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# On risk unbiased estimation after selection

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**Abstract.** In many practical situations, it is desired to compare several populations, find the best one and estimate some parametric functions associated with the selected population. This has been recognized as an important problem that arises in various applications in agricultural, industrial and medical studies. This paper concerns unbiased estimation of a general parametric function, say $\gamma(\theta)$, of selected populations under the squared error loss (SEL) function. Examples of $\gamma(\cdot)$ include reliability function, odds ratio and variance, among others. Also, we obtain the uniformly minimum risk unbiased estimators of the parameters of selected populations under some general class of loss functions other than the commonly used SEL function. Furthermore, we characterize some loss functions for which the risk unbiased estimators of parameters of selected populations do not exist. Theoretical results are augmented with various illustrations and examples.

## 1 Introduction

In many practical situations, the goal of the study is to compare several populations in order to make a decision in the form of ranking these populations, finding the best population and estimating the parameters of the selected population. Since the ranking and selection are done first, the preceding estimation problem is called estimation after selection. Estimation of a characteristic of the selected population has been recognized as an important problem that arises in various applications in agricultural, industrial and medical studies. For example,

(1) In industrial studies, researchers not only want to know which type of component system will last longest, but also want an estimate of the expected lifetime of the chosen system (Sackrowitz and Samuel-Cahn, 1986).
(2) In agricultural studies, researchers wish to select the best of $k$ fertilizers and estimate the mean yield produced by the selected fertilizer (Misra and Meulen, 2001).
(3) In microarray experiments, researchers are often interested in making inference for the parameters corresponding to the most extreme population (Qiu and Hwang, 2007).

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In many cases, the characteristic of interest is the population mean. For example, in clinical trial studies, one is interested in tracking the most promising compounds in drug development by choosing the best treatment out of $k$ which results in the maximum treatment mean and hence estimating the expected effect of selected treatment (Bowden and Glimm, 2008, Sill and Sampson, 2009).The common practice is to take independent samples from each population, compute respective sample means and select the population which results in the largest sample mean. Having selected the best population, one usually wants to estimate its mean. A natural estimator is the mean of the selected sample from the best population. Although the sample mean is a good estimator of the population mean in a regular setting, most of its optimal properties are no longer hold when the population is selected on the basis of the sample rather than being specified in advance. For example, the mean of the sample obtained from the selected population is a positively biased estimator of the selected population mean (see (1.3) below). For the clinical trial example, Bowden and Glimm (2008) showed that the naive estimator of a treatment's effect after selection can be severely hindered because selection mechanisms usually introduce bias.

Let $\Pi_1, \Pi_2, \ldots, \Pi_k$ be $k(\geq 2)$ independent populations with associated probability density functions (p.d.f.) $f(x|\theta_i), i = 1, 2, \ldots, k$, respectively, where $\theta_i$ is the unknown parameter. Suppose $X_{i1}, \ldots, X_{in_i}$, $n_i \geq 2$, are independent and identically distributed (i.i.d.) samples from the $i$th population $\Pi_i, i = 1, \ldots, k$. Define $X_i = g_i(X_{i1}, \ldots, X_{in_i})$, where $X_i$ is a suitable estimator of $\theta_i$, and let $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(k)}$ denote the order statistics of $X_1, X_2, \ldots, X_k$. Suppose we employ the natural *selection following ranking* rule, to select the best population, according to which the population corresponding to $X_{(1)}$ or $X_{(k)}$ is selected. Let $\theta_J$ and $\theta_M$ be the parameters associated with selected populations, respectively. Note that $\theta_J$ and $\theta_M$ are random variables which can be formally defined as

$$\theta_J = \sum_{i=1}^{k} \theta_i \left\{ \prod_{j \neq i} (1 - I(X_i, X_j)) \right\} \quad \text{and} \quad \theta_M = \sum_{i=1}^{k} \theta_i \left\{ \prod_{j \neq i} I(X_i, X_j) \right\}, \tag{1.1}$$

where

$$I(a, b) = \begin{cases} 1 & \text{if } a \geq b, \\ 0 & \text{if } a < b. \end{cases} \tag{1.2}$$

For the case where $E(X_i) = \theta_i$, since $E(X_{(k)}) \geq E(X_i) = \theta_i$, one can easily show that

$$E(X_{(k)}) \geq \max_i E(X_i) = E\Big(\max_i \theta_i\Big) \geq E(\theta_M). \tag{1.3}$$

In other word, $X_{(k)}$, as the naive estimator of $\theta_M$, by ignoring the fact that a prior selection has been made, is a positively biased estimator of $\theta_M$. In the past few years, many studies have been done to construct good estimators of $\theta_J$ and $\theta_M$ under the squared error loss (SEL) function. Most of these results are obtained based on the U.V. method of Robbins (1988) or some of its generalizations. For example, in estimation of the scale parameter $\theta$ after selection from a Gamma$(\alpha, \theta)$ distribution, Vellaisamy and Sharma (1989) derived the uniformly minimum variance unbiased (UMVU) estimator of $\theta_M$. Tappin (1992) derived the UMVU estimator of the parameter of selected binomial population. Vellaisamy (1993) considered the UMVU estimation of the parameter of one parameter continuous exponential family of distributions. Mishra and Singh (1994) considered the UMVU estimation of the location parameter of the selected exponential distribution. Misra and Meulen (2001) derived the UMVU estimator of $\theta_M$ and $\theta_J$ in non-regular family of distributions. Kumar and Gangopadhyay (2005) derived the UMVU estimator of the parameter of a selected Pareto distribution.

There are several situations where one is interested in estimating some parametric functions of selected populations other than the population mean. For example, in reliability testing one might be interested in estimating the reliability function associated with the variable of interest in the selected population (Kumar et al., 2009). In genomic studies, estimating the odds ratio is very important. In this application, samples are usually obtained from selected populations which are identified by using genome scans (Bowden and Dudbridge, 2009). Other examples include estimating quantiles (Kumar and Kar, 2001), variance and survival function based on the variable of interest for the selected population.

On the other hand, one can easily argue that for some parametric functions, over-estimation (under-estimation) could be more serious than under-estimation (over-estimation). The most prevalent loss function for the evaluation of estimators is the symmetric SEL function which assigns the same penalty to the over-estimation and under-estimation of the same magnitude. This functional form is assumed because mathematically is very tractable but from a practical point of view, it is not very realistic. The choice of the loss function is fundamental to the construction of an unbiased estimation of parametric functions of selected populations. There are only a few works regarding the problem of estimation after selection under loss functions other than SEL. For example, Nematollahi and Motamed-Shariati (2012, 2009) derived the risk unbiased and uniformly minimum risk unbiased (UMRU) estimators of $\theta_J$ and $\theta_M$ under the entropy and Stein loss functions. So, it is also important to consider the problem of estimation after selection under more general class of loss functions.

