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Estimates of the PDF and the CDF of the exponentiated Weibull distribution

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Abstract. Exponentiated Weibull distribution, introduced as an extension of the Weibull distribution, is characterized by bathtub shaped, unimodal failure rates besides a broader class of monotone failure rates. In this paper, we derive maximum likelihood estimators (MLEs), uniformly minimum variance unbiased estimators and three other estimators of the probability density function and the cumulative distribution function of the exponentiated Weibull distribution and compare their performances through numerical simulations. Simulation studies show that the MLE is more efficient than the others. Analysis of a real data set is presented for illustrative purposes.

1 Introduction

The exponentiated Weibull (EW) distribution introduced by Mudholkar and Srivastava (1993) as an extension of the Weibull distribution, is characterized by bathtub shaped, unimodal failure rates besides a broader class of monotone failure rates. The applications of the EW distribution in reliability and survival studies were illustrated by Mudholkar et al. (1995). Its properties were studied in detail by Mudholkar and Hutson (1996) and Nassar and Eissa (2003, 2004). These authors presented useful applications of the distribution in the modeling of flood data and in reliability. Practically, the EW distribution is more realistic than distributions exhibiting monotone failure rates.

Some researchers pointed out the use of the EW distribution in reliability estimation. Singh et al. (2002, 2005a, 2005b, 2006) obtained Bayes estimators of the parameters, reliability function and hazard function for EW distributed type II censored data under squared error and LINEX loss functions. Jaheen and Al Harbi (2011) discussed Bayesian estimation based on dual generalized order statistics from the EW distribution. Ashour and Afify (2007) analyzed EW distributed lifetime data observed under type I progressive interval censoring with random removals. They derived MLEs of the parameters and their asymptotic variances.

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Ashour and Afify (2008) derived MLEs of the parameters of the EW distribution and their asymptotic variances for type II progressive interval censoring with random removals. Kim et al. (2011) derived ML and Bayes estimators for the EW distribution using symmetric and asymmetric loss functions.

A random variable X is said to have a three-parameter EW distribution if its cumulative distribution function (CDF) and probability density function (PDF) are

$$F(x) = \left(1 - e^{-x^\beta/\theta^\beta}\right)^\alpha \tag{1.1}$$

and

$$f(x) = \alpha\beta \frac{x^{\beta-1}}{\theta^\beta} e^{-x^\beta/\theta^\beta} \left(1 - e^{-x^\beta/\theta^\beta}\right)^{\alpha-1}, \tag{1.2}$$

respectively, for $x > 0$, $\alpha > 0$, $\beta > 0$ and $\theta > 0$, where α and β are the shape parameters and θ is the scale parameter.

The recent applications of the EW distribution have been widespread. We mention: model for carbon fibrous composites (Surles and D'Ambrosio, 2004); modeling tree diameters of Chinese fir plantations located at Kaihua forestry farm, Zhejiang province, southeastern China (Wang and Rennolls, 2005); models for firmware system failure (Zhang et al., 2005); estimation of the number of ozone peaks in Mexico city (Achcar et al., 2009); mean residual life computation of $(n - k + 1)$-out-of-n systems for the case of independent but not necessarily identically distributed lifetimes of the components (Gurler and Capar, 2011).

Because of the numerous applications of the EW distribution, we feel the importance to investigate efficient estimation of its PDF and CDF. We consider several different estimation methods: uniformly minimum variance unbiased (UMVU) estimation, ML estimation, percentile (PC) estimation, least squares (LS) estimation and weighted least squares (WLS) estimation.

We have chosen these estimation methods because they are some of the most popular ones. In particular, ML estimation is the most widely used method. There are of course many other methods that can be used; for example, method of moments, generalized method of moments, Bayesian estimation, bootstrapping, jackknifing, empirical likelihood method and so on. We hope to consider some of these methods in a future work.

Studies similar to this paper have appeared in the recent literature for other distributions. For example, Dixit and Jabbari Nooghabi (2011) investigated efficient estimation of the PDF, the CDF and the rth moment of the exponentiated Pareto distribution in the presence of outliers. Bagheri et al. (2014) investigated efficient estimation of the PDF and the CDF of the generalized exponential-Poisson distribution. The former considered the following estimators: UMVUE, MLE, quantile estimator and LSE. They showed that the MLE is more efficient than the UMVUE and the UMVUE is more efficient than others.

The contents of this paper are organized as follows. The MLE and the UMVUE of the PDF and the CDF and their mean squared errors (MSEs) are derived in

Sections 2 and 3. Other estimation methods are considered in Sections 4 and 5. The estimators are compared by simulation and a real data application in Sections 6 and 7. Throughout the paper (except for Section 7), we assume α is unknown, but both β and θ are known. A future work is to extend the results of the paper to the case that all three parameters are unknown.

To the best of our knowledge, no one has considered estimation of the PDF and the CDF of the EW distribution. The results presented here in Sections 2 to 5 are all new.

2 MLEs of the PDF and the CDF

Let $X_1, X_2, \ldots, X_n$ denote a random sample from the EW distribution given by (1.1) and (1.2). The MLE of α is

$$\widetilde{\alpha} = -\frac{n}{\sum_{i=1}^{n} \log[1 - e^{-x_i^\beta/\theta^\beta}]}. \tag{2.1}$$

By the invariance property of MLEs, we can easily obtain the MLEs of the PDF and the CDF as

$$\widetilde{f}(x) = \widetilde{\alpha}\beta \frac{x^{\beta-1}}{\theta^\beta} e^{-x^\beta/\theta^\beta} \left[1 - e^{-x^\beta/\theta^\beta}\right]^{\widetilde{\alpha}-1} \tag{2.2}$$

and

$$\widetilde{F}(x) = \left(1 - e^{-x^\beta/\theta^\beta}\right)^{\widetilde{\alpha}} \tag{2.3}$$

for $x > 0$ provided that $\theta > 0$ and $\beta > 0$ are known. Let $T = -\sum_{i=1}^{n} \log(1 - e^{-x_i^\beta/\theta^\beta})$. We can write

$$T = -\frac{1}{\alpha} \sum_{i=1}^{n} \log F(x_i) \stackrel{\mathrm{d}}{=} -\frac{1}{\alpha} \sum_{i=1}^{n} \log U_i \stackrel{\mathrm{d}}{=} \sum_{i=1}^{n} E_i,$$

where U_i are independent uniform[0, 1] random variables, E_i are independent exponential random variables with scale parameter α and "$\stackrel{\mathrm{d}}{=}$" denotes equality in distribution. It is well known that the sum of independent exponential random variables is a gamma random variable: in fact, the moment generating function of T is

$$\mathrm{E}[\exp(sT)] = \mathrm{E}\left[\exp\left(s \sum_{i=1}^{n} E_i\right)\right] = \prod_{i=1}^{n} [\exp(sE_i)] = \prod_{i=1}^{n} \frac{\alpha}{\alpha - s} = \frac{\alpha^n}{(\alpha - s)^n},$$

the moment generating function of a gamma random variable with shape parameter n and scale parameter α. So,

$$f_T(t) = \frac{\alpha^n}{\Gamma(n)} t^{n-1} e^{-\alpha t} \tag{2.4}$$

for $t > 0$ and $\alpha > 0$. By using the transformation $\widetilde{\alpha} = W = \frac{n}{T}$, we obtain

$$f_W(w) = \frac{n^n \alpha^n}{\Gamma(n)} \frac{e^{-n\alpha/w}}{w^{n+1}} \tag{2.5}$$

for $w > 0$ and $\alpha > 0$.

Theorem 2.1 calculates $\mathrm{E}(\widetilde{f}(x))^r$ and $\mathrm{E}(\widetilde{F}(x))^r$.

