concept of solution they have in mind differs from the minimax concept of solution. This latter concept, in addition to the probability of gain with a pure strategy and the probability allocation required to form mixed strategies, requires that we know the probability that a player will play an inferior strategy. But, they assert, this cannot be analyzed by calculations, so the game cannot be solved.

This being said, there are a number of things that are said that suggest a certain level of confusion at a conceptual level. The two most important are these. First,



Bernoulli appears to have some difficulty with the relation between the knowledge of the players and the probabilities involved in mixing strategies. His circularity objection to Waldegrave is awkward and somewhat mystifying. Moreover, his argument that, if we allow randomized strategies with black and white tokens, it must be because neither player knows what the other player will do, and that as a result the only acceptable probability allocation of  $\frac{1}{2}$  is problematic. This kind of mistaken argument has been repeated over the centuries by some of the greatest minds in probability, statistics and game theory. Second, Montmort understands very well the idea of randomizing strategies, but he nonetheless claims that there is no optimal probability allocation that can be calculated. This claim, however, was made before consulting Waldegrave's letter in which he reveals the optimal probability.

#### 4. DISCUSSION OF THE GAME OF LE HER AFTER 1713

Referring to a letter from Bernoulli to Montmort dated February 20, 1714, Henny (1975), in his treatment of Le Her, mentions only that Bernoulli accepted Waldegrave's solution to the problem. However, Bernoulli had other things to say about Le Her in that same letter. Henny also refers to a letter of January 9, 1715, from Waldegrave to Bernoulli in which Waldegrave seemingly admits to Bernoulli that he does not have the mathematical skills to actually prove his results. What Henny leaves out is that the letter was written in reply to a detailed criticism of the solutions to Le Her that Bernoulli had sent earlier to Montmort.

After some personal news and apologies for not writing sooner, in his letter of February 20, 1714, to Montmort, Bernoulli initially thanks Montmort for correcting, editing and making clearer his letters that Montmort had printed in *Essay d'analyse*. Then follows the discussion of Le Her that Henny (1975) only briefly mentions. Initially, Bernoulli suggests that the controversy is essentially over:

Concerning Le Her, I seem to have foreseen that in the end we would all be right. However, I congratulate Mr. de Waldegrave who has the final decision on this question, and I willfully grant him the honor of closing this affair. . .

Despite this, Bernoulli still claims that he disagrees on a few minor points, and these point directly to the outstanding issues we mentioned above. The main concern is the relation between "establishing a maxim"

and solving the problem of Le Her posed by Montmort in his book. Bernoulli states:

One can establish a maxim and propose a rule to conduct one's game, without following it all the time. We sometimes play badly on purpose, to deceive the opponent, and that is what cannot be decided in such questions, when one should make a mistake on purpose.

This point was raised before by Montmort and Waldegrave, but they do not consider that such a play would be necessarily a mistake. Whether or not this kind of play is a mistake, we saw that from the same consideration, Montmort and Waldegrave conclude that solving the problem is impossible. However, Bernoulli now phrases things more carefully:

Mr. de Waldegrave wrongs me on p. 410 by claiming that I once said that the lot of Paul is to that of Pierre as 2828:2697. If you carefully read my letter from Oct. 10, 1711, you will find that I did not say it absolute and without restriction. I beg you to consider those words: once we have determined or rather supposed what are the cards to which the players will hold, etc. And the following words. You will see that I there supposed that the players want to hold to a fixed and determined card, and indeed I had not thought about the way of tokens, which, as Mr. de Waldegrave said, is not among the ordinary rules of the game.

Bernoulli essentially says that he was misinterpreted and that he only computed the best odds of winning with a pure strategy, not that he established what a player should do in an actual game. Moreover, if we grant his supposition, then he has found the most advantageous maxim. After this correction, Bernoulli thinks the discussion is over, saying, "We are thus all agreeing, and we have made peace; *canamus receptui* [sing retreat]."

In his response to Bernoulli, dated March 24, 1714, Montmort concurs by writing, "I am quite pleased that we are all together by and large agreeing." In this letter Montmort claims that he disagreed with Bernoulli on some aspect of the corrected interpretation of his position, but he leaves it to a later letter to explain. However, in his next letter to Bernoulli, November 21, 1714, Montmort does little to clarify. He says, "if it is

ever permitted to say to two persons maintaining contradictory claims that they are both right, it is assuredly at this occasion in our dispute.” Montmort emphasizes that what he seeks is the correct advice that should be given to the players, but the discussion does not go much further.

