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STRONG SUPERMARTINGALES AND LIMITS OF NONNEGATIVE MARTINGALES

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Given a sequence $(M^n)_{n=1}^{\infty}$ of nonnegative martingales starting at $M_0^n = 1$, we find a sequence of convex combinations $(\widetilde{M}^n)_{n=1}^{\infty}$ and a limiting process X such that $(\widetilde{M}_\tau^n)_{n=1}^{\infty}$ converges in probability to X_τ, for all finite stopping times τ. The limiting process X then is an optional strong supermartingale. A counterexample reveals that the convergence in probability cannot be replaced by almost sure convergence in this statement. We also give similar convergence results for sequences of optional strong supermartingales $(X^n)_{n=1}^{\infty}$, their left limits $(X_-^n)_{n=1}^{\infty}$ and their stochastic integrals $(\int \varphi\, dX^n)_{n=1}^{\infty}$ and explain the relation to the notion of the Fatou limit.

1. Introduction. Komlós's lemma (see [12, 18] and [4]) is a classical result on the convergence of random variables that can be used as a substitute for compactness. It has turned out to be very useful, similarly to the Bolzano–Weierstrass theorem, and has become a work horse of stochastic analysis in the past decades. In this paper, we generalize this result to work directly with nonnegative martingales and convergence in probability simultaneously at all finite stopping times.

Let us briefly explain this in more detail. Komlós's subsequence theorem states that given a bounded sequence $(f_n)_{n=1}^{\infty}$ of random variables in $L^1(P)$, there exists a random variable $f \in L^1(P)$ and a subsequence $(f_{n_k})_{k=1}^{\infty}$ such that the Cesàro means of any subsubsequence $(f_{n_{k_j}})_{j=1}^{\infty}$ converge almost surely to f. It quickly follows that there exists a sequence $(\tilde{f}_n)_{n=1}^{\infty}$ of convex combinations $\tilde{f}_n \in \operatorname{conv}(f_n, f_{n+1}, \dots)$ that converges to f almost surely that we refer to as Komlós's lemma.

Replacing the almost sure convergence by the concept of *Fatou convergence*, Föllmer and Kramkov [9] obtained the following variant of Komlós's lemma for stochastic processes. Given a sequence $(M^n)_{n=1}^{\infty}$ of nonnegative martingales $M^n = (M_t^n)_{0\le t\le 1}$ starting at $M_0^n = 1$, there exists a sequence $(\overline{M}^n)_{n=1}^{\infty}$ of convex

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combinations $\overline{M}^n \in \operatorname{conv}(M^n, M^{n+1}, \dots)$ and a nonnegative càdlàg supermartingale $\overline{X} = (\overline{X}_t)_{0 \le t \le 1}$ starting at $X_0 = 1$ such that $\overline{M}^n$ is Fatou convergent along the rationals $\mathbb{Q} \cap [0, 1]$ to $\overline{X}$ in the sense that

$$\overline{X}_t = \overline{\lim_{q \in \mathbb{Q} \cap [0,1], q \downarrow t}} \, \overline{\lim_{n \to \infty}} \, \overline{M}^n_q = \underline{\lim_{q \in \mathbb{Q} \cap [0,1], q \downarrow t}} \, \underline{\lim_{n \to \infty}} \, \overline{M}^n_q, \qquad P\text{-a.s.},$$

for all $t \in [0, 1)$ and $\overline{X}_1 = \lim_{n \to \infty} \overline{M}^n_1$.

In this paper, we are interested in a different version of Komlós's lemma for nonnegative martingales in the following sense. Given the sequence $(M^n)_{n=1}^\infty$ of nonnegative martingales as above and a finite stopping time τ defining $f_n := M^n_\tau$ gives a sequence of nonnegative random variables that is bounded in $L^1(P)$. By Komlós's lemma there exist convex combinations $\widetilde{M}^n \in \operatorname{conv}(M^n, M^{n+1}, \dots)$ such that $\widetilde{M}^n_\tau$ converges in probability to some random variable f_τ. The question is then, if we can find *one* sequence $(\widetilde{M}^n)_{n=1}^\infty$ of convex combinations $\widetilde{M}^n \in \operatorname{conv}(M^n, M^{n+1}, \dots)$ and a stochastic process $X = (X_t)_{0 \le t \le 1}$ such that we have that $\widetilde{M}^n_\tau$ converges to X_τ in probability for *all* finite stopping times τ.

Our first main result (Theorem 2.6) shows that this is possible and that the limiting process $X = (X_t)_{0 \le t \le 1}$ is an *optional strong supermartingale*. These supermartingales have been introduced by Mertens [14] and are optional processes that satisfy the supermartingale inequality for all finite stopping times. This indicates that optional strong supermartingales are the natural processes for our purpose to work with, and we expand in Theorem 2.7 our convergence result from martingales $(M^n)_{n=1}^\infty$ to optional strong supermartingales $(X^n)_{n=1}^\infty$.

In dynamic optimization problems our results can be used as substitute for compactness; compare, for example, [5, 9, 11, 13, 17]. Here the martingales M^n are usually a minimizing sequence of density processes of equivalent martingale measures for the dual problem or, as in [5] and [9], the wealth processes of self-financing trading strategies.

At a fixed stopping time the convergence in probability can always be strengthened to almost sure convergence by simply passing to a subsequence. By means of a counterexample (Proposition 4.1) we show that this is not possible for all stopping times simultaneously.

Conversely, one can ask what the smallest class of stochastic processes is that is closed under convergence in probability at all finite stopping times and contains all bounded martingales. Our second contribution (Theorem 2.8) is to show that this is precisely the class of all optional strong supermartingales provided the underlying probability space is sufficiently rich to support a Brownian motion.

As the limiting strong supermartingale of a sequence of martingales in the sense of convergence in probability at all finite stopping times is no longer a semimartingale, we need to restrict the integrands to be predictable finite variation processes $\varphi = (\varphi_t)_{0 \le t \le 1}$ to come up with a similar convergence result for stochastic integrals in Proposition 2.12. For this, we need to extend our convergence result to ensure the convergence of the left limit processes $(X^n_-)_{n=1}^\infty$ in probability at all finite stopping times to a limiting process $X^{(0)} = (X^{(0)})_{0\le t\le 1}$ as well after possibly passing once more to convex combinations. It turns out that $X^{(0)}$ is a *predictable strong supermartingale* that does, in general, *not* coincide with the left limit process X_- of the limiting optional strong supermartingale X. The notion of a predictable strong supermartingale has been introduced by Chung and Glover [2] and refers to predictable processes that satisfy the supermartingale inequality for all *predictable* stopping times. Using instead of the time interval $I = [0, 1]$ its *Alexandroff double arrow space* $\widetilde{I} = [0, 1] \times \{0, 1\}$ as index set we can merge both limiting strong supermartingales into one supermartingale $X = (X_{\tilde{t}})_{\tilde{t}\in\widetilde{I}}$ indexed by $\widetilde{I}$.

Our motivation for studying these questions comes from portfolio optimization under transaction costs in mathematical finance in [3]. While for the problem without transaction costs the solution to the dual problem is always attained as a Fatou limit, the dual optimizer under transaction costs is in general a truly làdlàg optional strong supermartingale. So we expect our results naturally to appear whenever one is optimizing over nonnegative martingales that are not uniformly integrable or stable under concatenation, and they might find other applications as well.

The paper is organized as follows. We formulate the problem and state our main results in Section 2. The proofs are given in Sections 3, 5, 6 and 7. Section 4 provides the counterexample that our convergence results cannot be strengthened to almost sure convergence.

2. Formulation of the problem and main results. Let $(\Omega, \mathcal{F}, P)$ be a probability space and $L^0(P) = L^0(\Omega, \mathcal{F}, P)$ the space of all real-valued random variables. As usual we equip $L^0(P)$ with the topology of convergence in probability and denote by $L^0_+(P) = L^0(\Omega, \mathcal{F}, P; \mathbb{R}_+)$ its positive cone. We call a subset A of $L^0(P)$ bounded in probability or simply bounded in $L^0(P)$, if $\lim_{m\to\infty} \sup_{f\in A} P(|f| > m) = 0$.

Komlós's subsequence theorem (see [12] and [18]) states the following.

THEOREM 2.1. *Let $(f_n)_{n=1}^\infty$ be a bounded sequence of random variables in $L^1(\Omega, \mathcal{F}, P)$. Then there exists a subsequence $(f_{n_k})_{k=1}^\infty$ and a random variable f such that the Cesàro means $\frac{1}{J}\sum_{j=1}^J f_{n_{k_j}}$ of any subsubsequence $(f_{n_{k_j}})_{j=1}^\infty$ converge P-almost surely to f, as $J \to \infty$.*

In applications this result is often used in the following variant that we also refer to as Komlós's lemma; compare Lemma A.1 in [4].

COROLLARY 2.2. *Let $(f_n)_{n=1}^\infty$ be a sequence of nonnegative random variables that is bounded in $L^1(P)$. Then there exists a sequence $(\tilde{f}_n)_{n=1}^\infty$ of convex combinations*

$$\tilde{f}_n \in \operatorname{conv}(f_n, f_{n+1}, \dots)$$

and a nonnegative random variable $f \in L^1(P)$ such that $\tilde{f}_n \overset{P\text{-a.s.}}{\longrightarrow} f$.

As has been illustrated by the work of Kramkov and Schachermayer [13] and Žitković [19] (see also [17]) Komlós's lemma can be used as a substitute for compactness, for example, in the derivation of minimax theorems for Lagrange functions, where the optimization is typically over convex sets. Replacing the P-almost sure convergence by the concept of *Fatou convergence* Föllmer and Kramkov [9] used Komlós's lemma to come up with a similar convergence result for stochastic processes. For this, we equip the probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le 1}$ satisfying the usual conditions of right continuity and completeness and let $(M^n)_{n=1}^\infty$ be a sequence of nonnegative martingales $M^n = (M_t^n)_{0\le t\le 1}$ starting at $M_0^n = 1$. For all unexplained notation from the general theory of stochastic processes and stochastic integration, we refer to the book of Dellacherie and Meyer [8].

The construction of the Fatou limit by Föllmer and Kramkov can be summarized as in the following proposition.

PROPOSITION 2.3 (Lemma 5.2 of [9]). *Let $(M^n)_{n=1}^\infty$ be a sequence of nonnegative martingales $M^n = (M_t^n)_{0\le t\le 1}$ starting at $M_0^n = 1$. Then there exists a sequence $(\overline{M}^n)_{n=1}^\infty$ of convex combinations*

$$\overline{M}^n \in \operatorname{conv}(M^n, M^{n+1}, \dots)$$

and nonnegative random variables Z_q for $q \in \mathbb{Q} \cap [0, 1]$ such that:

(1) $\overline{M}_q^n \overset{P\text{-a.s.}}{\longrightarrow} Z_q$ *for all* $q \in \mathbb{Q} \cap [0, 1]$;

(2) *the process $\overline{X} = (\overline{X}_t)_{0\le t\le 1}$ given by*

$$\overline{X}_t := \lim_{q\in\mathbb{Q}\cap[0,1], q\downarrow t} Z_q \quad \text{and} \quad \overline{X}_1 = Z_1 \tag{2.1}$$

is a càdlàg supermartingale;

(3) *the process $\overline{X} = (\overline{X}_t)_{0\le t\le 1}$ is the* Fatou limit *of the sequence $(\overline{M}^n)_{n=1}^\infty$ along $\mathbb{Q} \cap [0, 1]$, that is*,

$$\overline{X}_t = \varlimsup_{q\in\mathbb{Q}\cap[0,1], q\downarrow t} \varlimsup_{n\to\infty} \overline{M}_q^n = \varliminf_{q\in\mathbb{Q}\cap[0,1], q\downarrow t} \varliminf_{n\to\infty} \overline{M}_q^n, \qquad P\text{-a.s.}, \quad \text{and}$$

$$\overline{X}_1 = \lim_{n\to\infty} \overline{M}_1^n.$$

Here it is important to note that $\lim_{q\in\mathbb{Q}\cap[0,1], q\downarrow t}$ denotes the limit to t through all $q \in \mathbb{Q} \cap [0, 1]$ that are *strictly* bigger than t. Therefore we do not have in general that $\overline{X}_t = \lim_{n\to\infty} \overline{M}_t^n$ for $t \in [0, 1)$, not even for $t \in \mathbb{Q} \cap [0, 1]$, as is illustrated in the simple example below.

EXAMPLE 2.4. Let $(Y_n)_{n=1}^\infty$ be a sequence of random variables taking values in $\{0, n\}$ such that $P[Y_n = n] = \frac{1}{n}$ and define a sequence $(M^n)_{n=1}^\infty$ of martingales $M^n = (M_t^n)_{0\le t\le 1}$ by

$$M_t^n = 1 + (Y^n - 1)\mathbb{1}_{]\!]1/2(1+1/n),1]\!]}(t).$$

Then M_t^n converges to $\mathbb{1}_{[\![0,1/2]\!]}(t)$ for each $t \in [0,1]$. However, the càdlàg Fatou limit is $\overline{X} = \mathbb{1}_{[\![0,1/2[\![}(t)$.

The convergence, of course, also fails at stopping times in general. This motivates us to ask for a different extension of Komlós's lemma to nonnegative martingales in the following sense. Let $(M^n)_{n=1}^{\infty}$ be again a sequence of nonnegative martingales $M^n = (M_t^n)_{0 \le t \le 1}$ starting at $M_0^n = 1$ and τ a finite stopping time. Then defining $f_n := M_\tau^n$ gives a sequence $(f_n)_{n=1}^{\infty}$ of nonnegative random variables that are bounded in $L^1(P)$. By Komlós's lemma there exist convex combinations $\widetilde{M}^n \in \operatorname{conv}(M^n, M^{n+1}, \dots)$ and a nonnegative random variable f_τ such that

$$\widetilde{M}_\tau^n =: \tilde{f}_n \overset{P\text{-a.s.}}{\longrightarrow} f_\tau.$$

The questions are then:

(1) Can we find *one* sequence $(\widetilde{M}^n)_{n=1}^{\infty}$ of convex combinations

$$\widetilde{M}^n \in \operatorname{conv}\bigl(M^n, M^{n+1}, \dots\bigr)$$

such that, for *all* finite stopping times τ, we have

$$\widetilde{M}_\tau^n \overset{P\text{-a.s.}}{\longrightarrow} f_\tau \tag{2.2}$$

for some random variables f_τ that may depend on the stopping times τ?

(2) If (1) is possible, can we find a stochastic process $X = (X_t)_{0 \le t \le 1}$ such that $X_\tau = f_\tau$ for all finite stopping times τ?

(3) If such a process $X = (X_t)_{0 \le t \le 1}$ as in (2) exists, what kind of process is it?

Let us start with the last question. If such a process $X = (X_t)_{0 \le t \le 1}$ exists, it follows from Fatou's lemma that it is (up to optional measurability) an optional strong supermartingale.