Theorem 2.1. *We have*

$$\begin{aligned} \mathrm{E}(\widetilde{f}(x))^r = {} & \frac{2k^r (n\alpha)^{(r+n)/2}}{\Gamma(n)} \left(-\frac{1}{r\log(1-b)}\right)^{(r-n)/2} \\ & \times K_{r-n}\left(2\sqrt{-rn\alpha\log(1-b)}\right) \end{aligned} \tag{2.6}$$

and

$$\mathrm{E}(\widetilde{F}(x))^r = \frac{2(n\alpha)^{n/2}}{\Gamma(n)} \left(-\frac{1}{r\log(1-b)}\right)^{-n/2} K_{-n}\left(2\sqrt{-rn\alpha\log(1-b)}\right), \tag{2.7}$$

where $k = \beta \frac{x^{\beta-1}}{\theta^\beta} \frac{e^{-x^\beta/\theta^\beta}}{1-e^{-x^\beta/\theta^\beta}}$, $b = e^{-x^\beta/\theta^\beta}$, $\widetilde{\alpha} = w$ *and* $K_\nu(\cdot)$ *denotes the modified Bessel function of the second kind of order* ν.

Proof. Given the assumptions, we have $\widetilde{f}(x) = kw(1-b)^w$ and $\widetilde{F}(x) = (1-b)^w$ for $w > 0$. Therefore, by equation (2.5), we can write

$$\begin{aligned} \mathrm{E}(\widetilde{f}(x))^r &= \int_0^\infty k^r w^r (1-b)^{rw} \frac{n^n\alpha^n}{\Gamma(n)} \frac{e^{-n\alpha/w}}{w^{n+1}}\, dw \\ &= \frac{k^r n^n \alpha^n}{\Gamma(n)} \int_0^\infty w^r (1-b)^{rw} \frac{e^{-n\alpha/w}}{w^{n+1}}\, dw \\ &= \frac{k^r n^n \alpha^n}{\Gamma(n)} \int_0^\infty w^{r-n-1} e^{rw\log(1-b)} e^{-n\alpha/w}\, dw \\ &= \frac{2k^r (n\alpha)^{(r+n)/2}}{\Gamma(n)} \left(-\frac{1}{r\log(1-b)}\right)^{(r-n)/2} K_{r-n}\left(2\sqrt{-rn\alpha\log(1-b)}\right), \end{aligned}$$

where the last step follows by equation (3.471.9) in Gradshteyn and Ryzhik (2000). So, (2.6) follows. Also, by equation (2.5), we can write

$$\begin{aligned} \mathrm{E}(\widetilde{F}(x))^r &= \int_0^\infty (1-b)^{rw} \frac{n^n\alpha^n}{\Gamma(n)} \frac{e^{-n\alpha/w}}{w^{n+1}}\, dw \\ &= \frac{n^n\alpha^n}{\Gamma(n)} \int_0^\infty w^{-n-1} e^{rw\log(1-b)-n\alpha/w}\, dw \\ &= \frac{2(n\alpha)^{n/2}}{\Gamma(n)} \left(-\frac{1}{r\log(1-b)}\right)^{-n/2} K_{-n}\left(2\sqrt{-rn\alpha\log(1-b)}\right), \end{aligned}$$

where the last step follows by equation (3.471.9) in Gradshteyn and Ryzhik (2000). So, (2.7) follows. □

By using Theorem 2.1, we obtain the MSEs of $\widetilde{f}(x)$ and $\widetilde{F}(x)$.

Theorem 2.2. *The MSEs of* $\widetilde{f}(x)$ *and* $\widetilde{F}(x)$ *are*

$$\begin{aligned}\mathrm{MSE}\big(\widetilde{f}(x)\big) &= \frac{2k^2(n\alpha)^{(2+n)/2}}{\Gamma(n)}\left(-\frac{1}{2\log(1-b)}\right)^{(2-n)/2} K_{2-n}\big(2\sqrt{-2n\alpha\log(1-b)}\big) \\ &\quad - \frac{4kf(x)(n\alpha)^{(1+n)/2}}{\Gamma(n)}\left(-\frac{1}{\log(1-b)}\right)^{(1-n)/2} K_{1-n}\big(2\sqrt{-n\alpha\log(1-b)}\big) \\ &\quad + f^2(x)\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}\mathrm{MSE}\big(\widetilde{F}(x)\big) &= \frac{2(n\alpha)^{n/2}}{\Gamma(n)}\left(-\frac{1}{2\log(1-b)}\right)^{-n/2} K_{-n}\big(2\sqrt{-2n\alpha\log(1-b)}\big) \\ &\quad - \frac{4F(x)(n\alpha)^{n/2}}{\Gamma(n)}\left(-\frac{1}{\log(1-b)}\right)^{-n/2} K_{-n}\big(2\sqrt{-n\alpha\log(1-b)}\big) \\ &\quad + F^2(x),\end{aligned} \tag{2.9}$$

respectively.

Proof. One can easily find $\mathrm{E}(\widetilde{f}(x))$ and $\mathrm{E}(\widetilde{f}(x))^2$ by setting $r = 1$ and $r = 2$ in Theorem 2.1. Then by using $\mathrm{MSE}(\widetilde{f}(x)) = \mathrm{E}(\widetilde{f}(x))^2 - 2f(x)\mathrm{E}(\widetilde{f}(x)) + f^2(x)$, we obtain (2.8). The proof of (2.10) is similar. □

3 UMVUEs of the PDF and the CDF

In this section, we find the UMVUEs of the PDF and the CDF of the EW distribution. We also compute the MSEs of these estimators.

Let $X_1, X_2, \ldots, X_n$ be a random sample from the EW distribution given by (1.2). Then

$$T = -\sum_{i=1}^{n} \log\big(1 - e^{-x_i^\beta/\theta^\beta}\big) \tag{3.1}$$

is a complete sufficient statistic for the unknown parameter α (when β and θ are known). According to the Lehmann–Scheffe theorem, if $f_{X_1|T}(x_1|t) = f^*(t)$ say is the conditional PDF of X_1 given T, we have

$$\mathrm{E}\big[f^*(T)\big] = \int f_{X_1|T}(x_1|t) f_T(t)\, dt = \int f_{X_1,T}(x_1,t)\, dt = f_{X_1}(x_1),$$

where $f_{X_1,T}(x_1,t)$ denotes the joint PDF of X_1 and T. Therefore, $f^*(t)$ is the UMVUE of $f(x)$.

Lemma 3.1. *The joint PDF of X_1 and T can be expressed as*

$$f_{X_1,T}(x_1,t) = \frac{\alpha^n \beta x_1^{\beta-1}}{\theta^\beta \Gamma(n-1)} e^{-x_1^\beta/\theta^\beta}\big(1-e^{-x_1^\beta/\theta^\beta}\big)^{\alpha-1}\big[t+\log\big(1-e^{-x_1^\beta/\theta^\beta}\big)\big]^{n-2} \times e^{-\alpha[t+(1-e^{-x_1^\beta/\theta^\beta})]}.$$

Proof. The joint PDF of $(X_1, X_2, \ldots, X_n)$ is

$$\begin{aligned} f_{X_1,X_2,\ldots,X_n}(x_1,x_2,\ldots,x_n) &= \prod_{i=1}^n f(x_i) \\ &= \prod_{i=1}^n \Big[\alpha\beta \frac{x_i^{\beta-1}}{\theta^\beta} e^{-x_i^\beta/\theta^\beta}\big(1-e^{-x_i^\beta/\theta^\beta}\big)^{\alpha-1}\Big]. \end{aligned}$$

In order to find the joint PDF of (X_1, T), we set

$$\left\{\begin{aligned} Y_1 &= \log\big(1-e^{-X_1^\beta/\theta^\beta}\big), \\ Y_2 &= \log\big(1-e^{-X_2^\beta/\theta^\beta}\big), \\ &\vdots \\ Y_{n-1} &= \log\big(1-e^{-X_{n-1}^\beta/\theta^\beta}\big), \\ T &= -\sum_{i=1}^n \log\big(1-e^{-X_i^\beta/\theta^\beta}\big) \end{aligned}\right\}.$$

We obtain the result by integrating the joint pdf of $(Y_1, Y_2, \ldots, Y_{n-1}, T)$ with respect to $y_2, y_3, \ldots, y_{n-1}$. □

Theorem 3.2. *Given $T = t$,*

$$\begin{aligned} \widehat{f}(x) &= (n-1)\beta \frac{x^{\beta-1}}{\theta^\beta} \frac{e^{-x^\beta/\theta^\beta}}{1-e^{-x^\beta/\theta^\beta}} \frac{[t+\log(1-e^{-x^\beta/\theta^\beta})]^{n-2}}{t^{n-1}}, \\ & -\log\big(1-e^{-x^\beta/\theta^\beta}\big) < t < \infty \end{aligned} \tag{3.2}$$

is the UMVUE of $f(x)$ and

$$\widehat{F}(x) = \left[\frac{t + \log(1 - e^{-x^{\beta}/\theta^{\beta}})}{t}\right]^{n-1}, \quad -\log\left(1 - e^{-x^{\beta}/\theta^{\beta}}\right) < t < \infty \tag{3.3}$$

is the UMVUE of $F(x)$.