In this paper, we first consider the problem of unbiased estimation of some parametric functions, say $\gamma(\theta)$, of selected populations under the SEL function. In particular, we construct the UMVU estimators of general parametric functions $\gamma(\theta)$ for some selected non-regular family of distributions under SEL function. Examples of $\gamma(\cdot)$ include reliability function, odds ratio and variance, among others. Then we obtain the UMRU estimators of $\theta_M$ and $\theta_J$ under some general classes of loss functions other than the commonly used SEL function. Furthermore, we characterize some loss functions for which the risk unbiased estimators of $\theta_J$ or $\theta_M$ do not exist.

The outline of the paper is as follows. In Section 2, we give some preliminary results and obtain a useful relationship between the unbiasedness under SEL function and risk-unbiasedness under a general class of loss functions. In Section 3, for some non-regular family of distributions, we obtain the UMVU estimators of $\gamma(\theta_J)$ and $\gamma(\theta_M)$ under the SEL function and construct UMRU estimators of $\theta_J$ and $\theta_M$ under some general classes of loss functions which satisfy an easy to verify condition. In Section 4, we characterize some classes of loss functions for which the risk unbiased estimators of $\theta_J$ or $\theta_M$ do not exist. Finally, in Section 5 we give some concluding remarks.

## 2 Preliminary results

Consider the problem of estimating a real valued function, say $h(\boldsymbol{\theta})$, of $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_k)$. Dealing with unbiasedness, and following Lehmann (1951) an estimator $\delta$ of $h(\boldsymbol{\theta})$ is said to be risk unbiased if it satisfies

$$E_{\boldsymbol{\theta}}\big[L\big(h(\boldsymbol{\theta}), \delta(\mathbf{X})\big)\big] \leq E_{\boldsymbol{\theta}}\big[L\big(h(\boldsymbol{\theta}'), \delta(\mathbf{X})\big)\big] \qquad \forall \boldsymbol{\theta}' \neq \boldsymbol{\theta}, \tag{2.1}$$

where $\mathbf{X} = (X_1, \ldots, X_k)$. Under the SEL function $L(\boldsymbol{\theta}, \delta) = (\delta(\mathbf{X}) - h(\boldsymbol{\theta}))^2$, (2.1) reduces to the usual unbiasedness condition $E_{\boldsymbol{\theta}}[\delta(\mathbf{X})] = h(\boldsymbol{\theta})$. Also, if $h(\boldsymbol{\theta})$ is a random parameter such as $h(\boldsymbol{\theta}) = \theta_M$ or $\theta_J$ in (1.1), then the risk unbiasedness condition under the SEL function reduces to $E_{\boldsymbol{\theta}}[\delta(\mathbf{X})] = E_{\boldsymbol{\theta}}[h(\boldsymbol{\theta})]$. Hereafter, suppose that $h(\boldsymbol{\theta})$ is the random parameter $\theta_J$ or $\theta_M$. We are concern with

(A) Unbiased estimation of $\gamma(h(\boldsymbol{\theta}))$ under the SEL function

$$L_1(\boldsymbol{\theta}, \delta) = \big(\delta(\mathbf{X}) - \gamma\big(h(\boldsymbol{\theta})\big)\big)^2, \tag{2.2}$$

(B) Risk unbiased estimation of $h(\boldsymbol{\theta})$ under the $\gamma$-loss function

$$L_2(\boldsymbol{\theta}, \delta) = \big(\gamma\big(\delta(\mathbf{X})\big) - \gamma\big(h(\boldsymbol{\theta})\big)\big)^2, \tag{2.3}$$

where $\gamma(\cdot)$ is a monotone differentiable function. As we show in Proposition 1, and it was previously mentioned in Jafari Jozani and Marchand (2007), loss functions $L_1$ and $L_2$ are mathematically equivalent, but stem from separate practical problems. Loss $L_2$ produces a very large class of loss functions such as SEL with $\gamma(x) = x$, squared log error loss (SLEL) with $\gamma(x) = \ln(x)$ and exponential loss (EL) function with $\gamma(x) = \exp(x)$. Using (2.1), $\delta^*(\mathbf{X})$ is an unbiased estimator of the random parameter $\gamma(h(\boldsymbol{\theta}))$ under SEL function $L_1$ if

$$E_{\boldsymbol{\theta}}\big[\delta^*(\mathbf{X})\big] = E_{\boldsymbol{\theta}}\big[\gamma\big(h(\boldsymbol{\theta})\big)\big]. \tag{2.4}$$

Also, $\delta(\mathbf{X})$ is a risk unbiased estimator of $h(\boldsymbol{\theta})$ under the $\gamma$-loss function $L_2$ if

$$E_{\boldsymbol{\theta}}\big[\gamma\big(\delta(\mathbf{X})\big)\big] = E_{\boldsymbol{\theta}}\big[\gamma\big(h(\boldsymbol{\theta})\big)\big]. \tag{2.5}$$

Comparing (2.4) and (2.5), we have the following proposition.

**Table 1** *Some classes of loss functions with risk-unbiasedness condition as in* (2.5)

| Loss function | $L(\boldsymbol{\theta}, \delta(\mathbf{X}))$ | $\gamma(\cdot)$ |
|---|---|---|
| Stein's Loss (SL) | $\frac{\delta(\mathbf{X})}{h(\boldsymbol{\theta})} - \ln \frac{\delta(\mathbf{X})}{h(\boldsymbol{\theta})} - 1$ | $\gamma(x) = x$ |
| Entropy Loss (EL) | $\frac{h(\boldsymbol{\theta})}{\delta(\mathbf{X})} - \ln \frac{h(\boldsymbol{\theta})}{\delta(\mathbf{X})} - 1$ | $\gamma(x) = \frac{1}{x}$ |
| General Entropy Loss (GEL) | $(\frac{h(\boldsymbol{\theta})}{\delta(\mathbf{X})})^q - q \ln \frac{h(\boldsymbol{\theta})}{\delta(\mathbf{X})} - 1, q \neq 0$ | $\gamma(x) = \frac{1}{x^q}$ |
| LINEX loss | $e^{a(\delta(\mathbf{X}) - h(\boldsymbol{\theta}))} - a(\delta(\mathbf{X}) - h(\boldsymbol{\theta})) - 1, a \in \mathbb{R}$ | $\gamma(x) = e^{ax}$ |
| Intrinsic Loss (IL) | $\ln \frac{\beta(h(\boldsymbol{\theta}))}{\beta(\delta(\mathbf{X}))} + (\delta(\mathbf{X}) - h(\boldsymbol{\theta})) \frac{\beta'(h(\boldsymbol{\theta}))}{\beta(h(\boldsymbol{\theta}))}$ | $\gamma(x) = x$ |

