ZERO-SUM REPEATED GAMES: COUNTEREXAMPLES TO THE EXISTENCE OF THE ASYMPTOTIC VALUE AND THE CONJECTURE $\mathbf{maxmin = \lim v_n}$

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Mertens [In *Proceedings of the International Congress of Mathematicians* (*Berkeley*, *Calif.*, 1986) (1987) 1528–1577 Amer. Math. Soc.] proposed two general conjectures about repeated games: the first one is that, in any two-person zero-sum repeated game, the asymptotic value exists, and the second one is that, when Player 1 is more informed than Player 2, in the long run Player 1 is able to guarantee the asymptotic value. We disprove these two long-standing conjectures by providing an example of a zero-sum repeated game with public signals and perfect observation of the actions, where the value of the λ-discounted game does not converge when λ goes to 0. The aforementioned example involves seven states, two actions and two signals for each player. Remarkably, players observe the payoffs, and play in turn.

Notations. The notation "$X := Y$" means "X is defined by the expression Y."

The set of nonnegative integers is denoted by $\mathbb{N}$, and $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$. The set of real numbers is denoted by $\mathbb{R}$.

The complementary of B in A is denoted by $A \setminus B$.

If $x \in \mathbb{R}$, the integer part of x is denoted by $\lfloor x \rfloor$.

If $(C, \mathcal{C})$ is a measurable space, we denote by $\Delta(C)$ the set of probability measures on C, and $\Delta_f(C)$ the set of probability measures on C with finite support. We call δ_c the Dirac measure at c.

If $C_0 \subset C$ is a finite set and $(\alpha_c)_{c \in C_0} \in \Delta(C_0)$, then we write $\sum_{c \in C_0} \alpha_c \cdot c$ for $\sum_{c \in C} \alpha_c \delta_c$.

If X is a bounded real random variable, $\mathbb{E}(X)$ is the expectation of X.

If $f : A \to \mathbb{R}$ is a bounded function, $\|f\|_\infty$ is $\sup_{x \in \mathbb{R}} |f(x)|$, and if $B \subset A$, $\operatorname{argmax}_{x \in B} f := \{x \in B | \forall y \in B f(x) \geq f(y)\}$.

If $f : \mathbb{R} \to \mathbb{R}$ is differentiable, $\frac{df}{dx}$ is the derivative of f.

Let $a \in \mathbb{R}$ and f, g and h be real functions. If $f(x) - g(x)$ is little-o of $h(x)$ when x goes to a, we write $f(x) \underset{x \to a}{=} g(x) + o(h(x))$. If $f(x)$ is equivalent to $g(x)$ when x goes to a, we write $f(x) \underset{x \to a}{\sim} g(x)$.

Introduction.   The general model of two-person zero-sum repeated game was introduced in Mertens, Sorin and Zamir (1994), Chapter IV. Such a game is described by a finite set of states K, a finite set of actions I (resp., J) for Player 1 (resp., 2), a finite set of signals A (resp., B) for Player 1 (resp., 2), a payoff function $g : K \times I \times J \to \mathbb{R}$, and a transition function $q : K \times I \times J \to \Delta(K \times A \times B)$. The game proceeds as follows. Before the game starts, a triplet (k_1, a_0, b_0) is drawn according to an initial probability distribution $p \in \Delta(K \times A \times B)$. The state k_1 is the initial state, and Player 1 (resp., 2) receives the private signal a_0 (resp., b_0). At stage $m \geq 1$, both players choose an action simultaneously and independently, $i_m \in I$ (resp., $j_m \in J$) for Player 1 (resp., 2). The payoff at stage m is $g_m := g(k_m, i_m, j_m)$. A triplet (k_{m+1}, a_m, b_m) is drawn from the probability distribution $q(k_m, i_m, j_m)$. The signal a_m (resp., b_m) is announced to Player 1 (resp., 2). The game then moves on to state k_{m+1}, and enters the next stage.

For $\lambda \in (0, 1]$, in the λ-*discounted game*, the goal of Player 1 (resp., 2) is to maximize (resp., minimize) the expected Abel mean of stage payoffs $\sum_{m \geq 1} \lambda(1 - \lambda)^{m-1} g_m$. For $n \in \mathbb{N}^*$, in the n-*stage game*, the goal of Player 1 (resp., 2) is to maximize (resp., minimize) the expected Cesaro-mean of stage payoffs $1/n \sum_{m=1}^{n} g_m$. These two games have a value, denoted, respectively, by $v_\lambda(p)$ and $v_n(p)$. Two important conjectures were stated by Mertens [see Mertens (1987), page 1572 and Mertens, Sorin and Zamir (1994), Chapter VIII, pages 378 and 386].

CONJECTURE 1.   In a zero-sum repeated game, the asymptotic value exists, that is, the sequences of functions (v_n) and (v_λ) converge pointwise to the same limit, when n goes to infinity and λ goes to 0.

CONJECTURE 2.   In a zero-sum repeated game where Player 1 is more informed than Player 2 (i.e., Player 1 observes what Player 2 observes), (v_n) and (v_λ) converge pointwise to the maxmin of the game, which is the maximal amount that Player 1 can guarantee to herself in long games.

The Mertens' conjectures have been proven true in numerous particular classes of zero-sum repeated games [see Aumann and Maschler (1995), Bewley and Kohlberg (1976), Gensbittel, Oliu-Barton and Venel (2014), Mertens and Neyman (1981), Mertens and Zamir (1971), Neyman (2008), Renault (2006, 2012), Rosenberg (2000), Rosenberg, Solan and Vieille (2002, 2003, 2004), Rosenberg and Vieille (2000), Sorin (1984, 1985)].

In zero-sum repeated games with symmetric information [the particular class of zero-sum repeated games in which players observe the actions perfectly and receive a public signal about the state, i.e., for all $m \in \mathbb{N}^*$, $a_m = b_m$ and (i_m, j_m) is a_m-measurable], Conjectures 1 and 2 were not proven [see Forges (1982), Kohlberg and Zamir (1974), Venel (2015) for a proof of the conjectures under additional assumptions on the transition function]. Note that in this class of games,

Player 1 has the same information as Player 2, and in particular she is more informed than Player 2.

The main contribution of this paper is to provide an example of a zero-sum repeated game with symmetric information where (v_λ) does not converge when λ goes to 0, thus contradicting both conjectures. Our example also shows that there is no hope of obtaining an existence result for the asymptotic value in zero-sum repeated games, unless we make a very strong assumption on the transition function. Indeed, the structure of the example is very simple: at each stage, the action of one player only influences their stage payoff and the transition; moreover, players observe the payoff. In addition, since the information is symmetric, the belief hierarchy [belief of Player 1 about the state or actions played, belief of Player 2 about the belief of Player 1 about the state or actions played, and so on; see Mertens and Zamir (1985) for more details] is straightforward: players know the past actions and have the same belief about the state.

In addition, using the example we will construct a second example of a zero-sum repeated game with symmetric information where neither (v_λ) nor (v_n) converge.

Lastly, we will provide an example of a state-blind zero-sum repeated game (the particular class of zero-sum repeated games with symmetric information where players get no signals about the state) and an example of a zero-sum repeated game with one state-blind player (one player observes the state but the other gets no signal about it) without an asymptotic value. We will also give an example of a standard zero-sum stochastic game with compact action sets without an asymptotic value, providing an alternative counterexample to Vigeral (2013). Note that this last class of zero-sum repeated games does not concern the two conjectures, since the action sets are not finite. Nonetheless, the example is interesting because it has a simpler structure than the one given in Vigeral (2013) (players play in turn) and is thus easier to analyze.

The paper is organized as follows. In Section 1, we explain the model of zero-sum repeated game with symmetric information and some basic concepts. In Section 2, we present our main counterexample and show that (v_λ) does not converge. In Section 3, we construct a similar game where neither (v_λ) nor (v_n) converge. In Section 4, we show how our counterexample adapts to other classes of zero-sum repeated games.

1. Zero-sum repeated games with symmetric information.

1.1. *The model.* A zero-sum repeated game with symmetric information Γ is defined by the following elements:

– State space K.
– Action set I (resp., J) for Player 1 (resp., 2).
– Signal set A.
– Transition function $q : K \times I \times J \to \Delta(K \times A)$.
– Payoff function $g : K \times I \times J \to \mathbb{R}$.

We assume K, I, J and A to be finite. Both players know K, I, J, A, g, q.

Given an initial probability $p \in \Delta(K)$ known by both players, the game Γ^p proceeds as described below:

– Before the game starts, an initial state k_1 is drawn according to p. The quantity k_1 is the initial state. Players do not know k_1.
– At stage $m \geq 1$, both players choose an action simultaneously and independently, $i_m \in I$ (resp., $j_m \in J$) for Player 1 (resp., 2). The payoff at stage m is $g(k_m, i_m, j_m)$. A pair (k_{m+1}, a_m) is drawn from $q(k_m, i_m, j_m)$. Both players receive the public signal a_m, which contains the actions i_m and j_m. The game moves on to state k_{m+1}, and then continues to the next stage.

Compared to the model of general repeated game described in the Introduction, when the game has symmetric information, one has $a_m = b_m$ and (i_m, j_m) is a_m-measurable for all $m \geq 1$, and the players do not receive a signal at the outset of the game. The *history of the game before stage m* is the random sequence $(k_1, i_1, j_1, a_1, k_2, \ldots, i_{m-1}, j_{m-1}, a_{m-1}, k_m)$.

The set of all possible histories before stage m is $H_m := K \times (I \times J \times A \times K)^{m-1}$.

The set of all possible plays is $H_\infty := K \times (I \times J \times A \times K)^{\mathbb{N}^*}$.

At the beginning of stage m, both players know $(i_1, j_1, a_1, \ldots, i_{m-1}, j_{m-1}, a_{m-1})$.

A *pure strategy* for Player 1 (resp., 2) is a map $s : \bigcup_{m \geq 1}(I \times J \times A)^{m-1} \to I$ [resp., $t : \bigcup_{m \geq 1}(I \times J \times A)^{m-1} \to J$].

A *behavior strategy* for Player 1 (resp., 2) is a map $\sigma : \bigcup_{m \geq 1}(I \times J \times A)^{m-1} \to \Delta(I)$ [resp., $\tau : \bigcup_{m \geq 1}(I \times J \times A)^{m-1} \to \Delta(J)$]. The set of all behavior strategies for Player 1 (resp., 2) is denoted Σ (resp., $\mathcal{T}$).

