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Angular spectra for non-Gaussian isotropic fields

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Abstract. Cosmic microwave background (CMB) Anisotropies is a subject of intensive research in recent years, and therefore it is necessary to develop suitable theory and methods for the analysis of isotropic fields on spheres. The main object of our paper is to show that the polyspectra can be given as the coefficients of the orthogonal expansion of cumulants of the field in terms of irreducible tensor products of spherical harmonics. We obtain necessary and sufficient conditions for isotropy of a non-Gaussian field and the conditions are stated in terms of higher order spectra (polyspectra). The relation between cumulants and spectra gives a new method of estimating spectra.

1 Introduction

In the last decade or so there has been some growing interest in the study of space–time data measured on the surface of a sphere. The data include cosmic microwave background (CMB) anisotropies (Okamoto and Hu (2002), Adshead and Hu (2012)), medical imaging (Kakarala (2012)), global and land-based temperature data (Jones (1994), Subba Rao and Terdik (2006)), gravitational and geomagnetic data.

One of the problems in focus is the non-Gaussianity of the observed data, which leads to the investigation of higher order angular spectra called polyspectra (Hu (2001), Hu and Dodelson (2002), Benoit-Lévy et al. (2012)). Angular polyspectra, in particular the bispectrum and trispectrum, are shown to be an appropriate measure of non-Gaussianity since for a Gaussian process all higher (than second order) spectra are zero. Another important problem we consider is the Monte Carlo simulation of non-Gaussian isotropic maps with a given power spectrum and bispectrum (Contaldi and Magueijo (2001)).

Gaussian isotropic processes on the sphere have a long history since publication of Obukhov (1947) (some basic theory and references can be found in Yaglom (1961), Jones (1963), McLeod (1986), Yadrenko (1983), Leonenko (1999)). Due to many possible applications, several books (Gaetan and Guyon (2010), Cressie and Wikle (2011), Marinucci and Peccati (2011)) and papers have been published.

In this paper, we emphasize that the notion of spectra for isotropic stochastic fields on the sphere should be based on orthogonal expansion of corresponding cu-

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mulants. Our treatment follows in principle the basic theory of higher order spectra for non-Gaussian time series (Brillinger (1965), Brillinger (2001), Subba Rao and Gabr (1984), Terdik (1999)). Some general properties of the angular polyspectra will be given including the necessary and sufficient condition for isotropy. The Wigner theory of symmetries in quantum mechanics is extensively used for the symmetry relations and the appropriate series expansion of cumulants. The polyspectra has been given as the cumulants of the residual series in terms of Clebsch–Gordan coefficients Marinucci and Peccati (2011).

The polyspectra will be given as the coefficients of the orthogonal expansion of cumulants in terms of irreducible tensor products of spherical harmonics. The bispectrum is studied in more detail. The trispectrum is also considered mainly because it couples the methods used for bispectrum to the general polyspectra. In the sequel, we give a new proof for the Gaussianity of a linear isotropic field. The relation between cumulants and spectra allows us to obtain a new method of estimating the spectra. It seems more efficient to estimate the cumulants first rather than using the series expansion for estimation of spectra. The isotropy assumption implies a very particular form for the angular spectra and therefore a delicate question, we address, is the construction of isotropic stochastic maps on the sphere with some given structure of cumulants.

1.1 Gaussian isotropic fields

In this section, we consider a Gaussian stochastic process $X(L)$ on the unit sphere $\mathbb{S}_2$ in $\mathbb{R}^3$, where $L = (\vartheta, \varphi)$, with co-latitude $\vartheta \in [0, \pi]$ and longitude $\varphi \in [0, 2\pi]$. Let us suppose that $X(L)$ is continuous (in mean square sense), then it has a series expansion in terms of spherical harmonics Y_ℓ^m,

$$X(L) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Z_\ell^m Y_\ell^m(L). \tag{1.1}$$

The coefficients $\{Z_\ell^m\}$ are given by

$$Z_\ell^m = \int_{\mathbb{S}_2} X(L) Y_\ell^{m*}(L) \Omega(dL), \tag{1.2}$$

where $\Omega(dL) = \sin\vartheta \, d\vartheta \, d\varphi$ is the Lebesgue element of surface area on $\mathbb{S}_2$. The notation "$*$" is defined as the transpose and conjugate of a matrix and just conjugate for a scalar. Notice that the spherical harmonic $Y_0^0 = 1$, put $Z_0^0 = \mu$, otherwise let $\mathsf{E} Z_\ell^m = 0$, therefore $\mathsf{E} X(L) = \mu$, and the convergence is meant in mean square sense. The covariance function $\mathcal{C}_2(L_1, L_2) = \operatorname{Cov}(X(L_1), X(L_2))$ of an isotropic field on $\mathbb{S}_2$ depends on the angle between the locations, namely $\mathcal{C}_2(L_1, L_2) = \mathcal{C}(\cos\gamma)$. Note here that $\cos\gamma$ equals the inner product $L_1 \cdot L_2$. Since the spherical harmonics Y_ℓ^m are complex valued and orthonormal (see Appendix A.2), the coefficients in the serial expansion (1.1) are complex valued and

$Z_\ell^{-m} = (-1)^m \overline{Z_\ell^m}$. The Funk–Hecke formula (see Appendix A.4) gives us

$$\int_{\mathbb{S}_2} \mathcal{C}_2(L_1 \cdot L_2) Y_\ell^m(L_1)\Omega(dL_1) = f_\ell Y_\ell^m(L_2),$$

which implies that Z_ℓ^m are independent. Moreover $f_\ell = 2\pi \int_{-1}^1 \mathcal{C}(x) P_\ell(x)\, dx$, where P_ℓ denotes the standardized ($P_\ell(1) = 1$) Legendre polynomial of degree ℓ. Hence $\operatorname{Var}(Z_\ell^m) = f_\ell$, independent of m.

In turn, it is generally assumed in time series analysis that $X(L)$ is defined as linear if Z_ℓ^m are independent and for fixed ℓ they are identically distributed ($\mathsf{E}X(L) = 0$, $\mathsf{E}|Z_\ell^m|^2 = f_\ell$). Now it is straightforward that from the linearity and the addition formula for the spherical harmonics (see Appendix A.5, Müller (1966)), the covariance function is given by

$$\mathcal{C}_2(L_1, L_2) = \sum_{\ell=0}^{\infty} f_\ell \frac{2\ell + 1}{4\pi} P_\ell(\cos\gamma), \tag{1.3}$$

and all coefficients $f_\ell \geq 0$. Necessarily the covariance function depends on the central angle between the locations $\mathcal{C}_2(L_1, L_2) = \mathcal{C}(\cos\gamma)$. In other words $\mathcal{C}_2(L_1, L_2)$ is invariant under the group of rotations, that is, $X(L)$ is *isotropic*. Moreover the assumption of linearity, that is, the independence of the triangular array $\{Z_\ell^m\}$, is so strong that the Gaussianity also follows (Baldi and Marinucci (2007)). In that case, the distribution of $X(L)$ is isotropic as well. The only linear field on the sphere is the Gaussian one.

The convergence of the series $\sum_{\ell=0}^{\infty}(2\ell + 1) f_\ell/4\pi$ is equivalent to the continuity of $\mathcal{C}(\cdot)$ on $[-1, 1]$. The superposition (1.3) corresponds to the superposition of the covariance function of a time series on the real line in terms of its spectrum according to the orthogonal system $\{\exp(i2\pi\lambda k), k = 0, 1, 2, \ldots\}$, hence we treat ℓ as *frequency* and f_ℓ is the value of the *spectrum* at ℓ. Since $P_\ell(\cos 0) = 1$, the variance $\mathsf{E}X(L)^2$ is decomposed into a sum of spectra and therefore we have the analysis of variance interpretation. The orthogonal random "measure" $\{Z_\ell^m\}$ is a triangular array for each fixed ℓ; $m = -\ell, -\ell + 1, \ldots, \ell - 1, \ell$, that is, rows contain $2\ell + 1$ i.i.d. Gaussian random variables, $\mathsf{E}Z_\ell^m = 0$, $\mathsf{E}Z_\ell^m Z_k^{n*} = f_\ell \delta_{\ell,k}\delta_{m,n}$.

In general, a function $\mathcal{C}(L_1 \cdot L_2)$ of the form (1.3) is strictly positive definite if for all ℓ, $f_\ell \geq 0$, and only finitely many of them are zero (Schreiner (1997) and Schoenberg (1942)).

For a given covariance function $\mathcal{C}(\cos\gamma)$, there always exists a Gaussian isotropic fields $X(L)$ of the form (1.1) with this covariance function. As an example, consider an isotropic Gaussian field on $\mathbb{R}^3$, its restriction to the sphere $\mathbb{S}_2$ will be isotropic as well. The relation between the spectrum on $\mathbb{R}^3$ and the spectrum on $\mathbb{S}_2$ is usually called Poisson formula.

Example 1.1 (Poisson formula). For a homogeneous isotropic field on $\mathbb{R}^3$ we have the spectral representation

$$\mathcal{C}_0(r) = \int_0^\infty j_0(\lambda r)\Phi(d\lambda), \tag{1.4}$$

of a covariance function with spectral measure $\Phi(d\lambda)$ (see Yadrenko (1983), I.1.1), where j_0 is the Spherical Bessel function of the first kind (see Abramowitz and Stegun (1992), 10.1). If we consider two locations L_1 and L_2 on the sphere $\mathbb{S}_2$ with angle $\gamma \in [0, \pi]$, then the distance $r = \|L_1 - L_2\|$ between them in term of the angle is $2\sin(\gamma/2)$, and $L_1 \cdot L_2 = \cos\gamma$. Hence, we have the covariance function on sphere $\mathcal{C}(\cos\gamma) = \mathcal{C}_0(2\sin(\gamma/2))$. $\mathcal{C}(\cos\gamma)$ defines an isotropic field on the sphere $\mathbb{S}_2$ with spectrum

$$f_\ell = 2\pi^2 \int_0^\infty J_{\ell+1/2}^2(\lambda)\frac{1}{\lambda}\Phi(d\lambda), \tag{1.5}$$

where $J_{\ell+1/2}$ denotes the Bessel function of the first kind (see Abramowitz and Stegun (1992), 9.1).

Example 1.2 (Laplace–Beltrami model on $\mathbb{S}_2$). Consider the homogeneous isotropic field X on $\mathbb{R}^3$ according to the equation

$$(\Delta - c^2)X = \partial W,$$

where $\Delta = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2 + \partial^2/\partial x_3^2$ denotes the Laplace operator on $\mathbb{R}^3$, and ∂W is white noise. Its spectral density according to the measure $\lambda^2\, d\lambda$, is

$$S(\lambda) = \frac{2}{(2\pi)^2}\frac{1}{(\lambda^2 + c^2)^2}, \qquad \lambda^2 = \|(\lambda_1, \lambda_2, \lambda_3)\|^2,$$

with covariance of Matérn class

$$\begin{aligned}\mathcal{C}_0(r) &= \frac{2}{(2\pi)^2}\int_0^\infty j_0(\lambda r)\frac{\lambda^2\, d\lambda}{(\lambda^2 + c^2)^2}\\ &= \frac{1}{(2\pi)^{3/2}}\frac{(cr)^{1/2}K_{1/2}(cr)}{2c},\end{aligned}$$

see (1.4), where $K_{1/2}$ is the modified Bessel (Hankel) function (see Yadrenko (1983), I.1.6, Example 6. and Abramowitz and Stegun (1992), 10.2, 11.4.44). Now, consider the Laplace–Beltrami operator on sphere

$$\Delta_B = \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial}{\partial\vartheta}\right) + \frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2},$$

and the stochastic model

$$(\Delta_B - c^2)X_B = \partial W_B,$$

where ∂W_B is white noise on sphere and X_B is a solution of this equation. The covariance function $\mathcal{C}(\cdot)$ of X_B is the restriction of the covariance function $\mathcal{C}_0$ to the unit sphere and $\mathcal{C}(\cos\gamma) = \mathcal{C}_0(2\sin(\gamma/2))$, that is,

$$\mathcal{C}(\cos\gamma) = \frac{1}{(2\pi)^{3/2}}\sqrt{\frac{\sin(\gamma/2)}{2c}}K_{1/2}\big(2c\sin(\gamma/2)\big).$$

By applying the Poisson formula (1.5) with $\Phi(d\lambda) = S(\lambda)\,d\lambda$, we obtain the spectrum for X_B

$$f_\ell = \int_0^\infty J^2_{\ell+1/2}(\lambda) \frac{\lambda}{(\lambda^2 + c^2)^2}\,d\lambda.$$

From now on, we shall consider a weak linearity although it will be shown to be equivalent to the whole independence of Z_ℓ^m.

Definition 1.1. The field $X(L)$ is linear if the generating array $\{Z_\ell^m\}$ is uncorrelated and for fixed degree ℓ, $\{Z_\ell^m | m = -\ell, -\ell + 1, \dots, \ell - 1, \ell\}$ are independent.

This concept of linearity corresponds to the physical approach to the theory of angular momentum, that is, the subspaces of different ranks ℓ are orthogonal and $2\ell + 1$ projections inside these subspaces are independent.

2 Non-Gaussian isotropy and the angular spectrum

In a physical phenomenon, isotropy is treated as a principle. It means that there is no reason for making difference between directions. The corresponding property of a stochastic field is that all the finite dimensional distributions remain unchanged after rotating the space.

2.1 Isotropy on sphere

From now on, we do not assume Gaussianity and hence the covariance function will not be sufficient for describing the probability structure of a stochastic field. For simplicity, we suppose the existence of moments and that those determine the distribution as well. The series expansion (1.1) in terms of spherical harmonics for a mean square continuous field $X(L)$ remains valid here. Let SO(3) denote the 3D (special orthogonal) rotation group.

Definition 2.1. A stochastic field $X(L)$ on the unit sphere $\mathbb{S}_2$ is isotropic (in strict sense) if all finite dimensional distributions of $\{X(L), L \in \mathbb{S}_2\}$ are invariant under any rotation $g \in SO(3)$.

