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A recurrent random walk on the p-adic integers

M. Cruz-López and A. Murillo-Salas

Universidad de Guanajuato

Abstract. This paper describes a random walk on the p-adic integers, which turns out to be recurrent.

1 Introduction

Random walks on abstract groups have been studied during the last sixty years (see, e.g., Dudley (1962)). The description of stochastic processes on totally disconnected groups began with the revealing paper by Evans (1989) and, ever since, this area of study has been highly stimulated due to the wide range of applications of analysis on local fields to probability theory and physics.

In this paper it is done the description of a continuous time random walk with state space a compact Abelian group known as the p-adic integers $\mathbb{Z}_p$, which is the ring of integers of the field of p-adic numbers $\mathbb{Q}_p$. The random walk on $\mathbb{Z}_p$ is described by using the method developed in Albeverio and Karwowski (1994) for $\mathbb{Q}_p$. In the paper Lukierska-Walasek and Topolski (2006), the authors attempted to describe a random walk on $\mathbb{Z}_p$, but, the solution to Kolmogorov forward and backward equations is inaccurate, and the present authors do not know exactly where, since the ideas are only sketched. However, using the intensities suggested in the mentionated paper, the complete solution to the Kolmogorov forward and backward equations needed to describe the process is founded here. Thus, we are able to describe a Markov process Ξ which takes values in $\mathbb{Z}_p$. The precise statement of the main result in this paper can be found in Theorem 1 (see Section 3). By using the transition functions obtained for the process Ξ, we are able to prove that the process is recurrent (see Proposition 1 below).

Finally, it should be pointed out that different properties of p-adic integers have been used in several contexts as in Dragovich and Dragovich (2009), where $\mathbb{Z}_p$, for $p = 5$, has been proposed as a model to describe the genetic code. From this perspective, the authors guess that the process Ξ described here, suggests a new line of research to provide a random p-adic genetic code model. This will be the content of a future work.

General information and properties about p-adic integers can be consulted, v.g., in Koblitz (1984) and Robert (2000). In the first part of the paper, the basic objects of study are introduced, and, in the second part, the corresponding random walk is defined.

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2 The p-adic integers $\mathbb{Z}_p$

Denote by $\mathbb{P}$ the set of all prime numbers. Throughout the paper p will denote a prime number.

If $n < m$, then $p^m\mathbb{Z}$ is a subgroup of $p^n\mathbb{Z}$ and there is a well-defined surjective homomorphism

$$\rho_{nm} : \mathbb{Z}/p^m\mathbb{Z} \longrightarrow \mathbb{Z}/p^n\mathbb{Z},$$

given by

$$x \mod p^m\mathbb{Z} \longmapsto x \mod p^n\mathbb{Z}.$$

This determines a projective system $\{\mathbb{Z}/p^n\mathbb{Z}, \rho_{nm}\}$ of compact (finite) Abelian topological groups whose projective limit is the so called *p-adic integers*, denoted by $\mathbb{Z}_p$. That is,

$$\mathbb{Z}_p := \varprojlim \mathbb{Z}/p^n\mathbb{Z}.$$

$\mathbb{Z}_p$, with the profinite topology, is a compact Abelian topological group, which is also perfect and totally disconnected and therefore, homeomorphic to the p-adic Cantor group. A more complete description of $\mathbb{Z}_p$ as a projective limit can be found in Wilson (1998).

Remark 1. The following isomorphisms of topological groups are well known (see, e.g., Robert (2000)).

$$\mathbb{Z}_p \cong \prod_{j\geq 0}\{0, 1, \ldots, p-1\}$$
$$\cong \left\{\sum_{j\geq 0} a_j p^j : 0 \leq a_j < p\right\}.$$

According to the above remark, a p-adic integer x can be represented as the formal series

$$x = \sum_{j=0}^{\infty} a_j p^j,$$

with $a_j \in \{0, 1, \ldots, p-1\}$. The *$p$-adic order* of x, $\mathrm{ord}_p(x)$, is defined as the smallest index j_0 such that $a_{j_0} \neq 0$. Moreover, the norm of $x \in \mathbb{Z}_p$ is defined by

$$|x|_p := p^{-\mathrm{ord}_p(x)}.$$

This norm satisfies the non-archimedean inequality:

$$|x + y|_p \leq \max\{|x|_p, |y|_p\}.$$

$\mathbb{Z}_p$, with the metric d_p induced by this norm, is a complete metric space. More complete information about these properties can be found in Koblitz (1984), or, Robert (2000).