An initial probability $p \in \Delta(K)$ and a pair of (pure or behavior) strategies $(\sigma, \tau) \in \Sigma \times \mathcal{T}$ naturally induce a probability measure $\mathbb{P}^p_{\sigma,\tau}$ on the set of all finite histories $\bigcup_{m \geq 1} H_m$ [for more details, see Sorin (2002), Appendix D]. By the Kolmogorov extension theorem, this probability measure uniquely extends to H_∞. We denote $\mathbb{E}^p_{\sigma,\tau}$ the expectation with respect to the probability measure $\mathbb{P}^p_{\sigma,\tau}$. Let g_m be the random payoff at stage $m \geq 1$: $g_m := g(k_m, i_m, j_m)$.

For $\lambda \in (0, 1]$, the *λ-discounted game* is the strategic-form game Γ^p_λ with strategy set Σ for Player 1 and $\mathcal{T}$ for Player 2, and payoff function $\gamma^p_\lambda : \Sigma \times \mathcal{T} \to \mathbb{R}$ defined by

$$\gamma^p_\lambda(\sigma, \tau) := \mathbb{E}^p_{\sigma,\tau}\left(\sum_{m \geq 1} \lambda(1 - \lambda)^{m-1} g_m\right).$$

The goal of Player 1 (resp., 2) is to maximize (resp., minimize) γ^p_λ.

For $n \in \mathbb{N}^*$, the *n-stage repeated game* is the strategic-form game Γ^p_n with strategy set Σ for Player 1 and $\mathcal{T}$ for Player 2, and payoff function $\gamma^p_n : \Sigma \times \mathcal{T} \to \mathbb{R}$

defined by

$$\gamma_n^p(\sigma, \tau) := \mathbb{E}_{\sigma, \tau}^p \left(\frac{1}{n} \sum_{m=1}^n g_m \right).$$

The goal of Player 1 (resp., 2) is to maximize (resp., minimize) γ_n^p.

The games Γ_λ^p and Γ_n^p have a value, denoted, respectively, by $v_\lambda(p)$ and $v_n(p)$ [see Mertens, Sorin and Zamir (1994), Chapter IV]. That is, there are real numbers $v_\lambda(p)$ and $v_n(p)$ satisfying

$$v_\lambda(p) = \max_{\sigma \in \Sigma} \min_{\tau \in \mathcal{T}} \gamma_\lambda^p(\sigma, \tau) = \min_{\tau \in \mathcal{T}} \max_{\sigma \in \Sigma} \gamma_\lambda^p(\sigma, \tau),$$

$$v_n(p) = \max_{\sigma \in \Sigma} \min_{\tau \in \mathcal{T}} \gamma_n^p(\sigma, \tau) = \min_{\tau \in \mathcal{T}} \max_{\sigma \in \Sigma} \gamma_n^p(\sigma, \tau).$$

In the game Γ_λ^p, a strategy $\sigma^* \in \Sigma$ (resp., $\tau^* \in \mathcal{T}$) is *optimal* if for all $\tau \in \mathcal{T}$ (resp., $\sigma \in \Sigma$) we have $\gamma_\lambda(\sigma^*, \tau) \geq v_\lambda(p)$ [resp., $\gamma_\lambda(\sigma, \tau^*) \leq v_\lambda(p)$]. Optimal strategies in Γ_n^p are defined in the same way, replacing λ by n.

1.2. *Asymptotic approach.*

DEFINITION 1.1. Γ has an *asymptotic value* if the sequences of functions (v_n) and (v_λ) converge pointwise to the same limit (when $n \to +\infty$ and $\lambda \to 0$).

REMARK 1.2. For all $(n, \lambda) \in \mathbb{N}^* \times (0, 1]$, $v_n : \Delta(K) \to \mathbb{R}$ and $v_\lambda : \Delta(K) \to \mathbb{R}$ are $\|g\|_\infty$-Lipschitz [see Mertens, Sorin and Zamir (1994), Chapter V, page 184]. Thus, as far as these sequences are concerned, pointwise and uniform convergence are equivalent. In what follows, we will simply write "(v_n) converges" or "(v_λ) converges," whenever these sequences of functions converge pointwise.

1.3. *Equivalent repeated game with perfect observation of the state.* Let Γ be a repeated game with symmetric information. For $m \geq 1$, we denote p_m the conditional probability on the state k_m at stage m, given the random past history of the players $(i_1, j_1, a_1, \ldots, i_{m-1}, j_{m-1}, a_{m-1})$. The random variable p_m represents the common belief at stage m about the current state k_m. The triplet (p_{m+1}, i_m, j_m) is the only relevant information conveyed by the signal a_m, and p_m plays the role of a state variable. Indeed, extend q and g linearly to $\Delta(K) \times I \times J$: $q(p, i, j) := \sum_{k \in K} p(k) q(k, i, j)$ and $g(p, i, j) := \sum_{k \in K} p(k) g(k, i, j)$.

Assume that at some stage of the game, players have a common belief p about the current state. If they play (i, j) and receive the signal $a \in A$, then their posterior belief about the next state will be $q_{\Delta(K)|A}(p, i, j, a) \in \Delta(K)$, where

$$\forall k \in K \qquad q_{\Delta(K)|A}(p, i, j, a)(k) := \frac{q(p, i, j)(k, a)}{\sum_{k' \in K} q(p, i, j)(k', a)}.$$

Let $q_A(p, i, j) \in \Delta(A)$ be the marginal on A of $q(p, i, j)$. We define $\widetilde{q} : \Delta(K) \times I \times J \to \Delta_f(\Delta(K))$ by

$$(1.1) \qquad \widetilde{q}(p, i, j) := \sum_{a \in A} q_A(p, i, j)(a) \cdot q_{\Delta(K)|A}(p, i, j, a).$$

If players have a common belief p about the current state and play (i, j), then for all $a \in A$, their posterior belief about the next state will be $q_{\Delta(K)|A}(p, i, j, a)$ with probability $q_A(p, i, j)(a)$.

Fix $\lambda \in (0, 1]$. The function $v_\lambda : \Delta(K) \to \mathbb{R}$ is the unique solution of the following functional equation [see Mertens, Sorin and Zamir (1994), Chapter IV, Theorem 3.2, page 158]:

$$(1.2) \qquad f(p) = \max_{x \in \Delta(I)} \min_{y \in \Delta(J)} \left\{ \lambda g(p, x, y) + (1 - \lambda) \mathbb{E}_{x,y}^p(f) \right\}$$

$$(1.3) \qquad \quad = \min_{y \in \Delta(J)} \max_{x \in \Delta(I)} \left\{ \lambda g(p, x, y) + (1 - \lambda) \mathbb{E}_{x,y}^p(f) \right\},$$

where the unknown is a continuous function $f : \Delta(K) \to \mathbb{R}$,

$$\mathbb{E}_{x,y}^p(f) := \sum_{(p', i, j) \in \Delta(K) \times I \times J} x(i) y(j) \widetilde{q}(p, i, j)(p') f(p')$$

and

$$g(p, x, y) := \sum_{(i, j) \in I \times J} x(i) y(j) g(p, i, j).$$

The game that is equivalent to Γ is the repeated game $\widetilde{\Gamma}$ with state space $\Delta(K)$, action sets I and J, transition function $\widetilde{q} : \Delta(K) \times I \times J \to \Delta_f(\Delta(K))$ and payoff function g.

Given an initial state $p \in \Delta(K)$, the game $\widetilde{\Gamma}^p$ proceeds as follows. Players know p, and at each stage $m \geq 1$, they choose an action simultaneously and independently, $i_m \in I$ (resp., $j_m \in J$) for Player 1 (resp., 2). The payoff at stage m is $g(p_m, i_m, j_m)$, and p_{m+1} is drawn from $\widetilde{q}(p_m, i_m, j_m)$, and announced to both players. The game moves on to state p_{m+1}, and then continues to the next stage. Note that p_m can only take a countable number of values.

Given $\lambda \in (0, 1]$ and $n \in \mathbb{N}^*$, the λ-discounted game $\widetilde{\Gamma}_\lambda^p$ and the n-stage repeated game $\widetilde{\Gamma}_n^p$ are defined as in Section 1.1. The games $\widetilde{\Gamma}_\lambda^p$ (resp., $\widetilde{\Gamma}_n^p$) and Γ_λ^p (resp., Γ_n^p) have the same value, and optimal strategies in the first game induce optimal strategies in the second one, and vice versa.

DEFINITION 1.3. A strategy in $\widetilde{\Gamma}$ is *stationary* if it only depends on the state variable p_m. Such a strategy can be seen as a map from $\Delta(K)$ to $\Delta(I)$ or $\Delta(J)$.

There exists stationary strategies which are optimal in $\widetilde{\Gamma}_\lambda^p$ for any $p \in \Delta(K)$ [see Mertens, Sorin and Zamir (1994), Chapter VII, Proposition 1.4, page 326]. We have the following refinement, which follows from Shapley (1953) and the compactness of $\Delta(K)$:

DEFINITION 1.4. A player is said to *control* $p \in \Delta(K)$ if in this state the transition $\widetilde{q}(p, \cdot)$ and the payoff $g(p, \cdot)$ do not depend on the action of the other player.

LEMMA 1.5. *Assume that each state in $\Delta(K)$ is controlled by one player. Then both players have pure stationary strategies which are optimal in $\widetilde{\Gamma}^p_\lambda$ for any $p \in \Delta(K)$.*

2. A repeated game with symmetric information where (v_λ) does not converge. First we present the main counterexample of the paper. We then describe the equivalent game with perfect observation of the state and actions. This game might seem intricate, but it turns out that for each discount factor λ in $(0, 1]$, the discounted game is equivalent to a strategic-form game with strategy sets $\mathbb{N}$ for Player 1 and $2\mathbb{N}$ for Player 2. From the analysis of this last game, we deduce that the discounted value of the main counterexample does not converge.

