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PATHWISE NONUNIQUENESS FOR THE SPDES OF SOME SUPER-BROWNIAN MOTIONS WITH IMMIGRATION[1]

BY YU-TING CHEN

Harvard University

We prove pathwise nonuniqueness in the stochastic partial differential equations (SPDEs) for some one-dimensional super-Brownian motions with immigration. In contrast to a closely related case investigated by Mueller, Mytnik and Perkins [*Ann. Probab.* (2014) To appear], the solutions of the present SPDEs are assumed to be nonnegative and have very different properties including uniqueness in law. In proving possible separation of solutions, we derive delicate properties of certain correlated approximating solutions, which is based on a novel coupling method called continuous decomposition. In general, this method may be of independent interest in furnishing solutions of SPDEs with intrinsic adapted structure.

1. Introduction. In this work, we consider some one-dimensional super-Brownian motions with (continuous) immigration, and construct pairs of distinct *nonnegative* solutions to the associated stochastic partial differential equations (SPDEs). Hence, we resolve in the negative the long-standing open problem concerning the pathwise uniqueness in the SPDEs for one-dimensional super-Brownian motions, when additional immigration is present (cf. page 217 in Perkins [21]).

We start with some informal descriptions for the class of super-Brownian motions with immigration which are considered throughout this work. See Dawson [7], Dynkin [8], Le Gall [16], Perkins [21] and several others for super-Brownian motions as well as their connections with branching processes. Imagine that, in the barren territory $\mathbb{R}$, clouds of independent immigrants with infinitesimal initial mass land randomly in space and throughout time. The underlying immigration mechanism is time-homogeneous and gives a high intensity of arrivals of immigrants so that the inter-landing times are infinitesimal. After landing, each of the immigrant processes evolves independently of each other as a super-Brownian motion, obeying the SPDE

$$\frac{\partial X}{\partial t}(x,t)=\frac{\Delta X}{2}(x,t)+X(x,t)^{1/2}\dot{W}(x,t),\qquad X\geq 0,\tag{1.1}$$

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subject to infinitesimal initial mass, where W is (two-parameter) space–time white noise on $\mathbb{R} \times \mathbb{R}_+$. Superposition of their masses defines a super-Brownian motion with immigration and zero initial value. See Section 1.2 of Dawson [7], Konno and Shiga [15], Section III.4 of Perkins [21] and Reimers [22] for the connection between solutions to the SPDE (1.1) and super-Brownian motions. See Sections 1.2 and 3.2 in Chen [3] for some heuristic interpretations of the terms of the SPDE (1.1). Note that super-Brownian motions with immigration can also be constructed by Poisson point processes (see [4]).

We study the particular super-Brownian motions with immigration which have densities, and the density processes obey the SPDEs:

$$
\begin{aligned}
&\frac{\partial X}{\partial t}(x,t) = \frac{\Delta X}{2}(x,t) + \psi(x) + X(x,t)^{1/2}\dot{W}(x,t), \qquad X \geq 0,\\
&X(x,0) = 0.
\end{aligned}
\tag{1.2}
$$

Here, $\mathscr{C}_c^+(\mathbb{R})$ being the function space of nonnegative continuous functions on $\mathbb{R}$ with compact support, the *immigration functions* ψ satisfy

$$
\psi \in \mathscr{C}_c^+(\mathbb{R}) \qquad \text{with } \psi \neq 0 \tag{1.3}
$$

and can be thought informally as the density of immigrants landing within an infinitesimal amount of time.

To fix ideas, we give the precise definition of the pair (X, W) in the SPDE (1.2) before further discussions. We need a filtration $(\mathscr{G}_t)$ which satisfies the usual conditions, and it facilitates the following definitions of W and X. We require that W be a $(\mathscr{G}_t)$-space–time white noise in the sense that it is a family of $(\mathscr{G}_t)$-Brownian motions indexed by $L^2(\mathbb{R})$, and the Brownian motions satisfy the following properties: for any $d \in \mathbb{N}$, $\phi_1, \dots, \phi_d \in L^2(\mathbb{R})$ and $a_1, \dots, a_d \in \mathbb{R}$,

$$
W\left(\sum_{j=1}^{d} a_j \phi_j\right) = \sum_{j=1}^{d} a_j W(\phi_j) \qquad \text{a.s.} \tag{1.4}
$$

and $(W(\phi_1), \dots, W(\phi_d))$ is a d-dimensional $(\mathscr{G}_t)$-Brownian motion starting at zero with zero initial value and covariance matrix $[\langle \phi_i, \phi_j \rangle_{L^2(\mathbb{R})}]_{1 \leq i,j \leq d}$ (cf. Section 3 of Khoshnevisan [13] or Chapter 1 of Walsh [26] for the standard definition of space–time white noise). Since the immigration function under consideration has compact support, it can be shown that the density process of the corresponding super-Brownian motion with immigration takes values in $\mathscr{C}_c^+(\mathbb{R})$ (cf. Section III.4 of Perkins [21]). Let $\mathscr{C}_{\mathrm{rap}}(\mathbb{R})$ denote the function space of rapidly decreasing functions f:

$$
|f|_\lambda \triangleq \sup_{x \in \mathbb{R}} |f(x)| e^{\lambda |x|} < \infty \qquad \forall \lambda \in (0, \infty). \tag{1.5}
$$

Equip $\mathscr{C}_{\mathrm{rap}}(\mathbb{R})$ with the complete separable metric

$$
\|f\|_{\mathrm{rap}} \triangleq \sum_{\lambda=1}^{\infty} \frac{|f|_\lambda \wedge 1}{2^\lambda}. \tag{1.6}
$$

For convenience, we follow the convention in Shiga [25] and use $\mathscr{C}_{\mathrm{rap}}(\mathbb{R})$ as the underlying state space. Then by saying that $X = (X_t)$ is a *solution* to the SPDE (1.2), we require that X be a nonnegative $(\mathscr{G}_t)$-adapted continuous process with state space $\mathscr{C}_{\mathrm{rap}}(\mathbb{R})$ and satisfy the following weak formulation of (1.2):

$$X_t(\phi) = \int_0^t X_s\left(\frac{\Delta\phi}{2}\right) ds + t\langle\psi, \phi\rangle + \int_0^t \int_{\mathbb{R}} X(x,s)^{1/2}\phi(x)\, dW(x,s) \tag{1.7}$$

for any test function $\phi \in \mathscr{C}_c^\infty(\mathbb{R})$. Here, we identify any locally integrable function f on $\mathbb{R}$ as a signed measure on $\mathscr{B}(\mathbb{R})$ in the natural way and write

$$f(\phi) = \langle f, \phi\rangle \equiv \int_{\mathbb{R}} f(x)\phi(x)\, dx, \tag{1.8}$$

whenever there is no risk of confusion. For the last term in (1.7) and other two-parameter stochastic integrals in the sequel, see Section 5 of Khoshnevisan [13] or Chapter 2 of Walsh [26] for the construction.

A fundamental question for the SPDE (1.2) concerns its uniqueness theory, and the major difficulty arises from the presence of a non-Lipschitz diffusion coefficient. Uniqueness in law for the SPDE (1.2) holds and can be proved by the duality method via Laplace transforms (cf. Section 1.6 of Etheridge [9] or the proof of Lemma 2.10). In fact, it holds even if we impose general nonnegative initial conditions for the SPDE (1.1) for super-Brownian motion and the SPDEs (1.2) under consideration. Nonetheless, duality methods for more general SPDEs of the form

$$\frac{\partial X}{\partial t}(x,t) = \frac{\Delta X}{2}(x,t) + b\big(X(x,t)\big) + \sigma\big(X(x,t)\big)\dot{W}(x,t) \tag{1.9}$$

up to now seem only available when b and σ are of rather special forms, and hence are nonrobust. (See Mytnik [18] for the duality method for the case $b = 0$ and $\sigma(x) = x^p$, where $p \in (\frac{1}{2}, 1)$ and nonnegative solutions are assumed.) After all, duality is based on exactness and may become difficult to obtain by even slight changes of coefficients in the context of SPDEs.

Under the classical theory of stochastic differential equations (SDEs), uniqueness in law in an SDE is a consequence of pathwise uniqueness of its solutions (cf. Theorem IX.1.7 of Revuz and Yor [23]). The strength of this point of view is that it has emphasis on the *values* of the Hölder exponents of coefficients, instead of on the particular forms of coefficients. Then a natural question is whether the duality method can be circumvented by proving pathwise uniqueness in the SPDEs (1.9) instead. Here, *pathwise uniqueness* in an SPDE ensures that any two solutions subject to the same space–time white noise and initial value always coincide almost surely. Our objective in the present work is to settle the question of pathwise uniqueness in the particular SPDEs (1.2).

Let us discuss some results on pathwise uniqueness in various SDEs and SPDEs which are closely related to the SPDEs (1.2). We focus on the role of non-Lipschitz diffusion coefficients in determining pathwise uniqueness.

For one-dimensional SDEs with Hölder-p diffusion coefficients, the famous Girsanov example (see Section V.26 of Rogers and Williams [24]) shows the necessity of the condition $p \geq \frac{1}{2}$ for pathwise uniqueness of solutions. The sufficiency was later confirmed in the seminal work Yamada and Watanabe [27] as far as the cases with sufficiently regular drift coefficients are concerned. In fact, the work [27] showed that a finite-dimensional SDE defined by

$$dX_t^i = b_i(X_t)\,dt + \sigma_i\big(X_t^i\big)\,dB_t^i, \qquad 1 \leq i \leq d, \tag{1.10}$$

enjoys pathwise uniqueness as long as all b_i's are Lipschitz continuous and each σ_i is Hölder p-continuous, for any $p \geq \frac{1}{2}$.

In view of the complete results for SDEs and the strong parallels between (1.9) and (1.10), it had been hoped for decades that pathwise uniqueness would also hold in (1.9) if the diffusion coefficient σ is Hölder-p continuous whenever $p \geq \frac{1}{2}$. It was shown in Mytnik and Perkins [19] that this is the case if σ is Hölder-p for $p > \frac{3}{4}$, but in Burdzy, Mueller and Perkins [2] and Mueller, Mytnik and Perkins [17] that pathwise uniqueness in

$$\begin{aligned} &\frac{\partial X}{\partial t}(x,t) = \frac{\Delta X}{2}(x,t) + \big|X(x,t)\big|^p \dot{W}(x,t), \\ &X(x,0) = 0 \end{aligned} \tag{1.11}$$

fails for any $p \in (0, \frac{3}{4})$. Here, a nonzero solution to (1.11) exists and, as 0 is obviously another solution, both pathwise uniqueness and uniqueness in law fail. All these results point to the general conclusion that pathwise uniqueness of solutions holds for Hölder-p diffusion coefficients σ for $p > \frac{3}{4}$ but can fail for $p \in (0, \frac{3}{4})$. See also Mytnik, Perkins and Sturm [20] for the case of colored noises.

In this work, we confirm pathwise *nonuniqueness* in the SPDEs (1.2). We stress that by definition, only *nonnegative* solutions are considered in this regard and hence are unique in law by the duality argument mentioned above. Our main result is given by the following theorem.

THEOREM 1 (Pathwise nonuniqueness). *For any nonzero immigration function $\psi \in \mathscr{C}_c^+(\mathbb{R})$, there exists a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{G}_t), \mathbb{P})$ which carries a $(\mathscr{G}_t)$-space–time white noise W and two solutions X and Y of the SPDE* (1.2) *with respect to $(\mathscr{G}_t)$ such that $\mathbb{P}(X \neq Y) > 0$. Hence, there is pathwise nonuniqueness in the SPDE* (1.2) *for ψ as above.*

A comparison of diffusion coefficients may suggest that the construction in Mueller, Mytnik and Perkins [17] of a nonzero signed solution to the particular case

$$\begin{aligned} &\frac{\partial X}{\partial t}(x,t) = \frac{\Delta X}{2}(x,t) + \big|X(x,t)\big|^{1/2} \dot{W}(x,t), \\ &X(x,0) = 0 \end{aligned} \tag{1.12}$$

for (1.11) should be closely related to our case (1.2). Nonetheless, solutions to (1.2) are subject to the assumed nonnegativity, and uniqueness in law in the SPDEs (1.2) *does hold*. These facts mean that the goal will be to find two nonzero solutions which have the same law and are nontrivially correlated through the shared white noise. Although many features of our arguments will follow their counterparts in [17], a number of new problems, including the choice of solutions to work with, arise in dealing with these distinct properties.