2.1 The decomposition of $\mathbb{Z}_p$

The (closed) ball with centre at 0 and radius p^{-1} is denoted by

$$B := \overline{B}(0, p^{-1}) = \{x \in \mathbb{Z}_p : |x|_p \le p^{-1}\}.$$

By definition, $B = p\mathbb{Z}_p$ is a subgroup of $\mathbb{Z}_p$ and the quotient group $\mathbb{Z}_p/p\mathbb{Z}_p$ is isomorphic to the cyclic group of order p, $\mathbb{Z}/p\mathbb{Z}$. In particular, $\mathbb{Z}_p$ is a disjoint union of p cosets of $p\mathbb{Z}_p$, that is, of p balls of radius p^{-1}.

Similarly, for an integer $M \ge 0$, a (closed) ball with radius p^{-M} is denoted by

$$B_j^M := \overline{B}(j, p^{-M}) = j + p^M\mathbb{Z}_p \qquad (0 \le j < p^M).$$

This leads to a decomposition of $\mathbb{Z}_p$ as a (finite) disjoint union of balls of radius p^{-M}:

$$\mathbb{Z}_p = \bigcup_{j=0}^{p^M-1} B_j^M.$$

Each of these balls B_j^M can be decomposed as a disjoint union of balls of radius $p^{-(M+1)}$:

$$B_j^M = \bigcup_{k=0}^{p-1} B_{jk}^{M+1}.$$

For each $M \in \mathbb{N}$ write $\mathcal{B}^M := \{B_i^M : 0 \le i < p^M\}$. To simplify the notation, in what follows we write $B_i := B_i^M$. The distance between two elements in $\mathcal{B}^M$ is defined by

$$d_p(B_i, B_j) := d_p(x_i, x_j), \qquad i \ne j,$$

where $x_i \in B_i, x_j \in B_j$ and $0 \le i, j < p^M$. The distance $d_p(B_i, B_j)$ is well-defined due to the non-achimedean property and it is clearly independent of the chosen points. Putting in a different way, this is equivalent to say that any point in a ball is a centre:

$$x \in \overline{B}(y, p^{-M}) \Longrightarrow \overline{B}(y, p^{-M}) = \overline{B}(x, p^{-M}).$$

Lemma 1. *If $0 \le i, j < p^M$ with $i \ne j$, then $d_p(B_i, B_j) = p^{-(M-k)}$ for some $k \in \{1, 2, \dots, M\}$. Furthermore, if $0 \le j < p^M$, then*

$$\#\{i : d_p(B_i, B_j) = p^{-(M-k)}\} = (p-1)p^{k-1},$$

for each $k = 1, 2, \dots, M$.

Proof. To prove the first assertion, observe that if $0 \le i, j < p^M$ with $i \ne j$, then

$$p^{-M} < p^{-\operatorname{ord}_p(i-j)} \le p^{-N},$$

where $N = \min\{\operatorname{ord}_p(i), \operatorname{ord}_p(j)\} \le \operatorname{ord}_p(i-j) < M$. Since $\operatorname{ord}_p(i) \ge 0$ for all $i \in \mathbb{Z}_p$, it follows that $N \ge 0$. The second statement follows by a typical counting argument. $\square$