2.1. *Description of the example.* Consider the following repeated game with symmetric information Γ, with state space $K = \{1^*, 1^{++}, 1^T, 1^+, 0^*, 0^{++}, 0^+\}$, action sets $I = J = \{C, Q\}$, and signal set $A = \{D, D'\}$. The payoff function does not depend on the actions, and is equal to 1 in states $1^*, 1^{++}, 1^T$ and 1^+, and to 0 in states $0^*, 0^{++}$ and 0^+. Player 2 controls states $1^{++}, 1^T$ and 1^+. Player 1 controls states 0^{++} and 0^+. Lastly, states 1^* and 0^* are *absorbing* states: once 1^* or 0^* is reached, the game remains forever in this state, and the payoff does not depend on the actions (*absorbing* payoff). Figure 1 describes the transition function.

We have adopted the following notation: an arrow going from state $k \in K$ to state $k' \in K$ with the caption $(i, p, a) \in \{C, Q\} \times [0, 1] \times \{D, D'\}$ indicates that if the player who controls state k plays action i, then with probability p the state moves to state k' and the signal is a. For example, if the state is 1^{++} and Player 2 plays action C, then with probability $1/2$ the game moves to state 1^T and the signal is D, and with probability $1/2$ the game stays in state 1^{++} and the signal is D'.

The action Q causes absorption or switching from $\{1^{++}, 1^T, 1^+\}$ to $\{0^{++}, 0^+\}$ and vice versa. In particular, the players know in which of the following subsets of K the current state is: $\{1^{++}, 1^T, 1^+\}, \{0^{++}, 0^+\}, \{1^*\}$ or $\{0^*\}$.

2.2. *Equivalent repeated game with perfect observation.* In this subsection, we give the exact expression of the transition function $\widetilde{q}$ of the equivalent repeated game with perfect observation of the state and actions $\widetilde{\Gamma}$. We denote by $1_{2n}, 1_{2n+1}$, $0_n \in \Delta(K)$ ($n \in \mathbb{N}^*$) the possible beliefs of the players along the game. Starting from the prior belief 0^{++}, 0_n is the belief after n consecutive stages in which Player 1 played C and the signal was D; starting from the prior belief 1^{++}, 1_n is the belief after n consecutive stages in which Player 2 played C and the signal was D.
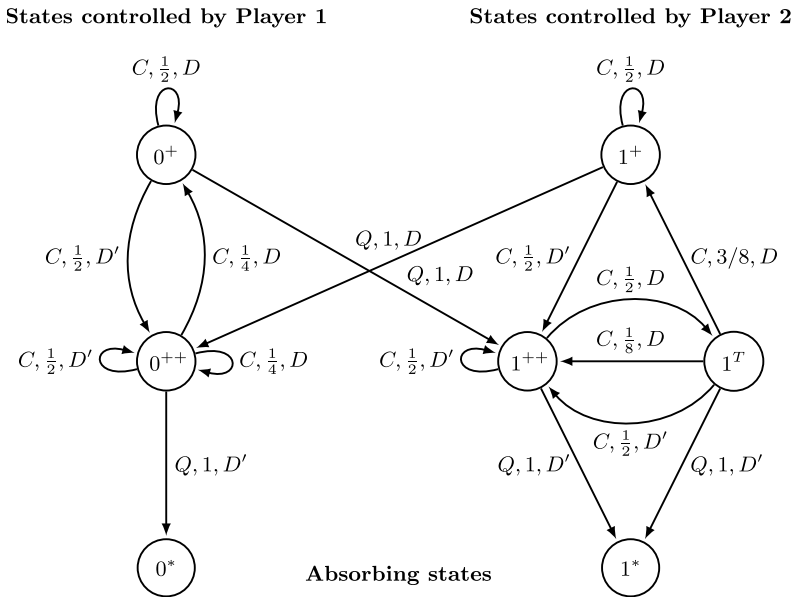
FIG. 1. *Transitions in the game* Γ.

Formally, given $n \in \mathbb{N}^*$, we define the beliefs $1_{2n}, 1_{2n+1}, 0_n \in \Delta(K)$ by:

$$1_{2n} := 2^{-2n} \cdot 1^{++} + (1 - 2^{-2n}) \cdot 1^+,$$

$$1_{2n+1} := 2^{-2n} \cdot 1^T + (1 - 2^{-2n}) \cdot 1^+,$$

$$0_n := 2^{-n} \cdot 0^{++} + (1 - 2^{-n}) \cdot 0^+.$$

Let us suppose that at some stage of Γ, the belief of the players about the current state is 1_n, for some $n \in \mathbb{N}$. Player 1's action has no influence on the transition and both players know it. If Player 2 plays C, then with probability $1/2$ the signal is D (resp., D'), and by Bayes rule the posterior belief about the next state will be 1_{n+1} (resp., $1_0 = 1^{++}$). Now let us suppose that the belief is 1_{2n} or 1_{2n+1}. If he plays Q, then with probability $1 - 2^{-2n}$ (resp., 2^{-2n}) the signal is D (resp., D'), and the posterior belief about the next state will be 0^{++} (resp., 1^*). Thus, in $\widetilde{\Gamma}$, the transition function $\widetilde{q}$ in the states 1_{2n} and 1_{2n+1} can be described by Figure 2.

Let us suppose that at some stage of Γ, the belief of the players about the current state is 0_n. Player 2's action has no influence on the transition and both players know it. If Player 1 plays C, then with probability $1/2$ the signal is D (resp., D'), and by Bayes rule the posterior belief about the next state will be 0_{n+1} (resp., $0_0 = 0^{++}$). Now let us suppose that the belief is 0_n. If she plays Q, then with probability $1 - 2^{-n}$ (resp., 2^{-n}) the signal is D (resp., D'), and the posterior belief about the next state will be 1^{++} (resp., 0^*). Thus, in $\widetilde{\Gamma}$, the transition function $\widetilde{q}$ in the state 0_n can be described by Figure 3.
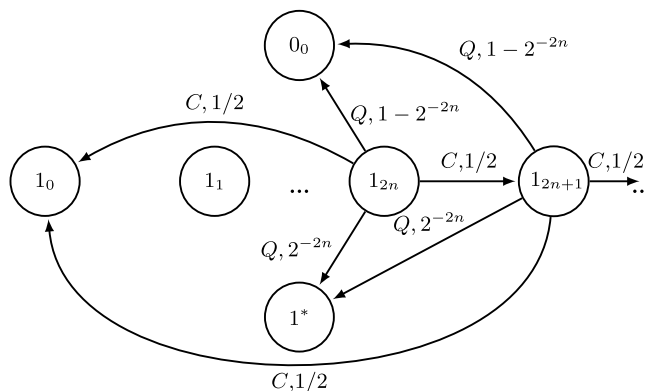
FIG. 2. *Transitions in the states 1_{2n} and 1_{2n+1}.*

Let $P_1 := \bigcup_{n \in \mathbb{N}}\{1_n\}$, $P_0 := \bigcup_{n \in \mathbb{N}}\{0_n\}$, and $P = P_1 \cup P_0 \cup \{1^*, 0^*\}$. Note that in $\widetilde{\Gamma}$, Player 1 controls all the states in P_0, and Player 2 controls all the states in P_1. The set of states which can be reached with positive probability under some strategy vector in $\widetilde{\Gamma}^{1^{++}}$ is exactly P.

Now let us explain the dynamics of the game informally. Assume that the game starts in state $p = 0_0 = 0^{++}$. Since the payoff is 1 in states lying in P_1 and 0 in states lying in P_0, and since Player 1 maximizes the payoff, Player 1 wants to go to state $1_0 = 1^{++}$. If she plays Q immediately, the game is absorbed in state 0^*, which is the worst state for her. If she never plays Q, the payoff is 0 forever, which is also an unfavorable outcome for her. If she plays C until the state is 0_n, and then she plays Q, then the game is absorbed in state 0^* with probability 2^{-n} (we will often call *absorbing risk taken by Player* 1 the probability that the game is absorbed in state 0^*), and the game goes to state 1^{++} with a probability of $(1 - 2^{-n})$.

To reach state 0_n from state 0^{++}, Player 1 needs 2^n stages on average. Hence, Player 1 has to make a trade-off between staying not too long in states of type 0,
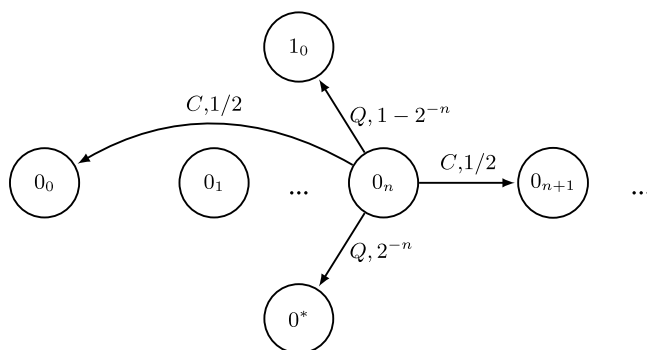

FIG. 3. *Transitions in the state 0_n.*

and having a low probability of being absorbed in 0^* when she plays Q. Basically, Player 1 needs to wait on average 2^n stages to reduce the absorbing risk to 2^{-n}.

The same principle applies to Player 2. Assume that the game starts in state $p = 1_0 = 1^{++}$. Player 2 plays C until reaching state 1_{2n}, and then Q. The game is absorbed in state 1^* with a probability of 2^{-2n}, and it goes to state state 0^{++} with a probability of $(1 - 2^{-2n})$.

To reach state 1_{2n}, Player 2 needs on average 2^{2n} stages. Player 2 can also play Q in state 1_{2n+1}, but this is not a good strategy, since such a state is harder to reach than state 1_{2n} (2^{2n+1} stages on average) but leads to the same absorbing risk 2^{-2n}. Note that the time needed by Player 2 to go from state 1^{++} to state 1_{2n} is on average the same as the time needed by Player 1 to go from state 0^{++} to state 0_{2n}. The state of the game oscillates between states of type 0 and states of type 1 as long as it does not reach an absorbing state. The only asymmetry of the game is that Player 1 can take any absorbing risks of the form 2^{-n}, whereas Player 2 can only take absorbing risks of the form 2^{-2n}.

2.3. *Equivalent strategic-form game.* Fix $\lambda \in (0, 1]$. Let $(a, b) \in \mathbb{N} \times 2\mathbb{N}$. Let $s(a) \in \Sigma$ be the following pure stationary strategy for Player 1: for every $n \geq a$, play Q in state 0_n; otherwise play C.