For a fixed nonzero immigration function $\psi \in \mathscr{C}_c^+(\mathbb{R})$, we construct the pair of distinct solutions to the corresponding SPDE (1.2) by approximation. Basic properties of the approximating solutions are as follows. An ε-approximating pair, still denoted by (X, Y) but under $\mathbb{P}_\varepsilon$, consists of super-Brownian motions with *intermittent immigration* and subject to the *same* space–time white noise. Here, a super-Brownian motion with intermittent immigration is defined as a discrete sum of certain immigrant processes. The immigrants land after intervals of deterministic and equal length and at i.i.d. targets, and then, along with their offspring, evolve independently as true super-Brownian motions. In more detail, the pairs (X, Y) satisfy the following properties. The initial masses of the immigrant processes are of the form $\psi(\mathbb{1})J_\varepsilon^a(\cdot)$ with a denoting the target, where

$$J_\varepsilon^a(x) \equiv \varepsilon^{1/2} J\big((a-x)\varepsilon^{-1/2}\big), \qquad x \in \mathbb{R}, \tag{1.13}$$

for a fixed even $\mathscr{C}_c^+(\mathbb{R})$-function J which is bounded by 1, has topological support contained in $[-1, 1]$, and satisfies $\int_{\mathbb{R}} J(x)\,dx = 1$. In addition, the landing times of the immigrants are interlaced as

$$s_i = \big(i - \tfrac{1}{2}\big)\varepsilon \quad \text{and} \quad t_i = i\varepsilon \qquad \text{for } i \in \mathbb{N}, \tag{1.14}$$

and the targets associated with the immigrants of X and Y are given by i.i.d. spatial variables x_i and y_i at s_i and t_i, respectively, where

$$\mathbb{P}_\varepsilon(x_i \in dx) = \mathbb{P}_\varepsilon(y_i \in dx) \equiv \frac{\psi(x)\,dx}{\psi(\mathbb{1})}. \tag{1.15}$$

Details of these approximating solutions and their convergence to true solutions of the SPDE (1.2) can be found in Sections 2.1 and 5.

At this point, we only describe two ε-approximating solutions which share the same space–time white noise, and what can be deduced from this relation in understanding their interactions seems limited. The perspective of the present work is to emphasize the role of immigrant processes, and the readers will see that they lead to a detailed comparison of local masses for the approximating pairs in particular. On the other hand, by adopting this point of view, we are faced with an issue of defining approximating solutions by appropriate immigrant processes, as will be discussed in more detail later on.

We notice that similar ε-approximating solutions appear in Mueller, Mytnik and Perkins [17] for the construction of a nonzero solution to the SPDE (1.12). In this case, each approximating solution is obtained by specifying its "positive part" and "negative part" as two super-Brownian motions with intermittent immigration, but now subject to pairwise annihilation upon collision. Both parts are in turn defined by sums of their own immigrant processes undergoing annihilation, and all of the summands can be seen as i.i.d. super-Brownian immigrant processes taken off annihilated individuals and their possible offspring (cf. equations (2.1), (2.4) and (2.6) of [17] and Lemma 2.10). The latter property implies fairly explicit stochastic calculus for the immigrant processes, and is the key to make further analysis possible in [17].

For our case, while a super-Brownian motion with intermittent immigration can be defined as a sum of independent immigrant processes, the question for the same construction of two, with interlacing immigrating times and subject to the *same* space–time white noise, lies in the interactions between immigrants through space–time white noises. The major difficulty here is in specifying a family of correlated immigrants so that the corresponding approximating solutions not only conform to the same space–time white noise but also generate two distinct solutions to the SPDE (1.2). After all, in contrast to the counterexample in Mueller, Mytnik and Perkins [17] which stems from annihilation of colliding individuals in two independent population processes, it is still not known whether a similar interpretation applies to the SPDE (1.2) under consideration, since in our case the existence of two different solutions means a special kind of coexistence of two population processes. We need a different point of view to choose approximating solutions.

The correlated immigrant processes which meet our needs are chosen through a reverse analysis for the ε-approximating solutions. The aim is to find immigrant processes subject a "tractable" correlation structure for every coupled pair of approximating solutions (readers interested in more details about the motivation may see Section 3.2 of [3] for a nonrigorous proof of our main result). Our main machinery is a novel coupling method called *continuous decomposition*. By this method, essentially we can *elicit* certain immigrant processes from any pair of ε-approximating solutions so that the integrals of $\mathscr{C}_c^\infty(\mathbb{R})$-functions against them define continuous semimartingales starting at the respective landing times, all with respect to the *same* filtration (see Theorem 2.6). Here, $\mathscr{C}_c^\infty(\mathbb{R})$ denotes the space of infinitely differentiable functions on $\mathbb{R}$ with compact support. We remark that this semimartingale property of immigrant processes does not follow directly from the general theory of coupling (see Section 2.2 for a discussion).

The immigrant processes from continuous decomposition satisfy natural distributional properties including the one that both families of immigrants, say $\{X^i\}$ and $\{Y^i\}$ for X and Y, respectively, consist of independent processes. Moreover, they can be chosen such that for all $\phi, \varphi \in \mathscr{C}_c^\infty(\mathbb{R})$, the "coarse" (predictable) covariations $\langle X^i(\phi), Y(\varphi)\rangle$ and $\langle X(\phi), Y^j(\varphi)\rangle$, rather than the covariations $\langle X^i(\phi), Y^j(\varphi)\rangle$ between immigrants, admit explicit expressions (see

Proposition 2.8). The expressions are simple enough for one to conjecture that the immigrant processes should satisfy the *coexistence condition*:

$$\left\langle X^i(\phi), Y^j(\varphi)\right\rangle_t = \int_0^t \int_{\mathbb{R}} \frac{X^i(x,s)Y^j(x,s)}{X(x,s)^{1/2}Y(x,s)^{1/2}}\phi(x)\varphi(x)\,dx\,ds, \qquad i,j\geq 1, \tag{1.16}$$

where $0/0$ is read as 0. Note that the covariations in (1.16) are given by *two-fold* integrals. See Section 2.3 for other possibilities of covariations for stochastic integrals with respect to space–time white noises, and also Remark 2.7 on related issues. By classical arguments, it can be verified that immigrant processes subject to the coexistence condition (1.16) do exist. See Theorem 2.12 for the precise result. We will restrict our attention to the corresponding ε-approximating solutions in the remaining of Section 1.

Let us explain why the ε-approximating solutions remain separated if we pass ε to zero. We switch to the conditional probability measure under which the total mass process of a generic immigrant, say X^i, hits 1. Let us call such a conditional probability measure $\mathbb{Q}^i_\varepsilon$ from now on. The motivation to invoke these conditional probabilities is that with high $\mathbb{P}_\varepsilon$-probability, there is at least one immigrant from X whose total mass will hit 1 by the independence of the immigrants for X, so whenever X and Y are separated with sufficiently high probability under every $\mathbb{Q}^i_\varepsilon$, we should be able to integrate these immigrant-wise phenomena of *conditional separation* of X and Y into a kind of separation under $\mathbb{P}_\varepsilon$.

The readers may notice that the above argument to obtain separation under $\mathbb{P}_\varepsilon$ is reminiscent of the use of excursion theory in studying pathwise uniqueness in SDEs and SPDEs (cf. Bass, Burdzy and Chen [1] and Burdzy, Mueller and Perkins [2]). The major difference, however, is that in the present case, the immigrant processes can overlap in time without waiting until the earlier ones die out. In order to use conditional separation of the approximating pairs, we resort to an inclusion–exclusion argument as in Mueller, Mytnik and Perkins [17]. The result is *uniform separation* of the approximating pairs under $\mathbb{P}_\varepsilon$. It states that for some constants $T,\Delta\in(0,\infty)$ independent of ε, $\sup_{0\leq s\leq T}\|X_s-Y_s\|_{\mathrm{rap}}$ under $\mathbb{P}_\varepsilon$ are uniformly bounded below by Δ with uniformly positive probability for all small $\varepsilon\in(0,1)$, or more precisely

$$\liminf_{\varepsilon\searrow 0}\mathbb{P}_\varepsilon\Big(\sup_{0\leq s\leq T}\|X_s-Y_s\|_{\mathrm{rap}}\geq\Delta\Big)>0.$$

Then it is not difficult to argue that any two true solutions to (1.2) as a limit of our approximation pairs separate with strictly positive probability. See Section 4 for the details.

The conditional separation under $\mathbb{Q}^i_\varepsilon$ of the two approximating solutions concerns a comparison of their local masses over a growing space–time region. We envelop the support processes of X^i and Y^j by *approximating* parabolas of the form

$$\mathcal{P}_\beta^{(a,s)}(t) = \{(x,r) \in \mathbb{R} \times [s,t]; |a-x| \le \varepsilon^{1/2} + (r-s)^\beta\} \tag{1.17}$$

for β near $1/2$ and consider the propagation of these parabolas instead of that of the support processes. The known modulus of continuity for the support of super-Brownian motion implies that, for example, the support of X^i satisfies

$$\operatorname{supp}(X^i) \cap (\mathbb{R} \times [s_i,t]) \subseteq \mathcal{P}_\beta^{(x_i,s_i)}(t) \qquad \text{for } t - s_i \text{ small}, \tag{1.18}$$

where $\operatorname{supp}(X^i)$ is the space–time support of the random function $(x,s) \longmapsto X^i(x,s)$, and x_i and s_i denote the landing target and landing time of X^i, respectively (see Section 3.2 and Proposition 7.1). As in Mueller, Mytnik and Perkins [17], the total mass process $X^i(\mathbb{1})$ under $\mathbb{Q}_\varepsilon^i$ can be shown to be a constant multiple of a 4-dimensional Bessel squared process near its landing time and hence has a known growth rate. Thanks to (1.18), this growth rate of the total mass is the same as the growth rate of the local mass of X^i over its support envelope, and then a lower bound of the associated local mass of X follows from the nonnegativity of the immigrant processes.

We prove that the local mass of Y over the envelope for X^i grows at a smaller rate. This involves a subcollection of immigrants from Y which we choose now. The $\mathbb{Q}_\varepsilon^i$-probability that one of the Y^j clusters preceding X^i ever invades the "territory" of X^i by time $t \in (s_i, \infty)$ can be made relatively small as long as $t - s_i$ small, which follows from an argument similar to the proof of Lemma 8.4 of Mueller, Mytnik and Perkins [17] (see Proposition 7.2). These Y^j clusters can henceforth be excluded from our consideration. Then the simple geometry of the approximating parabolas (1.17) yields the space–time rectangles

$$\mathcal{R}^i(t) = [x_i - 2(\varepsilon^{1/2} + (t-s_i)^\beta), x_i + 2(\varepsilon^{1/2} + (t-s_i)^\beta)] \times [s_i, t]$$

so that the immigrant processes Y^j landing inside $\mathcal{R}^i(t)$ are the only possible invaders of the support envelope for X^i by time t. This results in a family of clusters, say, $\{Y^j; j \in \mathcal{J}^i(t)\}$ to the effect that the local mass of Y over the growing envelope for X^i is dominated by the sum of total masses of these clusters. We further classify them into *critical clusters* and *lateral clusters*. In essence, the critical clusters land near the territory of X^i so the interactions between these clusters and X^i are significant. In contrast, the lateral clusters must evolve for relatively larger amounts of time before they start to interfere with X^i.

Up to this point, the framework we set for investigating conditional separation of approximating solutions is very similar to that in Mueller, Mytnik and Perkins [17]. The *interactions* between the approximating solutions considered in both cases are, however, very different in nature. For example, bounding the finite variation process of the semimartingale $Y^j(\mathbb{1})$ under $\mathbb{Q}_\varepsilon^i$ is the main source of difficulty in our case, but this creates no difficulty in [17]. Hence, our case calls for a new analysis again. Our result for the conditional separation can be captured quantitatively by saying that for arbitrarily small $\delta > 0$,

$$\begin{aligned}&\text{with high } \mathbb{Q}^i_\varepsilon\text{-probability, } X^i_t(\mathbb{1}) \geq \text{constant} \cdot (t - s_i)^{1+\delta} \text{ and}\\ &\textstyle\sum_{j\in\mathcal{J}^i(t)} Y^j_t(\mathbb{1}) \leq \text{constant} \cdot (t - s_i)^{3/2-\delta}, \text{ for } t \text{ close to } s_i+.\end{aligned} \tag{1.19}$$

Here, the initial behavior of $X^i(\mathbb{1})$ under $\mathbb{Q}^i_\varepsilon$ as a constant multiple of a 4-dimensional Bessel squared process readily gives the first part of (1.19) (see Section 6). On the other hand, the extra order, which is roughly $(t - s_i)^{1/2}$, for the sum of the (potential) invaders Y^j can be seen as the result of spatial structure.

In fact, the above framework needs to be further modified in a critical way due to a technical difficulty which arises in our setting (but not in Mueller, Mytnik and Perkins [17]). We must consider a slightly looser definition for critical clusters, and a slightly more stringent definition for lateral clusters. It will be convenient to consider this modified classification for the Y^j clusters, still indexed by $j \in \mathcal{J}^i(t)$, landing inside a slightly larger rectangle in place of $\mathcal{R}^i(t)$. Write

$$\mathcal{J}^i(t) = \mathcal{C}^i(t) \cup \mathcal{L}^i(t),$$

where $\mathcal{C}^i(t)$ and $\mathcal{L}^i(t)$ are the random index sets associated with critical clusters and lateral clusters, respectively. See Section 3.2 for the precise classification.