On August 15, 1714, Montmort sent a letter to Bernoulli containing a two-page “supplement” that reignites the debate. He makes six points. First, he claims that telling Paul always to switch with a seven is bad advice, since his minimum lot is then 2828. Second, that it would be better advice to tell him to do whatever he pleases with a seven, so that he can look at both options indifferently. Third, we cannot say that this would be the best advice either, for knowing that, Pierre would switch with an eight, in which case Paul should certainly have held on to a seven. This leads to a vicious circle. Fourth, if we admit the way of tokens, the best advice that he knows is to tell Paul to have the ratio 3 : 5 for switching with a seven. But even then, he does not think that we can demonstrate that it is the best advice. Fifth, he claims that it is impossible at this game to determine the lot of Paul, because one cannot determine what manner of playing is the most advantageous to each player, even when we admit a randomized strategy. This point makes explicit for the first time Montmort’s (and presumably Waldegrave’s) idea that you can only claim that you have found the lot of a player (which is what Montmort’s problem in *Essay d’analyse* demanded) if we can determine what is the best way to play. Moreover, determining the best way to play demands knowing more than the optimal token ratio for the randomized strategy. He adds that, of course, some methods of playing are better than others, as informed by the chances that have previously been calculated. He concludes, sixth, that he would not know what advice to give Paul if he had to. This letter sharpens the debate, in that it makes explicit the connection between “solving” a game and giving advice for play in actual situations.

In a long letter to Montmort dated August 28, 1714, along with a “supplement” dated November 1, 1714, Bernoulli replies to Montmort point by point. He asks Montmort a question that is meant to dismiss his argument:

If, admitting the way of tokens, the option of 3 to 5 for Paul to switch with a seven is the best you know, why do you want to give Paul other advice in article 6? It suffices for Paul to follow the best maxim that he could

know. It is not enough to claim that there is still a circle despite my reasons, one must fight my reasons.

And he continues: “It is not impossible at this game to determine the lot of Paul.” To counter Montmort’s previous argument, he once again insists that either Paul knows what Pierre will do, in which case his maxim is clear, or he does not, in which case Paul should use the probability  $\frac{1}{2}$  in the randomized strategy to determine what to do. As he admits, this is the exact same position he had at the beginning of the discussion, supported by the exact same argument. Thus, it seems that Bernoulli has missed the point Montmort made explicit in his August 15 letter.

It is at this point that Waldegrave reenters the debate at Montmort’s request. In a letter dated January 9, 1715, Waldegrave reiterates the six points that Montmort had laid out for Bernoulli in his letter of August 15. For each of the six points, Waldegrave’s arguments are longer and more detailed than what Montmort had previously given.

It is not until March 22, 1715, that Montmort replies to Bernoulli on this dispute. It is part of a very long letter that also contains the main topic for their further correspondence, infinite series. In this letter, Montmort writes once again about his views. They are the same as what we have seen already. However, Montmort stresses that a lot of what remains under discussion is based on inconsistent terminology and misinterpretation. In essence, he believes that the outstanding disagreements are only apparent contradictions. Nonetheless, he introduces one more element to clearly articulate his view. He distinguishes between the advice that he would put in print, or give to Paul publicly, and the advice he would give so that only Paul hears it. Montmort claims that, for the former, he would choose the mixed strategy with  $a = 3$  and  $b = 5$ , since it is the one that demonstrably brings about the lesser prejudice. However, he explains that, in practice, if Paul is playing against an ordinary player and not a mathematician, he would quietly give different advice that could allow Paul to take advantage of his opponent’s weakness. As he explains, the objective of this sort of analysis is not only to provide a maxim to otherwise ignorant players, but also to warn them about the potential advantages of using finesse. However, this latter part is not possible to establish, and it is in this sense that there is no possible solution to this problem.

The next letter, sent by Bernoulli to Montmort on May 4, 1715, disregards Montmort’s nuance. To begin,

Bernoulli “is forced to admit that he does not precisely know on what point [they] contradict each other.” Nonetheless, Bernoulli explains that, in his view, the distinction between public and private advice, the possibility of using finesse, or something similar, does not alter the fact that  $a = 3$  and  $b = 5$  is the best solution, and that it determines the lot of Paul (so that not only is the game solvable, but it is indeed solved).