Proof. By using (2.4) and Lemma 3.1, we see immediately that (3.2) is the UMVUE of $f(x)$. That (3.4) is the UMVUE of $F(x)$ follows from the fact (3.4) is the integral of (3.2) or the more easily verified fact that the right-hand side of (3.2) is the derivative of the right-hand side of (3.4). □

Theorem 3.2 calculates the MSEs of $\widehat{f}(x)$ and $\widehat{F}(x)$.

Theorem 3.3. *The MSEs of $\widehat{f}(x)$ and $\widehat{F}(x)$ are*

$$\mathrm{MSE}\left(\widehat{f}(x)\right) = D^2 \frac{\alpha^n}{\Gamma(n)} \sum_{j=0}^{2n-4} \binom{2n-4}{j} b_1^j \Gamma(n-j-2, -b\alpha) - f^2(x)$$

and

$$\mathrm{MSE}\left(\widehat{F}(x)\right) = \frac{\alpha^n}{\Gamma(n)} \sum_{j=0}^{2n-2} \binom{2n-2}{j} (\log b_1)^j \Gamma(n-j, -b\alpha) - F^2(x),$$

respectively, where $D = (n-1)\beta \frac{x^{\beta-1}}{\theta^{\beta}} \frac{e^{-x^{\beta}/\theta^{\beta}}}{1-e^{-x^{\beta}/\theta^{\beta}}}$, $b_1 = 1 - e^{-x^{\beta}/\theta^{\beta}}$ and $\Gamma(s,x) = \int_x^{\infty} t^{s-1} e^{-t}\,dt$ denotes the complementary incomplete gamma function.

Proof. Given the assumptions, we have $\widehat{f}(x) = \frac{D(t+b_1)^{n-2}}{t^{n-1}}$, $-b_1 < t < \infty$. So, we can write

$$\begin{aligned}
\mathrm{E}\left(\widehat{f}(x)\right)^2 &= \int \left(\widehat{f}(x)\right)^2 f_T(t)\,dt \\
&= \int_{-b_1}^{\infty} \frac{D^2 (t+b_1)^{2n-4}}{t^{2n-2}} f_T(t)\,dt \\
&= D^2 \frac{\alpha^n}{\Gamma(n)} \int_{-b_1}^{\infty} \frac{(t+b_1)^{2n-4}}{t^{2n-2}} t^{n-1} e^{-\alpha t}\,dt \\
&= D^2 \frac{\alpha^n}{\Gamma(n)} \int_{-b_1}^{\infty} \left(\frac{t+b_1}{t}\right)^{2n-4} t^{n-3} e^{-\alpha t}\,dt \\
&= D^2 \frac{\alpha^n}{\Gamma(n)} \int_{-b_1}^{\infty} \left(1 + \frac{b_1}{t}\right)^{2n-4} t^{n-3} e^{-\alpha t}\,dt
\end{aligned}$$

$$= D^2 \frac{\alpha^n}{\Gamma(n)} \int_{-b_1}^{\infty} \left[\sum_{j=0}^{2n-4} \binom{2n-4}{j} \left(\frac{b_1}{t}\right)^j \right] t^{n-3} e^{-\alpha t}\, dt$$

$$= D^2 \frac{\alpha^n}{\Gamma(n)} \sum_{j=0}^{2n-4} \binom{2n-4}{j} b_1^j \int_{-b}^{\infty} t^{n-j-3} e^{-\alpha t}\, dt$$

$$= D^2 \frac{\alpha^n}{\Gamma(n)} \sum_{j=0}^{2n-4} \binom{2n-4}{j} b_1^j \Gamma(n-j-2, -b\alpha),$$

where the last step follows by the definition of the complementary incomplete gamma function. The expression for the MSE for $\widehat{f}(x)$ follows by $\mathrm{MSE}(\widehat{f}(x)) = \mathrm{E}(\widehat{f}(x))^2 - f^2(x)$. The proof for the expression for the MSE for $\widehat{F}(x)$ is similar. □

In the following section, we present the PC method of estimation.

4 Estimators based on percentiles

Estimation based on percentiles was originally explored by Kao (1958, 1959); see also Mann et al. (1974) and Johnson et al. (1994). PCEs are based on the CDF. Since the EW distribution has a closed form CDF, PCEs are suited for this distribution.

Let $X_1, X_2, \dots, X_n$ denote a random sample from the EW distribution. Let $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ denote the corresponding order statistics in the ascending order. Then the PCE of α (when β and θ are known) say $\widetilde{\alpha}_{\mathrm{pc}}$ is obtained by minimizing

$$\sum_{i=1}^{n} \left[1 - p_i^{1/\alpha} - e^{-x_{(i)}^{\beta}/\theta^{\beta}}\right]^2, \tag{4.1}$$

where $p_i = \frac{i}{n+1}$. The PCEs of the PDF and the CDF are

$$\widetilde{f}_{\mathrm{pc}}(x) = \widetilde{\alpha}_{\mathrm{pc}} \beta \frac{x^{\beta-1}}{\theta^{\beta}} e^{-x^{\beta}/\theta^{\beta}} \left(1 - e^{-x^{\beta}/\theta^{\beta}}\right)^{\widetilde{\alpha}_{\mathrm{pc}}-1}, \tag{4.2}$$

$$\widetilde{F}_{\mathrm{pc}}(x) = \left(1 - e^{-x^{\beta}/\theta^{\beta}}\right)^{\widetilde{\alpha}_{\mathrm{pc}}}. \tag{4.3}$$

5 Least squares and weighted least squares estimators

In this section, we provide regression based estimators. This method was originally suggested by Swain et al. (1988) to estimate the parameters of beta distributions.

Suppose $X_1, X_2, \ldots, X_n$ is a random sample with common CDF $F(\cdot)$ and let $X_{(i)}, i = 1, 2, \ldots, n$ denote the corresponding order statistics in the ascending order. The proposed method uses the distribution of $F(X_{(i)})$ and the facts

$$\mathrm{E}\big[F(X_{(j)})\big] = \frac{j}{n+1},$$
$$\mathrm{Var}\big[F(X_{(j)})\big] = \frac{j(n-j-1)}{(n+1)^2(n+2)},$$

see Johnson et al. (1994, equation (12.20), page 9). Using the expectations and the variances, two variants of the least squares method follow.

5.1 Method 1: Least squares estimators

The LS estimators of the unknown parameter can be obtained by minimizing

$$\sum_{j=1}^{n}\left[F(X_{(j)}) - \frac{j}{n+1}\right]^2$$

with respect to the unknown parameter. For the EW distribution, the LSE of α (when β and θ are known) say $\widetilde{\alpha}_{\mathrm{ls}}$ can be obtained by minimizing

$$\sum_{j=1}^{n}\left[\big(1 - e^{-x_{(j)}^{\beta}/\theta^{\beta}}\big)^{\alpha} - \frac{j}{n+1}\right]^2. \tag{5.1}$$

So, the LS estimators of the PDF and the CDF are

$$\widetilde{f}_{\mathrm{ls}}(x) = \widetilde{\alpha}_{\mathrm{ls}}\beta\frac{x^{\beta-1}}{\theta^{\beta}}e^{-x^{\beta}/\theta^{\beta}}\big(1 - e^{-x^{\beta}/\theta^{\beta}}\big)^{\widetilde{\alpha}_{\mathrm{ls}}-1}, \tag{5.2}$$

$$\widetilde{F}_{\mathrm{ls}}(x) = \big(1 - e^{-x^{\beta}/\theta^{\beta}}\big)^{\widetilde{\alpha}_{\mathrm{ls}}}. \tag{5.3}$$

It is difficult to find the expectation and the MSE of these estimators analytically. We shall compute them by simulation.

