The Annals of Probability
2016, Vol. 44, No. 1, 360–398
DOI: 10.1214/14-AOP976

ON THE CAUCHY PROBLEM FOR BACKWARD STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS IN HÖLDER SPACES[1]

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This paper is concerned with solution in Hölder spaces of the Cauchy problem for linear and semi-linear backward stochastic partial differential equations (BSPDEs) of super-parabolic type. The pair of unknown variables are viewed as deterministic spatial functionals which take values in Banach spaces of random (vector) processes. We define suitable functional Hölder spaces for them and give some inequalities among these Hölder norms. The existence, uniqueness as well as the regularity of solutions are proved for BSPDEs, which contain new assertions even on deterministic PDEs.

1. Introduction. In this paper, we consider the Cauchy problem for backward stochastic partial differential equations (BSPDEs, for short) of super-parabolic type:

$$
\begin{cases} -du(t,x) = \big[a^{ij}(t,x)\,\partial^2_{ij}u(t,x) + b^i(t,x)\,\partial_i u(t,x) \\ \qquad\qquad + c(t,x)u(t,x) + f(t,x) + \sigma^l(t,x)v_l(t,x)\big]\,dt \\ \qquad\qquad - v_l(t,x)\,dW^l_t, \qquad (t,x)\in[0,T)\times\mathbb{R}^n; \\ u(T,x) = \Phi(x), \qquad\qquad x\in\mathbb{R}^n. \end{cases} \tag{1.1}
$$

Here, $T > 0$ is fixed, $W = \{W_t : t \in [0, T]\} := (W^1, \dots, W^d)'$ is a d-dimensional standard Brownian motion defined on some filtered complete probability space $(\Omega, \mathscr{F}, \mathbb{F}, P)$ with $\mathbb{F} := \{\mathscr{F}_t : t \in [0, T]\}$ being the augmented natural filtration generated by W, $a := (a^{ij})_{n\times n}$ is a symmetric and positive matrix-valued deterministic functions of the time and space variable (t, x), $b := (b^1, \dots, b^n)'$ and $\sigma := (\sigma^1, \dots, \sigma^d)'$ are random vector fields, and c, f, and terminal term Φ are scalar-valued random fields. Denote by $\mathscr{P}$ the predictable σ-algebra generated by $\mathbb{F}$. Here and after, we use the Einstein summation convention, the prime de-

Received April 2013; revised September 2014.

[1]Supported in part by the Natural Science Foundation of China (Grants #10325101 and #11171076), Basic Research Program of China (973 Program, No. 2007CB814904), WCU (World Class University) Program through the Korea Science and Engineering Foundation funded by the Ministry of Education, Science and Technology (R31-20007), the Science Foundation for Ministry of Education of China (No. 20090071110001), and the Chang Jiang Scholars Programme.

MSC2010 subject classifications. Primary 60H15; secondary 35R60.

Key words and phrases. Backward stochastic partial differential equations, backward stochastic differential equations, Hölder space, heat potential.

notes the transpose of a vector or a matrix, and denote

$$\partial_s := \frac{\partial}{\partial s}, \qquad \partial_i := \frac{\partial}{\partial x_i}, \qquad \partial^2_{ij} := \frac{\partial^2}{\partial x_i \, \partial x_j}.$$

Our aim is to find a pair of random fields $(u, v) : [0, T] \times \Omega \times \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^d$ in suitable Hölder spaces such that BSPDE (1.1) is satisfied in some sense, and to study the regularity of (u, v), particularly in the space variable x.

As a mathematically natural extension of backward stochastic differential equations (BSDEs) (see, e.g., [3–5, 12, 20, 28]), BSPDEs arise in many applications of probability theory and stochastic processes. For instance, in the optimal control problem of stochastic differential equations (SDEs) with incomplete information or stochastic partial differential equations (SPDEs), a linear BSPDE arises as the adjoint equation of SPDEs (or the Duncan–Mortensen–Zakai filtration equation) to formulate the maximum principle (see, e.g., [1, 2, 25, 26, 29, 30]). In the study of controlled non-Markovian SDEs by Peng [21], the so-called stochastic Hamilton–Jacobi–Bellman equation is a class of fully nonlinear BSPDEs. Solution of forward–backward stochastic differential equation (FBSDE) with random coefficients is also associated to that of a quasi-linear BSPDE, which gives the stochastic Feynman–Kac formula (see, e.g., [17]).

Weak and strong solutions of linear BSPDEs have already received an extensive attention in literature. Strong solution in the Sobolev space $W^{m,2}$ is referred to, for example, [7, 8, 10, 11, 14, 15, 17, 18, 23], and in L^p $[p \in (1, \infty)]$ is referred to, for example, [9]. The theory of linear BSPDEs in Sobolev spaces is rather complete now. Qiu and Tang [22] further discuss the maximum principle of BSPDEs in a domain. It is quite natural to consider now the Hölder solution of BSPDEs. We note that Tang [27] discusses the existence and uniqueness of a classical solution to semi-linear BSPDE using a probabilistic approach. However, the coefficients are required to be k-times (with $k \geq 2 + \frac{n}{2}$) continuously differentiable in the spatial variable x, which is much higher than the necessary regularity on the coefficients known in the theory of deterministic PDEs. In this paper, the pair of unknown variables are viewed as deterministic spatial functionals which take values in Banach spaces of random (vector) processes. We discuss BSPDE (1.1) in Hölder spaces, using the methods of deterministic PDEs (see Gilbarg and Trudinger [13], Ladyženskaja, Solonnikov and Ural'ceva [16]), and establish a Hölder theory for BSPDEs under the spatial Hölder-continuity assumption on the coefficients a, b, c and σ. The paper seems to be the first attempt at Hölder solution of BSPDEs.

As an alternative stochastic extension of deterministic second-order parabolic equations, (forward) SPDEs have been studied in Hölder spaces by Rozovskii [24] and Mikulevicius [19]. However, our BSPDE (1.1) is significantly different from an SPDE. Indeed, a BSPDE has an additional unknown variable v whose regularity is usually worse. It serves in our BSPDE as the diffusion, but it is not a priori given. Instead, it is endogenously determined by the given coefficients via a martingale representation theorem. It is crucial to choose a suitable Hölder space to describe its regularity. In light of the functional Hölder space introduced by Mikulevicius [19] for discussing a SPDE, we define in Section 2 the functional Hölder spaces such as $C^{m+\alpha}(\mathbb{R}^n; \mathscr{S}^2_{\mathbb{F}}[0,T])$ for u, and $C^{m+\alpha}(\mathbb{R}^n; \mathscr{L}^2_{\mathbb{F}}(0,T;\mathbb{R}^d))$ for v. That is, we only discuss the continuity of the unknown pair (u,v) in x by looking at $(u(\cdot,x), v(\cdot,x))$ as a functional stochastic process taking values in the space $\mathscr{S}^2_{\mathbb{F}}[0,T] \times \mathscr{L}^2_{\mathbb{F}}(0,T;\mathbb{R}^d)$.

We first study the following simpler BSPDE with space-invariant coefficients a and σ:

$$
(1.2) \qquad \begin{cases} -du(t,x) = \big[a^{ij}(t)\,\partial^2_{ij}u(t,x) + f(t,x) + \sigma^l(t)v_l(t,x)\big]\,dt \\ \qquad\qquad\qquad - v_l(t,x)\,dW^l_t, \qquad (t,x) \in [0,T) \times \mathbb{R}^n, \\ u(T,x) = \Phi(x), \qquad\qquad\qquad x \in \mathbb{R}^n. \end{cases}
$$

Here, the coefficients $a^{ij}(\cdot)$ and $\sigma^l(\cdot)$ $(i,j = 1,\cdot,n; l = 1,\cdot,d)$ do not depend on the space variable x. The advantage of the simpler case is that the solution (u,v) admits an explicit expression in terms of the terminal value Φ and the free term f via their convolution with the heat potential. We prove the existence and uniqueness result of this equation, and show that $(u,v)(t,\cdot) \in C^{2+\alpha} \times C^{\alpha}$, and $u(\cdot,x) \in C^{1/2}$, when $\Phi \in C^{1+\alpha}$ and $f(t,\cdot) \in C^{\alpha}$. These regularity results are extended to general space-variable BSPDE (1.1) by the freezing coefficients method and the standard continuity argument. Moreover, when all the coefficients are deterministic, BSPDE (1.1) becomes a deterministic PDE, and our results include new consequences on a deterministic PDE.

The rest of the paper is organized as follows. In Section 2, we define some functional Hölder spaces, and recall analytical properties of the heat potential. In Section 3, we study the existence, uniqueness and regularity of the solution of BSPDE (1.2). In Section 4, we extend the results in Section 3 to BSPDE (1.1) via the freezing coefficients method and the standard argument of continuity, and discuss their consequences on a deterministic PDE. In Section 5, we discuss a semi-linear BSPDE.

2. Preliminaries.

2.1. *Notation and Hölder spaces.* Define the set of multi-indices

$$\Gamma := \{\gamma = (\gamma_1, \dots, \gamma_n) : \gamma_1, \dots, \gamma_n \text{ are all nonnegative integers}\}.$$

For $\gamma \in \Gamma$ and $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, define

$$|\gamma| := \sum_{i=1}^n \gamma_i, \qquad D^\gamma := \frac{\partial^{|\gamma|}}{\partial x_1^{\gamma_1}\,\partial x_2^{\gamma_2} \cdots \partial x_n^{\gamma_n}}.$$

The inner product in an Euclidean space is denoted by $\langle \cdot, \cdot \rangle$, and the norm by $|\cdot|$.

The following are some spaces of random variables or stochastic processes. For $p \in [1, +\infty]$, $L^p(\Omega, P, Y) = L^p(\Omega, \mathscr{F}_T, P, Y)$ is the Banach space of Hilbert space Y-valued random variables ξ on a complete probability space $(\Omega, \mathscr{F}_T, P)$ with finite norm

$$\|\xi\|_{p,Y} = E\big[\|\xi\|_Y^p\big]^{1/p}, \qquad \|\xi\|_{\infty,Y} = \operatorname*{esssup}_{\omega} \big\|\xi(\omega)\big\|_Y;$$

$\mathscr{L}^p_{\mathbb{F},P}(0,T;Y)$ is the Banach space of Hilbert space Y-valued $\mathbb{F}$-adapted processes f with finite norm

$$\|f\|_{\mathscr{L}^p(Y)} := E\bigg[\int_0^T \|f(t)\|_Y^p\,dt\bigg]^{1/p}, \qquad \|f\|_{\mathscr{L}^\infty(Y)} := \operatorname*{esssup}_{(\omega,t)} \big\|f(\omega,t)\big\|_Y;$$

and $\mathscr{S}^p_{\mathbb{F},P}([0,T];Y)$ is the Banach space of Hilbert space Y-valued $\mathbb{F}$-adapted (path-wisely) continuous processes f with finite norm

$$\|f\|_{\mathscr{S}^p(Y)} := E\Big[\max_{t\in[0,T]} \|f(t)\|_Y^p\Big]^{1/p}, \qquad \|f\|_{\mathscr{S}^\infty(Y)} := \|f\|_{\mathscr{L}^\infty}.$$

If $Y = \mathbb{R}$ or there is no confusion on the underlying Hilbert space Y, we omit Y in these notations and simply write $L^p(\Omega, P)$, $\mathscr{L}^p_{\mathbb{F},P}(0,T)$, $\mathscr{S}^p_{\mathbb{F},P}[0,T]$; $\|\xi\|_p$, $\|f\|_{\mathscr{L}^p}$, $\|f\|_{\mathscr{S}^p}, \ldots$. Furthermore, the underlying probability P is omitted in these notations if there is no confusion and we simply write $L^p(\Omega)$, $\mathscr{L}^p_{\mathbb{F}}(0,T)$, $\mathscr{S}^p_{\mathbb{F}}[0,T], \ldots$.

Now we define our functional differentiable Hölder spaces. Let m be a nonnegative integer, $\alpha \in (0,1)$ a constant, and Y a Banach space. $C^m(\mathbb{R}^n, Y)$ is the Banach space of all Y-valued continuous functionals defined on $\mathbb{R}^n$ which are m-times continuously differentiable (strongly in Y) with all the derivatives up to order m being bounded in Y, equipped with the norm

$$\|\phi\|_{m,Y} := \sum_{k=0}^m [\phi]_{k,Y},$$

where

$$[\phi]_{k,Y} := \sum_{|\gamma|=k} \big[D^\gamma \phi\big]_{0,Y}, \qquad [\phi]_{0,Y} = \|\phi\|_{0,Y} := \sup_{x\in\mathbb{R}^n} \big\|\phi(x)\big\|_Y.$$

$C^{m+\alpha}(\mathbb{R}^n, Y)$ is the sub-space of all $\phi \in C^m(\mathbb{R}^n, Y)$ such that $[\phi]_{m+\alpha,Y} < +\infty$, where

$$[\phi]_{\alpha,Y} := \sup_{\substack{x,y\in\mathbb{R}^n\\ x\neq y}} \frac{\|\phi(x)-\phi(y)\|_Y}{|x-y|^\alpha}, \qquad [\phi]_{m+\alpha,Y} := \sum_{|\gamma|=m} \big[D^\gamma\phi\big]_{\alpha,Y}.$$

For $\phi \in C^{m+\alpha}(\mathbb{R}^n, Y)$, define the norm $\|\phi\|_{m+\alpha,Y} := \|\phi\|_{m,Y} + [\phi]_{m+\alpha,Y}$. If $Y = \mathbb{R}$, these spaces, semi-norms, and norms are classical differentiable and

Hölder ones on $\mathbb{R}^n$, and Y will be omitted in these notation and we simply write $C^m(\mathbb{R}^n)$, $C^{m+\alpha}(\mathbb{R}^n)$, $[\cdot]_\alpha$, $[\cdot]_{m+\alpha}$, and $|\cdot|_{m+\alpha}$. In this paper, we shall take $Y=\mathbb{R}, L^p(\Omega), \mathscr{L}^p_{\mathbb{F}}(0,T;\mathbb{R}^\iota), \mathscr{S}^p_{\mathbb{F}}([0,T])$ for $p\in[1,\infty]$. Moreover, we use the following abbreviations:

$$\|\cdot\|_{m+\alpha,\mathscr{S}^p}:=\|\cdot\|_{m+\alpha,\mathscr{S}^p_{\mathbb{F}}[0,T]},\qquad \|\cdot\|_{m+\alpha,\mathscr{L}^p}:=\|\cdot\|_{m+\alpha,\mathscr{L}^p_{\mathbb{F}}(0,T;\mathbb{R}^\iota)},$$

and similar abbreviations for semi-norms.

For $L^p(\Omega,\mathscr{F}_T,P,Y)$-valued functional u defined on $[0,T]\times\mathbb{R}^n$, we denote its partial derivatives in the space $L^p(\Omega,\mathscr{F}_T,P,Y)$ by $\partial_t u:=\frac{\partial u}{\partial t}$, $\partial_i u:=\frac{\partial u}{\partial x_i}$, $\partial^2_{ij}u:=\frac{\partial^2 u}{\partial x_i\,\partial x_j}$, etc.

$C=C(\cdot,\dots,\cdot)$ denotes a constant depending only on quantities appearing in parentheses. In a given context, the same letter will (generally) be used to denote different constants depending on the same set of arguments.

It can be verified that

$$[h\psi]_{\alpha,\mathscr{L}^p}\le[h]_{0,\mathscr{L}^\infty}[\psi]_{\alpha,\mathscr{L}^p}+[h]_{\alpha,\mathscr{L}^\infty}[\psi]_{0,\mathscr{L}^p} \tag{2.1}$$

for any $(h,\psi)\in C^\alpha(\mathbb{R}^n,\mathscr{L}^\infty_{\mathbb{F}}(0,T;\mathbb{R}^\iota))\times C^\alpha(\mathbb{R}^n,\mathscr{L}^p_{\mathbb{F}}(0,T;\mathbb{R}^\iota))$. This inequality will be used in Section 4.

Similar to classical Hölder spaces of scalar- or finite-dimensional vector-valued functions, we have the following interpolation inequalities.