3 A random walk on $\mathbb{Z}_p$

The aim of this section is to construct a $\mathbb{Z}_p$-valued stationary Markov process which is denoted by $\Xi \equiv \{\Xi(t), t \geq 0\}$. As a first step, we shall introduce a process on $\mathcal{B}^M$, for each $M \in \mathbb{N}$. The latter can be done by solving the classical system of forward Kolmogorov equations which read as follows:

$$\dot{P}_{B_i^M B_j^M}(t) = -q_i^M P_{B_i^M B_j^M}(t) + \sum_{0 \leq l < p^M, l \neq i} q_{il}^M P_{B_l^M B_j^M}(t), \tag{3.1}$$

for all $t \in [0, \infty)$ and $0 \leq i, j < p^M$ with initial condition $P_{B_i^M B_j^M}(0) = \delta_{ij}$. The parameter q_i^M can be thought of as the intensity of the state B_i^M and q_{ij}^M is the infinitesimal transition probability from the state B_i^M to the state B_j^M, for each $i \neq j$.

Remark 2. For each $M > 0$, denote by $P_{ij}^M(t)$ the transition function of the process Ξ restricted to move on $\mathcal{B}^M$. By using the decomposition of a ball of radius p^{-M} into balls of radii $p^{-(M+1)}$ the relation

$$P_{ij}^M(t) = p P_{ij}^{M+1}(t)$$

is obtained, and this implies the relation

$$q_{ij}^M = p q_{ij}^{M+1}$$

(see Lukierska-Walasek and Topolski (2006)).

For any sequence $(a(-n))_{n \in \mathbb{N} \cup \{0\}}$ of positive numbers such that

$$a(-n) \geq a(-n+1), \qquad a(0) = 0 \quad \text{and} \quad \lim_{n \to \infty} a(-n) \in (0, \infty), \tag{3.2}$$

define

$$u(-M, m) := (p-1)^{-1} p^{-m+1} \big[a(-M+m-1) - a(-M+m)\big]. \tag{3.3}$$

Given B_i^M and B_j^M any two balls in $\mathcal{B}^M$ such that $d_p(B_i^M, B_j^M) = p^{-n}$, define

$$q_{ij}^M := u(-M, M-n). \tag{3.4}$$

Then, for any $M > 0$ and $0 < m \leq M$ we have

$$u(-M+1, m-1) = p u(-M, m).$$

In fact, (3.4) tell us that the jump rate only depends on the distance between balls no matter what the balls are. Thus, the process Ξ will be spherically symmetric. Relation (3.3) together with Lemma 1 play a key role when solving the Kolmogorov equations.

3.1 Kolmogorov equations

From now on, we write $P_{B_j}(t)$ for $P_{B_j^M}(t)$, similarly for $P_{B_0^M B_j^M}(t)$. Setting $B_0 = B(0, p^{-M})$, we wish to solve Kolmogorov equations for $i = 0$ with initial condition $P_{B_0 B_j}(0) = \delta_{ij}$. It is enough to do it in this way due to the translation invariance of the p-adic topology. Write $P_{B_j}(t)$ instead of $P_{B_0 B_j}(t)$.

Observing that a ball of radius p^{-M} can be characterized by the values $\{a_0, a_1, \ldots, a_{M-1}\}$, which are the first coefficients of the p-adic expansion of the center of the ball, the equation (3.1) reads as follows:

$$
\begin{aligned}
\dot{P}_{\{a_0,\ldots,a_{M-1}\}}(t) = -a(-M+1) P_{\{a_0,\ldots,a_{M-1}\}}(t) \\
&+ u(-M+1,1) \sum_{a'_{M-1} \neq a_{M-1}} P_{\{a_0,\ldots,a'_{M-1}\}}(t) \\
&\vdots \\
&+ u(-M+1,M) \sum_{a'_{M-1},\ldots,a'_1} P_{\{a'_0,\ldots,a'_{M-1}\}}(t) \qquad (a'_0 \neq a_0).
\end{aligned}
$$