Let $t(b) \in \mathcal{T}$ be the following pure stationary strategy for Player 2: for every $n \geq b$, play Q in state 1_n; otherwise play C.

The aim of this section is to prove the following proposition.

PROPOSITION 2.1. *The game $\widetilde{\Gamma}_\lambda^p$ for $p = 1^{++}$ has the same value as the strategic-form game G_λ with action set $\mathbb{N}$ for Player 1, $2\mathbb{N}$ for Player 2, and payoff function*

$$(2.1) \qquad g_\lambda(a, b) := \frac{1 - f_\lambda(b)}{1 - f_\lambda(a) f_\lambda(b)},$$

where

$$f_\lambda(n) := \frac{(1 - 2^{-n})(1 - \lambda^2)}{1 + 2^{n+1}\lambda(1 - \lambda)^{-n} - \lambda}.$$

Moreover, $(a, b) \in \mathbb{N} \times 2\mathbb{N}$ are optimal strategies in G_λ if and only if $s(a)$ and $t(b)$ are optimal strategies in $\widetilde{\Gamma}_\lambda^{1^{++}}$.

PROOF. First note that by Lemma 1.5, there exists pure optimal stationary strategies in $\widetilde{\Gamma}_\lambda^{1^{++}}$. A pure stationary strategy for Player 1 corresponds to a strategy $s(a)$, where $a \in \mathbb{N}$ is the smallest integer for which Player 1 plays Q in state 1_a. Note that Player 2 is better off quitting in state 1_{2n} rather than in state 1_{2n+1}. Indeed, state 1_{2n+1} is harder to reach than state 1_{2n}, but the probability of being absorbed in state 1^* when playing Q is the same in both states. Thus, Player 2 has

an optimal strategy of the form $t(b)$, for some $b \in 2\mathbb{N}$. Fix now $a, b \in \mathbb{N} \times 2\mathbb{N}$, and let us compute the payoff $\gamma_\lambda^{1^{++}}(s(a), t(b))$ given by these strategies.

Let $T_b = \inf\{m \geq 1 | j_m = Q\}$ and $T_a = \inf\{m \geq T_b + 1 | i_m = Q\} - T_b$. The random variable T_b (resp., T_a) represents the time spent by Player 2 (resp., Player 1) in states of type 1 (resp., 0) before quitting. The payoff $\gamma_\lambda^{1^{++}}(s(a), t(b))$ is equal to

$$2^{-b} + (1 - 2^{-b})\mathbb{E}\left(\sum_{m=1}^{T_b} \lambda(1-\lambda)^{m-1}1 + \sum_{m=T_b+1}^{T_a+T_b} \lambda(1-\lambda)^{m-1}0\right)$$
$$+ (1 - 2^{-b})(1 - 2^{-a})((1-\lambda)^{T_a+T_b})\gamma_\lambda^{1^{++}}(s(a), t(b)),$$

where $\mathbb{E}_{s(a),t(b)}^{1^{++}}$ is denoted by $\mathbb{E}$. The quantity 2^{-b} corresponds to the probability that the game is absorbed in state 1^* when Player 2 plays Q: in this case the payoff is 1 at every stage. If the game is not absorbed at that point, then the payoff from stage 1 until the stage when Player 1 plays Q is the second term of the equation. When Player 1 plays Q, the game is absorbed in state 0^* with probability 2^{-a}, and goes back to state 1^{++} with probability $(1 - 2^{-a})$; this is the third term of the equation.

We deduce that

(2.2) $$\gamma_\lambda^{1^{++}}(s(a), t(b)) = \frac{1 - (1 - 2^{-b})\mathbb{E}((1-\lambda)^{T_b})}{1 - (1 - 2^{-a})(1 - 2^{-b})\mathbb{E}((1-\lambda)^{T_a+T_b})}.$$

Under $s(a)$ and $t(b)$, the stopping times T_a and T_b are independent. Thus, we have

(2.3) $$\mathbb{E}((1-\lambda)^{T_a+T_b}) = \mathbb{E}((1-\lambda)^{T_a})\mathbb{E}((1-\lambda)^{T_b}).$$

Under $s(a)$ [resp., $t(b)$], $T_a - 1$ (resp., $T_b - 1$) is a random variable denoting the number of trials needed to have a (resp., b) consecutive successes in independent trials with a success probability of $1/2$. Thus, this random variable follows the generalized geometric distribution of order a (resp., b) and parameter $1/2$ studied in Philippou, Georghiou and Philippou (1983). By Lemma 2.2 in Philippou, Georghiou and Philippou (1983), we get

(2.4) $$\mathbb{E}((1-\lambda)^{T_n}) = (1+\lambda)/(1 + 2^{n+1}\lambda(1-\lambda)^{-n} - \lambda).$$

Combining (2.2), (2.3) and (2.4), we get the desired result. $\square$

2.4. *Asymptotic study of G_λ and proof of the main theorem.* We first determine optimal strategies in G_λ.

PROPOSITION 2.2. *Let $(a^*, b^*) \in \operatorname{argmax}_{n \in \mathbb{N}} f_\lambda \times \operatorname{argmax}_{n \in 2\mathbb{N}} f_\lambda$. Then a^* (resp., b^*) is a dominant strategy for Player 1 (resp., 2) in G_λ. In particular, they are optimal strategies in $\widetilde{\Gamma}_\lambda^{1^{++}}$.*

PROOF.   We have $\lim_{n \to +\infty} f_\lambda(n) = 0$, therefore, a^* and b^* are well defined. Observe that the function $(x, y) \to (1 - y)(1 - xy)^{-1}$, defined on $[0, 1)^2$, is increasing in x for every fixed y, and decreasing in y for every fixed x, and that $f_\lambda(\mathbb{N}) \subset [0, 1)$. Hence, a^* and b^* are dominant strategies in G_λ, and by Proposition 2.1 they are optimal strategies in $\widetilde{\Gamma}_\lambda^{1++}$. $\square$

REMARK 2.3.   The existence of dominant strategies in G_λ can be explained without any computation. Indeed, in $\widetilde{\Gamma}_\lambda^{1++}$, the maximization problem faced by Player 2 does not depend on the payoff he receives once reaching state 1_0, as long as this payoff is positive. Therefore, whichever stationary strategy Player 1 chooses, the best-response for Player 2 is always the same. The same argument applies to Player 1.

To study f_λ, it is convenient to make the change of variables $r = 2^{-n}$. Let $\widehat{f}_\lambda : [0, 1] \to \mathbb{R}$ be the function defined for $r \in (0, 1]$ by

$$\widehat{f}_\lambda(r) := (1 - r)\big(1 + 2\lambda r^{-s} - \lambda\big)^{-1},$$

where $s := 1 - \ln(2)^{-1} \ln(1 - \lambda) > 1$, and $\widehat{f}_\lambda(0) := 0$. Note that for all $n \in \mathbb{N}$, $f_\lambda(n) = (1 - \lambda^2)\widehat{f}_\lambda(2^{-n})$.

LEMMA 2.4.   *The function $\widehat{f}_\lambda$ reaches its maximum at a unique point $r^*(\lambda)$, is strictly increasing on $[0, r^*(\lambda)]$, and strictly decreasing on $[r^*(\lambda), 1]$. Moreover, for all $c > 0$,*

$$(2.5) \qquad\qquad \widehat{f}_\lambda(c\sqrt{2\lambda}) \underset{\lambda \to 0}{=} 1 - \big(c + c^{-1}\big)\sqrt{2\lambda} + o(\sqrt{\lambda})$$

and

$$(2.6) \qquad\qquad r^*(\lambda) \underset{\lambda \to 0}{\sim} \sqrt{2\lambda}.$$

PROOF.   Differentiating $\widehat{f}_\lambda$ yields

$$\widehat{f}_\lambda'(r) = \frac{-(1 + 2\lambda r^{-s} - \lambda) - (1 - r)(-2\lambda s r^{-s-1})}{(1 + 2\lambda r^{-s} - \lambda)^2}.$$

The numerator of this expression is equal to $h_\lambda(r) := \lambda - 1 + 2\lambda(-(1 + s)r + s)r^{-s-1}$. Note that

$$h_\lambda'(r) = -2\lambda\big(s(1 + s)(1 - r)\big)r^{-s-2}.$$

We have $h_\lambda' < 0$ on $(0, 1)$, $\lim_{r \to 0} h_\lambda(r) = +\infty$, and $h_\lambda(1) = -(1 + \lambda)$. Hence there exists $r^*(\lambda) \in (0, 1)$ such that h_λ is strictly positive on $(0, r^*(\lambda)]$, and strictly negative on $[r^*(\lambda), 1]$. Thus, $\widehat{f}_\lambda$ is strictly increasing on $[0, r^*(\lambda)]$, and strictly decreasing on $[r^*(\lambda), 1]$.

If $c > 0$, we have

$$\widehat{f_\lambda}(c\sqrt{2\lambda}) \underset{\lambda \to 0}{=} 1 - (c + c^{-1})\sqrt{2\lambda} + o(\sqrt{\lambda}).$$

Let $\varepsilon > 0$. Applying the last relation to $c = 1 - \varepsilon$, $c = 1$, and $c = 1 + \varepsilon$ shows that for λ small enough, $\widehat{f_\lambda}(\sqrt{2\lambda}) > \widehat{f_\lambda}((1 - \varepsilon)\sqrt{2\lambda})$ and $\widehat{f_\lambda}(\sqrt{2\lambda}) > \widehat{f_\lambda}((1 + \varepsilon)\sqrt{2\lambda})$. Thus, for λ small enough, $r^*(\lambda) \in [(1 - \varepsilon)\sqrt{2\lambda}, (1 + \varepsilon)\sqrt{2\lambda}]$. We deduce that $r^*(\lambda) \underset{\lambda \to 0}{\sim} \sqrt{2\lambda}$. $\square$

We can now prove our main result.

THEOREM 2.5. *In Γ, (v_λ) does not converge when λ goes to 0.*

PROOF. We are going to show that the sequence $(v_\lambda(1^{++}))$ does not converge when λ goes to 0.