Some interesting distributional properties of Z_ℓ^m under the assumption of isotropy are given in Baldi and Marinucci (2007). The stochastic Peter–Weyl theorem (see Marinucci and Peccati (2011), 2.2.5) can also be used for studying the basic properties of X. In the sequel, we shall use the following weaker notion of isotropy.

Definition 2.2. In case the mth order cumulants of $X(L)$ are invariant under the rotation g, for every $g \in SO(3)$, then it will be called *isotropic in* mth order.

Naturally a strictly isotropic field with mth order moments is isotropic in mth order. Now, let us consider a rotation $g \in \mathrm{SO}(3)$. It is known that the spherical harmonics Y_ℓ^m at the rotated location are given in terms of the Wigner D-matrix (see Appendix A.1, item 7), more precisely

$$\Lambda(g) Y_\ell^m(L) = \sum_{k=-\ell}^{\ell} D_{k,m}^{(\ell)}(g) Y_\ell^k(L),$$

where $\Lambda(g)$ denotes the operator according to the rotation g, $\Lambda(g) Y_\ell^k(L) = Y_\ell^k(g^{-1}L)$. Hence, the rotated field has the following form

$$\Lambda(g) X(L) = \sum_{\ell=0}^{\infty} \sum_{k=-\ell}^{\ell} \sum_{m=-\ell}^{\ell} D_{k,m}^{(\ell)}(g) Z_\ell^m Y_\ell^k(L).$$

Actually, $\Lambda(g) X(L)$ can be expressed in terms of the rotated Z_ℓ^m. To see this, introduce

$$\mathcal{Z}_\ell^k(g) = \sum_{m=-\ell}^{\ell} D_{k,m}^{(\ell)}(g) Z_\ell^m, \tag{2.1}$$

and obtain

$$\Lambda(g) X(L) = \sum_{\ell=0}^{\infty} \sum_{k=-\ell}^{\ell} \mathcal{Z}_\ell^k(g) Y_\ell^k(L).$$

The isotropy assumption on X is equivalent to that the distribution of the variable $\mathcal{Z}_\ell^k$ is the same as the one of Z_ℓ^k for every $g \in \mathrm{SO}(3)$. This statement will be used frequently below. In matrix form $\mathcal{Z}_\ell(g) = D^{(\ell)}(g) Z_\ell$, where $Z_\ell = (Z_\ell^{-\ell}, Z_\ell^{-\ell+1}, \ldots, Z_\ell^{\ell})^\top$, see Appendix A.1, item 7 for $D^{(\ell)}$; the dependence on g will be omitted unless it is necessary. This relation provides an equation for the cumulant function Φ_Z^ℓ (log of the characteristic function) of Z_ℓ as well, in case of isotropy for each rotation g we have

$$\Phi_Z^\ell(\underline{\omega}_\ell) = \Phi_Z^\ell\big(\underline{\omega}_\ell D^{(\ell)}(g)\big),$$

where $\underline{\omega}_\ell = (\omega_{-\ell}, \omega_{-\ell+1}, \ldots, \omega_{\ell-1}, \omega_\ell)$. Hence, the distribution of Z_ℓ should be rotational invariant on $\mathbb{R}^{2\ell+1}$. For instance, the covariance matrix $\mathcal{C}_Z(\ell_1, \ell_2) = \mathrm{Cov}(Z_{\ell_1}, Z_{\ell_2})$ commutes with $D^{(\ell_j)}$, in the following sense

$$D^{(\ell_1)} \mathcal{C}_Z(\ell_1, \ell_2) = \mathcal{C}_Z(\ell_1, \ell_2) D^{(\ell_2)}.$$

If $\ell_1 = \ell_2 = \ell$, we have

$$D^{(\ell)} \mathcal{C}_Z(\ell, \ell) = \mathcal{C}_Z(\ell, \ell) D^{(\ell)}.$$

Since $D^{(\ell)}$ is unitary, the only matrix which commutes with $D^{(\ell)}(g)$, for any $g \in \mathrm{SO}(3)$ is a constant times unit matrix (Schur lemma) and it readily follows that the elements of Z_ℓ are uncorrelated with the same variances. We show below a bit stronger result, that from the isotropy $\mathrm{Cum}_2(\mathcal{Z}_{\ell_1}^k, \mathcal{Z}_{\ell_2}^{m*}) = \delta_{\ell_1,\ell_2}\delta_{k,m} f_{\ell_1}$, follows. Hence, we should restrict ourselves to an uncorrelated generating array $\{Z_\ell^m\}$, with $\mathsf{E} Z_\ell^m Z_k^{n*} = \delta_{\ell,k}\delta_{m,n}\sigma_{\ell,m}^2$, where $\sigma_{\ell,m}^2 = f_\ell$ does not depend on m. In other words, it will be seen that not only the second but higher order structure as well of the generating process Z_ℓ^m inside the same degree ℓ are hiding, they are not identifiable.

According to the angular momentum of degree ℓ, we rewrite X as a series of fields u_ℓ,

$$X(L) = \sum_{\ell=0}^{\infty} u_\ell(L),$$

where we define the angular momentum field u_ℓ as

$$u_\ell(L) = \sum_{m=-\ell}^{\ell} Z_\ell^m Y_\ell^m(L), \tag{2.2}$$

and we shall be interested in the invariance of the distribution of $u_\ell(L)$ under rotations. Our main interest is the comparison of the finite dimensional distributions of the field $u_\ell(L)$ to those of the rotated one. In case a rotation carries the location L to the North pole $N = (0, 0, 1)$, u_ℓ simplifies

$$u_\ell(N) = \sqrt{\frac{2\ell+1}{4\pi}} Z_\ell^0.$$

We consider real valued $X(L)$, therefore $Z_\ell^{m*} \stackrel{d}{=} (-1)^m Z_\ell^{-m}$, since $Y_\ell^m(L)^* = (-1)^m Y_\ell^{-m}(L)$. Moreover, if we reflect the location to the center then $Y_\ell^m(-L) = (-1)^\ell Y_\ell^m(L)$, hence

$$X(-L) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (-1)^\ell Z_\ell^m Y_\ell^m(L). \tag{2.3}$$

Therefore, we have the following:

Remark 2.1. The assumption of isotropy in combination with the parity relation $u_\ell(-L) = (-1)^\ell u_\ell(L)$ yields

$$\begin{aligned} &\mathrm{Cum}_p\big(u_{\ell_1}(L_1), \dots, u_{\ell_p}(L_p)\big) \\ &\quad = (-1)^{\ell_1+\cdots+\ell_p} \mathrm{Cum}_p\big(u_{\ell_1}(L_1), \dots, u_{\ell_p}(L_p)\big), \end{aligned} \tag{2.4}$$

hence for $p \geq 2$ either the sum $\mathcal{L}_p = \ell_1 + \ell_2 + \cdots + \ell_p$ is even or the cumulant (2.4) is zero.

2.2 Second order isotropy and spectrum

We use cumulants in particular for higher order spectra. Consider the second order cumulants

$$\operatorname{Cum}_2(\mathcal{Z}_{\ell_1}^k, \mathcal{Z}_{\ell_2}^{m*}) = \sum_{p,q=-\ell_1,-\ell_2}^{\ell_1,\ell_2} D_{k,p}^{(\ell_1)} D_{m,q}^{(\ell_2)*} \operatorname{Cum}_2(Z_{\ell_1}^p, Z_{\ell_2}^{q*}), \tag{2.5}$$

where $\mathcal{Z}_\ell^k$ is defined in (2.1). Note that the covariances between complex variables $Z_{\ell_1}^p$ and $Z_{\ell_2}^q$ is the second order cumulant of $Z_{\ell_1}^p$ and $Z_{\ell_2}^{q*}$. Now assume isotropy. Then $\operatorname{Cum}_2(\mathcal{Z}_{\ell_1}^k, \mathcal{Z}_{\ell_2}^{m*}) = \operatorname{Cum}_2(Z_{\ell_1}^k, Z_{\ell_2}^{m*})$ and upon integrating both sides of the above equation over the sphere according to the invariant Haar measure, we obtain (see (A.12)) that

$$\begin{aligned}\operatorname{Cum}_2(Z_{\ell_1}^k, Z_{\ell_2}^{m*}) &= \sum_{p,q=-\ell_1,-\ell_2}^{\ell_1,\ell_2} \frac{\delta_{\ell_1,\ell_2}\delta_{p,q}\delta_{k,m}}{2\ell_1+1} \operatorname{Cum}_2(Z_{\ell_1}^p, Z_{\ell_2}^{q*}) \\ &= \delta_{\ell_1,\ell_2}\delta_{k,m} C_2(\ell_1),\end{aligned}$$

where

$$C_2(\ell_1) = \frac{1}{2\ell_1+1} \sum_{p=-\ell_1}^{\ell_1} \operatorname{Cum}_2(Z_{\ell_1}^p, Z_{\ell_1}^{p*}).$$

Hence, $\operatorname{Cum}_2(Z_{\ell_1}^k, Z_{\ell_2}^{m*}) = \delta_{\ell_1,\ell_2}\delta_{k,m} f_{\ell_1}$, that is, the series Z_ℓ^k is uncorrelated.

If the series Z_ℓ^k is uncorrelated, then

$$\operatorname{Cum}_2(\mathcal{Z}_{\ell_1}^k, \mathcal{Z}_{\ell_2}^{m*}) = \delta_{\ell_1,\ell_2} f_{\ell_1} \sum_{p=-\ell_1}^{\ell_1} D_{k,p}^{(\ell_1)} D_{m,p}^{(\ell_1)*} = \delta_{\ell_1,\ell_2}\delta_{k,m} f_{\ell_1},$$

since $D_{k,m}^{(\ell)}$ is unitary (see (2.5) and (A.8)). Hence, $\operatorname{Cum}_2(\mathcal{Z}_{\ell_1}^k, \mathcal{Z}_{\ell_2}^{m*}) = \delta_{\ell_1,\ell_2} \times \delta_{k,m} f_{\ell_1} = \operatorname{Cum}_2(Z_{\ell_1}^k, Z_{\ell_2}^{m*})$.

Lemma 2.1. *The field $X(L)$ is isotropic in second order iff the triangular series Z_ℓ^k is uncorrelated with variance f_ℓ.*

We conclude that a field $X(L)$ with Gaussian i.i.d. Z_ℓ^m is strictly isotropic. Consider the covariance function $\mathcal{C}_2(L_1, L_2) = \operatorname{Cum}_2(X(L_1), X(L_2))$ of an isotropic field. Let the rotation $g_{L_2L_1}$ be the one which takes the location L_2 into the North pole N, and L_1 into the plane xOz. The Euler coordinates of $g_{L_2L_1}L_1$ are the co-latitude ϑ and 0, since the rotation does not change the angle ϑ between L_1 and L_2, such that $\cos\vartheta = L_1 \cdot L_2$. Under the isotropy assumption the joint distribution of $X(L_1)$ and $X(L_2)$ equals the joint distribution of $X(N)$ and $X(g_{L_2L_1}L_1)$, that is, $X(N)$ and $X(\vartheta, 0)$ contain all pairwise information. In other words, the covariance of an isotropic field depends on ϑ only, necessarily $\mathcal{C}_2(L_1, L_2) = \mathcal{C}(L_1 \cdot L_2) = \mathcal{C}(\cos\vartheta)$. Now

$$X(N) = \sum_{\ell=0}^{\infty} \sqrt{\frac{2\ell+1}{4\pi}} Z_\ell^0,$$

(see (A.3)), and

$$X(g_{L_2L_1}L_1) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_\ell^m(g_{L_2L_1}L_1) Z_\ell^m.$$

We have $\mathrm{Cum}_2(Z_\ell^0, Z_k^n) = \delta_{\ell,k}\delta_{0,n} f_\ell$, hence

$$\begin{aligned} \mathcal{C}_2(L_1, L_2) &= \sum_{\ell=0}^{\infty} f_\ell \sqrt{\frac{2\ell+1}{4\pi}} Y_\ell^0(g_{L_2L_1}L_1) \\ &= \sum_{\ell=0}^{\infty} f_\ell \frac{2\ell+1}{4\pi} P_\ell(\cos\vartheta). \end{aligned} \tag{2.6}$$

Similar to time series setup the covariance $\mathcal{C}_2(L_1, L_2)$ is expanded in terms of an orthonormal system $(2\ell+1)P_\ell(\cos\vartheta)/4\pi$ with coefficients f_ℓ. In particular the variance $\mathrm{Var}(X(L))$ is decomposed into the superposition of f_ℓ's. Hence, $S_2(\ell) = f_\ell$ is called *spectrum* of the field $X(L)$ with frequency ℓ.

3 Bispectrum

If the field $X(L)$ is non-Gaussian, then the first characteristic after the second order moments to be considered is the third order cumulant, referred to bicovariance or 3-point covariance also since it is the third order central moment. The corresponding quantity in frequency domain is the bispectrum. The use of the bispectrum for detecting non-Gaussianity and non-linearity is well known in time series analysis (Subba Rao and Gabr (1984), Hinich (1982) and Terdik and Máth (1998)), the similar question has been put and studied for CMB (Cosmic Microwave Background) analysis (see Marinucci (2004, 2006, 2008), Kamionkowski et al. (2011), Adshead and Hu (2012) and references therein). Similar to the spectrum when the covariance (2.6) is a sum of orthogonal functions, we consider the series expansion of bicovariances according to an orthonormal system and the coefficients will be called bispectrum. The Wigner 3j symbols

$$\begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$$

will be intensively used from now on (see Appendix A.1, item 6). They depend on the quantum numbers ℓ_1, ℓ_2, ℓ_3, called degrees, and orders m_1, m_2, m_3.

The invariance of distribution of the triangular array $\{Z_\ell^m\}$ under rotations, see (2.1), yields the necessary and sufficient condition for the third order isotropy.

