FURTHER RESULTS ON CONTROLLING THE FALSE DISCOVERY PROPORTION

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The probability of false discovery proportion (FDP) exceeding $\gamma \in [0, 1)$, defined as γ-FDP, has received much attention as a measure of false discoveries in multiple testing. Although this measure has received acceptance due to its relevance under dependency, not much progress has been made yet advancing its theory under such dependency in a nonasymptotic setting, which motivates our research in this article. We provide a larger class of procedures containing the stepup analog of, and hence more powerful than, the stepdown procedure in Lehmann and Romano [*Ann. Statist.* **33** (2005) 1138–1154] controlling the γ-FDP under similar positive dependence condition assumed in that paper. We offer better alternatives of the stepdown and stepup procedures in Romano and Shaikh [*IMS Lecture Notes Monogr. Ser.* **49** (2006a) 33–50, *Ann. Statist.* **34** (2006b) 1850–1873] using pairwise joint distributions of the null p-values. We generalize the notion of γ-FDP making it appropriate in situations where one is willing to tolerate a few false rejections or, due to high dependency, some false rejections are inevitable, and provide methods that control this generalized γ-FDP in two different scenarios: (i) only the marginal p-values are available and (ii) the marginal p-values as well as the common pairwise joint distributions of the null p-values are available, and assuming both positive dependence and arbitrary dependence conditions on the p-values in each scenario. Our theoretical findings are being supported through numerical studies.

1. Introduction. The idea of improving the traditional and often too conservative notion of familywise error rate (FWER) has been one of the main motivations behind much of the methodological developments taking place in modern multiple testing. One particular direction in which this idea has flourished is generalizing the FWER from its original definition of the probability of at least one false discovery or a nonzero fraction of false discoveries to one that allows more, yet tolerable, number or fraction of false discoveries and developing procedures that control these generalized error rates. The rationale behind taking this direction is that in many situations where a large number of hypotheses are tested one is often

willing to tolerate more than one false discovery, controlling of course too many of them. Moreover, due to high positive dependency among a group or groups of p-values corresponding to true null hypotheses, as in microarray experiments where the genes involved in the same biological process or pathway are highly dependent on each other and exhibit similar expression patterns, it is extremely unlikely that exactly one null p-value will be significant given that at least one of them will be significant. In such cases, a procedure controlling the probability of at least k false discoveries, the k-FWER, for some fixed $k > 1$, or the probability of the false discovery proportion (FDP) exceeding γ, the γ-FDP, for some fixed $0 < \gamma < 1$, will have a better ability to detect more false null hypotheses than the corresponding FWER procedure (i.e., when $k = 1$ or $\gamma = 0$).

Thus, the consideration of the k-FWER or γ-FDP seems more relevant than that of the FWER when controlling false discoveries in multiple testing of a large number of hypotheses under dependency. In fact, it has been noted that the dependency gets naturally factored into the constructions of procedures controlling the k-FWER or γ-FDP. For instance, the k-dimensional joint distributions of the null p-values can be explicitly used while constructing procedures controlling the k-FWER [Sarkar (2007, 2008a)]. Also, since the FDP becomes more variable and gets more skewed with increasing dependence among the p-values [Efron (2007), Kim and van de Wiel (2008), Korn et al. (2004), Owen (2005), and Schwartzman and Lin (2011)], by controlling the tail end probabilities of the FDP, the γ-FDP, one considers controlling a quantity that is more relevant under dependency than the expected FDP, the false discovery rate (FDR) [Benjamini and Hochberg (1995)], which is even less conservative than the FWER.

A number of papers have been written over the years on k-FWER and γ-FDP [Dudoit, van der Laan and Pollard (2004), Genovese and Wasserman (2004), Guo and Rao (2010), Guo and Romano (2007), Hommel and Hoffmann (1987), Korn and Freidlin (2008), Korn et al. (2004), Lehmann and Romano (2005), Romano and Shaikh (2006a, 2006b), Romano and Wolf (2005), Roquain and Villers (2011), Sarkar (2007, 2008a) and van der Laan, Dudoit and Pollard (2004)]. Among these, Lehmann and Romano (2005), and Romano and Shaikh (2006a, 2006b) are worth mentioning as they have made some fundamental contributions to the development of theory and methodology of γ-FDP. A part of our research is motivated by these papers, and aims at extending, and often improving, some results in those papers under certain dependence situations. The motivation of the other part of our research comes from the realization that if one indeed is willing to tolerate a few false rejections, the premise under which one would seek to use a generalized error rate, the notion of γ-FDP does not completely take that into account unless it is further generalized accordingly. In other words, one should consider in this case a generalized form of the FDP that accounts for k or more false rejections, and control the probability of this generalized FDP, rather than the original FDP, exceeding γ. So, we introduce such a generalized notion of γ-FDP, called the

γ-kFDP, and propose procedures that control it under different dependence scenarios in this paper.

The paper is organized as follows. We provide some preliminaries in Section 2, including the definition of our proposed notion of γ-kFDP. Section 3 contains our main results on controlling the γ-FDP and γ-kFDP, developed assuming both positive dependence (Section 3.1) and arbitrary dependence (Section 3.2) conditions on the p-values in each of the following two scenarios: (i) only the marginal p-values are available and (ii) the marginal p-values as well as the common pairwise joint distributions of the null p-values are available. We obtain a number of newer results on γ-FDP than what are available in the literature. We construct a larger class of procedures controlling the γ-FDP under positive dependence than the stepdown procedure given in Lehmann and Romano (2005). This class includes the stepup analog of, and hence more powerful than, this Lehmann–Romano stepdown procedure. We offer better alternatives of the stepdown and stepup procedures in Romano and Shaikh (2006a, 2006b), given pairwise joint distributions of the null p-values. Most of our main results have been obtained through a general framework that allows us not only to develop procedures controlling the newly proposed notion of γ-kFDP, for $k \geq 1$, but also to produce the aforementioned new results on γ-FDP by taking $k = 1$. The performances of the proposed γ-FDP and γ-kFDP procedures, individual as well as relative to relevant competitors, are numerically investigated through extensive simulations and reported in Section 4. Concluding remarks are made in Section 5. Proofs of some supporting results are given in the Appendix.

The supplementary material [Guo, He and Sarkar (2014)] is added due to space constraints to include some additional figures related to the numerical investigations in Section 4. Also presented in this section are the findings of simulation studies conducted to examine the effect of k on a γ-kFDP controlling procedure (see Remark 2.1) and to provide an insight into the choice of k under varying dependence.

2. Preliminaries. Suppose that $H_i, i = 1, \ldots, n$, are the n null hypotheses to be tested based on their respective p-values $P_i, i = 1, \ldots, n$. Let $P_{(1)} \leq \cdots \leq P_{(n)}$ be the ordered versions of all the p-values and $H_{(1)}, \ldots, H_{(n)}$ be their corresponding null hypotheses. There are n_0 null hypotheses that are true. For notational convenience, the p-values corresponding to these true null hypotheses will be denoted by $\widehat{P}_i, i = 1, \ldots, n_0$, and their ordered versions by $\widehat{P}_{(1)} \leq \cdots \leq \widehat{P}_{(n_0)}$.

Multiple testing is typically carried out using a stepwise or single-step procedure. Given a nondecreasing set of critical values $0 < \alpha_1 \leq \cdots \leq \alpha_n < 1$, a stepdown procedure rejects the set of null hypotheses $\{H_{(i)}, i \leq i^*_{\text{SD}}\}$, where $i^*_{\text{SD}} = \max\{1 \leq i \leq n : P_{(j)} \leq \alpha_j \ \forall j \leq i\}$ if the maximum exists, otherwise accepts all the null hypotheses. A stepup procedure, on the other hand, rejects the set of null hypotheses $\{H_{(i)}, i \leq i^*_{\text{SU}}\}$, where $i^*_{\text{SU}} = \max\{1 \leq i \leq n : P_{(i)} \leq \alpha_i\}$ if the

maximum exists, otherwise accepts all the null hypotheses. A stepdown or stepup procedure with the same critical values is referred to as a single-step procedure.

Let V be the number of falsely rejected and R be the total number of rejected null hypotheses. Then, with V/R, which is zero if $R = 0$, defining the false discovery proportion (FDP), and given a fixed $\gamma \in (0, 1)$, the γ-FDP is defined as the probability of the FDP exceeding γ; that is, γ-FDP = Pr(FDP > γ). Its generalized version introduced in this paper, which we call γ-kFDP, is defined as follows: let

$$
\text{kFDP} = \begin{cases} \dfrac{V}{R}, & \text{if } V \geq k, \\ 0, & \text{otherwise.} \end{cases}
$$

Then γ-kFDP = Pr(kFDP > γ). Since γ-kFDP is 0, and hence trivially controlled, for any procedure if $n_0 < k$, we assume throughout the paper that $k \leq n_0 \leq n$ when controlling this error rate. Also, while constructing a γ-kFDP controlling stepwise procedure, we will consider the first $k - 1$ critical constants to be the same as the kth one, as in k-FWER procedures, since their choice does not matter in calculating the γ-kFDP.

REMARK 2.1. It should be noted that since V and FDP are likely to be highly correlated the distribution of kFDP may be very similar to that of FDP with a small portion of its lower tail set to 0. Therefore, the difference between γ-kFDP and γ-FDP may be realized, with the control over γ-kFDP providing the stipulated power improvement, only when k/n exceeds a certain value. Of course, this value, given a specified γ, would depend on the type and strength of dependence. We did a numerical study to verify this intuition and offer an insight into the choice of k under different types and varying strengths of dependence, and report its findings in the supplementary material [Guo, He and Sarkar (2014)].

The following is the basic assumption regarding the marginal distributions of the p-values made throughout the paper.

ASSUMPTION 1. $\widehat{P}_i \sim U(0, 1)$.

3. Main results. In this section, we present the developments of our stepwise procedures controlling the γ-FDP and the newly proposed γ-kFDP under both positive dependence and arbitrary dependence conditions on the p-values. Typically, only the marginal distributions of the null p-values are used when constructing multiple testing procedures. However, in practice, the null p-values often have a known common pairwise joint distribution, and it would be worthwhile to consider developing γ-FDP or γ-kFDP stepwise procedures explicitly utilizing such additional dependence information, which could potentially produce more powerful procedures than just using the marginal p-values. With that in mind,

we construct our procedures in the following two different scenarios under each dependence condition: (i) only the marginal p-values are available, and (ii) the marginal p-values as well as the common pairwise joint distributions of the null p-values are available.

3.1. *Procedures under positive dependence.* We will make one of the following two commonly used assumptions characterizing a positive dependence structure among the p-values.

ASSUMPTION 2(a). The conditional expectation $E\{\phi(P_1, \ldots, P_n)|\widehat{P}_i \leq u\}$ is nondecreasing in $u \in (0, 1)$ for each $\widehat{P}_i$ and any nondecreasing (coordinatewise) function ϕ.

ASSUMPTION 2(b). The conditional expectation $E\{\phi(\widehat{P}_1, \ldots, \widehat{P}_{n_0})|\widehat{P}_i \leq u\}$ is nondecreasing in $u \in (0, 1)$ for each $\widehat{P}_i$ and any nondecreasing (coordinatewise) function ϕ.

Assumption 2(a) is slightly weaker than that characterized by the property: $E\{\phi(P_1, \ldots, P_n)|\widehat{P}_i = u\} \uparrow u \in (0, 1)$, referred to as the positive regression dependence on subset (PRDS) (of the null p-values); see, for example, Benjamini and Yekutieli (2001) or Sarkar (2002). Assumption 2(b), less restrictive than Assumption 2(a), is a weaker version of the property: $E\{\phi(\widehat{P}_1, \ldots, \widehat{P}_{n_0})|\widehat{P}_i = u\} \uparrow u \in (0, 1)$, known as the positive dependence (among the null p-values) through stochastic ordering (PDS) due to Block, Savits and Shaked (1985); see also Sarkar (2008b).