Equivalently,

$$
\begin{aligned}
\dot{P}_{\{a_0,\ldots,a_{M-1}\}}(t) &= -\big[a(-M+1) + u(-M+1,1)\big] P_{\{a_0,\ldots,a_{M-1}\}}(t) \\
&+ u(-M+1,1) \sum_{a'_{M-1}} P_{\{a_0,\ldots,a'_{M-1}\}}(t) \\
&+ u(-M+1,2) \sum_{a'_{M-1}} P_{\{a_0,\ldots,a'_{M-2}a'_{M-1}\}}(t) \qquad (a'_{M-2} \neq a_{M-2}) \\
&\vdots \\
&+ u(-M+1,M) \sum_{a'_{M-1},\ldots,a'_1} P_{\{a'_0,\ldots,a'_{M-1}\}}(t) \qquad (a'_0 \neq a_0).
\end{aligned}
$$

The above expression can be rewritten as:

$$
\begin{aligned}
\dot{P}_{\{a_0,\ldots,a_{M-1}\}}(t) &= -\big[a(-M+1) + u(-M+1,1)\big] P_{\{a_0,\ldots,a_{M-1}\}}(t) \\
&+ \sum_{m=1}^{M-1} \big[u(-M+1,m) - u(-M+1,m+1)\big] P_{\{a_0,\ldots,a_{M-m-1}\}}(t) \\
&+ u(-M+1,M).
\end{aligned}
$$

Summing over a_{M-1} in the equation above:

$$
\begin{aligned}
&\dot{P}_{\{a_0,\ldots,a_{M-2}\}}(t) \\
&\quad = -\big[a(-M+1) + u(-M+1,1)\big] P_{\{a_0,\ldots,a_{M-2}\}}(t)
\end{aligned}
$$

$$+ p \sum_{m=1}^{M-1} \big[u(-M+1,m) - u(-M+1,m+1)\big] P_{\{a_0,\dots,a_{M-m-1}\}}(t)$$
$$+ pu(-M+1,M).$$

Regrouping and using (3.3), the last equation can be written as:

$$\begin{aligned}
&\dot{P}_{\{a_0,\dots,a_{M-2}\}}(t)\\
&\quad = -\big[a(-M+2) + pu(-M+1,2)\big] P_{\{a_0,\dots,a_{M-2}\}}(t)\\
&\qquad + p \sum_{m=2}^{M-1} \big[u(-M+1,m) - u(-M+1,m+1)\big] P_{\{a_0,\dots,a_{M-m-1}\}}(t)\\
&\qquad + pu(-M+1,M).
\end{aligned}$$

Continuing in this fashion, we get, for any $1 \le k \le M-1$:

$$\begin{aligned}
&\dot{P}_{\{a_0,\dots,a_{M-k}\}}(t)\\
&\quad = -\big[a(-M+k) + p^{k-1}u(-M+1,k)\big] P_{\{a_0,\dots,a_{M-k}\}}(t)\\
&\qquad + p^{k-1} \sum_{m=k}^{M-1} \big[u(-M+1,m) - u(-M+1,m+1)\big] P_{\{a_0,\dots,a_{M-m-1}\}}(t)\\
&\qquad + p^{k-1}u(-M+1,M).
\end{aligned}$$

Define

$$P_{M,k}(t) := P_{\{a_0,\dots,a_{M-k}\}}(t) = P_{B(0,p^{-M+k-1})}(t) \qquad (1 \le k \le M-1).$$

Then the last equation can be written as:

$$\begin{aligned}
\dot{P}_{M,k}(t) &= -\big[a(-M+k) + p^{k-1}u(-M+1,k)\big] P_{M,k}(t)\\
&\quad + p^{k-1} \sum_{m=k}^{M-1} \big[u(-M+1,m) - u(-M+1,m+1)\big] P_{M,m+1}(t)\\
&\quad + p^{k-1}u(-M+1,M).
\end{aligned}$$

Direct computation implies:

$$\begin{aligned}
&(p\dot{P}_{M,k} - \dot{P}_{M,k+1})(t)\\
&\quad = -\big[a(-M+k+1) + p^k u(-M+1,k)\big](pP_{M,k} - P_{M,k+1})(t).
\end{aligned}$$