Lemma 3.1. *The field $X(L)$ is isotropic in third order iff the bicovariance* $\mathrm{Cum}_3(Z_{\ell_1}^{m_1}, Z_{\ell_2}^{m_2}, Z_{\ell_3}^{m_3})$ *of the triangular series $\{Z_\ell^m\}$ has the form*

$$\mathrm{Cum}_3\big(Z_{\ell_1}^{m_1}, Z_{\ell_2}^{m_2}, Z_{\ell_3}^{m_3}\big) = \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} B_3(\ell_1, \ell_2, \ell_3), \tag{3.1}$$

with

$$B_3(\ell_1, \ell_2, \ell_3) = \sum_{k_1,k_2,k_3} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ k_1 & k_2 & k_3 \end{pmatrix} \mathrm{Cum}_3(Z_{\ell_1}^{k_1}, Z_{\ell_2}^{k_2}, Z_{\ell_3}^{k_3}).$$

See Appendix A.2 for the proof. Note here that $\mathrm{Cum}_3(Z_{\ell_1}^{m_1}, Z_{\ell_2}^{m_2}, Z_{\ell_3}^{m_3}) = 0$, if $m_1 + m_2 + m_3 \neq 0$, that is not all the possible bicovariances come into the picture. Moreover the cumulants are depending on the orders m_1, m_2, m_3 through the Wigner 3j symbols only. In other words, the third order probabilistic property inside fixed quantum numbers ℓ_k does not show up. The function B_3 of the frequencies is an average of the cumulants $\mathrm{Cum}_3(Z_{\ell_1}^{k_1}, Z_{\ell_2}^{k_2}, Z_{\ell_3}^{k_3})$ by "probability" amplitudes, hence it is called "angle average bispectrum."

From now on, we *fix the order* of ℓ_1, ℓ_2, ℓ_3 such that $\ell_1 \leq \ell_2 \leq \ell_3$, and turn to the angular momentum field u_ℓ. First, observe

$$\begin{aligned}
&\mathrm{Cum}_3\big(u_{\ell_1}(L_1), u_{\ell_2}(L_2), u_{\ell_3}(L_3)\big) \\
&\quad = \sum_{m_1,m_2,m_3} Y_{\ell_1}^{m_1}(L_1) Y_{\ell_2}^{m_2}(L_2) Y_{\ell_3}^{m_3}(L_3)\, \mathrm{Cum}_3\big(Z_{\ell_1}^{m_1}, Z_{\ell_2}^{m_2}, Z_{\ell_3}^{m_3}\big) \\
&\quad = \sum_{m_1,m_2,m_3} Y_{\ell_1}^{m_1}(L_1) Y_{\ell_2}^{m_2}(L_2) Y_{\ell_3}^{m_3}(L_3) \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} B_3(\ell_1, \ell_2, \ell_3) \\
&\quad = B_3(\ell_1, \ell_2, \ell_3) \sum_{m_1,m_2,m_3} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} Y_{\ell_1}^{m_1}(L_1) Y_{\ell_2}^{m_2}(L_2) Y_{\ell_3}^{m_3}(L_3).
\end{aligned}$$

The above expression is invariant under both the rotation of L_j's and ordering of ℓ_1, ℓ_2, ℓ_3. The *3-product of spherical harmonics Y_ℓ^m*,

$$\widetilde{I}_{\ell_1,\ell_2,\ell_3}(L_1, L_2, L_3) = \sum_{m_1,m_2,m_3} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix} Y_{\ell_1}^{m_1}(L_1) Y_{\ell_2}^{m_2}(L_2) Y_{\ell_3}^{m_3}(L_3),$$

is *rotational invariant* (see Louck (2006), p. 14), therefore without any loss of generality we apply the rotation $g_{L_3L_2}$ which takes the location L_3 into the North pole N and at the same time takes L_2 into the zOx-plane

$$\begin{aligned}
&\mathrm{Cum}_3\big(u_{\ell_1}(L_1), u_{\ell_2}(L_2), u_{\ell_3}(L_3)\big) \\
&\quad = \mathrm{Cum}_3\big(u_{\ell_1}(g_{L_3L_2}L_1), u_{\ell_2}(g_{L_3L_2}L_2), u_{\ell_3}(N)\big)
\end{aligned}$$

$$= \sqrt{\frac{2\ell_3+1}{4\pi}} B_3(\ell_1,\ell_2,\ell_3)$$
$$\times \sum_{m_1,m_2} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & 0 \end{pmatrix} Y_{\ell_1}^{m_1}(g_{L_3L_2}L_1) Y_{\ell_2}^{m_2}(g_{L_3L_2}L_2).$$

The third order cumulants of u_ℓ contain an orthonormal system of functions

$$I_{\ell_1,\ell_2,\ell_3}(L_1,L_2,L_3)$$
$$= I_{\ell_1,\ell_2,\ell_3}(g_{L_3L_2}L_1, g_{L_3L_2}L_2, N)$$
$$= \sqrt{\frac{2\ell_3+1}{4\pi}} \sum_{m_1,m_2} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & 0 \end{pmatrix} Y_{\ell_1}^{m_1}(g_{L_3L_2}L_1) Y_{\ell_2}^{m_2}(g_{L_3L_2}L_2).$$

In this way $I_{\ell_1,\ell_2,\ell_3}(L_1,L_2,L_3)$ is connected to the *bipolar spherical harmonics*, that is, to the irreducible tensor products of the spherical harmonics with different arguments (see Varshalovich et al. (1988), 5.16.1). Rewrite I_{ℓ_1,ℓ_2,ℓ_3} in terms of Euler angles

$$\mathcal{I}_{\ell_1,\ell_2,\ell_3}(\vartheta_1,\varphi_1,\vartheta_2)$$
$$= I_{\ell_1,\ell_2,\ell_3}(g_{L_3L_2}L_1, g_{L_3L_2}L_2, N)$$
$$= \sqrt{\frac{2\ell_3+1}{4\pi}} \sum_{m_1,m_2} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & 0 \end{pmatrix} (-1)^{m_1} Y_{\ell_1}^{m_1}(\vartheta_1,\varphi_1) Y_{\ell_2}^{m_1*}(\vartheta_2,0).$$

The system of functions $\mathcal{I}_{\ell_1,\ell_2,\ell_3}$ forms an orthonormal system according to the usual measure $\Omega(dL_1)\Omega(dL_2) = \sin\vartheta_1\, d\vartheta_1\, d\varphi_1 \sin\vartheta_2\, d\vartheta_2\, d\varphi_2$ on $\vartheta \in [0,\pi]$, $\varphi \in [0,2\pi]$, since spherical harmonics Y_ℓ^m are orthogonal and

$$(2\ell_3+1) \sum_{m_1,m_2} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & 0 \end{pmatrix}^2 = 1.$$

We obtain

$$\iint_{\mathbb{S}_2} \mathcal{I}^*_{\ell_1,\ell_2,\ell_3}(\vartheta_1,\varphi_1,\vartheta_2) \mathcal{I}_{j_1,j_2,j_3}(\vartheta_1,\varphi_1,\vartheta_2) \Omega(dL_1)\Omega(dL_2) = \delta_{\ell_1,j_1}\delta_{\ell_2,j_2}\delta_{\ell_3,j_3}.$$

The third order cumulants of u_ℓ are invariant under rotation, hence

$$\operatorname{Cum}_3\big(u_{\ell_1}(L_1), u_{\ell_2}(L_2), u_{\ell_3}(L_3)\big) = B_3(\ell_1,\ell_2,\ell_3)\mathcal{I}_{\ell_1,\ell_2,\ell_3}(\vartheta_1,\varphi_1,\vartheta_2).$$

Introduce the notation $\operatorname{Cum}_3(u_{\ell_1}(L_1), u_{\ell_2}(L_2), u_{\ell_3}(L_3)) = C_{u,\ell_1,\ell_2,\ell_3}(\vartheta_1,\varphi_1,\vartheta_2)$. Then the orthogonality of $\mathcal{I}_{\ell_1,\ell_2,\ell_3}$ implies

$$\iint_{\mathbb{S}_2} C_{u,j_1,j_2,j_3}(\vartheta_1,\varphi_1,\vartheta_2) \mathcal{I}_{\ell_1,\ell_2,\ell_3}(\vartheta_1,\varphi_1,\vartheta_2) \Omega(dL_1)\Omega(dL_2)$$
$$= \delta_{\ell_1,j_1}\delta_{\ell_2,j_2}\delta_{\ell_3,j_3} B_3(\ell_1,\ell_2,\ell_3).$$

Consider now $\mathrm{Cum}_3(X(L_1), X(L_2), X(L_3))$. The value of the bicovariance does not change under the rotation $g_{L_3L_2}$, therefore

$$\begin{aligned}\mathrm{Cum}_3\big(X(L_1), X(L_2), X(L_3)\big) &= \mathrm{Cum}_3\big(X(g_{L_3L_2}L_1), X(g_{L_3L_2}L_2), X(N)\big)\\ &= \mathrm{Cum}_3\big(X(\vartheta_1, \varphi_1), X(\vartheta_2, 0), X(N)\big),\end{aligned}$$

the spherical coordinates of the locations $g_{L_3L_2}L_1 = (\vartheta_1, \varphi_1)$ and $g_{L_3L_2}L_2 = (\vartheta_2, 0)$ are defined by the locations L_1, L_2, L_3 as follows: $g_{L_3L_2}L_2 \cdot N = L_2 \cdot L_3 = \cos\vartheta_2$ and $g_{L_3L_2}L_1 \cdot N = L_1 \cdot L_3 = \cos\vartheta_1$. These angles $(\vartheta_1, \varphi_1, \vartheta_2)$ define uniquely, up to rotations, the triangle given by locations L_1, L_2, L_3. In general the bicovariance is written $C_3(\vartheta_1, \varphi_1, \vartheta_2) = \mathrm{Cum}_3(X(\vartheta_1, \varphi_1), X(\vartheta_2, 0), X(N))$, that is, it depends on a location $L(\vartheta_2, 0)$ from the main circle ($\varphi_2 = 0$) and a general location $L_0 = L_0(\vartheta_1, \varphi_1)$. The result is that

$$\begin{aligned}&\mathrm{Cum}_3\big(X(L_1), X(L_2), X(L_3)\big)\\ &\quad= \sum_{\ell_1,\ell_2,\ell_3=0}^{\infty} B_3(\ell_1, \ell_2, \ell_3)\mathcal{I}_{\ell_1,\ell_2,\ell_3}(\vartheta_1, \varphi_1, \vartheta_2).\end{aligned} \tag{3.2}$$

The above series expansion of $\mathrm{Cum}_3(X(L_1), X(L_2), X(L_3))$ leads to the following definition:

Definition 3.1. The bispectrum of the isotropic field $X(L)$ is given by

$$S_3(\ell_1, \ell_2, \ell_3) = B_3(\ell_1, \ell_2, \ell_3),$$

and the bicoherence of $X(L)$ is given by

$$\frac{S_3(\ell_1, \ell_2, \ell_3)}{\sqrt{S_2(\ell_1)S_2(\ell_2)S_2(\ell_3)}} = \frac{B_3(\ell_1, \ell_2, \ell_3)}{\sqrt{f_{\ell_1} f_{\ell_2} f_{\ell_3}}}.$$

Theorem 3.1. *The bicovariances* $\mathrm{Cum}_3(X(L_1), X(L_2), X(L_3))$ *of the isotropic field* $X(L)$ *have the series expansion* (3.2) *in terms of the bispectrum* $B_3(\ell_1, \ell_2, \ell_3)$ *and orthonormal system* $\mathcal{I}_{\ell_1,\ell_2,\ell_3}$*, hence*

$$\begin{aligned}&B_3(\ell_1, \ell_2, \ell_3)\\ &\quad= \iint_{\mathbb{S}_2} \mathrm{Cum}_3\big(X(\vartheta_1, \varphi_1), X(\vartheta_2, 0), X(N)\big)\\ &\qquad\quad \times \mathcal{I}_{\ell_1,\ell_2,\ell_3}(\vartheta_1, \varphi_1, \vartheta_2)\Omega(dL_1)\Omega(dL_2).\end{aligned}$$

3.1 Linear field

First, we assume that the rows of the triangle array $\{Z_\ell^m\}$ contain independent variables. In other words the angular momentum field u_ℓ is linear. Then for a fixed degree ℓ

$$\mathrm{Cum}_3(Z_\ell^{m_1}, Z_\ell^{m_2}, Z_\ell^{m_3}) = \begin{pmatrix} \ell & \ell & \ell \\ m_1 & m_2 & m_3 \end{pmatrix} B_3(\ell, \ell, \ell) \prod_i \delta_{m_i=m}.$$

An application of the selection rules (see Appendix A.1, item 6) yields

$$\mathrm{Cum}_3(Z_\ell^{m_1}, Z_\ell^{m_2}, Z_\ell^{m_3}) = \begin{pmatrix} \ell & \ell & \ell \\ 0 & 0 & 0 \end{pmatrix} B_3(\ell,\ell,\ell) \prod_i \delta_{m_i=0}.$$

Hence the only non-zero third order cumulant might be $\mathrm{Cum}_3(Z_\ell^0, Z_\ell^0, Z_\ell^0)$. Further the bispectrum

$$\begin{aligned} B_3(\ell,\ell,\ell) &= \sum_k \begin{pmatrix} \ell & \ell & \ell \\ k & k & k \end{pmatrix} \mathrm{Cum}_3(Z_\ell^k, Z_\ell^k, Z_\ell^k) \\ &= \begin{pmatrix} \ell & \ell & \ell \\ 0 & 0 & 0 \end{pmatrix} \mathrm{Cum}_3(Z_\ell^0, Z_\ell^0, Z_\ell^0), \end{aligned}$$

therefore

$$\mathrm{Cum}_3(Z_\ell^0, Z_\ell^0, Z_\ell^0) = \begin{pmatrix} \ell & \ell & \ell \\ 0 & 0 & 0 \end{pmatrix}^2 \mathrm{Cum}_3(Z_\ell^0, Z_\ell^0, Z_\ell^0).$$

We conclude from this that from the isotropy and independence assumptions of Z_ℓ^m follows that $\mathrm{Cum}_3(Z_\ell^m, Z_\ell^m, Z_\ell^m) = 0$. Moreover, if all the members of the triangle array $\{Z_\ell^m\}$ are independent then $\mathrm{Cum}_3(X(L_1), X(L_2), X(L_3)) = 0$. Third order cumulants vanish for instance when the distribution is Gaussian or symmetric.