LEMMA 2.1. *For $\varepsilon>0$, there is $C=C(\varepsilon,\alpha)>0$ such that for all $\psi\in C^{2+\alpha}(\mathbb{R}^n,\mathscr{L}^p_{\mathbb{F}}(0,T;\mathbb{R}^\iota))$*

$$\begin{aligned}
[\psi]_{2,\mathscr{L}^p}&\le\varepsilon[\psi]_{2+\alpha,\mathscr{L}^p}+C[\psi]_{0,\mathscr{L}^p},\\
[\psi]_{1+\alpha,\mathscr{L}^p}&\le\varepsilon[\psi]_{2+\alpha,\mathscr{L}^p}+C[\psi]_{0,\mathscr{L}^p},\\
[\psi]_{1,\mathscr{L}^p}&\le\varepsilon[\psi]_{2+\alpha,\mathscr{L}^p}+C[\psi]_{0,\mathscr{L}^p},\\
[\psi]_{\alpha,\mathscr{L}^p}&\le\varepsilon[\psi]_{2+\alpha,\mathscr{L}^p}+C[\psi]_{0,\mathscr{L}^p}.
\end{aligned}$$

Analogous inequalities also hold for elements of the Hölder functional space $C^{2+\alpha}(\mathbb{R}^n,L^p(\Omega))$ or $C^{2+\alpha}(\mathbb{R}^n,\mathscr{S}^p_{\mathbb{F}}[0,T])$.

The proof is similar to that of the interpolation inequalities in the classical Hölder spaces in Gilbarg and Trudinger [13], Lemma 6.32. It is omitted here.

2.2. *Linear BSPDEs.* Consider the Cauchy problem of linear BSPDE (1.1) in functional Hölder spaces. Denote by $\mathcal{S}^n$ the totality of all $n\times n$-symmetric matrices. Assume that all the coefficients:

$$\begin{aligned}
&a:[0,T]\times\mathbb{R}^n\to\mathcal{S}^n, &&b:[0,T]\times\Omega\times\mathbb{R}^n\to\mathbb{R}^n,\\
&c:[0,T]\times\Omega\times\mathbb{R}^n\to\mathbb{R}, &&\sigma:[0,T]\times\Omega\times\mathbb{R}^n\to\mathbb{R}^d,\\
&f:[0,T]\times\Omega\times\mathbb{R}^n\to\mathbb{R}, &&\Phi:\Omega\times\mathbb{R}^n\to\mathbb{R},
\end{aligned}$$

are random fields and jointly measurable, and are $\mathbb{F}$-adapted or $\mathscr{F}_T$-measurable at each $x \in \mathbb{R}^n$. We make the following assumptions.

ASSUMPTION 2.1 (Super-parabolicity). There are two positive constants λ and Λ such that

$$\lambda|\xi|^2 \leq \langle a(t,x)\xi, \xi\rangle \leq \Lambda|\xi|^2 \qquad \forall (t,x,\xi) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n.$$

ASSUMPTION 2.2 (Boundedness). The functionals

$$a \in C^\alpha(\mathbb{R}^n, L^\infty(0,T;\mathcal{S}^n)), \qquad b \in C^\alpha(\mathbb{R}^n, \mathscr{L}^\infty_{\mathbb{F}}(0,T;\mathbb{R}^n)),$$
$$c \in C^\alpha(\mathbb{R}^n, \mathscr{L}^\infty_{\mathbb{F}}(0,T)),$$

and $\sigma \in C^\alpha(\mathbb{R}^n, \mathscr{L}^\infty_{\mathbb{F}}(0,T;\mathbb{R}^d))$. Also, a, b, c and σ are bounded, that is, there is $\Lambda > 0$ such that $\|a\|_{\alpha,L^\infty} + \|b\|_{\alpha,\mathscr{L}^\infty} + \|c\|_{\alpha,\mathscr{L}^\infty} + \|\sigma\|_{\alpha,\mathscr{L}^\infty} \leq \Lambda$.

Note that throughout the paper a is assumed to be a deterministic $\mathcal{S}^n$-valued bounded function of the time–space variable (t,x).

A classical solution to BSPDE (1.1) in Hölder spaces is defined as follows.

DEFINITION 2.1. Let $\Phi \in C^{1+\alpha}(\mathbb{R}^n, L^2(\Omega))$ and $f \in C^\alpha(\mathbb{R}^n, \mathscr{L}^2_{\mathbb{F}}(0,T))$. We call (u,v) a classical solution to BSPDE (1.1) if

$$(u,v) \in C^\alpha(\mathbb{R}^n, \mathscr{S}^2_{\mathbb{F}}[0,T]) \cap C^{2+\alpha}(\mathbb{R}^n, \mathscr{L}^2_{\mathbb{F}}(0,T)) \times C^\alpha(\mathbb{R}^n, \mathscr{L}^2_{\mathbb{F}}(0,T;\mathbb{R}^d)),$$

and for all $(t,x) \in [0,T] \times \mathbb{R}^n$,

$$\begin{aligned} u(t,x) = \Phi(x) + \int_t^T \big[&a^{ij}(s,x)\,\partial^2_{ij}u(s,x) + b^i(s,x)\,\partial_i u(s,x) \\ &+ c(s,x)u(s,x) + f(s,x) + \sigma^l(s,x)v_l(s,x)\big]\,ds \\ &- \int_t^T v_l(s,x)\,dW^l_s, \qquad P\text{-a.s.} \end{aligned}$$

For simplicity of notation, define

$$\begin{aligned} &C^\alpha_{\mathscr{S}^2} \cap C^{2+\alpha}_{\mathscr{L}^2} \times C^\alpha_{\mathscr{L}^2} \\ &\quad := C^\alpha(\mathbb{R}^n; \mathscr{S}^2_{\mathbb{F}}[0,T]) \cap C^{2+\alpha}(\mathbb{R}^n; \mathscr{L}^2_{\mathbb{F}}(0,T)) C^\alpha(\mathbb{R}^n; \mathscr{L}^2_{\mathbb{F}}(0,T;\mathbb{R}^d)). \end{aligned}$$

2.3. *Estimates on the heat potential.* Consider the heat equation:

$$\partial_t u(t,x) = a^{ij}(t)\,\partial^2_{ij}u(t,x), \qquad (t,x) \in [0,T] \times \mathbb{R}^n, \tag{2.2}$$

where $a = (a^{ij})_{n\times n} : [0,T] \to \mathcal{S}^n$ satisfies the super-parabolic assumption. Define

$$G_{s,t}(x) := \frac{1}{(4\pi)^{n/2}(\det A_{s,t})^{1/2}} \exp\left(-\frac{1}{4}(A^{-1}_{s,t}x, x)\right) \qquad \forall 0 \leq t < s \leq T,$$

where $A_{s,t} := \int_t^s a(r)\,dr$. In a straightforward way, we have

$$\begin{aligned} \partial_s G_{s,t}(x) &= a^{ij}(s)\,\partial_{ij}^2 G_{s,t}(x), \qquad s > t; \\ \partial_t G_{s,t}(x) &= -a^{ij}(t)\,\partial_{ij}^2 G_{s,t}(x), \qquad s > t. \end{aligned} \tag{2.3}$$

REMARK 2.1. From Ladyženskaja, Solonnikov and Ural'ceva ([16], (1.7)) and (2.5) of Chapter IV, we have for $\gamma \in \Gamma$,

$$\int_{\mathbb{R}^n} D^\gamma G_{s,t}(x)\,dx = \begin{cases} 1, & \gamma = 0, \\ 0, & |\gamma| > 0 \end{cases} \tag{2.4}$$

and there are $C = C(\lambda, \Lambda, \gamma, n, T)$ and $c \in (0, \frac{1}{4})$ such that

$$\left|D^\gamma G_{s,t}(x)\right| \le C(s-t)^{-(n+|\gamma|)/2} \exp\left(-c\frac{|x|^2}{s-t}\right) \qquad \forall s > t. \tag{2.5}$$

Furthermore, we have

$$\int_0^s \left|D^\gamma G_{s,t}(x)\right| dt \le C|x|^{-(n+|\gamma|)+2} \int_0^\infty r^{(n+|\gamma|)/2+2} \exp(-cr)\,dr \tag{2.6}$$

for any $s \in [0, T]$ and $x \neq 0$, and

$$\int_{\mathbb{R}^n} \left|D^\gamma G_{s,t}(x)\right| |x|^\alpha\,dx \le C(s-t)^{(\alpha-|\gamma|)/2} \int_{\mathbb{R}^n} |x|^\alpha \exp\left(-c|x|^2\right) dx \tag{2.7}$$

for $s > t$ and $\alpha \in (0, 1)$.

The following lemmas will be used to derive a priori Hölder estimates in Section 3.

From Mikulevicius [19], Lemma 4, we have:

LEMMA 2.2. *For any multi-index $|\gamma| = 2$, there exists $C = C(\lambda, \Lambda, \gamma, n, T)$ such that for $0 \le \tau \le s \le T$ and $\eta > 0$,*

$$\int_\tau^s \left|\int_{|y|\le\eta} D^\gamma G_{s,t}(y)\,dy\right| dt = \int_\tau^s \left|\int_{|y|\ge\eta} D^\gamma G_{s,t}(y)\,dy\right| dt \le C.$$

LEMMA 2.3. *Let $\eta > 0$ be a constant. Then for $\gamma \in \Gamma$ such that $|\gamma| = 2$, there is a constant $C = C(\lambda, \Lambda, \gamma, \alpha, n, T)$ such that*

$$\int_{B_\eta(0)} \sup_{\tau \le s} \int_\tau^s \left|D^\gamma G_{s,t}(y)\right| |y|^\alpha\,dt\,dy \le C\eta^\alpha.$$

PROOF. In view of (2.6), we have

$$\int_{B_\eta(0)} \sup_{\tau \le s} \int_\tau^s \left|D^\gamma G_{s,t}(y)\right| |y|^\alpha\,dt\,dy \le C \int_{B_\eta(0)} |y|^{-n+\alpha}\,dy \le C\eta^\alpha. \qquad \square$$

LEMMA 2.4. *For any* $x, \bar{x} \in \mathbb{R}^n$ *and* $\gamma \in \Gamma$ *such that* $|\gamma| = 2$, *we have*

$$\int_{|y-x|>\eta} \sup_{\tau \le s} \int_\tau^s \left|D^\gamma G_{s,t}(x-y) - D^\gamma G_{s,t}(\bar{x}-y)\right| |\bar{x}-y|^\alpha \, dt \, dy \le C\eta^\alpha,$$

where $\eta := 2|x - \bar{x}|$ *and* $C = C(\lambda, \Lambda, \gamma, \alpha, n, T)$.

PROOF. Define $\widetilde{x} := x + 2(\bar{x} - x)$. Let ξ be any point on the segment joining x and $\bar{x}$. For $|y - x| > \eta$, we have

$$|\xi - x| \le \tfrac{1}{2}|x - \widetilde{x}| \le \tfrac{1}{2}|x - y|,$$
$$\tfrac{1}{2}|x - y| \le |\xi - y| = \left|(x - y) + (\xi - x)\right| \le \tfrac{3}{2}|x - y|.$$

In view of (2.5) and (2.6), we have

$$\begin{aligned}
&\int_{|y-x|>\eta} \sup_{\tau \le s} \int_\tau^s \left|D^\gamma G_{s,t}(x-y) - D^\gamma G_{s,t}(\bar{x}-y)\right| |\bar{x}-y|^\alpha \, dt \, dy \\
&\quad \le C\eta \int_{|y-x|>\eta} \sup_{\tau \le s} \int_\tau^s \int_0^1 \left|\partial_x D^\gamma G_{s,t}\big(r\bar{x} + (1-r)x - y\big)\right| dr |x-y|^\alpha \, dt \, dy \\
&\quad \le C\eta \int_{|y-x|>\eta} \int_0^T \int_0^1 t^{-(n+3)/2} \\
&\qquad\qquad \times \exp\left(-c\frac{|r\bar{x} + (1-r)x - y|^2}{t}\right) dr |x-y|^\alpha \, dt \, dy \\
&\quad \le C\eta \int_0^T \int_{|y-x|>\eta} t^{-(n+3)/2} \exp\left(-c\frac{|x-y|^2}{t}\right) |x-y|^\alpha \, dt \, dy \\
&\quad \le C\eta \int_{|y-x|>\eta} |x-y|^{-n-1+\alpha} \, dy \le C\eta^\alpha.
\end{aligned}$$

□

3. BSPDE with space-invariant coefficients *a* and *σ*. Consider the following linear BSPDE:

$$(3.1) \qquad \begin{cases} -du(t,x) = \left[a^{ij}(t)\, \partial_{ij}^2 u(t,x) + f(t,x) + \sigma^l(t) v_l(t,x)\right] dt \\ \qquad\qquad - v_l(t,x)\, dW_t^l, \qquad (t,x) \in [0,T) \times \mathbb{R}^n, \\ u(T,x) = \Phi(x), \qquad\qquad x \in \mathbb{R}^n, \end{cases}$$

where $a := (a^{ij})_{n \times n} : [0,T] \to \mathcal{S}^n$ is Borel measurable and $\sigma := (\sigma^1, \ldots, \sigma^d)' : \Omega \times [0,T] \to \mathbb{R}^d$ is $\mathbb{F}$-adapted. It is simpler than BSPDE (1.1), for both coefficients a and σ are assumed to be independent of the space variable x (hence not varying with the space variable, and hereafter called *space-invariant*). For this case, both Assumptions 2.1 and 2.2 can be combined into the following one.

ASSUMPTION 3.1. $a \in \mathscr{L}^\infty(0,T;\mathcal{S}^n)$ and $\sigma \in \mathscr{L}^\infty(0,T;\mathbb{R}^d)$. There are two positive constants λ and Λ such that $\lambda|\xi|^2 \le \langle a(t)\xi,\xi\rangle \le \Lambda|\xi|^2$ for any $(t,\xi) \in [0,T]\times\mathbb{R}^n$ and $\|\sigma\|_{\mathscr{L}^\infty} \le \Lambda$.

The special structural assumption on both coefficients a and σ allows us to give an explicit expression of the adapted solution (u,v) to BSPDE (3.1). To see this point, let us look at the respective contributions of both coefficients a and σ to the solution (u,v) of BSPDE (3.1).

Define

$$\widetilde{W}_t := -\int_0^t \sigma(s)\,ds + W_t, \qquad t \in [0,T] \tag{3.2}$$

and the equivalent probability Q by

$$dQ := \exp\biggl(\int_0^T \langle\sigma(t), dW_t\rangle - \frac{1}{2}\int_0^T |\sigma(t)|^2\,dt\biggr)\,dP. \tag{3.3}$$

It can be verified that $\widetilde{W}$ is a standard Brownian motion on $(\Omega,\mathscr{F}_T,\mathbb{F},Q)$. BSPDE (3.1) is written into the following form:

$$\begin{aligned} u(t,x) &= \Phi(x) + \int_t^T \bigl[a^{ij}(r)\,\partial_{ij}^2 u(r,x) + f(r,x)\bigr]\,dr \\ &\quad - \int_r^T v_l(r,x)\,d\widetilde{W}_t^l, \qquad (t,x) \in [0,T]\times\mathbb{R}^n. \end{aligned} \tag{3.4}$$

Furthermore, we have for $(t,x) \in [0,T]\times\mathbb{R}^n$

$$u(t,x) = E_Q^{\mathscr{F}_t}\biggl[\Phi(x) + \int_t^T \bigl(a^{ij}(r)\,\partial_{ij}^2 u(r,x) + f(r,x)\bigr)\,dr\biggr]. \tag{3.5}$$

Since a is deterministic and Q does not depend on the space variable x (in view of Assumption 3.1), we have for $(t,x) \in [0,T]\times\mathbb{R}^n$

$$u(t,x) = E_Q^{\mathscr{F}_t}\Phi(x) + \int_t^T \bigl[a^{ij}(r)\,\partial_{ij}^2\bigl(E_Q^{\mathscr{F}_t}u(r,x)\bigr) + E_Q^{\mathscr{F}_t} f(r,x)\bigr]\,dr, \qquad \text{a.s.}$$

Note that it is the integral on $[t,T]$ with respect to r of the following backward parabolic equation with $U(r,x;t) := E_Q^{\mathscr{F}_t}u(r,x)$:

$$\begin{cases} -\partial_r U(r,x;t) = a^{ij}(r)\,\partial_{ij}^2 U(r,x;t) + E_Q^{\mathscr{F}_t} f(r,x), & (r,x) \in [t,T)\times\mathbb{R}^n, \\ U(T,x) = E_Q^{\mathscr{F}_t}\Phi(x), & x \in \mathbb{R}^n. \end{cases}$$

Define for the convolution of the heat potential $G_{s,t}$ with a functional ϕ defined on $\mathbb{R}^n$ and a functional ψ defined on $[0,T]\times\mathbb{R}^n$ as follows: for $x \in \mathbb{R}^n$,

$$\begin{aligned} R_t^s\phi(x) &:= \int_{\mathbb{R}^n} G_{s,t}(x-y)\phi(y)\,dy \qquad \forall s > t, \\ R_t^s\psi(\cdot)(x) &:= \int_{\mathbb{R}^n} G_{s,t}(x-y)\psi(\cdot,y)\,dy \qquad \forall s > t. \end{aligned} \tag{3.6}$$