Set $\lambda_m = 2^{-4m-1}$ and $\mu_m = 2^{-4m-3}$. Hence $\sqrt{2\lambda_m} = 2^{-2m}$ and $\sqrt{2\mu_m} = 2^{-2m-1}$. By Lemma 2.4, for m large enough,

$$\underset{n \in \mathbb{N}}{\text{argmax}} \, f_{\lambda_m} = \underset{n \in 2\mathbb{N}}{\text{argmax}} \, f_{\lambda_m} = \{2m\}.$$

Thus, for the discount factors $(\lambda_m)_{m \in \mathbb{N}}$, the fact that Player 1 has a wider set of strategies than Player 2 does not affect the outcome of the game. By Proposition 2.2, we have

$$v_{\lambda_m}(1^{++}) = (1 - f_{\lambda_m}(2m))(1 - f_{\lambda_m}(2m)^2)^{-1} = (1 + f_{\lambda_m}(2m))^{-1}.$$

By Lemma 2.4, $(f_{\lambda_m}(2m))$ converges to 1 when m goes to infinity, thus $(v_{\lambda_m}(1^{++}))$ converges to $1/2$.

By Lemma 2.4 again, for m large enough, we have

$$\underset{n \in \mathbb{N}}{\text{argmax}} \, f_{\mu_m} = \{2m + 1\} \quad \text{and} \quad \underset{n \in 2\mathbb{N}}{\text{argmax}} \, f_{\mu_m} \subset \{2m, 2m + 2\}.$$

Contrary to the previous situation, Player 1 has an advantage over her opponent: Player 2 cannot choose $2m + 1$. In $\widetilde{\Gamma}_{\mu_m}^{1^{++}}$, choosing $t(2m)$ or $t(2m + 2)$ instead of $t(2m + 1)$ changes the dynamics of the state, and makes this advantage substantial. Formally, we have

$$v_{\mu_m}(1^{++}) = \min\left(\frac{1 - f_{\mu_m}(2m)}{1 - f_{\mu_m}(2m)f_{\mu_m}(2m + 1)}, \frac{1 - f_{\mu_m}(2m + 2)}{1 - f_{\mu_m}(2m + 2)f_{\mu_m}(2m + 1)}\right).$$

By Lemma 2.4, we have

$$f_{\mu_m}(2m + 1) \underset{m \to +\infty}{=} 1 - 2\sqrt{2\mu_m} + o(\sqrt{\mu_m}),$$

$$f_{\mu_m}(2m) \underset{m \to +\infty}{=} 1 - 5/2\sqrt{2\mu_m} + o(\sqrt{\mu_m}),$$

$$f_{\mu_m}(2m + 2) \underset{m \to +\infty}{=} 1 - 5/2\sqrt{2\mu_m} + o(\sqrt{\mu_m}).$$

Hence,

$$\frac{1 - f_{\mu_m}(2m)}{1 - f_{\mu_m}(2m) f_{\mu_m}(2m+1)} \underset{m \to +\infty}{\sim} \frac{5/2\sqrt{2\mu_m}}{(2+5/2)\sqrt{2\mu_m}} = 5/9$$

and similarly

$$\frac{1 - f_{\mu_m}(2m+2)}{1 - f_{\mu_m}(2m+2) f_{\mu_m}(2m+1)} \underset{m \to +\infty}{\sim} \frac{5/2\sqrt{2\mu_m}}{(2+5/2)\sqrt{2\mu_m}} = 5/9.$$

The sequences $(v_{\lambda_m}(1^{++}))$ and $(v_{\mu_m}(1^{++}))$ converge to different limits, hence (v_λ) does not converge. $\square$

REMARK 2.6.   More generally, for every initial state $p \in P \setminus \{1^*, 0^*\}$, $(v_\lambda(p))$ does not converge. Consider, for example, the case $p = 0_n$, for some $n \in \mathbb{N}$. Let $N \geq n$. Consider the following strategy σ for Player 1 in $\Gamma_\lambda^{0_n}$: play C until $p_m = 0_N$, then play Q, and then play optimally in $\Gamma_\lambda^{1^{++}}$ if the state 1^{++} is reached. For λ small enough, the strategy σ guarantees $v_\lambda(1^{++}) - 2/N$ in $\Gamma_\lambda^{0_n}$. We deduce that $\lim_{\lambda \to 0} |v_\lambda(0_n) - v_\lambda(1^{++})| = 0$. With a similar argument, one can show that for all $(p, p') \in P^2 \setminus \{1^*, 0^*\}$, $\lim_{\lambda \to 0} |v_\lambda(p) - v_\lambda(p')| = 0$, which gives the result.

In this section, we have shown that Conjectures 1 and 2 are false, by presenting an example of a repeated game with public signals and perfect observation of the actions where (v_λ) does not converge. In the following section, we construct a repeated game belonging to the same class, where neither (v_λ) nor (v_n) converge.

3. From (v_λ) to (v_n).

3.1. *Motivation of the example.*   The idea of the construction of the game is based on the following lemma, which can be deduced from the proof of Sorin (2002), Theorem C.8, page 177. We provide the proof for completeness.

LEMMA 3.1.   *Let Γ be any repeated game with symmetric information, and $p \in \Delta(K)$. Let $P \subset \Delta(K)$ be the set of states which can be reached with positive probability under some strategy vector in the game $\widetilde{\Gamma}^p$. Let $n_0, n \in \mathbb{N}^*$, and for $m \in \mathbb{N}^*$ set $w_m := v_{1/m}$. Let $\| \cdot \|$ denote the supremum over P. Then the following inequality holds*:

$$\|v_n - w_n\| \leq \frac{n_0}{n} \|v_{n_0} - w_{n_0}\| + \sum_{m=n_0}^{n-1} \|w_m - w_{m+1}\|.$$

PROOF. Let $m \geq 1$ and $p \in P$. We have the following dynamic programming principle [see Sorin (2002), Properties C.13, page 181]:

$$(3.1) \qquad v_m(p) = \max_{x \in \Delta(I)} \min_{y \in \Delta(J)} \left\{ m^{-1} g(p, x, y) + (m-1)m^{-1} \mathbb{E}^p_{x,y}(v_{m-1}) \right\}$$

$$(3.2) \qquad = \min_{y \in \Delta(J)} \max_{x \in \Delta(I)} \left\{ m^{-1} g(p, x, y) + (m-1)m^{-1} \mathbb{E}^p_{x,y}(v_{m-1}) \right\}$$

and

$$(3.3) \qquad w_m(p) = \max_{x \in \Delta(I)} \min_{y \in \Delta(J)} \left\{ m^{-1} g(p, x, y) + (m-1)m^{-1} \mathbb{E}^p_{x,y}(w_m) \right\}$$

$$(3.4) \qquad = \min_{y \in \Delta(J)} \max_{x \in \Delta(I)} \left\{ m^{-1} g(p, x, y) + (m-1)m^{-1} \mathbb{E}^p_{x,y}(w_m) \right\}.$$

Let $x \in \Delta(I)$ be optimal in (3.1) and $y \in \Delta(J)$ be optimal in (3.4). We have

$$v_m(p) \leq m^{-1} g(p, x, y) + (m-1)m^{-1} \mathbb{E}^p_{x,y}(v_{m-1}),$$

$$w_m(p) \geq m^{-1} g(p, x, y) + (m-1)m^{-1} \mathbb{E}^p_{x,y}(w_m).$$

The combination of these two inequalities gives

$$v_m(p) - w_m(p) \leq (m-1)m^{-1} \| v_{m-1} - w_m \|.$$

Taking $x' \in \Delta(I)$ optimal in (3.2) and $y' \in \Delta(J)$ optimal in (3.3) gives the symmetric inequality:

$$w_m(p) - v_m(p) \leq (m-1)m^{-1} \| v_{m-1} - w_m \|.$$

Hence,

$$\| v_m - w_m \| \leq (m-1)m^{-1} \| v_{m-1} - w_m \|$$

and

$$m \| v_m - w_m \| \leq (m-1) \| v_{m-1} - w_{m-1} \| + (m-1) \| w_{m-1} - w_m \|.$$

Let $n, n_0 \geq 1$. Summing the last inequality from $n_0 + 1$ to n yields

$$n \| v_n - w_n \| \leq n_0 \| v_{n_0} - w_{n_0} \| + \sum_{m=n_0}^{n-1} m \| w_m - w_{m+1} \|.$$

Dividing by n gives the desired result. $\square$

We construct a family of repeated games with symmetric information $(\Gamma(r))_{r \geq 1}$, such that the value (v^r_λ) of $\Gamma_\lambda(r)$ does not converge. Moreover, for all $p \in \Delta(K)$, the derivative function of $v^r_\lambda(p)$ with respect to λ is bounded by $C(r)\lambda^{-1}$ for λ sufficiently small, where $C(r) > 0$ is independent of p and goes to 0 as r goes to infinity. Applying Lemma 3.1, we are then able to show that for r large enough, (v^r_n) does not converge.
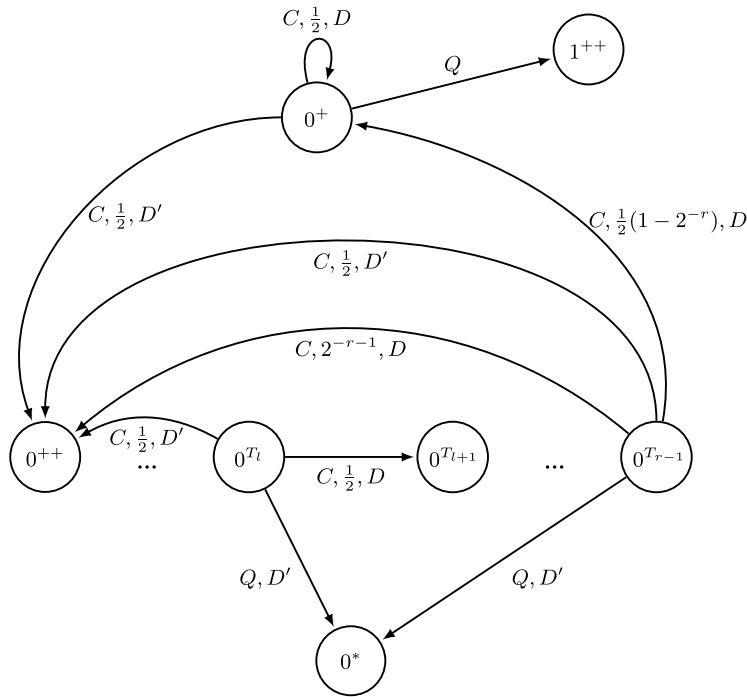
FIG. 4. *Transitions in the states controlled by Player* 1.