DEFINITION 2.5. A real-valued stochastic process $X = (X_t)_{0 \le t \le 1}$ is called an *optional strong supermartingale*, if:

(1) X is optional;
(2) X_τ is integrable for every $[0,1]$-valued stopping time τ;
(3) for all stopping times σ and τ with $0 \le \sigma \le \tau \le 1$, we have

$$X_\sigma \ge E[X_\tau | \mathcal{F}_\sigma].$$

These processes have been introduced by Mertens [14] as a generalization of the notion of a càdlàg (right continous with left limits) supermartingale that one is usually working with. Indeed, by the optional sampling theorem each càdlàg supermartingale is an optional strong supermartingale, but not every optional strong supermartingale has a càdlàg modification. For example, every *deterministic* decreasing function $(X_t)_{0\le t\le 1}$ is an optional strong supermartingale, but there is little reason why it should be càdlàg. However, by Theorem 4 in Appendix I in [8], every optional strong supermartingale is indistinguishable from a làdlàg (left and right limits) process, and so we can assume without loss of generality that all optional strong supermartingales we consider in this paper are làdlàg. Similarly to the Doob–Meyer decomposition in the càdlàg case, every optional strong supermartingale X has a unique decomposition

$$X = M - A \tag{2.3}$$

into a local martingale M and a nondecreasing predictable process A starting at 0. This decomposition is due to Mertens [14] (compare also Theorem 20 in Appendix I in [8]) and is therefore called the *Mertens decomposition*. Note that, under the usual conditions of completeness and right continuity of the filtration, we can and do choose a càdlàg modification of the local martingale M in (2.3). On the other hand, the nondecreasing process A is in particular làdlàg.

For làdlàg processes $X = (X_t)_{0\le t\le 1}$ we denote by $X_{t+} := \lim_{h\searrow 0} X_{t+h}$ and $X_{t-} := \lim_{h\searrow 0} X_{t-h}$ the right and left limits and by $\Delta_+ X_t := X_{t+} - X_t$ and $\Delta X_t := X_t - X_{t-}$ the right and left jumps. We also use the convention that $X_{0-} = 0$ and $X_{1+} = X_1$.

After these preparations we have now everything in place to formulate our main results. The proofs will be given in the Sections 3, 5, 6 and 7.

THEOREM 2.6. *Let $(M^n)_{n=1}^\infty$ be a sequence of nonnegative càdlàg martingales $M^n = (M_t^n)_{0\le t\le 1}$ starting at $M_0^n = 1$. Then there is a sequence $(\widetilde{M}^n)_{n=1}^\infty$ of convex combinations*

$$\widetilde{M}^n \in \operatorname{conv}(M^n, M^{n+1}, \dots)$$

and a nonnegative optional strong supermartingale $X = (X_t)_{0\le t\le 1}$ such that, for every $[0, 1]$-valued stopping time τ, we have that

$$\widetilde{M}_\tau^n \xrightarrow{P} X_\tau. \tag{2.4}$$

Combining the above with a similar convergence result for predictable finite variation processes by Campi and Schachermayer [1] allows us to extend our convergence result to optional strong supermartingales by using the Mertens decomposition. Theorem 2.6 is thus only a special case of the following result.

THEOREM 2.7. *Let $(X^n)_{n=1}^\infty$ be a sequence of nonnegative optional strong supermartingales $X^n = (X_t)_{0\le t\le 1}$ starting at $X_0^n = 1$. Then there is a sequence $(\widetilde{X}^n)_{n=1}^\infty$ of convex combinations*

$$\widetilde{X}^n \in \operatorname{conv}(X^n, X^{n+1}, \dots)$$

and a nonnegative optional strong supermartingale $X = (X_t)_{0 \le t \le 1}$ *such that, for every* $[0, 1]$*-valued stopping time* τ, *we have convergence in probability, that is,*

$$\widetilde{X}^n_\tau \xrightarrow{P} X_\tau. \tag{2.5}$$

We thank Kostas Kardaras for indicating that convergence (2.5) is *topological*. It corresponds to the weak topology that is generated on the space of optional processes by the topology of $L^0(P)$ and all evaluation mappings $e_\tau(X)(\omega) := X_{\tau(\omega)}(\omega)$ that evaluate an optional process $X = (X_t)_{0 \le t \le 1}$ at a finite stopping time τ. By the optional cross section theorem this topology is Hausdorff.

Given Theorem 2.6 and Theorem 2.7 above one can ask conversely what the smallest class of stochastic processes is that is closed under convergence in probability at all finite stopping times and contains the set of bounded martingales. Here the next result shows that this set is the set of optional strong supermartingales.

THEOREM 2.8. *Let* $X = (X_t)_{0 \le t \le 1}$ *be an optional strong supermartingale and suppose that its stochastic base* $(\Omega, \mathcal{F}, \mathbb{F}, P)$ *is sufficiently rich to support a Brownian motion* $W = (W_t)_{0 \le t \le 1}$. *Then there is a sequence of bounded càdlàg martingales* $(M^n)_{n=1}^\infty$ *such that, for every* $[0, 1]$*-valued stopping time* τ, *we have convergence in probability, that is,*

$$M^n_\tau \xrightarrow{P} X_\tau. \tag{2.6}$$

We thank Perkowski and Ruf for pointing out to us that they have independently obtained a similar result to Theorem 2.8 for càdlàg supermartingales in Proposition 5.9 of [15] by taking several limits successively. Moreover, we would like to thank Ruf for insisting on a clarification of an earlier version of Theorem 2.8 which led us to a correction of the statement [convergence in probability in (2.6) as opposed to almost sure convergence] as well as to a more detailed proof.

Let us now turn to the theme of stochastic integration. By Theorem 2.6 the limit of a sequence $(M^n)_{n=1}^\infty$ of martingales in the sense of (2.4) will, in general, be no longer a semimartingale. In order to come up with a similar convergence result for stochastic integrals $\varphi \cdot M^n = \int \varphi \, dM^n$, we therefore need to restrict the choice of integrands $\varphi = (\varphi_t)_{0 \le t \le 1}$ to predictable finite variation processes. As we shall explain in more detail in Section 7 below, this allows us to define stochastic integrals $\varphi \cdot X = \int \varphi \, dX$ with respect to optional strong supermartingales $X = (X_t)_{0 \le t \le 1}$ pathwise, since X is làdlàg. These integrals coincide with the usual stochastic integrals, if $X = (X_t)_{0 \le t \le 1}$ is a semimartingale. For a general predictable, finite variation process φ, the stochastic integral $\varphi \cdot X$ depends not only on the values of the integrator X but also explicitly on that of its left limits X_-; see (7.3) below. As a consequence, in order to obtain a satisfactory convergence result for the integrals

$\varphi \cdot X^n$ to a limit $\varphi \cdot X$, we have to take special care of the left limits of the integrators. (The convergence of stochastic integrals is crucially needed in applications in mathematical finance, where the integrals correspond to the gains from trading by using self-financing trading strategies.) More precisely: given the convergence $\widetilde{X}^n_\tau \xrightarrow{P} X_\tau$ as in (2.5), at all $[0,1]$-valued stopping times τ of a sequence $(\widetilde{X}^n)_{n=1}^\infty$ of optional strong supermartingales do we have the convergence of the left limits

$$\widetilde{X}^n_{\sigma-} \xrightarrow{P} X_{\sigma-} \tag{2.7}$$

for all $[0,1]$-valued stopping times σ as well?

For *totally inaccessible* stopping times σ, we are able to prove that (2.7) is actually the case.

PROPOSITION 2.9. *Let $(X^n)_{n=1}^\infty$ and X be nonnegative optional strong supermartingales $(X^n_t)_{0\le t\le 1}$ and $(X_t)_{0\le t\le 1}$ such that*

$$X^n_q \xrightarrow{P} X_q$$

for every rational number $q \in [0,1]$. Then

$$X^n_{\tau-} \xrightarrow{P} X_{\tau-}$$

for all $[0,1]$-*valued* totally inaccessible *stopping times τ.*

At accessible stopping times σ, the convergence $\widetilde{X}^n_\tau \xrightarrow{P} X_\tau$ for all finite stopping times τ does not necessarily imply convergence (2.7) of the left limits $\widetilde{X}^n_{\sigma-}$. Moreover, even if the left limits $\widetilde{X}^n_{\sigma-}$ converge to some random variable Y in probability, it may happen that $Y \neq X_{\sigma-}$. In order to take this phenomenon into account, we need to consider two processes $X^{(0)} = (X^{(0)}_t)_{0\le t\le 1}$ and $X^{(1)} = (X^{(1)}_t)_{0\le t\le 1}$ that correspond to the limiting processes of the left limits $\widetilde{X}^n_-$ and the processes $\widetilde{X}^n$ itself or, alternatively, replace the time interval $I = [0,1]$ by the set $\widetilde{I} = [0,1] \times \{0,1\}$ with the lexicographic order. The set $\widetilde{I}$ is motivated by the *Alexandroff double arrow space*. Equipping the set $\widetilde{I}$ with the lexicographic order simply means that we split every point $t \in [0,1]$ into a left and a right point $(t,0)$ and $(t,1)$, respectively, such that $(t,0) < (t,1)$, that $(t,0) \le (s,0)$ if and only if $t \le s$ and that $(t,1) < (s,0)$ if and only if $t < s$. Then we can merge both processes, $X^{(0)} = (X^{(0)}_t)_{0\le t\le 1}$ and $X^{(1)} = (X^{(1)}_t)_{0\le t\le 1}$, into one process,

$$X_{\tilde{t}} = \begin{cases} X^{(0)}_t, & \tilde{t} = (t,0), \\ X^{(1)}_t, & \tilde{t} = (t,1), \end{cases} \tag{2.8}$$

for $\tilde{t} \in \widetilde{I}$, which is by (2.11) below a supermartingale indexed by $\tilde{t} \in \widetilde{I}$. As the limit of the left limits, the process $X^{(0)} = (X^{(0)}_t)_{0\le t\le 1}$ will be predictable and it will turn out that it is even a predictable strong supermartingale. We refer to the article of

Chung and Glover [2] (see the second remark following the proof of Theorem 3 on page 243) as well as Definition 3 in Appendix I of the book of Dellacherie and Meyer [8] for the subsequent concept.

DEFINITION 2.10. A real-valued stochastic process $X = (X_t)_{0 \le t \le 1}$ is called a *predictable strong supermartingale* if:

(1) X is predictable;
(2) X_τ is integrable for every $[0, 1]$-valued *predictable* stopping time τ;
(3) for all *predictable* stopping times σ and τ with $0 \le \sigma \le \tau \le 1$, we have

$$X_\sigma \ge E[X_\tau | \mathcal{F}_{\sigma-}].$$

After these preparations we are able to extend Theorem 2.7 to hold also for left limits.

THEOREM 2.11. *Let $(X^n)_{n=1}^\infty$ be a sequence of nonnegative optional strong supermartingales starting at $X_0^n = 1$. Then there is a sequence $(\widetilde{X}^n)_{n=1}^\infty$ of convex combinations $\widetilde{X}^n \in \operatorname{conv}(X^n, X^{n+1}, \dots)$, a nonnegative optional strong supermartingale $X^{(1)} = (X_t^{(1)})_{0 \le t \le 1}$ and a nonnegative predictable strong supermartingale $X^{(0)} = (X_t^{(0)})_{0 \le t \le 1}$ such that*

$$\widetilde{X}^n_\tau \xrightarrow{P} X^{(1)}_\tau, \tag{2.9}$$

$$\widetilde{X}^n_{\tau-} \xrightarrow{P} X^{(0)}_\tau, \tag{2.10}$$

for all $[0, 1]$-*valued stopping times* τ, *and we have that*

$$X^{(1)}_{\tau-} \ge X^{(0)}_\tau \ge E\big[X^{(1)}_\tau | \mathcal{F}_{\tau-}\big] \tag{2.11}$$

for all $[0, 1]$-*valued* predictable *stopping times* τ.

With the above we can now formulate the following proposition. Note that, since $\varphi \cdot \widetilde{X}^n \in \operatorname{conv}(\varphi \cdot X^n, \varphi \cdot X^{n+1}, \dots)$, part (2) is indeed an analogous result to Theorem 2.7 for stochastic integrals.

PROPOSITION 2.12. *Let $(X^n)_{n=1}^\infty$ be a sequence of nonnegative optional strong supermartingales $X^n = (X_t^n)_{0 \le t \le 1}$ starting at $X_0^n = 1$. Then there exist convex combinations $\widetilde{X}^n \in \operatorname{conv}(X^n, X^{n+1}, \dots)$ as well as an optional and a predictable strong supermartingale $X^{(1)}$ and $X^{(0)}$ such that*:

(1) $\widetilde{X}^n_\tau \xrightarrow{P} X^{(1)}_\tau$ *and* $\widetilde{X}^n_{\tau-} \xrightarrow{P} X^{(0)}_\tau$ *for all* $[0, 1]$-*valued stopping times* τ;
(2) *for all predictable processes $\varphi = (\varphi_t)_{0 \le t \le 1}$ of finite variation, we have that*

$$\varphi \cdot \widetilde{X}^n_\tau \xrightarrow{P} \int_0^\tau \varphi^c_u \, dX^{(1)}_u + \sum_{0 < u \le \tau} \Delta\varphi_u \big(X^{(1)}_\tau - X^{(0)}_u\big) + \sum_{0 \le u < \tau} \Delta_+\varphi_u \big(X^{(1)}_\tau - X^{(1)}_u\big)$$

for all $[0,1]$-*valued stopping times* τ, *where* φ^c *denotes the continuous part of* φ, *that is*,

$$\varphi_t^c := \varphi_t - \sum_{0<u\leq t} \Delta\varphi_u - \sum_{0\leq u<t} \Delta_+\varphi_u \qquad \textit{for } t \in [0,1]. \tag{2.12}$$

3. Proof of Theorems 2.6 and 2.7. The basic idea for the proof of Theorem 2.6 is to consider the Fatou limit $\overline{X} = (\overline{X}_t)_{0\leq t\leq 1}$ as defined in (2.1). Morally speaking $\overline{X} = (\overline{X}_t)_{0\leq t\leq 1}$ should also be the limit of the sequence $(\overline{M})_{n=1}^{\infty}$ in the sense of (2.4). However, as we illustrated in Example 2.4, things may be more delicate. While we do not need to have convergence in probability at all finite stopping times in general, the next lemma shows that we always have one-sided P-almost sure convergence.

LEMMA 3.1. *Let* $\overline{X}$ *and* $(\overline{M}^n)_{n=1}^{\infty}$ *be as in Proposition* 2.3. *Then we have that*

$$\big(\overline{M}^n_\tau - \overline{X}_\tau\big)^- \overset{P\text{-a.s.}}{\longrightarrow} 0, \qquad \textit{as } n \to \infty, \tag{3.1}$$

for all $[0,1]$-*valued stopping times* τ, *where* $x^- = \max\{-x, 0\}$.

PROOF. Let σ_k be the kth dyadic approximation of the stopping time τ, that is,

$$\sigma_k := \inf\{t \in D_k | t > \tau\} \wedge 1, \tag{3.2}$$

where $D_k = \{j2^{-k} | j = 0, \ldots, 2^k\}$. As $\overline{M}^n$ is a martingale, we have $\overline{M}^n_\tau = E[\overline{M}^n_{\sigma_k} | \mathcal{F}_\tau]$, for every $n \in \mathbb{N}$, and therefore

$$\varliminf_{n\to\infty} \overline{M}^n_\tau = \varliminf_{n\to\infty} E[\overline{M}^n_{\sigma_k} | \mathcal{F}_\tau] \geq E\Big[\varliminf_{n\to\infty} \overline{M}^n_{\sigma_k} \Big| \mathcal{F}_\tau\Big] = E[Z_{\sigma_k} | \mathcal{F}_\tau]$$

for all k by Fatou's lemma, where Z_q is defined in Proposition 2.3, for every $q \in \mathbb{Q} \cap [0,1]$. Since $Z_{\sigma_k} \to \overline{X}_\tau$ P-a.s. and in $L^1(P)$ by backward supermartingale convergence (see Theorem V.30 and the proof of Theorem IV.10 in [8], e.g.), we obtain that

$$\varliminf_{n\to\infty} \overline{M}^n_\tau \geq \overline{X}_\tau,$$

which proves (3.1). □

For any sequence $(\widehat{M}^n)_{n=1}^{\infty}$ of convex combinations

$$\widehat{M}^n \in \operatorname{conv}\big(\overline{M}^n, \overline{M}^{n+1}, \ldots\big),$$

we can use the one-sided convergence (3.1) to show in the next lemma that at any given stopping time τ, we either have the convergence of $\widehat{M}^n_\tau$ to $\overline{X}_\tau$ in probability, or there exists a sequence $(\widetilde{M}^n)_{n=1}^{\infty}$ of convex combinations

$$\widetilde{M}^n \in \operatorname{conv}\big(\widehat{M}^n, \widehat{M}^{n+1}, \ldots\big)$$

and a nonnegative random variable Y such that $\widetilde{M}^n_\tau \xrightarrow{P} Y$. In the latter case, $Y \geq \overline{X}_\tau$ and $E[Y] > E[\overline{X}_\tau]$, as we shall now show.