Let us discuss the method to bound the sum of the total masses $Y^j_t(\mathbb{1})$, $j \in \mathcal{J}^i(t)$, under $\mathbb{Q}^i_\varepsilon$ [recall (1.19)]. As in Mueller, Mytnik and Perkins [17], this part plays a major role in the present work besides the selection of approximating solutions. The treatment of the sum is through an analysis of its first moment, or more precisely an analysis of the expected finite variation process of $Y^j(\mathbb{1})$ under $\mathbb{Q}^i_\varepsilon$ for $j \in \mathcal{J}^i(t)$.

For the critical clusters Y^j, the finite variation processes of their total masses under $\mathbb{Q}^i_\varepsilon$ have bounds given by

$$\int_{t_j}^t \frac{[Y^j_s(\mathbb{1})]^{1/2}}{[X^i_s(\mathbb{1})]^{1/2}}\, ds \tag{1.20}$$

for t sufficiently close to t_j+ (cf. Lemma 3.2 below), so only the *total masses* of the clusters need to be handled. In this direction, we use an *improved modulus of continuity* of the total mass processes $Y^j(\mathbb{1})$ and the lower bound of $X^i(\mathbb{1})$ in (1.19) to give deterministic bounds for the integrands in (1.20). The overall effect is a bound for the expected sum of the total masses $Y^j_t(\mathbb{1})$, $j \in \mathcal{C}^i(t)$, and this can be used to show that the corresponding random sum has growth similar to that in the second part of (1.19). See Section 3.4.

The lateral clusters pose an additional difficulty here which is not present in Mueller, Mytnik and Perkins [17] due to the possibly nontrivial covariations between these clusters and X^i. The question is whether or not conditioning on X^i being significant can pull along the nearby Y^j's at a greater rate, even though any of these Y^j does not interfere with X^i upon their landing. In order to help bound the contributions of these clusters, we argue that a lateral cluster Y^j is independent of X^i until they collide (cf. Lemma 3.15 and Proposition 3.16). This allows us to adapt the arguments for the critical clusters and furthermore bound the growth rate of the sums of the total masses $Y_t^j(\mathbb{1})$, $j \in \mathcal{L}^i(t)$, by the desired order. See the discussion in Section 3.5 for more on this issue.

We close our discussion in this section with an immediate corollary for the SPDE (1.2) in which the immigration function has small total mass $\psi(\mathbb{1})$ and the initial value is replaced by a nonzero nonnegative $\mathscr{C}_{\mathrm{rap}}(\mathbb{R})$-function. In this case, pathwise nonuniqueness remains true. This follows from the Markov property of super-Brownian motions with immigration and the recurrence of Bessel squared processes with small dimensions (cf. page 442 in Revuz and Yor [23]). In detail, we can run a copy of such a super-Brownian motion with immigration until its total mass first hits zero, and then the required distinct solutions can be obtained by concatenating this piece with the separating solutions in Theorem 1.

This paper is organized as follows. In Section 2.1, we give the precise definition of the pairs of approximating solutions from which we choose particular ones for the proof of our main result, and discuss their basic properties. In Section 2.2, we explain the idea of continuous decomposition of a super-Brownian motion with intermittent immigration and then give the rigorous proof for the continuous decompositions of the approximation solutions specified in Section 2.1. Covariations of the resulting immigrant processes are studied in Section 2.3. By the results in Sections 2.1–2.3, we identify a system of SPDEs for immigrant processes and prove the existence of its solutions in Section 2.4. Except in Section 5, we restrict our attention to the corresponding approximating solutions from Section 3 on.

In Section 3, we proceed to conditional separation of the approximating solutions. Some basic results are stated in Section 3.1, and the setup is given in Section 3.2. Due to the complexity, the main two lemmas of Section 3 are proved in Sections 3.4 and 3.5, respectively, with some preliminaries set in Section 3.3. In Section 4, we show the uniform separation of approximating solutions under $\mathbb{P}_\varepsilon$, which completes the proof of our main result.

In Sections 5 and 6, we prove Propositions 2.3 and 3.3, respectively, which are two technical results. In Section 7, we discuss some properties of the support processes for immigrants. Finally, in Section 8, we study the improved modulus of continuity for functions satisfying certain Gronwall-type integral inequalities.

2. Approximating solutions.

2.1. *Interlacing pairs of approximating solutions*. In this section, we give details for the approximating solutions of the SPDE (1.2) which are discussed in Section 1, and state their basic properties. Recall that we identify every locally integrable function f on $\mathbb{R}$ as a signed measure by (1.8). We will further write

$f(\Gamma) = f(\mathbb{1}_\Gamma)$ for Borel sets $\Gamma \in \mathscr{B}(\mathbb{R})$, whenever the right-hand side makes sense.

For $\varepsilon \in (0,1]$, the ε-approximating solutions X and Y in Section 1 obey the equations given as follows. The first solution X is a nonnegative càdlàg $\mathscr{C}_{\mathrm{rap}}(\mathbb{R})$-valued process and is continuous within each time interval $[s_i, s_{i+1})$ for $s_0 = 0$ and $s_1, s_2, \dots$ defined by (1.14). Its time evolution is given by

$$\begin{aligned} X_t(\phi) &= \int_0^t X_s\left(\frac{\Delta}{2}\phi\right) ds + \int_{(0,t]}\int_{\mathbb{R}} \phi(x)\, dA^X(x,s) \\ &\quad + \int_0^t \int_{\mathbb{R}} X(x,s)^{1/2}\phi(x)\, dW(x,s) \end{aligned} \tag{2.1}$$

for $\phi \in \mathscr{C}_c^\infty(\mathbb{R})$. In (2.1), the nonnegative measure A^X on $\mathbb{R} \times \mathbb{R}_+$ is defined by

$$A^X(\Gamma \times [0,t]) \triangleq \sum_{i\,:\,0<s_i\le t} \psi(\mathbb{1}) J_\varepsilon^{x_i}(\Gamma), \tag{2.2}$$

and W is a space–time white noise. Here, in (2.2), recall our notation J_ε^a in (1.13) and the i.i.d. spatial random points $\{x_i\}$ with law (1.15). In terms of the interpretation in Section 1, A^X can be thought of as being contributed by the initial masses of the underlying immigrant processes for X.

A similar characterization applies to the other approximating solution Y. It is a nonnegative càdlàg $\mathscr{C}_{\mathrm{rap}}(\mathbb{R})$-valued process satisfying

$$\begin{aligned} Y_t(\phi) &= \int_0^t Y_s\left(\frac{\Delta}{2}\phi\right) ds + \int_{(0,t]}\int_{\mathbb{R}} \phi(x)\, dA^Y(x,s) \\ &\quad + \int_0^t \int_{\mathbb{R}} Y(x,s)^{1/2}\phi(x)\, dW(x,s), \end{aligned} \tag{2.3}$$

for $\phi \in \mathscr{C}_c^\infty(\mathbb{R})$, and is continuous over each $[t_i, t_{i+1})$ for $t_0 = 0$ and $t_1, t_2, \dots$ defined by (1.14). The nonnegative measure A^Y on $\mathbb{R} \times \mathbb{R}_+$ is now defined by

$$A^Y(\Gamma \times [0,t]) \triangleq \sum_{j\,:\,0<t_i\le t} \psi(\mathbb{1}) J_\varepsilon^{y_i}(\Gamma).$$

We observe that the equations (2.1) and (2.3) for X and Y can be described completely in terms of the processes themselves. For X, each random point x_i in the definition (2.2) of A^X is a measurable function of the corresponding jump size ΔX_{s_i} and conversely, where we write $\Delta Z_s = Z_s - Z_{s-}$ with $Z_{0-} = 0$ for a càdlàg process Z taking values in a Polish space. Indeed, we have

$$x_i = \inf\left\{x \in \mathbb{R};\, \Delta X_{s_i}\big((-\infty, x]\big) > \frac{\psi(\mathbb{1})\varepsilon}{2}\right\} \quad \text{and} \quad \Delta X_{s_i} = \psi(\mathbb{1}) J_\varepsilon^{x_i}, \tag{2.4}$$

where the first equality follows since ΔX_{s_i} has total mass $\psi(\mathbb{1})\varepsilon$ and defines a measure symmetric about the center x_i of its topological support [cf. (1.13)].

For Y, similar relations between the random points $\{y_i\}$ and the jump sizes $\{\Delta Y_{t_i}\}$ hold.

As a summary of the above discussions, we give in Definition 2.1 below a minimal description of the approximating solutions considered throughout this paper. From Section 3.1 on, we will work with ε-approximating pairs subject to particular correlations. Here and in the sequel, we use the notation "$\mathscr{G} \perp\!\!\!\perp \xi$" to mean that the σ-field $\mathscr{G}$ and the random element ξ are independent, and analogous notation applies to other pairs of objects which allow probabilistic independence in the usual sense.

DEFINITION 2.1. Fix an immigration function $\psi \in \mathscr{C}_c^+(\mathbb{R}) \setminus \{0\}$. For any $\varepsilon \in (0, 1]$, an *interlacing pair* of ε-approximating solutions is a pair (X, Y) defined on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathbb{P}_\varepsilon)$, with $(\mathscr{F}_t)$ satisfying the usual conditions, which carries an $(\mathscr{F}_t)$-space–time white noise W, and such that:

(i) X and Y are two nonnegative $(\mathscr{F}_t)$-adapted $\mathscr{C}_{\mathrm{rap}}(\mathbb{R})$-valued processes satisfying (2.1) and (2.3) with respect to W for x_i defined by the first equation in (2.4) and y_i by the same equation with ΔX_{s_i} replaced by ΔY_{t_i}, and have paths which are càdlàg on $\mathbb{R}_+$ and continuous within each $[s_i, s_{i+1})$ and $[t_i, t_{i+1})$, respectively,

(ii) the jumps $\{\Delta X_{s_i}, \Delta Y_{t_i}; i \in \mathbb{N}\}$ are i.i.d. $\mathscr{C}_{\mathrm{rap}}(\mathbb{R})$-valued random elements with law given by (1.15) through the second equation in (2.4), and

(iii) the random variables x_i and y_i take values in the topological support of ψ with

$$\forall i \in \mathbb{N} \qquad \sigma(X_t, Y_t; t < s_i) \perp\!\!\!\perp x_i \quad \text{and} \quad \sigma(X_t, Y_t; t < t_i) \perp\!\!\!\perp y_i. \tag{2.5}$$

The existence of these pairs of approximation solutions can be obtained by considering the so-called mild forms of solutions of SPDEs and then resorting to the classical Peano's existence argument as in Theorem 2.6 of [25]. We omit the details.

NOTATION 2.2. The following convention will be in force throughout this paper unless otherwise mentioned. As before, we suppress the dependence on ε for quantities related to an interlacing pair of ε-approximating solutions except the underlying probability measure $\mathbb{P}_\varepsilon$. The subscript ε of $\mathbb{P}_\varepsilon$ is further omitted in cases where there is no ambiguity, although in this context we will remind the readers of this practice.

The processes described in Definition 2.1 are genuine approximating solutions to the SPDE (1.2) with respect to the same white noise, as the following proposition states.

PROPOSITION 2.3. *Equip $\mathscr{C}_{\mathrm{rap}}(\mathbb{R})$ with the norm $\|\cdot\|_{\mathrm{rap}}$ defined by (1.6) and $D(\mathbb{R}_+, \mathscr{C}_{\mathrm{rap}}(\mathbb{R}))$ with Skorokhod's J_1-topology. Let $(\varepsilon_n) \subseteq (0, 1]$ be such that*

$\varepsilon_n \searrow 0$, and $((X,Y),\mathbb{P}_{\varepsilon_n})$ be a sequence of interlacing pairs of ε_n-approximating solutions. Then the sequence of laws of $((X,Y),\mathbb{P}_{\varepsilon_n})$ is relatively compact in the space of probability measures on the product space $D(\mathbb{R}_+,\mathscr{C}_{\mathrm{rap}}(\mathbb{R}))\times D(\mathbb{R}_+,\mathscr{C}_{\mathrm{rap}}(\mathbb{R}))$ and every subsequential limit defines the law of a pair of solutions to the SPDE (1.2) subject to the same space–time white noise.

The proof of Proposition 2.3 is given in Section 5. At this point, the readers should be convinced of the result upon observing the limiting behavior of the random measures A^X: for any $t\in(0,\infty)$,

$$\mathbb{P}\text{-}\lim_{\varepsilon\searrow 0}\int_{(0,t]}\int_{\mathbb{R}}\phi(x)\,dA^X(x,s)=\mathbb{P}\text{-}\lim_{\varepsilon\searrow 0}\psi(\mathbb{1})\varepsilon\sum_{i=1}^{\lfloor t\varepsilon^{-1}\rfloor}\phi(x_i)=t\langle\psi,\phi\rangle \tag{2.6}$$

for any $\phi\in\mathscr{C}_c^\infty(\mathbb{R})$, by the law of large numbers. Here, $\mathbb{P}$-lim denotes convergence in probability, and $\lfloor t\rfloor$ is the greatest integer less than or equal to t.