3.2. *Description of the game.* Let $r \geq 1$. Consider the following repeated game with symmetric information $\Gamma(r)$. The state space is

$$K = \{1^{++}, 1^{T_1}, 1^{T_2}, \ldots, 1^{T_{2r-1}}, 1^+, 1^*, 0^{++}, 0^{T_1}, 0^{T_2}, \ldots, 0^{T_{r-1}}, 0^+, 0^*\},$$

the actions sets are $I = J = \{C, Q\}$, and the signal set is $A = \{D, D'\}$.

Payoffs are independent of actions, and are 1 in states belonging to $\{1^*, 1^{++}, 1^{T_1}, \ldots, 1^{T_{2r-1}}, 1^+\}$, and 0 in states belonging to $\{0^*, 0^{++}, 0^{T_1}, \ldots, 0^{T_{r-1}}, 0^+\}$.

Player 2 controls the states $1^{++}, 1^{T_1}, \ldots, 1^{T_{2r-1}}, 1^+$, and Player 1 controls the states $0^{++}, 0^{T_1}, \ldots, 0^{T_{r-1}}, 0^+$. Hence, q can be seen as a map from $K \times \{C, Q\}$ to $\Delta(K \times \{D, D'\})$. Lastly, states 1^* and 0^* are absorbing states. The next figure describes the transitions in the state 0^{T_l}, where $l \in \{0, 1, \ldots, r - 2\}$, and in the states $0^{T_{r-1}}$, 0^+ and 0^* (by convention $0^{T_0} := 0^{++}$). To simplify Figure 4, the transitions in states $0^{T_0} = 0^{++}$ and $0^{T_{l+1}}$ are not represented.

In the states controlled by Player 2, the transitions are analog: one replaces 0 by 1 and $r - 1$ by $2r - 1$ (with the convention $1^{T_0} := 1^{++}$).

REMARK 3.2. The case $r = 1$ corresponds to the example of Section 2.

We now argue that the game $\widetilde{\Gamma}(r)$ presents strong similarities with the game $\widetilde{\Gamma}$ introduced in Section 2. Let $m \in \mathbb{N}$ and $l \in \{0, 1, \ldots, r - 1\}$.

Let $0_{mr+l} := 2^{-mr} \cdot 0^{T_l} + (1 - 2^{-mr}) \cdot 0^+$, and for $l \in \{0, 1, \dots, 2r - 1\}$ let $1_{2mr+l} := 2^{-2mr} \cdot 1^{T_l} + (1 - 2^{-2mr}) \cdot 1^+$. Let

$$P := \bigcup_{n \in \mathbb{N}} \{1_n, 0_n\} \cup \{1^*, 0^*\}.$$

The set P is the set of all states which can be reached with positive probability in $\widetilde{\Gamma}(r)^{1++}$ (note that P has the same formal definition as the set P in Section 2.2). Moreover, when a player plays C in some state $p \in P$, the transition is identical to the transition in our first example $\widetilde{\Gamma}$ (see Section 2.2). When Player 1 (resp., 2) plays Q at state 0_{mr+l} (resp., 1_{2mr+l}), the state is absorbed in state 0^* (resp., 1^*) with probability 2^{-mr} (resp., 2^{-2mr}). Thus the only difference with $\widetilde{\Gamma}$ is that in $\widetilde{\Gamma}(r)$, Player 1 can only take absorbing risks in the set $\{2^{-mr}, m \in \mathbb{N}\}$, and Player 2 can only take absorbing risks in the set $\{2^{-2mr}, m \in \mathbb{N}\}$. The next proposition is the equivalent of Proposition 2.1, and its proof is identical.

PROPOSITION 3.3. *The game $\widetilde{\Gamma}(r)_\lambda^{1++}$ has the same value as the strategic-form game $G_\lambda(r)$, with action set $r\mathbb{N}$ for Player 1, $2r\mathbb{N}$ for Player 2, and payoff*

$$g_\lambda^r(a, b) := g_\lambda(a, b),$$

where g_λ is defined in (2.1). Moreover, optimal strategies in $G_\lambda(r)$ induce optimal strategies in $\widetilde{\Gamma}(r)_\lambda^{1++}$.

3.3. *Asymptotic study of $G_\lambda(r)$.* Let $r \geq 2$. For $m \geq 1$, let $\lambda_m := 2^{-4mr-1}$ and $\mu_m := 2^{-4mr-2r-1}$. Hence, $\sqrt{2\lambda_m} = 2^{-2mr}$ and $\sqrt{2\mu_m} = 2^{-(2m+1)r}$. Proceeding exactly the same way as in Section 2.4, we get the following proposition.

PROPOSITION 3.4.

$$\lim_{m \to +\infty} v_{\lambda_m}^r(1^{++}) = 1/2 \quad and \quad \lim_{m \to +\infty} v_{\mu_m}^r(1^{++}) = \frac{2^r + 2^{-r}}{2^r + 2^{-r} + 2}.$$

SKETCH OF PROOF. For m large enough, we have by Lemma 2.4

$$\operatorname*{argmax}_{n \in r\mathbb{N}} f_{\lambda_m} = \operatorname*{argmax}_{n \in 2r\mathbb{N}} f_{\lambda_m} = \{2mr\}.$$

We deduce (see the proof of Theorem 2.5) that $(v_{\lambda_m}^r)(1^{++})$ converges to $1/2$. By Lemma 2.4 once again, for m large enough, we have

$$\operatorname*{argmax}_{n \in r\mathbb{N}} f_{\mu_m} = \{(2m + 1)r\}$$

and

$$\operatorname*{argmax}_{n \in 2r\mathbb{N}} f_{\mu_m} \subset \{2mr, 2(m + 1)r\}.$$

We deduce (see the proof of Theorem 2.5) that $v_{\mu_m}^r(1^{++})$ converges to $(2^r + 2^{-r})/(2^r + 2^{-r} + 2)$. □

We now use Lemma 3.1 to show that the value (v_n^r) of the game $\Gamma(r)_n$ does not converge. We adopt the following notation: if $(\lambda, p) \in (0, 1] \times \Delta(K)$, we call $(v_\lambda^r)'(p)$ the derivative of $v_\lambda^r(p)$ with respect to λ, evaluated in λ. Analogously, $f_\lambda'(p)$ is the derivative of f_λ with respect to λ, evaluated in λ.

The following lemma gives a majorization of $(v_\lambda^r)'(p)$ on certain subintervals.

LEMMA 3.5.   *There exists $m_0 \geq 1$ such that for all $m \geq m_0$, there exists $\mu_m^0 \in [\mu_m, 2^{r/2-1}\mu_m]$ such that for all $p \in P$ and for all $\mu \in [\mu_m, 2^{r/2-1}\mu_m] \setminus \{\mu_m^0\}$, $v_\mu^r(p)$ is differentiable at μ and*

$$(3.5) \qquad\qquad |(v_\mu^r)'(p)| \leq 2^{-r/2}\mu^{-1},$$

and for all $\lambda_m \leq \lambda \leq 2^{r-1}\lambda_m$, $v_\lambda^r(p)$ is differentiable at λ and

$$(3.6) \qquad\qquad |(v_\lambda^r)'(p)| \leq 2^{r/2+2}\lambda^{-1/2}.$$

PROOF.   We start by proving inequality (3.5). The proof proceeds in three steps.

STEP 1 [*Computation of $v_\mu^r(1^{++})$*].   By Lemma 2.4, there exists $m_1 \in \mathbb{N}^*$ such that for all $m \geq m_1$ and $\mu \in [\mu_m, 2^{r/2-1}\mu_m]$,

$$\operatorname*{argmax}_{n \in r\mathbb{N}} f_\mu = \{2mr + r\}.$$

Hence, for such m and μ, $a(\mu) := 2mr + r$ is an optimal strategy for Player 1 in $G_\mu(r)$.

By Lemma 2.4 once again, there exists $m_2 \in \mathbb{N}^*$ such that for all $m \geq m_2$, there exists $\mu_m^0 \in [\mu_m, 2^{r/2-1}\mu_m]$ such that for all $\mu \in (\mu_m^0, 2^{r/2-1}\mu_m]$

$$\operatorname*{argmax}_{n \in 2r\mathbb{N}} f_\mu = \{2mr\},$$

and for all $\mu \in [\mu_m, \mu_m^0)$

$$\operatorname*{argmax}_{n \in 2r\mathbb{N}} f_\mu = \{2mr + 2r\}.$$

Hence, for all $m \geq m_2$ and $\mu \in (\mu_m^0, 2^{r/2-1}\mu_m]$, the integer $b_1(\mu) := 2mr$ is an optimal strategy for Player 2 in $G_\mu(r)$, and for all $\mu \in [\mu_m, \mu_m^0)$, the integer $b_2(\mu) := 2mr + 2r$ is an optimal strategy for Player 2 in $G_\mu(r)$. Thus, for all $m \geq \max(m_1, m_2) := m_3$ and $\mu \in (\mu_m^0, 2^{r/2-1}\mu_m]$, we have

$$(3.7) \qquad \begin{aligned} v_\mu^r(1^{++}) &= g_\lambda\big(a(\mu), b_1(\mu)\big) \\ &= \big[1 - f_\lambda(b_1(\mu))\big]\big[1 - f_\lambda(a(\mu))f_\lambda(b_1(\mu))\big]^{-1}, \end{aligned}$$

and for $\mu \in [\mu_m, \mu_m^0)$

$$(3.8) \qquad \begin{aligned} v_\mu^r(1^{++}) &= g_\lambda(a(\mu), b_2(\mu)) \\ &= [1 - f_\lambda(b_2(\mu))][1 - f_\lambda(a(\mu)) f_\lambda(b_2(\mu))]^{-1}. \end{aligned}$$

STEP 2 [*Asymptotic expansion of* $(v_\mu^r)'(1^{++})$ *as* $\mu \to 0$]. Fix $m \geq m_3$ and $\mu \in (\mu_m^0, 2^{r/2-1}\mu_m]$. Define $C_1 : (0, 1] \to \mathbb{R}_+^*$ and $C_2 : (0, 1] \to \mathbb{R}_+^*$ by

$$(3.9) \qquad C_1(\mu) := 2^{-2mr-r}(2\mu)^{-1/2} = (\mu_m)^{1/2}\mu^{-1/2}$$

and

$$(3.10) \qquad C_2(\mu) := 2^{-2mr}(2\mu)^{-1/2} = 2^r(\mu_m)^{1/2}\mu^{-1/2}.$$

Note that $2^{-a(\mu)} = C_1(\mu)\sqrt{2\mu}$ and $2^{-b_1(\mu)} = C_2(\mu)\sqrt{2\mu}$. Moreover, $1 \leq \|C_1\|_\infty \leq 2^r$ and $2^r \leq \|C_2\|_\infty \leq 2^{2r}$. The last two inequalities show that the functions C_1 and C_2 are bounded and bounded away from 0, which will be useful in the following asymptotic expansions.