3.1.1. *Based on marginal p-values.* Under a positive dependence assumption, Lehmann and Romano (2005) gave a stepdown procedure controlling the γ-FDP. We improve this work in two different ways. First, we consider the stepup analog of this stepdown procedure, which is known to be always more powerful in the sense of discovering more, and prove that it also controls the γ-FDP under the same assumption. Second, we offer larger class of stepdown and stepup procedures controlling the γ-FDP under similar assumption. The procedures in this larger class are presented in a general framework allowing us to propose procedures controlling not only the γ-FDP but also the γ-kFDP for $k \geq 2$.

THEOREM 3.1. *The stepup or stepdown procedure with the critical constants*

$$(1) \qquad \alpha_i = \frac{(\lfloor \gamma i \rfloor + 1)\alpha}{n + \lfloor \gamma i \rfloor + 1 - i}, \qquad i = 1, \ldots, n,$$

controls the γ-FDP at α under Assumptions 1 and 2(b).

PROOF.   Let $g(R) = \lfloor \gamma R \rfloor + 1$. Then first note that

$$\{V \geq g(R)\} = \bigcup_{v=1}^{n_0} \{\widehat{P}_{(v)} \leq \alpha_R, g(R) \leq v, V = v\}$$

$$= \bigcup_{v=1}^{n_0} \left\{\widehat{P}_{(v)} \leq \frac{g(R)\alpha}{n - R + g(R)}, g(R) \leq v, V = v\right\}$$

(2)

$$\subseteq \bigcup_{v=1}^{n_0} \left\{\widehat{P}_{(v)} \leq \frac{v\alpha}{n - R + v}, V = v\right\}$$

$$\subseteq \bigcup_{v=1}^{n_0} \left\{\widehat{P}_{(v)} \leq \frac{v\alpha}{n_0}, V = v\right\} \subseteq \bigcup_{v=1}^{n_0} \left\{\widehat{P}_{(v)} \leq \frac{v\alpha}{n_0}\right\}.$$

The probability of the event in the right-hand side of (2) is known to be less than or equal to α under Assumptions 1 and 2(b) from the so-called Simes' inequality [Simes (1986), Sarkar (1998), Sarkar and Chang (1997)]. Thus, we get the desired result noting that $\gamma\text{-FDP} = \Pr(V \geq g(R))$.   □

REMARK 3.1.   Lehmann and Romano (2005) proposed only the stepdown procedure considered in Theorem 3.1 under the same assumptions. Thus, Theorem 3.1 provides an improvement of the Lehmann–Romano result, since we now have an alternative procedure under the same assumptions, the stepup one, which is theoretically known to be more powerful. Moreover, not only our proof of the γ-FDP control is much simpler but also it covers both ours and the Lehmann–Romano original stepdown procedures. Our simulation studies indicate that this power improvement can be obvious when the underlying test statistics are highly correlated (see Figure 1 and Figures S.1–S.3 in the supplementary material [Guo, He and Sarkar (2014)]).

There are more general results than Theorem 3.1 in terms of deriving procedures controlling the γ-FDP under Assumptions 1 and 2(a) or 2(b). More specifically, we can start with any stepdown or stepup procedure, which may or may not control the γ-FDP to begin with, and rescale its critical values using a suitable upper bound for its γ-FDP derived under Assumptions 1 and 2(a) or 2(b) so that the γ-FDP based on these modified critical values is ultimately controlled. Romano and Shaikh (2006a, 2006b) first developed this idea, but they did it without any positive dependence assumption. We are now going to present these results in the general framework of controlling the γ-kFDP.

Our next main result is obtained with the idea of constructing a stepdown procedure controlling the γ-kFDP under Assumptions 1 and 2(a). The following lemma, to be proved in the Appendix, will provide the starting point for the development of this procedure.
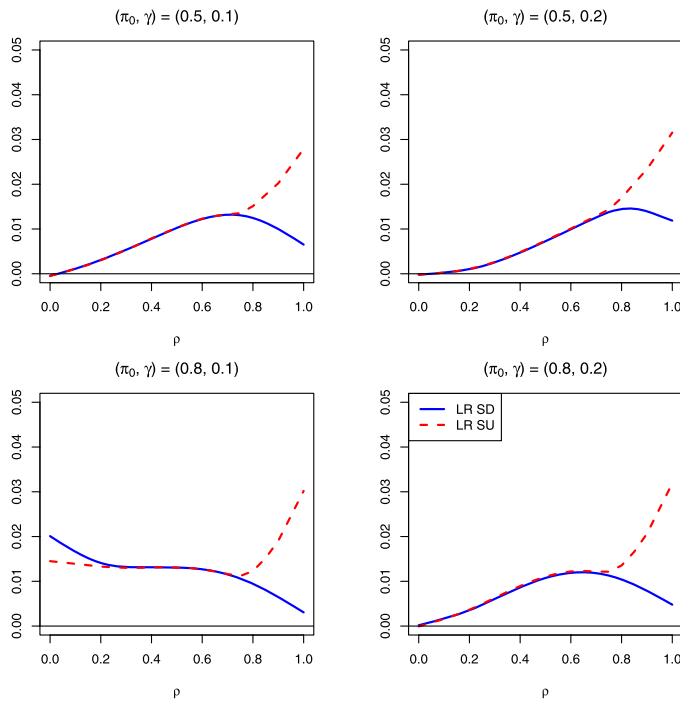
(a) Simulated γ-FDP

FIG. 1. *Simulated values of γ-FDP and average power of the original Lehmann–Romano stepdown procedure* (*LR SD*) *and its stepup analogue* (*LR SU*), *for $n = 100$ and $\alpha = 0.05$.*
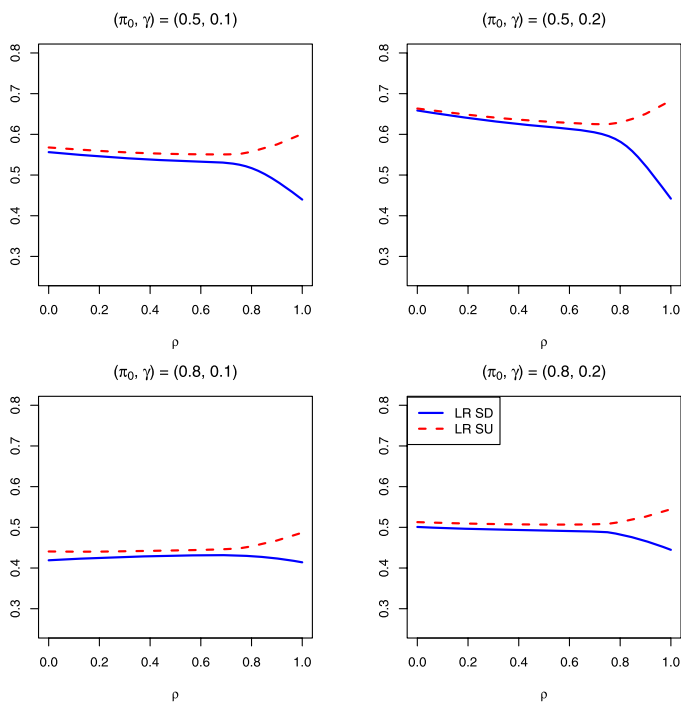
LEMMA 3.1. *With $n_1 = n - n_0$, let $M = \min\{n_0, \lfloor \gamma n_1/(1-\gamma) \rfloor + 1\}$, and $m(i) = \max\{0 \le j \le n_1 : \lfloor \gamma j/(1-\gamma) \rfloor + 1 = i\}$, for each $i = 1, \ldots, M$, where $m(0) = 0$. Consider a stepdown procedure with critical values $\alpha_1 \le \cdots \le \alpha_n$. Let S be the number of rejected false null hypotheses. Then*

$$\text{(3)} \qquad \begin{aligned} & I\bigl(V > \max[\gamma R, k-1]\bigr) \\ & \qquad \le \sum_{i=1}^{M} I\bigl(\widehat{P}_{(i \vee k)} \le \alpha_{i \vee k + m(i)}, \lfloor \gamma S/(1-\gamma) \rfloor + 1 = i\bigr), \end{aligned}$$

for any fixed $1 \le k \le n_0$.

Taking expectations of both sides in (3), we note that

$$\gamma\text{-kFDP} = \Pr\{V > \max[\gamma R, k-1]\}$$

$$\text{(4)} \qquad \le \sum_{i=1}^{M} \Pr\bigl(\widehat{P}_{(i \vee k)} \le \alpha_{i \vee k + m(i)}, \lfloor \gamma S/(1-\gamma) \rfloor + 1 = i\bigr)$$

(b) Simulated average power

FIG. 1. (*Continued*).

$$\leq \sum_{j=1}^{n_0} \sum_{i=1}^{M} \frac{1}{i \vee k} \Pr\big(\widehat{P}_j \leq \alpha_{i \vee k + m(i)}, \lfloor \gamma S/(1-\gamma) \rfloor + 1 = i\big)$$

$$= \sum_{j=1}^{n_0} \sum_{i=1}^{M} \frac{\alpha_{i \vee k + m(i)}}{i \vee k} \Pr\big(\lfloor \gamma S/(1-\gamma) \rfloor + 1 = i | \widehat{P}_j \leq \alpha_{i \vee k + m(i)}\big)$$

$$\leq \max_{1 \leq i \leq M} \left\{ \frac{\alpha_{i \vee k + m(i)}}{i \vee k} \right\}$$

$$\times \sum_{j=1}^{n_0} \sum_{i=1}^{M} \Pr\big(\lfloor \gamma S/(1-\gamma) \rfloor + 1 = i | \widehat{P}_j \leq \alpha_{i \vee k + m(i)}\big),$$

with the second inequality following from this:

$$(5) \qquad I(\widehat{P}_{(i)} \leq t) \leq \frac{1}{i} \sum_{j=1}^{n_0} I(\widehat{P}_j \leq t) \qquad \text{for any constant } 0 < t < 1,$$

which can be obtained from Markov's inequality.

Now, for each $1 \leq j \leq n_0$, we have

$$
\sum_{i=1}^{M} \Pr\big(\lfloor \gamma S/(1-\gamma) \rfloor + 1 = i \,|\, \widehat{P}_j \leq \alpha_{i \vee k + m(i)}\big)
$$

$$
= \sum_{i=1}^{M} \Pr\big(\lfloor \gamma S/(1-\gamma) \rfloor + 1 \geq i \,|\, \widehat{P}_j \leq \alpha_{i \vee k + m(i)}\big)
$$

$$
- \sum_{i=1}^{M} \Pr\big(\lfloor \gamma S/(1-\gamma) \rfloor + 1 \geq i + 1 \,|\, \widehat{P}_j \leq \alpha_{i \vee k + m(i)}\big)
$$

(6)

$$
\leq \sum_{i=1}^{M} \Pr\big(\lfloor \gamma S/(1-\gamma) \rfloor + 1 \geq i \,|\, \widehat{P}_j \leq \alpha_{i \vee k + m(i)}\big)
$$

$$
- \sum_{i=1}^{M} \Pr\big(\lfloor \gamma S/(1-\gamma) \rfloor + 1 \geq i + 1 \,|\, \widehat{P}_j \leq \alpha_{(i+1) \vee k + m(i+1)}\big)
$$

$$
\leq \Pr\big(\lfloor \gamma S/(1-\gamma) \rfloor + 1 \geq 1 \,|\, \widehat{P}_j \leq \alpha_{k + m(1)}\big) = 1.
$$

The first inequality follows from Assumption 2(a), since $I(\lfloor \gamma S/(1-\gamma) \rfloor + 1 \geq i)$ is a decreasing function of all the p-values. Applying (6) to (4), we finally note

$$
\gamma\text{-kFDP} \leq \max_{1 \leq i \leq M} \left\{ \frac{n_0 \alpha_{i \vee k + m(i)}}{i \vee k} \right\},
$$

and thus we have our next main result as follows.