5.2 Method 2: Weighted least squares estimators

The WLSE of the unknown parameter can be obtained by minimizing

$$\sum_{j=1}^{n} w_j\left[F(X_{(j)}) - \frac{j}{n+1}\right]^2$$

with respect to the unknown parameter, where $w_j = \frac{1}{\mathrm{Var}[F(X_{(j)})]} = \frac{(n+1)^2(n+2)}{j(n-j+1)}$. For the EW distribution, the WLS estimator of α (when β and θ are known) say $\widetilde{\alpha}_{\mathrm{wls}}$ can be obtained by minimizing

$$\sum_{j=1}^{n} w_j\left[\big(1 - e^{-x_{(j)}^{\beta}/\theta^{\beta}}\big)^{\alpha} - \frac{j}{n+1}\right]^2. \tag{5.4}$$

So, the WLS estimators of the PDF and the CDF are

$$\widetilde{f}_{\mathrm{wls}}(x) = \widetilde{\alpha}_{\mathrm{wls}}\beta\frac{x^{\beta-1}}{\theta^{\beta}}e^{-x^{\beta}/\theta^{\beta}}\left(1-e^{-x^{\beta}/\theta^{\beta}}\right)^{\widetilde{\alpha}_{\mathrm{wls}}-1}, \tag{5.5}$$

$$\widetilde{F}_{\mathrm{wls}}(x) = \left(1-e^{-x^{\beta}/\theta^{\beta}}\right)^{\widetilde{\alpha}_{\mathrm{wls}}}. \tag{5.6}$$

It is difficult to find the expectation and the MSE of these estimators by mathematical methods. We shall compute them by simulation.

6 Comparison of the UMVU, ML, PC, LS and WLS estimators

Here, we perform a simulation study to compare the performances of the following estimators: the MLE, the UMVUE, the PCE, the LSE and the WLSE of the PDF and the CDF. The comparison is based on the MSEs as follows:

1. Generate one thousand samples of size n from the EW distribution. The inversion method was used to generate samples, that is, variates of the EW distribution were generated as
$$X = \theta\left[-\log\left(1-U^{1/\alpha}\right)\right]^{1/\beta},$$
where $U \sim$ uniform(0, 1).
2. Compute the MLE $\widetilde{\alpha}$ in (2.1) and then $\widetilde{f}(1)$ and $\widetilde{F}(1)$ in (2.2) and (2.3), respectively, for each of the one thousand samples. If say $\widetilde{f}_i(1)$, $i = 1, 2, \ldots, 1000$ denote the one thousand values of $\widetilde{f}(1)$ the MSE is computed as
$$\mathrm{MSE}_1(n) = \frac{1}{1000}\sum_{i=1}^{1000}\left[\widetilde{f}_i(1) - f(1)\right]^2.$$
If say $\widetilde{F}_i(1)$, $i = 1, 2, \ldots, 1000$ denote the one thousand values of $\widetilde{F}(1)$, the MSE is computed as
$$\mathrm{MSE}_2(n) = \frac{1}{1000}\sum_{i=1}^{1000}\left[\widetilde{F}_i(1) - F(1)\right]^2.$$
3. Compute the T in (3.1) and then $\widehat{f}(1)$ and $\widehat{F}(1)$ in (3.2) and (3.4), respectively, for each of the one thousand samples. If say $\widehat{f}_i(1)$, $i = 1, 2, \ldots, 1000$ denote the one thousand values of $\widehat{f}(1)$ the MSE is computed as
$$\mathrm{MSE}_3(n) = \frac{1}{1000}\sum_{i=1}^{1000}\left[\widehat{f}_i(1) - f(1)\right]^2.$$
If say $\widehat{F}_i(1)$, $i = 1, 2, \ldots, 1000$ denote the one thousand values of $\widehat{F}(1)$ the MSE is computed as
$$\mathrm{MSE}_4(n) = \frac{1}{1000}\sum_{i=1}^{1000}\left[\widehat{F}_i(1) - F(1)\right]^2.$$

4. Compute the $\widetilde{\alpha}_{\mathrm{pc}}$ by minimizing (4.1) and then $\widetilde{f}_{\mathrm{pc}}(1)$ and $\widetilde{F}_{\mathrm{pc}}(1)$ in (4.2) and (4.3), respectively, for each of the one thousand samples. If say $\widetilde{f}_i(1)$, $i = 1, 2, \ldots, 1000$ denote the one thousand values of $\widetilde{f}_{\mathrm{pc}}(1)$ the MSE is computed as

$$\mathrm{MSE}_5(n) = \frac{1}{1000} \sum_{i=1}^{1000} \left[\widetilde{f}_i(1) - f(1) \right]^2.$$

If say $\widetilde{F}_i(1)$, $i = 1, 2, \ldots, 1000$ denote the one thousand values of $\widetilde{F}_{\mathrm{pc}}(1)$ the MSE is computed as

$$\mathrm{MSE}_6(n) = \frac{1}{1000} \sum_{i=1}^{1000} \left[\widetilde{F}_i(1) - F(1) \right]^2.$$

5. Compute the $\widetilde{\alpha}_{\mathrm{ls}}$ by minimizing (5.1) and then $\widetilde{f}_{\mathrm{ls}}(1)$ and $\widetilde{F}_{\mathrm{ls}}(1)$ in (5.2) and (5.3), respectively, for each of the one thousand samples. If say $\widetilde{f}_i(1)$, $i = 1, 2, \ldots, 1000$ denote the one thousand values of $\widetilde{f}_{\mathrm{ls}}(1)$ the MSE is computed as

$$\mathrm{MSE}_7(n) = \frac{1}{1000} \sum_{i=1}^{1000} \left[\widetilde{f}_i(1) - f(1) \right]^2.$$

If say $\widetilde{F}_i(1)$, $i = 1, 2, \ldots, 1000$ denote the one thousand values of $\widetilde{F}_{\mathrm{ls}}(1)$ the MSE is computed as

$$\mathrm{MSE}_8(n) = \frac{1}{1000} \sum_{i=1}^{1000} \left[\widetilde{F}_i(1) - F(1) \right]^2.$$

6. Compute the $\widetilde{\alpha}_{\mathrm{wls}}$ by minimizing (5.4) and then $\widetilde{f}_{\mathrm{wls}}(1)$ and $\widetilde{F}_{\mathrm{wls}}(1)$ in (5.5) and (5.6), respectively, for each of the one thousand samples. If say $\widetilde{f}_i(1)$, $i = 1, 2, \ldots, 1000$ denote the one thousand values of $\widetilde{f}_{\mathrm{wls}}(1)$ the MSE is computed as

$$\mathrm{MSE}_9(n) = \frac{1}{1000} \sum_{i=1}^{1000} \left[\widetilde{f}_i(1) - f(1) \right]^2.$$

If say $\widetilde{F}_i(1)$, $i = 1, 2, \ldots, 1000$ denote the one thousand values of $\widetilde{F}_{\mathrm{wls}}(1)$ the MSE is computed as

$$\mathrm{MSE}_{10}(n) = \frac{1}{1000} \sum_{i=1}^{1000} \left[\widetilde{F}_i(1) - F(1) \right]^2.$$