It is well known that the solution of the last PDE has the following representation: almost surely:

$$U(r,x;t) = R_r^T\big(E_Q^{\mathscr{F}_t}\Phi\big)(x) + \int_r^T R_r^s\big(E_Q^{\mathscr{F}_t} f(s,\cdot)\big)(x)\,ds, \qquad (r,x)\in[t,T]\times\mathbb{R}^n.$$

Setting $r=t$, we have almost surely

$$u(t,x) = R_t^T\big(E_Q^{\mathscr{F}_t}\Phi\big)(x) + \int_t^T R_t^s\big(E_Q^{\mathscr{F}_t} f(s,\cdot)\big)(x)\,ds, \qquad (t,x)\in[0,T]\times\mathbb{R}^n.$$

It is easy to see that $\{E_Q^{\mathscr{F}_t}\Phi(x), t\in[0,T]\}$ and $\{E_Q^{\mathscr{F}_t} f(s,x), t\in[0,T]\}$ are uniquely characterized by

$$E_Q^{\mathscr{F}_t}\Phi(x) = \varphi(t;x), \qquad E_Q^{\mathscr{F}_t} f(s,x) = Y(t;s,x), \qquad (t,x)\in[0,T]\times\mathbb{R}^n, \text{ a.s.},$$

where $(\varphi(\cdot;x),\psi(\cdot;x))$ and $(Y(\cdot;\tau,x), g(\cdot;\tau,x))$ are the unique adapted solution of the following two parameterized BSDEs:

$$\varphi(t;x) = \Phi(x) + \int_t^T \sigma^l(r)\psi_l(r;x)\,dr - \int_t^T \psi_l(r;x)\,dW_r^l, \qquad t\in[0,T] \tag{3.7}$$

and

$$Y(t;\tau,x) = f(\tau,x) + \int_t^\tau \sigma^l(r)g_l(r;\tau,x)\,dr - \int_t^\tau g_l(r;\tau,x)\,dW_r^l, \qquad t\in[0,\tau], \tag{3.8}$$

respectively. In this way, we have the desired representation of (u,v): for $(t,x)\in[0,T]\times\mathbb{R}^n$,

$$u(t,x) = R_t^T\varphi(x) + \int_t^T R_t^s Y(t;r,\cdot)(x)\,dr,$$

and further we expect from the linear structure of our BSPDE that

$$v(t,x) = R_t^T\psi(t,\cdot)(x) + \int_t^T R_t^s g(t;r,\cdot)(x)\,dr,$$

which is stated as the subsequent Theorem 3.3.

The rest of the section is structured as follows. In Section 3.1, we prove the above explicit expression for the classical solution (u,v) to BSPDE (3.1) in terms of the terminal term Φ and the free term f. In Section 3.2, we derive the a priori Hölder estimates. Finally, in Section 3.3, we prove the existence and uniqueness result of classical solution to BSPDE (3.1).

3.1. *Explicit expression of* (u, v).

LEMMA 3.1. *Suppose that Assumption* 3.1 *holds*, $\Phi \in C^{1+\alpha}(\mathbb{R}^n, L^2(\Omega))$ *and* $f \in C^\alpha(\mathbb{R}^n, \mathscr{L}^2_{\mathbb{F}}(0, T))$. *If* $(u, v) \in C^\alpha_{\mathscr{S}^2} \cap C^{2+\alpha}_{\mathscr{L}^2} \times C^\alpha_{\mathscr{L}^2}$ *is the classical solution of BSPDE* (3.1), *then for all* $(t, x) \in [0, T] \times \mathbb{R}^n$, *we have almost surely*

$$u(t, x) = R_t^T \Phi(x) + \int_t^T \big[R_t^s f(s)(x) + \sigma^l(s) R_t^s v_l(s)(x)\big]\, ds - \int_t^T R_t^s v_l(s)(x)\, dW_s^l.$$

PROOF. For fixed $(t, x) \in [0, T] \times \mathbb{R}^n$ and $s \in (t, T]$, using Itô's formula, we have

$$\begin{aligned} G_{s,t}(x-y)u(s, y) &= G_{T,t}(x-y)u(T, y) - \int_s^T G_{r,t}(x-y)\, du(r, y) - \int_s^T u(r, y)\, dG_{r,t}(x-y) \\ &= G_{T,t}(x-y)\Phi(y) + \int_s^T \big[G_{r,t}(x-y) f(r, y) + \sigma^l(r) G_{r,t}(x-y) v_l(r, y)\big]\, dr \\ &\quad - \int_s^T G_{r,t}(x-y) v_l(r, y)\, dW_r^l \\ &\quad - \int_s^T \big[a^{ij}(r)\, \partial^2_{ij} G_{r,t}(x-y) u(r, y) - G_{r,t}(x-y) a^{ij}(r)\, \partial^2_{ij} u(r, y)\big]\, dr. \end{aligned} \tag{3.9}$$

A direct computation shows that

$$\int_{\mathbb{R}^n} \int_s^T \big[a^{ij}(r)\, \partial^2_{ij} G_{r,t}(x-y) u(r, y) - G_{r,t}(x-y) a^{ij}(r)\, \partial^2_{ij} u(r, y)\big]\, dr\, dy = 0.$$

Stochastic Fubini theorem (see Da Prato [6], Theorem 4.18) gives that

$$\int_{\mathbb{R}^n} \int_s^T G_{r,t}(x-y) v_l(r, y)\, dW_r^l\, dy = \int_s^T \int_{\mathbb{R}^n} G_{r,t}(x-y) v_l(r, y)\, dy\, dW_r^l.$$

Thus, integrating w.r.t. y over $\mathbb{R}^n$ both sides of (3.9), we have

$$\begin{aligned} \int_{\mathbb{R}^n} G_{s,t}(x-y) u(s, y)\, dy &= R_t^T \Phi(x) + \int_s^T \big[R_t^r f(r)(x) + \sigma^l(r) R_t^r v_l(r)(x)\big]\, dr \\ &\quad - \int_s^T R_t^r v_l(r)(x)\, dW_r^l. \end{aligned} \tag{3.10}$$

In what follows, we compute the limit of each part of (3.10) as $s \to t$.

Since $u \in C^\alpha(\mathbb{R}^n, \mathscr{S}^2_{\mathbb{F}}[0,T])$, we have from estimates (2.4) and (2.5) on the heat potential that

$$
\begin{aligned}
&E\left|\int_{\mathbb{R}^n} G_{s,t}(x-y)u(s,y)\,dy - u(t,x)\right|^2 \\
&\quad = E\left|\int_{\mathbb{R}^n} G_{s,t}(x-y)\big[u(s,y)-u(t,x)\big]\,dy\right|^2 \\
&\quad \le CE\int_{\mathbb{R}^n} G_{s,t}(x-y)\big|u(s,y)-u(t,x)\big|^2\,dy \qquad (3.11)\\
&\quad \le CE\int_{\mathbb{R}^n} \exp\big(-c|z|^2\big)\big|u(s,x-\sqrt{s-t}z)-u(t,x)\big|^2\,dz \longrightarrow 0, \\
&\qquad\qquad \text{as } s\downarrow t.
\end{aligned}
$$

In view of (2.4), we have

$$
\begin{aligned}
&E\left|\int_t^s\int_{\mathbb{R}^n} G_{r,t}(x-y)v_l(r,y)\,dy\,dW_r^l\right|^2 \\
&\quad = E\int_t^s\left|\int_{\mathbb{R}^n} G_{r,t}(x-y)\big[v(r,y)-v(r,x)\big]\,dy + v(r,x)\right|^2 dr \\
&\quad \le 2E\int_t^s\left|\int_{\mathbb{R}^n} G_{r,t}(x-y)\big[v(r,y)-v(r,x)\big]\,dy\right|^2 dr + 2E\int_t^s\big|v(r,x)\big|^2\,dr \\
&\quad \le CE\int_t^s\int_{\mathbb{R}^n} G_{r,t}(x-y)\big|v(r,y)-v(r,x)\big|^2\,dy\,dr + 2E\int_t^s\big|v(r,x)\big|^2\,dr \\
&\quad \le C\int_{\mathbb{R}^n}\sup_{r\in[t,s]} G_{r,t}(x-y)|x-y|^{2\alpha}\,dy\cdot\sup_{y\in\mathbb{R}^n} E\int_t^s \frac{|v(r,y)-v(r,x)|^2}{|x-y|^{2\alpha}}\,dr \\
&\qquad + 2E\int_t^s\big|v(r,x)\big|^2\,dr.
\end{aligned}
$$

Since $v \in C^\alpha(\mathbb{R}^n, \mathscr{L}^2_{\mathbb{F}}(0,T;\mathbb{R}^d))$, we have

$$
\lim_{s\downarrow t} E\int_t^s\big|v(r,x)\big|^2\,dr = 0, \qquad \lim_{s\downarrow t}\sup_{y\in\mathbb{R}^n} E\int_t^s \frac{|v(r,y)-v(r,x)|^2}{|x-y|^{2\alpha}}\,dr = 0.
$$

In view of subsequent Lemma 3.2, we obtain

$$
\lim_{s\downarrow t} E\left|\int_t^s\int_{\mathbb{R}^n} G_{r,t}(x-y)v_l(r,y)\,dy\,dW_r^l\right|^2 = 0. \tag{3.12}
$$

In a similar way, we have

$$
\lim_{s\downarrow t} E\left|\int_t^s \big[R_t^r f(r)(x) + \sigma^l(r)R_t^r v_l(r)(x)\big]\,dr\right|^2 = 0. \tag{3.13}
$$

Letting $s \to t$ in equality (3.10), we have the desired result from (3.11), (3.12) and (3.13). □

LEMMA 3.2. *For* $0 \le t < s \le T$, *we have*

$$\int_{\mathbb{R}^n} \sup_{r\in[t,s]} G_{r,t}(x-y)|x-y|^{2\alpha}\,dy < +\infty.$$

PROOF. First, consider the following function ρ:

$$\rho(t,r) := t^{-n/2}\exp\left(-\frac{c}{t}r^2\right), \qquad (t,r)\in[0,T]\times\mathbb{R}.$$

We have

$$\partial_t\rho(t,r) = \left(cr^2 - \tfrac{1}{2}nt\right)t^{-2}\rho(t,r).$$

Therefore, the function $\rho(\cdot,r)$ increases on $[0,T]$ for any fixed r such that $r^2 > M^2 := \frac{nT}{2c}$, and we have

$$\sup_{t\in[0,T]} \rho(t,r) \le \rho(T,r).$$

In view of estimate (2.5) to the heat potential, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} |x-y|^{2\alpha} \sup_{r\in[t,s]} G_{r,t}(x-y)\,dy \\
&\quad = \int_{|x-y|\le M} |x-y|^{2\alpha} \sup_{r\in[t,s]} G_{r,t}(x-y)\,dy \\
&\qquad + \int_{|x-y|>M} |x-y|^{2\alpha} \sup_{r\in[t,s]} G_{r,t}(x-y)\,dy \\
&\quad \le C\int_{|z|\le M} |z|^{2\alpha} \sup_{r\in[t,s]} \rho\big(r-t,|z|\big)\,dz \\
&\qquad + C\int_{|z|>M} |z|^{2\alpha} \sup_{r\in[t,s]} \rho\big(r-t,|z|\big)\,dz \\
&\quad = I_1 + C\int_{|z|>M} |z|^{2\alpha}\rho\big(T,|z|\big)\,dz.
\end{aligned}$$

Since the second integral is easily verified to be finite, it remains to show $I_1 < \infty$. Noting that $\rho(t,|z|)$ is maximized at $t = \frac{2}{n}c|z|^2$ over $[0,T]$ for any fixed z such that $|z| \le M$, we have

$$\sup_{r\in[t,s]} \rho\big(r-t,|z|\big) \le \rho\left(\frac{2}{n}c|z|^2,|z|\right) \le C|z|^{-n}$$

and

$$I_1 \le C \int_{|z|\le M} |z|^{-n+2\alpha}\, dz < +\infty.$$

The proof is then complete. □

In Lemma 3.1, the expression of u still depends on v, which is unknown. Next, we construct an explicit expression of (u, v) only in terms of the terminal term Φ and the free term f.

Let $\Phi \in C^{1+\alpha}(\mathbb{R}^n, L^2(\Omega))$ and $f \in C^\alpha(\mathbb{R}^n, \mathscr{L}^2_{\mathbb{F}}(0,T))$. Consider the two family BSDEs (3.7) and (3.8): for any $x \in \mathbb{R}^n$ and almost all $\tau \in [0,T]$, their solutions are denoted by $(\varphi(\cdot,x), \psi(\cdot,x))$ and $(Y(\cdot;\tau,x), g(\cdot;\tau,x))$, respectively. From the theory of BSDEs, we have

$$(\varphi,\psi) \in C^{1+\alpha}\big(\mathbb{R}^n, \mathscr{S}^2_{\mathbb{F}}[0,T]\big) \times C^{1+\alpha}\big(\mathbb{R}^n, \mathscr{L}^2_{\mathbb{F}}(0,T;\mathbb{R}^d)\big),$$

and there is $C = C(\alpha, n, d)$ such that

$$\|\varphi\|_{1+\alpha,\mathscr{S}^2} + \|\psi\|_{1+\alpha,\mathscr{L}^2} \le C\|\Phi\|_{1+\alpha,L^2}, \tag{3.14}$$

$$\begin{aligned}
&\sup_x E\int_0^T \sup_{t\le r}\big|Y(t;r,x)\big|^2\,dr \\
&\qquad + \sup_{x\neq\bar{x}} \frac{E\int_0^T \sup_{t\le r}|Y(t;r,x)-Y(t;r,\bar{x})|^2\,dr}{|x-\bar{x}|^{2\alpha}} \\
&\qquad + \sup_x E\int_0^T\int_0^r \big|g(t;r,x)\big|^2\,dt\,dr \\
&\qquad + \sup_{x\neq\bar{x}} \frac{E\int_0^T\int_0^r |g(t;r,x)-g(t;r,\bar{x})|^2\,dt\,dr}{|x-\bar{x}|^{2\alpha}} \\
&\quad \le C\|f\|^2_{\alpha,\mathscr{L}^2}.
\end{aligned} \tag{3.15}$$

We have the following explicit expression of (u, v).

THEOREM 3.3. *Let Assumption 3.1 hold and $(\Phi, f) \in C^{1+\alpha}(\mathbb{R}^n, L^2(\Omega)) \times C^\alpha(\mathbb{R}^n, \mathscr{L}^2_{\mathbb{F}}(0,T))$. Let (φ,ψ) and (Y,g) be solutions of BSDEs (3.7) and (3.8), respectively, and $(u,v) \in C^\alpha_{\mathscr{S}^2} \cap C^{2+\alpha}_{\mathscr{L}^2} \times C^\alpha_{\mathscr{L}^2}$ solve BSPDE (3.1). Then for all $x \in \mathbb{R}^n$,*

$$u(t,x) = R_t^T\varphi(t)(x) + \int_t^T R_t^s Y(t;s)(x)\,ds \qquad \forall t\in[0,T],\ dP\text{-a.s.}, \tag{3.16}$$

$$v_l(s,x) = R_s^T\psi_l(s)(x) + \int_s^T R_s^r g_l(s;r)(x)\,dr, \qquad ds\times dP\text{-a.e., a.s.}, l=1,\dots,d, \tag{3.17}$$

where R_t^s is defined by (3.6).