**Proposition 1.** *The estimator $\delta^*(\mathbf{X}) = \gamma(\delta(\mathbf{X}))$ is an unbiased estimator of the random parameter $\gamma(h(\boldsymbol{\theta}))$ under SEL function $L_1$ if and only if $\delta(\mathbf{X}) = \gamma^{-1}(\delta^*(\mathbf{X}))$ is a risk unbiased estimator of $h(\boldsymbol{\theta})$ under $\gamma$-loss function $L_2$.*

In Sections 3 and 4, we show that the unbiasedness condition (2.4) is useful for deriving unbiased estimators of parametric functions $\gamma(\boldsymbol{\theta})$, such as the survival function $\gamma(\boldsymbol{\theta}) = P_{\boldsymbol{\theta}}(X > x)$, odds ratio $\gamma(\boldsymbol{\theta}) = \frac{P_{\boldsymbol{\theta}}(X<x)}{1-P_{\boldsymbol{\theta}}(X<x)}$ or the variance $\gamma(\boldsymbol{\theta}) = \mathrm{Var}_{\boldsymbol{\theta}}(X)$ of selected population, among others. Also, as the risk unbiased condition (2.5) holds for some famous loss functions, finding an unbiased estimator $\delta^*(\mathbf{X})$ of $\gamma(h(\boldsymbol{\theta}))$ which satisfy (2.4), leads to $\gamma^{-1}(\delta^*(\mathbf{X}))$ as a risk unbiased estimator of $h(\boldsymbol{\theta})$ under general classes of loss functions as in Table 1.

The intrinsic loss function in Table 1 is the Kullback–Leibler divergence between the true model $f(x|h(\boldsymbol{\theta}))$ and the model $f(x|\delta(\mathbf{x}))$ when the distribution of $X$ belongs to the one-parameter exponential family of distributions with p.d.f.

$$f(x|\theta) = \beta(\theta)t(x)e^{-\theta r(x)}, \tag{2.6}$$

where $r(x) > 0$, $\beta(\theta)t(x) > 0$ and $\theta$ is the unknown real-valued natural parameter of the model (Jafari Jozani and Jafari Tabrizi, 2013).

Before pursuing, we make a note about the relationship between the *selection after ranking* rules for the parameter $\theta$ and parametric function $\gamma(\theta)$ with monotone $\gamma(\cdot)$. Since for any function $\gamma(\cdot)$, $\theta_i \in \mathbb{R}$, and $Z_i \in \{0, 1\}$ with $\sum_{i=1}^n Z_i = 1$ we have

$$\gamma\left(\sum_{i=1}^n \theta_i Z_i\right) = \sum_{i=1}^n \gamma(\theta_i) Z_i,$$

one can easily verify that

$$\begin{aligned} \gamma(\theta_M) &= \sum_{i=1}^k \gamma(\theta_i)\left\{\prod_{j\neq i} I(X_i, X_j)\right\} \quad \text{and} \\ \gamma(\theta_J) &= \sum_{i=1}^k \gamma(\theta_i)\left\{\prod_{j\neq i} (1 - I(X_i, X_j))\right\}. \end{aligned} \tag{2.7}$$

So, for estimating $\gamma(\theta)$ after selection with monotone $\gamma(\cdot)$ one can either perform ranking and selection rule based on $X_i$ or $\gamma(X_i)$, $i = 1, \ldots, n$.

## 3 UMVU and UMRU estimation in non-regular family of distributions

In this section, we study the UMVU estimation of $\gamma(\theta_J)$ and $\gamma(\theta_M)$ under SEL function (2.2) for two non-regular family of distributions studied in Misra and Meulen (2001). Then we show how these results can be used to construct UMRU estimators of $\theta_J$ and $\theta_M$ under $\gamma$-loss function (2.3) or more generally any loss function which has a risk unbiasedness condition given by (2.5). Suppose $X_{i1}, \ldots, X_{in_i}$ is a random sample of size $n$ from a distribution with p.d.f. $f(\cdot|\theta_i)$ which belongs to one of the following non-regular family of densities:

(i) *Right Truncation Parameter Family* (*RTPF*):

$$f(x|\theta_i) = \begin{cases} r(\theta_i, \alpha)s(x, \alpha), & \text{if } a < x \le \theta_i, \\ 0, & \text{otherwise,} \end{cases} \tag{3.1}$$

where $\theta_i \in \Theta_1 = \{\mu : a < \mu < \tau\}$, $a, \alpha$ and $\tau$ are constants, and $r(\cdot, \cdot)$ and $s(\cdot, \cdot)$ are some non-negative functions.

(ii) *Left Truncation Parameter Family* (*LTPF*):

$$f(x|\theta_i) = \begin{cases} R(\theta_i, \alpha)S(x, \alpha), & \text{if } \theta_i < x \le b, \\ 0, & \text{otherwise,} \end{cases} \tag{3.2}$$

where $\theta_i \in \Theta_2 = \{\mu : \beta < \mu < b\}$, $b, \alpha$ and $\beta$ are constants, and $R(\cdot, \cdot)$ and $S(\cdot, \cdot)$ are some non-negative functions.

We assume $s(x, \alpha)$ and $S(x, \alpha)$ are continuous functions of $x$. Let $\mathbf{X} = (X_1, \ldots, X_k)$ denote the maximum likelihood estimator (MLE) of $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_k)$. One can easily show that under the RTPF model $X_i = \max(X_{i1}, \ldots, X_{in_i})$ and for the LTPF model $X_i = \min(X_{i1}, \ldots, X_{in_i})$, $i = 1, \ldots, k$, are distributed as

$$g_{\theta_i}(x_i) = \begin{cases} q(\theta_i, \alpha)p(x_i, \alpha), & \text{if } a < x_i \le \theta_i, \\ 0, & \text{otherwise,} \end{cases} \tag{3.3}$$

and

$$h_{\theta_i}(x_i) = \begin{cases} Q(\theta_i, \alpha)P(x_i, \alpha), & \text{if } \theta_i < x_i \le b, \\ 0, & \text{otherwise,} \end{cases} \tag{3.4}$$

respectively, where $q(\theta_i, \alpha) = \{r(\theta_i, \alpha)\}^{n_i}$ and $Q(\theta_i, \alpha) = \{R(\theta_i, \alpha)\}^{n_i}$ are normalizing constants,

$$p(x_i, \alpha) = \frac{n_i s(x_i, \alpha)}{\{r(x_i, \alpha)\}^{n_i - 1}} \quad \text{and} \quad P(x_i, \alpha) = \frac{n_i S(x_i, \alpha)}{\{R(x_i, \alpha)\}^{n_i - 1}}.$$

Note that $\mathbf{X} = (X_1, \ldots, X_k)$ or equivalently $\mathbf{U} = (X_{(1)}, \ldots, X_{(k)})$ is a complete sufficient statistic for $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_k)$. We pursue with the following main result.