3.2 Symmetries of the bispectrum

The cumulants $\mathrm{Cum}_3(X(L_1), X(L_2), X(L_3))$ according to different locations L_1, L_2 and L_3 are defined by the spherical triangle with vertices L_1, L_2 and L_3. To achieve efficiency in computations and redundancy in statistical procedures due to symmetry requires to determine the principal domain. The *principal domain* for the bispectrum with the frequencies ℓ_1, ℓ_2, ℓ_3 is given by

1. ℓ_1, ℓ_2, ℓ_3 is monotone: $\ell_1 \le \ell_2 \le \ell_3$,
2. $\ell_1 + \ell_2 + \ell_3$ is even (see Remark 2.1),
3. ℓ_1, ℓ_2, ℓ_3 fulfils the triangular inequality $|\ell_1 - \ell_2| \le \ell_3 \le \ell_1 + \ell_2$.

In addition, $\mathrm{Cum}_3(Z_{\ell_1}^{m_1}, Z_{\ell_2}^{m_2}, Z_{\ell_3}^{m_3}) = 0$, unless $m_1 + m_2 + m_3 = 0$.

Similarly, the *principal domain* for the bicovariance with the locations (L_1, L_2, L_3) is $\{(\vartheta_1, \varphi_1, \vartheta_2) | \vartheta_1 \in [0, \pi], \varphi_2 \in [0, \pi], \vartheta_2 \in [0, \pi]\}$. We apply the following notation of these angles $\cos\vartheta_1 = L_2 \cdot L_3$, $\cos\vartheta_2 = L_1 \cdot L_3$, and $\varphi_2 = \phi_3$ is the surface angle at L_3, see Figure 1. The third central angle is given by $\cos\vartheta_3 = L_1 \cdot L_2$. The surface angle ϕ_3 can be calculated for instance from the cosine formula $\cos\vartheta_3 = \cos\vartheta_1 \cos\vartheta_2 + \sin\vartheta_1 \sin\vartheta_2 \cos\varphi_2$.

4 Trispectrum

In time series analysis, the trispectrum (or fourth order cumulant spectrum) is the Fourier transform of the fourth order cumulants (see Brillinger (1965) and Molle

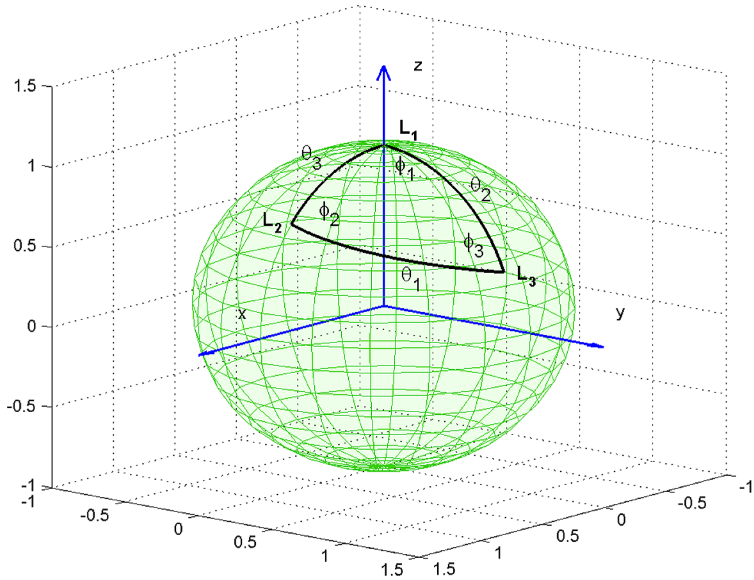

Figure 1 *Spherical triangle.*

and Hinich (1995)), which in fact can be written in terms of the fourth order moments, and all other lower order moments. The second and third order cumulants are equal to the central moments, but it is not so for fourth and higher order. Therefore, the fourth order cumulants can be considered as residual parts to explain any non-Gaussianity in the process. In the papers Hu (2001), Kogo and Komatsu (2006), for instance, the trispectrum was defined through fourth order moments, then after transformation the residual part was used correctly for the study of non-Gaussianity. The next lemma shows that the structure of the fourth order cumulant of Z_ℓ^m differs from the third order significantly.

Lemma 4.1. *The field $X(L)$ is isotropic in fourth order iff the cumulant* $\operatorname{Cum}_4(Z_{\ell_1}^{m_1}, Z_{\ell_2}^{m_2}, Z_{\ell_3}^{m_3}, Z_{\ell_4}^{m_4})$ *of the triangular array* $\{Z_\ell^m\}$ *has the form*

$$\begin{aligned}\operatorname{Cum}_4(Z_{\ell_1}^{m_1}, Z_{\ell_2}^{m_2}, Z_{\ell_3}^{m_3}, Z_{\ell_4}^{m_4}) &\\ = \sum_{\ell^1, m^1} (-1)^{m^1} \begin{pmatrix} \ell_1 & \ell_2 & \ell^1 \\ m_1 & m_2 & m^1 \end{pmatrix} \begin{pmatrix} \ell^1 & \ell_3 & \ell_4 \\ -m^1 & m_3 & m_4 \end{pmatrix} &\\ \times \sqrt{2\ell^1 + 1} T_4(\ell_1, \ell_2, \ell_3, \ell_4 | \ell^1), &\end{aligned} \tag{4.1}$$

where $m^1 = -m_1 - m_2$.

See Appendix A.3 for the proof. The function $T_4(\ell_1, \ell_2, \ell_3, \ell_4|\ell^1)$ is given by

$$T_4(\ell_1, \ell_2, \ell_3, \ell_4|\ell^1) = \sqrt{2\ell^1 + 1} \sum_{k_1,k_2,k_3,k_4,k^1} (-1)^{k^1} \begin{pmatrix} \ell_1 & \ell_2 & \ell^1 \\ k_1 & k_2 & k^1 \end{pmatrix} \times \begin{pmatrix} \ell^1 & \ell_3 & \ell_4 \\ -k^1 & k_3 & k_4 \end{pmatrix} \mathrm{Cum}_4(Z_{\ell_1}^{k_1}, Z_{\ell_2}^{k_2}, Z_{\ell_3}^{k_3}, Z_{\ell_4}^{k_4}), \tag{4.2}$$

where $k^1 = -k_1 - k_2$. We notice here, equation (4.2) shows that the cumulants of Z_ℓ^m and the function $T_4(\ell_1, \ell_2, \ell_3, \ell_4|\ell^1)$ define each other uniquely.

Remark 4.1. If we assume that $T_4(\ell_1, \ell_2, \ell_3, \ell_4|\ell^1)$ is known for $\ell_1 \le \ell_2 \le \ell_3 \le \ell_4$, then the cumulants $\mathrm{Cum}_4(Z_{\ell_1}^{m_1}, Z_{\ell_2}^{m_2}, Z_{\ell_3}^{m_3}, Z_{\ell_4}^{m_4})$ can be evaluated and vice versa for any $(\ell_1, \ell_2, \ell_3, \ell_4)$, $T_4(\ell_1, \ell_2, \ell_3, \ell_4|\ell^1)$ can be calculated by (4.2).

The sum in (4.2) works for k_1, k_2, k_3, k_4 and $k^1 = k_1 + k_2 = -k_3 - k_4$. Hence the summation contains those k_m's when equation $k_1 + k_2 + k_3 + k_4 = 0$ is satisfied. Similarly, $\mathrm{Cum}_4(Z_{\ell_1}^{m_1}, Z_{\ell_2}^{m_2}, Z_{\ell_3}^{m_3}, Z_{\ell_4}^{m_4}) = 0$, if $m_1 + m_2 + m_3 + m_4 \neq 0$. The triangular inequality for ℓ_1, ℓ_2, ℓ^1 and ℓ_3, ℓ_4, ℓ^1 suggests that the "quadrilateral" with edges $(\ell_1, \ell_2, \ell_3, \ell_4)$ consists of two triangles ℓ_1, ℓ_2, ℓ^1 and ℓ^1, ℓ_3, ℓ_4.

Note that parity transformation (2.4) implies that $\ell_1 + \ell_2 + \ell_3 + \ell_4$ must be even and we consider the field u_ℓ, given by (2.2), and observe

$$\begin{aligned} &\mathrm{Cum}_4(u_{\ell_1}(L_1), u_{\ell_2}(L_2), u_{\ell_3}(L_3), u_{\ell_4}(L_4)) \\ &\quad = \sum_{m_{1:4}} \prod_{j=1}^{4} Y_{\ell_j}^{m_j}(L_j) \mathrm{Cum}_4(Z_{\ell_1}^{m_1}, Z_{\ell_2}^{m_2}, Z_{\ell_3}^{m_3}, Z_{\ell_4}^{m_4}) \\ &\quad = \sum_{\ell^1, m^1} T_4(\ell_1, \ell_2, \ell_3, \ell_4|\ell^1) \widetilde{I}_{\ell_1,\ell_2,\ell_3,\ell_4|\ell^1}(L_1, L_2, L_3, L_4), \end{aligned}$$

where

$$\widetilde{I}_{\ell_1,\ell_2,\ell_3,\ell_4|\ell^1}(L_1, L_2, L_3, L_4) = \sqrt{2\ell^1 + 1} \sum_{m_1,\dots,m_4,m^1} \prod_{j=1}^{4} Y_{\ell_j}^{m_j}(L_j) \times \begin{pmatrix} \ell_1 & \ell_2 & \ell^1 \\ m_1 & m_2 & -m^1 \end{pmatrix} \begin{pmatrix} \ell^1 & \ell_3 & \ell_4 \\ m^1 & m_3 & m_4 \end{pmatrix}.$$

The cumulant in the left-hand side is symmetric in $\ell_1, \ell_2, \ell_3, \ell_4$ and invariant under rotation. The function $\widetilde{I}_{\ell_1,\ell_2,\ell_3,\ell_4|\ell^1}$ is invariant under rotation, since, if we apply a rotation g on each L_j, then we can use Lemma A.1, for $p = 4$, to show that the right-hand side does not change. Therefore, we apply the rotation $g_{L_4L_3}$, see Appendix A.1, item 1,

$$\begin{aligned}
&\mathrm{Cum}_4\big(u_{\ell_1}(L_1), u_{\ell_2}(L_2), u_{\ell_3}(L_3), u_{\ell_4}(L_4)\big)\\
&\quad = \mathrm{Cum}_4\big(u_{\ell_1}(g_{L_4L_3}L_1), u_{\ell_2}(g_{L_4L_3}L_2), u_{\ell_3}(g_{L_4L_3}L_3), u_{\ell_4}(N)\big)\\
&\quad = \sum_{\ell^1} T_4\big(\ell_1, \ell_2, \ell_3, \ell_4|\ell^1\big)\mathcal{I}_{\ell_1,\ell_2,\ell_3,\ell_4|\ell^1}(\vartheta_1, \vartheta_2, \vartheta_3, \varphi_1, \varphi_2),
\end{aligned}$$

where the function $\mathcal{I}_{\ell_1,\ell_2,\ell_3,\ell_4|\ell^1}$ is given by

$$\begin{aligned}
\mathcal{I}_{\ell_1,\ell_2,\ell_3,\ell_4|\ell^1}(\vartheta_1, \vartheta_2, \vartheta_3, \varphi_1, \varphi_2) = \sqrt{\frac{2\ell_4+1}{4\pi}(2\ell^1+1)} \sum_{m_{1:3},m^1} \prod_{j=1}^{3} Y_{\ell_j}^{m_j}(g_{L_4L_3}L_j)\\
\times \begin{pmatrix} \ell_1 & \ell_2 & \ell^1 \\ m_1 & m_2 & -m^1 \end{pmatrix} \begin{pmatrix} \ell^1 & \ell_3 & \ell_4 \\ m^1 & m_3 & 0 \end{pmatrix}.
\end{aligned}$$

We note $\mathcal{I}_{\ell_1,\ell_2,\ell_3,\ell_4|\ell^1}$ is the rotated version of $\widetilde{I}_{\ell_1,\ell_2,\ell_3,\ell_4|\ell^1}$, and it corresponds to the tripolar spherical harmonics defined as irreducible tensor products of the spherical harmonics with different arguments (see Varshalovich et al. (1988), p. 160). We define the spherical coordinates $(\vartheta_1, \vartheta_2, \vartheta_3, \varphi_1, \varphi_2)$ as follows $g_{L_4L_3}L_3 = (\vartheta_3, 0)$, $g_{L_4L_3}L_j = (\vartheta_j, \varphi_j)$, $j = 1, 2$. Note the orthonormality of the system according to the measure $\prod_{k=1}^{3} \Omega(dL_k) = \prod_{k=1}^{3} \sin\vartheta_k \, d\vartheta_k \, d\varphi_k$, $\vartheta_k \in [0, \pi]$, $\varphi_k \in [0, 2\pi]$, that is,

$$\iiint_{\mathbb{S}_2} \mathcal{I}_{\ell_1,\ell_2,\ell_3,\ell_4|\ell^1}\mathcal{I}_{j_1,j_2,j_3,j_4|\ell^2} \prod_{k=1}^{3} \Omega(dL_k) = \delta_{\ell^1,\ell^2} \prod_{k=1:4} \delta_{\ell_k,j_k},$$

since collecting the coefficients after integration, we have

$$(2\ell_4+1) \sum_{m^1,m_3} \begin{pmatrix} \ell^1 & \ell_3 & \ell_4 \\ m^1 & m_3 & 0 \end{pmatrix}^2 (2\ell^1+1) \sum_{m_1,m_2,m^2} \begin{pmatrix} \ell_1 & \ell_2 & \ell^1 \\ m_1 & m_2 & -m^2 \end{pmatrix}^2 = 1.$$

The expression for the $\mathrm{Cum}_4(X(L_1), X(L_2), X(L_3), X(L_4)) = \mathcal{C}_4(\vartheta_1, \vartheta_2, \vartheta_3, \varphi_1, \varphi_2)$ is straightforward

$$\begin{aligned}
&\mathcal{C}_4(\vartheta_1, \vartheta_2, \vartheta_3, \varphi_1, \varphi_2)\\
&\quad = \sum_{\ell_1,\ell_2,\ell_3,\ell_4=0}^{\infty} \mathrm{Cum}_4\big(u_{\ell_1}(L_1), \ldots, u_{\ell_4}(L_4)\big) \qquad (4.3)\\
&\quad = \sum_{\ell_1,\ldots,\ell_4=0}^{\infty} \sum_{\ell^1} T_4\big(\ell_1, \ell_2, \ell_3, \ell_4|\ell^1\big)\mathcal{I}_{\ell_1,\ell_2,\ell_3,\ell_4|\ell^1}(\vartheta_1, \vartheta_2, \vartheta_3, \varphi_1, \varphi_2).
\end{aligned}$$