PROOF. In view of Lemma 3.1 and the definition (3.2), we see that for all $(t,x) \in [0,T] \times \mathbb{R}^n$,

$$u(t,x) = R_t^T \Phi(x) + \int_t^T R_t^s f(s)(x)\, ds - \int_t^T R_t^s v_l(s)(x)\, d\widetilde{W}_s^l, \qquad P\text{-a.s.},$$

$$\varphi(t;x) = \Phi(x) - \int_t^T \psi_l(s;x)\, d\widetilde{W}_s^l, \qquad P\text{-a.s.},$$

and for almost all $\tau \in [0,T]$ and any $s \le \tau$,

$$Y(s;\tau,x) = f(\tau,x) - \int_s^\tau g_l(r;\tau,x)\, d\widetilde{W}_r^l, \qquad P\text{-a.s.}$$

In view of the stochastic Fubini theorem and semi-group property of $G_{s,t}$, we have

$$\begin{aligned}
&R_t^T \Phi(x) + \int_t^T R_t^s f(s)(x)\, ds \\
&\quad = R_t^T \varphi(t)(x) + \int_t^T R_t^s Y(t;s)(x)\, ds \\
&\qquad + \int_t^T R_t^T \psi_l(s)(x)\, d\widetilde{W}_s^l + \int_t^T R_t^s \int_t^s g_l(r;s)(x)\, d\widetilde{W}_r^l\, ds \\
&\quad = R_t^T \varphi(t)(x) + \int_t^T R_t^s Y(t;s)(x)\, ds \\
&\qquad + \int_t^T R_t^s \biggl(R_s^T \psi_l(s) + \int_s^T R_s^r g_l(s;r)\, dr \biggr)(x)\, d\widetilde{W}_s^l.
\end{aligned} \tag{3.18}$$

Therefore,

$$\begin{aligned}
&u(t,x) + \int_t^T R_t^s v_l(s)(x)\, d\widetilde{W}_s^l \\
&\quad = R_t^T \varphi(t)(x) + \int_t^T R_t^s Y(t;s)(x)\, ds \\
&\qquad + \int_t^T R_t^s \biggl(R_s^T \psi_l(s) + \int_s^T R_s^r g_l(s;r)\, dr \biggr)(x)\, d\widetilde{W}_s^l.
\end{aligned} \tag{3.19}$$

In view of (3.14) and (3.15), we have for each $x \in \mathbb{R}^n$,

$$\int_t^T \biggl| R_t^s \biggl(R_s^T \psi(s) + \int_s^T R_s^r g(s;r)\, dr \biggr)(x) - R_t^s v(s)(x) \biggr|^2 ds < +\infty, \qquad P\text{-a.s.}$$

Taking on both sides of (3.19) the expectation with respect to the new probability Q [see (3.3) for the definition] conditioned on $\mathscr{F}_t$, we have almost surely

$$u(t,x) = R_t^T \varphi(t)(x) + \int_t^T R_t^s Y(t;s)(x)\, ds \qquad \forall x \in \mathbb{R}^n;$$

and

$$\int_t^T R_t^s\Big(R_s^T\psi_l(s)+\int_s^T R_s^r g_l(s;r)\,dr-v_l(s)\Big)(x)\,d\widetilde{W}_s^l=0$$

for any $(t,x)\in[0,T]\times\mathbb{R}^n$, which implies the following:

$$E\bigg[\int_t^T\bigg|R_t^s\Big(R_s^T\psi(s)+\int_s^T R_s^r g(s;r)\,dr-v(s)\Big)(x)\bigg|^2 ds\bigg]=0 \tag{3.20}$$

for any $(t,x)\in[0,T]\times\mathbb{R}^n$. Then, for all $l=1,\dots,d$, we have almost surely

$$V_l(t,x):=\int_t^T R_t^s\Big(R_s^T\psi_l(s)+\int_s^T R_s^r g_l(s;r)\,dr-v_l(s)\Big)(x)\,ds=0$$

for any $(t,x)\in[0,T]\times\mathbb{R}^n$, which almost surely solves a deterministic PDE and, therefore, the nonhomogeneous term (the sum in the bigger pair of parentheses in the last equality) of this PDE is equal to zero. Consequently, we have for each $x\in\mathbb{R}^n$,

$$v_l(s,x)=R_s^T\psi_l(s)(x)+\int_s^T R_s^r g_l(s;r)(x)\,dr,\qquad ds\times dP\text{-a.e., a.s.}$$

The proof is complete. □

REMARK 3.1. Let Assumption 3.1 hold and $(\Phi,f)\in C^{1+\alpha}(\mathbb{R}^n,L^2(\Omega))\times C^{\alpha}(\mathbb{R}^n,\mathscr{L}_{\mathbb{F}}^2(0,T))$. Let (φ,ψ) and (Y,g) be solutions of BSDEs (3.7) and (3.8), respectively. Then, for all $x\in\mathbb{R}^n$, $R_t^T\varphi(t)(x)$ and $\int_t^T R_t^s Y(t;s)(x)\,ds$ are twice continuously differentiable in x as $\mathscr{L}_{\mathbb{F}}^2(0,T)$-valued functionals. Moreover, we have for all $(t,x)\in[0,T)\times\mathbb{R}^n$,

$$\partial_i R_t^T\varphi(t)(x)=\int_{\mathbb{R}^n}G_{T,t}(x-y)\,\partial_i\varphi(t,y)\,dy,\qquad P\text{-a.s.,}$$

$$\partial_{ij}^2 R_t^T\varphi(t)(x)=\int_{\mathbb{R}^n}\partial_i G_{T,t}(x-y)\big[\partial_j\varphi(t,y)-\partial_j\varphi(t,x)\big]\,dy,\qquad P\text{-a.s.}$$

and for all $x\in\mathbb{R}^n$, $dt\times dP$-a.e., a.s.,

$$\partial_i\int_t^T R_t^s Y(t;s)(x)\,ds=\int_t^T\int_{\mathbb{R}^n}\partial_i G_{s,t}(x-y)Y(t;s,y)\,dy\,ds,$$

$$\partial_{ij}^2\int_t^T R_t^s Y(t;s)(x)\,ds$$
$$=\int_t^T\int_{\mathbb{R}^n}\partial_{ij}^2 G_{s,t}(x-y)\big[Y(t;s,y)-Y(t;s,x)\big]\,dy\,ds.$$

3.2. *Hölder estimates.* Using the explicit expression of (u,v) in Theorem 3.3, we shall derive Hölder estimates for (u,v).

LEMMA 3.4. *Let Assumption 3.1 be satisfied and suppose that*

$$(\Phi, f) \in C^{1+\alpha}(\mathbb{R}^n, L^2(\Omega)) \times C^{\alpha}(\mathbb{R}^n, \mathscr{L}^2_{\mathbb{F}}(0, T)).$$

If $(u, v) \in C^{\alpha}_{\mathscr{S}^2} \cap C^{2+\alpha}_{\mathscr{L}^2} \times C^{\alpha}_{\mathscr{L}^2}$ *solves BSPDE* (3.1), *then we have*

$$\|u\|_{2+\alpha,\mathscr{L}^2} \le C(\|\Phi\|_{1+\alpha,L^2} + \|f\|_{\alpha,\mathscr{L}^2}),$$

where $C = C(\lambda, \Lambda, \alpha, n, d, T)$.

PROOF. In view of (3.16) and (3.17), we need to prove

$$\|R^T_{\cdot}\varphi(\cdot)\|_{2+\alpha,\mathscr{L}^2} \le C\|\Phi\|_{1+\alpha,L^2},$$

$$\left\|\int_{\cdot}^{T} R^s_{\cdot} Y(\cdot; s)\, ds\right\|_{2+\alpha,\mathscr{L}^2} \le C\|f\|_{\alpha,\mathscr{L}^2}.$$

It is sufficient to prove the second inequality, and the first one can be proved in a similar way.

For $\gamma \in \Gamma$ such that $|\gamma| \le 1$, in view of (2.5), (2.7), (3.15) and Remark 3.1, we have

$$\begin{aligned}
E\int_0^T & \left|D^\gamma \int_t^T R_{s,t} Y(t; s)(x)\, ds\right|^2 dt \\
&= E\int_0^T \left|\int_t^T \int_{\mathbb{R}^n} D^\gamma G_{s,t}(x-y) Y(t; s, y)\, dy\, ds\right|^2 dt \\
&\le E\int_0^T \int_t^T \int_{\mathbb{R}^n} |D^\gamma G_{s,t}(x-y)||Y(t; s, y)|^2\, dy\, ds \\
&\qquad \times \int_t^T \int_{\mathbb{R}^n} |D^\gamma G_{s,t}(x-y)|\, dy\, ds\, dt
\end{aligned}$$

and, therefore,

$$\begin{aligned}
E\int_0^T & \left|D^\gamma \int_t^T R_{s,t} Y(t; s)(x)\, ds\right|^2 dt \\
&\le \int_{\mathbb{R}^n} E\int_0^T \sup_{t\le s}|Y(t; s, y)|^2 \int_0^s |D^\gamma G_{s,t}(x-y)|\, dt\, ds\, dy \\
&\qquad \times \sup_\tau \int_\tau^T \int_{\mathbb{R}^n} |D^\gamma G_{s,\tau}(x-y)|\, dy\, ds \\
&\le C \sup_y E\int_0^T \sup_{t\le s}|Y(t; s, y)|^2\, ds \cdot \int_{\mathbb{R}^n} \sup_s \int_0^s |D^\gamma G_{s,t}(x-y)|\, dt\, dy \\
&\qquad \times \sup_\tau \int_\tau^T \int_{\mathbb{R}^n} |D^\gamma G_{s,\tau}(x-y)|\, dy\, ds
\end{aligned}$$

$$\le C \sup_y E \int_0^T \sup_{t\le s} \big|Y(t;s,y)\big|^2 \, ds \cdot \left| \int_{\mathbb{R}^n} \sup_{\tau \le s} \int_\tau^s \big|D^\gamma G_{s,t}(x-y)\big| \, dt \, dy \right|^2$$

$$\le C \|f\|_{0,\mathscr{L}^2}^2 \left| \int_0^T \int_{\mathbb{R}^n} t^{-(n+|\gamma|)/2} \exp\left(-c\frac{|x-y|^2}{t}\right) dy \, dt \right|^2$$

$$\le C \|f\|_{0,\mathscr{L}^2}^2.$$

That is,

$$\left\| \int_\cdot^T R_\cdot^s Y(\cdot;s) \, ds \right\|_{1,\mathscr{L}^2} \le C \|f\|_{0,\mathscr{L}^2}. \tag{3.21}$$

For $|\gamma| = 2$, in view of (2.5), (2.7), (3.15) and Remark 3.1, we have

$$E \int_0^T \left| D^\gamma \int_t^T R_t^s Y(t;s) \, ds \right|^2 dt$$

$$= C E \int_0^T \left| \int_t^T \int_{\mathbb{R}^n} D^\gamma G_{s,t}(x-y)|x-y|^\alpha \frac{|Y(t;s,y) - Y(t;s,x)|}{|x-y|^\alpha} \, dy \, ds \right|^2 dt$$

$$\le C \sup_y E \int_0^T \sup_{t \le s} \frac{|Y(t;s,y) - Y(t;s,x)|^2}{|x-y|^{2\alpha}} \, ds$$

$$\times \left| \int_{\mathbb{R}^n} \sup_{\tau \le s} \int_\tau^s \big|D^\gamma G_{s,t}(x-y)\big| |x-y|^\alpha \, dt \, dy \right|^2$$

$$\le C [f]_{\alpha,\mathscr{L}^2}^2 \left| \int_0^T \int_{\mathbb{R}^n} t^{-(n+|\gamma|)/2} \exp\left(-c\frac{|x-y|^2}{t}\right) |x-y|^\alpha \, dy \, dt \right|^2$$

$$\le C [f]_{\alpha,\mathscr{L}^2}^2.$$

Thus,

$$\left[\int_\cdot^T R_\cdot^s Y(t;s) \, ds \right]_{2,\mathscr{L}^2} \le C [f]_{\alpha,\mathscr{L}^2}. \tag{3.22}$$

Define $\upsilon := 2|x - \bar{x}|$ for $x \ne \bar{x}$. By Remark 3.1, we have for $|\gamma| = 2$,

$$D^\gamma \int_t^T R_t^s Y(t;s)(x) \, ds - D^\gamma \int_t^T R_t^s Y(t;s)(\bar{x}) \, ds$$

$$= \int_t^T \int_{\mathbb{R}^n} D^\gamma G_{s,t}(x-y) \big[Y(t;s,y) - Y(t;s,x)\big] \, dy \, ds$$

$$- \int_t^T \int_{\mathbb{R}^n} D^\gamma G_{s,t}(\bar{x}-y) \big[Y(t;s,y) - Y(t;s,\bar{x})\big] \, dy \, ds$$

$$= \sum_{i=1}^4 I_i(t,x,\bar{x})$$

with

$$
\begin{aligned}
I_1 &:= \int_t^T \int_{B_\upsilon(x)} D^\gamma G_{s,t}(x-y)\big[Y(t;s,y)-Y(t;s,x)\big]\,dy\,ds,\\
I_2 &:= -\int_t^T \int_{B_\upsilon(x)} D^\gamma G_{s,t}(\bar{x}-y)\big[Y(t;s,y)-Y(t;s,\bar{x})\big]\,dy\,ds,\\
I_3 &:= -\int_t^T \int_{|y-x|>\upsilon} D^\gamma G_{s,t}(x-y)\big[Y(t;s,x)-Y(t;s,\bar{x})\big]\,dy\,ds,\\
I_4 &:= \int_t^T \int_{|y-x|>\upsilon} \big[D^\gamma G_{s,t}(x-y)-D^\gamma G_{s,t}(\bar{x}-y)\big]\\
&\qquad\qquad \times \big[Y(t;s,y)-Y(t;s,\bar{x})\big]\,dy\,ds.
\end{aligned}
\tag{3.23}
$$

Next, we estimate $I_i(t,x,\bar{x})$ for $i=1,2,3,4$. In view of (3.15), Lemma 2.3, and Remark 3.1, we have

$$
\begin{aligned}
&E\int_0^T \big|I_1(t,x,\bar{x})\big|^2\,dt\\
&\quad = E\int_0^T \bigg|\int_t^T \int_{B_\upsilon(x)} \big|D^\gamma G_{s,t}(x-y)\big||x-y|^\alpha\\
&\qquad\qquad\qquad \times \frac{|Y(t;s,y)-Y(t;s,x)|}{|x-y|^\alpha}\,dy\,ds\bigg|^2\,dt\\
&\quad \le C\sup_y E\int_0^T \sup_{t\le s}\frac{|Y(t;s,y)-Y(t;s,x)|^2}{|x-y|^{2\alpha}}\,ds\\
&\qquad \times \bigg|\int_{B_\upsilon(x)} \sup_{\tau\le s}\int_\tau^s \big|D^\gamma G_{s,t}(x-y)\big||x-y|^\alpha\,dt\,dy\bigg|^2\\
&\quad \le C[f]^2_{\alpha,\mathscr{L}^2}|x-\bar{x}|^{2\alpha}.
\end{aligned}
$$

In the same way, we have

$$
E\int_0^T \big|I_2(t,x,\bar{x})\big|^2\,dt \le C[f]^2_{\alpha,\mathscr{L}^2}|x-\bar{x}|^{2\alpha}.
$$

From Lemma 2.2, we have

$$
\begin{aligned}
&E\int_0^T \big|I_3(t,x,\bar{x})\big|^2\,dt\\
&\quad \le E\int_0^T \bigg|\int_t^T \big|Y(t;s,x)-Y(t;s,\bar{x})\big|\bigg|\int_{|y-x|>\upsilon} D^\gamma G_{s,t}(x-y)\,dy\bigg|\,ds\bigg|^2\,dt\\
&\quad \le E\int_0^T \int_t^T \big|Y(t;s,x)-Y(t;s,\bar{x})\big|^2\bigg|\int_{|y-x|>\upsilon} D^\gamma G_{s,t}(x-y)\,dy\bigg|\,ds\,dt\\
&\qquad \times \sup_t \int_t^T \bigg|\int_{|y-x|>\upsilon} D^\gamma G_{s,t}(\bar{x}-y)\,dy\bigg|\,ds
\end{aligned}
$$

$$\leq CE\int_0^T \sup_{t\leq s}\big|Y(t;s,x)-Y(t;s,\bar{x})\big|^2\,ds$$

$$\times \sup_s \int_0^s \left|\int_{|y-x|>\upsilon} D^\gamma G_{s,t}(x-y)\,dy\right|dt$$

$$\leq C[f]^2_{\alpha,\mathscr{L}^2}|x-\bar{x}|^{2\alpha}.$$

For $I_4(t,x,\bar{x})$, in view of (2.5), (3.15) and Lemma 2.4, we have

$$E\int_0^T \big|I_4(t,x,\bar{x})\big|^2\,dt$$

$$\leq CE\int_0^T \left|\int_t^T \int_{|y-x|>\upsilon} \big|D^\gamma G_{s,t}(x-y)-D^\gamma G_{s,t}(\bar{x}-y)\big| \times \big|Y(t;s,y)-Y(t;s,\bar{x})\big|\,dy\,ds\right|^2 dt$$

$$\leq C\sup_y E\int_0^T \sup_{t\leq s}\frac{|Y(t;s,y)-Y(t;s,\bar{x})|^2}{|\bar{x}-y|^{2\alpha}}\,ds \times \left[\int_{|y-x|>\upsilon}\sup_{\tau\leq s}\int_\tau^s \big|D^\gamma G_{s,t}(x-y)-D^\gamma G_{s,t}(\bar{x}-y)\big||\bar{x}-y|^\alpha\,dt\,dy\right]^2$$

$$\leq C[f]^2_{\alpha,\mathscr{L}^2}|x-\bar{x}|^{2\alpha}.$$

In summary, we have

$$\sum_{i=1}^4 E\int_0^T \big|I_i(t,x,\bar{x})\big|^2\,dt \leq C[f]^2_{\alpha,\mathscr{L}^2}|x-\bar{x}|^{2\alpha}. \tag{3.24}$$

Combining (3.21), (3.22) and (3.24), we have

$$\left\|\int_\cdot^T R^s_\cdot Y(\cdot;s)\,ds\right\|_{2+\alpha,\mathscr{L}^2} \leq C\|f\|_{\alpha,\mathscr{L}^2}.$$

□

LEMMA 3.5. *Let Assumption 3.1 be satisfied and suppose that*

$$(\Phi,f)\in C^{1+\alpha}\big(\mathbb{R}^n,L^2(\Omega)\big)\times C^\alpha\big(\mathbb{R}^n,\mathscr{L}^2_{\mathbb{F}}(0,T)\big).$$

If $(u,v)\in C^\alpha_{\mathscr{S}^2}\cap C^{2+\alpha}_{\mathscr{L}^2}\times C^\alpha_{\mathscr{L}^2}$ *solves BSPDE* (3.1), *then we have*

$$\|v\|_{\alpha,\mathscr{L}^2}\leq C\big(\|\Phi\|_{1+\alpha,L^2}+\|f\|_{\alpha,\mathscr{L}^2}\big),$$

where $C=C(\lambda,\Lambda,\alpha,n,d,T)$.