LEMMA 3.2. *Let $\overline{X}$ and $(\overline{M}^n)_{n=1}^\infty$ be as in Proposition 2.3, let τ be a* $[0, 1]$-*valued stopping time and $(\widehat{M}^n)_{n=1}^\infty$ a sequence of convex combinations $\widehat{M}^n \in$* $\mathrm{conv}(\overline{M}^n, \overline{M}^{n+1}, \dots)$. *Then we have either*

$$(\widehat{M}^n_\tau - \overline{X}_\tau)^+ \xrightarrow{P} 0, \qquad \text{as } n \to \infty, \tag{3.3}$$

with $x^+ = \max\{x, 0\}$, or there exists a sequence $(\widetilde{M})_{n=1}^\infty$ of convex combinations

$$\widetilde{M}^n \in \mathrm{conv}(\widehat{M}^n, \widehat{M}^{n+1}, \dots) \subseteq \mathrm{conv}(\overline{M}^n, \overline{M}^{n+1}, \dots)$$

and a nonnegative random variable Y such that

$$\widetilde{M}^n_\tau \xrightarrow{P} Y, \qquad \text{as } n \to \infty, \tag{3.4}$$

and

$$E[Y_\tau] > E[\overline{X}_\tau]. \tag{3.5}$$

PROOF. If (3.3) does not hold, there exists $\alpha > 0$ and a subsequence $(\widehat{M}^n)$, still denoted by $(\widehat{M}^n)_{n=1}^\infty$ again indexed by n, such that

$$P(\widehat{M}^n_\tau - \overline{X}_\tau > \alpha) \geq \alpha \tag{3.6}$$

for all n. Since $E[\widehat{M}^n_\tau] = 1$, there exists by Komlós's lemma a sequence $(\widetilde{M}^n)_{n=1}^\infty$ of convex combinations $\widetilde{M}^n \in \mathrm{conv}(\widehat{M}^n, \widehat{M}^{n+1}, \dots)$ and a nonnegative random variable Y such that (3.4) holds. To see (3.5), we observe that, for each $\varepsilon > 0$,

$$\mathbb{1}_{\{\widehat{M}^n_\tau \geq \overline{X}_\tau - \varepsilon\}} \xrightarrow{P} 1, \qquad \text{as } n \to \infty,$$

by (3.1). From the inequality

$$\widehat{M}^n_\tau \mathbb{1}_{A_n} \geq \overline{X}_\tau \mathbb{1}_{A_n} + \alpha \mathbb{1}_{A_n},$$

where $A_n := \{\widehat{M}^n_\tau \geq \overline{X}_\tau + \alpha\}$, we obtain

$$\widehat{M}^n_\tau \mathbb{1}_{\{\widehat{M}^n_\tau \geq \overline{X}_\tau - \varepsilon\}} \geq \overline{X}_\tau \mathbb{1}_{\{\widehat{M}^n_\tau \geq \overline{X}_\tau - \varepsilon\}} + \alpha \mathbb{1}_{A_n}.$$

Now taking the convex combinations leading to $\widetilde{M}^n$ and then

$$\widetilde{Y}^n \in \mathrm{conv}(\alpha \mathbb{1}_{A_n}, \alpha \mathbb{1}_{A_{n+1}}, \dots)$$

such that $\widetilde{Y}^n \xrightarrow{P} \widetilde{Y}$, as $n \to \infty$, we derive

$$Y \geq \overline{X}_\tau + \widetilde{Y} - \varepsilon \tag{3.7}$$

by passing to limits. Since $|\widetilde{Y}^n| \leq 1$ and $E[\widetilde{Y}^n] \geq \alpha^2$, we deduce from Lebesgue's theorem that $\widetilde{Y}^n \xrightarrow{L^1(P)} \widetilde{Y}$, as $n \to \infty$, and $E[\widetilde{Y}] \geq \alpha^2$. Therefore (3.7) implies that

$$E[Y] \geq E[\overline{X}_\tau] + E[\widetilde{Y}] - \varepsilon \geq E[\overline{X}_\tau] + \alpha^2 - \varepsilon$$

for each $\varepsilon > 0$ and hence (3.5) by sending $\varepsilon \to 0$. $\square$

By the previous lemma we either already have the convergence of $\widehat{M}^n_\tau$ to $\overline{X}_\tau$ in probability at a given stopping time τ, or we can use Komlós's lemma once again to find convex combinations $\widetilde{M}^n \in \operatorname{conv}(\widehat{M}^n, \widehat{M}^{n+1}, \dots)$ and a random variable Y such that $\widetilde{M}^n_\tau \xrightarrow{P} Y$. The next lemma shows that we can exhaust this latter phenomenon by a countable number of stopping times $(\tau_m)_{m=1}^\infty$ and that we can use the random variables $Y_m := P - \lim_{n\to\infty} \widetilde{M}^n_{\tau_m}$ to redefine the càdlàg supermartingale $\overline{X}$ at the stopping times τ_m to obtain a limiting process $\widetilde{X} = (\widetilde{X}_t)_{0\le t\le 1}$. The limiting process $\widetilde{X}$ will be an optional strong supermartingale, and we can relate the loss of mass $Y_m - \overline{X}_{\tau_m}$ to the right jumps $\Delta_+ \widetilde{A}_{\tau_m}$ of the predictable part of the Mertens decomposition $\widetilde{X} = \widetilde{M} - \widetilde{A}$.

LEMMA 3.3. *In the setting of Proposition 2.3, let $(\tau_m)_{m=1}^\infty$ be a sequence of $[0,1]\cup\{\infty\}$-valued stopping times with disjoint graphs, that is, $[\![\tau_m]\!] \cap [\![\tau_k]\!] = \varnothing$ for $m \neq k$. Then there exists a sequence $(\widetilde{M}^n)_{n=1}^\infty$ of convex combinations $\widetilde{M}^n \in \operatorname{conv}(\overline{M}^n, \overline{M}^{n+1}, \dots)$ such that, for each $m \in \mathbb{N}$, the sequence $(\widetilde{M}^n_{\tau_m})_{n=1}^\infty$ converges P-a.s. to a random variable Y_m on $\{\tau_m < \infty\}$. The process $\widetilde{X} = (\widetilde{X}_t)_{0\le t\le 1}$ given by*

$$\widetilde{X}_t(\omega) = \begin{cases} Y_m(\omega), & t = \tau_m(\omega) < \infty \text{ and } m \in \mathbb{N}, \\ \overline{X}_t(\omega), & \text{elsewhere} \end{cases} \tag{3.8}$$

is an optional strong supermartingale with the following properties:

(1) *$\widetilde{X}_+ = \overline{X}$, where $\widetilde{X}_+$ denotes the process of the right limits of $\widetilde{X}$;*
(2) *denoting by $\widetilde{X} = \widetilde{M} - \widetilde{A}$, the Mertens decomposition of $\widetilde{X}$, we have*

$$\widetilde{X}_{\tau_m} - \overline{X}_{\tau_m} = -\Delta_+ \widetilde{X}_{\tau_m} = \Delta_+ \widetilde{A}_{\tau_m} := \widetilde{A}_{\tau_{m+}} - \widetilde{A}_{\tau_m} \tag{3.9}$$

for each $m \in \mathbb{N}$.

PROOF. Combining Komlós's lemma with a diagonalization procedure we obtain nonnegative random variables Y_m and convex combinations $\widetilde{M}^n \in \operatorname{conv}(\overline{M}^n, \overline{M}^{n+1}, \dots)$ such that

$$\widetilde{M}^n_{\tau_m} \xrightarrow{P\text{-a.s.}} Y_m,$$

for all $m \in \mathbb{N}$, and we can define the process $\widetilde{X}$ via (3.8). This process $\widetilde{X}$ is clearly optional.

To show that $\widetilde{X}$ is indeed an optional strong supermartingale, we need to verify that

$$\widetilde{X}_{\varrho_1} \ge E[\widetilde{X}_{\varrho_2} | \mathcal{F}_{\varrho_1}] \tag{3.10}$$

for every pair of $[0,1]$-valued stopping times ϱ_1 and ϱ_2 such that $\varrho_1 \le \varrho_2$. For this, we observe that it is sufficient to consider (3.10) on the set $\{\varrho_1 < \varrho_2\}$. For $i = 1, 2$

denote by $(\varrho_{i,k})_{k=1}^{\infty}$ the kth dyadic approximation of ϱ_i as in (3.2) above. Then we have

$$
\begin{aligned}
&E[\widetilde{X}_{\varrho_2}|\mathcal{F}_{\varrho_1}]\\
&\quad = E\Bigg[\lim_{n\to\infty}\sum_{m=1}^{\infty}\widetilde{M}^n_{\tau_m}\mathbb{1}_{\{\tau_m=\varrho_2\}} + \lim_{k\to\infty}\Big(\lim_{n\to\infty}\overline{M}^n_{\varrho_{2,k}}\Big)\mathbb{1}_{\{\tau_m\neq\varrho_2,\forall m\}}\Big|\mathcal{F}_{\varrho_1}\Bigg]\\
&\quad = E\Bigg[\lim_{n\to\infty}\sum_{m=1}^{\infty}\widetilde{M}^n_{\tau_m}\mathbb{1}_{\{\tau_m=\varrho_2\}} + \lim_{k\to\infty}\Big(\lim_{n\to\infty}\widetilde{M}^n_{\varrho_{2,k}}\Big)\mathbb{1}_{\{\tau_m\neq\varrho_2,\forall m\}}\Big|\mathcal{F}_{\varrho_1}\Bigg]\\
&\quad \le E\Bigg[\lim_{n\to\infty}\sum_{m=1}^{\infty}\widetilde{M}^n_{\tau_m}\mathbb{1}_{\{\tau_m=\varrho_2\}}\\
&\qquad\quad + \lim_{k\to\infty}\Big(\lim_{n\to\infty}E\big[\widetilde{M}^n_{\varrho_{2,k}}|\mathcal{F}_{\varrho_2}\big]\Big)\mathbb{1}_{\{\tau_m\neq\varrho_2,\forall m\}}\Big|\mathcal{F}_{\varrho_1}\Bigg] &(3.11)\\
&\quad = E\Big[\lim_{n\to\infty}\widetilde{M}^n_{\varrho_2}|\mathcal{F}_{\varrho_1}\Big] &(3.12)\\
&\quad \le E\Big[\lim_{k\to\infty}\lim_{n\to\infty}E\big[\widetilde{M}^n_{\varrho_2}|\mathcal{F}_{\varrho_{1,k}}\big]|\mathcal{F}_{\varrho_1}\Big] &(3.13)\\
&\quad = E\Big[\lim_{k\to\infty}\lim_{n\to\infty}\widetilde{M}^n_{\varrho_{1,k}}|\mathcal{F}_{\varrho_1}\Big] &(3.14)\\
&\quad = E\Bigg[\lim_{k\to\infty}\lim_{n\to\infty}\sum_{m=1}^{\infty}\widetilde{M}^n_{\varrho_{1,k}}\mathbb{1}_{\{\tau_m=\varrho_1\}} + \lim_{k\to\infty}\lim_{n\to\infty}\widetilde{M}^n_{\varrho_{1,k}}\mathbb{1}_{\{\tau_m\neq\varrho_1,\forall m\}}\Big|\mathcal{F}_{\varrho_1}\Bigg]\\
&\quad \le \lim_{k\to\infty}\lim_{n\to\infty}\sum_{m=1}^{\infty}E\big[\widetilde{M}^n_{\varrho_{1,k}}|\mathcal{F}_{\varrho_1}\big]\mathbb{1}_{\{\tau_m=\varrho_1\}}\\
&\qquad\quad + E\Big[\lim_{k\to\infty}\lim_{n\to\infty}\overline{M}^n_{\varrho_{1,k}}|\mathcal{F}_{\varrho_1}\Big]\mathbb{1}_{\{\tau_m\neq\varrho_1,\forall m\}} &(3.15)\\
&\quad = \lim_{n\to\infty}\sum_{m=1}^{\infty}\widetilde{M}^n_{\tau_m}\mathbb{1}_{\{\tau_m=\varrho_1\}} + E\Big[\lim_{k\to\infty}Z_{\varrho_{1,k}}|\mathcal{F}_{\varrho_1}\Big]\mathbb{1}_{\{\tau_m\neq\varrho_1,\forall m\}} &(3.16)\\
&\quad = \sum_{m=1}^{\infty}\widetilde{X}_{\tau_m}\mathbb{1}_{\{\tau_m=\varrho_1\}} + \overline{X}_{\varrho_1}\mathbb{1}_{\{\tau_m\neq\varrho_1,\forall m\}} = \widetilde{X}_{\varrho_1} &(3.17)
\end{aligned}
$$

by using Fatou's lemma in (3.11), (3.13) and (3.15), the martingale property of the $\widetilde{M}^n$ and the convergence in probability of the M^n in (3.12), (3.14) and (3.16) and exploiting the backward supermartingale convergence of $(Z_{\varrho_{1,k}})_{k=1}^{\infty}$ in (3.17).

(1) We argue by contradiction and assume that $G := \{\widetilde{X}_+ \neq \overline{X}\}$ has $P(\pi(G)) > 0$, where $\pi : \Omega \times [0, 1] \to \Omega$ is given by $\pi((\omega, t)) = \omega$. As the set G is optional, there exists by the optional cross-section theorem (Theorem IV.84 in [8]) a $[0, 1] \cup$

$\{\infty\}$-valued stopping time σ such that $[\![\sigma_{\{\sigma<\infty\}}]\!] \subseteq G$ and $P(\sigma < \infty) > 0$, which is equivalent to the assumption that the set $F := \{\widetilde{X}_{\sigma+} \neq \overline{X}_\sigma\}$ has strictly positive measure $P(F) > 0$. Without loss of generality we can assume that there exists $\delta > 0$ such that $F \subseteq \{\sigma + \delta < 1\}$. Let $(h_i)_{i=1}^\infty$ be a sequence of real numbers decreasing to 0 that are no atoms of the laws $\tau_m - \sigma$ for all $m \in \mathbb{N}$. Then defining $\sigma_i := (\sigma + h_i)_F \wedge 1$ for each $i \in \mathbb{N}$ gives a sequence of stopping times such that $\widetilde{X}_{\sigma_i} = \overline{X}_{\sigma_i}$ for each i and $\sigma_i \searrow \sigma$ on F. But this implies that

$$\widetilde{X}_{\sigma+} = \lim_{i\to\infty} \widetilde{X}_{\sigma_i} = \lim_{i\to\infty} \overline{X}_{\sigma_i} = \overline{X}_\sigma \qquad \text{on } F, \tag{3.18}$$

which contradicts $P(F) > 0$ and hence also $P(\pi(G)) > 0$.