We repeated these steps for $n = 5, 6, \ldots, 100$ and for various choices for (α, β, θ), so computing $\mathrm{MSE}_j(n)$ for $j = 1, 2, \ldots, 10$ and $n = 5, 6, \ldots, 100$. Plots of the MSEs of the UMVUEs, the PCEs, the LSEs and the WLSEs relative to the MSEs for the MLEs are shown in Figures 1–3. That is, Figure 1 plots $\mathrm{MSE}_3(n) - \mathrm{MSE}_1(n)$, $\mathrm{MSE}_4(n) - \mathrm{MSE}_2(n)$, $\mathrm{MSE}_5(n) - \mathrm{MSE}_1(n)$, $\mathrm{MSE}_6(n) - \mathrm{MSE}_2(n)$, $\mathrm{MSE}_7(n) - \mathrm{MSE}_1(n)$, $\mathrm{MSE}_8(n) - \mathrm{MSE}_2(n)$, $\mathrm{MSE}_9(n) - \mathrm{MSE}_1(n)$ and $\mathrm{MSE}_{10}(n) - \mathrm{MSE}_2(n)$ versus $n = 5, 6, \ldots, 100$ for $(\alpha, \beta, \theta) = (1, 1, 1)$, $(2, 2, 2)$, $(3, 3, 3)$; Figure 2 plots $\mathrm{MSE}_3(n) - \mathrm{MSE}_1(n)$, $\mathrm{MSE}_4(n) - \mathrm{MSE}_2(n)$, $\mathrm{MSE}_5(n) - \mathrm{MSE}_1(n)$, $\mathrm{MSE}_6(n) - \mathrm{MSE}_2(n)$, $\mathrm{MSE}_7(n) - \mathrm{MSE}_1(n)$, $\mathrm{MSE}_8(n) - \mathrm{MSE}_2(n)$, $\mathrm{MSE}_9(n) - \mathrm{MSE}_1(n)$ and $\mathrm{MSE}_{10}(n) - \mathrm{MSE}_2(n)$ versus $n = 5, 6, \ldots, 100$ for $(\alpha, \beta, \theta) = (0.5, 2, 0.5)$, $(1, 2, 0.5)$, $(0.5, 2, 1)$; Figure 3 plots $\mathrm{MSE}_3(n) - \mathrm{MSE}_1(n)$, $\mathrm{MSE}_4(n) - \mathrm{MSE}_2(n)$, $\mathrm{MSE}_5(n) - \mathrm{MSE}_1(n)$, $\mathrm{MSE}_6(n) - \mathrm{MSE}_2(n)$, $\mathrm{MSE}_7(n) - \mathrm{MSE}_1(n)$, $\mathrm{MSE}_8(n) - \mathrm{MSE}_2(n)$, $\mathrm{MSE}_9(n) - \mathrm{MSE}_1(n)$ and $\mathrm{MSE}_{10}(n) - \mathrm{MSE}_2(n)$ versus $n = 5, 6, \ldots, 100$ for $(\alpha, \beta, \theta) = (1, 2, 3)$, $(1, 3, 2)$, $(3, 2, 1)$.

We can see from the figures that the ML estimators of the PDF and the CDF have the smallest MSEs. The UMVU estimators of the PDF and the CDF have the second smallest MSEs. The PC estimators of the PDF and the CDF have the third smallest MSEs. The WLS estimators of the PDF and the CDF have the fourth smallest MSEs. The LS estimators of the PDF and the CDF have the largest MSEs.

Although not shown (because of space concerns), the MSEs of the estimators of α follow the same pattern. That is, the ML estimators of α have the smallest MSEs, followed by the UMVU estimators, the PC estimators, the WLS estimators and the LS estimators in that order. Moreover, the MSEs for each estimator decrease with increasing sample size.

In this simulation study, we have reported the results for $x = 1$ and $(\alpha, \beta, \theta) = (1, 1, 1)$, $(2, 2, 2)$, $(3, 3, 3)$, $(0.5, 2, 0.5)$, $(1, 2, 0.5)$, $(0.5, 2, 1)$, $(1, 2, 3)$, $(1, 3, 2)$, $(3, 2, 1)$. The results for other choices were similar.

7 Data analysis

Here, we use a real data set to compare the performances of the MLE, the PCE, the LSE and the WLSE for the PDF and the CDF. The data gives one hundred observations on breaking stress of carbon fibers (in Gba): 3.7, 2.74, 2.73, 2.5, 3.6, 3.11, 3.27, 2.87, 1.47, 3.11, 4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.9, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53, 2.67, 2.93, 3.22, 3.39, 2.81, 4.2, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59, 2.38, 2.81, 2.77, 2.17, 2.83, 1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59, 3.19, 1.57, 0.81, 5.56, 1.73, 1.59, 2, 1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.7, 2.03, 1.8, 1.57, 1.08, 2.03, 1.61, 2.12, 1.89, 2.88, 2.82, 2.05, 3.65. The data was obtained from Nichols and Padgett (2006). Pal et al. (2006) fitted the EW distribution to this data set and found it to be very good.

In practical applications, all of the parameters of a model are usually unknown. Let $x_1, x_2, \ldots, x_n$ denote the observations of the real data set, assumed to be a

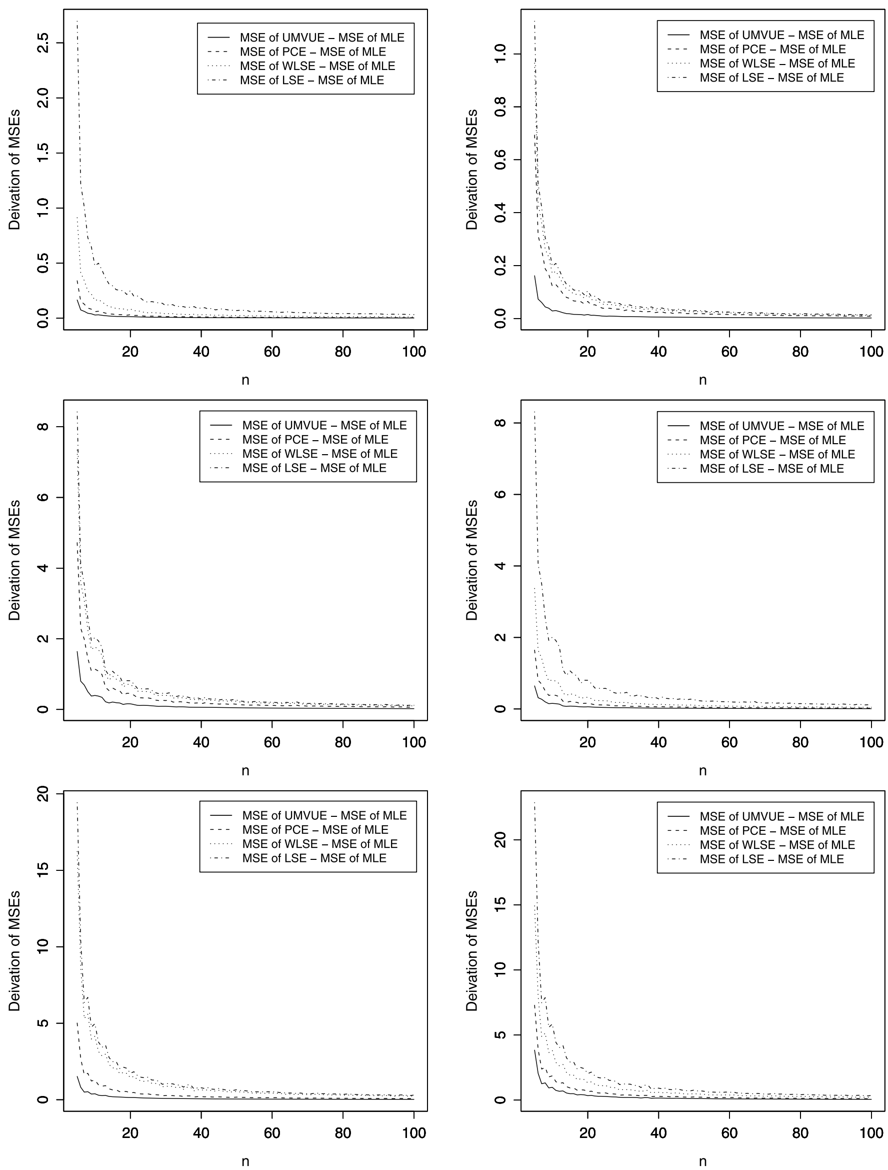

Figure 1 *Deviations of the MSEs of the UMVUE, the PCE, the WLSE and the LSE from the MSE of the MLE for $f(1)$ and $(\alpha, \beta, \theta) = (1, 1, 1)$ (top left); Deviations of the MSEs of the UMVUE, the PCE, the WLSE and the LSE from the MSE of the MLE for $F(1)$ and $(\alpha, \beta, \theta) = (1, 1, 1)$ (top right); Deviations of the MSEs of the UMVUE, the PCE, the WLSE and the LSE from the MSE of the MLE for $f(1)$ and $(\alpha, \beta, \theta) = (2, 2, 2)$ (middle left); Deviations of the MSEs of the UMVUE, the PCE, the WLSE and the LSE from the MSE of the MLE for $F(1)$ and $(\alpha, \beta, \theta) = (2, 2, 2)$ (middle right); Deviations of the MSEs of the UMVUE, the PCE, the WLSE and the LSE from the MSE of the MLE for $f(1)$ and $(\alpha, \beta, \theta) = (3, 3, 3)$ (bottom left); Deviations of the MSEs of the UMVUE, the PCE, the WLSE and the LSE from the MSE of the MLE for $F(1)$ and $(\alpha, \beta, \theta) = (3, 3, 3)$ (bottom right).*