THEOREM 3.2. *Let M and $m(i)$, for $i = 1, \ldots, M$, be defined as in Lemma* 3.1. *Then, given any set of constants $0 = \alpha_0' \leq \alpha_1' \leq \cdots \leq \alpha_n'$, the stepdown procedure with the critical values $\alpha_{i \vee k} = \alpha \alpha_{i \vee k}' / C_{k,n,\mathrm{SD}}^{(1)}, i = 1, \ldots, n$, where*

$$
C_{k,n,\mathrm{SD}}^{(1)} = \max_{k \leq n_0 \leq n} \max_{1 \leq i \leq M} \left\{ \frac{n_0 \alpha_{i \vee k + m(i)}'}{i \vee k} \right\},
$$

controls the γ-kFDP at α under Assumptions 1 *and* 2(a).

A stepup analog of Theorem 3.2 can be developed starting from the following lemma, whose proof again is given in the Appendix.

LEMMA 3.2. *Let $\tilde{m}(i) = \min\{m^*(i), i + n_1\}$, where $m^*(i) = \max\{1 \leq j \leq n : \lfloor \gamma j \rfloor + 1 \leq i\}$, for each $i = 1, \ldots, n_0$, and $m^*(0) = 0$. Consider a stepup proce-*

dure with critical values $\alpha_1 \leq \cdots \leq \alpha_n$. Then, for any fixed $1 \leq k \leq n_0$,

$$I\left(V > \max[\gamma R, k-1]\right) \leq \sum_{j=1}^{n_0} \sum_{i=k}^{n_0} \frac{I(\widehat{P}_j \leq \alpha_{\tilde{m}(i)}, \widehat{R}_2 = i)}{i}$$

(7)
$$\leq \sum_{j=1}^{n_0} \frac{I(\widehat{P}_j \leq \alpha_{\tilde{m}(k)}, \widehat{R}_2 \geq k)}{k}$$

$$+ \sum_{j=1}^{n_0} \sum_{i=k+1}^{n_0} \frac{I(\alpha_{\tilde{m}(i-1)} < \widehat{P}_j \leq \alpha_{\tilde{m}(i)}, \widehat{R}_2 \geq i)}{i},$$

with the double summation in the right-hand side of the second inequality being zero if $n_0 = k$, where $\widehat{R}_2$ is the number of rejections in a stepup procedure based on the p-values $\widehat{P}_i$, $i = 1, \ldots, n_0$, and the critical values $\alpha_{\tilde{m}(i)}$, $i = 1, \ldots, n_0$.

REMARK 3.2. If we let $n_0 = n$ in the above lemma, we note that $I(V > \max[\gamma R, k-1]) = I(V \geq k)$ and $\tilde{m}(i) = i$. In other words, the above lemma produces inequalities similar to (7) for $I(\widehat{R}_2 \geq k)$, with $\widehat{R}_2$ representing the number of rejections in a stepup procedure based on the null p-values $\widehat{P}_i$, $i = 1, \ldots, n_0$, and critical values α_i, $i = 1, \ldots, n_0$. For instance, from the second inequality in (7), we have

(8) $$I(\widehat{R}_2 \geq k) \leq \sum_{j=1}^{n_0} \frac{I(\widehat{P}_j \leq \alpha_k)}{k} + \sum_{j=1}^{n_0} \sum_{i=k+1}^{n_0} \frac{I(\alpha_{i-1} < \widehat{P}_j \leq \alpha_i)}{i},$$

which will be of use later. Of course, the first inequality in this case becomes an equality.

Taking expectations of both sides of the first inequality in (7), we get

$$\gamma\text{-kFDP} \leq \sum_{j=1}^{n_0} \sum_{i=k}^{n_0} \frac{\alpha_{\tilde{m}(i)}}{i} \Pr(\widehat{R}_2 = i | \widehat{P}_j \leq \alpha_{\tilde{m}(i)})$$

(9)
$$\leq \max_{k \leq i \leq n_0} \left\{ \frac{\alpha_{\tilde{m}(i)}}{i} \right\} \sum_{j=1}^{n_0} \sum_{i=k}^{n_0} \Pr(\widehat{R}_2 = i | \widehat{P}_j \leq \alpha_{\tilde{m}(i)}).$$

Making the same kind of arguments as in (6), we note that

$$\sum_{i=k}^{n_0} \Pr(\widehat{R}_2 = i | \widehat{P}_j \leq \alpha_{\tilde{m}(i)}) \leq \Pr(\widehat{R}_2 \geq k | \widehat{P}_j \leq \alpha_{\tilde{m}(k)}) \leq 1,$$

for each $1 \leq j \leq n_0$, using the fact that $I(\widehat{R}_2 \geq i)$ is a decreasing function of the null p-values and applying Assumption 2(b). Hence,

$$\gamma\text{-kFDP} \leq \max_{k \leq i \leq n_0} \left\{ \frac{n_0 \alpha_{\tilde{m}(i)}}{i} \right\},$$

which provides the following result.

THEOREM 3.3. *Let $\tilde{m}(i)$ be defined as in Lemma 3.2 for $i = 1, \ldots, n_0$. Then, given any set of constants $0 = \alpha'_0 \leq \alpha'_1 \leq \cdots \leq \alpha'_n$, the stepup procedure with the critical values $\alpha_{i \vee k} = \alpha \alpha'_{i \vee k} / C^{(1)}_{k,n,\mathrm{SU}}, i = 1, \ldots, n$, where*

$$C^{(1)}_{k,n,\mathrm{SU}} = \max_{k \leq n_0 \leq n} \max_{k \leq i \leq n_0} \left\{ \frac{n_0 \alpha'_{\tilde{m}(i)}}{i} \right\},$$

controls the γ-kFDP at α under Assumptions 1 and 2(b).

REMARK 3.3. Theorems 3.2 and 3.3 not only provide general approaches to constructing stepdown and stepup γ-kFDP controlling procedures, respectively, using only the marginal p-values under independence or certain positive dependence condition on the p-values, but also produce results when $k = 1$ that improve some previous works on controlling the γ-FDP [Lehmann and Romano (2005), Romano and Shaikh (2006a, 2006b)]. For instance, if we choose the α'_i's in these theorems as follows: $\alpha'_i = \{\lfloor \gamma i \rfloor + 1\}\alpha/\{n + \lfloor \gamma i \rfloor + 1 - i\}, i = 1, \ldots, n$, then we get the original Lehmann–Romano procedure and its stepup analog, since both $C^{(1)}_{1,n,\mathrm{SD}}$ and $C^{(1)}_{1,n,\mathrm{SU}}$ are equal to α (see Proposition A.1 and its proof in the Appendix). However, there are other stepdown and stepup procedures controlling the γ-FDP under these assumptions, such as those obtained by re-scaling the critical values, $\alpha'_i = i\alpha/n, i = 1, \ldots, n$, of the BH [Benjamini and Hochberg (1995)] stepup or the critical values, $\alpha'_i = i\alpha/[n - i(1 - \alpha) + 1], i = 1, \ldots, n$, of the GBS [Gavrilov, Benjamini and Sarkar (2009)] stepdown methods, that can be constructed using the above theorems. Our simulation studies indicate that a stepwise procedure based on the rescaled versions of the BH or GBS critical values is less powerful than that based on the rescaled version of the Lehmann–Romano critical values (see Figures S.4–S.7 in the supplementary material [Guo, He and Sarkar (2014)]). Therefore, the interest of Theorems 3.2 and 3.3 with respect to Theorem 3.1 seems to be mainly theoretical when $k = 1$.

3.1.2. *Based on marginal and pairwise null distributions of the p-values.* In practice, the null p-values often have a known common pairwise joint distribution, and by explicitly utilizing such correlation information better adjustments can be made, potentially resulting in more powerful γ-FDP stepwise procedures. So, with that in mind, we present some results here and in Section 3.2.2 under the following assumption along with Assumptions 1 and 2(b) or only Assumption 1.

ASSUMPTION 3. The null p-values $\widehat{P}_1, \ldots, \widehat{P}_{n_0}$ have a known common pairwise joint distribution function $F(u, v) = \Pr(\widehat{P}_i \leq u, \widehat{P}_j \leq v)$.

We consider generalizing the Lehmann–Romano stepwise procedure in Theorem 3.1, for any fixed $2 \leq k \leq n_0$. The γ-kFDP of this procedure is given by

$$\gamma\text{-kFDP} = \Pr\{V \geq \max[g(R), k]\}$$

(10)
$$= \Pr\left(\bigcup_{v=k}^{n_0} \{\widehat{P}_{(v)} \leq \alpha_R, g(R) \leq v, V = v\}\right)$$

$$\leq \Pr\left(\bigcup_{v=k}^{n_0} \left\{\widehat{P}_{(v)} \leq \frac{v\alpha}{n_0}\right\}\right) = \Pr(\widehat{R}_{n_0} \geq k),$$

where $\widehat{R}_{n_0}$ is the number of rejections in the stepup procedure based on all the n_0 null p-values and the critical values $\beta_i = i\alpha/n_0$, $i = 1, \ldots, n_0$. The γ-kFDP can be bounded above using the following inequality which holds under Assumptions 1 and 2(b):

(11)
$$\Pr(\widehat{R}_{n_0} \geq k) \leq \frac{\alpha}{n_0} \sum_{i=1}^{n_0} \Pr(\widehat{R}_{n_0-1}^{(-i)} \geq k - 1 | \widehat{P}_i \leq \beta_k),$$

for any fixed $1 \leq k \leq n_0$, where $\widehat{R}_{n_0-1}^{(-i)}$ is the number of rejections in the stepup procedure based on the $n_0 - 1$ null p-values $\{\widehat{P}_1, \ldots, \widehat{P}_{n_0}\} \setminus \{\widehat{P}_i\}$ and the critical values β_i, $i = 2, \ldots, n_0$. This can be proved using arguments similar to those used above while proving Theorems 3.2 or 3.3; see the Appendix, for a proof.

As seen from (11), if we rely only on the marginal distributions of the null p-values, we simply get γ-kFDP $\leq \alpha$, and thus our attempt to generalize the Lehmann–Romano procedure to a γ-kFDP controlling procedure under Assumption 2(b) does not work in the sense that it takes us back to the original Lehmann–Romano procedure, which is trivially known to control the γ-kFDP. Hence, we consider utilizing also the pairwise distributions of the null p-values to obtain a nontrivial generalization of the Lehmann–Romano procedure. More specifically, we use the following inequality provided by (8):

(12)
$$I\left(\widehat{R}_{n_0-1}^{(-i)} \geq k - 1\right)$$
$$\leq \sum_{j(\neq i)=1}^{n_0} \frac{I(\widehat{P}_j \leq \beta_k)}{k-1} + \sum_{j(\neq i)=1}^{n_0} \sum_{l=k}^{n_0-1} \frac{I(\beta_l < \widehat{P}_j \leq \beta_{l+1})}{l},$$

and apply it to the right-hand side of (11) to get the following upper bound for the γ-kFDP of the Lehmann–Romano stepwise procedure under Assumption 2(b):

(13)
$$\gamma\text{-kFDP} \leq \frac{\alpha}{n_0} \sum_{i=1}^{n_0} \sum_{j(\neq i)=1}^{n_0} \left(\frac{\Pr(\widehat{P}_j \leq \beta_k | \widehat{P}_i \leq \beta_k)}{k-1}\right.$$
$$\left. + \sum_{l=k}^{n_0-1} \frac{\Pr(\beta_l < \widehat{P}_j \leq \beta_{l+1} | \widehat{P}_i \leq \beta_k)}{l}\right).$$

Based on this upper bound and that the γ-kFDP of the Lehmann–Romano stepwise procedure is $\leq \alpha$ under Assumptions 1 and 2(b), we now have the following theorem providing the desired generalized version of the Lehmann–Romano procedure controlling the γ-kFDP.