(2) By property (1), modifying $\overline{X}$ at countably many stopping times $(\tau_m)_{m=1}^\infty$ to obtain $\widetilde{X}$ leaves right limits of the làdlàg optional strong supermartingale $\widetilde{X}$ invariant so that these remain

$$\widetilde{X}_{\tau_m+} = \overline{X}_{\tau_m^+} = \overline{X}_{\tau_m} \qquad \text{on } \{\tau_m < 1\} \text{ for each } m. \tag{3.19}$$

Since $\widetilde{M}$ is càdlàg, this implies that

$$\widetilde{X}_{\tau_m} - \overline{X}_{\tau_m} = -\Delta_+ \widetilde{X}_{\tau_m} = \Delta_+ \widetilde{A}_{\tau_m} \tag{3.20}$$

for each m, thus proving property (2). $\square$

Continuing with the proof of Theorem 2.6, the idea is to define the limiting supermartingale X by (3.8) and to use Lemma 3.3 to enforce the convergence at a well-chosen *countable number* of stopping times $(\tau_m)_{m=1}^\infty$ to obtain the convergence in (2.5) for *all* stopping times. It is rather intuitive that one has to take special care of the jumps of the limiting process X. As these can be exhausted by a sequence $(\tau_k)_{k=1}^\infty$ of stopping times, the previous lemma can take care of this issue. However, the subsequent example shows that there also may be a problem with the convergence in (2.4) at a stopping time τ at which $\overline{X}$ is *continuous*.

EXAMPLE 3.4. Let $\sigma : \Omega \longrightarrow [0, 1]$ be a *totally inaccessible* stopping time, $(A_t)_{0<t\le 1}$ its compensator so that $(\mathbb{1}_{[\![\sigma,1]\!]}(t) - A_t)_{0\le t\le 1}$ is a martingale. Let $(Y_n)_{n=1}^\infty$ be a sequence of random variables independent of σ such that Y_n takes values in $\{0, n\}$ and $P[Y_n = n] = \frac{1}{n}$. Define the *continuous* supermartingale

$$X_t^1 = 1 - A_t, \qquad 0 \le t \le 1,$$

and the optional strong supermartingale

$$X_t^2 = 1 - A_t + \mathbb{1}_{[\![\sigma]\!]}(t), \qquad 0 \le t \le 1.$$

Define the sequences $(M^{1,n})_{n=1}^\infty$ and $(M^{2,n})_{n=1}^\infty$ of martingales by

$$M_t^{1,n} = 1 - A_t + Y_n \mathbb{1}_{[\![\sigma,1]\!]}(t),$$

$$M_t^{2,n} = 1 - A_t + \mathbb{1}_{[\![\sigma,1]\!]}(t) + (Y_n - 1)\mathbb{1}_{[\![\sigma+1/n,1]\!]}(t)$$

for $t \in [0, 1]$ and $n \in \mathbb{N}$. Then we have that

$$\begin{aligned} M_\tau^{1,n} &\xrightarrow{P} X_\tau^1, \\ M_\tau^{2,n} &\xrightarrow{P} X_\tau^2 \end{aligned} \tag{3.21}$$

for all $[0, 1]$-valued stopping times τ. The left and right limits of X^1 and X^2 coincide, that is, $X_-^1 = X_-^2$ and $X_+^1 = X_+^2$, but $X^1 \neq X^2$. As $X^1 = X_-^1 = X_+^1 = X_+^2$ coincides with the Fatou limits $\overline{X}^1$ (and $\overline{X}^2$, resp.) of $(M^{1,n})_{n=1}^\infty$ [and $(M^{2,n})_{n=1}^\infty$, resp.], this example illustrates that we cannot deduce from the Fatou limits $\overline{X}^1$ and $\overline{X}^2$, where it is necessary to correct the convergence by using Lemma 3.3. Computing the Mertens decompositions $X^1 = M^1 - A^1$ and $X^2 = M^2 - A^2$, we obtain

$$\begin{aligned} M^1 &= 1, \\ A^1 &= \sigma \wedge t, \\ M^2 &= 1 - \sigma \wedge t + \mathbb{1}_{[\![\sigma,1]\!]}, \\ A^2 &= \mathbb{1}_{]\!]\sigma,1]\!]}. \end{aligned}$$

This shows that using X^2 instead of $\overline{X}^2 = X^1$ changes the compensator of M^2 not only after the correction in the sense of Lemma 3.3 on $]\!]\sigma, 1]\!]$ but on all of $[0, 1]$.

As the previous example shows, it might be difficult to identify the stopping times $(\tau_m)_{m=1}^\infty$, where one needs to enforce the convergence in probability by using Lemma 3.3. Therefore we combine the previous lemmas with an exhaustion argument to prove Theorem 2.6.

PROOF OF THEOREM 2.6. Let $\mathbb{T}$ be the collection of all families $\mathcal{T} = (\tau_m)_{m=1}^{N(\mathcal{T})}$ of finitely many $[0, 1] \cup \{\infty\}$-valued stopping times τ_m with disjoint graphs. For each $\mathcal{T} \in \mathbb{T}$, we consider an optional strong supermartingale $X^{\mathcal{T}}$ that is obtained by taking convex combinations $\widetilde{X}^{n,\mathcal{T}} \in \operatorname{conv}(\overline{M}^n, \overline{M}^{n+1}, \dots)$ such that $\widetilde{X}_{\tau_m}^{n,\mathcal{T}} \xrightarrow{P} Y_m^{\mathcal{T}}$ on $\{\tau_m < \infty\}$ for each $m = 1, \dots, N(\mathcal{T})$ and then setting

$$X_t^{\mathcal{T}}(\omega) = \begin{cases} Y_m^{\mathcal{T}}(\omega), & t = \tau_m(\omega) < \infty \text{ and } m = 1, \dots, N(\mathcal{T}), \\ \overline{X}_t(\omega), & \text{else,} \end{cases} \tag{3.22}$$

as explained in Lemma 3.3. Then each $X^{\mathcal{T}}$ has a Mertens decomposition

$$X^{\mathcal{T}} = M^{\mathcal{T}} - A^{\mathcal{T}}, \tag{3.23}$$

and we have by part (2) of Lemma 3.3 that

$$E\left[\sum_{m=1}^{N(\mathcal{T})} \left(X_{\tau_m \wedge 1}^{\mathcal{T}} - \overline{X}_{\tau_m \wedge 1}\right)\right] = E\left[\sum_{m=1}^{N(\mathcal{T})} \Delta_+ A_{\tau_m \wedge 1}^{\mathcal{T}}\right] \le 1.$$

Therefore

$$\widehat{\vartheta} := \sup_{\mathcal{T}\in\mathbb{T}} E\left[\sum_{m=1}^{N(\mathcal{T})} \left(X^{\mathcal{T}}_{\tau_m\wedge 1} - \overline{X}_{\tau_m\wedge 1}\right)\right] \le 1, \tag{3.24}$$

and there exists a maximizing sequence $(\mathcal{T}_k)_{k=1}^{\infty}$ such that

$$E\left[\sum_{m=1}^{N(\mathcal{T}_k)} \left(X^{\mathcal{T}_k}_{\tau_m\wedge 1} - \overline{X}_{\tau_m\wedge 1}\right)\right] \nearrow \sup_{\mathcal{T}\in\mathbb{T}} E\left[\sum_{m=1}^{N(\mathcal{T})} \left(X^{\mathcal{T}}_{\tau_m\wedge 1} - \overline{X}_{\tau_m\wedge 1}\right)\right] = \widehat{\vartheta}. \tag{3.25}$$

It is easy to see that we can assume that $(\mathcal{T}_k)_{k=1}^{\infty}$ can be chosen to be increasing, that is, $\mathcal{T}_k \subseteq \mathcal{T}_{k+1}$ for each k. This means that $\mathcal{T}_{k+1}$ just adds some stopping times to those which appear in $\mathcal{T}_k$. Then $\widetilde{\mathcal{T}} := \bigcup_{k=1}^{\infty} \mathcal{T}_k$ is a countable collection of stopping times $(\tau_m)_{m=1}^{\infty}$ with disjoint graphs, and by Lemma 3.3 there exists an optional strong supermartingale $X^{\widetilde{\mathcal{T}}}$ and convex combinations $X^{n,\widetilde{\mathcal{T}}} \in \operatorname{conv}(\overline{M}^n, \overline{M}^{n+1}, \dots)$ such that $X^{n,\widetilde{\mathcal{T}}}_{\widetilde{\tau}_m} \xrightarrow{P} Y^{\widetilde{\mathcal{T}}}_m$ for all m and

$$X^{\widetilde{\mathcal{T}}}_t(\omega) := \begin{cases} Y^{\widetilde{\mathcal{T}}}_m(\omega), & t = \tau_m(\omega) < \infty, \\ \overline{X}_t(\omega), & \text{else.} \end{cases} \tag{3.26}$$

As we can suppose without loss of generality that $X^{n,\mathcal{T}_{k+1}} \in \operatorname{conv}(X^{n,\mathcal{T}_k}, X^{n+1,\mathcal{T}_k}, \dots)$ and $X^{n,\widetilde{\mathcal{T}}} \in \operatorname{conv}(X^{n,\mathcal{T}_k}, X^{n+1,\mathcal{T}_{n+1}}, \dots)$, we have that $Y^{\mathcal{T}_k}_m = Y^{\mathcal{T}_{k+1}}_m = Y^{\widetilde{\mathcal{T}}}_m$ on $\{\tau_m < 1\}$ for all $k \ge m$. Let $X^{\widetilde{\mathcal{T}}} = M^{\widetilde{\mathcal{T}}} - A^{\widetilde{\mathcal{T}}}$ be the Mertens decomposition of $X^{\widetilde{\mathcal{T}}}$. Then

$$\Delta_+ A^{\widetilde{\mathcal{T}}}_{\tau_m} = X^{\widetilde{\mathcal{T}}}_{\tau_m} - \overline{X}_{\tau_m} = X^{\mathcal{T}_k}_{\tau_m} - \overline{X}_{\tau_m} = \Delta_+ A^{\mathcal{T}_k}_{\tau_m} \tag{3.27}$$

on $\{\tau_m < 1\}$ for $m \le N(\mathcal{T}_k)$, since, as we explained in the proof of Lemma 3.3, modifying $\overline{X}$ at countably many stopping times does not change the right limits, and these remain

$$X^{\widetilde{\mathcal{T}}}_{\tau_m+} = \overline{X}_{\tau_m} = X^{\mathcal{T}_k}_{\tau_m+} \qquad \text{on } \{\tau_m < 1\} \text{ for } m \le N(\mathcal{T}_k). \tag{3.28}$$

This implies that

$$\sum_{m=1}^{N(\mathcal{T}_k)} \left(X^{\mathcal{T}_k}_{\tau_m\wedge 1} - \overline{X}_{\tau_m\wedge 1}\right) = \sum_{m=1}^{N(\mathcal{T}_k)} \left(X^{\widetilde{\mathcal{T}}}_{\tau_m\wedge 1} - \overline{X}_{\tau_m\wedge 1}\right) = \sum_{m=1}^{N(\mathcal{T}_k)} \Delta_+ A^{\widetilde{\mathcal{T}}}_{\tau_m\wedge 1} \tag{3.29}$$

and therefore

$$E\left[\sum_{m=1}^{\infty} \Delta_+ A^{\widetilde{\mathcal{T}}}_{\tau_m\wedge 1}\right] = E\left[\sum_{m=1}^{\infty} \left(X^{\widetilde{\mathcal{T}}}_{\tau_m\wedge 1} - \overline{X}_{\tau_m\wedge 1}\right)\right] = \widehat{\vartheta} \tag{3.30}$$

by the monotone convergence theorem.

Now suppose that there exists a $[0, 1]$-valued stopping time τ such that $X^{n,\widetilde{\mathcal{T}}}_\tau$ does not converge in probability to $X^{\widetilde{\mathcal{T}}}_\tau$. By Lemma 3.2 we can then pass once more to convex combinations $\widetilde{M}^n \in \operatorname{conv}(X^{n,\widetilde{\mathcal{T}}}, X^{n+1,\widetilde{\mathcal{T}}}, \dots)$ such that there exists a random variable Y such that $\widetilde{M}^n_\tau \xrightarrow{P} Y$, $\widetilde{M}^n_{\tau_m} \xrightarrow{P} Y^{\widetilde{\mathcal{T}}}_m$ and an optional strong supermartingale $\widetilde{X}$ such that

$$\widetilde{X}_t(\omega) = \begin{cases} Y(\omega), & t = \tau(\omega) \le 1, \\ X^{\widetilde{\mathcal{T}}}_t(\omega), & \text{else.} \end{cases} \tag{3.31}$$

However, since $E[\widetilde{X}_\tau - \overline{X}_\tau] > 0$ by Lemma 3.2, setting $\widetilde{\mathcal{T}}_k := \mathcal{T}_k \cup \{\mathcal{T}\}$ gives a sequence in $\mathbb{T}$ such that

$$\begin{aligned} &\lim_{k\to\infty} E\left[\sum_{m=1}^{N(\widetilde{\mathcal{T}}_k)} \left(X^{\widetilde{\mathcal{T}}_k}_{\tau_m\wedge 1} - \overline{X}^{\widetilde{\mathcal{T}}_k}_{\tau_m\wedge 1}\right)\right] \\ &\quad = \lim_{k\to\infty} E\left[\sum_{m=1}^{N(\mathcal{T}_k)} \left(X^{\mathcal{T}_k}_{\tau_m\wedge 1} - \overline{X}_{\tau_m\wedge 1}\right)\right] + E[\widetilde{X}_\tau - \overline{X}_\tau] \\ &\quad = \widehat{\vartheta} + E[\widetilde{X}_\tau - \overline{X}_\tau] > \widehat{\vartheta}, \end{aligned}$$

and therefore a contradiction to the definition of $\widehat{\vartheta}$ as supremum. Here we can take the convex combinations $\widetilde{M}^n \in \operatorname{conv}(X^{n,\widetilde{\mathcal{T}}}, X^{n+1,\widetilde{\mathcal{T}}}, \dots)$ for all $\widetilde{\mathcal{T}}_k$. $\square$

Combining Theorem 2.6 with a similar convergence result for predictable finite variation processes by Campi and Schachermayer [1], we now deduce Theorem 2.7 from Theorem 2.6.

PROOF OF THEOREM 2.7. We consider the extension of Theorem 2.6 to local martingales first. For this, let $(X^n)_{n=1}^\infty$ be a sequence of nonnegative local martingales $X^n = (X^n_t)_{0\le t\le 1}$ and $(\sigma^n_m)_{m=1}^\infty$ a localizing sequence of $[0,1]$-valued stopping times for each X^n. Then, for each $n \in \mathbb{N}$, there exists $m(n) \in \mathbb{N}$ such that $P(\sigma^n_m < 1) < 2^{-(n+1)}$ for all $m \ge m(n)$. Define the martingales

$$M^n := (X^n)^{\sigma^n_{m(n)}} \tag{3.32}$$

that satisfy $M^k = X^k$ for all $k \ge n$ on $F_n := \bigcap_{k\ge n}\{\sigma^k_{m(k)} = 1\}$ with $P(F_n) > 1 - 2^{-n}$. By Theorem 2.6 there exist a sequence of convex combinations $\widetilde{M}^n \in \operatorname{conv}(M^n, M^{n+1}, \dots)$ and an optional strong supermartingale X such that

$$\widetilde{M}^k_\tau \xrightarrow{P} X_\tau \qquad \text{on } F_n$$

for all $[0,1]$-valued stopping times τ. Therefore taking $\widetilde{X}^n \in \operatorname{conv}(X^n, X^{n+1}, \dots)$ with the same weights as $\widetilde{M}^n \in \operatorname{conv}(M^n, M^{n+1}, \dots)$ gives

$$\widetilde{X}^k_\tau \xrightarrow{P} X_\tau \qquad \text{on } F_n$$

for all $[0,1]$-valued stopping times τ and for each n and, since $\widetilde{X}^k = \widetilde{M}^k$ for all $k \geq n$. But, since $P(F_n^c) < 2^{-n} \to 0$, as $n \to \infty$ this implies that $\widetilde{X}_\tau^k \xrightarrow{P} X_\tau$ for all $[0,1]$-valued stopping times τ. This finishes the proof in the case when the X^n are local martingales.