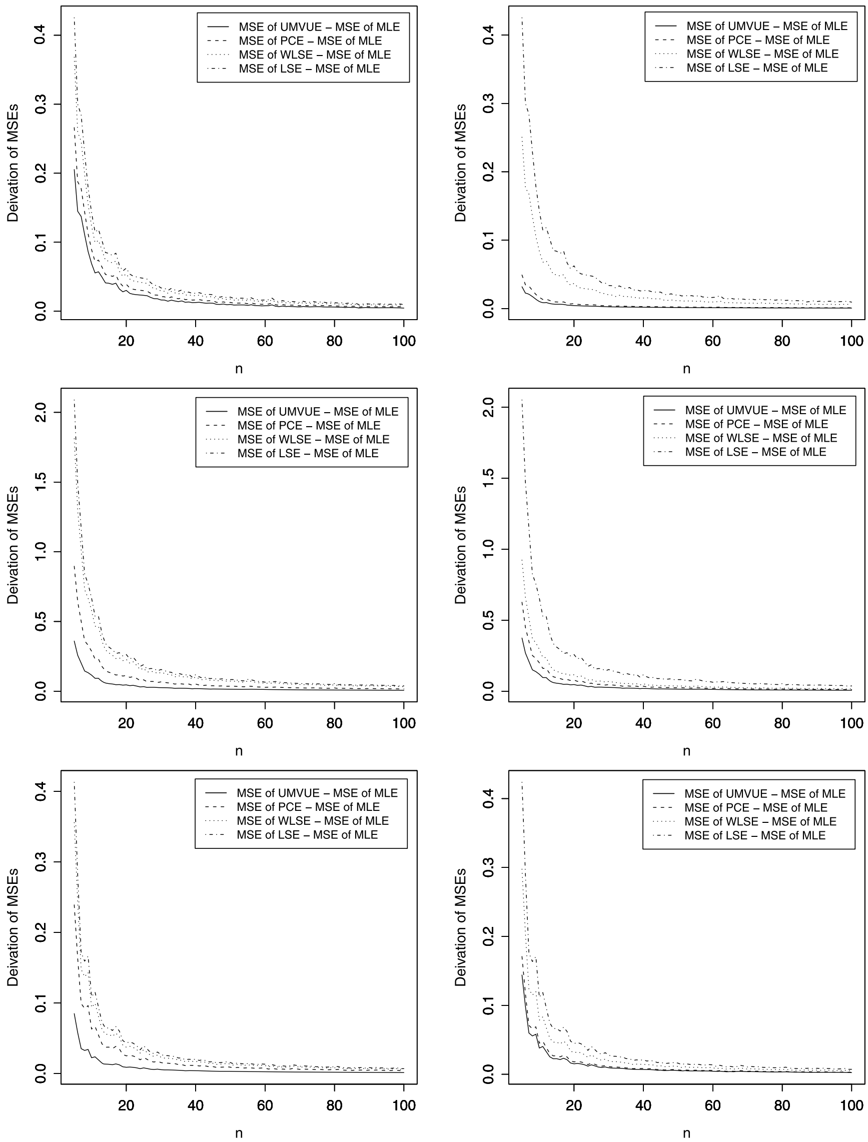

Figure 2 *Deviations of the MSEs of the UMVUE, the PCE, the WLSE and the LSE from the MSE of the MLE for $f(1)$ and $(\alpha, \beta, \theta) = (0.5, 2, 0.5)$ (top left); Deviations of the MSEs of the UMVUE, the PCE, the WLSE and the LSE from the MSE of the MLE for $F(1)$ and $(\alpha, \beta, \theta) = (0.5, 2, 0.5)$ (top right); Deviations of the MSEs of the UMVUE, the PCE, the WLSE and the LSE from the MSE of the MLE for $f(1)$ and $(\alpha, \beta, \theta) = (1, 2, 0.5)$ (middle left); Deviations of the MSEs of the UMVUE, the PCE, the WLSE and the LSE from the MSE of the MLE for $F(1)$ and $(\alpha, \beta, \theta) = (1, 2, 0.5)$ (middle right); Deviations of the MSEs of the UMVUE, the PCE, the WLSE and the LSE from the MSE of the MLE for $f(1)$ and $(\alpha, \beta, \theta) = (0.5, 2, 1)$ (bottom left); Deviations of the MSEs of the UMVUE, the PCE, the WLSE and the LSE from the MSE of the MLE for $F(1)$ and $(\alpha, \beta, \theta) = (0.5, 2, 1)$ (bottom right).*