THEOREM 3.4. *Let $2 \leq k \leq n_0$ and Assumption 3 hold. Given $\beta_i = i\alpha/n_0$, $i = 1, \ldots, n_0$, let*

$$(14) \qquad C_{k,n} = \max_{k \leq n_0 \leq n} \left\{ (n_0 - 1) \left(\frac{F(\beta_k|\beta_k)}{k-1} + \sum_{l=k}^{n_0-1} \frac{F(\beta_{l+1}|\beta_k) - F(\beta_l|\beta_k)}{l} \right) \right\}$$

with the summation within parentheses being zero if $n_0 = k$, where $F(u|v) = F(u, v)/v$. Then the stepup or stepdown procedure with the critical constants $\alpha_{i \vee k}$, $i = 1, \ldots, n$, where

$$(15) \qquad \alpha_i = \frac{(\lfloor \gamma i \rfloor + 1)\alpha}{(C_{k,n} \wedge 1)(n + \lfloor \gamma i \rfloor + 1 - i)}, \qquad i = 1, \ldots, n,$$

controls the γ-kFDP at α under Assumptions 1 and 2(b).

3.2. *Procedures under arbitrary dependence.* We now present some γ-kFDP controlling procedures under arbitrary dependence of the p-values. By arbitrary dependence, we mean that these p-values are not known to have any specific type of dependence structure, like positive or other, even though their joint distributions of some particular orders might be known. We will assume, as in Section 3.1.2, that the null p-values have a common pairwise joint distribution of a known form $F(u, v)$. Our procedures are developed relying either only on the marginal p-values or on the marginal as well as this common pairwise joint null distribution of the p-values. We can obtain some new results on controlling the γ-FDP by taking $k = 1$.

3.2.1. *Based on marginal p-values.* First, let us consider a stepdown procedure with critical values $\alpha_{i \vee k}$, $i = 1, \ldots, n$. Starting from Lemma 3.1 and proceeding as in proving Theorem 3.2, we have, with $i \vee k + m(i)$ defined as $\bar{m}(i)$ [where $\bar{m}(0) = 0$] for notational convenience,

$$I(V > \max[\gamma R, k - 1])$$

$$(16) \qquad \leq \sum_{j=1}^{n_0} \sum_{i=1}^{M} \frac{I(\widehat{P}_j \leq \alpha_{\bar{m}(i)})}{i \vee k} I(\lfloor \gamma S/(1 - \gamma) \rfloor + 1 = i)$$

$$\leq \sum_{j=1}^{n_0} \sum_{i=1}^{M} \left[\frac{I(\widehat{P}_j \leq \alpha_{\bar{m}(i)})}{i \vee k} - \frac{I(\widehat{P}_j \leq \alpha_{\bar{m}(i-1)})}{(i-1) \vee k} \right] I(\lfloor \gamma S/(1 - \gamma) \rfloor + 1 \geq i)$$

$$\leq \sum_{j=1}^{n_0} \sum_{i=1}^{M} \frac{I(\alpha_{\bar{m}(i-1)} < \widehat{P}_j \leq \alpha_{\bar{m}(i)})}{i \vee k}.$$

Taking expectations of both sides in (16), we get

$$(17) \qquad \gamma\text{-kFDP} = \Pr\{V \geq \max[g(R), k]\} \leq n_0 \sum_{i=1}^{M} \frac{\alpha_{\bar{m}(i)} - \alpha_{\bar{m}(i-1)}}{i \vee k}.$$

From this, we get the following theorem.

THEOREM 3.5. *Let M and $m(i)$, for $i = 1, \ldots, M$, be defined as in Lemma* 3.1, *and $\bar{m}(i) = i \vee k + m(i)$ [where $\bar{m}(0) = 0$]. Then, given any set of constants $\alpha'_k \leq \cdots \leq \alpha'_n$, the stepdown procedure with the critical values $\alpha_i = \alpha \alpha'_{i \vee k} / C^{(2)}_{k,n,\text{SD}}, i = 1, \ldots, n$, where*

$$C^{(2)}_{k,n,\text{SD}} = \max_{k \leq n_0 \leq n} \left\{ n_0 \left(\sum_{i=1}^{M} \frac{\alpha'_{\bar{m}(i)} - \alpha'_{\bar{m}(i-1)}}{i \vee k} \right) \right\},$$

controls the γ-kFDP at α under Assumption 1.

We now present the development of a stepup analog of Theorem 3.5. From Lemma 3.2, we note that for a stepup procedure with critical values $\alpha_{i \vee k}$, $i = 1, \ldots, n$,

$$(18) \qquad \begin{aligned} &I\big(V > \max[\gamma R, k-1]\big) \\ &\leq \sum_{j=1}^{n_0} \frac{I(\widehat{P}_j \leq \alpha_{\tilde{m}(k)})}{k} + \sum_{j=1}^{n_0} \sum_{i=k+1}^{n_0} \frac{I(\alpha_{\tilde{m}(i-1)} < \widehat{P}_j \leq \alpha_{\tilde{m}(i)})}{i}. \end{aligned}$$

Taking expectations of both sides in (18), we get

$$(19) \qquad \begin{aligned} \gamma\text{-kFDP} &= \Pr\{V \geq \max[\gamma R, k-1]\} \\ &\leq n_0 \left(\frac{\alpha_{\tilde{m}(k)}}{k} + \sum_{i=k+1}^{n_0} \frac{\alpha_{\tilde{m}(i)} - \alpha_{\tilde{m}(i-1)}}{i} \right), \end{aligned}$$

which gives the following theorem.

THEOREM 3.6. *Let $\tilde{m}(i)$ be defined as in Lemma* 3.2 *for $i = 1, \ldots, n_0$. Then, given any set of constants $\alpha'_k \leq \cdots \leq \alpha'_n$, the stepup procedure with the critical values $\alpha_i = \alpha \alpha'_{i \vee k} / C^{(2)}_{k,n,\text{SU}}, i = 1, \ldots, n$, where*

$$C^{(2)}_{k,n,\text{SU}} = \max_{k \leq n_0 \leq n} \left\{ n_0 \left(\frac{\alpha'_{\tilde{m}(k)}}{k} + \sum_{i=k+1}^{n_0} \frac{\alpha'_{\tilde{m}(i)} - \alpha'_{\tilde{m}(i-1)}}{i} \right) \right\},$$

controls the γ-kFDP at α under Assumption 1.

REMARK 3.4. When $k = 1$, the results in Theorems 3.5 and 3.6 reduce to those given by Romano and Shaikh in (2006a) and (2006b), respectively, although

our expressions of the upper bounds given in these theorems are different from theirs. Thus, our results generalize those of Romano and Shaikh from controlling the γ-FDP to γ-kFDP under arbitrary dependence and relying only on the marginal null distributions of the p-values. However, we should emphasize that we provide alternative, much simpler proofs for these results.

3.2.2. *Based on marginal and pairwise distributions of the null p-values.* We will start again from Lemma 3.1 towards constructing a stepdown procedure. Consider splitting the sum in the right-hand side of (3) in two parts, with the summation taken over i from 1 to K in the first part and over i from $K + 1$ to M in the second, for some fixed K, where $1 \le K \le M$. The idea behind this splitting is to utilize the marginal distributions of the null p-values from the first part through the inequality (5), as we did before, and the pairwise joint distributions of these p-values from the second part through the following new inequality (to be proved in the Appendix):

$$(20) \qquad I(\widehat{P}_{(i)} \le t) \le \frac{1}{i(i-1)} \sum_{j=1}^{n_0} \sum_{j'(\ne j)=1}^{n_0} I(\max\{\widehat{P}_j, \widehat{P}_{j'}\} \le t),$$

where $0 < t < 1$ is fixed, for all i such that $2 \le i \le n_0$,

$$I(V > \max[\gamma R, k-1])$$

$$\le \sum_{i=1}^{K} \sum_{j=1}^{n_0} \frac{1}{i \vee k} I(\widehat{P}_j \le \alpha_{\bar{m}(i)}) I(\lfloor \gamma S/(1-\gamma) \rfloor + 1 = i)$$

$$(21)$$

$$+ \sum_{i=K+1}^{M} \sum_{j=1}^{n_0} \sum_{l(\ne j)=1}^{n_0} \frac{1}{(i \vee k)(i \vee k - 1)} I(\max(\widehat{P}_j, \widehat{P}_l) \le \alpha_{\bar{m}(i)})$$

$$\times I(\lfloor \gamma S/(1-\gamma) \rfloor + 1 = i).$$

Now, for each $j = 1, \dots, n_0$, the summation over i in the double-summation in (21) is equal to

$$\sum_{i=1}^{K} \left[\frac{I(\widehat{P}_j \le \alpha_{\bar{m}(i)})}{i \vee k} - \frac{I(\widehat{P}_j \le \alpha_{\bar{m}(i-1)})}{(i-1) \vee k} \right] I(\lfloor \gamma S/(1-\gamma) \rfloor + 1 \ge i)$$

$$- \frac{I(\widehat{P}_j \le \alpha_{\bar{m}(K)})}{K \vee k} I(\lfloor \gamma S/(1-\gamma) \rfloor + 1 \ge K + 1)$$

$$(22)$$

$$\le \sum_{i=1}^{K} \frac{I(\alpha_{\bar{m}(i-1)} < \widehat{P}_j \le \alpha_{\bar{m}(i)})}{i \vee k}$$

$$- \frac{I(\widehat{P}_j \le \alpha_{\bar{m}(K)})}{K \vee k} I(\lfloor \gamma S/(1-\gamma) \rfloor + 1 \ge K + 1, M \ge K + 1)$$

with $I(\widehat{P}_j \le \alpha_{\bar{m}(0)})/0 \vee k = 0$, and similarly for each $j \ne l$, the summation over i in the triple-summation in (21) is less than or equal to

$$
\sum_{i=K+2}^{M} \left[\frac{I(\max(\widehat{P}_j, \widehat{P}_l) \le \alpha_{\bar{m}(i)})}{(i \vee k)(i \vee k - 1)} - \frac{I(\max(\widehat{P}_j, \widehat{P}_l) \le \alpha_{\bar{m}(i-1)})}{((i-1) \vee k)((i-1) \vee k - 1)} \right]
$$

$$
\times I\big(\lfloor \gamma S/(1-\gamma) \rfloor + 1 \ge i\big)
$$

$$
+ \frac{I(\max(\widehat{P}_j, \widehat{P}_l) \le \alpha_{\bar{m}(K+1)})}{((K+1) \vee k)[(K+1) \vee k - 1]}
$$

(23)
$$
\times I\big(\lfloor \gamma S/(1-\gamma) \rfloor + 1 \ge K+1, M \ge K+1\big)
$$

$$
\le \sum_{i=K+2}^{M} \frac{I(\alpha_{\bar{m}(i-1)} < \max(\widehat{P}_j, \widehat{P}_l) \le \alpha_{\bar{m}(i)})}{(i \vee k)(i \vee k - 1)}
$$

$$
+ \frac{I(\max(\widehat{P}_j, \widehat{P}_l) \le \alpha_{\bar{m}(K+1)})}{((K+1) \vee k)[(K+1) \vee k - 1]}
$$

$$
\times I\big(\lfloor \gamma S/(1-\gamma) \rfloor + 1 \ge K+1, M \ge K+1\big).
$$

In addition, by simple algebraic calculation, we have

$$
\frac{I(\widehat{P}_j \vee \widehat{P}_l \le \alpha_{\bar{m}(K+1)})}{((K+1) \vee k)((K+1) \vee k - 1)} - \frac{I(\widehat{P}_j \le \alpha_{\bar{m}(K)})}{(K \vee k)(n_0 - 1)}
$$