**Theorem 3.1.** *Suppose $X_1, \ldots, X_k$ are $k$ independent random variables from a distribution with a p.d.f. given by (3.3). Let $U_1(\mathbf{X}), \ldots, U_k(\mathbf{X})$ be $k$ real-valued functions and $\gamma(\cdot)$ be differentiable, such that:*

(a) $E_{\boldsymbol{\theta}}[|\gamma(X_i)U_i(\mathbf{X})|] < \infty, \forall \theta \in \Theta_1, i = 1, \ldots, k,$
(b) $\int_a^{x_i} \gamma(x_i)U_i(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_k)p(y, \alpha)\,dy < \infty, \forall a < x_i < \tau, i = 1, \ldots, k,$
(c) $\lim_{x_i \to a}[\gamma(x_i)\int_a^{x_i} U_i(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_k)p(y, \alpha)\,dy] = 0, \quad \forall a < x_j < \tau, j \neq i, i = 1, \ldots, k.$

*Then, the functions $V_1(\mathbf{X}), \ldots, V_k(\mathbf{X})$, defined by*

$$V_i(\mathbf{X}) = \gamma(X_i)U_i(\mathbf{X}) + \frac{\gamma'(X_i)}{p(X_i, \alpha)} \int_a^{X_i} U_i(X_1, \ldots, X_{i-1}, y, X_{i+1}, \ldots, X_k)p(y, \alpha)\,dy, \tag{3.5}$$

*satisfy*

$$E_{\boldsymbol{\theta}}\left[\sum_{i=1}^{k} V_i(\mathbf{X})\right] = E_{\boldsymbol{\theta}}\left[\sum_{i=1}^{k} \gamma(\theta_i)U_i(\mathbf{X})\right]. \tag{3.6}$$

**Proof.** First, we show that for a given $U(X_i) = U_i(\mathbf{X})$, the function $V(X_i) = \gamma(X_i)U(X_i) + \frac{\gamma'(X_i)}{p(X_i,\alpha)}\int_a^{X_i} U(y)p(y, \alpha)\,dy$ satisfies $E_{\boldsymbol{\theta}}[V(X_i)] = \gamma(\theta_i) \times E_{\boldsymbol{\theta}}[U(X_i)]$. Using the integration by part, from (a)–(c) we have

$$\begin{aligned} E_{\boldsymbol{\theta}}&[V(X_i)] \\ &= q(\theta_i, \alpha)\left(\int_a^{\theta_i} \gamma(t)U(t)p(t, \alpha)\,dt + \int_a^{\theta_i} \gamma'(t)\left[\int_a^t U(y)p(y, \alpha)\,dy\right]dt\right) \\ &= q(\theta_i, \alpha)\left(\int_a^{\theta_i} \gamma(t)U(t)p(t, \alpha)\,dt \right. \\ &\qquad \left. + \left[\gamma(t)\int_a^t U(y)p(y, \alpha)\,dy\right]_a^{\theta_i} - \int_a^{\theta_i} \gamma(t)U(t)p(t, \alpha)\,dt\right) \\ &= q(\theta_i, \alpha)\gamma(\theta_i)\int_a^t U(y)p(y, \alpha)\,dy \\ &= E_{\boldsymbol{\theta}}\left[\gamma(\theta_i)U(X_i)\right]. \end{aligned}$$

Similarly, it can be shown that $V_i(\mathbf{X})$ in (3.5) satisfies

$$\begin{aligned} &E_{\boldsymbol{\theta}}\left[V_i(\mathbf{X})|X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_k\right] \\ &\quad = \gamma(\theta_i)E_{\boldsymbol{\theta}}\left[U_i(\mathbf{X})|X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_k\right]. \end{aligned}$$

Therefore, $E_{\boldsymbol{\theta}}[V_i(\mathbf{X})] = \gamma(\theta_i)E_{\boldsymbol{\theta}}[U_i(\mathbf{X})]$, and the result follows immediately.

Theorem 3.1 helps to obtain UMVU estimators of $\gamma(\theta_J)$ and $\gamma(\theta_M)$ under the SEL function (2.2) and then use them to construct UMRU estimators of $\theta_J$ and $\theta_M$ for a general class of loss functions with a risk unbiasedness condition given by (2.5) such as the $\gamma$-loss function (2.3) or loss functions in Table 1. □

**Theorem 3.2.** *Under the assumptions of Theorem 3.1, let*

$$\text{(i)}\quad \delta^*_{1,1}(\mathbf{X}) = \gamma(X_{(k)}) + \frac{\gamma'(X_{(k)})}{p(X_{(k)},\alpha)} \int_{X_{(k-1)}}^{X_{(k)}} p(y,\alpha)\,dy \tag{3.7}$$

*and*

$$\text{(ii)}\quad \delta^*_{1,2}(\mathbf{X}) = \gamma(X_{(1)}) + \left(\sum_{i=1}^{k} \frac{\gamma'(X_{(i)})}{p(X_{(i)},\alpha)}\right) \int_{a}^{X_{(1)}} p(y,\alpha)\,dy. \tag{3.8}$$

*Then $\delta^*_{1,1}(\mathbf{X})$ and $\delta^*_{1,2}(\mathbf{X})$ are, respectively, the UMVU estimators of $\gamma(\theta_M)$ and $\gamma(\theta_J)$ under the SEL function (2.2). Also, $\delta_{1,1}(\mathbf{X}) = \gamma^{-1}(\delta^*_{1,1}(\mathbf{X}))$ and $\delta_{1,2}(\mathbf{X}) = \gamma^{-1}(\delta^*_{1,2}(\mathbf{X}))$ are the UMRU estimators of $\theta_M$ and $\theta_J$ under any loss function with a risk unbiasedness condition given by (2.5).*