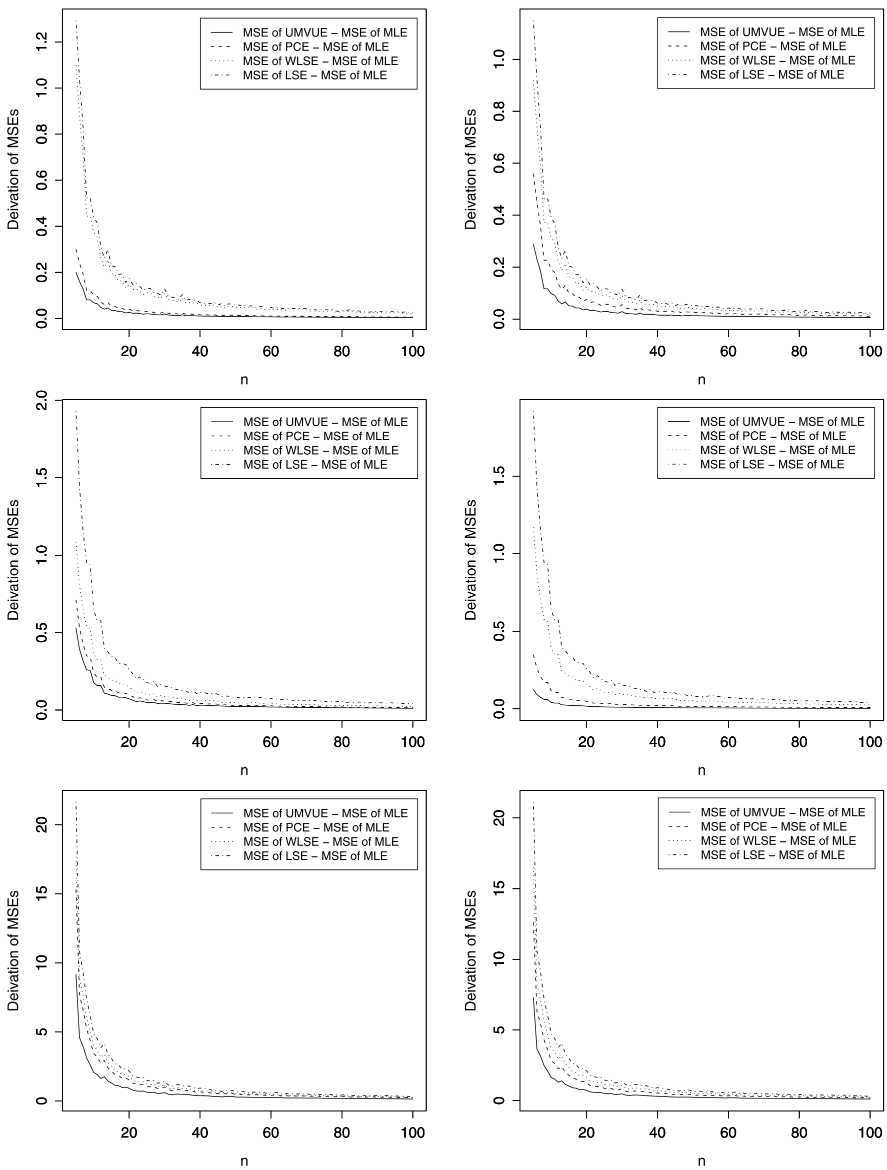

Figure 3 *Deviations of the MSEs of the UMVUE, the PCE, the WLSE and the LSE from the MSE of the MLE for* $f(1)$ *and* $(\alpha, \beta, \theta) = (1, 2, 3)$ (*top left*); *Deviations of the MSEs of the UMVUE, the PCE, the WLSE and the LSE from the MSE of the MLE for* $F(1)$ *and* $(\alpha, \beta, \theta) = (1, 2, 3)$ (*top right*); *Deviations of the MSEs of the UMVUE, the PCE, the WLSE and the LSE from the MSE of the MLE for* $f(1)$ *and* $(\alpha, \beta, \theta) = (1, 3, 2)$ (*middle left*); *Deviations of the MSEs of the UMVUE, the PCE, the WLSE and the LSE from the MSE of the MLE for* $F(1)$ *and* $(\alpha, \beta, \theta) = (1, 3, 2)$ (*middle right*); *Deviations of the MSEs of the UMVUE, the PCE, the WLSE and the LSE from the MSE of the MLE for* $f(1)$ *and* $(\alpha, \beta, \theta) = (3, 2, 1)$ (*bottom left*); *Deviations of the MSEs of the UMVUE, the PCE, the WLSE and the LSE from the MSE of the MLE for* $F(1)$ *and* $(\alpha, \beta, \theta) = (3, 2, 1)$ (*bottom right*).

random sample from the EW distribution. The parameters α, β and θ are unknown for this data set. We use the following procedures to estimate them:

- The log-likelihood function of the parameters is

$$
\begin{aligned}
L(\alpha, \beta, \theta) = n \log \alpha + n \log \beta + (\beta - 1) \sum_{i=1}^{n} \log x_i - n\beta \log \theta \\
- \sum_{i=1}^{n} \left(\frac{x_i}{\theta}\right)^{\beta} + (\alpha - 1) \sum_{i=1}^{n} \log\left[1 - e^{-(x_i/\theta)^{\beta}}\right].
\end{aligned}
\tag{7.1}
$$

  The MLEs of α, β and θ say $\widetilde{\alpha}$, $\widetilde{\beta}$ and $\widetilde{\theta}$, respectively, can be obtained as the simultaneous solutions of

$$
\begin{aligned}
\frac{\partial L}{\partial \alpha} &= \frac{n}{\alpha} + \sum_{i=1}^{n} \log\left[1 - e^{-(x_i/\theta)^{\beta}}\right] = 0, \\
\frac{\partial L}{\partial \beta} &= \frac{n}{\beta} + \sum_{i=1}^{n} \log x_i - n \log \theta - \sum_{i=1}^{n} \left(\frac{x_i}{\theta}\right)^{\beta} \log\left(\frac{x_i}{\theta}\right) \\
&\quad + (\alpha - 1) \sum_{i=1}^{n} \frac{(x_i/\theta)^{\beta} \log(x_i/\theta) e^{-(x_i/\theta)^{\beta}}}{1 - e^{-(x_i/\theta)^{\beta}}} = 0, \\
\frac{\partial L}{\partial \theta} &= -\frac{n\beta}{\theta} + \frac{\beta}{\theta} \sum_{i=1}^{n} \left(\frac{x_i}{\theta}\right)^{\beta} \\
&\quad - \frac{(\alpha - 1)\beta}{\theta} \sum_{i=1}^{n} \frac{(x_i/\theta)^{\beta} e^{-(x_i/\theta)^{\beta}}}{1 - e^{-(x_i/\theta)^{\beta}}} = 0.
\end{aligned}
$$

  It is often easier to obtain the MLEs by maximizing (7.1) numerically or by minimizing the minus of (7.1) numerically.
- The PCEs of α, β and θ say $\widetilde{\alpha}_{\mathrm{pc}}$, $\widetilde{\beta}_{\mathrm{pc}}$ and $\widetilde{\theta}_{\mathrm{pc}}$, respectively, can be obtained by minimizing

$$
\sum_{i=1}^{n} \left[1 - p_i^{1/\alpha} - e^{-x_{(i)}^{\beta}/\theta^{\beta}}\right]^2.
$$

- The LSEs of α, β and θ say $\widetilde{\alpha}_{\mathrm{ls}}$, $\widetilde{\beta}_{\mathrm{ls}}$ and $\widetilde{\theta}_{\mathrm{ls}}$, respectively, can be obtained by minimizing

$$
\sum_{j=1}^{n} \left[\left(1 - e^{-x_{(j)}^{\beta}/\theta^{\beta}}\right)^{\alpha} - \frac{j}{n+1}\right]^2.
$$

Table 1 *Estimates of the parameters, standard errors and log-likelihoods*

	Estimate of α	Estimate of β	Estimate of θ	Log-likelihood
MLE	1.317 (0.001)	2.409 (0.024)	2.682435 (0.003)	−141.332
PCE	1.560 (0.007)	2.151 (0.042)	2.518 (0.112)	−141.4632
LSE	0.410 (0.022)	5.498 (0.178)	3.500 (0.191)	−153.0544
WLSE	0.847 (0.014)	3.200 (0.056)	3.0588529 (0.149)	−142.1879

- The WLSEs of α, β and θ say $\widetilde{\alpha}_{\mathrm{wls}}$, $\widetilde{\beta}_{\mathrm{wls}}$ and $\widetilde{\theta}_{\mathrm{wls}}$, respectively, can be obtained by minimizing

$$\sum_{j=1}^{n} w_j \left[\left(1 - e^{-x_{(j)}^{\beta}/\theta^{\beta}}\right)^{\alpha} - \frac{j}{n+1} \right]^2,$$

where $w_j = \frac{1}{\mathrm{Var}[F(X_{(j)})]} = \frac{(n+1)^2(n+2)}{j(n-j+1)}$.

The EW distribution was fitted to the fiber data by the MLE, the PCE, the LSE and the WLSE. The MLEs were computed by minimizing the minus of (7.1) numerically. For the minimization, we used the nlm function in the R software (R Development Core Team, 2014). Table 1 gives the estimates of α, β, θ, their standard errors and the corresponding log-likelihoods.

The standard errors were computed by simulation as follows:

- simulate one thousand samples each of size 100 from the EW distribution fitted by the MLE, the PCE, the LSE or the WLSE;
- refit the EW distribution by the respective estimation method (MLE, PCE, LSE or WLSE) for each of the one thousand samples;
- compute the sampling distribution from the one thousand estimates of α, that from the one thousand estimates of β and that from the one thousand estimates of θ for each method;
- compute the standard error as the standard deviation of the sampling distribution divided by 10.

The standard errors for the MLEs could have been computed by inverting the observed information matrix or by inverting the expected information matrix. But for consistency with other estimation methods, we have used the simulation method.

We see from Table 1 that the log-likelihood value is the largest for the MLE, second largest for the PCE, third largest for the WLSE and the smallest for the LSE. We also see that the standard errors are the smallest for the MLE, second smallest for the PCE, third smallest for the WLSE and the largest for the LSE.