For the case of optional strong supermartingales, let $(X^n)_{n=1}^\infty$ be a sequence of nonnegative optional strong supermartingales $X^n = (X_t^n)_{0 \leq t \leq 1}$ and $X^n = M^n - A^n$ their Mertens decompositions into a càdlàg local martingale M^n and a predictable, nondecreasing, làdlàg process A^n. As the local martingales $M^n \geq X^n + A^n \geq X^n$ are nonnegative, there exists by the first part of the proof a sequence of convex combinations $\widehat{M}^n \in \operatorname{conv}(M^n, M^{n+1}, \dots)$ and an optional strong supermartingale $\widehat{X}$ with Mertens decomposition $\widehat{X} = \widehat{M} - \widehat{A}$ such that

$$\widehat{M}_\tau^n \xrightarrow{P} \widehat{X}_\tau \tag{3.33}$$

for all $[0,1]$-valued stopping times τ. Now let $\widehat{A}^n \in \operatorname{conv}(A^n, A^{n+1}, \dots)$ be the convex combinations that are obtained with the same weights as the $\widehat{M}^n$. Then there exists a sequence $(\widetilde{A}^n)_{n=1}^\infty$ of convex combinations $\widetilde{A}^n \in \operatorname{conv}(\widehat{A}^n, \widehat{A}^{n+1}, \dots)$ and a predictable, nondecreasing, làdlàg process $\widetilde{A}$ such that

$$P\Big[\lim_{n\to\infty} \widetilde{A}_t^n = \widetilde{A}_t, \forall t \in [0,1]\Big] = 1. \tag{3.34}$$

Indeed, we only need to show that $(\widetilde{A}_1^n)_{n\in\mathbb{N}}$ is bounded in $L^0(P)$; then (3.34) follows from Proposition 3.4 of Campi and Schachermayer in [1]. By monotone convergence we obtain

$$E\big[\widetilde{A}_1^n\big] = \lim_{m\to\infty} E\big[\widetilde{A}_{1\wedge\sigma_m^n}^n\big] = \lim_{m\to\infty} E\big[\widetilde{M}_{1\wedge\sigma_m^n}^n - \widetilde{X}_{1\wedge\sigma_m^n}^n\big] \leq 1$$

for all $n \in \mathbb{N}$ and therefore the boundedness in $L^0(P)$. Here $\widetilde{M}^n \in \operatorname{conv}(\widehat{M}^n, \widehat{M}^{n+1}, \dots)$ and $\widetilde{X}^n \in \operatorname{conv}(\widehat{X}^n, \widehat{X}^{n+1}, \dots)$ denote convex combinations having the same weights as the $\widetilde{A}^n$ and $(\sigma_m^n)_{m=1}^\infty$ is a localizing sequence of stopping times for the local martingale $\widetilde{M}^n$.

Taking convex combinations does not change the convergence (3.33), and so $\widetilde{X}^n \in \operatorname{conv}(X^n, X^{n+1}, \dots)$ is a sequence of convex combinations and $\widetilde{X} := \widehat{X} - \widehat{A}$ an optional strong supermartingale such that

$$\widetilde{X}_\tau^n \xrightarrow{P} \widetilde{X}_\tau \tag{3.35}$$

for all $[0,1]$-valued stopping times τ. □

REMARK 3.5. (1) Observe that the proof of Theorem 2.7 actually shows that the limiting optional strong supermartingale X is equal to $\overline{X}$ up to a set that is included in the graphs of countably many stopping times $(\tau_m)_{m=1}^\infty$.

(2) Replacing Komlós's lemma (Corollary 2.2) by Komlós's subsequence theorem (Theorem 2.1) in the proof of Theorems 2.6 and 2.7, we obtain, by taking subsequences of subsequences rather than convex combinations of convex combinations, the following stronger assertion: Given a sequence $(X^n)_{n=1}^\infty$ of nonnegative

optional strong supermartingales $X^n = (X_t^n)_{0\le t\le 1}$ starting at $X_0^n = 1$, there exists a subsequence $(X^{n_k})_{k=1}^\infty$ and an optional strong supermartingale $X = (X_t)_{0\le t\le 1}$ such that the Cesàro means $\frac{1}{J}\sum_{j=1}^J X^{n_{k_j}}$ of any subsubsequence $(X^{n_{k_j}})_{j=1}^\infty$ converge to X in probability at all finite stopping times, as $J \to \infty$.

4. A counterexample. At a *single* finite stopping time τ we may, of course, pass to a subsequence to obtain that $\widetilde{M}_\tau^n$ converges not only in probability but also P-almost surely to $\widetilde{X}_\tau$. The next proposition shows that we cannot strengthen Theorem 2.6 to obtain P-almost sure convergence for *all* finite stopping times simultaneously. The obstacle is, of course, that the set of all stopping times is far from being countable.

PROPOSITION 4.1. *Let $(M^n)_{n=1}^\infty$ be a sequence of independent nonnegative continuous martingales $M^n = (M_t^n)_{0\le t\le 1}$ starting at $M_0^n = 1$ such that*

$$M_\tau^n \xrightarrow{P} 1 - \tau \tag{4.1}$$

for all $[0,1]$-valued stopping times τ. Then we have for all $\varepsilon > 0$ and all sequences $(\widetilde{M}^n)_{n=1}^\infty$ of convex combinations $\widetilde{M}^n \in \operatorname{conv}(M^n, M^{n+1}, \dots)$ that there exists a stopping time τ such that

$$P\Big[\varlimsup_{n\to\infty} \widetilde{M}_\tau^n = +\infty\Big] > 1 - \varepsilon.$$

REMARK 4.2. If $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le 1}, P)$ supports a sequence $(W^n)_{n=1}^\infty$ of independent Brownian motions $W^n = (W_t^n)_{0\le t\le 1}$, the existence of a sequence $(M^n)_{n=1}^\infty$ verifying (4.1) follows similarly as in the proof of Theorem 2.8 in Section 5 below.

For the proof of Proposition 4.1 we will need the following auxiliary lemma.

LEMMA 4.3. *In the setting of Proposition 4.1, let τ and σ be two $[0,1]$-valued stopping times such that $\tau \le \sigma$ and $\tau < \sigma$ on some $A \in \mathcal{F}_\tau$ with $P(A) > 0$. Then there exists, for all $c > 1$, a constant $\gamma = \gamma(c, \tau, \sigma) > 0$ and a number $N = N(\tau, \sigma) \in \mathbb{N}$ such that*

$$P\Big(\sup_{t\in[\tau,\sigma]} \widetilde{M}_t^n > c + 1\Big) \ge \gamma$$

for all $n \ge N$.

PROOF. Let $\alpha = \frac{E[(\sigma-\tau)\mathbb{1}_A]}{P(A)}$ and $\varepsilon \in (0,1)$ such that $\alpha > (c+4)\varepsilon$ and

$$P(B_n) \ge (1-\varepsilon)P(A)$$

for all $n \geq N$, where

$$A_n := \{|\widetilde{M}^n_\tau - (1-\tau)| < \varepsilon\} \cap A,$$
$$B_n := \{|\widetilde{M}^n_\sigma - (1-\sigma)| < \varepsilon\} \cap A_n.$$

Then setting $\varrho_n := \inf\{t \in [\tau, \sigma] | \widetilde{M}^n_t > c+1\}$ we can estimate

$$\begin{aligned} E[\widetilde{M}^n_\tau \mathbb{1}_{A_n}] &= E[\widetilde{M}^n_{\varrho_n \wedge 1} \mathbb{1}_{A_n}] \\ &= E[\widetilde{M}^n_{\varrho_n \wedge 1}(\mathbb{1}_{A_n \cap \{\varrho_n \leq 1\}} + \mathbb{1}_{\{\varrho_n > 1\} \cap B_n} + \mathbb{1}_{\{\varrho_n > 1\} \cap B^c_n \cap A_n})] \\ &\leq (c+1)P(\varrho_n \leq 1, A_n) + E[(1-\sigma+\varepsilon)\mathbb{1}_{B_n}] + (c+1)P(B^c_n \cap A_n) \end{aligned}$$

by the optional sampling theorem and the continuity of $\widetilde{M}^n$. Since

$$E[\widetilde{M}^n_\tau \mathbb{1}_{A_n}] \geq E[(1-\tau-\varepsilon)\mathbb{1}_{A_n}] \geq E[(1-\tau-\varepsilon)\mathbb{1}_{B_n}],$$

we obtain that

$$\begin{aligned} E[((1-\tau-\varepsilon) - (1-\sigma+\varepsilon))\mathbb{1}_{B_n}] - (c+1)(P(A) - P(B_n)) \\ \leq (c+1)P(\varrho_n \leq 1, A_n) \\ \leq (c+1)P(\varrho_n \leq 1) \end{aligned}$$

and therefore that

$$\gamma := \frac{\alpha - 3\varepsilon - (c+1)\varepsilon}{c+1} P(A) \leq P(\varrho_n \leq 1) = P\left(\sup_{t \in [\tau,\sigma]} \widetilde{M}^n_\tau > c+1\right)$$

for all $n \geq N$, where $\gamma > 0$ by our choice of ε, as $E[(\sigma-\tau)\mathbb{1}_{B_n}] \geq (\alpha - \varepsilon)P(A)$. □

PROOF OF PROPOSITION 4.1. We shall define τ as an increasing limit of a sequence of stopping times τ_m. For this, we set $n_0 = 0$, $\tau_0 = 0$ and $\sigma_0 = \frac{1}{2}$ and then define for $m \in \mathbb{N}$ successively

$$\begin{aligned} n_m(\omega) &:= \inf\Big\{n \in \mathbb{N} \Big| n > n_{m-1}(\omega) \text{ and } \exists t \in [\tau_{m-1}(\omega), \sigma_{m-1}(\omega)] \\ &\qquad\qquad \text{with } \widetilde{M}^n_t(\omega) \geq 2^m + 1\Big\}, \\ \tau_m(\omega) &:= \inf\{t \in (\tau_{m-1}(\omega), \sigma_{m-1}(\omega)) | \widetilde{M}^{n_m(\omega)}_t(\omega) \geq 2^m + 1\} \wedge 1, \\ \sigma_m(\omega) &:= \inf\{t > \tau_m(\omega) | \widetilde{M}^{n_m(\omega)}_t(\omega) < 2^m\} \wedge \sigma_{m-1}(\omega). \end{aligned}$$

By construction and the continuity of $\widetilde{M}^n$ we then have, for all $k \geq m$, that

$$\widetilde{M}^{n_m(\omega)}_t(\omega) \geq 2^m \qquad \text{for all } t \in [\tau_k(\omega), \sigma_k(\omega)]$$

on $\{\tau_k < 1\}$. Therefore setting $\tau := \lim_{m\to\infty} \tau_m$ gives that

$$\widetilde{M}^{n_m(\omega)}_\tau(\omega) \geq 2^m \qquad \text{for all } m$$

on $\{\tau < 1\}$. So it only remains to show that

$$P(\tau < 1) \geq 1 - \varepsilon. \tag{4.2}$$

We prove (4.2) by induction. For this, assume that there exists for each $m \in \mathbb{N}_0$, some $\alpha_m > 0$ and $N_m \in \mathbb{N}_0$ such that $P(D_m) < 1 - \varepsilon 2^{-m}$ for

$$D_m := \{\sigma_m > \tau_m + \alpha_m, n_m \in (N_{m-1}, N_m]\}. \tag{4.3}$$

Indeed, for $m = 0$, we can choose $\alpha_0 = \frac{1}{2}$, $N_{-1} = 0$ and $N_0 = 1$. Regarding the induction step we first show that $n_m < \infty$ P-a.s. on D_{m-1}. To that end, we can assume w.l.o.g. that the $(\widetilde{M}^n)_{n=1}^{\infty}$ are also independent by choosing the blocks of which we take the convex combinations disjoint and passing to a subsequence. As we are only making an assertion about the limes superior, this will be sufficient. Moreover, we observe that

$$F := \{n_m < \infty\} \cap D_{m-1} = \bigcup_{n=N_{m-1}}^{\infty} F_n \cap D_{m-1}$$

with $F_n := \{\exists t \in (\tau_{m-1}(\omega), \sigma_{m-1}(\omega)] | \widetilde{M}_t^n(\omega) \geq 2^m + 1\}$. Then using the estimate $1 - x \leq \exp(-x)$ and the independence of the F_n of each other and D_{m-1} gives

$$\begin{aligned} P(D_{m-1} \cap F^c) &= \lim_{k\to\infty} P\left(\bigcap_{n=N_{m-1}}^{k} F_n^c\right) P(D_{m-1}) \\ &= \lim_{k\to\infty} \prod_{n=N_{m-1}}^{k} (1 - P(F_n)) P(D_{m-1}) \\ &\leq \lim_{k\to\infty} \exp\left(-\sum_{n=N_{m-1}}^{k} P(F_n)\right) P(D_{m-1}). \end{aligned}$$

Since $\sum_{n=N_{m-1}}^{\infty} P(F_n) = \infty$ by Lemma 4.3, this implies that $P(D_{m-1} \cap F^c) = 0$ and hence that $n_m < \infty$ P-a.s. on D_{m-1}. More precisely, by applying Lemma 4.3 for $c = 2^m$ with $\tau = \tau_{m-1}$, $\sigma = \sigma_{m-1}$ and $A = D_{m-1}$ to $\widetilde{M}^n$ for $n \geq N_{m-1}$, we get that $P(F_n) \geq \gamma > 0$ for all $n \geq N_{m-1}$. Therefore $\tau_m < 1$ P-a.s. on D_{m-1} as well. By the continuity of the $\widetilde{M}^n$ and, as $\tau_m < \frac{1}{2}$ on D_{m-1}, we obtain that $\frac{1}{2} \geq \sigma_m > \tau_m$ P-a.s. on D_{m-1}, which finishes the induction step.

Now, since $\{\tau < 1\} \supseteq \bigcap_{m=1}^{\infty} D_m =: D$ and

$$P(D) \geq 1 - \sum_{m=1}^{\infty} P(D_m^c) = 1 - \sum_{m=1}^{\infty} \frac{\varepsilon}{2^m} = 1 - \varepsilon,$$

we have established (4.3), which completes the proof of the proposition. □

5. Proof of Theorem 2.8. We now pass to the proof of Theorem 2.8. The following lemma yields a building block.

LEMMA 5.1. *Let $W = (W_t)_{0\le t\le 1}$ be a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{F}, P)$ and ϱ a $[0,1]\cup\{\infty\}$-valued stopping time. Then there exists a sequence $(\varphi^n)_{n=1}^\infty$ of predictable integrands of finite variation such that $M^n := \varphi^n \cdot W \ge -1$ is a bounded martingale for each $n \in \mathbb{N}$ and*

$$M^n_\tau \overset{P\text{-a.s.}}{\longrightarrow} -\mathbb{1}_{]\!]\varrho,1]\!]}(\tau) = -\mathbb{1}_{\{\tau>\varrho\}}, \qquad \text{as } n\to\infty, \tag{5.1}$$

for all $[0,1]$-valued stopping times τ.