The solution of this differential equation, according with the initial conditions is:

$$pP_{M,k}(t) - P_{M,k+1}(t) = (p-1)\exp\big\{-\big[a(-M+k+1) + p^k u(-M+1,k)\big]t\big\}.$$

For any i, the following relation holds

$$p^{-i}P_{M,k+i}(t) - p^{-(i+1)}P_{M,k+i+1}(t) = \frac{p-1}{p^{i+1}}\exp\{-[a(-M+k+i+1) + p^{k+i}u(-M+1,k+i)]t\}.$$

Summing up on both sides of equation from $i=0$ to $i=m-1$:

$$P_{M,k}(t) - p^{-m}P_{M,k+m}(t) = \frac{p-1}{p}\sum_{i=0}^{m-1} p^{-i}\exp\{-[a(-M+k+i+1) + p^{k+i}u(-M+1,k+i)]t\}.$$

Since $1 \leq m \leq M-k+1$, writing recursively the last equality for each m, we get

$$P_{M,k}(t) = p^{-(M-k+1)} + \frac{p-1}{p}\sum_{i=0}^{M-k} p^{-i}\exp\{-[a(-M+k+i+1) + p^{k+i}u(-M+1,k+i)]t\}.$$

Finally, using the definition of u (see equation (3.3)), we get:

$$P_{M,k}(t) = p^{-(M-k+1)} + \frac{p-1}{p}\sum_{i=0}^{M-k} p^{-i}\exp\left\{-\frac{1}{p-1}[pa(-M+k+i) - a(-M+k+i+1)]t\right\}. \tag{3.5}$$

3.2 Transition probabilities

The transition probabilities of a Markov process on $\mathbb{Z}_p$ can now be described in the following way.

Denote by $P_N(t)$ the complete solution of the system (3.1) with initial condition

$$P_N(0) = P_{B(0,p^{-N})}(0) = 1.$$

For any $M > 0$, denote by $P_t(B^M, B(0,p^{-N}))$ the transition probability of moving from the ball B^M to the ball $B(0,p^{-N})$. Then,

$$P_N(t) = P_t(B^M, B(0,p^{-N})) \qquad (M > N).$$

Using the fact that the p-adic topology is invariant under translations and the Finite Intersection Property for closed balls in $\mathbb{Z}_p$, we can define

$$P_t(x, B(0,p^{-N})) := P_N(t),$$

for any $x \in B(0,p^{-N})$.

If $x \notin B(0, p^{-N})$, then the transition probability $P_t(x, B(0, p^{-N}))$, can be calculated as follows:

$$P_t\big(x, B\big(0, p^{-N}\big)\big) = \frac{1}{p^{k-1}(p-1)}\big[P_{N-1}(t) - P_N(t)\big].$$

Remark 3. In this case, $N = M - k + 1$ according with the complete solution (3.5).

To calculate explicitly this solution, observe that

$$P_N(t) = p^{-N} + \frac{p-1}{p}\sum_{i=0}^{N-1} p^{-i} \exp\bigg\{-\frac{1}{p-1}\big[pa(-N+i+1) \\ - a(-N+i+2)\big]t\bigg\},$$

and

$$P_{N-1}(t) = p^{-N+1} + \frac{p-1}{p}\sum_{i=0}^{N-2} p^{-i} \exp\bigg\{-\frac{1}{p-1}\big[pa(-N+i+2) \\ - a(-N+i+3)\big]t\bigg\}.$$

Taking the difference $P_{N-1}(t) - P_N(t)$, separating the first term in the sum of $P_N(t)$ and regrouping the remaining terms in both series, we get:

$$\begin{aligned} &P_{N-1}(t) - P_N(t) \\ &\quad = \big(p^{-N+1} - p^{-N}\big) \\ &\qquad + \frac{(p-1)^2}{p}\sum_{i=1}^{N-1} p^{-i} \exp\bigg\{-\frac{1}{p-1}\big[pa(-N+i+1) - a(-N+i+2)\big]t\bigg\} \\ &\qquad - \frac{p-1}{p}\exp\bigg\{-\frac{1}{p-1}\big[pa(-N+1) - a(-N+2)\big]t\bigg\}. \end{aligned}$$