(24)
$$
\le \frac{(n_0 - (K+1) \vee k) I(\widehat{P}_j \le \alpha_{\bar{m}(K)}, \widehat{P}_l \le \alpha_{\bar{m}(K+1)})}{((K+1) \vee k)((K+1) \vee k - 1)(n_0 - 1)}
$$

$$
+ \frac{I(\alpha_{\bar{m}(K)} < \widehat{P}_j \le \alpha_{\bar{m}(K+1)}, \widehat{P}_l \le \alpha_{\bar{m}(K+1)})}{((K+1) \vee k)((K+1) \vee k - 1)}.
$$

Applying (22)–(24) to (21) and taking expectations of both sides in (21), we get

$$
\gamma\text{-kFDP} \le \sum_{i=1}^{K} \frac{n_0(\alpha_{\bar{m}(i)} - \alpha_{\bar{m}(i-1)})}{i \vee k}
$$

$$
+ \sum_{i=K+2}^{M} \frac{n_0(n_0 - 1)[F(\alpha_{\bar{m}(i)}, \alpha_{\bar{m}(i)}) - F(\alpha_{\bar{m}(i-1)}, \alpha_{\bar{m}(i-1)})]}{(i \vee k)(i \vee k - 1)}
$$

(25)
$$
+ \frac{n_0(n_0 - 1) F(\alpha_{\bar{m}(K+1)}, \alpha_{\bar{m}(K+1)})}{((K+1) \vee k)((K+1) \vee k - 1)} I(M \ge K+1)
$$

$$
- \frac{n_0 F(\alpha_{\bar{m}(K)}, \alpha_{\bar{m}(K+1)})}{(K+1) \vee k} I(M \ge K+1).
$$

This inequality produces the next theorem, one of our main results in this subsection, with $C_{n,\text{SD}}^{(3)}(\beta)$ in that theorem being defined as follows:

$$C_{k,n,\text{SD}}^{(3)}(\beta)$$

$$= \max_{k \leq n_0 \leq n} \min_{1 \leq K \leq M} \left\{ \sum_{i=1}^{K} \frac{n_0[\alpha'_{\tilde{m}(i)}(\beta) - \alpha'_{\tilde{m}(i-1)}(\beta)]}{i \vee k} \right.$$

$$+ \sum_{i=K+2}^{M} \left(n_0(n_0 - 1)[F(\alpha'_{\tilde{m}(i)}(\beta), \alpha'_{\tilde{m}(i)}(\beta)) \right.$$

$$\left. - F(\alpha'_{\tilde{m}(i-1)}(\beta), \alpha'_{\tilde{m}(i-1)}(\beta))] \right)$$

$$/((i \vee k)(i \vee k - 1))$$

$$+ \frac{n_0(n_0 - 1)F(\alpha'_{\tilde{m}(K+1)}(\beta), \alpha'_{\tilde{m}(K+1)}(\beta))}{((K+1) \vee k)((K+1) \vee k - 1)}$$

$$\times I(M \geq K + 1)$$

$$\left. - \frac{n_0 F(\alpha'_{\tilde{m}(K)}(\beta), \alpha'_{\tilde{m}(K+1)}(\beta))}{(K+1) \vee k} I(M \geq K + 1) \right\},$$

given a sequence of constants $0 = \alpha'_0(\beta) \leq \alpha'_1(\beta) \leq \cdots \leq \alpha'_n(\beta)$ with a fixed $\beta \in (0, 1)$.

THEOREM 3.7. *Given any sequence of critical constants* $0 = \alpha'_0(\beta) \leq \alpha'_1(\beta) \leq \cdots \leq \alpha'_n(\beta)$, *for a fixed* $\beta \in (0, 1)$, *the stepdown procedure with the critical values* $\alpha_{i \vee k}$, $i = 1, \ldots, n$, *satisfying* $\alpha_{i \vee k} = \alpha'_{i \vee k}(\beta^*_{\text{SD}})$ *and* $C_{k,n,\text{SD}}^{(3)}(\beta^*_{\text{SD}}) = \alpha$, *controls the* γ-*kFDP at* α *under Assumptions* 1 *and* 3.

We now derive a stepup analog of Theorem 3.7 starting from the following inequality, which is obtained from Lemma 3.2 by splitting the right-hand sum in the second inequality of that lemma into two parts, as before, for a fixed $1 \leq k \leq K \leq n_0$:

$$I(V > \max[\gamma R, k - 1])$$

$$\leq \sum_{j=1}^{n_0} \frac{I(\widehat{P}_j \leq \alpha_{\tilde{m}(k-1)})}{k} + \sum_{j=1}^{n_0} \sum_{i=k}^{K} \frac{I(\alpha_{\tilde{m}(i-1)} < \widehat{P}_j \leq \alpha_{\tilde{m}(i)})}{i}$$

$$+ \sum_{j=1}^{n_0} \sum_{i=K+1}^{n_0} \frac{I(\widehat{R}_2 \geq i)I(\alpha_{\tilde{m}(i-1)} < \widehat{P}_j \leq \alpha_{\tilde{m}(i)})}{i}.$$

Again, the idea behind this splitting is to capture the pairwise joint distributions of the null p-values from the second part, and for that, we use the following inequal-

ity, which can be seen to follow from Lemma 3.2 (see Remark 3.2):

$$(26) \quad I(\widehat{R}_2 \geq r) \leq \sum_{l=1}^{n_0} \left(\frac{I(\widehat{P}_l \leq \alpha_{\tilde{m}(r)})}{r} + \sum_{s=r+1}^{n_0} \frac{I(\alpha_{\tilde{m}(s-1)} < \widehat{P}_l \leq \alpha_{\tilde{m}(s)})}{s} \right).$$

Thus, we get

$$I(V > \max[\gamma R, k-1])$$

$$\leq \sum_{j=1}^{n_0} \frac{I(\widehat{P}_j \leq \alpha_{\tilde{m}(k-1)})}{k}$$

$$+ \sum_{j=1}^{n_0} \sum_{r=k}^{K} \frac{I(\alpha_{\tilde{m}(r-1)} < \widehat{P}_j \leq \alpha_{\tilde{m}(r)})}{r}$$

$$(27)$$

$$+ \sum_{j=1}^{n_0} \sum_{r=K+1}^{n_0} \frac{I(\alpha_{\tilde{m}(r-1)} < \widehat{P}_j \leq \alpha_{\tilde{m}(r)})}{r^2}$$

$$+ \sum_{j=1}^{n_0} \sum_{l(\neq j)=1}^{n_0} \sum_{r=K+1}^{n_0} \sum_{s=r+1}^{n_0} \frac{I(\alpha_{\tilde{m}(r-1)} < \widehat{P}_j \leq \alpha_{\tilde{m}(r)}, \alpha_{\tilde{m}(s-1)} < \widehat{P}_l \leq \alpha_{\tilde{m}(s)})}{rs}$$

$$+ \sum_{j=1}^{n_0} \sum_{l(\neq j)=1}^{n_0} \sum_{r=K+1}^{n_0} \frac{I(\alpha_{\tilde{m}(r-1)} < \widehat{P}_j \leq \alpha_{\tilde{m}(r)}, \widehat{P}_l \leq \alpha_{\tilde{m}(r)})}{r^2}.$$

Taking expectations of both sides in (27), we finally have

$$\gamma\text{-kFDP} \leq \frac{n_0 \alpha_{\tilde{m}(k-1)}}{k}$$

$$+ \sum_{r=k}^{K} \frac{n_0(\alpha_{\tilde{m}(r)} - \alpha_{\tilde{m}(r-1)})}{r}$$

$$(28)$$

$$+ \sum_{r=K+1}^{n_0} \frac{n_0(\alpha_{\tilde{m}(r)} - \alpha_{\tilde{m}(r-1)})}{r^2}$$

$$+ \sum_{r=K+1}^{n_0} \sum_{s=r+1}^{n_0} \frac{n_0(n_0 - 1)G(\alpha_{\tilde{m}(r)}, \alpha_{\tilde{m}(s)})}{rs}$$

$$+ \sum_{r=K+1}^{n_0} \frac{n_0(n_0 - 1)(F(\alpha_{\tilde{m}(r)}, \alpha_{\tilde{m}(r)}) - F(\alpha_{\tilde{m}(r)}, \alpha_{\tilde{m}(r-1)}))}{r^2},$$

where

$$G(\alpha_r, \alpha_s) = F(\alpha_r, \alpha_s) - F(\alpha_{r-1}, \alpha_s) - F(\alpha_r, \alpha_{s-1}) + F(\alpha_{r-1}, \alpha_{s-1}).$$

Our next main result of this subsection follows from the inequality (28), with $C_{n,\mathrm{SU}}^{(3)}(\beta)$ in that result being defined as follows:

$$
\begin{aligned}
C_{k,n,\mathrm{SU}}^{(3)}&(\beta) \\
&= \max_{k \le n_0 \le n} \min_{k \le K \le n_0} \Bigg\{ \frac{n_0 \alpha'_{\tilde{m}(k-1)}(\beta)}{k} + \sum_{r=k}^{K} \frac{n_0[\alpha'_{\tilde{m}(r)}(\beta) - \alpha'_{\tilde{m}(r-1)}(\beta)]}{r} \\
&\qquad + \sum_{r=K+1}^{n_0} \Bigg(\frac{n_0[\alpha'_{\tilde{m}(r)}(\beta) - \alpha'_{\tilde{m}(r-1)}(\beta)]}{r^2} \\
&\qquad\qquad + \sum_{s=r+1}^{n_0} \frac{n_0(n_0-1)G(\alpha'_{\tilde{m}(r)}(\beta), \alpha'_{\tilde{m}(s)}(\beta))}{rs} \\
&\qquad\qquad + (n_0(n_0-1) \\
&\qquad\qquad\quad \times [F(\alpha'_{\tilde{m}(r)}(\beta), \alpha'_{\tilde{m}(r)}(\beta)) \\
&\qquad\qquad\qquad - F(\alpha'_{\tilde{m}(r)}(\beta), \alpha'_{\tilde{m}(r-1)}(\beta))])/r^2 \Bigg) \Bigg\},
\end{aligned}
$$

for any given sequence of constants $0 = \alpha'_0(\beta) \le \alpha'_1(\beta) \le \cdots \le \alpha'_n(\beta)$, for a fixed $\beta \in (0, 1)$.

THEOREM 3.8. *Given any sequence of critical constants* $0 = \alpha'_0(\beta) \le \alpha'_1(\beta) \le \cdots \le \alpha'_n(\beta)$, *for a fixed* $\beta \in (0, 1)$, *the stepup procedure with the critical values* $\alpha_{i \vee k}$, $i = 1, \ldots, n$, *satisfying* $\alpha_i = \alpha'_i(\beta_{\mathrm{SU}}^*)$ *and* $C_{k,n,\mathrm{SU}}^{(3)}(\beta_{\mathrm{SU}}^*) = \alpha$, *controls the* γ-*kFDP at* α *under Assumptions* 1 *and* 3.

REMARK 3.5. Romano and Shaikh proved the following two results in (2006a) and (2006b), respectively, based on marginal p-values under arbitrary dependence: the γ-FDP of the stepdown procedure with critical values α_i, $i = 1, \ldots, n$, satisfies

$$
(29) \qquad \gamma\text{-FDP} \le \max_{1 \le n_0 \le n} \left\{ n_0 \sum_{i=1}^{M} \frac{\alpha_{\tilde{m}(i)} - \alpha_{\tilde{m}(i-1)}}{i} \right\};
$$

whereas the γ-FDP of the stepup procedure with the same set of critical values satisfies

$$
(30) \qquad \gamma\text{-FDP} \le \max_{1 \le n_0 \le n} \left\{ n_0 \sum_{i=1}^{n_0} \frac{\alpha_{\tilde{m}(i)} - \alpha_{\tilde{m}(i-1)}}{i} \right\}.
$$

These upper bounds are always larger than the corresponding upper bounds of the γ-FDP we derive here, as seen by letting $k = 1$, $K = M$ in (25) and $k = 1$, $K = n_0$ in (28), respectively. Thus, theoretically, the stepdown and stepup γ-FDP controlling procedures introduced in Theorems 3.7 (with $k = 1$) and 3.8 (with $k = 1$), respectively, are always more powerful than the corresponding ones given by Romano and Shaikh in (2006a) and (2006b), respectively.