PROOF. We consider the case $\varrho \equiv 0$ first. There are many possible choices for the integrands $(\varphi^n)_{n=1}^\infty$. To come up with one, we use the deterministic functions

$$\psi^n_t := \frac{1}{2^{-n}-t}\mathbb{1}_{(0,2^{-n})}(t).$$

Then the continuous martingales $N^n := (\psi^n \cdot W_t)_{0\le t<2^{-n}}$ are well defined, for each $n\in\mathbb{N}$. It follows from the Dambis–Dubins–Schwarz theorem that the stopping times

$$\tau_n := \inf\{t \in (0,2^{-n}) | N^n_t = -1\},$$
$$\sigma_{n,k} := \inf\{t \in (0,2^{-n}) | N^n_t > k\}$$

are P-a.s. strictly smaller than 2^{-n} for all $n,k\in\mathbb{N}$, since

$$\langle N^n\rangle_t = \frac{1}{2^{-n}-t} - \frac{1}{2^{-n}} \qquad \text{for } t\in[0,2^{-n})$$

and $\lim_{t\nearrow 2^{-n}}\langle N^n\rangle_t = \infty$. Therefore setting $\widetilde{\psi}^{n,k} = \psi^n \mathbb{1}_{[\![0,\tau_n\wedge\sigma_{n,k}]\!]}$ gives a sequence

$$\widetilde{N}^{n,k} = \widetilde{\psi}^{n,k}\cdot W = (\psi^n\cdot W)^{\tau_n\wedge\sigma_{n,k}}$$

of bounded martingales such that, for all $[0,1]$-valued stopping times τ,

$$\widetilde{N}^{n,k}_\tau \overset{P\text{-a.s.}}{\longrightarrow} -1 \qquad \text{on } \{\tau \ge 2^{-n}\}, \text{ as } k\to\infty,$$

since $\sigma_{n,k}\nearrow 2^{-n}$ P-a.s, as $k\to\infty$. Defining $\varphi^n := \widetilde{\psi}^{n,k(n)}$ and $M^n = \widetilde{N}^{n,k(n)}$ as a suitable diagonal sequence such that $M^n_{2^{-n}} = \widetilde{N}^{n,k(n)}_{2^{-n}} \to -1$, as $n\to\infty$, then yields the assertion for $\varrho\equiv 0$, as $M^n_0 = 0$ for all $n\in\mathbb{N}$ and $\mathbb{1}_{\{\tau\ge 2^{-n}\}} \overset{P\text{-a.s.}}{\longrightarrow} \mathbb{1}_{\{\tau>0\}}$, as $n\to\infty$.

Next we observe that if we consider for some $[0,1]\cup\{\infty\}$-valued stopping time σ the stopped Brownian notion $W^\sigma = (W_{\sigma\wedge t})_{0\le t\le 1}$, then we obtain by the above argument that

$$(M^n)^\sigma_\tau = M^n_{\sigma\wedge\tau} = (\varphi^n \cdot (W^\sigma))_\tau \overset{P\text{-a.s.}}{\longrightarrow} \mathbb{1}_{(0,1)}(\sigma\wedge\tau)$$

for every $[0,1]$-valued stopping time τ.

For the general case $\varrho \not\equiv 0$, consider the process $\overline{W}_t := (W_{t+\varrho} - W_\varrho)_{0\le t\le 1}$ which is a Brownian motion with respect to the filtration $\overline{\mathbb{F}} := (\overline{\mathcal{F}}_t)_{0\le t\le 1} := (\mathcal{F}_{(t+\varrho)\wedge 1})_{0\le t\le 1}$ that is independent of $\mathcal{F}_\varrho$ and stopped at the $\overline{\mathbb{F}}$-stopping time $\bar{\sigma} := (1-\varrho)$. Then the general case $\varrho \not\equiv 0$ follows by applying the result for $\varrho \equiv 0$ for the stopped Brownian motion $\overline{W}$ and the stopping time $\bar{\tau} = (\tau - \varrho)_{\{\tau > \varrho\}}$ which is always smaller than $\bar{\sigma}$. Indeed, as the corresponding martingales $\overline{M}^n$ obtained for $\overline{W}$ with respect to $(\overline{\mathcal{F}}_t)_{0\le t\le 1}$ start at 0, the processes

$$M_t^n(\omega) = \begin{cases} 0, & t \le \varrho(\omega)\wedge 1, \\ \overline{M}^n_{t+\varrho(\omega)}(\omega), & \varrho(\omega) < t \le 1, \end{cases}$$

are martingales with respect to the filtration $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le 1}$ that converge to $\mathbb{1}_{[\![\varrho,1]\!]}(\tau)$ P-a.s. for every $[0,1]$-valued $\mathbb{F}$-stopping time τ. $\square$

PROOF OF THEOREM 2.8. Let $X = M - A$ be the Mertens decomposition of the optional strong supermartingale X. It is then sufficient to show the assertion for M and A separately.

(1) We begin with the local martingale M. As any localizing sequence $(\tau_m)_{m=1}^\infty$ of stopping times for M gives a sequence $\widetilde{M}^m := M^{\tau_m}$ of martingales that converges uniformly in probability, we obtain a sequence $\overline{M}^n$ of martingales that converges P-a.s. uniformly to M by passing to a subsequence $(\widetilde{M})_{n=1}^\infty$ such that $P(\tau_n < 1) < 2^{-n}$. To see that we can choose the M^n to be bounded, we observe that setting

$$\overline{M}_t^{n,k} := E\big[\overline{M}_1^n \wedge k \vee -k \,|\, \mathcal{F}_t\big]$$

for $t \in [0,1]$ gives for every martingale $\overline{M}^n$ a sequence of bounded martingales $\overline{M}^{n,k} = (\overline{M}_t^{n,k})_{0\le t\le 1}$ such that $\overline{M}_1^{n,k} \xrightarrow{L^1(P)} \overline{M}_1^n$, as $k \to \infty$, and therefore locally in $\mathcal{H}^1(P)$ by Theorem 4.2.1 in [10]. By the Burkholder–Davis–Gundy inequality (see, e.g., Theorem IV.48 in [16]), this also implies uniform convergence in probability and hence P-a.s. uniform convergence by passing to a subsequence, again indexed by k. Then taking a diagonal sequence $(\overline{M}^{n,k(n)})_{n=1}^\infty$ gives a sequence of martingales $(M^n)_{n=1}^\infty = (\overline{M}^{n,k(n)})_{n=1}^\infty$ that converges P-a.s. uniformly to M and therefore also satisfies (2.6) for every $[0,1]$-valued stopping time τ.

(2) To prove the assertion for the predictable part A, we decompose

$$A = A^c + \sum_{i=1}^\infty \Delta_+ A_{\sigma_i} \mathbb{1}_{]\!]\sigma_i,1]\!]} + \sum_{j=1}^\infty \Delta A_{\varrho_j} \mathbb{1}_{[\![\varrho_j,1]\!]}$$

into its continuous part A^c, its totally right-discontinuous part $A^{rd} := \sum_{i=1}^\infty \Delta_+ A_{\sigma_i} \mathbb{1}_{]\!]\sigma_i,1]\!]}$ and totally left-discontinuous part $A^{ld} := \sum_{j=1}^\infty \Delta A_{\varrho_j} \mathbb{1}_{[\![\varrho_j,1]\!]}$. By superposition it is sufficient to approximate $-A^c$, each single right jump process $-A_{\sigma_i}\mathbb{1}_{]\!]\sigma_i,1]\!]}$ for $i\in\mathbb{N}$ and each single left jump process $-\Delta A_{\varrho_j}\mathbb{1}_{[\![\varrho_j,1]\!]}$ for $j\in\mathbb{N}$ separately. Indeed, let $(M^{c,n})_{n=1}^\infty$, $(M^{rd,i,n})_{n=1}^\infty$ for each $i\in\mathbb{N}$ and $(M^{ld,j,n})_{n=1}^\infty$ for each $j\in\mathbb{N}$ be sequences of bounded martingales such that

$$M_\tau^{c,n}\xrightarrow{P}-A_\tau^c, \tag{5.2}$$

$$M_\tau^{rd,i,n}\xrightarrow{P}-\Delta_+A_{\sigma_i}\mathbb{1}_{]\!]\sigma_i,1]\!]}(\tau), \tag{5.3}$$

$$M_\tau^{ld,j,n}\xrightarrow{P}-\Delta A_{\varrho_j}\mathbb{1}_{[\![\varrho_j,1]\!]}(\tau), \tag{5.4}$$

as $n\to\infty$, for all $[0,1]$-valued stopping times τ. Then setting

$$M^n:=M^{c,n}+\sum_{i=1}^n M^{rd,i,n}+\sum_{j=1}^n M^{ld,j,n}$$

gives a sequence of bounded martingales such that $M_\tau^n\xrightarrow{P}-A_\tau$, as $n\to\infty$, for all $[0,1]$-valued stopping times τ.

(2a) We begin with showing the existence of $(M^{rd,i,n})_{n=1}^\infty$ for some fixed $i\in\mathbb{N}$. For this, we set

$$\vartheta_t^{i,n}:=(\Delta_+A_{\sigma_i}\wedge n)\mathbb{1}_{]\!]\sigma_i,1]\!]}\varphi_t^n\in L^2(W),$$

where $(\varphi^n)_{n=1}^\infty$ is a sequence of integrands as obtained in Lemma 5.1 for the stopping time $\varrho=\sigma_i$. Then it follows immediately from Lemma 5.1 that $\vartheta^{i,n}\bullet W_\tau\xrightarrow{P\text{-a.s.}}\Delta_+A_{\sigma_i}\mathbb{1}_{]\!]\sigma_i,1]\!]}(\tau)$, as $n\to\infty$, for every $[0,1]$-valued stopping time τ and therefore that

$$M^{rd,i,n}:=\vartheta^{i,n}\bullet W$$

gives a sequence of bounded martingales such that (5.3) holds. Note that by the construction of the integrands φ^n in Lemma 5.1 the approximating martingales $M^{rd,i,n}$ are 0 on $[\![0,\sigma_i]\!]$, constant to either $-\Delta_+A_{\sigma_i}\wedge n$ or $(\Delta_+A_{\sigma_i}\wedge n)k(n)$ on $[\![\sigma_i+2^{-n},1]\!]$. Therefore they converge P-a.s. uniformly to $-\Delta_+A_{\sigma_i}$ on $[\![\sigma_i+2^{-m},1]\!]$ for each $m\in\mathbb{N}$.

(2b) To obtain the approximating sequence $(M^{ld,j,n})_{n=1}^\infty$ for some fixed $j\in\mathbb{N}$, we observe that the stopping time ϱ_j is predictable and let $(\varrho_{j,k})_{k=1}^\infty$ be an announcing sequence of stopping times, that is, a nondecreasing sequence of stopping times such that $\varrho_{j,k}<\varrho_j$ on $\{\varrho_j>0\}$ and $\varrho_{j,k}\xrightarrow{P\text{-a.s.}}\varrho_j$, as $k\to\infty$. Since $\Delta A_{\varrho_j}\in L^1(P)$ is $\mathcal{F}_{\varrho_j-}$-measurable by Theorem IV.67.b) in [7] and $\mathcal{F}_{\varrho_j-}=\bigvee_{k=1}^\infty\mathcal{F}_{\varrho_{j,k}}$ by Theorem IV.56.d) in [7], we have that

$$E[\Delta A_{\varrho_j}|\mathcal{F}_{\varrho_{j,k}}]\xrightarrow{P\text{-a.s.}}\Delta A_{\varrho_j},\qquad\text{as } k\to\infty, \tag{5.5}$$

by martingale convergence. Therefore setting

$$\widetilde{A}^{ld,j,k}:=E[\Delta A_{\varrho_j}|\mathcal{F}_{\varrho_{j,k}}]\mathbb{1}_{]\!]\varrho_{j,k},1]\!]} \tag{5.6}$$

gives a sequence of single right jump processes that converges to $\Delta A_{\varrho_j}\mathbb{1}_{[\![\varrho_j,1]\!]}$ P-a.s. at each $[0,1]$-valued stopping time τ, since $\mathbb{1}_{]\!]\varrho_{j,k},1]\!]}(\tau) \xrightarrow{P\text{-a.s.}} \mathbb{1}_{[\![\varrho_j,1]\!]}(\tau)$, as $k\to\infty$, for all $[0,1]$-valued stopping times τ.

By part (2a) there exists for each $k\in\mathbb{N}$ a sequence $(\widetilde{M}^{j,k,n})_{n=1}^{\infty}$ of bounded martingales such that $\widetilde{M}^{j,k,n}_\tau \xrightarrow{P\text{-a.s.}} -\widetilde{A}^{ld,j,k}_\tau$, as $n\to\infty$, for all $[0,1]$-valued stopping times τ. For the stopping time ϱ_j we can therefore find a diagonal sequence $(\widetilde{M}^{j,k,n(k)})_{k=1}^{\infty}$ such that $\widetilde{M}^{j,k,n(k)}_{\varrho_j} \xrightarrow{P\text{-a.s.}} -\widetilde{A}^{ld,j,k}_{\varrho_j}$, as $k\to\infty$. By the proof of Lemma 5.1 and part (2a) above we can choose the martingales $\widetilde{M}^{j,k,n(k)}$ such that $\widetilde{M}^{j,k,n(k)}\equiv 0$ on $[\![0,\varrho_{j,k}]\!]$ and $\widetilde{M}^{j,k,n(k)}\equiv -(E[\Delta A_{\varrho_j}|\mathcal{F}_{\varrho_{j,k}}]\wedge n(k))$ on $[\![(\varrho_{j,k}+2^{-n(k)})_{F_k},1]\!]$, where the set

$$F_k := \Big\{\widetilde{M}^{j,k,n(k)}_{\varrho_j+2^{-n(k)}} = -\big(E[\Delta A_{\varrho_j}|\mathcal{F}_{\varrho_{j,k}}]\wedge n(k)\big)\Big\}$$

has probability $P(F_k)>1-2^{-k}$. This sequence $(\widetilde{M}^{j,k,n(k)})_{k=1}^{\infty}$ therefore already satisfies $\widetilde{M}^{j,k,n(k)}_\tau \xrightarrow{P\text{-a.s.}} -\Delta A_{\varrho_j}\mathbb{1}_{[\![\varrho_j,1]\!]}(\tau)$ for all $[0,1]$-valued stopping times τ and we have (5.4).

(2c) For the approximation of the continuous part A^c, we observe that by the left-continuity and adaptedness of A^c there exists a sequence $(\widetilde{A}^n)_{n=1}^{\infty}$ of nondecreasing integrable simple predictable processes that converges uniformly in probability to A^c and hence P-a.s. uniform by passing to a fast convergent subsequence again indexed by n; see for example Theorem II.10 in [16]. Recall that a simple predictable process is a predictable process $\widetilde{A}$ of the form

$$\widetilde{A} = \sum_{i=1}^{m} \Delta_+ A_{\sigma_i}\mathbb{1}_{]\!]\sigma_i,1]\!]}, \tag{5.7}$$

where $(\sigma_i)_{i=1}^{m}$ are $[0,1]\cup\{\infty\}$-valued stopping times such that $\sigma_i<\sigma_{i+1}$ for $i=1,\dots,m-1$ and $\Delta_+A_{\sigma_i}$ is $\mathcal{F}_{\sigma_i}$-measurable.