PROOF. In view of (3.16) and (3.17), we need to prove

$$\|R^T_\cdot\psi(\cdot)\|_{\alpha,\mathscr{L}^2}\leq C\|\Phi\|_{1+\alpha,L^2},$$

$$\left\|\int_\cdot^T R^s_\cdot g(\cdot;s)\,ds\right\|_{\alpha,\mathscr{L}^2}\leq C\|f\|_{\alpha,\mathscr{L}^2}.$$

It is sufficient to prove the second inequality, and the first one can be proved in a similar way.

For all $x \in \mathbb{R}^n$, and almost all $r \in [0, T]$,

$$R_t^r Y(t;r)(x) = f(r,x) + \int_t^r \sigma(s) R_s^r g(s;r)(x)\,ds - \int_t^r R_s^r g(s;r)(x)\,dW_s + \int_t^r a_{ij}(s) \int_{\mathbb{R}^n} \partial_{ij}^2 G_{r,s}(x-y)\big[Y(s;r,y) - Y(s;r,x)\big]\,ds \qquad \forall t \le r.$$

From the theory of BSDEs, we have

$$\begin{aligned} E\Big[\int_0^r \big|R_s^r g(s;r)(x)\big|^2\,ds\Big] &\le CE\Big[\big|f(r,x)\big|^2 \\ &\qquad + \int_0^r \Big|a_{ij}(s)\int_{\mathbb{R}^n} \partial_{ij}^2 G_{r,s}(x-y)\big[Y(s;r,y)-Y(s;r,x)\big]\,dy\Big|^2\,ds\Big] \\ &\le CE\Big[\big|f(r,x)\big|^2 + \int_0^r \Big|\int_{\mathbb{R}^n} \partial_{ij}^2 G_{r,s}(x-y)\big[Y(s;r,y)-Y(s;r,x)\big]\,dy\Big|^2\,ds\Big]. \end{aligned}$$

Integrating both sides on $[0, T]$, we have

$$\begin{aligned} E\Big[\int_0^T \int_s^T \big|R_s^r g(s;r)(x)\big|^2\,dr\,ds\Big] &= E\Big[\int_0^T \int_0^r \big|R_s^r g(s;r)(x)\big|^2\,ds\,dr\Big] \\ &\le CE\Big[\int_0^T \big|f(r,x)\big|^2\,dr \\ &\qquad + \int_0^T \int_0^r \Big|\int_{\mathbb{R}^n} \partial_{ij}^2 G_{r,s}(x-y)\big[Y(s;r,y)-Y(s;r,x)\big]\,dy\Big|^2\,ds\,dr\Big] \\ &\le CE\Big[\int_0^T \big|f(r,x)\big|^2\,dr \\ &\qquad + \int_0^T \int_s^T \Big|\int_{\mathbb{R}^n} \partial_{ij}^2 G_{r,s}(x-y)\big[Y(s;r,y)-Y(s;r,x)\big]\,dy\Big|^2\,dr\,ds\Big]. \end{aligned}$$

Similarly,

$$\begin{aligned} E\Big[\int_0^T \int_s^T \big|R_s^r g(s;r)(x) - R_s^r g(s;r)(\bar{x})\big|^2\,dr\,ds\Big] \\ \le CE\Big[\int_0^T \big|f(r,x) - f(r,\bar{x})\big|^2\,dr\Big] \end{aligned}$$

$$+ CE\left[\int_0^T \int_s^T \left| \int_{\mathbb{R}^n} \big(\partial_{ij}^2 G_{r,s}(x-y)[Y(s;r,y) - Y(s;r,x)] \right.\right.$$
$$- \partial_{ij}^2 G_{r,s}(\bar{x}-y)$$
$$\left.\left. \times [Y(s;r,y) - Y(s;r,\bar{x})]\big)\,dy \right|^2 dr\,ds\right].$$

In view of the proof in Lemma 3.4, we have

$$\left\| \int_\cdot^T R_\cdot^s g(\cdot; s)\,ds \right\|_{\alpha,\mathscr{L}^2} \le C\|f\|_{\alpha,\mathscr{L}^2}. \qquad \square$$

We have the following Hölder estimate for (u, v).

THEOREM 3.6. *Let Assumption 3.1 be satisfied and*

$$(\Phi, f) \in C^{1+\alpha}\big(\mathbb{R}^n, L^2(\Omega)\big) \times C^\alpha\big(\mathbb{R}^n, \mathscr{L}^2_{\mathbb{F}}(0,T)\big).$$

If (u, v) is a classical solution to BSPDE (3.1), then

$$\|u\|_{\alpha,\mathscr{S}^2} + \|u\|_{2+\alpha,\mathscr{L}^2} + \|v\|_{\alpha,\mathscr{L}^2} \le C\big(\|\Phi\|_{1+\alpha,L^2} + \|f\|_{\alpha,\mathscr{L}^2}\big),$$

where $C = C(\lambda, \Lambda, \alpha, n, d, T)$.

PROOF. From Theorem 3.3 and the Lemmas 3.4 and 3.5, we have

$$\|u\|_{2+\alpha,\mathscr{L}^2} + \|v\|_{\alpha,\mathscr{L}^2} \le C\big(\|\Phi\|_{1+\alpha,L^2} + \|f\|_{\alpha,\mathscr{L}^2}\big).$$

Since (u, v) is the solution of BSPDE (3.1), for all $(t, x) \in [0, T] \times \mathbb{R}^n$, the equality holds almost surely:

$$u(t,x) = \Phi(x) + \int_t^T \big[a^{ij}(s)\,\partial_{ij}^2 u(s,x) + f(s,x) + \sigma(s)v(s,x)\big]\,ds$$
$$- \int_t^T v(s,x)\,dW_s.$$

For each x, it is a BSDE of terminal value $\Phi(x)$ and generator $a^{ij}(t)\,\partial_{ij}^2 u(t,x) + f(t,x) + \sigma(t)V$. From the theory of BSDEs, we have

$$E\Big[\sup_t |u(t,x)|^2\Big] \le CE\left[|\Phi(x)|^2 + \int_0^T |a^{ij}(s)\,\partial_{ij}^2 u(s,x) + f(s,x)|^2\,ds\right]$$
$$\le CE\left[|\Phi(x)|^2 + \int_0^T \big(|\partial_{ij}^2 u(s,x)|^2 + |f(s,x)|^2\big)\,ds\right]$$

and for all $\bar{x} \neq x$,

$$
\begin{aligned}
E\Big[\sup_t \big|u(t,x) - u(t,\bar{x})\big|^2\Big]
&\le CE\big[\big|\Phi(x) - \Phi(\bar{x})\big|^2\big] \\
&\quad + CE\bigg[\int_0^T \big(\big|\partial_{ij}^2 u(s,x) - \partial_{ij}^2 u(s,\bar{x})\big|^2 + \big|f(s,x) - f(s,\bar{x})\big|^2\big)\,ds\bigg].
\end{aligned}
$$

Then

$$
\begin{aligned}
\|u\|_{\alpha,\mathscr{S}^2} &\le C\big(\|\Phi\|_{\alpha,L^2} + \|u\|_{2+\alpha,\mathscr{L}^2} + \|f\|_{\alpha,\mathscr{L}^2}\big) \\
&\le C\big(\|\Phi\|_{1+\alpha,L^2} + \|f\|_{\alpha,\mathscr{L}^2}\big).
\end{aligned}
$$

The proof is complete. □

3.3. *Existence and uniqueness.*

THEOREM 3.7. *Let Assumption 3.1 be satisfied and*

$$
(\Phi, f) \in C^{1+\alpha}\big(\mathbb{R}^n, L^2(\Omega)\big) \times C^{\alpha}\big(\mathbb{R}^n, \mathscr{L}^2_{\mathbb{F}}(0,T)\big).
$$

Let (φ, ψ) and (Y, g) be solutions of BSDEs (3.7) and (3.8), respectively. Then, the pair (u, v) of random fields defined by

$$
u(t,x) = R_t^T \varphi(t)(x) + \int_t^T R_t^s Y(t;s)(x)\,ds,
$$

$$
v(t,x) = R_t^T \psi(t)(x) + \int_t^T R_s^T g(t;s)(x)\,ds
$$

is the unique classical solution to BSPDE (3.1). Moreover, $(u,v) \in (C^{\alpha}_{\mathscr{S}^2} \cap C^{2+\alpha}_{\mathscr{L}^2}) \times C^{\alpha}_{\mathscr{L}^2}$, and

$$
\|u\|_{\alpha,\mathscr{S}^2} + \|u\|_{2+\alpha,\mathscr{L}^2} + \|v\|_{\alpha,\mathscr{L}^2} \le C\big(\|\Phi\|_{1+\alpha,L^2} + \|f\|_{\alpha,\mathscr{L}^2}\big),
$$

where $C = C(\lambda, \Lambda, \alpha, n, d, T)$.

PROOF. In view of Remark 3.1, for all $(t,x) \in [0,T] \times \mathbb{R}^n$, we have

$$
\begin{aligned}
&\int_t^T a^{ij}(s)\,\partial_{ij}^2 R_s^T \varphi(s)(x)\,ds \\
&\qquad = \int_t^T \int_{\mathbb{R}^n} a^{ij}(s)\,\partial_{ij}^2 G_{T,s}(x-y)\big[\varphi(s,y) - \varphi(s,x)\big]\,dy\,ds \\
&\qquad = \int_{\mathbb{R}^n} \int_t^T -\frac{\partial}{\partial s} G_{T,s}(x-y)\big[\varphi(s,y) - \varphi(s,x)\big]\,dy\,ds \\
&\qquad = \int_{\mathbb{R}^n} G_{T,t}(x-y)\big[\varphi(t,y) - \varphi(t,x)\big]\,dy \qquad (3.25)
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^n}\int_t^T G_{T,s}(x-y)\,d\big[\varphi(s,y)-\varphi(s,x)\big]\,dy\\
&=R_t^T\varphi(t)(x)-\varphi(t,x)+\int_t^T R_s^T\psi_l(s)(x)\,d\widetilde{W}_s^l\\
&\quad-\int_t^T \psi_l(s,x)\,d\widetilde{W}_s^l\\
&=R_t^T\varphi(t)(x)-\Phi(x)+\int_t^T R_s^T\psi_l(s)(x)\,d\widetilde{W}_s^l.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\int_t^T a^{ij}(s)\,\partial_{ij}^2\int_s^T R_s^r Y(s;r)(x)\,dr\,ds\\
&=\int_t^T\int_s^T\int_{\mathbb{R}^n} a^{ij}(s)\,\partial_{ij}^2 G_{r,s}(x-y)\big[Y(s;r,y)-Y(s;r,x)\big]\,dy\,dr\,ds\\
&=\int_t^T\int_s^T\int_{\mathbb{R}^n} -\frac{\partial}{\partial s}G_{r,s}(x-y)\big[Y(s;r,y)-Y(s;r,x)\big]\,dy\,dr\,ds\\
&=\int_t^T\int_{\mathbb{R}^n}\int_t^r -\frac{\partial}{\partial s}G_{r,s}(x-y)\big[Y(s;r,y)-Y(s;r,x)\big]\,ds\,dy\,dr\\
&=\int_t^T\int_{\mathbb{R}^n} G_{r,t}(x-y)\big[Y(t;r,y)-Y(t;r,x)\big]\,dy\,dr\\
&\quad+\int_t^T\int_{\mathbb{R}^n}\int_t^r G_{r,s}(x-y)\,d_s\big[Y(s;r,y)-Y(s;r,x)\big]\,dy\,dr\\
&=-\int_t^T f(r,x)\,dr+\int_t^T R_t^r Y(t;r)(x)\,dr\\
&\quad+\int_t^T\int_s^T R_s^r g_l(s;r)(x)\,dr\,d\widetilde{W}_s^l.
\end{aligned}
\tag{3.26}
$$

In view of (3.25) and (3.26), we have

$$
\begin{aligned}
&\int_t^T a^{ij}(s)\,\partial_{ij}^2 u(s,x)\,ds\\
&=\int_t^T a^{ij}(s)\,\partial_{ij}^2\big[R_s^T\varphi(s)(x)\big]\,ds+\int_t^T a^{ij}(s)\,\partial_{ij}^2\bigg[\int_s^T R_s^r Y(s;r)(x)\,dr\bigg]\,ds\\
&=R_t^T\varphi(t)(x)-\Phi(x)+\int_t^T R_s^T\psi_l(s)(x)\,d\widetilde{W}_s^l-\int_t^T f(r,x)\,dr\\
&\quad+\int_t^T R_t^r Y(t;r,y)\,dr+\int_t^T\int_s^T R_s^r g_l(s;r)(x)\,dr\,d\widetilde{W}_s^l\\
&=-\Phi(x)-\int_t^T f(r,x)\,dr+u(t,x)+\int_t^T v_l(s,x)\,d\widetilde{W}_s^l.
\end{aligned}
$$

Thus, (u, v) solves BSPDE (3.1), that is,

$$u(t,x) = \Phi(x) + \int_t^T \big[a^{ij}(s)\,\partial_{ij}^2 u(s,x) + f(s,x) + \sigma(s)v(s,x)\big]\,ds - \int_t^T v(s,x)\,dW_s.$$

The desired estimate follows from Theorem 3.6. The proof is complete. □

Moreover, we have the following Hölder continuity of u in time t. For any $\tau \in [0, T]$, denote by $\|\cdot\|_{m+\alpha,\mathscr{S}^2,\tau}$ and $\|\cdot\|_{m+\alpha,\mathscr{L}^2,\tau}$ the obvious Hölder norms of a process restricted to the time interval $[\tau, T]$.

PROPOSITION 3.8. *Let Assumption 3.1 be satisfied and*

$$(\Phi, f) \in C^{1+\alpha}\big(\mathbb{R}^n, L^2(\Omega)\big) \times C^{\alpha}\big(\mathbb{R}^n, \mathscr{L}^2_{\mathbb{F}}(0,T)\big).$$

Let $(u, v) \in C^{\alpha}_{\mathscr{S}^2} \cap C^{2+\alpha}_{\mathscr{L}^2} \times C^{\alpha}_{\mathscr{L}^2}$ *be the classical solution to BSPDE* (3.1). *Then for any* $\tau \in [0, T]$, *we have*

$$\big\|u(\cdot,\cdot) - u(\cdot - \tau, \cdot)\big\|_{\alpha,\mathscr{L}^2,\tau} \le C\sqrt{\tau}\big(\|\Phi\|_{1+\alpha,L^2} + \|f\|_{\alpha,\mathscr{L}^2}\big),$$

where $C = C(\lambda, \Lambda, \alpha, T, n, d)$.