4. Simulation studies. We ran extensive simulations numerically examining the performances of different procedures proposed in the above section in comparison with their relevant competitors under different settings for the parameters, π_0, γ, k and the strength of positive dependence, and having considered all or just one of three special types of positive dependence structure—uniform pairwise dependence, clumpy dependence and autoregressive of order one [AR(1)] dependence. The results were graphically summarized in twelve figures, and the main findings in those graphs are described in this section. However, we present here the figures that pertain to the uniform pairwise dependence, while the rest are presented, for lack of space here, in the supplementary material [Guo, He and Sarkar (2014)].

Note that, except in the procedures in Theorems 3.1 and 3.4, which have been developed directly from the Lehmann–Romano (LR) critical values $\alpha_i'(\beta) = \frac{(\lfloor \gamma i \rfloor + 1)\beta}{n + \lfloor \gamma i \rfloor + 1 - i}$, $i = 1, \ldots, n$, the critical values in all other procedures can be chosen arbitrarily before being rescaled appropriately to ensure a control over the γ-FDP or γ-kFDP. In many of our simulations, we had chosen the same LR critical values $\alpha_i'(\beta)$ in these other procedures with β being rescaled according to the formulas in the corresponding theorems. We will refer to a procedure, except the stepwise one in Theorem 3.1, as simply LR-type procedure whenever it is directly or indirectly based on the LR critical values. Similarly, by BH- and GBS-type stepwise procedures that we will use in some simulations, we mean that the critical values of the procedure in that procedure are obtained by rescaling the original BH or GBS critical values according to the formula given in the corresponding theorem.

A part of our simulation study was geared toward answering the following two questions:

(Q1) When controlling the γ-FDP assuming positive dependence, how good is the improvement supposedly offered by the newly proposed LR stepup procedure in Theorem 3.1 over the original LR stepdown procedure?

(Q2) When controlling the γ-FDP assuming arbitrary dependence, how do the newly suggested LR-type stepdown and stepup procedures in Theorems 3.7 and 3.8, respectively, with $k = 1$, incorporating pairwise correlation information perform compared to the corresponding existing LR-type stepdown and stepup procedures in Romano and Shaikh (2006a, 2006b) that do not incorporate such pairwise correlation information?

The performance of each procedure is judged, while answering (Q1) and (Q2), in terms of how well the γ-FDP is controlled at the desired level and also the average power, which is the expected proportion of false nulls that are rejected, under varying π_0, γ, and the strength of positive dependence.

To simulate the values of γ-FDP and average power for each of the methods referred to in (Q1) and (Q2), we first generated n dependent normal random variables $N(\mu_i, 1)$, $i = 1, \ldots, n$, with $\pi_0 n$ of the μ_i's being equal to 0 and the rest being equal to $d = \sqrt{10}$, and a correlation matrix Γ. The following three different types of Γ were considered for (Q1): (i) $\Gamma = (1 - \rho)I_n + \rho 1_n 1_n'$, in case of uniform

pairwise dependence, (ii) $\gamma = I_{n/s} \otimes [(1-\rho)I_s + \rho 1_s 1_s']$, in case of block dependence with the block size s, and (iii) $\Gamma = ((\rho^{|i-j|}))$, in case of AR(1) dependence, where $1_n = (1, \ldots, 1)'$; whereas, for (Q2), the Γ of the type (i) was considered. In each case, ρ was nonnegative. We then applied each method to the generated data to test $H_i : \mu_i = 0$ against $K_i : \mu_i \neq 0$ simultaneously for $i = 1, \ldots, n$, at level $\alpha = 0.05$. We repeated the above two steps 2000 times.

Figure 1 and Figures S.1–S.3 (in the supplementary material [Guo, He and Sarkar (2014)]) provide an answer to (Q1) and Figure 2 answers (Q2). As seen from Figure 1, when the underlying test statistics have a common positive correlation, the newly introduced stepup γ-FDP procedure in Theorem 3.1 improves the power of the original Lehmann–Romano stepdown procedure. This improvement is quite noticeable when the correlation is high. When the underlying test statistics are block or AR(1) dependent, the stepup procedure, as expected, does still have better power, as seen from Figures S.1–S.3. However, in case of block dependence, the larger the block size, the more significant seems to be the power improvement; whereas, in case of AR(1) dependence, the power improvement seems to be only
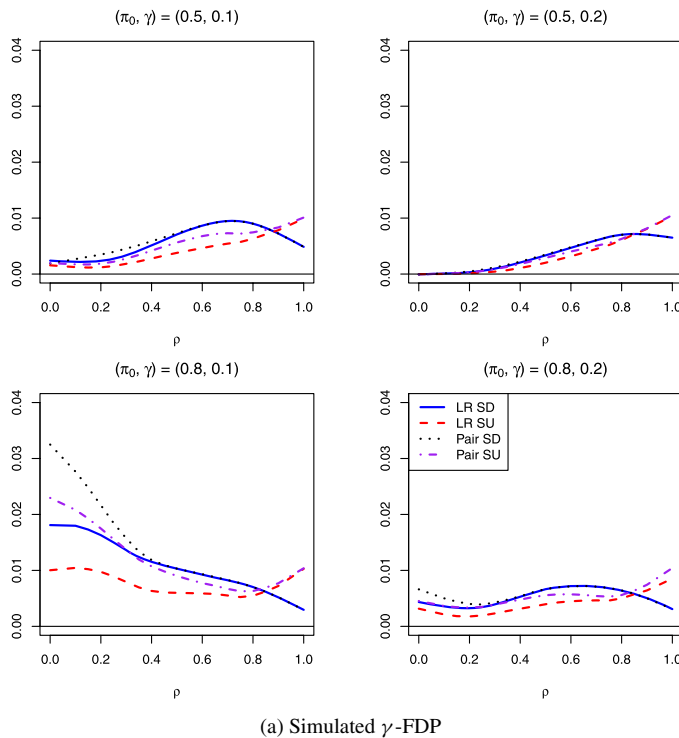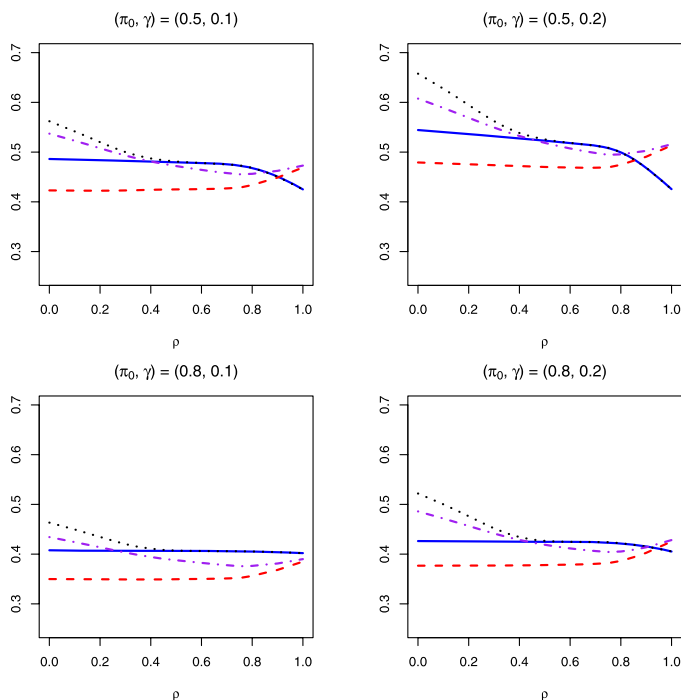

(a) Simulated γ-FDP

FIG. 2. *Simulated values of γ-FDP and average power of the existing LR-type stepdown (LR SD) and stepup (LR SU) γ-FDP procedures in Theorems 3.5 and 3.6 (with $k = 1$) and the newly suggested LR-type stepdown (Pair SD) and stepup (Pair SU) γ-FDP procedures in Theorems 3.7 and 3.8 (with $k = 1$), all developed assuming arbitrary dependence, for $n = 50$ and $\alpha = 0.05$.*

(b) Simulated average power

FIG. 2. (*Continued*).

significant when the dependence is high and the proportion of true nulls is not large. In addition, as seen from Figure 1, for the original LR stepdown procedure and its stepup analogue, they behave very differently when correlation ρ is close to one, which corroborates the observation of Roquain and Villers (2011), and their apparent worst performances in terms of the γ-FDP control seem to depend on the values of π_0 and γ.

From Figure 2, we see that when controlling the γ-FDP assuming arbitrary dependence, the performances of the existing LR-type Romano–Shaikh stepdown and stepup procedures can be significantly improved by utilizing the pairwise correlation information via the use of the newly suggested LR-type stepdown and stepup γ-FDP procedures in Theorems 3.7 and 3.8, respectively, with $k = 1$, when the underlying test statistics are slightly or moderately correlated with a common correlation.

Our next set of simulations was run with a view to investigating the performances of the proposed stepwise γ-kFDP controlling procedures in the setting of a common pairwise positive dependence. Specifically, we investigated the following two questions:

(Q3) When controlling the γ-kFDP assuming positive dependence, how well the LR-type stepdown and stepup procedures in Theorem 3.4 incorporating pair-

wise correlation information perform compared to the LR-type stepdown and stepup procedures in Theorems 3.2 and 3.3, respectively, that do not incorporate such pairwise correlation information?

(Q4) When controlling the γ-kFDP assuming arbitrary dependence, how well do the LR-type stepdown and stepup procedures in Theorems 3.7 and 3.8, respectively, incorporating pairwise correlation information perform compared to the LR-type stepdown and stepup procedures in Theorems 3.5 and 3.6, respectively, that do not incorporate such pairwise correlation information?

The performance of each procedure is judged, while answering (Q3) and (Q4), in terms of how well the γ-kFDP is controlled at the desired level and also the average power under varying π_0, k and the strength of positive dependence. We used the simulation settings for (Q3) and (Q4) that are same as in answering (Q1) and (Q2), respectively, but considering only the equi-correlated normal case.