By part (2a) there exists, for each $n\in\mathbb{N}$, a sequence $(\widetilde{M}^{n,k})_{k=1}^{\infty}$ of martingales such that $\widetilde{M}^{n,k}_\tau \xrightarrow{P\text{-a.s.}} -\widetilde{A}^n_\tau$, as $k\to\infty$, for all $[0,1]$-valued stopping times τ. Therefore we can pass to a diagonal sequence $\widetilde{M}^{n,k(n)}$ such that

$$P\Big[\lim_{n\to\infty}\widetilde{M}^{n,k(n)}_q = -A^c_q, \forall q\in\mathbb{Q}\cap[0,1]\Big]=1. \tag{5.8}$$

By Theorem 2.7 there exists a sequence $(M^n)_{n=1}^{\infty}$ of convex combinations

$$M^n\in\operatorname{conv}\big(\widetilde{M}^{n,k(n)},\widetilde{M}^{n+1,k(n+1)},\dots\big)$$

and an optional strong supermartingale X such that $M^n_\tau \xrightarrow{P} X_\tau$ for all $[0,1]$-valued stopping times τ.

To complete the proof it therefore only remains to show that $X=-A^c$. For this, we argue by contradiction and assume that the optional set $G:=\{X\neq -A^c\}$ is not evanescent, that is, that $P(\pi(G)) > 0$, where $\pi((\omega,t)) = \omega$ denotes the projection on the first component. By the optional cross-section theorem (Theorem IV.84 in [8]) there then exists a $[0,1] \cup \{\infty\}$-valued stopping time τ such that $X_\tau \neq -A^c_\tau$ on $F := \{\tau < \infty\}$ with $P(F) > 0$, which we can decompose into an accessible stopping time τ^A and a totally inaccessible stopping time τ^I such that $\tau = \tau^A \wedge \tau^I$ by Theorem IV.81.c) in [7]. On $\{\tau^I < \infty\}$ we obtain that $M^n_{\tau^I -} = M^n_{\tau^I} \xrightarrow{P} X_{\tau^I}$ and $A^c_{\tau^I -} = A^c_{\tau^I}$ from the continuity of M^n and A^c. Therefore $X_{\tau^I} = -A^c_{\tau^I}$, as $M^n_{\tau^I -} \xrightarrow{P} X_{\tau^I -}$ by Proposition 2.9 and $X_{\tau^I -} = -A^c_{\tau^I -}$ by (5.8). This implies that $P(\tau^I < \infty) = 0$ and hence $P(\tau^A < \infty) = P(F) > 0$. Since τ^A is accessible, there exists a predictable stopping time σ such that $P(\tau^A = \sigma < \infty) > 0$. By the strong supermartingale property of X we have that

$$X_{\sigma-} \geq E[X_\sigma | \mathcal{F}_{\sigma-}] \geq E[X_{\sigma+} | \mathcal{F}_{\sigma-}] \qquad \text{on } \{\sigma < \infty\},$$

as σ is predictable. Since $X_- = -A^c_-$ and $X_+ = -A^c_+$ by (5.8), this implies that $X_\sigma = -A^c_\sigma$ by the continuity of A^c. However, this contradicts $P(F) > 0$ and therefore shows (5.2), which completes the proof. □

6. Proof of Theorem 2.11. We begin with the proof of Proposition 2.9, and for this, we will use the following variant of Doob's up-crossing inequality that holds uniformly over the set $\mathfrak{X}$ of nonnegative optional strong supermartingales $X = (X_t)_{0 \leq t \leq 1}$ starting at $X_0 = 1$.

LEMMA 6.1. *For each* $\varepsilon > 0$ *and* $\delta > 0$, *there exists a constant* $C = C(\varepsilon, \delta) \in \mathbb{N}$ *such that*

$$\sup_{X \in \mathfrak{X}} P\big[M_\varepsilon(X) > C\big] < \delta,$$

where the random variable $M_\varepsilon(X)$ *is pathwise defined as the maximal amount of moves of the process* X *of size bigger than* ε, *that is,*

$$M_\varepsilon(X)(\omega) := \sup\Big\{m \in \mathbb{N} \Big| \big|X_{t_i}(\omega) - X_{t_{i-1}}(\omega)\big| > \varepsilon, \text{ for } 0 \leq t_0 < t_1 < \cdots < t_m \leq 1\Big\}.$$

PROOF. Choose $n \in \mathbb{N}$ such that $\frac{1}{n} \leq \frac{\varepsilon}{2}$, fix some $X \in \mathfrak{X}$ and denote by $X = M - A$ its Mertens decomposition. Then $M = X + A$ is a nonnegative càdlàg local martingale and hence a càdlàg supermartingale such that

$$E[M_t] \leq 1$$

for all $t \in [0,1]$. Letting $C_1 \in \mathbb{N}$ with $C_1 \geq \frac{2}{\delta}$ we obtain from Doob's maximal inequality that

$$P\Big(M^*_1 := \sup_{0 \leq s \leq 1} M_s > C_1\Big) \leq \frac{1}{C_1} \leq \frac{\delta}{2}.$$

Then we divide the interval $[0, C_1]$ into $nC_1 =: N$ subintervals $I_k := [\frac{k}{N}, \frac{k+1}{N}]$ of equal length of at most $\frac{\varepsilon}{2}$ for $k = 0, \dots, N-1$. The basic intuition behind this is that whenever the nonnegative (càdlàg) local martingale $M = (M_t)_{0\le t\le 1}$ moves more than ε, while its supremum stays below C_1, it has at least to cross one of the subintervals I_k. For each interval I_k we can estimate the number $U(M; I_k)$ of up-crossings of the interval I_k by the process $M = (M_t)_{0\le t\le 1}$ up to time 1 by Doob's up-crossing inequality by

$$P\bigl[U(M; I_k) > C_2\bigr] \le \frac{N}{C_2} E\bigl[U(M; I_k)\bigr] \le \frac{N}{C_2} \sup_{0\le t\le 1} E[M_t] \le \frac{N}{C_2}.$$

Choosing $\tilde{C}_2 = \frac{2N^2}{\delta}$ we obtain that

$$P\bigl[U(M; I_k) > \tilde{C}_2\bigr] \le \frac{\delta}{2N}.$$

Then summing over all intervals gives for the number $U_\varepsilon(M)$ of up-moves of the process M of size ε that

$$\begin{aligned} &P\bigl[U_\varepsilon(M) > \tilde{C}_2 N\bigr] \\ &\quad\le P\bigl[M_1^* \le C_1, \exists k \in \{1, \dots, N\} \text{ with } U(M; I_k) > \tilde{C}_2\bigr] + P\bigl[M_1^* > C_1\bigr] \le \delta. \end{aligned}$$

Since $X = M - A$ is nonnegative starting at $X_0 = 1$ and A is nondecreasing, the number $M_\varepsilon(X)$ of moves of X of size ε is smaller than $2(U_\varepsilon(X) + N)$. Therefore we can conclude that

$$P\bigl[M_\varepsilon(X) > C\bigr] \le \delta \tag{6.1}$$

for $C = 2(\tilde{C}_2 + 1)N$. To complete the proof, we observe that the constants C_1 and $C = 2(\tilde{C}_2 + 1)N$ are independent of the choice of the optional strong supermartingale $X \in \mathfrak{X}$, and we can therefore take the supremum over all $X \in \mathfrak{X}$ in the inequality (6.1). □

Let $X = (X_t)_{0\le t\le 1}$ be a làg (existence of left limits) process and τ be a $(0, 1]$-valued stopping time. For $m \in \mathbb{N}$, let τ_m be the mth dyadic approximation of the stopping time τ as defined in (3.2). Note that τ_m is $\{\frac{1}{2^m}, \dots, 1\}$-valued, as $\tau > 0$. As $(X_t)_{0\le t\le 1}$ is assured to have làg trajectories, we obtain

$$X_{\tau_m - 2^{-m}} \xrightarrow{P\text{-a.s.}} X_{\tau-}, \qquad \text{as } m \to \infty, \tag{6.2}$$

and therefore in probability. The next lemma gives a quantitative version of this rather obvious fact.

LEMMA 6.2. *Let τ be a totally inaccessible $(0, 1]$-valued stopping time. Then the convergence in (6.2) above holds true in probability uniformly over all nonnegative optional strong supermartingales $X \in \mathfrak{X}$, that is, $X = (X_t)_{0\le t\le 1}$, starting at $X_0 = 1$. More precisely, we have for each $\varepsilon > 0$ that*

$$\lim_{m\to\infty} \sup_{X\in\mathfrak{X}} P\bigl[|X_{\tau_m - 2^{-m}} - X_{\tau-}| > \varepsilon\bigr] = 0. \tag{6.3}$$

PROOF. Denote by $A = (A_t)_{0\le t\le 1}$ the compensator of τ, which is the unique continuous increasing process such that $(\mathbb{1}_{[\![\tau,1]\!]} - A_t)_{0\le t\le 1}$ is a martingale. For every predictable set $G \subseteq \Omega \times [0,1]$, we then have

$$P[\tau \in G] = E[\mathbb{1}_G \mathbb{1}_{[\![\tau]\!]}] = E\left[\int_0^1 \mathbb{1}_G(t)\, d\mathbb{1}_{[\![\tau,1]\!]}(t)\right] = E\left[\int_0^1 \mathbb{1}_G(t)\, dA_t\right]. \tag{6.4}$$

Here we used that the predictable σ-algebra on $\Omega \times [0,1]$ is generated by the left-open stochastic intervals, that is, intervals of the form $]\!]\sigma_1, \sigma_2]\!]$ for stopping times σ_1 and σ_2 and a monotone class argument to deduce the second equality in (6.4). The third equality is the definition of the compensator. Fix $X \in \mathfrak{X}$, $\varepsilon > 0$, $\delta > 0$ and apply Lemma 6.1 and the integrability of A_1 to find $c = c(\varepsilon, \delta, \tau)$ such that the exceptional set

$$F_1 = \{M_\varepsilon(X) \ge c\} \tag{6.5}$$

satisfies

$$E[\mathbb{1}_{F_1} A_1] < \delta. \tag{6.6}$$

Find m large enough such that

$$E[\mathbb{1}_{F_2} A_1] < \delta, \tag{6.7}$$

where F_2 is the exceptional set

$$F_2 = \left\{\exists k \in \{1, \dots, 2^m\} \text{ such that } A_{k/2^m} - A_{(k-1)/2^m} > \frac{\delta}{c}\right\}. \tag{6.8}$$

Define G to be the predictable set

$$\begin{aligned} G = \bigcup_{k=1}^{2^m} \bigg\{(\omega, t) \bigg| \frac{k-1}{2^m} < t \le \frac{k}{2^m} \text{ and} \\ \sup_{(k-1)/2^m \le u \le t} \big|X_{u-}(\omega) - X_{(k-1)/2^m}(\omega)\big| \le \varepsilon\bigg\}. \end{aligned} \tag{6.9}$$

We then have $P[\tau \notin G] < 3\delta$. Indeed, applying (6.4) to the complement G^c of G we get

$$P[\tau \notin G] = E\left[(\mathbb{1}_{F_1 \cup F_2} + \mathbb{1}_{\Omega \setminus (F_1 \cup F_2)}) \int_0^1 \mathbb{1}_{G^c}\, dA_t\right],$$

where F_1 and F_2 denote the exceptional sets in (6.5) and (6.8). By (6.6) and (6.7),

$$E\left[\mathbb{1}_{F_1 \cup F_2} \int_0^1 \mathbb{1}_{G^c}\, dA_t\right] \le 2\delta. \tag{6.10}$$

On the set $\Omega \setminus (F_1 \cup F_2)$ we deduce from (6.5), (6.8) and (6.9) that

$$\int_0^1 \mathbb{1}_{G^c}\, dA_t \le c\frac{\delta}{c} = \delta$$

so that

$$P[\tau \notin G] \leq 3\delta. \tag{6.11}$$

For $(\omega, t) \in G$ such that $\frac{k-1}{2^m} < t \leq \frac{k}{2^m}$, we have

$$\left|X_{t-}(\omega) - X_{(k-1)/2^m}(\omega)\right| \leq \varepsilon$$

so that by (6.11) we get

$$P\left[|X_{\tau-} - X_{\tau_m - 2^{-m}}| > \varepsilon\right] < 3\delta,$$

which shows (6.3). □

PROOF OF PROPOSITION 2.9. Fix $\varepsilon > 0$, and apply Lemma 6.2 to find $m \in \mathbb{N}$ such that

$$P\left[|\widetilde{X}_{\tau_m - 2^{-m}} - \widetilde{X}_{\tau-}| > \varepsilon\right] < \varepsilon, \tag{6.12}$$

for each $\widetilde{X} \in \mathfrak{X}$. As $(X_q^n)_{n=1}^{\infty}$ converges to X_q in probability, for every rational number $q \in \mathbb{Q} \cap [0, 1]$ we have

$$P\left[\max_{0 \leq k \leq 2^m} |X_{k/2^m}^n - X_{k/2^m}| > \varepsilon\right] < \varepsilon,$$

for all $n \geq N(\varepsilon)$. We then may apply (6.12) to X^n and X to conclude that

$$P\left[\left|X_{\tau-}^n - X_{\tau-}\right| > 3\varepsilon\right] < 3\varepsilon. \qquad \square$$

With Proposition 2.9 we have now everything in place to prove Theorem 2.11.

PROOF OF THEOREM 2.11. The existence of the optional strong supermartingale $X^{(1)}$ is the assertion of Theorem 2.7. To obtain the predictable strong supermartingale $X^{(0)}$, we observe that, since $\widetilde{X}^n$ and $X^{(1)}$ are làdlàg, the optional set

$$F := \bigcup_{n=1}^{\infty} \{\widetilde{X}^n \neq \widetilde{X}_-^n\} \cup \{X^{(1)} \neq X_-^{(1)}\}$$

has at most countably many sections, and therefore there exists by Theorem 117 in Appendix IV of [7] a countable number of $[0, 1] \cup \{\infty\}$-valued stopping times $(\sigma_m)_{m=1}^{\infty}$ with disjoint graphs such that $F = \bigcup_{m=1}^{\infty} [\![\sigma_m]\!]$. By Theorem IV.81.c) in [7] we can decompose each stopping time σ_m into an accessible stopping time σ_m^A and a totally inaccessible stopping time σ_m^I such that $\sigma_m = \sigma_m^A \wedge \sigma_m^I$. Again combining Komlós's lemma with a diagonalization procedure we obtain a sequence of convex combinations $\widetilde{X}^n \in \operatorname{conv}(X^n, X^{n+1}, \dots)$ such that $\widetilde{X}_\tau^n \xrightarrow{P} X_\tau^{(1)}$ for all $[0, 1]$-valued stopping times τ as well as

$$\widetilde{X}_{\tau_m -}^n \xrightarrow{P\text{-a.s.}} Y_m^{(0)}, \qquad \text{as } n \to \infty,$$

for all stopping times $\tau_m := \sigma_m^A \wedge 1$ and suitable nonnegative random variables $Y_m^{(0)}$ for $m \in \mathbb{N}$. Now we can define $X^{(0)}$ by

$$X_t^{(0)}(\omega) = \begin{cases} Y_m^{(0)}(\omega), & t = \sigma_m^A(\omega) \text{ and } m \in \mathbb{N}, \\ X_{t-}^{(1)}(\omega) = X_t^{(1)}(\omega), & \text{else.} \end{cases}$$