Since the functions $\lambda \to a(\lambda)$ and $\lambda \to b_1(\lambda)$ are constant on $(\mu_m^0, 2^{r/2-1}\mu_m]$, $\lambda \to f_\lambda(a(\lambda))$ and $\lambda \to f_\lambda(b_1(\lambda))$ are differentiable at μ, and we have

$$\begin{aligned} f_\mu'(a(\mu)) &= (1 - 2^{-a(\mu)})(1 + 2^{a(\mu)+1}\mu(1-\mu)^{-a(\mu)} - \mu)^{-2} \\ &\quad \times [-2\mu(1 + 2^{a(\mu)+1}\mu(1-\mu)^{-a(\mu)} - \mu) \\ &\quad - (1-\mu^2)(2^{a(\mu)+1}[(1-\mu)^{-a(\mu)} - a(\mu)\mu(1-\mu)^{a(\mu)-1}] - 1)]. \end{aligned}$$

We deduce that

$$(3.11) \qquad f_\mu'(a(\mu)) \underset{\mu \to 0}{=} -2C_1(\mu)^{-1}(2\mu)^{-1/2} + o(\mu^{-1/2}),$$

and likewise

$$(3.12) \qquad f_\mu'(b_1(\mu)) \underset{\mu \to 0}{=} -2C_2(\mu)^{-1}(2\mu)^{-1/2} + o(\mu^{-1/2}).$$

The same computation as in Lemma 2.4 gives

$$f_\mu(a(\mu)) \underset{\mu \to 0}{=} 1 - (C_1(\mu) + C_1(\mu)^{-1})(2\mu)^{1/2} + o(\mu^{1/2}),$$

$$f_\mu(b_1(\mu)) \underset{\mu \to 0}{=} 1 - (C_2(\mu) + C_2(\mu)^{-1})(2\mu)^{1/2} + o(\mu^{1/2}).$$

Now we differentiate $v_\mu^r(1^{++})$ in (3.7) (we omit the dependence of a and b_1 on μ). The derivative $(v_\mu^r)'(1^{++}) = g_\mu'(a, b_1)$ is equal to

$$\begin{aligned} &\frac{-f_\mu'(b_1)(1 - f_\mu(a)f_\mu(b_1)) + (1 - f_\mu(b_1))(f_\mu'(a)f_\mu(b_1) + f_\mu(a)f_\mu'(b_1))}{(1 - f_\mu(a)f_\mu(b_1))^2} \\ &= \frac{f_\mu'(a)f_\mu(b_1)(1 - f_\mu(b_1)) - f_\mu'(b_1)(1 - f_\mu(a))}{(1 - f_\mu(a)f_\mu(b_1))^2}. \end{aligned}$$

When μ goes to 0, the numerator of this expression is (we omit the dependence on μ)

$$-2C_1^{-1}(C_2 + C_2^{-1}) + 2C_2^{-1}(C_1 + C_1^{-1}) + o(1) = 2(C_2^{-1}C_1 - C_1^{-1}C_2) + o(1).$$

Hence,

$$(v_\mu^r)'(1^{++}) \underset{\mu \to 0}{=} \frac{(C_2^{-1}C_1 - C_1^{-1}C_2)}{(C_1 + C_1^{-1} + C_2 + C_2^{-1})^2} \mu^{-1} + o(\mu^{-1}).$$

Since $\mu \le 2^{r/2-1}\mu_m$, we have

$$\left| \frac{(C_2^{-1}C_1 - C_1^{-1}C_2)}{(C_1 + C_1^{-1} + C_2 + C_2^{-1})^2} \right| \le C_1^{-1}C_2^{-1} = 2^{-r}\frac{\mu}{\mu_m} \le 2^{-r/2-1}.$$

The last two relations show that for m large enough and $\mu_m^0 < \mu \le 2^{r/2-1}\mu_m$

$$(3.13) \qquad\qquad |(v_\mu^r)(1^{++})| \le 2^{-r/2-1}\mu^{-1}.$$

Equation (3.8) and similar computations show that the last inequality is also true for m large enough and $\mu \in [\mu_m, \mu_m^0)$.

STEP 3 [*Computation of $(v_\mu^r)'(p)$ for $p \in P$ and proof of inequality* (3.5)]. For $a \in \mathbb{N}$ and $n \le a$, let T_a^n be the random time needed by Player 1 to go from state 0_n to state 0_a, when she plays strategy $s(a)$. Let $m \ge m_3$ and $\mu \in (\mu_m^0, 2^{r/2-1}\mu_m]$.

If $n < a(\mu)$, then

$$(3.14) \qquad v_\mu^r(0_n) = (1 - 2^{-a(\mu)})\mathbb{E}((1-\mu)^{T_{a(\mu)}^n + 1})v_\mu^r(1^{++}).$$

If $n \ge a(\mu)$, then

$$(3.15) \qquad v_\mu^r(0_n) = (1 - 2^{-n})(1 - \mu)v_\mu^r(1^{++}).$$

Thus, for all $n \in \mathbb{N}$

$$(3.16) \qquad \begin{aligned} |(v_\mu^r)'(0_n)| &\le \mathbb{E}(T_{a(\mu)}^n + 1) + |(v_\mu^r)'(1^{++})| \\ &\le \mathbb{E}(T_{a(\mu)}) + |(v_\mu^r)'(1^{++})|. \end{aligned}$$

Similar arguments lead to

$$(3.17) \qquad\qquad |v_\mu^r(1_n)| \le \mathbb{E}(T_{b_1(\mu)}) + |(v_\mu^r)'(0^{++})|.$$

By (2.4), $\mathbb{E}(T_{a(\mu)}) = o(\mu^{-1})$ and $\mathbb{E}(T_{b_1(\mu)}) = o(\mu^{-1})$ when μ goes to 0.

Combining inequalities (3.13), (3.16) and (3.17), we get inequality (3.5) for m large enough and $\mu \in (\mu_m^0, 2^{r/2-1}\mu_m]$. For m large enough and $\mu \in (\mu_m, \mu_m^0]$, the computations are similar.

Let us now prove inequality (3.6). By Lemma 2.4, there exists $m_4 \in \mathbb{N}^*$ such that for all $\lambda \in [\lambda_m, 2^{r-1}\lambda_m]$, $a(\lambda) := 2mr$ is an optimal strategy for both players in $G_\lambda(r)$, and

$$(v_\lambda^r)'(1^{++}) = -\frac{f_\lambda(a(\lambda))'}{(1 + f_\lambda(a(\lambda)))^2}.$$

For $m \geq m_4$ and $\lambda \in [\lambda_m, 2^{r-1}\lambda_m]$, let $C(\lambda) := \sqrt{\frac{\lambda_m}{\lambda}}$. Then as in (3.11)

$$f_\lambda'(a(\lambda)) \underset{\lambda \to 0}{=} -2C(\lambda)^{-1}(2\lambda)^{-1/2} + o(\lambda^{-1/2}).$$

Since $C(\lambda)^{-1} \leq 2^{(r-1)/2}$ and $f_\lambda(a(\lambda))$ goes to 1 when λ goes to 0, we get inequality (3.6) for $(v_\lambda^r)'(1^{++})$ and it extends to $(v_\lambda^r)'(p)$ with the method used in Step 3. □

THEOREM 3.6. *There exists $r_0 \in \mathbb{N}^*$ such that for all $r \geq r_0$, (v_n^r) and (v_λ^r) do not converge.*

PROOF. Let $r \geq 2$. Recall that from Proposition 3.4,

$$\lim_{m \to +\infty} v_{\lambda_m}^r(1^{++}) = 1/2 \quad \text{and} \quad \lim_{m \to +\infty} v_{\mu_m}^r(1^{++}) = \frac{2^r + 2^{-r}}{2^r + 2^{-r} + 2} := w(r).$$

In particular, (v_λ^r) does not converge. We are going to show that for r big enough, $(v_n^r(1^{++}))$ does not converge.

Let m_0 be as in Lemma 3.5 and let $m \geq m_0$. Let $n(m) := \mu_m^{-1} = 2^{4mr+2r+1}$ and $n_0(m) := 2^{-\lfloor r/2 \rfloor + 1} n(m)$. We now compare $v_{n(m)}^r$ and $v_{\mu_m}^r$. Using Lemma 3.1, we get

$$\|v_{n(m)}^r - v_{\mu_m}^r\| \leq \frac{n_0(m)}{n(m)} \|v_{n_0(m)}^r - v_{2^{\lfloor r/2 \rfloor - 1}\mu_m}^r\| + \sum_{m'=n_0(m)}^{n(m)-1} \|v_{1/m'}^r - v_{1/(m'+1)}^r\|.$$

Since the payoff function is bounded by 1, we have

$$\frac{n_0(m)}{n(m)} \|v_{n_0(m)}^r - v_{2^{\lfloor r/2 \rfloor - 1}\mu_m}^r\| \leq 2^{-\lfloor r/2 \rfloor + 1}.$$

By inequality (3.5) in Lemma 3.5 and the mean value theorem, we have

$$\sum_{m'=n_0(m)}^{n(m)-1} \|v_{1/m'}^r - v_{1/(m'+1)}^r\| \leq 2^{-r/2+1} \sum_{m'=n_0(m)}^{n(m)-1} \int_{1/(m'+1)}^{1/m'} \frac{1}{x} dx$$

$$= 2^{-r/2+1} \int_{(n(m))^{-1}}^{(n_0(m))^{-1}} \frac{1}{x} dx$$

$$= 2^{-r/2+1}(\lfloor r/2 \rfloor - 1).$$

Letting m going to infinity, we deduce that

$$\limsup_{n \to +\infty} v_n^r(1^{++}) \geq w(r) - 2^{-\lfloor r/2 \rfloor + 1} - 2^{-r/2 + 1}(\lfloor r/2 \rfloor - 1).$$

Note that $\lim_{r \to +\infty} w(r) = 1$ and that the term on the right goes to 0 when r goes to infinity.