Definition 4.1. The trispectrum of the isotropic field $X(L)$ is given by

$$S_4\big(\ell_1, \ell_2, \ell_3, \ell_4|\ell^1\big) = T_4\big(\ell_1, \ell_2, \ell_3, \ell_4|\ell^1\big).$$

Theorem 4.1. *The fourth order cumulant* $\operatorname{Cum}_4(X(L_1), X(L_2), X(L_3), X(L_4))$ *of the isotropic field* $X(L)$ *has the series expansion* (4.3) *in terms of the trispectrum* $S_4(\ell_1, \ell_2, \ell_3, \ell_4|\ell^1)$ *and orthonormal system* $\mathcal{I}_{\ell_1,\ell_2,\ell_3,\ell_4|\ell^1}$, *hence*

$$S_4(\ell_1, \ell_2, \ell_3, \ell_4|\ell^1) = \iiint_{\mathbb{S}_2} \mathcal{C}_4(\vartheta_1, \vartheta_2, \vartheta_3, \varphi_1, \varphi_2)\mathcal{I}_{\ell_1,\ell_2,\ell_3,\ell_4|\ell^1}(\vartheta_1, \vartheta_2, \vartheta_3, \varphi_1, \varphi_2) \prod_{k=1}^{3} \Omega(dL_k).$$

4.1 Linear field

If the angular momentum field u_ℓ is linear, then for a fixed degree ℓ

$$\operatorname{Cum}_4(Z_\ell^{m_1}, Z_\ell^{m_2}, Z_\ell^{m_3}, Z_\ell^{m_4}) = \delta_{m_i=m} \sum_{\ell^1,m^1} \begin{pmatrix} \ell_1 & \ell_2 & \ell^1 \\ m_1 & m_2 & m^1 \end{pmatrix} \begin{pmatrix} \ell^1 & \ell_3 & \ell_4 \\ -m^1 & m_3 & m_4 \end{pmatrix} \times (-1)^{m^1} \sqrt{2\ell^1+1} T_4(\ell, \ell, \ell, \ell|\ell^1),$$

from the selection rules it follows $m^1 = -(m_1 + m_2) = -2m$, $m^1 = m_3 + m_4 = 2m$, therefore $m_i = 0$. For similar reason, we get

$$T_4(\ell, \ell, \ell, \ell|\ell^1) = \sqrt{2\ell^1+1} \operatorname{Cum}_4(Z_\ell^0, Z_\ell^0, Z_\ell^0, Z_\ell^0) \begin{pmatrix} \ell & \ell & \ell^1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell^1 & \ell & \ell \\ 0 & 0 & 0 \end{pmatrix}.$$

The only non-zero cumulants are

$$\operatorname{Cum}_4(Z_\ell^0, Z_\ell^0, Z_\ell^0, Z_\ell^0) = \sum_{\ell^1} \sqrt{2\ell^1+1} \begin{pmatrix} \ell & \ell & \ell^1 \\ 0 & 0 & 0 \end{pmatrix}^2 \begin{pmatrix} \ell^1 & \ell & \ell \\ 0 & 0 & 0 \end{pmatrix}^2 \times \operatorname{Cum}_4(Z_\ell^0, Z_\ell^0, Z_\ell^0, Z_\ell^0),$$

hence for an isotropic linear field u_ℓ $\operatorname{Cum}_4(Z_\ell^{m_1}, Z_\ell^{m_2}, Z_\ell^{m_3}, Z_\ell^{m_4}) = 0$, for all ℓ and m_i. If additionally the series of u_ℓ, is independent then $\operatorname{Cum}_4(X(L_1), X(L_2), X(L_3), X(L_4)) = 0$.

5 Higher order spectra for isotropic fields

The generalization of the bispectrum and trispectrum is possible for arbitrary higher order. In Marinucci and Peccati (2010) and Marinucci and Peccati (2011), the polyspectrum is defined as the higher order cumulants of the residual triangle $\{Z_\ell^m\}$ in terms of Clebsch–Gordan coefficients. If the bispectrum and trispectrum are zero, then the field can not be Gaussian, in case all polyspectra, except the second order one, are zero then the isotropic field is necessarily Gaussian. First, we show that the characterization of the isotropy in pth order can be given following the methods given earlier.

Lemma 5.1. *The field*

$$X(L) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Z_\ell^m Y_\ell^m(L),$$

is isotropic in pth order ($p > 3$) *iff the cumulant* $\mathrm{Cum}_p(Z_{\ell_1}^{m_1}, Z_{\ell_2}^{m_2}, \ldots, Z_{\ell_p}^{m_p})$ *of the triangular array* $\{Z_\ell^m\}$ *has the form*

$$\begin{aligned}\mathrm{Cum}_p&(Z_{\ell_1}^{m_1}, \ldots, Z_{\ell_p}^{m_p}) \\ &= \sum_{\substack{\ell^1,\ldots,\ell^{p-3} \\ m^1,\ldots,m^{p-3}}} (-1)^{\sum_{a=1}^{p-3} m^a} \prod_{a=0}^{p-3} \begin{pmatrix} \ell^a & \ell_{a+2} & \ell^{a+1} \\ -m^a & m_{a+2} & m^{a+1} \end{pmatrix} \\ &\quad \times \prod_{a=1}^{p-3} \sqrt{2\ell^a + 1}\, \widetilde{S}_p(\ell_1, \ldots, \ell_p | \ell^1, \ldots, \ell^{p-3}),\end{aligned} \tag{5.1}$$

where $\ell^0 = \ell_1$, $\ell^{p-2} = \ell_p$, $k^0 = -k_1$, $k^{p-2} = k_p$, $m^0 = -m_1$, $m^{p-2} = m_p$. *The function* $\widetilde{S}_p(\ell_1, \ldots, \ell_p | \ell^1, \ldots, \ell^{p-3})$ *has the form*

$$\begin{aligned}\widetilde{S}_p&(\ell_1, \ldots, \ell_p | \ell^1, \ldots, \ell^{p-3}) \\ &= \sum_{\substack{k_1,\ldots,k_p \\ k^1,\ldots,k^{p-3}}} (-1)^{\sum_{a=1}^{p-3} k^a} \prod_{a=0}^{p-3} \begin{pmatrix} \ell^a & \ell_{a+2} & \ell^{a+1} \\ -k^a & k_{a+2} & k^{a+1} \end{pmatrix} \\ &\quad \times \prod_{a=1}^{p-3} \sqrt{2\ell^a + 1}\, \mathrm{Cum}_p(Z_{\ell_1}^{k_1}, Z_{\ell_2}^{k_2}, \ldots, Z_{\ell_p}^{k_p}).\end{aligned} \tag{5.2}$$

See Appendix A.4 for the proof.

Consider a convex polygon with vertices $A_{1:p-1} = (A_1, \ldots, A_{p-1})$, and edges $\ell_{1:p} = (\ell_1, \ldots, \ell_p)$. The diagonals denoted by ℓ^j, $j = 1, 2, \ldots, p-3$, starting from the vertex A_1 divide this polygon into $p-2$ triangles. The first triangle has sides (angular momentum) ℓ_1, ℓ_2 and ℓ^1, the next one has sides ℓ^1, ℓ_3 and ℓ^2, the general one is ℓ^a, ℓ_{a+2} and ℓ^{a+1}, finally the last one ℓ^{p-3}, ℓ_{p-1} and ℓ_p. For each a, the sides of the triangle $(\ell^a, \ell_{a+2}, \ell^{a+1})$ should fulfil the triangle inequality $|\ell^a - \ell^{a+1}| \le \ell_{a+2} \le \ell^a + \ell^{a+1}$. The coefficients in (5.1) will differ from zero if orders $-m^a, m_{a+2}, m^{a+1}$, fulfil the assumption $-m^a + m_{a+2} + m^{a+1} = 0$, for all a. This implies $m_1 + m_2 = -m^1$, $m_3 + m^2 = m^1$, …, $m_{p-1} + m_p = m^{p-3}$. Let us plug in consecutively m^a and we shall arrive at the result $m_1 + m_2 + \cdots + m_p = 0$. Hence, $\mathrm{Cum}_p(Z_{\ell_1}^{m_1}, \ldots, Z_{\ell_p}^{m_p}) = 0$, unless $m_1 + m_2 + \cdots + m_p = 0$.

Remark 5.1. The cumulant $\operatorname{Cum}_p(Z_{\ell_1}^{m_1}, \dots, Z_{\ell_p}^{m_p})$ is invariant under the order of the quantum numbers $\ell_{1:p}$, hence the right-hand side of (5.1) is invariant as well. The result is that the function $\widetilde{S}_p(\ell_1, \dots, \ell_p|\ell^1, \dots, \ell^{p-3})$ given on values $\ell_1 \leq \ell_2 \leq \dots \leq \ell_p$ will determine all cumulants $\operatorname{Cum}_p(Z_{\ell_1}^{m_1}, \dots, Z_{\ell_p}^{m_p})$ by (5.1).

Repeat the representation of the field

$$X(L) = \sum_{\ell=0}^{\infty} u_\ell(L),$$

where $u_\ell(L)$is given by (2.2). Our main interest is the comparison of the finite dimensional distributions of the process $u_\ell(L)$ to the rotated one in case the rotation carries one location to the North pole N, since then u_ℓ simplifies to

$$u_\ell(N) = \sqrt{\frac{2\ell+1}{4\pi}} Z_\ell^0.$$

The pth order cumulant is

$$\begin{aligned}
&\operatorname{Cum}_p\big(u_{\ell_1}(L_1), u_{\ell_2}(L_2), \dots, u_{\ell_p}(L_p)\big) \\
&\quad = \sum_{m_1,\dots,m_p} \prod_{j=1}^{p} Y_{\ell_j}^{m_j}(L_j) \operatorname{Cum}_p\big(Z_{\ell_1}^{m_1}, Z_{\ell_2}^{m_2}, \dots, Z_{\ell_p}^{m_p}\big) \\
&\quad = \sum_{m_1,\dots,m_p} \prod_{j=1}^{p} Y_{\ell_j}^{m_j}(L_j) \sum_{\substack{\ell^1,\dots,\ell^{p-3} \\ m^1,\dots,m^{p-3}}} (-1)^{\sum_{a=1}^{p-3} m^a} \prod_{a=0}^{p-3} \begin{pmatrix} \ell^a & \ell_{a+2} & \ell^{a+1} \\ -m^a & m_{a+2} & m^{a+1} \end{pmatrix} \\
&\qquad \times \prod_{a=1}^{p-3} \sqrt{2\ell^a+1} \widetilde{S}_p\big(\ell_1, \dots, \ell_p|\ell^1, \dots, \ell^{p-3}\big) \\
&\quad = \sum_{\ell^1,\dots,\ell^{p-3}} \widetilde{I}_{\ell_{1:p},\ell^{1:p-3}}(L_1, \dots, L_p) \widetilde{S}_p\big(\ell_1, \dots, \ell_p|\ell^1, \dots, \ell^{p-3}\big),
\end{aligned}$$

where $\widetilde{I}_{\ell_1,\dots,\ell_p|\ell^1,\dots,\ell^{p-3}}$ is the p-product *of spherical harmonics* Y_ℓ^m

$$\begin{aligned}
&\widetilde{I}_{\ell_1,\dots,\ell_p|\ell^1,\dots,\ell^{p-3}}(L_1, \dots, L_p) \\
&\quad = \sum_{\substack{m_1,\dots,m_p \\ m^1,\dots,m^{p-3}}} \prod_{j=1}^{p} Y_{\ell_j}^{m_j}(L_j) \\
&\qquad \times (-1)^{\Sigma k^{1:p-3}} \prod_{a=0}^{p-3} \begin{pmatrix} \ell^a & \ell_{a+2} & \ell^{a+1} \\ -m^a & m_{a+2} & m^{a+1} \end{pmatrix} \prod_{a=1}^{p-3} \sqrt{2\ell^a+1}.
\end{aligned}$$

$\widetilde{I}$ is rotation invariant, see Lemma A.1, and $k^{1:p-3}$ denotes $(k^1,\dots,k^{p-3})$ and $\Sigma k^{1:p-3}=\sum_{j=1}^{p-3}k^j$, for short. Now we can apply the rotation $g_{L_pL_{p-1}}$, see Appendix A.1, item 1, hence $\operatorname{Cum}_p(u_{\ell_1}(L_1),u_{\ell_2}(L_2),\dots,u_{\ell_p}(L_p))=\mathcal{C}_{u,p}(\vartheta_{1:p-1},\varphi_{1:p-3})$ such that

$$\begin{aligned}\mathcal{C}_{u,p}&(\vartheta_{1:p-1},\varphi_{1:p-3})\\&=\sqrt{\frac{2\ell_p+1}{4\pi}}\sum_{m_1,\dots,m_p}\prod_{j=1}^{p-1}Y_{\ell_j}^{m_j}(g_{L_pL_{p-1}}L_j)\operatorname{Cum}_p(Z_{\ell_1}^{m_1},Z_{\ell_2}^{m_2},\dots,Z_{\ell_p}^{0})\\&=\sum_{\ell_1,\dots,\ell_p|\ell^1,\dots,\ell^{p-3}}\mathcal{I}_{\ell_1,\dots,\ell_p|\ell^1,\dots,\ell^{p-3}}(\vartheta_{1:p-1},\varphi_{1:p-3})\\&\qquad\times\widetilde{S}_p(\ell_1,\dots,\ell_p|\ell^1,\dots,\ell^{p-3}),\end{aligned}$$

we have

$$\begin{aligned}\mathcal{I}_{\ell_1,\dots,\ell_p|\ell^1,\dots,\ell^{p-3}}&(\vartheta_{1:p-1},\varphi_{1:p-3})\\&=\sqrt{\frac{2\ell_p+1}{4\pi}}\sum_{\substack{m_1,\dots,m_p\\m^1,\dots,m^{p-3}}}\prod_{j=1}^{p-1}Y_{\ell_j}^{m_j}(g_{L_pL_{p-1}}L_j)\\&\quad\times(-1)^{\Sigma k^{1:p-3}}\prod_{a=0}^{p-3}\begin{pmatrix}\ell^a&\ell_{a+2}&\ell^{a+1}\\-m^a&m_{a+2}&m^{a+1}\end{pmatrix}\\&\quad\times\prod_{a=1}^{p-3}\sqrt{2\ell^a+1}\begin{pmatrix}\ell^{p-3}&\ell_{p-1}&\ell_p\\-m^{p-3}&m_{p-1}&0\end{pmatrix}.\end{aligned}$$