For all $[0, 1]$-valued stopping times τ, we then have convergence (2.10), that is,

$$\begin{aligned} \widetilde{X}^n_{\tau-}(\omega) &= \widetilde{X}^n_\tau(\omega)\mathbb{1}_F\big(\omega, \tau(\omega)\big) + \sum_{m=1}^\infty \widetilde{X}^n_{\tau_m^-}\mathbb{1}_{\{\sigma_m^A=\tau\}} + \sum_{m=1}^\infty \widetilde{X}^n_{\sigma_m^I-}\mathbb{1}_{\{\sigma_m^I=\tau\}} \\ &\xrightarrow{P} X_\tau^{(0)}(\omega)\mathbb{1}_F\big(\omega, \tau, (\omega)\big) + \sum_{m=1}^\infty Y_m^{(0)}\mathbb{1}_{\{\sigma_m^A=\tau\}} + \sum_{m=1}^\infty X^{(1)}_{\sigma_m^I-}\mathbb{1}_{\{\sigma_m^I=\tau\}}, \end{aligned}$$

since $\widetilde{X}^n = \widetilde{X}^n_-$ for all $n \in \mathbb{N}$ on F and $\widetilde{X}^n_{\sigma-}\mathbb{1}_{\{\sigma=\tau\}} \xrightarrow{P} X_{\sigma-}\mathbb{1}_{\{\sigma=\tau\}}$ for all $[0, 1]$-valued totally inaccessible stopping times τ by Proposition 2.9. As all stopping times σ_m^A are accessible and each $Y_m^{(0)}$ is $\mathcal{F}_{\tau_m-}$-measurable, we have that $X^{(0)}$ is an accessible process such that $X_\tau^{(0)}\mathbb{1}_{\{\tau<\infty\}}$ is $\mathcal{F}_{\tau-}$-measurable for every stopping time τ. Therefore $X^{(0)}$ is by Theorem 3.20 in [6] even predictable. By Remark 5.(c) in Appendix I of [8] the left limit process $\widetilde{X}^n_-$ of each optional strong supermartingale $\widetilde{X}^n$ is a predictable strong supermartingale satisfying

$$\widetilde{X}^n_{\tau-} \geq E\big[\widetilde{X}^n_\tau|\mathcal{F}_{\tau-}\big]$$

for all $[0, 1]$-valued predictable stopping times. Therefore the predictable strong supermartingale property [part (3) of Definition 2.10] and $X_\tau^{(0)} \geq E[X_\tau^{(1)}|\mathcal{F}_{\tau-}]$ follow immediately from (2.9) and (2.10) by Fatou's lemma. To see $X_{\tau-}^{(1)} \geq X_\tau^{(0)}$, let $(\tau_m)_{m=1}^\infty$ be a foretelling sequence of stopping times for the predictable stopping time τ. Then we have

$$\widetilde{X}^n_{\tau_m} \geq E\big[\widetilde{X}^n_{\tau_{m+k}}|\mathcal{F}_{\tau_m}\big]$$

for all $n, m, k \in \mathbb{N}$. Applying Fatou's lemma we then obtain

$$\widetilde{X}^n_{\tau_m} \geq E\big[\widetilde{X}^n_{\tau-}|\mathcal{F}_{\tau_m}\big]$$

by sending $k \to \infty$,

$$X^{(1)}_{\tau_m} \geq E\big[X^{(0)}_{\tau-}|\mathcal{F}_{\tau_m}\big]$$

by sending also $n \to \infty$ and finally $X_{\tau-}^{(1)} \geq X_\tau^{(0)}$ by sending $m \to \infty$. $\square$

7. Proof of Proposition 2.12. One application of Theorem 2.11 is a convergence result for stochastic integrals of predictable integrands of finite variation with respect to nonnegative optional strong supermartingales.

Fix a nonnegative optional strong supermartingale $X \in \mathfrak{X}$, and let $\varphi = (\varphi_t)_{0\le t\le 1}$ be a predictable process of finite variation, so that it has làdlàg paths. We then define

$$\begin{aligned}\int_0^t X_u(\omega)\,d\varphi_u(\omega) := \int_0^t X_u(\omega)\,d\varphi_u^c(\omega) + \sum_{0<u\le t} X_{u-}(\omega)\Delta\varphi_u(\omega)\\ + \sum_{0\le u<t} X_u(\omega)\Delta_+\varphi_u(\omega)\end{aligned}\tag{7.1}$$

for all $t \in [0,1]$, which is P-a.s. pathwise well defined, as X is ládlág and φ of finite variation. Here the integral $\int_0^t X_u(\omega)\,d\varphi_u^c(\omega)$ with respect to the continuous part φ^c [see (2.12)] can be defined as a pathwise Riemann–Stieltjes integral or a pathwise Lebesgue–Stieltjes integral, as both integrals coincide.

To ensure the integration by parts formula

$$\varphi_t(\omega)X_t(\omega) - \varphi_0(\omega)X_0(\omega) = \int_0^t \varphi_u(\omega)\,dX_u(\omega) + \int_0^t X_u(\omega)\,d\varphi_u(\omega),\tag{7.2}$$

we define the stochastic integral $\varphi \cdot X_t := \int_0^t \varphi_u\,dX_u$ by

$$\begin{aligned}\int_0^t \varphi_u(\omega)\,dX_u(\omega) := \int_0^t \varphi_u^c(\omega)\,dX_u(\omega) + \sum_{0<u\le t} \Delta\varphi_u(\omega)\big(X_t(\omega) - X_{u-}(\omega)\big)\\ + \sum_{0\le u<t} \Delta_+\varphi_u(\omega)\big(X_t(\omega) - X_u(\omega)\big)\end{aligned}\tag{7.3}$$

for $t \in [0,1]$ that is again pathwise well defined. The integral $\int_0^t \varphi_u^c(\omega)\,dX_u(\omega)$ can again be defined as a pathwise Riemann–Stieltjes integral or a pathwise Lebesgue–Stieltjes integral. If $X = (X_t)_{0\le t\le 1}$ is a semimartingale, the definition of $(\int_0^t \varphi_u\,dX_u)_{0\le t\le 1}$ via (7.3) coincides with the classical stochastic integral.

We first derive an auxiliary result.

LEMMA 7.1. *Let $(X^n)_{n=1}^\infty$, $X^{(0)}$ and $X^{(1)}$ be làdlàg stochastic processes such that*:

(i) $X_\tau^n \xrightarrow{P} X_\tau^{(1)}$ *and* $X_{\tau-}^n \xrightarrow{P} X_\tau^{(0)}$ *for all* $[0,1]$*-valued stopping times* τ;

(ii) *for all* $\varepsilon > 0$ *and* $\delta > 0$, *there are constants* $C_1(\delta) > 0$ *and* $C_2(\varepsilon,\delta) > 0$ *such that*

$$\sup_{X\in\mathcal{X}^0} P\Big[\sup_{0\le s\le 1} |X_s| > C_1(\delta)\Big] \le \delta,\tag{7.4}$$

$$\sup_{X\in\mathcal{X}^1} P\big[M_\varepsilon(X) > C_2(\varepsilon,\delta)\big] \le \delta,\tag{7.5}$$

where $\mathcal{X}^0 = \{X^{(0)}, X^{(1)}, X^n, X^n_- \text{ for } n \in \mathbb{N}\}$, $\mathcal{X}^1 = \{X^{(1)}, X^n \text{ for } n \in \mathbb{N}\}$ *and*

$$M_\varepsilon(X) := \sup\Big\{m \in \mathbb{N} \Big| \big|X_{t_i}(\omega) - X_{t_{i-1}}(\omega)\big| > \varepsilon \text{ for } 0 \le t_0 < t_1 < \cdots < t_m \le 1\Big\}$$

for $X \in \mathcal{X}^1$.

Then we have, for all predictable processes $\varphi = (\varphi_t)_{0\le t\le 1}$ *of finite variation, that*:

$$\int_0^\tau X^n_u \, d\varphi_u \xrightarrow{P} \int_0^\tau X^{(1)}_u \, d\varphi^c_u + \sum_{0<u\le\tau} X^{(0)}_u \Delta\varphi_u + \sum_{0\le u<\tau} X^{(1)}_u \Delta_+\varphi_u; \tag{1}$$

$$\begin{aligned}\int_0^\tau \varphi_u \, dX^n_u \xrightarrow{P} {} & \int_0^\tau \varphi^c_u \, dX^{(1)}_u + \sum_{0<u\le\tau} \Delta\varphi_u\big(X^{(1)}_\tau - X^{(0)}_u\big) \\ & + \sum_{0\le u<\tau} \Delta_+\varphi_u\big(X^{(1)}_\tau - X^{(1)}_u\big)\end{aligned} \tag{2}$$

for all $[0, 1]$*-valued stopping times* τ. *Convergence* (1) *is even uniformly in probability.*

PROOF. (1) We first show that

$$\sup_{0\le t\le 1}\left|\sum_{0<u\le t} X^n_{u-}\Delta\varphi_u - \sum_{0<u\le t} X^{(0)}_{u-}\Delta\varphi_u\right| \xrightarrow{P} 0, \qquad \text{as } n \to \infty, \tag{7.6}$$

that is, uniformly in probability. The proof of the convergence

$$\sup_{0\le t\le 1}\left|\sum_{0<u\le t} X^n_u \Delta_+\varphi_u - \sum_{0<u\le t} X^{(1)}_{u-}\Delta_+\varphi_u\right| \xrightarrow{P} 0, \qquad \text{as } n \to \infty,$$

is completely analog and therefore omitted.

Since φ is predictable and of finite variation and hence làdlàg, there exists a sequence $(\tau_m)_{m=1}^\infty$ of $[0, 1] \cup \{\infty\}$-valued stopping times exhausting the jumps of φ. Using the stopping times $(\tau_m)_{m=1}^\infty$ we can write

$$\sum_{0<u\le t} X_u \Delta\varphi_u = \sum_{m=1}^\infty X_{\tau_m}\Delta\varphi_{\tau_m}\mathbb{1}_{\{\tau_m\le t\}}$$

for all $X \in \mathcal{X}^0$ and estimate

$$\begin{aligned}&\sup_{0\le t\le 1}\left|\sum_{m=1}^\infty X^n_{\tau_m-}\Delta\varphi_{\tau_m}\mathbb{1}_{\{\tau_m\le t\}} - \sum_{m=1}^\infty X^{(0)}_{\tau_m}\Delta\varphi_{\tau_m}\mathbb{1}_{\{\tau_m\le t\}}\right| \\ &\quad \le \sum_{m=1}^N \big|X^n_{\tau_m-} - X^{(0)}_{\tau_m}\big|\big|\Delta\varphi_{\tau_m}\big| + \sup_{m\in\mathbb{N}}\big|X^n_{\tau_m-} - X^{(0)}_{\tau_m}\big| \sum_{m=N+1}^\infty |\Delta\varphi_{\tau_m}|.\end{aligned} \tag{7.7}$$

Combining (7.7) with the fact that φ is of finite variation we obtain (7.6), as

$$\sup_{m\in\mathbb{N}}|X^n_{\tau_m-}-X^{(0)}_{\tau_m}|\sum_{m=N+1}^{\infty}|\Delta\varphi_{\tau_m}|\xrightarrow{P}0,\qquad \text{as } N\to\infty,$$

by (7.4) and $\sum_{m=1}^{N}|X^n_{\tau_m-}-X^{(0)}_{\tau_m}||\Delta\varphi_{\tau_m}|\xrightarrow{P}0$, as $n\to\infty$, for each N by assumption (i).

The key observation for the proof of the convergence

$$\sup_{0\le t\le 1}\left|\int_0^t X^n_u\,d\varphi^c_u-\int_0^t X^{(1)}_u\,d\varphi^c_u\right|\xrightarrow{P}0,\qquad \text{as } n\to\infty, \tag{7.8}$$

is that we can use assumption (ii) to approximate the stochastic Riemann–Stieltjes integrals by Riemann sums in probability uniformly for all $X\in\mathcal{X}^1$, as either the integrator or the integrand moves very little. Indeed, for $\varepsilon>0$ and $c_1,c_2>0$, we have that

$$\begin{aligned}
&\sup_{0\le t\le 1}\left|\int_0^t X_u\,d\varphi^c_u-\sum_{m=1}^{N}X_{\sigma_{m-1}}(\varphi^c_{\sigma_m\wedge t}-\varphi^c_{\sigma_{m-1}\wedge t})\right|\\
&\quad\le\sum_{m=1}^{N}\sup_{u\in[\sigma_{m-1},\sigma_m]}|X_u-X_{\sigma_{m-1}}|\big(|\varphi^c|_{\sigma_m}-|\varphi^c|_{\sigma_{m-1}}\big)\\
&\quad\le c_2 2c_1\frac{\varepsilon}{4c_1c_2}+\frac{\varepsilon}{2c_1}c_1=\varepsilon
\end{aligned}$$

on $\{|\varphi|_1\le c_1\}\cap\{X^*_1\le c_1\}\cap\{M_{\varepsilon/(2c_1)}(X)\le c_2\}$, where the stopping times $(\sigma_m)_{m=0}^{\infty}$ are given by $\sigma_0=0$ and

$$\sigma_m:=\inf\left\{t>\sigma_{m-1}\,\middle|\,|\varphi^c|_t-|\varphi^c|_{\sigma_{m-1}}>\frac{\varepsilon}{4c_1c_2}\right\}\wedge 1$$

and $N=\frac{4c_1c_2}{\varepsilon}$. Choosing $c_1,c_2>0$ and hence N sufficiently large we therefore obtain

$$\sup_{X\in\mathcal{X}^1}P\left(\sup_{0\le t\le 1}\left|\int_0^t X_n\,d\varphi^c_u-\sum_{m=1}^{N}X_{\sigma_{m-1}}(\varphi^c_{\sigma_m\wedge t}-\varphi^c_{\sigma_{m-1}\wedge t})\right|>\varepsilon\right)<\delta$$

for any $\delta>0$ by assumption (ii). Combing this with the estimate

$$\begin{aligned}
&\sup_{0\le t\le 1}\left|\int_0^t X^n_u\,d\varphi^c_u-\int_0^t X^{(1)}_u\,d\varphi^c_u\right|\\
&\quad\le\sup_{0\le t\le 1}\left|\int_0^t X^n_u\,d\varphi^c_u-\sum_{m=1}^{N}X^n_{\sigma_{m-1}}(\varphi^c_{\sigma_m\wedge t}-\varphi^c_{\sigma_{m-1}\wedge t})\right|
\end{aligned}$$

$$+\sum_{m=1}^{N}\left|X^n_{\sigma_{m-1}}-X^{(1)}_{\sigma_{m-1}}\right|\left(|\varphi^c|_{\sigma_m}-|\varphi^c|_{\sigma_{m-1}}\right)$$

$$+\sup_{0\le t\le 1}\left|\int_0^t X^{(1)}_u\,d\varphi^c_u-\sum_{m=1}^{N}X^{(1)}_{\sigma_{m-1}}\left(\varphi^c_{\sigma_m\wedge t}-\varphi^c_{\sigma_{m-1}\wedge t}\right)\right|$$

then implies (7.8), as

$$\max_{m=0,\dots,N-1}\left|X^n_{\sigma_m}-X^{(1)}_{\sigma_m}\right|\xrightarrow{P}0,\qquad \text{as } n\to\infty,$$

for each fixed N by assumption (i).

(2) As $X^n_\tau\varphi_\tau\xrightarrow{P}X^{(1)}_\tau\varphi_\tau$ for all [0, 1]-valued stopping times, this assertion follows immediately from (1) and the integration by parts formula (7.2). $\square$

Combining the previous lemma with Lemma 6.1 allows us now to complete the proof of Proposition 2.12.

PROOF OF PROPOSITION 2.12. Part (1) is Theorem 2.11, and part (2) follows from Lemma 7.1 as soon as we have shown that its assumptions are satisfied. Assumption (i) is (1) and for the set $\mathcal{X}^1$ assumption (ii) can be derived from Lemma 6.1. Therefore it only remains to show (7.4) for $X^{(0)}$ and X^n_- for $n\in\mathbb{N}$. For the left limits (7.4) follows from the validity of the latter for the processes X^n for $n\in\mathbb{N}$ and for the predictable strong supermartingale $X^{(0)}$ from (3.1) in Appendix I of [8]. $\square$

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