**Proof.** To show (i) let $U_i(\mathbf{X}) = I(X_i, \max_{j\neq i} X_j)$ and note that $\sum_{i=1}^{k} \gamma(\theta_i) \times U_i(\mathbf{X}) = \gamma(\theta_M)$. Let $\delta^*_{1,1}(\mathbf{X}) = \sum_{i=1}^{k} V_i(\mathbf{X})$ where $V_i(\mathbf{X})$ is defined in (3.5). Then

$$\begin{aligned}
\delta^*_{1,1}(\mathbf{X}) &= \sum_{i=1}^{k} \gamma(X_i) I\Big(X_i, \max_{j\neq i} X_j\Big) + \sum_{i=1}^{k} \frac{\gamma'(X_i)}{p(X_i,\alpha)} \int_a^{X_i} I\Big(y, \max_{j\neq i} X_j\Big) p(y,\alpha)\,dy \\
&= \sum_{i=1}^{k} \left(\gamma(X_i) + \frac{\gamma'(X_i)}{p(X_i,\alpha)} \int_{\max_{j\neq i} X_j}^{X_i} p(y,\alpha)\,dy\right) I\Big(X_i, \max_{j\neq i} X_j\Big) \\
&= \gamma(X_{(k)}) + \frac{\gamma'(X_{(k)})}{p(X_{(k)},\alpha)} \int_{X_{(k-1)}}^{X_{(k)}} p(y,\alpha)\,dy.
\end{aligned}$$

Therefore from Theorem 3.1, $\delta^*_{1,1}(\mathbf{X})$ is an unbiased estimator of $\gamma(\theta_M)$. Since $\delta^*_{1,1}(\mathbf{X})$ is a function of the complete and sufficient statistics $X_{(1)}, \ldots, X_{(k)}$, it is the UMVU estimator of $\gamma(\theta_M)$. See Mishra and Singh (1994) as well as Misra (1994) for more details. Now, from Proposition 1 it follows that $\delta_{1,1}(\mathbf{X}) = \gamma^{-1}(\delta^*_{1,1}(\mathbf{X}))$ is the UMRU estimator of $\theta_M$ under any loss functions with a risk unbiasedness condition given by (2.5). Note that part (ii) follows similarly by taking $U_i(\mathbf{X}) = I(\min_{j\neq i} X_j, X_i)$ and the same calculations as in (i) which we do not present here. □

**Remark 1.** Similar results as in Theorems 3.1 and 3.2 can be obtained for LTPF model which we present here without proof. Under suitable conditions, one can construct functions $V_i(\mathbf{X})$, $i=1,\ldots,k$, as follow

$$V_i(\mathbf{X}) = \gamma(X_i)U_i(\mathbf{X}) - \frac{\gamma'(X_i)}{P(X_i,\alpha)}\int_{X_i}^{b} U_i(X_1,\ldots,X_{i-1},y,X_{i+1},\ldots,X_k)P(y,\alpha)\,dy,$$

which satisfy $E_{\boldsymbol{\theta}}[\sum_{i=1}^{k} V_i(\mathbf{X})] = E_{\boldsymbol{\theta}}[\sum_{i=1}^{k}\gamma(\theta_i)U_i(\mathbf{X})]$. Now,

$$\delta_{2,1}^{*}(\mathbf{X}) = \gamma(X_{(k)}) - \left(\sum_{i=1}^{k}\frac{\gamma'(X_{(i)})}{P(X_{(i)},\alpha)}\right)\int_{X_{(k)}}^{b} P(y,\alpha)\,dy \tag{3.9}$$

and

$$\delta_{2,2}^{*}(\mathbf{X}) = \gamma(X_{(1)}) - \frac{\gamma'(X_{(1)})}{P(X_{(1)},\alpha)}\int_{X_{(1)}}^{X_{(2)}} P(y,\alpha)\,dy, \tag{3.10}$$

are the UMVU estimators of $\gamma(\theta_M)$ and $\gamma(\theta_J)$ under the SEL function (2.2), respectively. Also, $\delta_{2,1}(\mathbf{X}) = \gamma^{-1}(\delta_{2,1}^{*}(\mathbf{X}))$ and $\delta_{2,2}(\mathbf{X}) = \gamma^{-1}(\delta_{2,2}^{*}(\mathbf{X}))$ are the UMRU estimators of $\theta_M$ and $\theta_J$, respectively, for the LTPF model and under any loss function having the risk unbiasedness condition as in (2.5).

**Remark 2.** Taking $\gamma(x)=x$ in (3.9) and (3.10) one can easily obtain the results of Misra and Meulen (2001) for unbiased estimation of $\theta_M$ and $\theta_J$ under the SEL function. However, their results neither hold for estimating $\gamma(\theta_M)$ and $\gamma(\theta_J)$ under SEL function nor for estimating $\theta_J$ and $\theta_M$ under general class of loss functions such as the $\gamma$-loss function (2.3) or the loss functions in Table 1.

**Example 1.** Let $X_{i1},\ldots,X_{in}$ be a random sample from a Pareto$(\alpha,\theta_i)$ distribution with p.d.f. $f(x_i|\theta_i) = \frac{\alpha\theta_i^{\alpha}}{x_i^{\alpha+1}}$, $x_i > \theta_i$, $i=1,\ldots,k$. Let $X_i = \min(X_{i1},\ldots,X_{in})$, and note that $X_i$ is distributed according to a Pareto$(n\alpha,\theta_i)$ distribution which is a member of the LTPF model with $P(y,\alpha) = \frac{n\alpha}{y^{n\alpha+1}}$. Therefore, for suitable choices of $\gamma(\cdot)$,

$$\delta_{2,1}(\mathbf{X}) = \gamma^{-1}\left(\gamma(X_{(k)}) - \frac{1}{n\alpha}\sum_{i=1}^{k}\gamma'(X_{(i)})X_{(i)}\left(\frac{X_{(i)}}{X_{(k)}}\right)^{n\alpha}\right)$$

and

$$\delta_{2,2}(\mathbf{X}) = \gamma^{-1}\left(\gamma(X_{(1)}) - \gamma'(X_{(1)})\frac{X_{(1)}}{n\alpha}\left(1-\left(\frac{X_{(1)}}{X_{(2)}}\right)^{n\alpha}\right)\right),$$

are UMRU estimators of $\theta_M$ and $\theta_J$, respectively, under any loss function with a risk unbiasedness condition as in (2.5). In particular, under the entropy loss function we have $\gamma(t)=\frac{1}{t}$, and hence

$$\delta_{2,1}(\mathbf{X}) = \frac{n\alpha X_{(k)}}{n\alpha + \sum_{i=1}^{k}(X_{(i)}/X_{(k)})^{n\alpha-1}}$$