Figures 4–6 show the Q–Q plots (observed quantiles versus expected quantiles), the density plots (fitted PDFs versus empirical PDF) and the P–P plots (observed

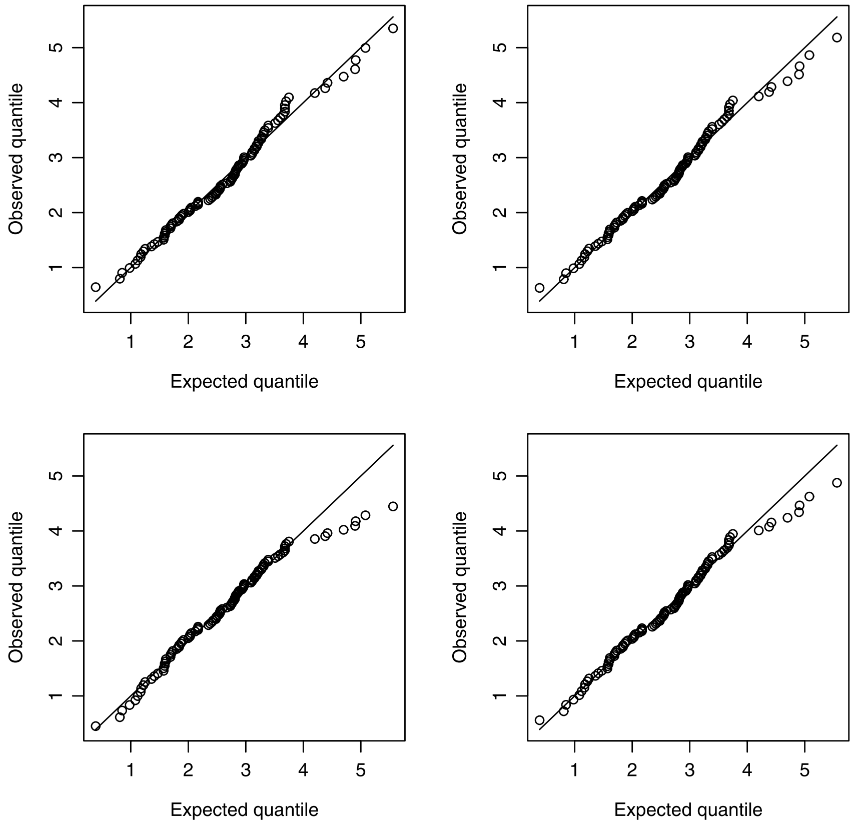

Figure 4 *Q–Q plots for the fit of the MLE (top left), the PCE (top right), the LSE (bottom left) and the WLSE (bottom right).*

probabilities versus expected probabilities) for the four different estimation methods. The points are closest to the diagonal line for the MLE, second closest for the PCE, third closest for the WLSE and furthest for the LSE. The fitted PDF for the MLE appears to best capture the pattern in the histogram. Hence, the figures show that the MLE provides the best fit.

We also compared the estimation methods by means of model selection criteria. The ones we considered were: 'pure' maximum likelihood, Akaike information criterion (Akaike, 1974), corrected AIC (Hurvich and Tsai, 1989), Bayes information criterion (Schwarz, 1978) and Hannan–Quinn criterion (Hannan and Quinn, 1979) defined by

$$\mathrm{ML} = -2\log L(\theta),$$

$$\mathrm{AIC} = -2\log L(\theta) + 2k,$$

$$\mathrm{AICc} = -2\log L(\theta) + 2k\frac{n}{n-k-1},$$

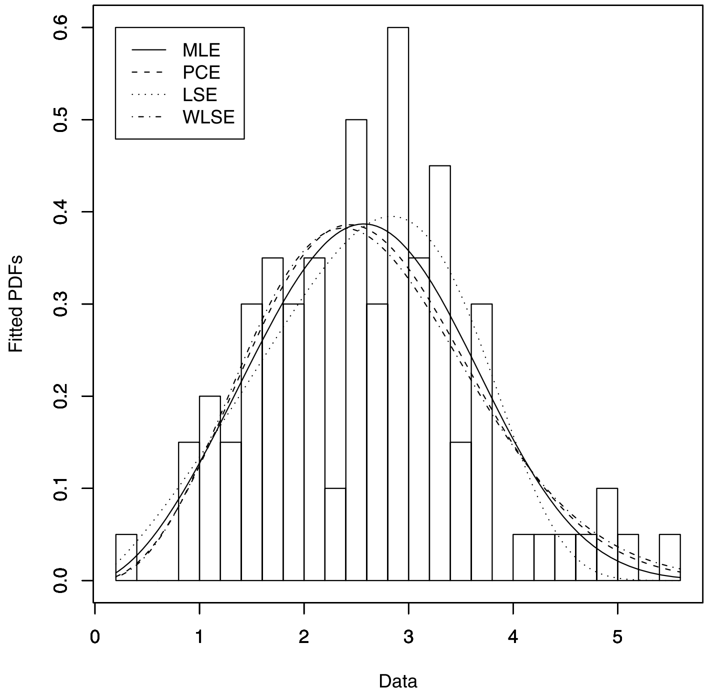

Figure 5 *Histogram of the data and the PDFs for the fit of the MLE, the PCE, the LSE and the WLSE.*

$$\mathrm{BIC} = -2\log L(\theta) + k\log n,$$
$$\mathrm{HQC} = -2\log L(\theta) + 2k\log\log n,$$

respectively, where $\log L(\theta)$ denotes the log-likelihood, n denotes the number of observations and k denotes the number of parameters in the distribution. The smaller the values of these criteria the better the fit. For more on these criteria, see Burnham and Anderson (2004) and Fang (2011).

Table 2 gives values of the model selection criteria for the four different estimation methods. We can see that the MLEs give the smallest values for all five model selection criteria. The second smallest values for all five criteria are for the PCE. The third smallest values for all five criteria are for the WLSE. The largest values for all five criteria are for the LSE.

Hence, evidence based on the MSEs in the simulation study, the log-likelihood values, the standard errors, the Q–Q plots, the density plots, the P–P plots and the model selection criteria show that the ML estimators for the PDF and the CDF are the best.

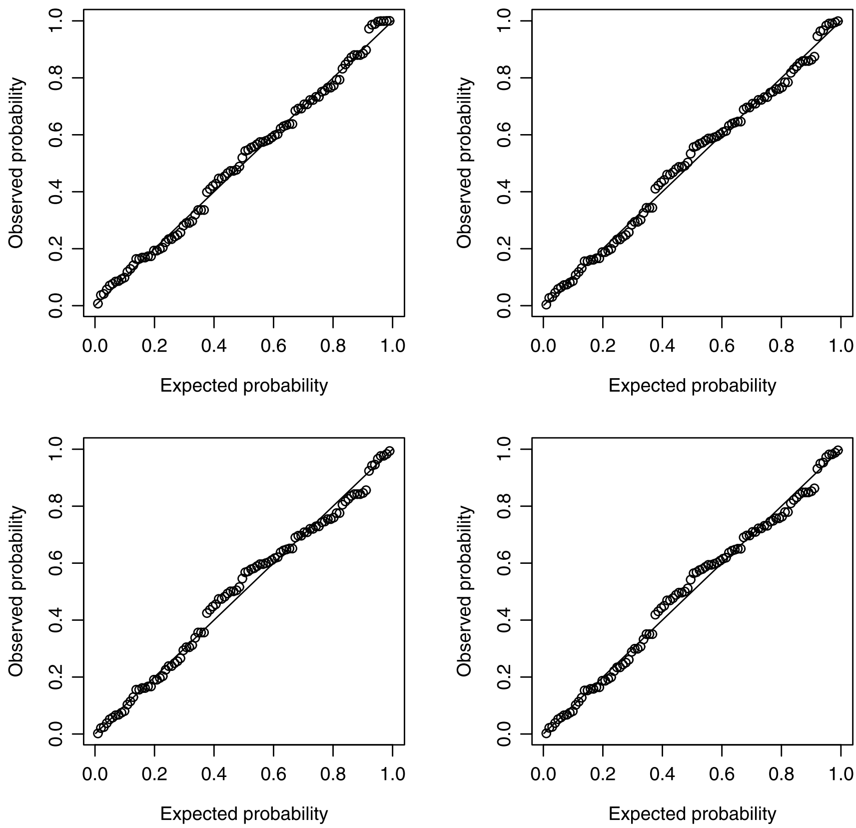

Figure 6 *P–P plots for the fit of the MLE* (*top left*), *the PCE* (*top right*), *the LSE* (*bottom left*) *and the WLSE* (*bottom right*).

Table 2 *Model selection criteria for the fiber data*

	ML	AIC	BIC	AICc	HQC
MLE	282.6641	288.6641	296.4796	288.9141	291.8271
PCE	282.9265	288.9265	296.7420	289.1765	292.0895
LSE	306.1087	312.1087	319.9242	312.3587	315.2718
WLSE	284.3759	290.3759	298.1914	290.6259	293.5390

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