Figures 3 and 4 provide answers to (Q3) and (Q4), respectively. From Figure 3, we see that when controlling the γ-kFDP assuming positive dependence,
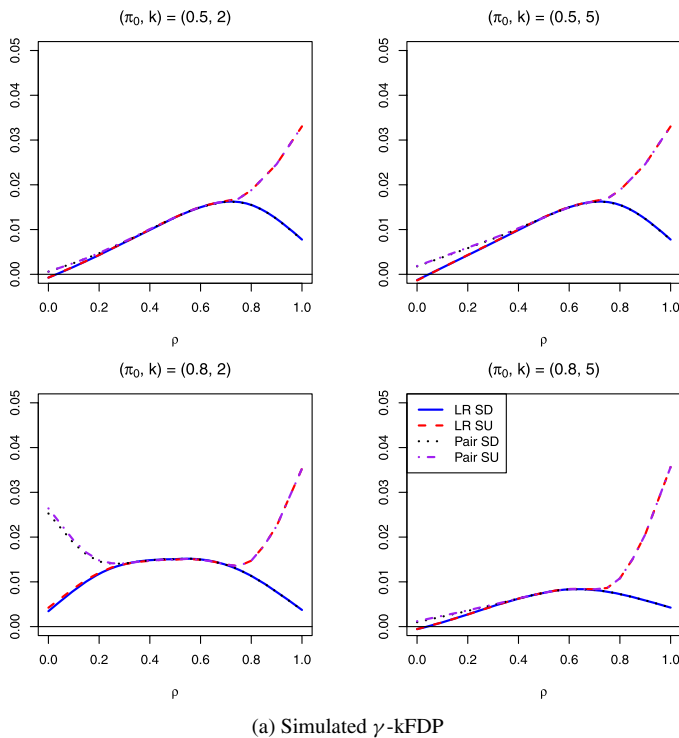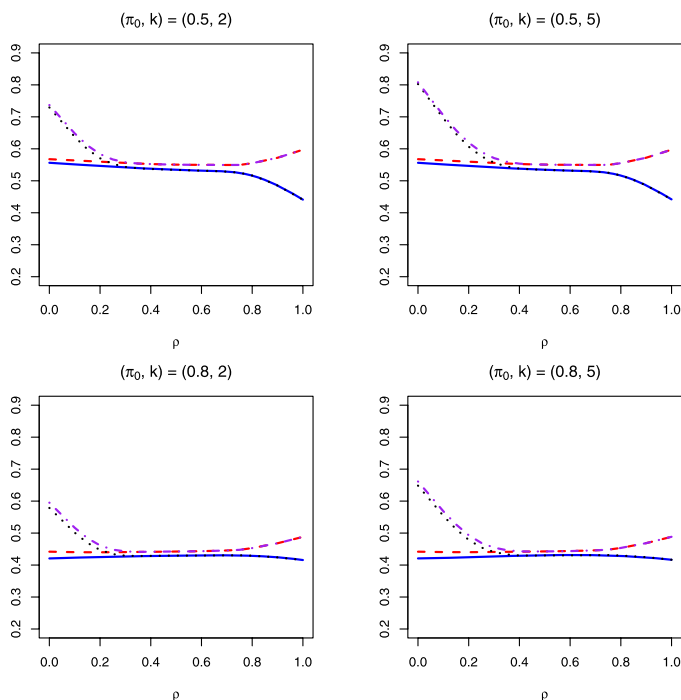

(a) Simulated γ-kFDP

FIG. 3. *Simulated values of γ-kFDP and average power of the LR stepdown (LR SD) and stepup (LR SU) γ-kFDP procedures in Theorems* 3.2 *and* 3.3 *and the LR-type stepdown (Pair SD) and stepup (Pair SU) γ-kFDP procedures in Theorem* 3.4, *all developed assuming positive dependence, for $n = 100$, $\gamma = 0.1$ and $\alpha = 0.05$.*
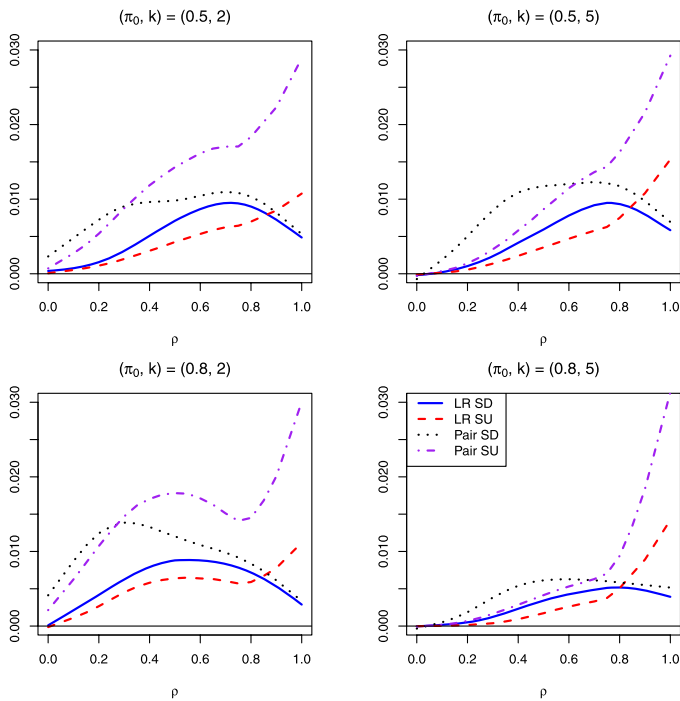
(b) Simulated average power

FIG. 3.   (*Continued*).

the stepwise γ-kFDP procedure in Theorems 3.2 or 3.3, which is based only on the marginal p-values, seem to perform well, but it can be significantly improved by utilizing the pairwise correlation information via the use of the corresponding stepwise procedure in Theorem 3.4 when the underlying test statistics are weakly correlated. However, when the test statistics are strongly correlated, this stepwise procedure incorporating such pairwise correlations has almost the same power performance as the corresponding stepwise procedure based only on the marginal p-values. Of course, such phenomenon has been noted before in the context of other generalized error rates [Sarkar and Guo (2010)]. Figure 4, however, reveals an interesting picture. It seems to say that when controlling the γ-kFDP assuming arbitrary dependence, the LR-type stepwise procedure in Theorems 3.5 or 3.6 based only on the marginal p-values can be made consistently more powerful by utilizing the pairwise correlation information through the use of the corresponding LR-type stepwise procedure in Theorems 3.7 or 3.8, with the power gaps still being quite significant even when the test statistics are highly correlated.

Looking at all the these seven figures, it becomes clear that given a choice of γ, the performance of an LR-type stepwise procedure, particulary in terms of controlling the γ-FDP or γ-kFDP, is affected not only by dependence but also by π_0.

We also did some simulations to examine the following question:

(a) Simulated γ-kFDP

FIG. 4.   *Simulated values of γ-kFDP and average power of the LR-type stepdown (LR SD) and stepup (LR SU) γ-kFDP procedures in Theorems 3.5 and 3.6 and the LR-type stepdown (Pair SD) and stepup (Pair SU) γ-kFDP procedures in Theorems 3.7 and 3.8, all developed assuming arbitrary dependence, for $n = 50$, $\gamma = 0.1$ and $\alpha = 0.05$.*

(Q5) How do the newly suggested BH- and GBS-type γ-FDP stepup proce-dures assuming positive dependence in Theorem 3.3 with $k = 1$ perform compared to the corresponding BH- and GBS-type γ-FDP stepdown procedures obtained from Theorem 3.2?

We used the same simulation settings involving three different types of positive dependence structure as in answering (Q1). From Figures S.4–S.7 (in the supple-mentary material [Guo, He and Sarkar (2014)]) that answers (Q5), we see that the BH- or GBS-type stepup and stepdown procedures have the similar behaviors as the LR-type procedures. Generally, when the underlying test statistics are highly correlated, the power improvements of the stepup procedures over the correspond-ing stepdown procedures are always quite significant. For other cases, the power improvement depends on the dependence structure of the test statistics. In addi-tion, an interesting observation is that the BH-type stepwise procedures are always more powerful than the corresponding GBS-type procedures.

Our last set of simulations was carried out to investigate the following:
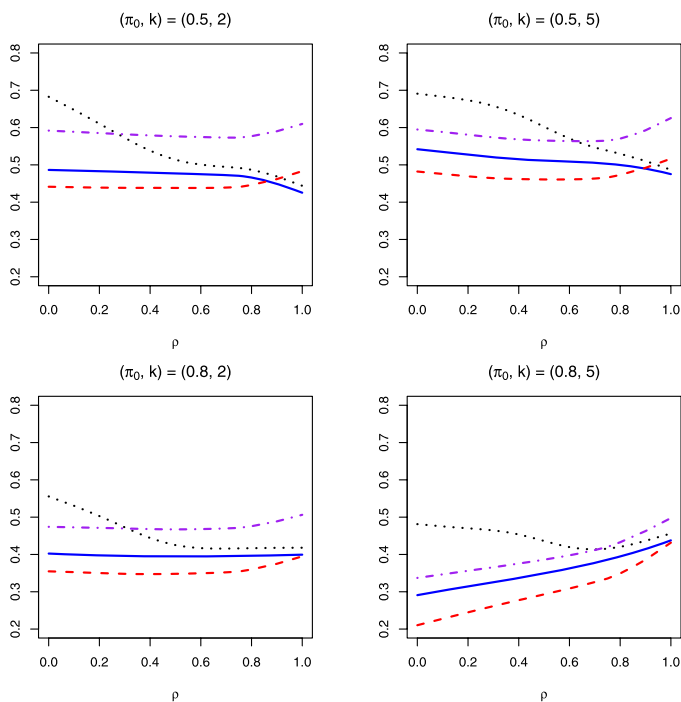
(b) Simulated average power

FIG. 4. (*Continued*).

(Q6) As a γ-kFDP procedure under positive dependence, how does the LR-type stepwise procedure in Theorem 3.4 incorporating pairwise correlation perform in terms of power with increasing k and strength of dependence, compared to the corresponding LR-type stepwise procedure in Theorems 3.2 or 3.3 that do not incorporate such pairwise correlation information?

We used the same simulation setting as in answering (Q3). From Figure 5 that answers this question, we see that the power of each of these LR-type stepwise γ-kFDP procedures increases with k, as expected. The power gap between the stepwise γ-kFDP procedure in Theorem 3.4 and the corresponding stepwise γ-kFDP procedure in Theorems 3.2 or 3.3 gets wider with increasing k. The stepwise procedures in Theorem 3.4 are more powerful than the corresponding stepwise procedures in Theorems 3.2 and 3.3, irrespective of choice of k if the underlying test statistics are weakly correlated and with properly chosen k if these statistics are moderately correlated.

5. Concluding remarks. The paper is motivated by the need to advance the theory of FDP control which is still underdeveloped despite being well accepted
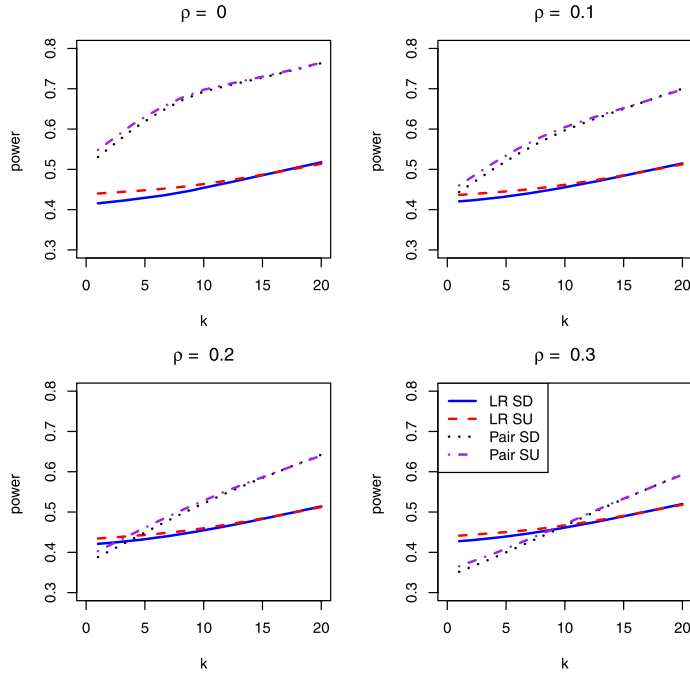
FIG. 5.   *Simulated average power of the LR stepdown (LR SD) and stepup (LR SU) γ-kFDP procedures in Theorems 3.2 and 3.3 and the LR-type stepdown (Pair SD) and stepup (Pair SU) γ-kFDP procedures in Theorem 3.4 with respect to different values of k, all developed assuming positive dependence, for $n = 100$, $\pi_0 = 0.8$, $\gamma = 0.1$ and $\alpha = 0.05$.*

by the multiple testing research community. Our focus has been two-fold: (i) enlarging the class of procedures controlling the γ-FDP, the existing notion of FDP control, and (ii) generalizing this notion to one that is often more appropriate and powerful—with improving some of the currently available results under certain dependence situations being the overreaching goal. We have given a large class of procedures controlling the γ-FDP and its generalization under different dependence assumptions, and numerical evidences showing superior performances of the proposed procedures compared to those they intend to improve under some dependence cases, although these proposed procedures themselves, like their competitors, are still quite conservative.