We close this section with a property of the above approximating solutions. Here and in the sequel, we use the following notation. For any real-valued random function $Z:(x,s)\longmapsto Z(x,s)$, we write

$$(Z\in\Gamma)\triangleq\{(x,s)\in\mathbb{R}\times\mathbb{R}_+;\,Z(x,s)\in\Gamma\},\qquad \Gamma\in\mathscr{B}(\mathbb{R}). \tag{2.7}$$

In addition, for a space–time white noise W', we write $L^2_{\mathrm{loc}}(W')$ for the set of functions $Z=Z(\omega,x,s)$, product measurable in (ω,s) and x with respect to the underlying predictable σ-field and $\mathscr{B}(\mathbb{R})$, so that

$$\int_0^t\int_{\mathbb{R}}Z(x,s)^2\,dx\,ds<\infty\qquad \forall t\in(0,\infty)\text{ a.s.}, \tag{2.8}$$

and define processes of stochastic integrals as

$$Z\bullet W'(\phi)\equiv\int_0^\cdot\int_{\mathbb{R}}Z(x,s)\phi(x)\,dW(x,s) \tag{2.9}$$

for $Z\in L^2_{\mathrm{loc}}(W')$ and $\phi\in L^2(\mathbb{R})$.

PROPOSITION 2.4 (Cherny's substitution). *For $\varepsilon\in(0,1]$, let (X,Y) be an interlacing pair of ε-approximating solutions. By enlarging the underlying filtered probability space $(\Omega,\mathscr{F},(\mathscr{F}_t),\mathbb{P}_\varepsilon)$ if necessary, we can find random elements*

$$V^{X,i}=\{(V_t^{X,i}(\phi))_{t\in[s_i,\infty)};\,\phi\in L^2(\mathbb{R})\}$$

and

$$V^{Y,i}=\{(V_t^{Y,i}(\phi))_{t\in[t_i,\infty)};\,\phi\in L^2(\mathbb{R})\},$$

for $i\in\mathbb{N}$, which satisfy the following properties:

(i) *Every* $V^{X,i}$ *is an* $(\mathscr{F}_t)_{t\in[s_i,\infty)}$*-space–time white noise and satisfies*

$$V^{X,i} \perp\!\!\!\perp \{\mathscr{F}_{s_i}, (X_t)_{t\in[s_i,s_{i+1})}\}. \tag{2.10}$$

Here, an $(\mathscr{F}_t)_{t\in[s_i,\infty)}$*-space–time white noise is defined as an* $(\mathscr{F}_t)$*-space–time white noise except that its components are* $(\mathscr{F}_t)_{t\in[s_i,\infty)}$*-Brownian motions started at* s_i *with zero initial value.*

(ii) *Every* $V^{Y,i}$ *satisfies the same properties in* (*i*) *with* (X, s_i) *replaced by* (Y, t_i).

(iii) *The following substitution identities of space–time white noises hold*: *for all* $i \in \mathbb{N}$,

$$\mathbb{1}_{[s_i,s_{i+1})}\mathbb{1}_{(X=0)} \bullet W = \mathbb{1}_{[s_i,s_{i+1})}\mathbb{1}_{(X=0)} \bullet V^{X,i}, \tag{2.11}$$

$$\mathbb{1}_{[t_i,t_{i+1})}\mathbb{1}_{(Y=0)} \bullet W = \mathbb{1}_{[t_i,t_{i+1})}\mathbb{1}_{(Y=0)} \bullet V^{Y,i}. \tag{2.12}$$

Proposition 2.4 will be used in Section 2.2 to reinforce immigrant processes from continuous decomposition with analogous properties [condition (vi) of Theorem 2.6]. From these properties, we will deduce some key equations for covariations of the immigrant processes (see Proposition 2.8).

SKETCH OF PROOF OF PROPOSITION 2.4. The proof is a generalization of the proof of Theorem 3.1 in Cherny [6] to the context of the SPDE (1.1), and so we only give a sketch. Below we consider the assertions for $V^{X,i}$ for $i \in \mathbb{N}$. The assertions for $V^{Y,i}$ follow similarly.

We define $V^{X,i}$ as a mixture of the original noise W and another space–time white noise, say $U^{X,i}$, which is independent of (X, Y, W) and adapted to the same filtration, by

$$V^{X,i} \triangleq \mathbb{1}_{[s_i,\infty)}\mathbb{1}_{(X=0)} \bullet W + \mathbb{1}_{[s_i,\infty)}\mathbb{1}_{(X>0)} \bullet U^{X,i}. \tag{2.13}$$

Then $V^{X,i}$ is an $(\mathscr{F}_t)_{t\in[s_i,\infty)}$-space–time white noise by Lévy's theorem (cf. Theorem IV.3.6 of [23]) and gives the required substitution (2.11). We have proved the first assertion in (i) and the assertion in (iii) for $V^{X,i}$.

It remains to prove the independence (2.10). Consider the counterpart of $V^{X,i}$ (2.13):

$$\widetilde{V}^{X,i} \triangleq \mathbb{1}_{[s_i,\infty)}\mathbb{1}_{(X>0)} \bullet W + \mathbb{1}_{[s_i,\infty)}\mathbb{1}_{(X=0)} \bullet U^{X,i}.$$

By Lévy's theorem again, $\widetilde{V}^{X,i}$ is an $(\mathscr{F}_t)_{t\in[s_i,\infty)}$-space–time white noise and $V^{X,i} \perp\!\!\!\perp \widetilde{V}^{X,i}$. The latter property implies that $(X, \widetilde{V}^{X,i})$ over $[s_i, s_{i+1})$ solves the SPDE (1.1) of super-Brownian motion with respect to $(\mathscr{F}_t \vee \sigma(V^{X,i}))_{t\in[s_i,s_{i+1})}$. Recall that the martingale problem for super-Brownian motion is well-posed (cf. Lemma 2.10), and note that $V^{X,i} \perp\!\!\!\perp \mathscr{F}_{s_i}$ by a standard property of Brownian motion. We deduce that for any $\Gamma_0 \in \sigma(V^{X,i})$ and $\Gamma_1 \in \mathscr{F}_{s_i}$ with $\mathbb{P}_\varepsilon(\Gamma_0 \cap \Gamma_1) > 0$,

$$\frac{\mathbb{P}_\varepsilon(\Gamma_0 \cap \Gamma_1 \cap \{(X_t)_{t\in[s_i,s_{i+1})} \in \cdot\})}{\mathbb{P}_\varepsilon(\Gamma_0 \cap \Gamma_1)} = \frac{\mathbb{P}_\varepsilon(\Gamma_1 \cap \{(X_t)_{t\in[s_i,s_{i+1})} \in \cdot\})}{\mathbb{P}_\varepsilon(\Gamma_1)}. \tag{2.14}$$

Indeed, under the conditional probabilities $\mathbb{P}_\varepsilon(\cdot|\Gamma_0 \cap \Gamma_1)$ and $\mathbb{P}_\varepsilon(\cdot|\Gamma_1)$, the laws of X_{s_i} are the same and $((\mathscr{F}_t \vee \sigma(V^{X,i}))_{t\in[s_i,s_{i+1})}, \mathbb{P}_\varepsilon)$-martingales remain martingales. See the proof of Theorem 4.4.2 of [10]. Using $V^{X,i} \perp\!\!\!\perp \mathscr{F}_{s_i}$ again, we can substitute $\mathbb{P}_\varepsilon(\Gamma_0 \cap \Gamma_1)$ on the left-hand side of (2.14) with $\mathbb{P}_\varepsilon(\Gamma_0)\mathbb{P}_\varepsilon(\Gamma_1)$. The required property (2.10) follows. $\square$

2.2. *Continuous decomposition.* For every $\varepsilon \in (0, 1]$, consider an interlacing pair (X, Y) of ε-approximating solutions to the SPDE (1.2) (recall Definition 2.1). From their informal descriptions in Section 1, it is reasonable to expect that they can be *decomposed* into

$$X = \sum_{i=1}^{\infty} X^i \quad \text{and} \quad Y = \sum_{i=1}^{\infty} Y^i, \tag{2.15}$$

where the summands X^i and Y^i are super-Brownian motions started at s_i and t_i and with starting measures $\Delta X_{s_i} = \psi(\mathbb{1})J_\varepsilon^{x_i}$ and $\Delta Y_{t_i} = \psi(\mathbb{1})J_\varepsilon^{y_i}$, respectively, for each i, and each of the families $\{X^i\}$ and $\{Y^i\}$ consists of independent random elements.

Let us give an elementary discussion on obtaining the decompositions in (2.15). Later on, we will require additional properties of the decompositions. It follows from the uniqueness in law of super-Brownian motions and the defining equation (2.1) that X is a (time-inhomogeneous) Markov process and, for each $i \in \mathbb{N}$, $(X_t)_{t\in[s_i,s_{i+1})}$ defines a super-Brownian motion with initial distribution X_{s_i} (cf. the proof of Theorem 4.4.2 in [10] and the martingale problem characterization of super-Brownian motion in [21]). Hence, each of the equalities in (2.15) holds in the sense of being *identical in distribution*. Then we recall the following general theorem (see Theorem 6.10 in [12]).

THEOREM 2.5. *Fix any measurable space E_1 and Polish space E_2, and let $\xi \overset{(\mathrm{d})}{=} \widetilde{\xi}$ and η be random elements taking values in E_1 and E_2, respectively. Here, we only assume that ξ and η are defined on the same probability space. Then there exists a measurable function $F : E_1 \times [0, 1] \longrightarrow E_2$ such that for any random variable $\widetilde{U}$ uniformly distributed over $[0, 1]$ with $\widetilde{U} \perp\!\!\!\perp \widetilde{\xi}$, the random element $\widetilde{\eta} = F(\widetilde{\xi}, \widetilde{U})$ solves $(\xi, \eta) \overset{(\mathrm{d})}{=} (\widetilde{\xi}, \widetilde{\eta})$.*

By the preceding discussions and Theorem 2.5, we can immediately construct the summands X^i and Y^i by introducing additional independent uniform variables and validate the equalities (2.15) as almost-sure equalities. Such decompositions, however, are too crude because, for example, we are unable to say that all the resulting random processes perform their defining properties with respect to the *same* filtration. This difficulty implies in particular that we cannot do semimartingale calculations for them. A finer decomposition method, however, does yield a solution to this problem. The result is stated in Theorem 2.6 below. See also Figure 1 for a sketch of the decomposition of X along a particular value x.

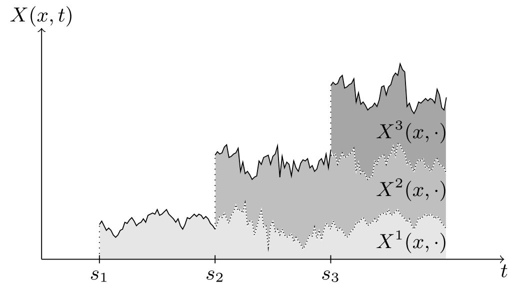

FIG. 1. *Decomposition of X along x.*

THEOREM 2.6 (Continuous decomposition). *Fix $\varepsilon \in (0,1]$. Let (X,Y,W) be an interlacing pair of ε-approximating solutions, and $\{V^{X,i}\}$, $\{V^{Y,i}\}$ be two families of space–time white noises chosen in Theorem 2.4. By changing the underlying probability space if necessary, we can find a filtration $(\mathscr{G}_t)$ satisfying the usual conditions and two families $\{X^i\}$ and $\{Y^i\}$ of nonnegative $\mathscr{C}_{\mathrm{rap}}(\mathbb{R})$-valued processes, such that the following conditions are satisfied*:

(i) *The processes X^i, $i \in \mathbb{N}$, are independent.*

(ii) *The equality in (2.15) for X and X^i holds almost surely.*

(iii) *Each $(X^i_t)_{t\in[s_i,\infty)}$ has sample paths in $C([s_i,\infty),\mathscr{C}_{\mathrm{rap}}(\mathbb{R}))$ and is a $(\mathscr{G}_t)_{t\geq s_i}$-super-Brownian motion started at time s_i with starting measure $\psi(\mathbb{1})J^{x_i}_{\varepsilon}$. Also, $X^i_t \equiv 0$ for every $t \in [0,s_i)$.*

(iv) *The processes Y^i, $i \in \mathbb{N}$, satisfy the same properties as* (i)–(iii) *with the roles of X and $\{(X^i,x_i,s_i)\}$ replaced by Y and $\{(Y^i,y_i,t_i)\}$, respectively.*

(v) *Conditions* (i)–(iii) *of Definition 2.1 hold with $(\mathscr{F}_t)$ replaced by $(\mathscr{G}_t)$, and* (2.5) *in the same definition is replaced by the stronger independent landing property*:

$$\sigma\big(X^j_t, Y^j_t; t < s_i, j \in \mathbb{N}\big) \perp\!\!\!\perp x_i \quad \textit{and} \quad \sigma\big(X^j_t, Y^j_t; t < t_i, j \in \mathbb{N}\big) \perp\!\!\!\perp y_i \qquad \forall i \in \mathbb{N}. \tag{2.16}$$

(vi) *Condition* (iii) *of Proposition 2.4 holds. In addition,*

$$\begin{aligned} &\big\{\big(X^j_t\big)_{t\in[0,s_{i+1})}; j \in \mathbb{N} \text{ satisfying } s_j < s_{i+1}\big\} \perp\!\!\!\perp V^{X,i} \quad \textit{and} \\ &\big\{\big(Y^j_t\big)_{t\in[0,t_{i+1})}; j \in \mathbb{N} \text{ satisfying } t_j < t_{i+1}\big\} \perp\!\!\!\perp V^{Y,i} \qquad \forall i \in \mathbb{N}. \end{aligned} \tag{2.17}$$

Due to the length of the proof of Theorem 2.6, we first outline its *informal* idea for the convenience of readers. Recall that the first immigration event for X and Y occurs at $s_1 = \frac{\varepsilon}{2}$. Take a grid of $[\frac{\varepsilon}{2},\infty)$ containing all the points s_i and t_i for $i \in \mathbb{N}$ and with "infinitesimal" mesh size. Here, the mesh size of a grid is the supremum of the distances between consecutive grid points. The key observation in this construction is that, over any subinterval $[t, t+\Delta t] \subseteq [s_i, s_{i+1})$ from this grid, $(X_r; r \in [t, t + \Delta t])$ has the same distribution as the sum of i independent super-Brownian motions started at t over $[t, t + \Delta t]$, *whenever* the sum of the initial conditions of these independent super-Brownian motions has the same distribution as X_t.