Finally, recalling that $N = M - k + 1$ and dividing the above difference by $p^{k-1}(p-1)$ we get:

$$\begin{aligned} &\frac{1}{p^{k-1}(p-1)}\big[P_{N-1}(t) - P_N(t)\big] \\ &\quad = \frac{1}{p^M} \\ &\qquad + \frac{p-1}{p^k}\sum_{i=1}^{N-1} p^{-i} \exp\bigg\{-\frac{1}{p-1}\big[pa(-N+i+1) - a(-N+i+2)\big]t\bigg\} \\ &\qquad - \frac{1}{p^k}\exp\bigg\{-\frac{1}{p-1}\big[pa(-N+1) - a(-N+2)\big]t\bigg\}. \end{aligned}$$

3.3 The Markov process

According with the last analysis, we have:

$$\begin{aligned} P_t(x, B(0, p^{-M})) \\ &= p^{-M} + \frac{p-1}{p}\sum_{i=1}^{M} p^{-i}\exp\Big\{-\frac{1}{p-1}\big[pa(-M+i) \\ &\qquad - a(-M+i+1)\big]t\Big\}, \end{aligned} \tag{3.6}$$

if $x \in B(0, p^{-M})$. If $d_p(x, B(0, p^{-M})) = p^{-M+k}$, then

$$\begin{aligned} P_t(x, B(0, p^{-M})) \\ &= p^{-M+1} \\ &\quad + \frac{p-1}{p^k}\sum_{i=1}^{M-k-1} p^{-i}\exp\Big\{-\frac{1}{p-1}\big[pa(-M+k+i) \\ &\qquad - a(-M+k+i+1)\big]t\Big\} \\ &\quad - \frac{1}{p^k}\exp\Big\{-\frac{1}{p-1}\big[pa(-M+k) - a(-M+k+1)\big]t\Big\}. \end{aligned} \tag{3.7}$$

Theorem 1. *For any sequence $\{a(-n)\}_{n\in\mathbb{N}\cup\{0\}}$ satisfying (3.2) there exists a spherically symmetric Markov process $\Xi = \{\Xi_t, t \geq 0\}$ on $\mathbb{Z}_p$ with transition functions given by (3.6) and (3.7).*

Proof. The proof can be done following the same lines as in Albeverio and Karwowski (1994). Let A be a disjoint union of balls of the same radii p^{-M}, for some $M > 0$. Then, with the obvious notation, $P_t(x, A)$ defines a symmetric (with respect to the Haar measure on $\mathbb{Z}_p$) Markov transition function. Thus, it defines a strongly continuous Markov semigroup $\{T_t, t \geq 0\}$ on $L^2(\mathbb{Z}_p)$. Hence, we have constructed a continuous time Markov process $\Xi = \{\Xi_t, t \geq 0\}$ with state space $\mathbb{Z}_p$ and transition function $P_t(x, A)$, $t \geq 0$, $x \in \mathbb{Z}_p$, as defined before. □

As a by-product of the above result, we get that the process Ξ just defined is recurrent. This is the content of the following proposition.

Proposition 1. *The process Ξ is recurrent.*

Proof. For given $x \in \mathbb{Z}_p$ consider the ball $B(x, r) = \{y \in \mathbb{Z}_p : d_p(y, x) \leq r\}$, for each $r > 0$. Now, note that there exists $M \in \mathbb{Z}$ such that $B(x, p^{-M}) \subset B(x, r)$.

Hence,

$$\int_0^\infty P_t\big(x, B(x,r)\big)\,dt \geq \int_0^\infty P_t\big(x, B\big(x, p^{-M}\big)\big)\,dt = \infty,$$

where (3.6) is used in the last equality. Thus, Ξ is recurrent. □

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Departamento de Matemáticas
Universidad de Guanajuato
Jalisco S/N
Mineral de Valenciana
Guanajuato, Gto. C.P. 36240
México
E-mail: manuelcl@ugto.mx
amurillos@ugto.mx