The system of functions $\mathcal{I}_{\ell_1,\dots,\ell_p|\ell^1,\dots,\ell^{p-3}}$ forms an orthogonal system according to the measure $\prod_{k=1}^{p-1}\Omega(dL_k)=\prod_{k=1}^{p-1}\sin\vartheta_k\,d\vartheta_k\,d\varphi_k$, $\vartheta_k\in[0,\pi]$, $\varphi_k\in[0,2\pi]$. Consider now $\widetilde{S}_p(\ell_1,\dots,\ell_p|\ell^1,\dots,\ell^{p-3})$, see (5.2), where $\ell^0=\ell_1$, and $\ell^{p-2}=\ell_p$. One might fix an order for the entries of $\ell_1,\dots,\ell_p$ to get a unique representation for the cumulant (5.1). We consider a monotone ordering $\ell_1\le\ell_2\le\cdots\le\ell_p$ and refer to it as canonical representation. Note that parity transformation (2.3) implies that $\ell_1+\ell_2+\ell_3+\cdots+\ell_p$ must be even. Let $\operatorname{Cum}_p(X(L_1),X(L_2),X(L_3),\dots,X(L_p))=\mathcal{C}_p(\vartheta_{1:p-1},\varphi_{1:p-3})$, then

$$\begin{aligned}\mathcal{C}_p&(\vartheta_{1:p-1},\varphi_{1:p-3})\\&=\sum_{\ell_{1:p}=0}^{\infty}\widetilde{S}_p(\ell_1,\dots,\ell_p|\ell^1,\dots,\ell^{p-3})\mathcal{I}_{\ell_1,\dots,\ell_p|\ell^1,\dots,\ell^{p-3}}(\vartheta_{1:p-1},\varphi_{1:p-3}).\end{aligned}\tag{5.3}$$

Definition 5.1. The pth order polyspectrum of the isotropic field $X(L)$ is $S_p(\ell_1,\dots,\ell_p|\ell^1,\dots,\ell^{p-3})=\widetilde{S}_p(\ell_1,\dots,\ell_p|\ell^1,\dots,\ell^{p-3})$.

We have proved the following theorem.

Theorem 5.1. *The pth order cumulant* $\operatorname{Cum}_p(X(L_1), X(L_2), \dots, X(L_p))$ *of the isotropic field* $X(L)$ *have the series expansion* (5.3) *in terms of the polyspectrum* $S_p(\ell_1, \dots, \ell_p|\ell^1, \dots, \ell^{p-3})$ *and orthonormal system* $\mathcal{I}_{\ell_1,\dots,\ell_p|\ell^1,\dots,\ell^{p-3}}$, *hence*

$$S_p(\ell_1, \dots, \ell_p|\ell^1, \dots, \ell^{p-3}) = \int_{\mathbb{S}_2} \cdots \int \mathcal{C}_p(\vartheta_{1:p-1}, \varphi_{1:p-3}) \mathcal{I}_{\ell_1,\dots,\ell_p|\ell^1,\dots,\ell^{p-3}}(\vartheta_{1:p-1}, \varphi_{1:p-3}) \prod_{k=1}^{p-1} \Omega(dL_k).$$

5.1 Linear field

Let us consider the particular case when Z_ℓ^m are independent if ℓ is fixed. We have

$$\begin{aligned}
&\operatorname{Cum}_p(Z_\ell^{m_1}, Z_\ell^{m_2}, \dots, Z_\ell^{m_p}) \\
&\quad = \delta_{m_i=m} \sum_{\substack{\ell^1,\dots,\ell^{p-3} \\ m^1,\dots,m^{p-3}}} (-1)^{\sum_{a=1}^{p-3} m^a} \prod_{a=0}^{p-3} \begin{pmatrix} \ell^a & \ell & \ell^{a+1} \\ -m^a & m_{a+2} & m^{a+1} \end{pmatrix} \\
&\qquad \times S_p(\ell|\ell^1, \dots, \ell^{p-3}) \prod_{a=1}^{p-3} \sqrt{2\ell^a + 1}.
\end{aligned}$$

We have seen that if $p = 3, 4$, in the above expression then all $m_i = 0$. Let us consider the case $p = 5$,

$$\begin{pmatrix} \ell & \ell^1 \\ m_{1:2} & m^1 \end{pmatrix} \begin{pmatrix} \ell^1 & \ell & \ell^2 \\ -m^1 & m_3 & m^2 \end{pmatrix} \begin{pmatrix} \ell^2 & \ell & \ell \\ -m^2 & m_4 & m_5 \end{pmatrix}$$

then from the selection rules it follows that $m^1 = -(m_1 + m_2) = -2m$, $m^2 = m^1 - m_3 = -3m$, $m^2 = m_4 + m_5 = 2m$, hence $m = 0$. In general it is easy to see that $m^k = -(k+1)m$, for $k = 1, 2, \dots, p-3$, and at the same time $m^{p-3} = 2m$, hence $m = 0$, and $m^j = 0$ as well. Consider the polyspectrum

$$\begin{aligned}
&\widetilde{S}_p(\ell|\ell^{1:p-3}) \\
&\quad = \sum_{\substack{k_1,\dots,k_p \\ k^1,\dots,k^{p-3}}} (-1)^{\Sigma k^{1:p-2}} \prod_{a=0}^{p-3} \begin{pmatrix} \ell^a & \ell & \ell^{a+1} \\ -k^a & k_{a+2} & k^{a+1} \end{pmatrix} \\
&\qquad \times \prod_{a=1}^{p-3} \sqrt{2\ell^a + 1} \operatorname{Cum}_p(Z_\ell^{k_{1:p}}),
\end{aligned}$$

from the independence it follows that $\ell_j = \ell$ and $k_j = k$, for $j = 1, 2, \ldots, p$, where $\ell^{1:p-3} = (\ell^1, \ldots, \ell^{p-3})$. Hence, a similar argument to the previous one leads to the result: $k_j = 0$, and $k^j = 0$ as well. We have

$$\widetilde{S}_p(\ell|\ell^{1:p-3}) = \prod_{a=0}^{p-3} \begin{pmatrix} \ell^a & \ell & \ell^{a+1} \\ 0 & 0 & 0 \end{pmatrix} \prod_{a=1}^{p-3} \sqrt{2\ell^a + 1}\, \mathrm{Cum}_p(Z_\ell^0, Z_\ell^0, \ldots, Z_\ell^0).$$

Now by the Lemma 5.1 we get

$$\begin{aligned} &\mathrm{Cum}_p(Z_\ell^0, Z_\ell^0, \ldots, Z_\ell^0) \\ &\quad = \sum_{\ell^1,\ldots,\ell^{p-3}} \prod_{a=0}^{p-3} \begin{pmatrix} \ell^a & \ell & \ell^{a+1} \\ 0 & 0 & 0 \end{pmatrix}^2 \prod_{a=1}^{p-3} \sqrt{2\ell^a + 1}\, \mathrm{Cum}_p(Z_\ell^0, Z_\ell^0, \ldots, Z_\ell^0). \end{aligned}$$

Hence, $\mathrm{Cum}_p(Z_{\ell_1}^{k_1}, Z_{\ell_2}^{k_2}, \ldots, Z_{\ell_p}^{k_p}) = 0$, for all k_j. Now we have a general conclusion because the only case when all cumulants vanish except the second order one is the Gaussian. Once the rows of $\{Z_\ell^m\}$ are Gaussian then all the entries of $\{Z_\ell^m\}$ are independent. Indeed the isotropy implies that all the entries of $\{Z_\ell^m\}$ are uncorrelated and now since they are Gaussian, they are independent.

Lemma 5.2. *If the isotropic field $X(L)$ is linear, that is, the generating array $\{Z_\ell^m\}$ is uncorrelated and inside the rows of $\{Z_\ell^m\}$ all random variables are independent, then the whole array contains independent Gaussian entries and $X(L)$ is Gaussian.*

6 Construction of isotropic field

We have seen that the bispectrum, the trispectrum and in general higher order spectra of an isotropic field on sphere should have very special form and fulfil particular equations like (3.1), (4.1) and (5.1). In this section, we give a transformation in terms of Wigner D-transforms of an uncorrelated triangular array with arbitrary cumulants such that the necessary restrictions will be fulfilled as a result. If one starts with a non-Gaussian continuous (in mean square) $X(L)$, then the triangular array $\{Z_\ell^m\}$ is given by the inversion formula

$$Z_\ell^m = \int_{\mathbb{S}_2} X(L) Y_\ell^{m*}(L) \Omega(dL),$$

and it fulfils the assumptions of Lemma 5.1, say. Now we consider the converse question of construction of a triangular array $\{Z_\ell^m\}$ with the desired cumulant properties. Let us start with triangular array $\{\widetilde{Z}_\ell^m\}$, assume it is uncorrelated and all moments exist. Consider the vectors $\widetilde{Z}_\ell = (\widetilde{Z}_\ell^{-\ell}, \widetilde{Z}_\ell^{-\ell+1}, \ldots, \widetilde{Z}_\ell^{\ell})^\top$, $\ell = 0, 1, 2, \ldots,$

according to the rows of $\{\widetilde{Z}_\ell^m\}$. The finite dimensional distribution is characterized by its cumulant function (logarithm of the characteristic function)

$$\Phi_{\widetilde{Z}}(\omega_\ell|\ell=0,1,2,\ldots)=\ln \mathsf{E}\exp\left(i\sum_\ell \omega_\ell^\top \widetilde{Z}_\ell\right),$$

such that only finitely many coordinates of the variables $(\omega_\ell|\ell=0,1,2,\ldots)$ are different from zero. Consider the transformed series $\widehat{Z}_\ell = D^{(\ell)}\widetilde{Z}_\ell$, where $D^{(\ell)}$ is the Wigner matrix of rotations (see Appendix A.1, item 7), and define a triangular array $\{Z_\ell^m\}$ through the cumulant function

$$\Phi_{\widehat{Z}}(\omega_\ell|\ell=0,1,2,\ldots)=\int_{\mathrm{SO}(3)}\ln \mathsf{E}\exp\left(i\sum_\ell \omega_\ell^\top D^{(\ell)}\widetilde{Z}_\ell\right)dg,$$

where again only finitely many coordinates of the variables $(\omega_\ell|\ell=0,1,2,\ldots)$ are different from zero and $dg=\sin\vartheta\, d\vartheta\, d\varphi\, d\gamma/8\pi^2$, is the Haar measure with unit mass. This new triangular array $\{\widehat{Z}_\ell^m\}$ will be called Wigner D-transform of $\{\widetilde{Z}_\ell^m\}$. The third order cumulants of $\widehat{Z}_\ell^m$ for instance

$$\begin{aligned}
&\mathrm{Cum}_3(\widehat{Z}_{\ell_1}^{k_1},\widehat{Z}_{\ell_2}^{k_2},\widehat{Z}_{\ell_3}^{k_3})\\
&\quad=\int_{\mathrm{SO}(3)}\sum_{m_1,m_2,m_3} D_{k_1,m_1}^{(\ell_1)}D_{k_2,m_2}^{(\ell_2)}D_{k_3,m_3}^{(\ell_3)}\,dg\,\mathrm{Cum}_3(\widetilde{Z}_{\ell_1}^{m_1},\widetilde{Z}_{\ell_2}^{m_2},\widetilde{Z}_{\ell_3}^{m_3})\\
&\quad=\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ k_1 & k_2 & k_3\end{pmatrix}\sum_{m_1,m_2,m_3}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ m_1 & m_2 & m_3\end{pmatrix}\mathrm{Cum}_3(\widetilde{Z}_{\ell_1}^{m_1},\widetilde{Z}_{\ell_2}^{m_2},\widetilde{Z}_{\ell_3}^{m_3})\\
&\quad=\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ k_1 & k_2 & k_3\end{pmatrix}B_3(\ell_1,\ell_2,\ell_3),
\end{aligned}$$

which fulfils (3.1). The function $B_3(\ell_1,\ell_2,\ell_3)$ is given by

$$B_3(\ell_1,\ell_2,\ell_3)=\sum_{m_1,m_2,m_3}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ m_1 & m_2 & m_3\end{pmatrix}\mathrm{Cum}_3(\widetilde{Z}_{\ell_1}^{m_1},\widetilde{Z}_{\ell_2}^{m_2},\widetilde{Z}_{\ell_3}^{m_3}),$$

and also

$$B_3(\ell_1,\ell_2,\ell_3)=\sum_{m_1,m_2,m_3}\begin{pmatrix}\ell_1 & \ell_2 & \ell_3\\ m_1 & m_2 & m_3\end{pmatrix}\mathrm{Cum}_3(\widehat{Z}_{\ell_1}^{m_1},\widehat{Z}_{\ell_2}^{m_2},\widehat{Z}_{\ell_3}^{m_3}).$$

The conclusion is that a subset of cumulants $\mathrm{Cum}_3(\widetilde{Z}_{\ell_1}^{m_1},\widetilde{Z}_{\ell_2}^{m_2},\widetilde{Z}_{\ell_3}^{m_3})$, that is, $m_1+m_2+m_3=0$, is used in the construction and the superposition with "probability" amplitudes is applied. In general, we also have

$$\begin{aligned}
\mathrm{Cum}_p(\widehat{Z}_{\ell_1}^{m_1},\widehat{Z}_{\ell_2}^{m_2},\ldots,\widehat{Z}_{\ell_p}^{m_p}) &= \sum_{\substack{\ell^1,\ldots,\ell^{p-3}\\ m^1,\ldots,m^{p-3}}}(-1)^{\sum_{a=1}^{p-3}m^a}\prod_{a=0}^{p-3}\begin{pmatrix}\ell^a & \ell_{a+2} & \ell^{a+1}\\ -m^a & m_{a+2} & m^{a+1}\end{pmatrix}\\
&\quad\times\prod_{a=1}^{p-3}\sqrt{2\ell^a+1}\,S_p(\ell_1,\ldots,\ell_p|\ell^1,\ldots,\ell^{p-3}).
\end{aligned}$$

The conclusion now is that for any system of triangular array $\{\widetilde{Z}_\ell^m\}$ (uncorrelated and the series (1.3) converges) with cumulants $\operatorname{Cum}_p(\widetilde{Z}_{\ell_1}^{k_1}, \widetilde{Z}_{\ell_2}^{k_2}, \dots, \widetilde{Z}_{\ell_p}^{k_p})$ there exist a stochastic isotropic field with polyspectra

$$S_p(\ell_1, \dots, \ell_p | \ell^{1:p-3}) = \sum_{\substack{m_1,\dots,m_p \\ m^1,\dots,m^{p-3}}} (-1)^{\sum_{a=1}^{p-3} m^a} \prod_{a=0}^{p-3} \begin{pmatrix} \ell^a & \ell_{a+2} & \ell^{a+1} \\ -m^a & m_{a+2} & m^{a+1} \end{pmatrix}$$
$$\times \prod_{a=1}^{p-3} \sqrt{2\ell^a + 1}\operatorname{Cum}_p(\widetilde{Z}_{\ell_1}^{k_1}, \widetilde{Z}_{\ell_2}^{k_2}, \dots, \widetilde{Z}_{\ell_p}^{k_p}),$$

where $\ell^{1:p-3} = (\ell^1, \dots, \ell^{p-3})$.