This fact allows us to inductively decompose X over the intervals of infinitesimal lengths from this grid, such that the resulting infinitesimal pieces of super-Brownian motions can be concatenated in the natural way to obtain the desired immigrating super-Brownian motions. More precisely, we apply Theorem 2.5 by bringing in independent uniform variables as "allocators" to obtain these infinitesimal pieces. A similar method applies to continuously decompose Y into the desired independent super-Brownian motions by another family of independent allocators.

Finally, because the path regularity of these concatenated processes and W allows us to characterize their laws over the entire time horizon $\mathbb{R}_+$ by their laws over $[0, \varepsilon/2]$ and their probabilistic transitions on this grid with infinitesimal mesh size, the filtration obtained by sequentially adding the σ-fields of the independent allocators will be the desired one. In particular, the time evolutions of these stochastic processes are now consistent with the "progression" of the enlarged filtration.

PROOF OF THEOREM 2.6. Fix $\varepsilon \in (0, 1]$ and we shall drop the subscript ε of $\mathbb{P}_\varepsilon$. Throughout the proof, we take, for each $m \in \mathbb{N}$, a countable subset D_m of $[\frac{\varepsilon}{2}, \infty)$ which contains s_i and t_i for any $i \in \mathbb{N}$ and satisfies $\#(D_m \cap K) < \infty$ for any compact subset K of $\mathbb{R}_+$. We further assume that (1) $D_{m+1} \subseteq D_m$ for each m, (2) between any two points s_i and t_i there is another point belonging to D_1, and hence to each D_m, and (3) the mesh size of D_m goes to zero as $m \longrightarrow \infty$. In addition, we will write $\{\mathrm{SBM}_t(\mu, d\nu); t \in \mathbb{R}_+\}$ for the semigroup of super-Brownian motion on $\mathbb{R}$. When the *density* of the super-Brownian motion on $\mathbb{R}$ started at time s and with starting measure $f(x)\,dx$ for a nonnegative $\mathscr{C}_{\mathrm{rap}}(\mathbb{R})$-function f is concerned, we write $\mathrm{SBM}_{f,[s,t]}$ for the law of its $C([s, \infty), \mathscr{C}_{\mathrm{rap}}(\mathbb{R}))$-valued density restricted to the time interval $[s, t]$.

Step 1. Fix $m \in \mathbb{N}$ and write $\frac{\varepsilon}{2} = \tau_0 < \tau_1 < \cdots$ as the consecutive points of D_m. Assume, by an enlargement of the underlying probability space where $(X, Y, W, \{V^{X,i}\}, \{V^{Y,i}\})$ lives if necessary, the existence of i.i.d. variables $\{U_j^X, U_j^Y; j \in \mathbb{N}\}$ with

$$U_1^X \text{ is uniformly distributed over } [0, 1] \text{ and } \{U_j^X, U_j^Y; j \in \mathbb{N}\} \perp\!\!\!\perp \mathscr{F}. \tag{2.18}$$

In this step, we will decompose X and Y into the random elements

$$\mathcal{X}^m = (X^{m,1}, X^{m,2}, \ldots) \quad \text{and} \quad \mathcal{Y}^m = (Y^{m,1}, Y^{m,2}, \ldots),$$

respectively, according to the grid D_m. Here,

$$\begin{aligned} &X^{m,i} \in C([s_i, \infty), \mathscr{C}_{\mathrm{rap}}(\mathbb{R})) \quad \text{and} \quad Y^{m,i} \in C([t_i, \infty), \mathscr{C}_{\mathrm{rap}}(\mathbb{R})) \\ &\qquad\qquad \text{with } X^{m,i} \equiv 0 \text{ on } [0, s_i) \text{ and } Y^{m,i} \equiv 0 \text{ on } [0, t_i), \end{aligned} \tag{2.19}$$

so we need to specify $X^{m,i}$ over $[s_i, \infty)$ and $Y^{m,i}$ over $[t_i, \infty)$.

Consider the construction of $\mathcal{X}^m$. The decomposition of X over $[s_1, s_2]$ should be evident. Over this interval, set $X^{m,1} \equiv X$ on $[s_1, s_2)$ with $X^{m,1}_{s_2} = X_{s_2-}$ and

$$X^{m,2}_{s_2} = \psi(\mathbb{1}) J^{x_2}_{\varepsilon} = \Delta X_{s_2}.$$

We define $\mathcal{X}^m$ over $[s_2, \tau_j]$ by an induction on integers $j \geq j^X_*$, where $j^X_* \in \mathbb{N}$ satisfies $s_2 = \tau_{j^X_*}$, such that:

(1) the following measurability condition holds:

$$\begin{aligned}(X^{m,i}_s; s \in [0, \tau_k], i \in \mathbb{N}) \in \sigma(X_s; s \in [0, \tau_k]) \vee \sigma(U^X_i, 1 \leq i \leq k)& \\ \forall k \in \{0, \dots, j\},& \end{aligned} \tag{2.20}$$

with $\sigma(U^X_i, 1 \leq i \leq 0)$ understood to be the trivial σ-field $\{\Omega, \varnothing\}$,

(2) the laws of $X^{m,i}$ obey

$$\begin{cases} \text{(a)} \quad \mathscr{L}(X^{m,i}_s; s \in [s_i, \tau_j]) \sim \mathrm{SBM}_{\psi(\mathbb{1}) J^{x_i}_{\varepsilon}, [s_i, \tau_j]} \text{ if } s_i \leq \tau_j \text{ and} \\ \text{(b)} \quad (X^{m,i}_s; s \in [s_i, \tau_j]) \text{ for } i \text{ satisfying } s_i \leq \tau_j, \text{ are independent,} \end{cases} \tag{2.21}$$

and

(3) a preliminary decomposition of X up to time τ_j holds:

$$X_s = \sum_{i=1}^{\infty} X^{m,i}_s \qquad \forall s \in [0, \tau_j] \text{ a.s.} \tag{2.22}$$

By the foregoing definition of $\mathcal{X}^m$ over $[s_1, s_2]$ and (2.5), we have handled the case that $j = j^X_*$, that is, the first step of our inductive construction.

Assume that $\mathcal{X}^m$ has been defined up to time τ_j for some integer $j \geq j^X_*$ such that (2.20)–(2.22) are all satisfied. Then our goal is to extend $\mathcal{X}^m$ over $[\tau_j, \tau_{j+1}]$ so that all of (2.20)–(2.22) hold with j replaced by $j+1$. First, consider the case that

$$[\tau_j, \tau_{j+1}] \subseteq [s_k, s_{k+1}) \tag{2.23}$$

for some $k \geq 2$. In this case, we need to extend $X^{m,1}, \dots, X^{m,k}$ up to time τ_{j+1}. Take an auxiliary nonnegative random element

$$\xi = (\xi^1, \dots, \xi^k) \in C\left([\tau_j, \infty), \prod_{i=1}^{k} \mathscr{C}_{\mathrm{rap}}(\mathbb{R})\right)$$

such that the coordinates $(\xi^i_s; s \in [\tau_j, \infty))$ are independent processes and each of them defines a super-Brownian motion started at time τ_j with initial law

$$\mathscr{L}(\xi^i_{\tau_j}) = \mathscr{L}(X^{m,i}_{\tau_j}) \qquad \forall i \in \{1, \dots, k\}. \tag{2.24}$$

We claim that it is possible to extend $X^{m,1},\dots,X^{m,k}$ continuously over $[\tau_j,\tau_{j+1}]$ so that

$$
\begin{aligned}
&\big(X^{m,1}_{\tau_j},\dots,X^{m,k}_{\tau_j},(X_r)_{r\in[\tau_j,\tau_{j+1}]},\big(X^{m,1}_r\big)_{r\in[\tau_j,\tau_{j+1}]},\dots,\big(X^{m,k}_r\big)_{r\in[\tau_j,\tau_{j+1}]}\big)\\
&\quad\overset{\text{(d)}}{=}\left(\xi^1_{\tau_j},\dots,\xi^k_{\tau_j},\left(\sum_{i=1}^k\xi^i_r\right)_{r\in[\tau_j,\tau_{j+1}]},\big(\xi^1_r\big)_{r\in[\tau_j,\tau_{j+1}]},\dots,\big(\xi^k_r\big)_{r\in[\tau_j,\tau_{j+1}]}\right).
\end{aligned}
\tag{2.25}
$$

To prove our claim that (2.25) can be done, first we consider

$$
\begin{aligned}
&\mathbb{P}\big((X_r)_{r\in[\tau_j,\tau_{j+1}]}\in\Gamma,X^{m,1}_{\tau_j}\in A_1,\dots,X^{m,k}_{\tau_j}\in A_k\big)\\
&\qquad=\mathbb{E}\big[\mathbb{P}\big((X_r)_{r\in[\tau_j,\tau_{j+1}]}\in\Gamma|X_{\tau_j}\big);X^{m,1}_{\tau_j}\in A_1,\dots,X^{m,k}_{\tau_j}\in A_k\big]\\
&\qquad=\mathbb{E}\big[\mathrm{SBM}_{X_{\tau_j},[\tau_j,\tau_{j+1}]}(\Gamma);X^{m,1}_{\tau_j}\in A_1,\dots,X^{m,k}_{\tau_j}\in A_k\big]\\
&\qquad=\mathbb{E}\big[\mathrm{SBM}_{\sum_{i=1}^k X^{m,i}_{\tau_j},[\tau_j,\tau_{j+1}]}(\Gamma);X^{m,1}_{\tau_j}\in A_1,\dots,X^{m,k}_{\tau_j}\in A_k\big],
\end{aligned}
\tag{2.26}
$$

where the first and the second equalities use the (time-inhomogeneous) Markov property of X and (2.20), and the last equality follows from the equality (2.22) by induction. Second, by (2.21) from induction and (2.24), we have

$$
\big(X^{m,1}_{\tau_j},\dots,X^{m,k}_{\tau_j}\big)\overset{\text{(d)}}{=}\big(\xi^1_{\tau_j},\dots,\xi^k_{\tau_j}\big).
$$

Hence, from (2.26), we get

$$
\begin{aligned}
&\mathbb{P}\big((X_r)_{r\in[\tau_j,\tau_{j+1}]}\in\Gamma,X^{m,1}_{\tau_j}\in A_1,\dots,X^{m,k}_{\tau_j}\in A_k\big)\\
&\quad=\mathbb{E}\big[\mathrm{SBM}_{\sum_{i=1}^k \xi^i_{\tau_j},[\tau_j,\tau_{j+1}]}(\Gamma);\xi^1_{\tau_j}\in A_1,\dots,\xi^k_{\tau_j}\in A_k\big]\\
&\quad=\mathbb{E}\left[\mathbb{P}\left(\left(\sum_{i=1}^k\xi^i_r\right)_{r\in[\tau_j,\tau_{j+1}]}\in\Gamma\,\middle|\,\xi^1_{\tau_j},\dots,\xi^k_{\tau_j}\right);\xi^1_{\tau_j}\in A_1,\dots,\xi^k_{\tau_j}\in A_k\right]\\
&\quad=\mathbb{P}\left(\left(\sum_{i=1}^k\xi^i_r\right)_{r\in[\tau_j,\tau_{j+1}]}\in\Gamma,\xi^1_{\tau_j}\in A_1,\dots,\xi^k_{\tau_j}\in A_k\right).
\end{aligned}
\tag{2.27}
$$