Despite Bernoulli’s explanation, Montmort’s next letter, dated June 8, 1715, once again reiterates that “you have badly solved the proposed question, or you have not solved it at all.” He makes explicit what he takes the proposed question to be:

The question is and has always been to know whether we can establish the lots and as a result the advantage of playing first under the supposition not that Pierre and Paul follow this or that maxim (this would have no utility, no difficulty), but that both of them having the same skills, each follow the conduct that is the most advantageous.

Montmort then says that this dispute is beginning to bore him. He considers that furthering it will not make them learn anything new and that in the end the dispute must be about some other thing.

Our presentation of the correspondence makes it clear that they are using different concepts of solution; Bernoulli’s in essence is the concept of the minimax solution, whereas Montmort’s further depends on the probability of imperfect play (i.e., on the skill level of the players).

Around this time, Montmort’s interest shifts from probability and its applications to infinite series. In fact, most of the remaining correspondence with Bernoulli turns to that topic. At the same time, Montmort began an extensive correspondence with Brook Taylor, also mainly on infinite series (St. John’s College Library, Cambridge, TaylorB/E4). Although the dispute with Bernoulli seems to have petered out, Montmort was not yet done with it. In a letter dated July 4, 1716, Montmort asked Taylor to examine his dispute with Bernoulli about Le Her and to express his opinion on who was right. He referred Taylor only to the correspondence that appears in *Essay d’analyse*. Taylor apparently wrote back but with the wrong impression about what Montmort wanted. Montmort replied to Taylor on August 4, 1716, that he did not want any new research into the problem but only to examine, at his leisure, which of Bernoulli or Montmort was right. In a letter to Taylor dated November 10,

1717, Montmort thanked Taylor for his opinion on the dispute and concluded the letter by saying that Waldegrave would write him about Le Her as well as another game. Unfortunately, neither Taylor’s reply expressing his opinion nor Waldegrave’s letter to Taylor are extant.

## 5. THE PROBLEM OF THE POOL AND OTHER PROBABILITY PROBLEMS

Compared to the discussion of Le Her, the remaining discussion in the post-1713 correspondence with regard to probability problems is relatively minor. For example, after the remarks on Le Her that Bernoulli made in his letter of February 20, 1714, to Montmort, Bernoulli comments that he thinks there is an error in Montmort’s solution to a problem related to the jeu du petit palet in *Essay d’analyse*. He asks Montmort to check his solution. The problem appears to be Problème IV in Montmort (1713), page 254. The jeu du petit palet is a game in which players toss coins or flat stones (the “palets”) toward a target set on the ground or a table. The player with the most coins or stones on the target wins. The English equivalent game is called chuck-farthing or chuck-penny.

What takes up much of the discussion, other than Le Her, is news about Abraham De Moivre’s work. De Moivre corresponded with both Montmort and Bernoulli until about 1715 when he ceased corresponding with either of them. Prior to this discussion, Bernoulli had sent De Moivre a general solution to the problem of the pool on December 30, 1713 (Bellhouse, 2011, pages 106–107).

A report on De Moivre’s activities in probability takes up part of a letter from Bernoulli to Montmort dated April 4, 1714. Bernoulli mentions that De Moivre has sent him a long letter with reports of new solutions that will appear in a much expanded version of his treatise *De mensura sortis* (De Moivre, 1711). De Moivre’s new work, which was entitled *The Doctrine of Chances*, did not appear until 1718 (De Moivre, 1718). No details are given to Montmort other than that De Moivre has made inroads in three areas. First, De Moivre used his own method for the solution of the problem of the pool to generalize it to more than three players. Second, he developed a new kind of algebra to solve probability problems. Finally, Bernoulli reports that De Moivre considered that nearly all problems in probability can be reduced to series summations. Not only did De Moivre report that he had generalized the problem of the pool, but he also sent Bernoulli his solution to the problem. At the time of his writing to Montmort, Bernoulli had not read the solution and did not

pass the solution on to Montmort. The new algebra is probably the one that De Moivre developed for finding probabilities of compound events. See, for example, Hald [(1990), pages 336–338] for a modern discussion of this topic. This part of the letter ends with what might be interpreted as a nasty comment about De Moivre:

I will share here in confidence what he wrote to me concerning you. Here is what he told me about your comments that I had sent him. ‘I cannot stop myself etc. Our Society etc. I just received etc. kind [regards].’ After the letter I find written there these words: ‘in a sense,’ that made me laugh.