7 Conclusion

The rotational invariance of the probability structure of non-Gaussian stochastic fields on 2-spheres $\mathbb{S}_2$ implies very interesting behavior of cumulants and spectra. The orthogonal system of functions on sphere in terms of irreducible tensor products of the spherical harmonics plays important role. The higher order cumulants are decomposed according to these functions and the coefficients provide the angular polyspectra. The bispectrum is studied in details. The trispectrum proved to be the first polyspectrum which shows the general properties of all higher order polyspectra. Polyspectra of linear fields have particular forms, naturally, and it has been shown that from linearity follows Gaussianity. Finally, we constructed polyspectra corresponding to triangular arrays with arbitrary structure of cumulants.

Appendix

A.1 Basics

1. The rotation $g_{L'L}$ of the locations L and L' is defined such that it takes the location L' into the North pole N, and L into the plane xOz.

2. *Standardized Legendre polynomial* $P_0(x) = 1$,

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{d^\ell (x^2 - 1)^\ell}{dx^\ell}, \qquad x \in [-1, 1],$$

$P_\ell(1) = 1$ (Erdélyi et al. (1981), vol. 2, p. 180) it is orthogonal and

$$\int_{\mathbb{S}_2} \left[P_\ell(\cos\vartheta)\right]^2 \Omega(dL) = \frac{4\pi}{2\ell + 1}. \tag{A.1}$$

3. *Orthonormal spherical harmonics with complex values* $Y_\ell^m(\vartheta,\varphi)$, $\ell = 0, 1, 2, \ldots$, $m = -\ell, -\ell+1, \ldots, -1, 0, 1, \ldots, \ell-1, \ell$ of *degree* ℓ and *order* m (rank ℓ and projection m)

$$Y_\ell^m(\vartheta,\varphi) = (-1)^m \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}} P_\ell^m(\cos\vartheta)e^{im\varphi},$$
$$\varphi \in [0, 2\pi], \vartheta \in [0, \pi], \tag{A.2}$$

where P_ℓ^m is the *associated normalized Legendre function of the first kind* (Gegenbauer polynomial at particular indices) of *degree* ℓ and *order* m

$$P_\ell^m(x) = (-1)^m\left(1-x^2\right)^{m/2}\frac{d^m P_\ell(x)}{dx^m}.$$

In particular $P_\ell^m(1) = \delta_{m,0}$, $P_\ell^0(x) = P_\ell(x)$,

$$Y_\ell^m(\vartheta,\varphi)^* = (-1)^m Y_\ell^{-m}(\vartheta,\varphi),$$

and $Y_\ell^0(\vartheta,\varphi) = \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\cos\vartheta)$, $Y_0^0(\vartheta,\varphi) = \sqrt{\frac{1}{4\pi}}$, moreover

$$Y_\ell^m(N) = \delta_{m,0}\sqrt{\frac{2\ell+1}{4\pi}}. \tag{A.3}$$

Y_ℓ^m is fully normalized

$$\int_0^{2\pi}\int_0^\pi \left|Y_\ell^m(\vartheta,\varphi)\right|^2 \sin\vartheta \, d\vartheta \, d\varphi = 1.$$

Some detailed account of spherical harmonics Y_ℓ^m can be found in Varshalovich et al. (1988) and Stein and Weiss (1971).

4. *Funk–Hecke formula*, Müller (1966). Suppose G is continuous on $[-1, 1]$, then for any spherical harmonic $Y_\ell(L)$

$$\int_{\mathbb{S}_2} G(L_1 \cdot L)Y_\ell(L)\Omega(dL) = cY_\ell(L_1),$$
$$c = 2\pi\int_{-1}^1 G(x)P_\ell(x)\,dx,$$

where $\Omega(dL) = \sin\vartheta\, d\vartheta\, d\varphi$ is Lebesgue element of surface area on $\mathbb{S}_2$, $L_1 \cdot L_2 = \cos\vartheta$. In particular

$$\int_{\mathbb{S}_2} G(L_1 \cdot L)Y_\ell^m(L)\Omega(dL) = g_\ell Y_\ell^m(L_1),$$
$$g_\ell = 2\pi\int_{-1}^1 G(x)P_\ell(x)\,dx. \tag{A.4}$$

5. *Addition formula* (see Gradshteyn and Ryzhik (2000), 8.814, Erdélyi et al. (1981), 11.4(8)),

$$\sum_{m=-\ell}^{\ell} Y_\ell^{m*}(L_1) Y_\ell^m(L_2) = \frac{2\ell+1}{4\pi} P_\ell(\cos\vartheta), \tag{A.5}$$

where $\cos\vartheta = L_1 \cdot L_2$.

6. *Wigner 3j-symbols* (see Louck (2006)), notation

$$\begin{pmatrix} \ell_{1:3} \\ m_{1:3} \end{pmatrix} = \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \end{pmatrix}.$$

Selection rules: a Wigner 3j symbols vanishes unless

- $m_1 + m_2 + m_3 = 0$.
- Integer perimeter rule: $\mathcal{L} = \ell_1 + \ell_2 + \ell_3$ is an integer (if $m_1 = m_2 = m_3 = 0$, then $\mathcal{L}$ is even).
- Triangular inequality $|\ell_1 - \ell_2| \le \ell_3 \le \ell_1 + \ell_2$ is fulfilled.

Permutations

$$\begin{aligned} \begin{pmatrix} \ell_{1:3} \\ m_{1:3} \end{pmatrix} &= (-1)^{\ell_1+\ell_2+\ell_3} \begin{pmatrix} \ell_2 & \ell_1 & \ell_3 \\ m_2 & m_1 & m_3 \end{pmatrix} = \begin{pmatrix} \ell_2 & \ell_3 & \ell_1 \\ m_2 & m_3 & m_1 \end{pmatrix}, \\ \begin{pmatrix} \ell_{1:3} \\ m_{1:3} \end{pmatrix} &= (-1)^{\ell_1+\ell_2+\ell_3} \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix}. \end{aligned} \tag{A.6}$$

Orthogonality relation

$$(2\ell+1) \sum_{m_{1:2}} \begin{pmatrix} \ell_{1:2} & \ell \\ m_{1:2} & m \end{pmatrix} \begin{pmatrix} \ell_{1:2} & j \\ m_{1:2} & k \end{pmatrix} = \delta_{m,k} \delta_{\ell,j} \tag{A.7}$$

(see Edmonds (1957), (3.7.8)).

7. *Wigner D-functions* (see Varshalovich et al. (1988), Edmonds (1957)). If ℓ is fixed $D_{m,k}^{(\ell)}(g)$ is unitary

$$\sum_{k=-\ell}^{\ell} D_{m_1,k}^{(\ell)}(g) D_{m_2,k}^{(\ell)*}(g) = \delta_{m_1,m_2} \tag{A.8}$$

(see Edmonds (1957), (4.3.3)), also

$$\sum_{m_1,m_2,m_3} D_{m_1,k_1}^{(\ell_1)} D_{m_2,k_2}^{(\ell_2)} D_{m_3,k_3}^{(\ell_3)} \begin{pmatrix} \ell_{1:3} \\ m_{1:3} \end{pmatrix} = \begin{pmatrix} \ell_{1:3} \\ k_{1:3} \end{pmatrix}. \tag{A.9}$$

The Wigner matrix of rotations is $D^{(\ell)} = (D_{k,m}^{(\ell)})_{k,m=-\ell}^{\ell}$. Singly coupled form

$$D_{m_1,k_1}^{(\ell_1)} D_{m_2,k_2}^{(\ell_2)} = \sum_{\ell,m,k} (2\ell+1) \begin{pmatrix} \ell_{1:2} & \ell \\ m_{1:2} & m \end{pmatrix} D_{m,k}^{(\ell)*} \begin{pmatrix} \ell_{1:2} & \ell \\ k_{1:2} & k \end{pmatrix}, \tag{A.10}$$

$-m = m_1 + m_2$, $-k = k_1 + k_2$ (see Edmonds (1957), (4.3.4)). One can prove the following generalization of (A.9) using induction.

Lemma A.1. *Define* $\ell^0 = \ell_1$, $\ell^{p-2} = \ell_p$, $k^0 = -k_1$, $k^{p-2} = k_p$, $m^0 = -m_1$, $m^{p-2} = m_p$, *then for* $p > 3$ *and for any* $\ell^{1:p-3} = (\ell^1, \ell^2, \dots, \ell^{p-3})$ *we have*

$$\begin{aligned}
&\mathcal{H}_{k_{1:p},m_{1:p}}^{\ell_{1:p}}(\ell^{1:p-3}) \\
&\quad = \sum_{m_{1:p}} \prod_{a=1}^{p} D_{k_a,m_a}^{(\ell_a)} (-1)^{\Sigma m^{1:p-3}} \prod_{a=0}^{p-3} \begin{pmatrix} \ell^a & \ell_{a+2} & \ell^{a+1} \\ -m^a & m_{a+2} & m^{a+1} \end{pmatrix} \\
&\quad = (-1)^{\Sigma k^{1:p-3}} \prod_{a=0}^{p-3} \begin{pmatrix} \ell^a & \ell_{a+2} & \ell^{a+1} \\ -k^a & k_{a+2} & k^{a+1} \end{pmatrix},
\end{aligned}$$

where $\Sigma m^{1:p-3}$ *denotes the sum of entries of the vector* $m^{1:p-3} = (m^1, m^2, \dots, m^{p-3})$.

The integral

$$\int_{\mathrm{SO}(3)} D_{m,k}^{(\ell)}\, dg = \delta_{\ell,0}\delta_{m,0}\delta_{k,0}. \tag{A.11}$$

The Gaunt type integrals

$$\begin{aligned}
\mathcal{G}_{k_1,k_2;m_1,m_2,}^{\ell_1,\ell_2} &= \int_{\mathrm{SO}(3)} D_{m_1,k_1}^{(\ell_1)*} D_{m_2,k_2}^{(\ell_2)}\, dg \\
&= \delta_{\ell_1,\ell_2}\delta_{m_1,m_2}\delta_{k_1,k_2}\frac{1}{2\ell_1+1},
\end{aligned} \tag{A.12}$$

$$\begin{aligned}
\mathcal{G}_{k_1,k_2,k_3;m_1,m_2,m_3}^{\ell_1,\ell_2,\ell_3} &= \int_{\mathrm{SO}(3)} D_{m_1,k_1}^{(\ell_1)} D_{m_2,k_2}^{(\ell_2)} D_{m_3,k_3}^{(\ell_3)}\, dg \\
&= \begin{pmatrix} \ell_{1:3} \\ m_{1:3} \end{pmatrix} \begin{pmatrix} \ell_{1:3} \\ k_{1:3} \end{pmatrix},
\end{aligned} \tag{A.13}$$

where the Haar measure: $dg = \sin\vartheta\, d\vartheta\, d\varphi\, d\gamma / 8\pi^2$ (see Varshalovich et al. (1988), Edmonds (1957), (4.6.3)). Notation: $\mathcal{G}_{k_{1:p},m_{1:p}}^{\ell_{1:p}} = \mathcal{G}_{k_1,k_2,\dots,k_p,m_1,m_2,\dots,m_p}^{\ell_1,\ell_2,\dots,\ell_{3p}}$.

Lemma A.2. *Define* $\ell^0 = \ell_1$, $\ell^{p-2} = \ell_p$, $k^0 = -k_1$, $k^{p-2} = k_p$, $m^0 = -m_1$, $m^{p-2} = m_p$, *then for* $p > 3$

$$\begin{aligned}
\mathcal{G}_{k_{1:p},m_{1:p}}^{\ell_{1:p}} &= \int_{\mathrm{SO}(3)} \prod_{a=1}^{p} D_{k_a,m_a}^{(\ell_a)}\, dg \\
&= \sum_{\ell^{1:p-3},m^{1:p-3},k^{1:p-3}} (-1)^{\Sigma(m^{1:p-3}-k^{1:p-3})} \prod_{a=0}^{p-3} \begin{pmatrix} \ell^a & \ell_{a+2} & \ell^{a+1} \\ -k^a & k_{a+2} & k^{a+1} \end{pmatrix} \\
&\quad \times \prod_{a=0}^{p-3} \begin{pmatrix} \ell^a & \ell_{a+2} & \ell^{a+1} \\ -m^a & m_{a+2} & m^{a+1} \end{pmatrix} \prod_{a=1}^{p-3} (2\ell^a + 1).
\end{aligned}$$

8. *Condon and Shortley phase convention* (see Edmonds (1957), (4.3.3)),

$$\begin{aligned} Y_\ell^m(\vartheta,\varphi) &= \sqrt{\frac{2\ell+1}{4\pi}} D_{0,-m}^{(\ell)}(\gamma,\vartheta,\varphi) \\ &= \sqrt{\frac{2\ell+1}{4\pi}} D_{m,0}^{(\ell)*}(\varphi,\vartheta,\gamma), \end{aligned} \tag{A.14}$$

where γ is arbitrary angle. This form is referred to as passive convention as well (see Morrison and Parker (1987)).