Here, the second equality follows from the convolution property of the laws of super-Brownian motions:

$$
\mathrm{SBM}_{f_1,[s,t]}\star\cdots\star\mathrm{SBM}_{f_k,[s,t]}=\mathrm{SBM}_{\sum_{i=1}^k f_i,[s,t]}.
$$

Then (2.27) implies that

$$
\big(X^{m,1}_{\tau_j},\dots,X^{m,k}_{\tau_j},(X_r)_{r\in[\tau_j,\tau_{j+1}]}\big)\overset{\text{(d)}}{=}\left(\xi^1_{\tau_j},\dots,\xi^k_{\tau_j},\left(\sum_{i=1}^k\xi^i_r\right)_{r\in[\tau_j,\tau_{j+1}]}\right).
\tag{2.28}
$$

Using the "boundary condition" (2.28) and Theorem 2.5, we can solve the stochastic equation on the left-hand side of (2.25) by a Borel measurable function

$$\begin{aligned} F_j^m : \prod_{i=1}^k \mathscr{C}_{\mathrm{rap}}(\mathbb{R}) \times C\big([\tau_j, \tau_{j+1}], \mathscr{C}_{\mathrm{rap}}(\mathbb{R})\big) \times [0, 1] \\ \longrightarrow \prod_{i=1}^k C\big([\tau_j, \tau_{j+1}], \mathscr{C}_{\mathrm{rap}}(\mathbb{R})\big) \end{aligned}$$

such that the desired extension of $\mathcal{X}^m$ over $[\tau_j, \tau_{j+1}]$ can be defined by

$$\begin{aligned} &\big((X_r^{m,1})_{r\in[\tau_j,\tau_{j+1}]}, \dots, (X_r^{m,k})_{r\in[\tau_j,\tau_{j+1}]}\big) \\ &\quad = F_j^m\big(X_{\tau_j}^{m,1}, \dots, X_{\tau_j}^{m,k}, (X_r)_{r\in[\tau_j,\tau_{j+1}]}, U_{j+1}^X\big), \end{aligned} \tag{2.29}$$

where the independent uniform variable U_{j+1}^X now plays its role to decompose $(X_r)_{r\in[\tau_j,\tau_{j+1}]}$. This proves our claim on the continuous extension of $X^{m,1}, \dots, X^{m,k}$ over $[\tau_j, \tau_{j+1}]$ satisfying (2.25). As a consequence of the equality (2.25) in distribution, the following equalities hold almost surely:

$$X_r^{m,1} + \cdots + X_r^{m,k} = X_r \qquad \forall r \in [\tau_j, \tau_{j+1}] \tag{2.30}$$

and

$$\begin{aligned} &\mathscr{L}\big((X_r^{m,1})_{r\in[\tau_j,\tau_{j+1}]}, \dots, (X_r^{m,k})_{r\in[\tau_j,\tau_{j+1}]} | X_{\tau_j}^{m,1}, \dots, X_{\tau_j}^{m,k}, X_{\tau_j}\big) \\ &\quad = \mathscr{L}\big((X_r^{m,1})_{r\in[\tau_j,\tau_{j+1}]}, \dots, (X_r^{m,k})_{r\in[\tau_j,\tau_{j+1}]} | X_{\tau_j}^{m,1}, \dots, X_{\tau_j}^{m,k}\big) \\ &\quad = \mathrm{SBM}_{X_{\tau_j}^{m,1},[\tau_j,\tau_{j+1}]} \otimes \cdots \otimes \mathrm{SBM}_{X_{\tau_j}^{m,k},[\tau_j,\tau_{j+1}]}, \end{aligned} \tag{2.31}$$

where the first equality of (2.31) follows from (2.30), and the second from the definition of ξ. By induction and (2.29), the extension of $\mathcal{X}^m$ over $[\tau_j, \tau_{j+1}]$ satisfies (2.20) with j replaced by $j+1$; by induction and (2.30), it satisfies (2.22) with j replaced by $j+1$.

Let us verify that (2.21) is satisfied with j replaced by $j+1$. By (2.20), we can write

$$\begin{aligned} &\mathbb{P}\big((X_r^{m,i})_{r\in[\tau_j,\tau_{j+1}]} \in A_i, (X_r^{m,i})_{r\in[s_i,\tau_j]} \in B_i, \forall i \in \{1, \dots, k\}\big) \\ &\quad = \mathbb{E}\big[\mathbb{P}\big((X_r^{m,i})_{r\in[\tau_j,\tau_{j+1}]} \in A_i, \forall i \in \{1, \dots, k\} | \mathscr{F}_{\tau_j} \vee \sigma\big(U_1^X, \dots, U_j^X\big)\big); \\ &\qquad\qquad (X_r^{m,i})_{r\in[s_i,\tau_j]} \in B_i, \forall i \in \{1, \dots, k\}\big]. \end{aligned} \tag{2.32}$$

To reduce the conditional probability on the right-hand side of (2.32) to a probability conditioned on $X_{\tau_j}^{m,1}, \dots, X_{\tau_j}^{m,k}$, we review the defining equation (2.29) of $\mathcal{X}^m$

over $[\tau_j, \tau_{j+1}]$ and consider the calculation:

$$
\begin{aligned}
&\mathbb{E}\big[g_1\big(X^{m,1}_{\tau_j}, \dots, X^{m,k}_{\tau_j}\big) g_2\big((X_r)_{r\in[\tau_j,\tau_{j+1}]}\big) g_3\big(U^X_{j+1}\big) | \mathscr{F}_{\tau_j} \vee \sigma\big(U^X_1, \dots, U^X_j\big)\big] \\
&\quad = g_1\big(X^{m,1}_{\tau_j}, \dots, X^{m,k}_{\tau_j}\big) \mathbb{E}\big[g_2\big((X_r)_{r\in[\tau_j,\tau_{j+1}]}\big) | \mathscr{F}_{\tau_j} \vee \sigma\big(U^X_1, \dots, U^X_j\big)\big] \\
&\qquad \times \mathbb{E}\big[g_3\big(U^X_{j+1}\big)\big] \\
&\quad = g_1\big(X^{m,1}_{\tau_j}, \dots, X^{m,k}_{\tau_j}\big) \mathbb{E}\big[g_2\big((X_r)_{r\in[\tau_j,\tau_{j+1}]}\big) | X^{m,1}_{\tau_j}, \dots, X^{m,k}_{\tau_j}\big] \\
&\qquad \times \mathbb{E}\big[g_3\big(U^X_{j+1}\big)\big] \\
&\quad = \mathbb{E}\big[g_1\big(X^{m,1}_{\tau_j}, \dots, X^{m,k}_{\tau_j}\big) g_2\big((X_r)_{r\in[\tau_j,\tau_{j+1}]}\big) g_3\big(U^X_{j+1}\big) | X^{m,1}_{\tau_j}, \dots, X^{m,k}_{\tau_j}\big],
\end{aligned}
\tag{2.33}
$$

where the first equality follows again from (2.20) and the second equality follows by using the $(\mathscr{F}_t)$-Markov property of X and considering the "sandwich" of σ-fields:

$$
\sigma(X_{\tau_j}) \subseteq \sigma\big(X^{m,1}_{\tau_j}, \dots, X^{m,k}_{\tau_j}\big) \vee \mathscr{N} \subseteq \mathscr{F}_{\tau_j} \vee \sigma\big(U^X_1, \dots, U^X_j\big)
$$

with $\mathscr{N}$ being the collection of $\mathbb{P}$-null sets, and the last equality (2.33) follows since U^X_{j+1} is not yet used in the construction of $\mathcal{X}^m$ up to time τ_j. Hence, by (2.29) and (2.33), we can continue our calculation in (2.32) as follows:

$$
\begin{aligned}
&\mathbb{P}\big(\big(X^{m,i}_r\big)_{r\in[\tau_j,\tau_{j+1}]} \in A_i, \big(X^{m,i}_r\big)_{r\in[s_i,\tau_j]} \in B_i, \forall i \in \{1, \dots, k\}\big) \\
&\quad = \mathbb{E}\big[\mathbb{P}\big(\big(X^{m,i}_r\big)_{r\in[\tau_j,\tau_{j+1}]} \in A_i, \forall i \in \{1, \dots, k\} | X^{m,1}_{\tau_j}, \dots, X^{m,k}_{\tau_j}\big); \\
&\qquad\qquad \big(X^{m,i}_r\big)_{r\in[s_i,\tau_j]} \in B_i, \forall i \in \{1, \dots, k\}\big] \\
&\quad = \mathbb{E}\left[\prod_{i=1}^k \mathrm{SBM}_{X^{m,i}_{\tau_j}, [\tau_j,\tau_{j+1}]}(A_i); \big(X^{m,i}_r\big)_{r\in[s_i,\tau_j]} \in B_i, \forall i \in \{1, \dots, k\}\right],
\end{aligned}
$$

where the second equality follows from (2.31). By (2.21) and induction, the foregoing equality implies that (2.21) with j replaced by $j + 1$ still holds. This completes our inductive construction for the case (2.23).

We also need to handle the case complementary to (2.23) that $[\tau_j, \tau_{j+1}] \subseteq (s_k, s_{k+1}]$ and $\tau_{j+1} = s_{k+1}$ for some $k \geq 2$. In this case, the construction of $X^{m,1}, \dots, X^{m,k}$ over the time interval $[\tau_j, \tau_{j+1}]$ is the same as before, but the extra coordinate $X^{m,k+1}$ is defined to be $\psi(\mathbb{1}) J^{x_{k+1}}_\varepsilon$ at time $\tau_{j+1} = s_{k+1}$. The properties (2.20) and (2.22) with j replaced by $j + 1$ remain true, by the argument for the previous case. The property (2.21) with j replaced by $j + 1$ follows too, if we notice that the coordinate $X^{m,k+1}$ is independent of the others by time τ_{j+1} by (iii) of Definition 2.1. This completes our inductive construction of $\mathcal{X}^m$.

The construction of $\mathcal{Y}^m$ is similar to that of $\mathcal{X}^m$. We use $\{U^Y_j\}$ to validate decompositions, and the immigration times $\{t_j; j \in \mathbb{N}\}$ are taken into consideration for the construction instead. We omit other details.

We observe that $\mathcal{X}^m$ and $\mathcal{Y}^m$ satisfy properties analogous to (2.16) and (2.17). First, from the constructions of $\mathcal{X}^m$ and $\mathcal{Y}^m$, (2.18), and the property (iii) in Definition 2.1, we see that the following independent landing property is satisfied by $\mathcal{X}^m$ and $\mathcal{Y}^m$:

$$\begin{aligned} &\sigma\bigl(X_s^{m,j}, Y_s^{m,j}; s < s_i, j \in \mathbb{N}\bigr) \perp\!\!\!\perp x_i \quad \text{and} \\ &\sigma\bigl(X_s^{m,j}, Y_s^{m,j}; s < t_i, j \in \mathbb{N}\bigr) \perp\!\!\!\perp y_i \qquad \forall i \in \mathbb{N}. \end{aligned} \tag{2.34}$$

Second, since for all $j \in \mathbb{N}$ satisfying $s_j < s_{i+1}$ and $i \in \mathbb{N}$, $(X_r^{m,j})_{r\in[0,s_{i+1})}$ is given by a measurable function of the random elements $(X_r)_{r\in[0,s_{i+1})}$ and $\{U_k^X\}$ [cf. (2.29) for the case (2.23) and use the path regularity of X in the complementary case], we deduce from Theorem 2.4(i) and (2.18) that

$$\begin{aligned} &\bigl\{\bigl(X_r^{m,j}\bigr)_{r\in[0,s_{i+1})}; j \in \mathbb{N} \text{ satisfying } s_j < s_{i+1}\bigr\} \perp\!\!\!\perp V^{X,i} \quad \text{and} \\ &\bigl\{\bigl(Y_r^{m,j}\bigr)_{r\in[0,t_{i+1})}; j \in \mathbb{N} \text{ satisfying } t_j < t_{i+1}\bigr\} \perp\!\!\!\perp V^{Y,i} \qquad \forall i \in \mathbb{N}. \end{aligned} \tag{2.35}$$

Step 2. Let us define a filtration $(\mathscr{G}_t^{(m)})$ with respect to which the processes $X^{m,i}$, $Y^{m,i}$, and W perform their defining properties *on the grid* D_m. The filtration $(\mathscr{G}_t^{(m)})$ is larger than $(\mathscr{F}_t)$ and is defined by

$$\begin{cases} \mathscr{G}_t^{(m)} = \mathscr{F}_t, & t \in [0, \tau_0], \\ \mathscr{G}_t^{(m)} = \mathscr{F}_{\tau_{j+1}} \vee \sigma\bigl(U_k^X, U_k^Y; 1 \le k \le j+1\bigr), & t \in (\tau_j, \tau_{j+1}], j \in \mathbb{Z}_+. \end{cases}$$

In particular, it follows from (2.20) and the analogue for $\mathcal{Y}^m$ that the processes $X^{m,i}$ and $Y^{m,i}$ are all $(\mathscr{G}_t^{(m)})$-adapted. Also, it is obvious that X, Y and $W(\phi)$ for any $\phi \in L^2(\mathbb{R})$ are $(\mathscr{G}_t^{(m)})$-adapted.