There is scope of doing further research in the context of what we discuss in this paper. We have defined the γ-kFDP, for the first time in this paper, with the idea of introducing a more powerful notion of error rate than the γ-FDP under dependence. We have proposed several procedures controlling the γ-kFDP and given numerical evidence of their power superiority over the corresponding γ-FDP controlling procedures for some specific values of k and under certain dependence situations. Although a deeper understanding of γ-kFDP under dependence, particularly, how the choice of k depends on correlations, would require studying distri-

butional properties of FDP or kFDP under dependence, an area still less developed, we have provided some insight into it through additional simulations whose findings are reported in the supplementary material [Guo, He and Sarkar (2014)]. In particular, it has been noted that the difference between controlling γ-kFDP and γ-FDP and the stipulated power gain in using a γ-kFDP procedure over the corresponding γ-FDP procedure may not be realized until k/n reaches a certain critical point. Once this point is reached, the power gain can be expected to steadily increase with k/n. Some idea about the choice of k relative to n under different types and varying strengths of dependence has also been provided.

APPENDIX

PROOF OF LEMMA 3.1.   First, note that

$$
\begin{aligned}
I\big(V &> \max[\gamma R, k-1]\big) \\
&= I\big(V > \max[\gamma(V+S), k-1]\big) \\
&= I\big(V \geq \max\{\lfloor \gamma S/(1-\gamma)\rfloor + 1, k\}\big) \\
&= \sum_{i=1}^{M} I\big(V \geq i \vee k, \lfloor \gamma S/(1-\gamma)\rfloor + 1 = i\big).
\end{aligned}
$$

(31)

Also, for a stepdown procedure with the critical constants α_i's, we have

$$
\begin{aligned}
I\big(V &\geq i \vee k, \lfloor \gamma S/(1-\gamma)\rfloor + 1 = i\big) \\
&= I\big(R \geq i \vee k + S, \lfloor \gamma S/(1-\gamma)\rfloor + 1 = i\big) \\
&= I\big(P_{(1)} \leq \alpha_1, \ldots, P_{(i\vee k+S)} \leq \alpha_{i\vee k+S}, \lfloor \gamma S/(1-\gamma)\rfloor + 1 = i\big) \\
&\leq I\big(\widehat{P}_{(1)} \leq \alpha_{1+S}, \ldots, \widehat{P}_{(i\vee k)} \leq \alpha_{i\vee k+S}, \lfloor \gamma S/(1-\gamma)\rfloor + 1 = i\big) \\
&\leq I\big(\widehat{P}_{(1)} \leq \alpha_{1+m(i)}, \ldots, \widehat{P}_{(i\vee k)} \leq \alpha_{i\vee k+m(i)}, \lfloor \gamma S/(1-\gamma)\rfloor + 1 = i\big) \\
&\leq I\big(\widehat{P}_{(i\vee k)} \leq \alpha_{i\vee k+m(i)}, \lfloor \gamma S/(1-\gamma)\rfloor + 1 = i\big).
\end{aligned}
$$

(32)

Combining (31) and (32), we get the lemma.   □

PROOF OF LEMMA 3.2.   Since $V \geq R - n_1$, we have

$$
\begin{aligned}
I\big(V &> \max[\gamma R, k-1]\big) \\
&= I\big(V \geq \max\{\lfloor \gamma R\rfloor + 1, k\}, V \geq R - n_1\big) \\
&= I\left(\bigcup_{j=k}^{n_0}\{\widehat{P}_{(j)} \leq \alpha_R, V = j, \lfloor \gamma R\rfloor + 1 \leq j, R \leq j + n_1\}\right) \\
&\leq I\left(\bigcup_{j=k}^{n_0}\{\widehat{P}_{(j)} \leq \alpha_R, R \leq \tilde{m}(j)\}\right)
\end{aligned}
$$

$$\leq I\left(\bigcup_{j=k}^{n_0}\{\widehat{P}_{(j)} \leq \alpha_{\tilde{m}(j)}\}\right)$$

$$= \sum_{i=1}^{n_0}\sum_{j=k}^{n_0}\frac{I(\widehat{P}_i \leq \alpha_{\tilde{m}(j)}, \widehat{R}_2 = j)}{j}.$$

This is the first inequality. The second inequality can be proved as follows:

$$\sum_{i=1}^{n_0}\sum_{j=k}^{n_0}\frac{I(\widehat{P}_i \leq \alpha_{\tilde{m}(j)}, \widehat{R}_2 = j)}{j}$$

$$= \sum_{i=1}^{n_0}\sum_{j=k}^{n_0}\frac{I(\widehat{P}_i \leq \alpha_{\tilde{m}(j)}, \widehat{R}_2 \geq j)}{j} - \sum_{i=1}^{n_0}\sum_{j=k}^{n_0}\frac{I(\widehat{P}_i \leq \alpha_{\tilde{m}(j)}, \widehat{R}_2 \geq j+1)}{j}$$

$$\leq \sum_{i=1}^{n_0}\frac{I(\widehat{P}_i \leq \alpha_{\tilde{m}(k)}, \widehat{R}_2 \geq k)}{k} + \sum_{i=1}^{n_0}\sum_{j=k+1}^{n_0}\frac{I(\alpha_{\tilde{m}(j-1)} < \widehat{P}_i \leq \alpha_{\tilde{m}(j)}, \widehat{R}_2 \geq j)}{j}.$$

Thus, the lemma is proved. $\square$

PROPOSITION A.1. *Let M and $m(i)$, for $i = 1, \ldots, M$, be defined as in Lemma* 3.1 *and $\tilde{m}(i)$ for $i = 1, \ldots, n_0$ be defined as in Lemma* 3.2. *Then, for given set of critical constants,*

$$\alpha_i' = \frac{(\lfloor \gamma i \rfloor + 1)\alpha}{n + \lfloor \gamma i \rfloor + 1 - i}, \qquad i = 1, \ldots, n,$$

we have $C_{k,n,\mathrm{SD}}^{(1)} = C_{k,n,\mathrm{SU}}^{(1)} = \alpha$ when $k = 1$, where $C_{k,n,\mathrm{SD}}^{(1)}$ and $C_{k,n,\mathrm{SU}}^{(1)}$ are, respectively, defined as in Theorems 3.2 *and* 3.3.

PROOF. We first prove that $C_{k,n,\mathrm{SD}}^{(1)} = \alpha$ when $k = 1$. From the definition of $m(i)$, we have

$$i - 1 \leq \frac{\gamma m(i)}{1 - \gamma} < i.$$

Thus,

$$i - 1 \leq i - (1 - \gamma) \leq \gamma(i + m(i)) < i.$$

Hence,

(33) $$\lfloor \gamma(i + m(i)) \rfloor + 1 = i.$$

Based on (33), we have

(34) $$\frac{n_0\alpha_{i+m(i)}'}{i} = \frac{n_0(\lfloor \gamma(i + m(i)) \rfloor + 1)\alpha}{i(n + \lfloor \gamma(i + m(i)) \rfloor + 1 - i - m(i))} = \frac{n_0\alpha}{n - m(i)} \leq \alpha.$$

Here, the inequality follows from the facts that $m(i) \le n_1$ and $n_0 + n_1 = n$. Note that when $n_0 \ge \lfloor \gamma n_1/(1-\gamma) \rfloor + 1$, $M = \lfloor \gamma n_1/(1-\gamma) \rfloor + 1$, and hence $\max_{1 \le i \le M} m(i) = n_1$. Combining (34) with the above fact, we have

$$C_{1,n,\mathrm{SD}}^{(1)} = \max_{1 \le n_0 \le n} \max_{1 \le i \le M} \left\{ \frac{n_0 \alpha'_{i+m(i)}}{i} \right\} = \alpha.$$

Second, we prove that $C_{k,n,\mathrm{SU}}^{(1)} = \alpha$ when $k = 1$. Note that for $i = 1, \ldots, n_0$,

(35) $$\lfloor \gamma \tilde{m}(i) \rfloor + 1 \le \lfloor \gamma m^*(i) \rfloor + 1 \le i.$$

Thus,

(36) $$\frac{n_0 \alpha'_{\tilde{m}(i)}}{i} = \frac{n_0 (\lfloor \gamma \tilde{m}(i) \rfloor + 1)\alpha}{i(n + \lfloor \gamma \tilde{m}(i) \rfloor + 1 - \tilde{m}(i))} \le \frac{n_0 \alpha}{n + i - \tilde{m}(i)} \le \alpha.$$

Here, the first inequality follows from (35) and the second follows from the fact $\tilde{m}(i) \le i + n_1$. In addition, it is easy to see that when $i = \lfloor \gamma n \rfloor + 1$ and $i + n_1 = n$, we have $m^*(i) = n$ and $n_0 = i$. Thus, $\tilde{m}(i) = n$ and $\lfloor \gamma \tilde{m}(i) \rfloor + 1 = i$. By using the first equality of (36), $n_0 \alpha'_{\tilde{m}(i)}/i = \alpha$. Combining (36) and the above fact, we have

$$C_{1,n,\mathrm{SU}}^{(1)} = \max_{1 \le n_0 \le n} \max_{1 \le i \le n_0} \left\{ \frac{n_0 \alpha'_{\tilde{m}(i)}}{i} \right\} = \alpha. \qquad \square$$

PROOF OF (11). As in proving Lemma 3.2,

$\Pr(\widehat{R}_{n_0} \ge k)$

$$= \Pr\left(\bigcup_{v=k}^{n_0} \{\widehat{P}_{(v)} \le \beta_v\} \right) = \sum_{i=1}^{n_0} \sum_{r=k}^{n_0} \frac{\Pr(\widehat{R}_{n_0} = r, \widehat{P}_i \le \beta_r)}{r}$$

$$= \sum_{i=1}^{n_0} \sum_{r=k}^{n_0} \frac{\Pr(\widehat{R}_{n_0-1}^{(-i)} = r - 1, \widehat{P}_i \le \beta_r)}{r}$$

$$= \frac{\alpha}{n_0} \sum_{i=1}^{n_0} \left\{ \sum_{r=k}^{n_0} \Pr(\widehat{R}_{n_0-1}^{(-i)} \ge r - 1 | \widehat{P}_i \le \beta_r) - \sum_{r=k}^{n_0-1} \Pr(\widehat{R}_{n_0-1}^{(-i)} \ge r | \widehat{P}_i \le \beta_r) \right\}$$

$$\le \frac{\alpha}{n_0} \sum_{i=1}^{n_0} \left\{ \sum_{r=k}^{n_0} \Pr(\widehat{R}_{n_0-1}^{(-i)} \ge r - 1 | \widehat{P}_i \le \beta_r) - \sum_{r=k}^{n_0-1} \Pr(\widehat{R}_{n_0-1}^{(-i)} \ge r | \widehat{P}_i \le \beta_{r+1}) \right\}$$

$$= \frac{\alpha}{n_0} \sum_{i=1}^{n_0} \Pr(\widehat{R}_{n_0-1}^{(-i)} \ge k - 1 | \widehat{P}_i \le \beta_k),$$

where the first inequality follows from (A.3) and (A.4) of Sarkar and Guo (2010) and the second follows from Assumption 2(b). $\square$

PROOF OF (20). Consider a single-step test based on the p-values $\widehat{P}_1, \ldots, \widehat{P}_{n_0}$ and the constant threshold t. Let $\widehat{R}_1$ denote the number of rejections. Then we have for each $i = 2, \ldots, n_0$,

$$I(\widehat{P}_{(i)} \leq t) \leq I(\widehat{R}_1(\widehat{R}_1 - 1) \geq i(i-1)) \leq \frac{1}{i(i-1)} \sum_{j=1}^{n_0} \sum_{l(\neq j)=1}^{n_0} I(\widehat{P}_j \leq t, \widehat{P}_l \leq t),$$

which proves the desired inequality. $\square$

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SUPPLEMENTARY MATERIAL

Supplement to "Further results on controlling the false discovery proportion" (DOI: 10.1214/14-AOS1214SUPP; .pdf). Due to space constraints, we have relegated to the supplemental article [Guo, He and Sarkar (2014)] the remaining figures generated from the simulations in Section 4 and the findings of additional simulations mentioned in Remark 2.1.

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