and

$$\delta_{2,2}(\mathbf{X}) = \frac{n\alpha X_{(1)}}{n\alpha + 1 - (X_{(1)}/X_{(2)})^{n\alpha}},$$

are UMRU estimators of $\theta_M$ and $\theta_J$, respectively, which are obtained by Nematollahi and Motamed-Shariati (2009). Also for estimating the reliability function of selected population, by taking $\gamma(\theta_i) = P(X_{ij} > x) = (\frac{\theta_i}{x})^\alpha$, and using Remark 1, we obtain

$$\delta^*_{2,1}(\mathbf{X}) = \left(\frac{X_{(k)}}{x}\right)^\alpha \left(1 - \frac{1}{n}\sum_{i=1}^{k}\left(\frac{X_{(i)}}{X_{(k)}}\right)^{(n+1)\alpha}\right)$$

and

$$\delta^*_{2,2}(\mathbf{X}) = \left(\frac{X_{(1)}}{x}\right)^\alpha \left(1 - \frac{1}{n}\left(1 - \left(\frac{X_{(1)}}{X_{(2)}}\right)^{n\alpha}\right)\right),$$

as the UMVU estimators of $\gamma(\theta_M) = (\frac{\theta_M}{x})^\alpha$ and $\gamma(\theta_J) = (\frac{\theta_J}{x})^\alpha$, respectively, under SEL function (2.2).

**Example 2.** Let $X_{i1}, \ldots, X_{in}$ be a random sample from a Uniform$(0, \theta_i)$ distribution with unknown $\theta_i > 0$, $i = 1, \ldots, k$. Let $X_i = \max(X_{i1}, \ldots, X_{in})$ which has the p.d.f. $f_{X_i}(y) = \frac{ny^{n-1}}{\theta_i^n}, 0 < y < \theta_i$ that is a member of the RTPF model with $p(y, \alpha) = ny^{n-1}$. Therefore, for suitable choices of $\gamma(\cdot)$,

$$\delta_{1,1}(\mathbf{X}) = \gamma^{-1}\left(\gamma(X_{(k)}) + \frac{\gamma'(X_{(k)})}{n} X_{(k)}\left(1 - \left(\frac{X_{(k-1)}}{X_{(k)}}\right)^n\right)\right)$$

and

$$\delta_{1,2}(\mathbf{X}) = \gamma^{-1}\left(\gamma(X_{(1)}) + \sum_{i=1}^{k}\frac{\gamma'(X_{(i)})}{n} X_{(i)}\left(\frac{X_{(1)}}{X_{(i)}}\right)^n\right),$$

are UMRU estimators of $\theta_M$ and $\theta_J$, respectively under any loss function having a risk unbiasedness condition as in (2.5). In particular, under the squared log error loss function we have $\gamma(t) = \ln(t)$, and hence

$$\delta_{1,1}(\mathbf{X}) = X_{(k)} \exp\left\{\frac{1}{n}\left(1 - \left(\frac{X_{(k-1)}}{X_{(k)}}\right)^n\right)\right\}$$

and

$$\delta_{1,2}(\mathbf{X}) = X_{(1)} \exp\left\{\frac{1}{n}\sum_{i=1}^{k}\left(\frac{X_{(1)}}{X_{(i)}}\right)^n\right\},$$

are UMRU estimators of $\theta_M$ and $\theta_J$, respectively. Also, for estimating the variance of the selected population, one can take $\gamma(\theta_i) = \mathrm{Var}(X_{ij}) = \frac{\theta_i^2}{12}$, to obtain, using Theorem 3.2,

$$\delta^*_{1,1}(\mathbf{X}) = \frac{X^2_{(k)}}{12}\left\{1 + \frac{2}{n}\left(1 - \left(\frac{X_{(k-1)}}{X_{(k)}}\right)^n\right)\right\}$$

and

$$\delta^*_{1,2}(\mathbf{X}) = \frac{X^2_{(1)}}{12}\left\{1 + \frac{2}{n}\sum_{i=1}^{k}\left(\frac{X_{(1)}}{X_{(i)}}\right)^{n-2}\right\},$$

as UMRU estimators of $\gamma(\theta_M) = \frac{\theta_M^2}{12}$ and $\gamma(\theta_J) = \frac{\theta_J^2}{12}$, respectively, under SEL function (2.2).

**Example 3.** Let $X_{i1}, \ldots, X_{in}$ be a random sample from an $\mathrm{Exp}(\theta_i, \alpha)$ distribution with known $\alpha$ and unknown parameter $\theta_i$ and p.d.f. $f(x_i|\theta_i) = \frac{1}{\alpha}e^{-(x_i-\theta_i)/\alpha}$, $x_i > \theta_i, i = 1, \ldots, k$. Let $X_i = \min(X_{i1}, \ldots, X_{in})$ which has an $\mathrm{Exp}(\theta_i, \frac{\alpha}{n})$ distribution that is a member of the LTPF model with $P(y, \alpha) = \frac{n}{\alpha}e^{-ny/\alpha}$. Therefore, for suitable choices of $\gamma(\cdot)$,

$$\delta_{2,1}(\mathbf{X}) = \gamma^{-1}\left(\gamma(X_{(k)}) - \frac{\alpha}{n}\sum_{i=1}^{k}\gamma'(X_{(i)})e^{n/\alpha(X_{(i)}-X_{(k)})}\right)$$

and

$$\delta_{2,2}(\mathbf{X}) = \gamma^{-1}\left(\gamma(X_{(1)}) - \frac{\alpha}{n}\gamma'(X_{(1)})\{1 - e^{n/\alpha(X_{(1)}-X_{(2)})}\}\right),$$

are UMRU estimators of $\theta_M$ and $\theta_J$, respectively, under any loss function with a risk unbiasedness condition as in (2.5). In particular, under the LINEX loss function we have $\gamma(t) = e^{at}$, and hence

$$\delta_{2,1}(\mathbf{X}) = X_{(k)} + \frac{1}{a}\ln\left(1 - \frac{a\alpha}{n}\sum_{i=1}^{k}e^{a(1+n/(a\alpha))(X_{(i)}-X_{(k)})}\right)$$

and

$$\delta_{2,2}(\mathbf{X}) = X_{(1)} + \frac{1}{a}\ln\left(1 - \frac{a\alpha}{n}\{1 - e^{n/\alpha(X_{(1)}-X_{(2)})}\}\right),$$

are UMRU estimators of $\theta_M$ and $\theta_J$, respectively. Also, for estimating the reliability function for selected population one can take $\gamma(\theta_i) = P(X_{ij} > x) = e^{-1/\alpha(x-\theta_i)}$, and use Remark 1 to obtain