PROOF. Since (u, v) satisfies BSPDE (3.1), we have

$$\begin{aligned}
&E\int_\tau^T \big|u(t,x) - u(t-\tau,x)\big|^2\,dt \\
&\quad\le CE\int_\tau^T \left|\int_{t-\tau}^t \big(a^{ij}(s)\,\partial_{ij}^2 u(s,x) + f(s,x) + \sigma(s)v(s,x)\big)\,ds\right|^2 dt \\
&\qquad + CE\int_\tau^T \left|\int_{t-\tau}^t v(s,x)\,dW_s\right|^2 dt \\
&\quad\le CE\int_0^T \int_{s\vee\tau}^{T\wedge(s+\tau)} \big(\big|\partial_{ij}^2 u(s,x)\big|^2 + \big|f(s,x)\big|^2 + \big|v(s,x)\big|^2\big)\,dt\,ds \\
&\qquad + CE\int_0^T \int_{s\vee\tau}^{T\wedge(s+\tau)} \big|v(s,x)\big|^2\,dt\,ds \\
&\quad\le C\tau\big([u]^2_{2,\mathscr{L}^2} + [f]^2_{0,\mathscr{L}^2} + [v]^2_{0,\mathscr{L}^2}\big) \\
&\quad\le C\tau\big(\|\Phi\|^2_{1+\alpha,L^2} + \|f\|^2_{\alpha,\mathscr{L}^2}\big).
\end{aligned}$$

Similarly, for any $x \ne \bar{x}$,

$$\begin{aligned}
&E\int_\tau^T \big|u(t,x) - u(t-\tau,x) - \big[u(t,\bar{x}) - u(t-\tau,\bar{x})\big]\big|^2\,dt \\
&\quad\le C\tau\big(\|\Phi\|^2_{1+\alpha,L^2} + \|f\|^2_{\alpha,\mathscr{L}^2}\big)|x-\bar{x}|^{2\alpha}.
\end{aligned}$$

Therefore, we have the desired result. □

4. BSPDEs with space-variable coefficients. In this section, using the conventional combinational techniques of the freezing coefficients method and the parameter continuation argument well developed in the theory of deterministic PDEs, we extend the a priori Hölder estimates as well as the existence and uniqueness result for BSPDE of the preceding section to the more general BSPDE (1.1).

Consider a smooth function $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that

$$0 \le \varphi \le 1 \quad \text{and} \quad \varphi(x) = \begin{cases} 1, & |x| \le 1, \\ 0, & |x| > 2. \end{cases}$$

For any $z \in \mathbb{R}^n$ and $\theta > 0$ fixed, define

$$\eta_\theta^z(x) := \varphi\left(\frac{x - z}{\theta}\right).$$

We easily see that for $\gamma \in \Gamma$, there is a constant $C = C(\gamma, n)$ such that

$$[D^\gamma \eta_\theta^z]_0 \le C\theta^{-|\gamma|}, \qquad [D^\gamma \eta_\theta^z]_\alpha \le C\theta^{-|\gamma|-\alpha}.$$

LEMMA 4.1. *Let $h \in C^{m+\alpha}(\mathbb{R}^n; \mathscr{L}_{\mathbb{F}}^2(0, T; \mathbb{R}^\iota))$ with $m = 0, 1, 2$. Then there is a positive constant $C(\theta, \alpha)$ such that*

$$\|h\|_{m+\alpha,\mathscr{L}^2} \le 2 \sup_{z\in\mathbb{R}^n} \|\eta_\theta^z h\|_{m+\alpha,\mathscr{L}^2} + C(\theta, \alpha)\|h\|_{0,\mathscr{L}^2}.$$

PROOF. It is sufficient to prove

$$[h]_{2+\alpha,\mathscr{L}^2} \le 2 \sup_{z\in\mathbb{R}^n} [\eta_\theta^z h]_{2+\alpha,\mathscr{L}^2} + C(\theta, \alpha)\|h\|_{0,\mathscr{L}^2}.$$

The proof of the rest is similar.

For any $\theta > 0$ fixed, we have

$$[h]_{2+\alpha,\mathscr{L}^2} \le I_1 + I_2$$

with

$$I_1 := \sum_{|\gamma|=2} \sup_{|x-\bar{x}|<\theta} \frac{E[\int_0^T |D^\gamma h(t, x) - D^\gamma h(t, \bar{x})|^2\, dt]^{1/2}}{|x - \bar{x}|^\alpha}$$

and

$$I_2 := \sum_{|\gamma|=2} \sup_{|x-\bar{x}|\ge\theta} \frac{E[\int_0^T |D^\gamma h(t, x) - D^\gamma h(t, \bar{x})|^2\, dt]^{1/2}}{|x - \bar{x}|^\alpha}.$$

For any $x, \bar{x} \in \mathbb{R}^n$, if $|x - \bar{x}| < \theta$, choose $z = x$,

$$\begin{aligned} I_1 &\le \sum_{|\gamma|=2} \sup_{|x-\bar{x}|<\theta} \frac{E[\int_0^T |D^\gamma(\eta_\theta^x(x)h(t, x)) - D^\gamma(\eta_\theta^x(\bar{x})h(t, \bar{x}))|^2]^{1/2}}{|x - \bar{x}|^\alpha} \\ &\le \sup_{z\in\mathbb{R}^n} [\eta_\theta^z h]_{2+\alpha,\mathscr{L}^2}. \end{aligned}$$

If $|x-\bar{x}|\geq\theta$, using the interpolation inequality in Lemma 2.1, we have

$$
\begin{aligned}
I_2 &\leq \sum_{|\gamma|=2}\sup_{|x-\bar{x}|\geq\theta} E\biggl[\int_0^T \bigl|D^\gamma h(t,x)-D^\gamma h(t,\bar{x})\bigr|^2\,dt\biggr]^{1/2}\theta^{-\alpha}\\
&\leq C(\theta,\alpha)[h]_{2,\mathscr{L}^2}\leq \frac{1}{2}[h]_{2+\alpha,\mathscr{L}^2}+C(\theta,\alpha)[h]_{0,\mathscr{L}^2}.
\end{aligned}
$$

Then

$$
[h]_{2+\alpha,\mathscr{L}^2}\leq 2\sup_{z\in\mathbb{R}^n}\bigl[\eta_\theta^z h\bigr]_{2+\alpha,\mathscr{L}^2}+C\|h\|_{0,\mathscr{L}^2}. \qquad \square
$$

We have the following a priori Hölder estimate on the solution (u,v) to BSPDE (1.1).

THEOREM 4.2. *Let the Assumptions 2.1 and 2.2 be satisfied and* $(\Phi,f)\in C^{1+\alpha}(\mathbb{R}^n,L^2(\Omega))\times C^\alpha(\mathbb{R}^n,\mathscr{L}^2_{\mathbb{F}}(0,T))$. *If* $(u,v)\in C^\alpha_{\mathscr{S}^2}\cap C^{2+\alpha}_{\mathscr{L}^2}\times C^\alpha_{\mathscr{L}^2}$ *solves BSPDE (1.1), we have*

$$
\|u\|_{\alpha,\mathscr{S}^2}+\|u\|_{2+\alpha,\mathscr{L}^2}+\|v\|_{\alpha,\mathscr{L}^2}\leq C\bigl(\|\Phi\|_{1+\alpha,L^2}+\|f\|_{\alpha,\mathscr{L}^2}\bigr),
$$

where $C=C(\lambda,\Lambda,\alpha,n,d,T)$.

PROOF. For any $z\in\mathbb{R}^n$ and $\theta>0$, denote

$$
\begin{aligned}
&u_\theta^z(t,x):=\eta_\theta^z(x)u(t,x), \qquad v_\theta^z(t,x):=\eta_\theta^z(x)v(t,x),\\
&\Phi_\theta^z(x):=\eta_\theta^z(x)\Phi(x),
\end{aligned}
$$

and

$$
\begin{aligned}
f_\theta^z(t,x):={}&\bigl[a^{ij}(t,x)-a^{ij}(t,z)\bigr]\,\partial^2_{ij}u(t,x)\eta_\theta^z(x)\\
&+\bigl[\sigma(t,x)-\sigma(t,z)\bigr]v(t,x)\eta_\theta^z(x)\\
&-2a^{ij}(t,z)\,\partial_i u(t,x)\,\partial_j\eta_\theta^z(x)-a^{ij}(t,z)u(t,x)\,\partial^2_{ij}\eta_\theta^z(x)\\
&+b^i(t,x)\,\partial_i u(t,x)\eta_\theta^z(x)+c(t,x)u(t,x)\eta_\theta^z(x)+f(t,x)\eta_\theta^z(x)\\
={}&\sum_{i=1}^7\mathscr{A}_i(t,x,z,\theta),
\end{aligned}
$$

with $\mathscr{A}_i(t,x,z,\theta)$ denoting the obvious ith term $(i=1,2,\ldots,7)$ in the three lines of sum. Then we have $\Phi_\theta^z\in C^{1+\alpha}(\mathbb{R}^n,L^2(\Omega))$, $f_\theta^z\in C^\alpha(\mathbb{R}^n,\mathscr{L}^2_{\mathbb{F}}(0,T))$, and $(u_\theta^z,v_\theta^z)\in C^\alpha_{\mathscr{S}^2}\cap C^{2+\alpha}_{\mathscr{L}^2}\times C^\alpha_{\mathscr{L}^2}$. Moreover, (u_θ^z,v_θ^z) solves the following BSPDE:

$$
\begin{cases}
-du_\theta^z(t,x)=\bigl[a^{ij}(t,z)\,\partial^2_{ij}u_\theta^z(t,x)+f_\theta^z(t,x)+\sigma^l(t,z)\bigl(v_\theta^z(t,x)\bigr)_l\bigr]\,dt\\
\qquad\qquad -\bigl(v_\theta^z(t,x)\bigr)_l\,dW_t^l, \qquad (t,x)\in[0,T]\times\mathbb{R}^n;\\
u_\theta^z(T,x)=\Phi_\theta^z(x), \qquad x\in\mathbb{R}^n.
\end{cases}
$$

To simplify notation, define the following two types of universal constants:

$$C := C(\lambda, \Lambda, \alpha, n, d, T),$$
$$C(\cdot) := C(\cdot, \lambda, \Lambda, \alpha, n, d, T).$$

In view of Theorem 3.6, we have

$$\|u_\theta^z\|_{2+\alpha,\mathscr{L}^2} + \|v_\theta^z\|_{\alpha,\mathscr{L}^2} \le C\big(\|\Phi_\theta^z\|_{1+\alpha,L^2} + \|f_\theta^z\|_{\alpha,\mathscr{L}^2}\big).$$

From Lemma 4.1, we have

$$\begin{aligned}\|u\|_{2+\alpha,\mathscr{L}^2} + \|v\|_{\alpha,\mathscr{L}^2} &\le C\Big(\sup_z\|\Phi_\theta^z\|_{1+\alpha,L^2} + \sup_z\|f_\theta^z\|_{\alpha,\mathscr{L}^2}\Big)\\ &\quad + C(\theta)\big(\|u\|_{0,\mathscr{L}^2} + \|v\|_{0,\mathscr{L}^2}\big).\end{aligned} \tag{4.1}$$

Thus, to estimate (u, v), we need to estimate Φ_θ^z and $\mathscr{A}_i, i = 1, \dots, 7$, in terms of f_θ^z.

$$\begin{aligned}\|\Phi_\theta^z\|_{1+\alpha,L^2} &= [\eta_\theta^z\Phi]_{0,L^2} + [\eta_\theta^z\Phi]_{1,L^2} + [\eta_\theta^z\Phi]_{1+\alpha,L^2}\\ &\le C\Big(1 + \frac{1}{\theta} + \frac{1}{\theta^{1+\alpha}}\Big)[\Phi]_{0,L^2} + C\Big(1 + \frac{1}{\theta^\alpha}\Big)[\Phi]_{1,L^2}\\ &\quad + \frac{C}{\theta}[\Phi]_{\alpha,L^2} + [\Phi]_{1+\alpha,L^2}\\ &\le C(\theta)\|\Phi\|_{1+\alpha,L^2}.\end{aligned}$$

Denote by $[\cdot]_{m+\alpha,\mathscr{L}^2,A}$ and $\|\cdot\|_{m+\alpha,\mathscr{L}^2,A}$ the semi-norm and norm of functionals on subset $A \subset \mathbb{R}^n$ instead of on the whole space $\mathbb{R}^n$. It is obvious that $\mathscr{A}_1(t, x, z, \theta) \equiv 0$ for $x \notin B_{2\theta}(z)$. In view of inequality (2.1) and the interpolation inequality in Lemma 2.1, we have

$$\begin{aligned}\|\mathscr{A}_1(\cdot,\cdot,z,\theta)\|_{\alpha,\mathscr{L}^2} &= \|\mathscr{A}_1(\cdot,\cdot,z,\theta)\|_{\alpha,\mathscr{L}^2,B_{2\theta}(z)}\\ &\le \big[a^{ij}(\cdot,\cdot) - a^{ij}(\cdot,z)\big]_{0,\mathscr{L}^\infty,B_{2\theta}(z)}\|\partial_{ij}^2 u\|_{\alpha,\mathscr{L}^2}\\ &\quad + \big[a^{ij}(\cdot,\cdot) - a^{ij}(\cdot,z)\big]_{\alpha,\mathscr{L}^\infty}[u]_{2,\mathscr{L}^2}\\ &\quad + \big[a^{ij}(\cdot,\cdot) - a^{ij}(\cdot,z)\big]_{0,\mathscr{L}^\infty,B_{2\theta}(z)}[\eta_\theta^z]_\alpha[u]_{2,\mathscr{L}^2}\\ &\le \Lambda(2\theta)^\alpha\big([u]_{2+\alpha,\mathscr{L}^2} + [u]_{2,\mathscr{L}^2}\big) + C[u]_{2,\mathscr{L}^2}\\ &\le \Lambda(2\theta)^\alpha\big([u]_{2+\alpha,\mathscr{L}^2} + \varepsilon[u]_{2+\alpha,\mathscr{L}^2} + C(\varepsilon)[u]_{0,\mathscr{L}^2}\big)\\ &\quad + C\big(\varepsilon[u]_{2+\alpha,\mathscr{L}^2} + C(\varepsilon)[u]_{0,\mathscr{L}^2}\big)\\ &\le C\big(\theta^\alpha(1+\varepsilon) + \varepsilon\big)[u]_{2+\alpha,\mathscr{L}^2} + C(\varepsilon,\theta)[u]_{0,\mathscr{L}^2}.\end{aligned}$$

Similarly, we have

$$
\begin{aligned}
\|\mathscr{A}_2(\cdot,\cdot,z,\theta)\|_{\alpha,\mathscr{L}^2} &\le C\theta^\alpha\|v\|_{\alpha,\mathscr{L}^2}+C[v]_{0,\mathscr{L}^2},\\
\|\mathscr{A}_3(\cdot,\cdot,z,\theta)\|_{\alpha,\mathscr{L}^2} &\le C[\partial_i u]_{0,\mathscr{L}^2}|\partial_j\eta_\theta^z|_\alpha + C[\partial_i u]_{\alpha,\mathscr{L}^2}[\partial_j\eta_\theta^z]_0\\
&\le C\Big(\frac{1}{\theta}+\frac{1}{\theta^{1+\alpha}}\Big)\big[\varepsilon[u]_{2+\alpha,\mathscr{L}^2}+C(\varepsilon)[u]_{0,\mathscr{L}^2}\big]\\
&\quad+\frac{C}{\theta}\big[\varepsilon[u]_{2+\alpha,\mathscr{L}^2}+C(\varepsilon)[u]_{0,\mathscr{L}^2}\big]\\
&\le C\Big(\frac{1}{\theta}+\frac{1}{\theta^{1+\alpha}}\Big)\varepsilon[u]_{2+\alpha,\mathscr{L}^2}+C(\varepsilon,\theta)[u]_{0,\mathscr{L}^2},\\
\|\mathscr{A}_4(\cdot,\cdot,z,\theta)\|_{\alpha,\mathscr{L}^2} &\le C[u]_{0,\mathscr{L}^2}|\partial_{ij}^2\eta_\theta^z|_\alpha + C[u]_{\alpha,\mathscr{L}^2}[\partial_{ij}^2\eta_\theta^z]_0\\
&\le C\Big(\frac{1}{\theta^2}+\frac{1}{\theta^{2+\alpha}}\Big)[u]_{0,\mathscr{L}^2}\\
&\quad+\frac{C}{\theta^2}\big(\varepsilon[u]_{2+\alpha,\mathscr{L}^2}+C_\varepsilon[u]_{0,\mathscr{L}^2}\big)\\
&\le \frac{C}{\theta^2}\varepsilon[u]_{2+\alpha,\mathscr{L}^2}+C(\varepsilon,\theta)[u]_{0,\mathscr{L}^2},\\
\|\mathscr{A}_5(\cdot,\cdot,z,\theta)\|_{\alpha,\mathscr{L}^2} &\le C[\partial_i u]_{0,\mathscr{L}^2}|\eta_\theta^z|_\alpha + C[\partial_i u]_{\alpha,\mathscr{L}^2}+C[\partial_i u]_{0,\mathscr{L}^2}\\
&\le C\Big(1+\frac{1}{\theta^\alpha}\Big)\varepsilon[u]_{2+\alpha,\mathscr{L}^2}+C(\varepsilon,\theta)[u]_{0,\mathscr{L}^2},\\
\|\mathscr{A}_6(\cdot,\cdot,z,\theta)\|_{\alpha,\mathscr{L}^2} &\le C\big([u]_{0,\mathscr{L}^2_T}|\eta_\theta^z|_\alpha+[u]_{\alpha,\mathscr{L}^2}+[u]_{0,\mathscr{L}^2}\big)\\
&\le C\varepsilon[u]_{2+\alpha,\mathscr{L}^2}+C(\varepsilon,\theta)[u]_{0,\mathscr{L}^2},\\
\|\mathscr{A}_7(\cdot,\cdot,z,\theta)\|_{\alpha,\mathscr{L}^2} &\le \|f\|_{\alpha,\mathscr{L}^2}+[f]_{0,\mathscr{L}^2}[\eta_\theta^z]_\alpha \le \Big(1+\frac{1}{\theta^\alpha}\Big)\|f\|_{\alpha,\mathscr{L}^2}.
\end{aligned}
$$