A.2 Proofs: Bispectrum

We use the following notations $Z_{\ell_{1:3}}^{m_{1:3}} = (Z_{\ell_1}^{m_1}, Z_{\ell_2}^{m_2}, Z_{\ell_3}^{m_3})$, $B_3(\ell_{1:3}) = B_3(\ell_1, \ell_2, \ell_3)$.

Proof of Lemma 3.1. Let

$$\operatorname{Cum}_3(Z_{\ell_{1:3}}^{m_{1:3}}) = \begin{pmatrix} \ell_{1:3} \\ m_{1:3} \end{pmatrix} B_3(\ell_{1:3}),$$

then

$$\begin{aligned} \operatorname{Cum}_3(\mathcal{Z}_{\ell_{1:3}}^{k_{1:3}}) &= \sum_{m_1,m_2,m_3} D_{k_1,m_1}^{(\ell_1)} D_{k_2,m_2}^{(\ell_2)} D_{m_3,k_3}^{(\ell_3)} \operatorname{Cum}_3(Z_{\ell_{1:3}}^{m_{1:3}}) \\ &= \sum_{m_{1:3}} D_{k_1,m_1}^{(\ell_1)} D_{k_2,m_2}^{(\ell_2)} D_{m_3,k_3}^{(\ell_3)} \begin{pmatrix} \ell_{1:3} \\ m_{1:3} \end{pmatrix} B_3(\ell_{1:3}) \\ &= \begin{pmatrix} \ell_{1:3} \\ k_{1:3} \end{pmatrix} B_3(\ell_{1:3}) = \operatorname{Cum}_3(Z_{\ell_1}^{k_1}, Z_{\ell_2}^{k_2}, Z_{\ell_3}^{k_3}), \end{aligned} \tag{A.15}$$

where $\mathcal{Z}_{\ell_{1:3}}^{k_{1:3}} = (\mathcal{Z}_{\ell_1}^{k_1}, \mathcal{Z}_{\ell_2}^{k_2}, \mathcal{Z}_{\ell_3}^{k_3})$ see (A.9). Hence, the assumption of isotropy is satisfied.

If (3.1) is not assumed then under the assumption of isotropy we have $\operatorname{Cum}_3(\mathcal{Z}_{\ell_{1:3}}^{k_{1:3}}) = \operatorname{Cum}_3(Z_{\ell_{1:3}}^{k_{1:3}})$, and integrate both sides of (A.15) (see (A.13)),

$$\operatorname{Cum}_3(Z_{\ell_{1:3}}^{k_{1:3}}) = \begin{pmatrix} \ell_{1:3} \\ k_{1:3} \end{pmatrix} \sum_{m_{1:3}} \begin{pmatrix} \ell_{1:3} \\ m_{1:3} \end{pmatrix} \operatorname{Cum}_3(Z_{\ell_{1:3}}^{m_{1:3}}) = \begin{pmatrix} \ell_{1:3} \\ k_{1:3} \end{pmatrix} B_3(\ell_{1:3}),$$

where

$$B_3(\ell_{1:3}) = \sum_{m_{1:3}} \begin{pmatrix} \ell_{1:3} \\ m_{1:3} \end{pmatrix} \operatorname{Cum}_3(Z_{\ell_{1:3}}^{m_{1:3}}).$$

Hence, (3.1) is a necessary and sufficient assumption for the third order isotropy. In this case, the bispectrum is a linear combination of the cumulants of the angular projections by the probability amplitude of coupling three angular momenta $\ell_{1:3}$. □

Proof of 3-product of spherical harmonics is rotation invariant. Indeed

$$I_{\ell_{1:3}}(gL_{1:3}) = \sum_{m_{1:3}} \binom{\ell_{1:3}}{m_{1:3}} \sum_{k_{1:3}} D^{(\ell_1)}_{k_1,m_1} D^{(\ell_2)}_{k_2,m_2} D^{(\ell_3)}_{m_3,k_3} Y^{k_1}_{\ell_1}(L_1) Y^{k_2}_{\ell_2}(L_2) Y^{k_3}_{\ell_3}(L_3)$$

$$= \sum_{k_{1:3}} \binom{\ell_{1:3}}{k_{1:3}} Y^{k_1}_{\ell_1}(L_1) Y^{k_2}_{\ell_2}(L_2) Y^{k_3}_{\ell_3}(L_3) = I_{\ell_{1:3}}(L_{1:3}),$$

see (A.9). □

A.3 Proofs: Trispectrum

Repeat the notation $\operatorname{Cum}_4(X(L_1), X(L_2), X(L_3), X(L_4)) = \operatorname{Cum}_4(X(L_{1:4}))$.

Proof of Lemma 4.1. We have

$$\operatorname{Cum}_4(\mathcal{Z}^{m_{1:4}}_{\ell_{1:4}}) = \sum_{k_{1:4}} D^{(\ell_1)}_{m_1,k_1} D^{(\ell_2)}_{m_2,k_2} D^{(\ell_3)}_{m_3,k_3} D^{(\ell_4)}_{m_4,k_4} \operatorname{Cum}_4(Z^{k_{1:4}}_{\ell_{1:4}}).$$

Under the assumption of isotropy $\operatorname{Cum}_4(\mathcal{Z}^{m_{1:4}}_{\ell_{1:4}}) = \operatorname{Cum}_4(Z^{m_{1:4}}_{\ell_{1:4}})$, now integrate both sides by the Haar measure, see Lemma A.2 for $p = 4$, and obtain

$$\operatorname{Cum}_4(Z^{m_{1:4}}_{\ell_{1:4}}) = \sum_{k_{1:4}} \sum_{\ell^1,k^1,m^1} \begin{pmatrix} \ell_{1:2} & \ell^1 \\ m_{1:2} & -m^1 \end{pmatrix} \begin{pmatrix} \ell^1 & \ell_{3:4} \\ m^1 & m_{3:4} \end{pmatrix} (-1)^{m^1-k^1}$$
$$\times (2\ell^1+1) \begin{pmatrix} \ell_{1:2} & \ell^1 \\ k_{1:2} & -k^1 \end{pmatrix} \begin{pmatrix} \ell^1 & \ell_{3:4} \\ k^1 & k_{3:4} \end{pmatrix} \operatorname{Cum}_4(Z^{k_{1:4}}_{\ell_{1:4}}).$$

Note that each term according to summation $k_{1:4}$ is *symmetric* in $\ell_{1:4}$. Now, define

$$T_4(\ell_{1:4}|\ell^1) = \sqrt{2\ell^1+1} \sum_{k^1,k_{1:4}} (-1)^{k^1} \begin{pmatrix} \ell_{1:2} & \ell^1 \\ k_{1:2} & -k^1 \end{pmatrix} \begin{pmatrix} \ell^1 & \ell_{3:4} \\ k^1 & k_{3:4} \end{pmatrix} \operatorname{Cum}_4(Z^{k_{1:4}}_{\ell_{1:4}}),$$

with this notation we have the cumulant in the form

$$\operatorname{Cum}_4(Z^{m_{1:4}}_{\ell_{1:4}}) = \sum_{\ell^1,m^1} \sqrt{2\ell^1+1} \begin{pmatrix} \ell_{1:2} & \ell^1 \\ m_{1:2} & -m^1 \end{pmatrix} \begin{pmatrix} \ell^1 & \ell_{3:4} \\ m^1 & m_{3:4} \end{pmatrix} (-1)^{m^1} T_4(\ell_{1:4}|\ell^1).$$

If instead of isotropy (4.1) is assumed, then

$$\operatorname{Cum}_4(\mathcal{Z}^{m_{1:4}}_{\ell_{1:4}})$$
$$= \sum_{k_{1:4}} D^{(\ell_1)}_{m_1,k_1} D^{(\ell_2)}_{m_2,k_2} D^{(\ell_3)}_{m_3,k_3} D^{(\ell_4)}_{m_4,k_4} \operatorname{Cum}_4(Z^{k_{1:4}}_{\ell_{1:4}})$$
$$= \sum_{k_{1:4}} D^{(\ell_1)}_{m_1,k_1} D^{(\ell_2)}_{m_2,k_2} D^{(\ell_3)}_{m_3,k_3} D^{(\ell_4)}_{m_4,k_4}$$

$$\times \sum_{\ell} \begin{pmatrix} \ell_{1:2} & \ell \\ k_{1:2} & -k \end{pmatrix} \begin{pmatrix} \ell & \ell_{3:4} \\ k & k_{3:4} \end{pmatrix} (-1)^k \sqrt{2\ell+1} T_4(\ell_{1:4}|\ell)$$

$$= \sum_{\ell} T_4(\ell_{1:4}|\ell)\sqrt{2\ell+1}$$

$$\times \sum_{k_{1:4}} D^{(\ell_1)}_{m_1,k_1} D^{(\ell_2)}_{m_2,k_2} D^{(\ell_3)}_{m_3,k_3} D^{(\ell_4)}_{m_4,k_4} (-1)^k \begin{pmatrix} \ell_{1:2} & \ell \\ k_{1:2} & -k \end{pmatrix} \begin{pmatrix} \ell & \ell_{3:4} \\ k & k_{3:4} \end{pmatrix},$$

the Lemma A.1 can be applied for $p = 4$, and we get

$$\operatorname{Cum}_4(\mathcal{Z}^{m_{1:4}}_{\ell_{1:4}}) = \sum_{m,\ell} \begin{pmatrix} \ell_{1:2} & \ell \\ m_{1:2} & -m \end{pmatrix} \begin{pmatrix} \ell & \ell_{3:4} \\ m & m_{3:4} \end{pmatrix} (-1)^m \sqrt{2\ell+1} T_4(\ell_{1:4}|\ell)$$

$$= \operatorname{Cum}_4(Z^{m_{1:4}}_{\ell_{1:4}}). \qquad \square$$

A.4 Proofs: Polyspectrum

Proof of Lemma 5.1. We repeat the proof given for the trispectrum above. We have that

$$\operatorname{Cum}_p(\mathcal{Z}^{m_{1:p}}_{\ell_{1:p}}) = \sum_{k_{1:p}} \prod_{a=1}^{p} D^{(\ell_a)}_{m_a,k_a} \operatorname{Cum}_p(Z^{k_{1:p}}_{\ell_{1:p}}).$$

Under isotropy assumption

$$\operatorname{Cum}_p(Z^{m_{1:p}}_{\ell_{1:p}})$$

$$= \sum_{k_{1:p}} \int_{SO(3)} \prod_{a=1}^{p} D^{(\ell_a)}_{m_a,k_a} \, dg \operatorname{Cum}_p(Z^{k_{1:p}}_{\ell_{1:p}})$$

$$= \sum_{\ell^{1:p-3},k^{1:p-3},m^{1:p-3}} (-1)^{\Sigma m^{1:p-3}} \prod_{a=0}^{p-3} \begin{pmatrix} \ell^a & \ell_{a+2} & \ell^{a+1} \\ -m^a & m_{a+2} & m^{a+1} \end{pmatrix}$$

$$\times \sum_{k_{1:p}} (-1)^{\Sigma k^{1:p-3}} \prod_{a=0}^{p-3} \begin{pmatrix} \ell^a & \ell_{a+2} & \ell^{a+1} \\ -k^a & k_{a+2} & k^{a+1} \end{pmatrix} (2\ell^{a+1}+1) \operatorname{Cum}_p(Z^{k_{1:p}}_{\ell_{1:p}})$$

$$= \sum_{\ell^{1:p-3},k^{1:p-3},m^{1:p-3}} (-1)^{\Sigma m^{1:p-3}}$$

$$\times \prod_{a=0}^{p-3} \sqrt{2\ell^a+1} \begin{pmatrix} \ell^a & \ell_{a+2} & \ell^{a+1} \\ -m^a & m_{a+2} & m^{a+1} \end{pmatrix} \widetilde{S}_p(\ell_{1:p}|\ell^{1:p-3}).$$

If isotropy is not assumed, then from

$$\operatorname{Cum}_p(Z_{\ell_{1:p}}^{k_{1:p}}) = \sum_{\ell^{1:p-3},k^{1:p-3}} (-1)^{\Sigma k^{1:p-3}} \prod_{a=0}^{p-3} \begin{pmatrix} \ell^a & \ell_{a+2} & \ell^{a+1} \\ -k^a & k_{a+2} & k^{a+1} \end{pmatrix} \times \prod_{a=1}^{p-3} \sqrt{2\ell^a+1}\widetilde{S}_p(\ell_{1:p}|\ell^{1:p-3}),$$

we obtain

$$\begin{aligned}\operatorname{Cum}_p(\mathcal{Z}_{\ell_{1:p}}^{m_{1:p}}) &= \sum_{k_{1:p}} \prod_{a=1}^{p} D_{m_a,k_a}^{(\ell_a)} (-1)^{\Sigma k^{1:p-3}} \prod_{a=0}^{p-3} \begin{pmatrix} \ell^a & \ell_{a+2} & \ell^{a+1} \\ -k^a & k_{a+2} & k^{a+1} \end{pmatrix} \\ &\quad \times \prod_{a=1}^{p-3} \sqrt{2\ell^a+1}\widetilde{S}_p(\ell_{1:p}|\ell^{1:p-3}) \\ &= (-1)^{\Sigma m^{1:p-3}} \prod_{a=0}^{p-3} \begin{pmatrix} \ell^a & \ell_{a+2} & \ell^{a+1} \\ -m^a & m_{a+2} & m^{a+1} \end{pmatrix} \\ &\quad \times \prod_{a=1}^{p-3} \sqrt{2\ell^a+1}\widetilde{S}_p(\ell_{1:p}|\ell^{1:p-3}),\end{aligned}$$

see Lemma A.1. □

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