We observe a key feature of $\mathcal{X}^m$:

$$\begin{aligned} &\mathbb{P}\bigl(X_t^{m,i} \in \Gamma | \mathscr{G}_{\tau_j}^{(m)}\bigr) = \mathrm{SBM}_{t-\tau_j}\bigl(X_{\tau_j}^{m,i}, \Gamma\bigr) \\ &\qquad\qquad \forall t \in (\tau_j, \tau_{j+1}] \text{ for } s_i \le \tau_j \text{ and } i \in \mathbb{N}, \end{aligned} \tag{2.36}$$

for any Borel measurable subset Γ of the space of finite measures on $\mathbb{R}$. To see (2.36), we consider a slight generalization of the proof of (2.33) by adding $\sigma(U_1^Y, \dots, U_j^Y)$ to the σ-field $\mathscr{F}_{\tau_j} \vee \sigma(U_1^X, \dots, U_j^X)$ in the first line therein and then apply (2.21) to obtain

$$\begin{aligned} \mathbb{P}\bigl(X_t^{m,i} \in \Gamma | \mathscr{G}_{\tau_j}^{(m)}\bigr) &= \mathbb{P}\bigl(X_t^{m,i} \in \Gamma | X_{\tau_j}^{m,1}, X_{\tau_j}^{m,2}, \dots\bigr) \\ &= \mathbb{P}\bigl(X_t^{m,i} \in \Gamma | X_{\tau_j}^{m,i}\bigr) = \mathrm{SBM}_{t-\tau_j}\bigl(X_{\tau_j}^{m,i}, \Gamma\bigr) \\ &\qquad \forall t \in (\tau_j, \tau_{j+1}]. \end{aligned}$$

In particular, we deduce from iteration and the semigroup property of $\{\mathrm{SBM}_t\}$ that the following *grid Markov property* is satisfied:

$$\mathbb{P}\big(X_t^{m,i} \in \Gamma | \mathscr{G}_{\tau_j}^{(m)}\big) = \mathrm{SBM}_{t-\tau_j}\big(X_{\tau_j}^{m,i}, \Gamma\big) \qquad \forall t \in (\tau_k, \tau_{k+1}] \text{ when } s_i \leq \tau_j \leq \tau_k. \tag{2.37}$$

We note that the foregoing display does *not* say that $X^{m,i}$ is a $(\mathscr{G}_s^{(m)})_{s\geq s_i}$-super-Brownian motion because the σ-fields which we can use in verifying the $(\mathscr{G}_s^{(m)})_{s\geq s_i}$-Markov property are only $\mathscr{G}_{\tau_j}^{(m)}$, rather than *any* σ-field $\mathscr{G}_s^{(m)}$. With a similar argument, we also have the grid Markov property of $Y^{m,i}$ stated as

$$\mathbb{P}\big(Y_t^{m,i} \in \Gamma | \mathscr{G}_{\tau_j}^{(m)}\big) = \mathrm{SBM}_{t-\tau_j}\big(Y_{\tau_j}^{m,i}, \Gamma\big) \qquad \forall t \in (\tau_k, \tau_{k+1}] \text{ when } t_i \leq \tau_j \leq \tau_k. \tag{2.38}$$

With a much simpler argument, the space–time white noise W has the same grid Markov property:

$$\mathscr{L}\big(W_t(\phi) | \mathscr{G}_{\tau_j}^{(m)}\big) = \mathcal{N}\big(W_{\tau_j}(\phi), (t-\tau_j)\|\phi\|_{L^2(\mathbb{R})}^2\big) \qquad \forall t \in (\tau_k, \tau_{k+1}] \text{ for } \tau_j \leq \tau_k \text{ and } \phi \in L^2(\mathbb{R}), \tag{2.39}$$

where $\mathcal{N}(\mu, \sigma^2)$ denotes the normal distribution with mean μ and variance σ^2. Similar results hold for the substituting space–time white noises $V^{X,i}$ and $V^{Y,i}$.

Step 3. To facilitate our argument in the next step, we digress to a general property of space–time white noises.

Let W^1 denote a space–time white noise, and suppose that $\{W^2(\phi_n)\}$ is a family of Brownian motions indexed by a countable dense subset $\{\phi_n\}$ of $L^2(\mathbb{R})$ such that $\{W^1(\phi_n)\}$ and $\{W^2(\phi_n)\}$ have the same law as random elements taking values in $\prod_{n=1}^{\infty} C(\mathbb{R}_+, \mathbb{R})$. Then, whenever (ϕ_{n_k}) is a subsequence converging to some ϕ in $L^2(\mathbb{R})$, the linearity of W^1 gives

$$\begin{aligned}\mathbb{E}\Big[\sup_{0\leq s\leq T}\big|W_s^2(\phi_{n_k}) - W_s^2(\phi_{n_\ell})\big|^2\Big] &= \mathbb{E}\Big[\sup_{0\leq s\leq T}\big|W_s^1(\phi_{n_k} - \phi_{n_\ell})\big|^2\Big] \\ &\leq 4T\|\phi_{n_k} - \phi_{n_\ell}\|_{L^2(\mathbb{R})}^2 \xrightarrow[k,\ell\to\infty]{} 0 \\ &\qquad \forall T \in (0, \infty),\end{aligned} \tag{2.40}$$

where the inequality follows from Doob's L^2-inequality and the fact that, for any $\phi \in L^2(\mathbb{R})$, $W^1(\phi)$ is a Brownian motion with $\mathscr{L}(W_1^1(\phi)) = \mathcal{N}(0, \|\phi\|_{L^2(\mathbb{R})}^2)$. The convergence in (2.40) implies that, for some continuous process, say $W^2(\phi)$, we have

$$W^2(\phi_{n_k}) \longrightarrow W^2(\phi) \qquad \text{uniformly on } [0, T] \text{ a.s., } \forall T \in (0, \infty).$$

The same holds with W^2 replaced by W^1. Hence, making comparisons with the reference space–time white noise W^1, we obtain an extension of the map $\phi \longmapsto W^2(\phi)$ to the entire space $L^2(\mathbb{R})$ such that $\{W^2(\phi); \phi \in L^2(\mathbb{R})\}$ is a space–time white noise and, in fact, is uniquely defined by $\{W^2(\phi_n)\}$.

Step 4. In this step, we formalize the infinitesimal description outlined before by shrinking the mesh size of D_m, that is, by passing $m \longrightarrow \infty$, and then work with the limiting objects. To use our observation in step 3, we work with a fixed countable dense subset $\{\phi_n\}$ of $L^2(\mathbb{R})$.

We have constructed in step 1 random elements $\mathcal{X}^m$ and $\mathcal{Y}^m$, and hence determined the laws

$$\mathscr{L}\big(X, Y, W, \{V^{X,i}\}, \{V^{Y,i}\}, \mathcal{X}^m, \mathcal{Y}^m, \{x_i\}, \{y_i\}\big), \qquad m \in \mathbb{N}, \tag{2.41}$$

as probability measures on a countably infinite product of Polish spaces. More precisely, our choice of the Polish spaces is through the following identifications of state spaces. We identify X as a random element taking values in the closed subset of $D(\mathbb{R}_+, \mathscr{C}_{\mathrm{rap}}(\mathbb{R}))$ consisting of paths having continuity over each interval $[s_i, s_{i+1})$ for $i \in \mathbb{Z}_+$ (recall $s_0 = 0$), with a similar identification applied to Y (cf. Proposition 5.3 and Remark 5.4 of [10]). By step 3, we identify W as the infinite-dimensional vectors $(W(\phi_1), W(\phi_2), \ldots)$ whose coordinates are $C(\mathbb{R}_+, \mathbb{R})$-valued random elements. Similarly, $V^{X,i}$ and $V^{Y,i}$ are infinite-dimensional vectors of $C([s_i, \infty), \mathbb{R})$ and $C([t_i, \infty), \mathbb{R})$-valued random elements. We identify each coordinate $X^{m,i}$ of $\mathcal{X}^m$ as a random element taking values in $C([s_i, \infty), \mathscr{C}_{\mathrm{rap}}(\mathbb{R}))$, with a similar identification applied to $\mathcal{Y}^m$. Finally, the Polish spaces for the infinite sequences $\{x_i\}$ and $\{y_i\}$ are obvious.

We make an observation for the sequence of laws in (2.41). Note that $\mathscr{L}(\mathcal{X}^m)$ does not depend on m, because, by (2.21), any of its ith coordinate $X^{m,i}$ is a super-Brownian motion with initial measure $\psi(\mathbb{1})J_\varepsilon^{x_i}$ and started at s_i, and the coordinates are independent. Similarly, $\mathscr{L}(\mathcal{Y}^m)$ does not depend on m. This implies that the sequence of laws in (2.41) is tight in the space of probability measures on the above infinite product of Polish spaces. Hence, by taking a subsequence if necessary, we may assume that this sequence converges in distribution. By Skorokhod's representation, we may assume the existence of the vectors of random elements in the following display as well as the almost-sure convergence therein:

$$\begin{aligned} &\big(\widetilde{X}^{(m)}, \widetilde{Y}^{(m)}, \widetilde{W}^m, \{\widetilde{V}^{X,i,m}\}, \{\widetilde{V}^{Y,i,m}\}, \widetilde{\mathcal{X}}^m, \widetilde{\mathcal{Y}}^m, \{\widetilde{x}_i^m\}, \{\widetilde{y}_i^m\}\big) \\ &\qquad \xrightarrow[m\to\infty]{\text{a.s.}} \big(\widetilde{X}, \widetilde{Y}, \widetilde{W}, \{\widetilde{V}^{X,i}\}, \{\widetilde{V}^{Y,i}\}, \widetilde{\mathcal{X}}, \widetilde{\mathcal{Y}}, \{\widetilde{x}_i\}, \{\widetilde{y}_i\}\big). \end{aligned} \tag{2.42}$$

Here, $\widetilde{x}_i$ and $\widetilde{y}_i$ take values in the topological support of ψ and

$$\begin{aligned} &\mathscr{L}\big(\widetilde{X}^{(m)}, \widetilde{Y}^{(m)}, \widetilde{W}^m, \{\widetilde{V}^{X,i,m}\}, \{\widetilde{V}^{Y,i,m}\}, \widetilde{\mathcal{X}}^m, \widetilde{\mathcal{Y}}^m, \{\widetilde{x}_i^m\}, \{\widetilde{y}_i^m\}\big) \\ &\quad = \mathscr{L}\big(X, Y, W, \{V^{X,i}\}, \{V^{Y,i}\}, \mathcal{X}^m, \mathcal{Y}^m, \{x_i\}, \{y_i\}\big) \qquad \forall m \in \mathbb{N}. \end{aligned}$$

Step 5. We define $(\widetilde{\mathscr{G}}_t)$ to be the minimal filtration satisfying the usual conditions to which the limiting objects $\widetilde{X}, \widetilde{Y}, \widetilde{W}, \{\widetilde{V}^{X,i}\}, \{\widetilde{V}^{Y,i}\}, \widetilde{\mathscr{X}}, \widetilde{\mathscr{Y}}$ on the right-hand side of (2.42) are adapted. We will complete the proof in this step by verifying that, with an obvious adaptation of notation, all the limiting objects on the right-hand side of (2.42) along with the filtration $(\widetilde{\mathscr{G}}_t)$ are the required objects satisfying conditions (i)–(vi) of Theorem 2.6.

First, let us verify the easier properties (i) and (ii) for $\{\widetilde{X}^i\}$ and the analogues for $\{\widetilde{Y}^i\}$. The statement (i) and its analogue for $\{\widetilde{Y}^i\}$ obviously hold, by the analogous properties of $\widetilde{\mathscr{X}}^m$ and $\widetilde{\mathscr{Y}}^m$ [see (b) of (2.21)]. To verify the statement (ii), we use the property (2.22) possessed by $(\widetilde{X}^{(m)}, \widetilde{\mathscr{X}}^m)$ and then pass limit, as is legitimate because the infinite series in (2.22) are always finite sums on compact time intervals. Similarly, the analogue of (ii) holds for $(\widetilde{Y}, \widetilde{\mathscr{Y}})$.