Choosing first θ and then ε to be sufficiently small, in view of inequality (4.1), we have

$$
\begin{aligned}
&\|u\|_{2+\alpha,\mathscr{L}^2}+\|v\|_{\alpha,\mathscr{L}^2}\\
&\quad\le \tfrac{1}{2}\big([u]_{2+\alpha,\mathscr{L}^2}+\|v\|_{\alpha,\mathscr{L}^2}\big)\\
&\qquad+C\big(\|\Phi\|_{1+\alpha,L^2}+\|f\|_{\alpha,\mathscr{L}^2}+\|u\|_{0,\mathscr{L}^2}+\|v\|_{0,\mathscr{L}^2}\big).
\end{aligned}
$$

Then

$$
\begin{aligned}
&\|u\|_{2+\alpha,\mathscr{L}^2}+\|v\|_{\alpha,\mathscr{L}^2}\\
&\quad\le C\big(\|\Phi\|_{1+\alpha,L^2}+\|f\|_{\alpha,\mathscr{L}^2}+\|u\|_{0,\mathscr{L}^2}+\|v\|_{0,\mathscr{L}^2}\big).
\end{aligned}\tag{4.2}
$$

Next, we estimate $\|v\|_{0,\mathscr{L}^2}$. BSPDE (1.1) can be written into the integral form:

$$u(t,x) = \Phi(x) + \int_t^T \big[a^{ij}(s,x)\,\partial^2_{ij}u(s,x) + b^i(s,x)\,\partial_i u(s,x) + c(s,x)u(s,x) + f(s,x) + \sigma(s,x)v(s,x)\big]\,ds - \int_t^T v(s,x)\,dW_s, \qquad dP\text{-a.s.} \tag{4.3}$$

For any fixed $x \in \mathbb{R}^n$, it is a BSDE with terminal condition $\Phi(x)$ and generator

$$a^{ij}(t,x)\,\partial^2_{ij}u(t,x) + b^i(t,x)\,\partial_i u(t,x) + c(t,x)U + f(t,x) + \sigma(t,x)V.$$

We have

$$\begin{aligned} E\int_0^T |v(t,x)|^2\,dt &\le CE\bigg[|\Phi(x)|^2 + \int_0^T |a^{ij}(t,x)\,\partial^2_{ij}u(t,x) + b^i(t,x)\,\partial_i u(t,x) + f(t,x)|^2\,dt\bigg] \\ &\le CE\bigg[|\Phi(x)|^2 + \int_0^T \big[|\partial^2_{ij}u(t,x)|^2 + |\partial_i u(t,x)|^2 + |f(t,x)|^2\big]\,dt\bigg]. \end{aligned}$$

By the interpolation inequalities in Lemma 2.1,

$$\begin{aligned} \|v\|_{0,\mathscr{L}^2} &\le C\big(\|\Phi\|_{0,L^2} + \|u\|_{2,\mathscr{L}^2} + \|u\|_{1,\mathscr{L}^2} + \|f\|_{0,\mathscr{L}^2}\big) \\ &\le C\varepsilon[u]_{2+\alpha,\mathscr{L}^2} + C(\varepsilon)\big(\|\Phi\|_{0,L^2} + \|u\|_{0,\mathscr{L}^2} + \|f\|_{0,\mathscr{L}^2}\big). \end{aligned}$$

In view of (4.2), choosing ε to be sufficiently small, we have

$$\|u\|_{2+\alpha,\mathscr{L}^2} + \|v\|_{\alpha,\mathscr{L}^2} \le C\big(\|\Phi\|_{1+\alpha,L^2} + \|f\|_{\alpha,\mathscr{L}^2} + \|u\|_{0,\mathscr{L}^2}\big). \tag{4.4}$$

We now establish a maximum principle of u. In BSDE (4.3), for any $(t,x) \in [0,T] \times \mathbb{R}^n$,

$$\begin{aligned} E\big[|u(t,x)|^2\big] &\le CE\bigg[|\Phi(x)|^2 + \int_t^T |a^{ij}(t,x)\,\partial^2_{ij}u(t,x) + b^i(t,x)\,\partial_i u(t,x) + f(t,x)|^2\,dt\bigg] \\ &\le CE\bigg[\int_t^T \big(|\partial^2_{ij}u(t,x)|^2 + |\partial_i u(t,x)|^2\big)\,dt\bigg] \\ &\quad + C\big(\|\Phi(x)\|^2_{0,L^2} + \|f\|^2_{0,\mathscr{L}^2}\big). \end{aligned} \tag{4.5}$$

For any $t \in [0, T]$, repeating all the preceding arguments on $[t, T]$, we see that the estimate (4.4) still holds for $\|\cdot\|_{m+\alpha,\mathscr{L}^2,t}$, that is,

$$\|u\|_{2+\alpha,\mathscr{L}^2,t} \le C\big(\|\Phi\|_{1+\alpha,L^2} + \|f\|_{\alpha,\mathscr{L}^2} + \|u\|_{0,\mathscr{L}^2,t}\big).$$

Taking supremum on both sides of (4.5), we have

$$\begin{aligned}\sup_x E\big[|u(t,x)|^2\big] &\le C\big(\|u\|_{2+\alpha,\mathscr{L}^2,t} + \|\Phi(x)\|_{0,L^2}^2 + \|f\|_{0,\mathscr{L}^2}^2\big)\\ &\le C\big(\|u\|_{0,\mathscr{L}^2,t} + \|\Phi(x)\|_{1+\alpha,L^2}^2 + \|f\|_{\alpha,\mathscr{L}^2}^2\big)\\ &\le C\bigg(\int_t^T \sup_x E|u(s,x)|^2\,ds + \|\Phi(x)\|_{1+\alpha,L^2}^2 + \|f\|_{\alpha,\mathscr{L}^2}^2\bigg).\end{aligned}$$

From Gronwall's inequality, we have

$$\|u\|_{0,\mathscr{L}^2} \le \int_0^T \sup_x E\big[|u(t,x)|^2\big]\,dt \le C\big(\|\Phi(x)\|_{1+\alpha,L^2}^2 + \|f\|_{\alpha,\mathscr{L}^2}^2\big). \tag{4.6}$$

By (4.4) and (4.6), we conclude that

$$\|u\|_{2+\alpha,\mathscr{L}^2} + \|v\|_{\alpha,\mathscr{L}^2} \le C\big(\|\Phi\|_{1+\alpha,L^2} + \|f\|_{\alpha,\mathscr{L}^2}\big). \tag{4.7}$$

In a similar way, we have

$$\begin{aligned}\|u\|_{\alpha,\mathscr{S}^2} &\le C\big[\|\Phi\|_{\alpha,L^2} + \|u\|_{2+\alpha,\mathscr{L}^2} + \|v\|_{\alpha,\mathscr{L}^2} + \|f\|_{\alpha,\mathscr{L}^2}\big]\\ &\le C\big[\|\Phi\|_{1+\alpha,L^2} + \|f\|_{\alpha,\mathscr{L}^2}\big].\end{aligned} \tag{4.8}$$

The proof is complete. □

Using the method of continuation (see Gilbarg and Trudinger [13], Theorem 5.2), we have from the Theorems 3.7 and 4.2 the following existence and uniqueness result for BSPDE (1.1).

THEOREM 4.3. *Let the Assumptions 2.1 and 2.2 be satisfied, and*

$$(\Phi, f) \in C^{1+\alpha}\big(\mathbb{R}^n, L^2(\Omega)\big) \times C^{\alpha}\big(\mathbb{R}^n, \mathscr{L}^2_{\mathbb{F}}(0,T)\big).$$

Then BSPDE (1.1) has a unique solution $(u, v) \in (C^{\alpha}_{\mathscr{S}^2} \cap C^{2+\alpha}_{\mathscr{L}^2}) \times C^{\alpha}_{\mathscr{L}^2}$. Moreover, there is a positive constant $C = C(\lambda, \Lambda, \alpha, n, d, T)$ such that

$$\|u\|_{\alpha,\mathscr{S}^2} + \|u\|_{2+\alpha,\mathscr{L}^2} + \|v\|_{\alpha,\mathscr{L}^2} \le C\big(\|\Phi\|_{1+\alpha,L^2} + \|f\|_{\alpha,\mathscr{L}^2}\big).$$

PROOF. Define

$$Lu := a^{ij}\,\partial^2_{ij}u + b^i\,\partial_i u + cu, \qquad Mv := \sigma v;$$

and for $\tau \in [0,1]$,

$$L_\tau u := (1-\tau)Lu + \tau\Delta u, \qquad M_\tau v := (1-\tau)Mv + \tau v,$$

with Δ being the Laplacian of $\mathbb{R}^n$.

Consider the following space:

$$\mathscr{J}^\alpha := \Big\{(u,v) \in \big(C^\alpha_{\mathscr{S}^2} \cap C^{2+\alpha}_{\mathscr{L}^2}\big) \times C^\alpha_{\mathscr{L}^2} : \forall t \in [0,T],$$

$$u(t,x) = \Phi(x) + \int_t^T F(s,x)\,ds - \int_t^T v(s,x)\,dW_s;$$

$$\text{for some } (\Phi, F) \in C^{1+\alpha}\big(\mathbb{R}^n, L^2(\Omega)\big) \times C^\alpha\big(\mathbb{R}^n, \mathscr{L}^2_{\mathbb{F}}(0,T)\big)\Big\},$$

equipped with the norm of $(u,v) \in \mathscr{J}^\alpha$:

$$\|(u,v)\|_{\mathscr{J}^\alpha} := \|u\|_{\alpha,\mathscr{S}^2} + \|u\|_{2+\alpha,\mathscr{L}^2} + \|v\|_{\alpha,\mathscr{L}^2} + \|\Phi\|_{1+\alpha,L^2} + \|F\|_{\alpha,\mathscr{L}^2}.$$

Then $\mathscr{J}^\alpha$ is a Banach space.

Define the mapping $\Pi_\tau : \mathscr{J}^\alpha \to C^{1+\alpha}(\mathbb{R}^n, L^2(\Omega)) \times C^\alpha(\mathbb{R}^n, \mathscr{L}^2_{\mathbb{F}}(0,T))$ as follows:

$$\Pi_\tau(u,v) := (\Phi, F - L_\tau u - M_\tau v), \qquad (u,v) \in \mathscr{J}^\alpha.$$

We have

$$\begin{aligned}
\|\Pi_\tau(u,v)\| &:= \|\Phi\|_{1+\alpha,L^2} + \|F - L_\tau u - M_\tau v\|_{\alpha,\mathscr{L}^2} \\
&\le \|\Phi\|_{1+\alpha,L^2} + \|F\|_{\alpha,\mathscr{L}^2} + \|L_\tau u\|_{\alpha,\mathscr{L}^2} + \|M_\tau v\|_{\alpha,\mathscr{L}^2} \\
&\le C\big(\|\Phi\|_{1+\alpha,L^2} + \|F\|_{\alpha,\mathscr{L}^2} + \|u\|_{2+\alpha,\mathscr{L}^2} + \|v\|_{\alpha,\mathscr{L}^2}\big) \\
&= C\|(u,v)\|_{\mathscr{J}^\alpha}.
\end{aligned}$$

On the other hand, for all $(t,x) \in [0,T] \times \mathbb{R}^n$, we have almost surely

$$u(t,x) = \Phi(x) + \int_t^T \big[L_\tau u + M_\tau v + (F - L_\tau u - M_\tau v)\big]\,ds - \int_t^T v(s,x)\,dW_s.$$

Then we have from Theorem 4.2 the following estimate:

$$\|u\|_{\alpha,\mathscr{S}^2} + \|u\|_{2+\alpha,\mathscr{L}^2} + \|v\|_{\alpha,\mathscr{L}^2} \le C\big(\|\Phi\|_{1+\alpha,L^2} + \|F - L_\tau u - M_\tau v\|_{\alpha,\mathscr{L}^2}\big).$$

Thus, we obtain the following inverse inequality:

$$\begin{aligned}
\|(u,v)\|_{\mathscr{J}^\alpha} &= \|u\|_{\alpha,\mathscr{S}^2} + \|u\|_{2+\alpha,\mathscr{L}^2} + \|v\|_{\alpha,\mathscr{L}^2} + \|\Phi\|_{1+\alpha,L^2} + \|F\|_{\alpha,\mathscr{L}^2} \\
&\le \|u\|_{\alpha,\mathscr{S}^2} + \|u\|_{2+\alpha,\mathscr{L}^2} + \|v\|_{\alpha,\mathscr{L}^2} + \|\Phi\|_{1+\alpha,L^2} \\
&\quad + \|F - L_\tau u - M_\tau v\|_{\alpha,\mathscr{L}^2} + \|L_\tau u\|_{\alpha,\mathscr{L}^2} + \|M_\tau v\|_{\alpha,\mathscr{L}^2} \\
&\le C\big(\|\Phi\|_{1+\alpha,L^2} + \|F - L_\tau u - M_\tau v\|_{\alpha,\mathscr{L}^2}\big) \\
&= C\|\Pi_\tau(u,v)\|.
\end{aligned}$$

Theorem 3.7 implies that Π_1 is onto. Then, in view of the method of continuation in Gilbarg and Trudinger [13], Theorem 5.2, page 75, Π_τ is also onto for all $\tau \in [0, 1)$. In particular, Π_0 is onto. The desired result follows. □

Similar to Proposition 3.8, we have the following Hölder time-continuity of u.

PROPOSITION 4.4. *Let Assumptions 2.1 and 2.2 be satisfied and* $(\Phi, f) \in C^{1+\alpha}(\mathbb{R}^n, L^2(\Omega)) \times C^\alpha(\mathbb{R}^n, \mathscr{L}^2_{\mathbb{F}}(0, T))$. *Let* $(u, v) \in (C^\alpha_{\mathscr{S}^2} \cap C^{2+\alpha}_{\mathscr{L}^2}) \times C^\alpha_{\mathscr{L}^2}$ *solve BSPDE (1.1). Then, for any* $\tau \in [0, T]$,

$$\|u(\cdot, \cdot) - u(\cdot - \tau, \cdot)\|_{\alpha, \mathscr{L}^2, \tau} \le C\tau^{1/2}\big(\|\Phi\|_{1+\alpha, L^2} + \|f\|_{\alpha, \mathscr{L}^2}\big),$$

where $C = C(\lambda, \Lambda, \alpha, T, n, d)$.

At the end of the section, we discuss the consequence of the preceding results on a deterministic PDE. Consider the deterministic functions

$$\begin{aligned} \Phi : \mathbb{R}^n \to \mathbb{R}, &\qquad a : [0, T] \times \mathbb{R}^n \to \mathcal{S}^n, \\ b : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n, &\qquad \sigma : [0, T] \times \mathbb{R}^n \to \mathbb{R}^d, \\ c, f : [0, T] \times \mathbb{R}^n \to \mathbb{R}. & \end{aligned}$$

As we know, a BSPDE with deterministic coefficients is in fact a deterministic PDE. Then the second unknown variable of BSPDE (1.1) turns out to be 0, and BSPDE (1.1) is in fact the following deterministic PDE:

$$\begin{cases} \partial_t u(t, x) = a^{ij}(t, x)\,\partial^2_{ij} u(t, x) + b^i(t, x)\,\partial_i u(t, x) & \\ \qquad\qquad + c(t, x)u(t, x) + f(t, x), & (t, x) \in [0, T) \times \mathbb{R}^n; \\ u(T, x) = \Phi(x), & x \in \mathbb{R}^n, \end{cases} \tag{4.9}$$

which does not involve the coefficient σ anymore.