Lemma 3.1 for $n(m) := \lambda_m^{-1}$ and $n_0(m) := 2^{-\lfloor r/2 \rfloor + 1} n(m)$ gives also an inequality of the form

$$\liminf_{n \to +\infty} v_n^r(1^{++}) \leq 1/2 + t(r),$$

where $\lim_{r \to +\infty} t(r) = 0$. Hence, for r large enough, $(v_n^r(1^{++}))$ does not converge, thus (v_n^r) does not converge. $\square$

We have proved in this section that in a repeated game with symmetric information, the value of the n-stage repeated game might not converge. To do so, we have exploited the very flexible structure of the first example of Section 2. Indeed, we have managed to slow down the oscillations of v_λ, without changing much the dynamics of the game.

In the next section, we again take advantage of the flexibility of the game, to provide other examples of repeated games without an asymptotic value.

4. Extension to other classes of repeated games.

4.1. *State-blind repeated games.*   Consider the following state-blind repeated game Γ, with state space $K = \{1^*, 1^{++}, 1^T, 1^+, 0^*, 0^{++}, 0^+\}$, action sets $I = \{T, B, Q\}$ for Player 1 and $J = \{L, R, Q\}$ for Player 2. The states 0^* and 1^* are absorbing states. The payoff is 1 in states $1^{++}, 1^T, 1^+$ and 1^*, and 0 in states 0^{++}, 0^+ and 0^*. The transitions are described in Tables 1–5.

Recall that in this model, the players do not observe any signal about the state, and only observe past actions.

The idea of this example is to artificially recreate the dynamics of the example of Section 2, replacing signals by the mixed actions of one player.

Formally, let $a^*(\lambda)$ [resp., $b^*(\lambda)$] be defined as in Proposition 2.2.

Let $\sigma^* \in \Sigma$ be the following strategy for Player 1 in $\Gamma_\lambda^{1^{++}}$: play $1/2 \cdot T + 1/2 \cdot B$ if $\mathbb{P}(k_m = 0^+ | k_m \notin \{0^*, 1^*\}) \leq 1 - 2^{-a^*(\lambda)}$, otherwise play Q.

Let $\tau^* \in \mathcal{T}$ be the following strategy for Player 2: play $1/2 \cdot L + 1/2 \cdot R$ if $\mathbb{P}(k_m = 1^+ | k_m \notin \{0^*, 1^*\}) \leq 1 - 2^{-b^*(\lambda)}$, otherwise play Q.

Proceeding as in Section 2, one can show that σ^* and τ^* are optimal strategies, respectively, for Player 1 and 2 in $\Gamma_\lambda^{1^{++}}$. Moreover, the probability measure on the histories of the game induced by these strategies is the same as the probability measure induced by $s(a^*(\lambda))$ and $t(b^*(\lambda))$ in the example of Section 2. In particular, the two examples have the same discounted value, thus $(v_\lambda(1^{++}))$ does not converge.

TABLE 1
State 1^{++}

	L	R	Q
T	1^{++}	1^T	1^*
B	1^T	1^{++}	1^*
Q	0^*	0^*	0^*

TABLE 2
State 1^T

	L	R	Q
T	1^{++}	$3/4 \cdot 1^+ + 1/4 \cdot 1^{++}$	1^*
B	$3/4 \cdot 1^+ + 1/4 \cdot 1^{++}$	1^{++}	1^*
Q	0^*	0^*	0^*

TABLE 3
State 1^+

	L	R	Q
T	1^{++}	1^+	0^{++}
B	1^+	1^{++}	0^{++}
Q	0^*	0^*	0^*

TABLE 4
State 0^{++}

	L	R	Q
T	$1/2 \cdot 0^{++} + 1/2 \cdot 0^+$	0^{++}	1^*
B	0^{++}	$1/2 \cdot 0^{++} + 1/2 \cdot 0^+$	1^*
Q	0^*	0^*	1^*

TABLE 5
State 0^+

	L	R	Q
T	0^+	0^{++}	1^*
B	0^{++}	0^+	1^*
Q	1^{++}	1^{++}	1^*

4.2. *Repeated games with one informed player.*   We now investigate a repeated game with perfect observation of the actions, where Player 2 is fully informed about the state, while Player 1 has no information about it. As usual, both players observe past actions.

The state space is $K = \{1^*, 1, 0^*, 0^{++}, 0^+\}$, action sets are $I = \{T, B, Q\}$ for Player 1 and $J = \{L, R\}$ for Player 2. The states 0^* and 1^* are absorbing states. The payoff is 1 in states 1 and 1^*, and 0 in states 0^{++}, 0^+ and 0^*. The transitions are described in the following Tables 6–8.

Compared to the game of Section 4.1, states 1^{++}, 1^T and 1^+ have been replaced by a single state 1, which is similar to the state ω^+ in Vigeral (2013). The other states have not been changed.

Let σ^* be the following strategy for Player 1: in state 1, play $(1 - \sqrt{\lambda}) \cdot T + \sqrt{\lambda} \cdot B$, and when the belief is in $\Delta(\{0^{++}, 0^+\})$, play the same strategy as in the preceding example.

Let τ^* be the following strategy for Player 2: in state 1, play $(1 - \sqrt{\lambda}) \cdot L + \sqrt{\lambda} \cdot R$, and in states 0^{++} and 0^+, play $(1/2 \cdot L + 1/2 \cdot R)$. Proceeding the same way

TABLE 6
State 1

	L	R
T	1	0^{++}
B	0^{++}	1^*
Q	0^*	0^*

TABLE 7
State 0^{++}

	L	R
T	$1/2 \cdot 0^{++} + 1/2 \cdot 0^+$	0^{++}
B	0^{++}	$1/2 \cdot 0^{++} + 1/2 \cdot 0^+$
Q	0^*	0^*

TABLE 8
State 0^+

	L	R
T	0^+	0^{++}
B	0^{++}	0^+
Q	1	1

as in Section 2, one can show that σ^* and τ^* are asymptotically optimal strategies in Γ_λ^1, and that $(v_\lambda(1))$ does not converge.

4.3. *Stochastic games with compact action sets.* We now study a repeated game with perfect observation (states and actions are known by both players) but where I and J are compact. As mentioned in the Introduction, this example does not relate to the two conjectures, because I and J are not finite. But it yields a simpler alternative counterexample to Vigeral (2013). This example is similar to the example of Section 2 in terms of dynamics.

The state space is $K = \{1^*, 1, 0^*, 0\}$, and actions sets are $I = [0, 1]$ and $J = \{0\} \cup \bigcup_{m \in \mathbb{N}} 4^{-m}$. The transition q is defined by

$$q(1, x, y) := (1 - y) \cdot 1 + (y - y^2) \cdot 0 + y^2 \cdot 1^*,$$

$$q(0, x, y) := (1 - x) \cdot 0 + (x - x^2) \cdot 1 + x^2 \cdot 0^*.$$

Hence, Player 1 controls state 0 and Player 2 controls state 1.

Let $\lambda \in (0, 1]$. A pure stationary strategy in Γ_λ for Player 1 (resp., 2) can be seen as an element of I (resp., J).

REMARK 4.1. The real number $x \in I$ corresponds to the absorbing risk 2^{-a} in the example of Section 2. Indeed, when Player 1 plays x in state 0, she waits on average x^{-1} stages before switching to state 1, and the probability of absorbing in 0^* before reaching state 1 is approximately x. Recall that in the example of Section 2, when Player 1 plays $a \in \mathbb{N}$, she waits on average 2^a stages before quitting, and when she plays Q the game is absorbed in state 0^* with probability 2^{-a}. It is the same for Player 2. As in our first example, Player 2 cannot take any absorbing risk: only $y = 4^{-m}$ for some $m \in \mathbb{N}$, or $y = 0$. But Player 1 can take any absorbing risk in $[0, 1]$. That is why we expect (v_λ) to oscillate, just as in the first example.

The payoff in Γ_λ^1 given by a pair of strategies $(x, y) \in I \times J$ is

$$\gamma_\lambda^1(x, y) = \frac{(1 - (1 - \lambda)(1 - y^2))(1 - (1 - \lambda)(1 - x))}{(1 - (1 - \lambda)(1 - xy))(1 - (1 - \lambda)(1 - x)(1 - y))}.$$

For any $x \in [0, 1]$ (resp., $y \in [0, 1]$) $\gamma_\lambda^1(x, \cdot)$ [resp., $\gamma_\lambda^1(\cdot, y)$] is convex (resp., concave) and reaches its minimum (resp., its maximum) at y^* (resp., x^*) such that

$$x^* = y^* = (\sqrt{\lambda} - \lambda)(1 - \lambda)^{-1}.$$

For $m \geq 1$, we define $\lambda_m := 2^{-2m}$ and $\mu_m := 2^{-2m-1}$. Then for m large enough, $x_m = y_m = \sqrt{\lambda_m}$ are asymptotically optimal strategies in $\Gamma_{\lambda_m}^1$. Thus,

$$\lim_{m \to +\infty} v_{\lambda_m}(1) = \lim_{m \to +\infty} \gamma_{\lambda_m}^1(x_m, y_m) = 1/2.$$

For m large enough, $x_m = \sqrt{\mu_m}$ is an asymptotically optimal strategy for Player 1 in $\Gamma^1_{\mu_m}$, and either $y_m = 2\sqrt{\mu_m}$ or $y'_m = 1/2\sqrt{\mu_m}$ is an optimal strategy for Player 2 in $\Gamma^1_{\mu_m}$. We have

$$\lim_{m \to +\infty} \gamma^1_{\mu_m}(x_m, y_m) = \lim_{m \to +\infty} \gamma^1_{\mu_m}(x_m, y'_m) = 5/9.$$

Thus, $\lim_{m \to +\infty} v_{\mu_m}(1) = 5/9$, and $(v_\lambda(1))$ does not converge.

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