It is difficult to know what exactly Bernoulli is saying here. It appears that he sent De Moivre Montmort’s severe criticism of *De mensura sortis* that Montmort published in *Essay d’analyse* (Montmort, 1713, pages 363–369).

Later that month, Montmort reported back to Bernoulli that he received a very polite and fair letter from De Moivre in which De Moivre announced that he had found a new solution to the problem of the duration of play. See Bellhouse [(2011), pages 111–114] for a discussion of the publication of this solution. De Moivre sent reports about more of his results in probability to Montmort and Montmort sent on a précis of these results to Bernoulli in a letter dated August 15, 1714. Many of the results that Montmort mentions found their way into *The Doctrine of Chances*, including what is called Woodcock’s problem discussed in Bellhouse [(2011), pages 125–126].

On August 28, 1714, Bernoulli finally wrote to Montmort enclosing a copy of De Moivre’s general solution to the problem of the pool. In the letter, Bernoulli asks Montmort to tell him what he thinks of the solution. He further states that it appears that De Moivre is using Bernoulli’s approach to the solution for three and four players that appears in *Essay d’analyse* (Montmort, 1713, pages 380–387). At the same time he is using an analytical approach rather than infinite series (De Moivre actually used a recursive method for his general solution). Montmort replied on March 22, 1715, that he agrees with Bernoulli’s assessment. On returning from a trip to England, Montmort reported to Bernoulli in a letter dated June 8, 1715, that one of Bernoulli’s solutions to the problem of the pool had just been printed in the *Philosophical Transactions* (Bernoulli, 1714). Bernoulli had sent De Moivre two solutions; De Moivre claimed he had found an error in the first solution.

## 6. WALDEGRAVE IDENTIFIED

Many in the past have tried unsuccessfully to identify the Waldegrave who solved the problem of Le Her and who suggested the problem of the pool, often called Waldegrave’s problem. Bellhouse (2007) reviewed these attempts at identification and narrowed the field down to Charles, Edward or Francis Waldegrave, the three brothers of Henry Waldegrave, 1st Baron Waldegrave. Bellhouse argued for Charles Waldegrave, but in view of new information his choice was incorrect. Key to the proper identification is that several Waldegraves—siblings, cousins and at least one uncle of Henry—followed King James II into exile in France after James was deposed in 1688.

The proper identification of the Waldegrave of interest may be found in legal papers in the Archive nationales de France in conjunction with a letter from Waldegrave to Nicolaus Bernoulli; the letter to Bernoulli is signed only “Waldegrave” and is the only known letter in Waldegrave’s hand that is extant (Universitätsbibliothek Basel L Ia 22, Nr. 261). Other Waldegrave signatures to compare to the one on Bernoulli’s letter can be found on various legal documents, two in France (Archives nationales de France MC/ET/XVII/486 and 514) and one in England (House of Lords Record Office HL/PO/JO/10/1/439/481). See Figure 1. From the signatures, it is obvious that Francis is the Waldegrave of interest. From these records, it is also apparent that Charles Waldegrave handled the family’s affairs in England while Francis Waldegrave took charge of them in France.

What little is known of the life of Francis Waldegrave comes mostly from Montmort’s correspondence with Brook Taylor and Nicolaus Bernoulli. Montmort reported to Taylor one of Waldegrave’s political activities. Waldegrave was planning to take part in the Jacobite uprising in England in 1715. He was to be part of an invasion force led by the son of James II, James Stuart. The uprising in England fizzled out, James Stuart remained in France and Waldegrave fell ill just prior to the time when the planned invasion was to occur. Montmort called Waldegrave’s illness apoplexy; it was probably a stroke. From time to time, Montmort commented to Taylor and Bernoulli about Waldegrave’s illness, recovery and setbacks. At one point, for a cure or a rest, Waldegrave took the waters at a spa in France. He also spent time at Montmort’s chateau. Though ill, he was alive in France in 1719 when Montmort died so that the flow of information to Bernoulli and Taylor about Waldegrave stopped. Presumably, Waldegrave died in France.



FIG. 1. Waldegrave signatures from various sources.