$$\delta^*_{2,1}(\mathbf{X}) = e^{-1/\alpha(x-X_{(k)})} - \frac{1}{n}\sum_{i=1}^{k}e^{-1/\alpha(x-(n+1)X_{(i)}+nX_{(k)})}$$

and

$$\delta^{**}_{2,1}(\mathbf{X}) = e^{X_{(k)}} - \frac{1}{n}\sum_{i=1}^{k} e^{(n+1)X_{(i)}-nX_{(k)}}$$

as UMVU estimators of $\gamma(\theta_M) = e^{-1/\alpha(x-\theta_M)}$ and $\gamma^*(\theta_M) = e^{\theta_M}$, respectively, under the SEL function (2.2). Note that the estimator $\delta^{**}_{2,1}(\mathbf{X})$ is obtained by Kumar et al. (2009). Similarly, to estimate the odds ratio, under SEL function (2.2), associated with the selected population given by $\gamma(\theta_M) = e^{1/\alpha(x-\theta_M)} - 1$ and $\gamma(\theta_J) = e^{1/\alpha(x-\theta_J)} - 1$, respectively, one can simply take $\gamma(\theta_i) = \frac{P(X_{ij}\le x)}{1-P(X_{ij}\le x)} = e^{1/\alpha(x-\theta_i)} - 1$ and use Remark 1 to get

$$\delta^{*}_{2,1}(\mathbf{X}) = e^{1/\alpha(x-X_{(k)})}\left\{1 + \frac{1}{n}\sum_{i=1}^{k} e^{(n-1)/\alpha(X_{(i)}-X_{(k)})}\right\} - 1$$

and

$$\delta^{*}_{2,2}(\mathbf{X}) = e^{1/\alpha(x-X_{(1)})}\left\{1 + \frac{1}{n} - \frac{1}{n}e^{n/\alpha(X_{(1)}-X_{(2)})}\right\} - 1.$$

## 4 Non-existence of unbiased estimator in exponential family

In this section, we first characterize random parametric functions $\gamma(\theta_M)$ and $\gamma(\theta_J)$ for a selected population distributed according to the one-parameter exponential family of distributions that are not unbiasedly estimable (NUBE) under the SEL function (2.2). Then, we expand our results to characterize $\gamma$-loss functions of the form (2.3) as well as loss functions having the risk unbiasedness condition as in (2.5) for which, under these loss functions, $\theta_J$ and $\theta_M$ are not risk unbiasedly estimable (NRUBE).

Suppose $X$ is distributed according to the one-parameter exponential family of distributions with p.d.f.

$$f(x|\theta) = \beta(\theta)t(x)e^{-\theta r(x)}, \qquad x \in (a,b), \theta \in \Theta \subset \Re, \tag{4.1}$$

where $\beta(\theta)t(x) > 0$ and $a$ and $b$ do not depend on $\theta$. In the following result, similar to Vellaisamy (2009), we first determine some conditions for which $\gamma(\theta)P_\theta(X > c) = \gamma(\theta)\overline{F}_\theta(c)$ is NRUBE under the SEL function (2.2).

**Lemma 1.** *Suppose $X$ is distributed according to a distribution with a p.d.f. $f(x|\theta)$ given by (4.1), where $f(x|\theta)$ satisfies the following conditions*

(i) $A(c,\theta) = \frac{\beta(\theta)}{\gamma(\theta)\overline{F}_\theta(c)} < \infty, \forall\theta \in \Theta$,
(ii) $A'(c,\theta) - A(c,\theta)r(x) = g_1(\theta,c)g_2(x,c)$,

*for some $g_1 \neq 0$ and $g_2 \neq 0$ for almost all $x$. Then $\gamma(\theta)P_\theta(X > c)$ is NUBE.*

**Proof.** Suppose there exists a $k(\cdot)$ such that

$$E_\theta\big[k(X)\big] = \gamma(\theta)P_\theta(X > c) = \frac{\beta(\theta)}{A(c,\theta)}.$$

Then $\int_{-\infty}^{\infty} k(x)A(c,\theta)t(x)e^{-\theta r(x)}\,dx = 1$, $\forall\theta \in \Theta$. Differentiating with respect to $\theta$, and using (ii) we get, for all $\theta \in \Theta$,

$$\int_a^b k(x)\big(A'(c,\theta) - A(c,\theta)r(x)\big)t(x)e^{-\theta r(x)}\,dx = 0 \Leftrightarrow E_\theta\big[k(X)g_2(X,c)\big] = 0.$$

Since $X$ is complete, this results in $k(x) = 0$ for almost all $x \in \mathbb{R}$ which is a contradiction. Therefore $\gamma(\theta)P_\theta(X > c)$ is NUBE and this completes the proof.

Using Lemma 1, one can easily verify that condition (ii) is satisfied if $\frac{A'(c,\theta)}{A(c,\theta)} = m(c)$ and

$$A(c,\theta)\left[\frac{A'(c,\theta)}{A(c,\theta)} - r(x)\right] = g_1(\theta,c)g_2(x,c).$$

Or equivalently,

$$\gamma(\theta) = \frac{B\beta(\theta)e^{-m(c)\theta}}{\overline{F}_\theta(c)}, \tag{4.2}$$

for some constants $m(c)$ and $B$. Therefore, any parametric function $\gamma^*(\theta) = B\beta(\theta)e^{-m(c)\theta}$ is NUBE for a selected one-parameter exponential family of distributions under SEL function. □

**Example 4.** Suppose $X \sim \text{Weibull}(p,\theta)$ with known $p > 0$ and the p.d.f. $f(x|\theta) = p\theta x^{p-1}e^{-\theta x^p}$, $x > 0$. Here $\beta(\theta) = \theta$, $P_\theta(X > c) = e^{-\theta c^p}$ and

$$\gamma(\theta) = B\theta e^{-(m(c)-c^p)\theta}.$$

So, $\gamma^*(\theta) = B\theta e^{-m(c)\theta}$ is NUBE for estimating after selection under SEL function (2.2). Note that $\gamma^*(\theta)$ does not depend on $p$.

Now, we consider the estimation of the parameter of a selected population when $k = 2$. Let $X_1$ and $X_2$ be independent random variables, where $X_i$ has p.d.f. (4.1) with parameter $\theta_i$. Similar to (2.7), for $k = 2$, we have

$$\gamma(\theta_M) = \begin{cases} \gamma(\theta_1), & X_1 \ge X_2, \\ \gamma(\theta_2), & X_1 < X_2, \end{cases} \quad \text{and} \quad \gamma(\theta_J) = \begin{cases} \gamma(\theta_2), & X_1 \ge X_2, \\ \gamma(\theta_1), & X_1 < X_2. \end{cases}$$

The NUBE results in this case are given in the following theorem.