Note that the classical Hölder space $C^{m+\alpha}(\mathbb{R}^n)$ consists of all the deterministic elements of the Hölder space $C^{m+\alpha}(\mathbb{R}^n, L^p(\Omega))$, and the two Hölder functional spaces $C^{m+\alpha}(\mathbb{R}^n, L^p(0, T; \mathbb{R}^\iota))$ and $C^{m+\alpha}(\mathbb{R}^n, C[0, T])$ consist of all the deterministic elements of the two Hölder functional spaces

$$C^{m+\alpha}\big(\mathbb{R}^n, \mathscr{L}^p_{\mathbb{F}}(0, T; \mathbb{R}^\iota)\big) \quad \text{and} \quad C^{m+\alpha}\big(\mathbb{R}^n, \mathscr{S}^p_{\mathbb{F}}[0, T]\big),$$

respectively. Assumption 2.2 is replaced with the following one.

ASSUMPTION 4.1. The functions

$$a \in C^\alpha\big(\mathbb{R}^n, L^\infty(0, T; \mathbb{R}^{n\times n})\big), \qquad b \in C^\alpha\big(\mathbb{R}^n, L^\infty(0, T; \mathbb{R}^n)\big),$$

and $c \in C^\alpha(\mathbb{R}^n, L^\infty(0, T))$. There is a constant $\Lambda > 0$ such that

$$\|a\|_{\alpha, L^\infty} + \|b\|_{\alpha, L^\infty} + \|c\|_{\alpha, L^\infty} \le \Lambda.$$

In view of Theorem 4.3, we have the following existence, uniqueness and regularity result for PDE (4.9).

PROPOSITION 4.5. *Let the Assumptions 2.1 and 4.1 be satisfied, and*

$$(\Phi, f) \in C^{1+\alpha}(\mathbb{R}^n) \times C^{\alpha}(\mathbb{R}^n, L^2(0,T)).$$

Then PDE (4.9) has a unique solution

$$u \in C^{\alpha}(\mathbb{R}^n, C[0,T]) \cap C^{2+\alpha}(\mathbb{R}^n, L^2(0,T))$$

such that

$$\|u\|_{\alpha,C} + \|u\|_{2+\alpha,L^2} \le C(|\Phi|_{1+\alpha} + \|f\|_{\alpha,L^2}),$$

where $C = C(\lambda, \Lambda, \alpha, n, d, T)$.

The preceding proposition shows that the solution u to PDE (4.9) is $(2+\alpha)$-Hölder continuous if Φ is $(1+\alpha)$-Hölder continuous and f is α-Hölder continuous. It seems to have a novelty as explained in the following remark.

REMARK 4.1. Mikulevicius [19] studies the Cauchy problem of an SPDE in a functional Hölder space, and includes the following a priori estimate for PDE (4.9): if $\Phi = 0$, $f(t,\cdot) \in C^{\alpha}(\mathbb{R}^n)$ for $t \in [0,T]$, and $\sup_t |f(t,\cdot)|_{\alpha} < +\infty$, then PDE (4.9) has a unique solution u such that

$$u(t,\cdot) \in C^{2+\alpha}(\mathbb{R}^n) \qquad \forall t \in [0,T] \quad \text{and} \quad \sup_t |u(t,\cdot)|_{2+\alpha} < C \sup_t |f(t,\cdot)|_{\alpha}.$$

In contrast, in Proposition 4.5 we require $f \in C^{\alpha}(\mathbb{R}^n, L^2(0,T))$ and assert $u \in C^{2+\alpha}(\mathbb{R}^n, L^2(0,T))$.

5. Semi-linear BSPDEs. In this section, consider the following semi-linear BSPDE:

$$\text{(5.1)} \quad \begin{cases} -du(t,x) = [a^{ij}(t,x)\,\partial^2_{ij}u(t,x) + f(t,x,\nabla u(t,x), u(t,x), v(t,x))]\,dt \\ \qquad\qquad - v(t,x)\,dW_t, \qquad (t,x) \in [0,T) \times \mathbb{R}^n, \\ u(T,x) = \Phi(x), \qquad\qquad x \in \mathbb{R}^n. \end{cases}$$

Here, $a : [0,T] \times \mathbb{R}^n \to \mathcal{S}^n$ satisfies both super-parabolicity and boundedness Assumptions 2.1 and 2.2, $f : [0,T] \times \Omega \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ is jointly measurable, and $f(\cdot, x, q, u, v)$ is $\mathbb{F}$-adapted for any $(x,q,u,v) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d$.

We make the following Lipschitz assumption on f.

ASSUMPTION 5.1. $f_0(\cdot,\cdot) := f(\cdot,\cdot,0,0,0) \in C^{\alpha}(\mathbb{R}^n, \mathscr{L}^2_{\mathbb{F}}(0,T))$, and there is a constant $L > 0$ such that

$$\begin{aligned} &|f(t,x,q_1,u_1,v_1) - f(t,x,q_2,u_2,v_2)| \\ &\quad \le L(|q_1 - q_2| + |u_1 - u_2| + |v_1 - v_2|), \qquad dt \times dP\text{-a.e., a.s.} \end{aligned}$$

for any $(q_1,u_1,v_1), (q_2,u_2,v_2) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d$ and $x \in \mathbb{R}^d$.

Then we have the following existence, uniqueness and regularity on semi-linear BSPDE (5.1).

THEOREM 5.1. *Let the Assumptions 2.1, 2.2 and 5.1 be satisfied, and* $\Phi \in C^{1+\alpha}(\mathbb{R}^n, L^2(\Omega))$. *Then the semi-linear BSPDE* (5.1) *has a unique solution* $(u, v) \in (C^{\alpha}_{\mathscr{S}^2} \cap C^{2+\alpha}_{\mathscr{L}^2}) \times C^{\alpha}_{\mathscr{L}^2}$. *Moreover,*

$$\|u\|_{\alpha,\mathscr{S}^2} + \|u\|_{2+\alpha,\mathscr{L}^2} + \|v\|_{\alpha,\mathscr{L}^2} \le C\big(\|\Phi\|_{1+\alpha,L^2} + \|f_0\|_{\alpha,\mathscr{L}^2}\big),$$

where $C = C(\lambda, \Lambda, \alpha, n, d, T)$.

The proof requires the following two additional preliminary lemmas. Consider the following linear BSPDE:

$$\begin{cases} -du(t,x) = \big[a^{ij}(t,x)\,\partial^2_{ij}u(t,x) - \beta u(t,x) + f(t,x)\big]\,dt \\ \qquad\qquad\qquad - v(t,x)\,dW_t, \qquad (t,x) \in [0,T) \times \mathbb{R}^n, \\ u(T,x) = 0, \qquad\qquad\qquad\qquad x \in \mathbb{R}^n, \end{cases} \tag{5.2}$$

where $a : [0,T] \times \mathbb{R}^n \to \mathcal{S}^n$ is the same as before, and $\beta > 0$ is a constant. When $a(t,x) \equiv a(t)$, define

$$G^{\beta}_{s,t}(x) := e^{-\beta(s-t)} G_{s,t}(x), \qquad 0 \le t \le s \le T.$$

LEMMA 5.2. *For a universal constant* $C = C(\lambda, \Lambda, \alpha, \gamma, n, T)$, *we have*:

(i) *For* $\alpha \in (0,1)$ *and* $\gamma \in \Gamma$ *such that* $|\gamma| \le 2$,

$$\int_\tau^s \int_{\mathbb{R}^n} \big|D^\gamma G^{\beta}_{s,t}(x)\big| |x|^\alpha \,dx\,dt \le C\beta^{-1+(|\gamma|-\alpha)/2}, \qquad T \ge s > \tau \ge 0. \tag{5.3}$$

(ii) *For* $\gamma \in \Gamma$ *such that* $|\gamma| = 2$ *and* $0 \le \tau \le s \le T$,

$$\begin{aligned} \int_\tau^s \left| \int_{|y| \le \eta} D^\gamma G^{\beta}_{s,t}(y)\,dy \right| dt &= \int_\tau^s \left| \int_{|y| \ge \eta} D^\gamma G^{\beta}_{s,t}(y)\,dy \right| dt \\ &\le C\beta^{-1} \qquad \forall \eta > 0. \end{aligned} \tag{5.4}$$

(iii) *For* $\gamma \in \Gamma$ *such that* $|\gamma| = 2$,

$$\int_{|y| \le \eta} \sup_{\tau \le s} \int_\tau^s \big|D^\gamma G^{\beta}_{s,t}(y)\big| |y|^\alpha \,dt\,dy \le C\beta^{-1}\eta^\alpha \qquad \forall \eta > 0. \tag{5.5}$$

(iv) *For any* $x, \bar{x} \in \mathbb{R}^n$ *and* $\gamma \in \Gamma$ *such that* $|\gamma| = 2$,

$$\begin{aligned} &\int_{|y-x| > \eta} \sup_{\tau \le s} \int_\tau^s \big|D^\gamma G^{\beta}_{s,t}(x-y) - D^\gamma G^{\beta}_{s,t}(\bar{x}-y)\big| |\bar{x}-y|^\alpha \,dt\,dy \\ &\qquad \le C\beta^{-1}|x-\bar{x}|^\alpha \qquad \forall \eta > 0. \end{aligned} \tag{5.6}$$

LEMMA 5.3. *Let $f \in C^\alpha(\mathbb{R}^n, \mathscr{L}^2_{\mathbb{F}}(0,T))$. If $(u,v) \in (C^\alpha_{\mathscr{S}^2} \cap C^{2+\alpha}_{\mathscr{L}^2}) \times C^\alpha_{\mathscr{L}^2}$ is the solution of BSPDE* (5.2), *then*

$$\|u\|_{\alpha,\mathscr{S}^2} + \|u\|_{2+\alpha,\mathscr{L}^2} + \|v\|_{\alpha,\mathscr{L}^2} \le C(\beta)\|f\|_{\alpha,\mathscr{L}^2},$$

where $C(\beta) := C(\beta,\lambda,\Lambda,\alpha,n,d,T) > 0$, and converges to zero as $\beta \to \infty$.

PROOF. *Step* 1 [$a(t,x) \equiv a(t)$]. Proceeding similarly as in the proof of Lemma 3.1 and the Theorems 3.3 and 3.7, we have that the pair (u,v) defined for each $x \in \mathbb{R}^d$ by

$$u(t,x) := \int_t^T \int_{\mathbb{R}^n} G^\beta_{s,t}(x-y) Y(t;s,y)\,dy\,ds \qquad \forall t \in [0,T], dP\text{-a.s.},$$

and

$$v_l(t,x) := \int_t^T \int_{\mathbb{R}^n} G^\beta_{r,t}(x-y) g_l(s;r,y)\,dy\,dr,$$

$$dt \times dP\text{-a.e., a.s., } l = 1,\dots,d,$$

is the unique solution to the linear BSPDE (5.2) with

$$Y(t;\tau,x) := f(\tau,x) - \int_t^\tau g_l(r;\tau,x)\,dW^l_r \qquad \forall t \le \tau.$$

In view of the estimates of Lemma 5.2, proceeding similarly as in the proof of the Lemmas 3.4 and 3.5 and Theorem 3.6, we have

$$\|u\|_{\alpha,\mathscr{S}^2} + \|u\|_{2+\alpha,\mathscr{L}^2} + \|v\|_{\alpha,\mathscr{L}^2} \le C(\beta)\|f\|_{\alpha,\mathscr{L}^2},$$

where $C(\beta) := C(\beta,\lambda,\Lambda,\alpha,n,d,T) > 0$ is sufficiently small for sufficiently large β.

Step 2 [$(a^{ij})_{n\times n}$ depends on x]. Using the freezing coefficients method as in Theorem 4.2, we have the desired result. □

PROOF OF THEOREM 5.1. For any $(U_1,V_1) \in (C^\alpha_{\mathscr{S}^2} \cap C^{2+\alpha}_{\mathscr{L}^2}) \times C^\alpha_{\mathscr{L}^2}$, $f(\cdot,\cdot,\nabla U_1(\cdot,\cdot), U_1(\cdot,\cdot), V_1(\cdot,\cdot)) \in C^\alpha(\mathbb{R}^n, \mathscr{L}^2_{\mathbb{F}}(0,T))$ because of Assumption 5.1 for f. In view of Theorem 4.3,

$$\begin{cases} -du_1(t,x) = \big[a^{ij}(t,x)\,\partial^2_{ij}u_1(t,x) \\ \qquad\qquad + f\big(t,x,\nabla U_1(t,x), U_1(t,x), V_1(t,x)\big)\big]\,dt \\ \qquad\qquad - v_1(t,x)\,dW_t, \qquad (t,x) \in [0,T) \times \mathbb{R}^n, \\ u_1(T,x) = \Phi(x), \qquad x \in \mathbb{R}^n \end{cases} \tag{5.7}$$

has a unique solution $(u_1,v_1) \in (C^\alpha_{\mathscr{S}^2} \cap C^{2+\alpha}_{\mathscr{L}^2}) \times C^\alpha_{\mathscr{L}^2}$. For any $(U_2,V_2) \in (C^\alpha_{\mathscr{S}^2} \cap C^{2+\alpha}_{\mathscr{L}^2}) \times C^\alpha_{\mathscr{L}^2}$, denote $(u_2,v_2) \in (C^\alpha_{\mathscr{S}^2} \cap C^{2+\alpha}_{\mathscr{L}^2}) \times C^\alpha_{\mathscr{L}^2}$ as the solution of equation (5.7) with (U_1,V_1) replaced by (U_2,V_2). Define

$$\bar u(t,x) := u_1(t,x) - u_2(t,x), \qquad \bar U(t,x) := U_1(t,x) - U_2(t,x),$$
$$\bar v(t,x) := v_1(t,x) - v_2(t,x), \qquad \bar V(t,x) := V_1(t,x) - V_2(t,x)$$

and

$$\bar{f}(t,x) := f\big(t,x,\nabla U_1(t,x),U_1(t,x),V_1(t,x)\big) \\ - f\big(t,x,\nabla U_2(t,x),U_2(t,x),V_2(t,x)\big).$$

Then we have

$$\begin{cases} -d\big[e^{\beta t}\bar{u}(t,x)\big] = \big(a^{ij}(t,x)\,\partial^2_{ij}\big[e^{\beta t}\bar{u}(t,x)\big] \\ \qquad\qquad - \beta e^{\beta t}\bar{u}(t,x) + e^{\beta t}\bar{f}(t,x)\big)\,dt \\ \qquad\qquad - e^{\beta t}\bar{v}(t,x)(t,x)\,dW_t, \qquad (t,x)\in[0,T)\times\mathbb{R}^n, \\ e^{\beta T}\bar{u}(T,x) = 0, \qquad x\in\mathbb{R}^n. \end{cases} \tag{5.8}$$

In view of Lemma 5.3, we have

$$\begin{aligned} &\|e^{\beta\cdot}\bar{u}\|_{\alpha,\mathscr{S}^2} + \|e^{\beta\cdot}\bar{u}\|_{2+\alpha,\mathscr{L}^2} + \|e^{\beta\cdot}\bar{v}\|_{\alpha,\mathscr{L}^2} \\ &\quad\le C(\beta)\|e^{\beta\cdot}\bar{f}\|_{\alpha,\mathscr{L}^2} \\ &\quad\le C(\beta)L\big[\|e^{\beta\cdot}\bar{U}\|_{\alpha,\mathscr{S}^2} + \|e^{\beta\cdot}\bar{U}\|_{2+\alpha,\mathscr{L}^2} + \|e^{\beta\cdot}\bar{V}\|_{\alpha,\mathscr{L}^2}\big], \end{aligned}$$

with $C(\beta)L < 1$ for a sufficiently large β. Since the weighted norm $\|e^{\beta\cdot}u\|_{\alpha,\mathscr{S}^2} + \|e^{\beta\cdot}u\|_{2+\alpha,\mathscr{L}^2} + \|e^{\beta\cdot}v\|_{\alpha,\mathscr{L}^2}$ is equivalent to the original one $\|u\|_{\alpha,\mathscr{S}^2} + \|u\|_{2+\alpha,\mathscr{L}^2} + \|v\|_{\alpha,\mathscr{L}^2}$ in $(C^\alpha_{\mathscr{S}^2}\cap C^{2+\alpha}_{\mathscr{L}^2})\times C^\alpha_{\mathscr{L}^2}$, the semi-linear BSPDE (5.1) has a unique solution $(u,v)\in(C^\alpha_{\mathscr{S}^2}\cap C^{2+\alpha}_{\mathscr{L}^2})\times C^\alpha_{\mathscr{L}^2}$. The desired estimate is proved in a similar way. □

Acknowledgments. The second author thanks Xicheng Zhang and Jinniao Qiu for their valuable comments. Both authors would like to thank both anonymous reviewers and the Associate Editor for their careful reading and very helpful comments and suggestions.

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