How Waldegrave obtained his mathematical training is unknown. In whatever way he was educated, he was an adept amateur mathematician. This is contrary to Henny's interpretation of Waldegrave's skills. For example, Henny [(1975), page 502] claims that Waldegrave did not have the mathematical skills to work out a general method of calculation in Le Her. On the contrary, there is a hint of the fairly high level of Waldegrave's mathematical abilities in a letter from Montmort to Bernoulli dated March 24, 1714. There Montmort says that he is getting Waldegrave to read L'Hôpital's (1696) calculus book *Analyse des infiniment petits* and that Waldegrave has a natural aptitude for mathematics.

At the time that Montmort was sending *Essay d'analyse* to his publisher, Francis Waldegrave was living in Rue Princesse near Église Saint-Sulpice in Paris. In modern Paris, it is a 1.1 kilometer walk to Montmort's publisher in Rue Galande. In Section 1 it was mentioned that Montmort was staying only 350 meters from his publisher. This was not the only time that Montmort enlisted a colleague to take on some of the tedious parts of getting results to print. After Montmort sent Brook Taylor a number of theorems about infinite series, they decided that the results should be published in the Royal Society's *Philosophical Transactions* (Montmort, 1717). In a letter dated June 15, 1717, Montmort gave Taylor complete editorial control over the paper that included having Taylor translate the results from French into Latin (St. John's College Library, Cambridge TaylorB/E4). Taylor replied August 9, 1717, saying that he had made many changes and corrections to the paper (St. John's College Library, Cambridge, TaylorB/E5).

## 7. DISCUSSION AND CONCLUSIONS

The unpublished letters between Bernoulli and Montmort reveal a much more complex story than either Henny (1975) or Hald (1990) have described. The entire group—Bernoulli, Montmort and Waldegrave—were for the most part clear about the issues at the conceptual level. In the end it came down to a disagreement about what it meant to solve a problem. Further, Henny recognized many modern game theory concepts, but we show that the group's understanding of the modern notions is deeper than what Henny realized.

Apart from the technical and conceptual aspects of Le Her and other probability problems, we also get a glimpse into the social side of a rich amateur mathematician at work. Montmort was a good mathematician, but mathematics was his hobby and at times he did not have time to pursue his hobby. There is a bit of quid pro quo in his relationships with Bernoulli, Taylor and Waldegrave. Montmort acquires some status through his connections to artists, philosophers and scientists. He can impose on his scientific friends to do some of the more menial work for him in getting his research to print. On the other side, his scientific friends enjoy his hospitality, his gifts and the benefits of any political and scientific connections that he may have.

Traditionally, the mixed strategy solution with  $a = 3$  and  $b = 5$  for Le Her has been attributed to Waldegrave. It certainly appears to be the correct attribution based on the correspondence in the second edition of *Essay d'analyse*. However, in the long letter from Montmort to Bernoulli dated March 22, 1715, that covers discussions of Le Her, De Moivre and other topics, Montmort appears to claim priority of solution. As part of the discussion of Le Her, he says, "although I first found the determination of the numbers  $a$  and  $b$ ,

$c$  and  $d$ ...” Montmort’s suggestion of priority could have come about as a result of a conversation between Waldegrave and Montmort, with Waldegrave putting pen to paper. This illustrates Fasolt’s (2004) claims about the limits of history. Our data from the past is what has been written, not what has been spoken. Further, we can never know the tone behind what was written, such as Bernoulli’s apparently nasty comments to Montmort about De Moivre in his letter of April 14, 1714. Instead of coming up in conversation, Montmort may be claiming priority because he found the general formula in  $a$ ,  $b$ ,  $c$  and  $d$ ; the numbers were only a special case. Or it could be something else. Like Le Her itself, depending on how the problem is approached, the assignment of priority is a problem with no solution.

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The originals of the letters of Bernoulli, Montmort and Waldgrave are in Universitätsbibliothek Basel. The letters from Montmort to Bernoulli are catalogued Handschriften L Ia 22:2 Nr.187–206 and from Bernoulli to Montmort are L Ia 21:2 Bl.209–275. The letter from Bernoulli to Waldegrave is catalogued L Ia 21:2 Bl.229v–232r and the letter from Waldegrave to Bernoulli is L Ia 22, Nr. 261. When referencing these letters, we have to do so by the date, writer and recipient, rather than the catalog numbers.

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