**Theorem 4.1.** *Let $X_1$ and $X_2$ be independent random variables, where $X_i$ has p.d.f. (4.1) with parameter $\theta_i$, and $\boldsymbol{\theta} = (\theta_1,\theta_2)$. Assume the conditions of Lemma 1 hold. Then*

(i) $\gamma(\theta_1)P_{\boldsymbol{\theta}}(X_1 > X_2)$ *is NUBE if and only if* $\gamma(\theta_1)P_{\theta_1}(X > c)$ *is NUBE for all* $c \in \mathbb{R}$.

(ii) $E_{\boldsymbol{\theta}}[\gamma(\theta_M)] = \gamma(\theta_1)P_{\boldsymbol{\theta}}(X_1 > X_2) + \gamma(\theta_2)P_{\boldsymbol{\theta}}(X_2 > X_1)$ *is NUBE.*

**Proof.** The proof is similar to the proof of Lemma 2.4 and Theorem 2.2 of Vellaisamy (2009). □

**Remark 3.** From part (ii) of Theorem 4.1 we conclude that, there does not exist an estimator $\delta^*(\mathbf{X})$ such that

$$E_{\boldsymbol{\theta}}[\delta^*(\mathbf{X})] = E_{\boldsymbol{\theta}}[\gamma(\theta_M)], \tag{4.3}$$

where $\mathbf{X} = (X_1, X_2)$. Therefore $\gamma(\theta_M)$ in the form of (4.2) is NUBE under SEL function (2.2). If we define $\delta(\mathbf{X}) = \gamma^{-1}(\delta^*(\mathbf{X}))$, then (4.3) implies that, there does not exist an estimator $\delta(\mathbf{X}) = \gamma^{-1}(\delta^*(\mathbf{X}))$ such that $E_{\boldsymbol{\theta}}[\gamma(\delta(\mathbf{X})] = E_{\boldsymbol{\theta}}[\gamma(\theta_M)]$. In other word $\theta_M$ is NRUBE for selected population under $\gamma$-loss function (2.3) where $\gamma(\cdot)$ is in the form of (4.2). Also, $\theta_M$ is NRUBE under any loss function having a risk unbiasedness condition (2.5) as long as $\gamma(\cdot)$ is in the form of (4.2). For example, for the case of the exponential distribution given in Example 3, if $B = 1$ and $m(c) = c$, then $\gamma(t) = t$. Therefore $\theta_M$ is NUBE under SEL function, which is shown by Vellaisamy (2009). Also $\theta_M$ is NRUBE under the Stein and Intrinsic loss functions defined in Table 1 which satisfy the risk unbiasedness condition (2.5) with $\gamma(t) = t$. Note that by taking $\gamma(t) = \frac{1}{t}$ (which does not satisfy (4.2)), the estimator $\delta(\mathbf{X}) = \frac{1}{X_{(2)} - X_{(1)}}$ is a risk unbiased estimator of $\theta_M$ for selected population under the entropy loss function. This can be verified as

$$E_{\boldsymbol{\theta}}\left[\frac{1}{\delta(\mathbf{X})}\right] = E_{\boldsymbol{\theta}}[X_{(2)} - X_{(1)}] = \frac{\theta_1}{\theta_2(\theta_1 + \theta_2)} + \frac{\theta_2}{\theta_1(\theta_1 + \theta_2)}$$

and

$$E_{\boldsymbol{\theta}}\left[\frac{1}{\theta_M}\right] = \frac{1}{\theta_1}P_{\boldsymbol{\theta}}(X_1 > X_2) + \frac{1}{\theta_2}P_{\boldsymbol{\theta}}(X_2 > X_1) = \frac{\theta_2}{\theta_1(\theta_1 + \theta_2)} + \frac{\theta_1}{\theta_2(\theta_1 + \theta_2)}.$$

Therefore $E_{\boldsymbol{\theta}}[\frac{1}{\delta(\mathbf{X})}] = E_{\boldsymbol{\theta}}[\frac{1}{\theta_M}]$, that is, $\delta(\mathbf{X})$ is a risk unbiased estimator of $\theta_M$ under the entropy loss function given in Table 1.

Similar results can be obtained for estimating $\theta_J$ which we present here without a proof.

**Theorem 4.2.** *Let $X_1$ and $X_2$ be independent random variables, where $X_i$ are distributed according to a distribution with a p.d.f. given by* (4.1) *with parameter $\theta_i$, and $\boldsymbol{\theta} = (\theta_1, \theta_2)$. Assume $f(x|\theta)$ satisfies the following conditions*

(i) $A(c, \theta) = \frac{\beta(\theta)}{\gamma(\theta)F_\theta(c)} < \infty, \forall \theta \in \Theta$,

(ii) $A'(c, \theta) - A(c, \theta)r(x) = g_1(\theta, c)g_2(x, c)$, *for some* $g_1 \neq 0$ *and* $g_2 \neq 0$ *for almost all* $x$. *Then*

$$E_{\boldsymbol{\theta}}[\gamma(\theta_J)] = \gamma(\theta_1) P_{\boldsymbol{\theta}}(X_1 < X_2) + \gamma(\theta_2) P_{\boldsymbol{\theta}}(X_2 < X_1),$$

*is NUBE under SEL for selected population.*

## 5 Concluding remarks

The problem of estimation after selection arises in many applications. There are many situations where one is interested in estimating some parametric functions of selected populations. For example, in genomic studies, samples are obtained after populations that are selected using genome scans and then they are used to estimate the odds ratio. Intuitive estimators of parametric functions of selected populations constructed as if there were no prior selection are usually biased. In this paper, we have considered the problem of unbiased estimation of a general parametric function, say $\gamma(\theta)$, of selected populations under the SEL function as well as some general class of loss functions. To this end, we first obtained the UMVU estimators of $\gamma(\theta)$ for some non-regular family of distributions under SEL function. Examples of $\gamma(\cdot)$ include reliability function, odds ratio and variance, among others. Then, we obtained the UMRU estimators of $\theta_M$ and $\theta_J$ under some general classes of loss functions other than the commonly used SEL function. Furthermore, we characterized some loss functions for which the risk unbiased estimators of $\theta_J$ or $\theta_M$ do not exist. It would naturally be of interest to extend the results of this paper to the minimax and admissible estimation of parametric functions of selected population under the SEL as well as the general class of loss functions in (2.3).

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