Condition (iii) holds by the property (a) in (2.21) of $\widetilde{\mathscr{X}}^m$, except that we still need to verify that each $\widetilde{X}^i$ defines a $(\widetilde{\mathscr{G}}_t)_{t\geq s_i}$-super Brownian motion, not just a super-Brownian motion in itself. From this point on, we will use the continuity of the underlying objects and the fact that $\bigcup_m D_m$ is dense in $[\frac{\varepsilon}{2}, \infty)$. Let $\frac{\varepsilon}{2} \leq s < t < \infty$ with $s, t \in \bigcup_m D_m$. Then $s, t \in D_m$ from some large m on by the nesting property of the sequence $\{D_m\}$. For any bounded continuous function g on the path space of

$$(\widetilde{X}^{(m)}, \widetilde{Y}^{(m)}, \widetilde{W}^m, \{\widetilde{V}^{X,i,m}\}, \{\widetilde{V}^{Y,i,m}\}, \widetilde{\mathscr{X}}^m, \widetilde{\mathscr{Y}}^m)$$

restricted to the time interval $[0, s]$, $\phi \in \mathscr{C}_c^+(\mathbb{R})$, and index i such that $s_i \leq s$, the grid Markov property (2.37) entails that

$$\begin{aligned}
&\mathbb{E}[g(\widetilde{X}^{(m)}, \widetilde{Y}^{(m)}, \widetilde{W}^m, \{\widetilde{V}^{X,i,m}\}, \{\widetilde{V}^{Y,i,m}\}, \widetilde{\mathscr{X}}^m, \widetilde{\mathscr{Y}}^m)e^{-\langle \widetilde{X}_t^{(m),i}, \phi\rangle}] \\
&\quad = \mathbb{E}\Big[g(\widetilde{X}^{(m)}, \widetilde{Y}^{(m)}, \widetilde{W}^m, \{\widetilde{V}^{X,i,m}\}, \{\widetilde{V}^{Y,i,m}\}, \widetilde{\mathscr{X}}^m, \widetilde{\mathscr{Y}}^m) \\
&\qquad \times \int \mathrm{SBM}_{t-s}(\widetilde{X}_s^{(m),i}, d\nu)e^{-\langle \nu, \phi\rangle}\Big].
\end{aligned} \tag{2.43}$$

The formula of Laplace transforms of super-Brownian motion shows that the map

$$f \longmapsto \int \mathrm{SBM}_{t-s}(f, d\nu)e^{-\langle \nu, \phi\rangle}$$

has a natural extension to $\mathscr{C}_{\mathrm{rap}}(\mathbb{R})$ which is continuous (cf. Proposition II.5.10 of [21]). Hence, passing $m \longrightarrow \infty$ for both sides of (2.43) leads to

$$\begin{aligned}
&\mathbb{E}[g(\widetilde{X}, \widetilde{Y}, \widetilde{W}, \{\widetilde{V}^{X,i}\}, \{\widetilde{V}^{Y,i}\}, \widetilde{\mathscr{X}}, \widetilde{\mathscr{Y}})e^{-\langle \widetilde{X}_t^i, \phi\rangle}] \\
&\quad = \mathbb{E}\Big[g(\widetilde{X}, \widetilde{Y}, \widetilde{W}, \{\widetilde{V}^{X,i}\}, \{\widetilde{V}^{Y,i}\}, \widetilde{\mathscr{X}}, \widetilde{\mathscr{Y}}) \\
&\qquad \times \int \mathrm{SBM}_{t-s}(\widetilde{X}_s^i, d\nu)e^{-\langle \nu, \phi\rangle}\Big].
\end{aligned} \tag{2.44}$$

By the continuity of super-Brownian motion and the denseness of $\bigcup_m D_m$ in $[\frac{\varepsilon}{2},\infty)$, the foregoing display implies that each coordinate $\widetilde{X}^i$ is truly a $(\widetilde{\mathscr{G}}_t)_{t\geq s_i}$-super-Brownian motion. A similar argument shows that each $\widetilde{Y}^i$ is a $(\widetilde{\mathscr{G}}_t)_{t\geq t_i}$-super-Brownian motion. We have proved the statement (iii) and its analogue for $\widetilde{Y}^i$ in (iv).

Next, we consider the assertions of (v) concerning conditions analogous to (i) and (ii) of Definition 2.1. By definition,

$$
\begin{aligned}
&\mathscr{L}\big(\widetilde{X}^m,\widetilde{Y}^m,\widetilde{W}^m,\big\{\widetilde{V}^{X,i,m}\big\},\big\{\widetilde{V}^{Y,i,m}\big\},\{\widetilde{x}_i^m\},\{\widetilde{y}_i^m\}\big)\\
&\quad=\mathscr{L}\big(X,Y,W,\big\{V^{X,i}\big\},\big\{V^{Y,i}\big\},\{x_i\},\{y_i\}\big)\qquad\forall m\in\mathbb{N},
\end{aligned}\tag{2.45}
$$

and this stationarity gives

$$
\begin{aligned}
&\mathscr{L}\big(\widetilde{X},\widetilde{Y},\widetilde{W},\big\{\widetilde{V}^{X,i}\big\},\big\{\widetilde{V}^{Y,i}\big\},\{\widetilde{x}_i\},\{\widetilde{y}_i\}\big)\\
&\quad=\mathscr{L}\big(X,Y,W,\big\{V^{X,i}\big\},\big\{V^{Y,i}\big\},\{x_i\},\{y_i\}\big).
\end{aligned}\tag{2.46}
$$

Arguing as in the proof of (2.43) and using the grid Markov property (2.39) of $\widetilde{W}^m$, we see that each $\widetilde{W}(\phi_n)$ is a $(\widetilde{\mathscr{G}}_t)$-Brownian motion with

$$
\mathscr{L}\big(\widetilde{W}_1(\phi_n)\big)=\mathcal{N}\big(0,\|\phi_n\|^2_{L_2(\mathbb{R})}\big).
$$

It follows from (2.46) and our discussion in step 3 that $\widetilde{W}$ extends uniquely to a $(\widetilde{\mathscr{G}}_t)$-space–time white noise. In addition, one more application of (2.46) shows that the defining equations (2.1) and (2.3) of X and Y by $\{(x_i,y_i)\}$ and W carry over to the analogous equations for $\widetilde{X}$ and $\widetilde{Y}$ by $\{(\widetilde{x}_i,\widetilde{y}_i)\}$ and $\widetilde{W}$, respectively [recall (2.4) as well]. This proves that $(\widetilde{X},\widetilde{Y},\widetilde{W})$ satisfies the analogous property described in (i) and (ii) of Definition 2.1 with $(\mathscr{F}_t)$ replaced by $(\widetilde{\mathscr{G}}_t)$.

By construction, $\widetilde{x}_i$ and $\widetilde{y}_i$ take values in the topological support of ψ. Hence, to complete the proof of (v), it remains to obtain the independent landing property (2.16). We recall that an analogous property is satisfied by $(\widetilde{\mathcal{X}}^m,\widetilde{\mathcal{Y}}^m,\{\widetilde{x}_i^m\},\{\widetilde{y}_i^m\})$ in (2.34). Then arguing in the standard way as in the proof of (2.43) with the use of bounded continuous functions shows that the required independent landing property (2.16) is satisfied by $(\widetilde{\mathcal{X}},\widetilde{\mathcal{Y}},\{\widetilde{x}_i\},\{\widetilde{y}_i\})$. The proof of (v) is complete.

Finally, we explain the proof of the assertions in (vi). The proof of (iii) of Proposition 2.4 uses again the stationarity (2.45). The assertion that (2.17) holds follows from its discrete version (2.35) and a limiting argument as in the proof of (iii). We have proved that all of the conditions (i)–(vi) of Theorem 2.6 hold. The proof is complete. $\square$

2.3. *Covariations of immigrant processes.* In this section, we study the covariations $\langle X^i(\phi_1),Y^j(\phi_2)\rangle$ of the immigrant processes $\{X^i\}$ and $\{Y^i\}$ in Theorem 2.6 for $\varepsilon\in(0,1]$. Here, the test functions ϕ_1,ϕ_2 belong to $\mathscr{C}_c^\infty(\mathbb{R})$. Our goal is to understand how explicit $\langle X^i(\phi_1), Y^j(\phi_2)\rangle$ can be in terms of the immigrant processes.

For convenience, we attach space–time white noises to the immigrant processes $\{X^i\}$ and $\{Y^i\}$. Recall that by (iii) of Theorem 2.6, $(X^i_t)_{t\in[s_i,\infty)}$ is a $(\mathscr{G}_t)_{t\geq s_i}$-super-Brownian motion for any $i \in \mathbb{N}$, and similarly, each $(Y^i_t)_{t\in[t_i,\infty)}$ is a $(\mathscr{G}_t)_{t\geq t_i}$-super-Brownian motion. By a classical argument, we can find, by enlarging the filtered probability space if necessary, two families of $(\mathscr{G}_t)$-white noises $\{W^{X^i}\}$ and $\{W^{Y^i}\}$ such that (X^i, W^{X^i}) and (Y^i, W^{Y^i}) are solutions to the SPDE (1.1) of super-Brownian motion (up to appropriate translations of starting time). See Theorem III.4.2 of [21] for details. Moreover, by (i) and (vi) of Theorem 2.6, we can assume that each of the families $\{W^{X^i}\}$ and $\{W^{Y^i}\}$ consists of independent adapted space–time white noises, and in addition, the following independence holds:

$$\begin{aligned}&\{X^i_t, W^{X^i}_t(\phi); t \in [0, s_{j+1}), 1 \leq i \leq j, \phi \in L^2(\mathbb{R})\} \perp\!\!\!\perp V^{X,j} \quad \text{and}\\ &\{Y^i_t, W^{Y^i}_t(\phi); t \in [0, t_{j+1}), 1 \leq i \leq j, \phi \in L^2(\mathbb{R})\} \perp\!\!\!\perp V^{Y,j} \qquad \forall j \in \mathbb{N},\end{aligned} \tag{2.47}$$

where $V^{X,j}$ and $V^{Y,j}$ are the adapted space–time white noises which substitute W and satisfy (2.11) and (2.12).

REMARK 2.7. Let us point out an issue for the covariations $\langle X^i(\phi_1), Y^j(\phi_2)\rangle$. An implication of the classical Kunita–Watanabe inequality (cf. Proposition IV.1.15 of [23]) is that for any two $(\mathscr{G}_t)$-Brownian motions B^1, B^2 and nonnegative locally bounded predictable processes H^1 and H^2, the covariation of the (ordinary) stochastic integrals $H^i \bullet B^i$ of H^i with respect to B^i satisfies

$$\left|\langle H^1 \bullet B^1, H^2 \bullet B^2\rangle_t\right| \leq \int_0^t H^1_s H^2_s\, ds \tag{2.48}$$

(recall $d\langle B^i, B^i\rangle_s \equiv ds$). On the other hand, let W^1 and W^2 be two $(\mathscr{G}_t)$-space–time white noises, and $J^1, J^2 \in L^2_{\mathrm{loc}}(W^1) = L^2_{\mathrm{loc}}(W^2)$ be nonnegative. [Recall the notation (2.8) and (2.9).] In this case, the measure $dx\, ds$ determines quadratic variations of stochastic integrals with respect to a space–time white noise in the sense that $\langle J^i \bullet W^i(\mathbb{1}), J^i \bullet W^i(\mathbb{1})\rangle_t = \int_0^t \int_{\mathbb{R}} J^i(x,s)^2\, dx\, ds$. In bounding the covariations of $J^1 \bullet W^1(\mathbb{1})$ and $J^2 \bullet W^2(\mathbb{1})$, however, the following inequality, analogous to (2.48), is not always true:

$$\left|\langle J^1 \bullet W^1(\mathbb{1}), J^2 \bullet W^2(\mathbb{1})\rangle_t\right| \leq \int_0^t \int_{\mathbb{R}} J^1(x,s) J^2(x,s)\, dx\, ds. \tag{2.49}$$

A counterexample is given by taking W^2 to be a nonidentity spatial translation of W^1. Hence, the conclusion of Proposition 3.5 in [3] is incorrect in general, as pointed by the anonymous referee, and there it is used to bound covariations of general adapted immigrant processes. Nevertheless, we will show in Section 2.4 that there do exist *some* immigrant processes whose covariations satisfy the concluding inequality of Proposition 3.5 in [3] (see Theorem 2.12 and Proposition 2.13), and so the arguments from Section 3.5 on in [3] remain valid if these particular immigrant processes are in force.

To facilitate the forthcoming computation of covariations, we give some terminology and notation. For a locally bounded signed measure μ on $\mathbb{R}\times\mathbb{R}_+$, we define a measure-valued process $J\bullet\mu$ by

$$J\bullet\mu(\phi)\triangleq\int_{(0,\cdot]}\int_{\mathbb{R}}J(x,s)\phi(x)\,d\mu(x,s),\qquad \phi\in\mathscr{C}_c^\infty(\mathbb{R}),$$

whenever J is a two-parameter random function satisfying $\int_{(0,t]}\int_{\mathbb{R}}|J(x,s)\phi(x)|\times|d\mu(x,s)|<\infty$ a.s. for all t and $\phi\in\mathscr{C}_c^\infty(\mathbb{R})$, and we put $\mu(\phi)\equiv\mathbb{1}\bullet\mu(\phi)$. Let U^i,V^i be $(\mathscr{G}_t)$-space–time white noises and $J^i,K^i\in L^2_{\mathrm{loc}}(U^i)=L^2_{\mathrm{loc}}(V^i)$ for $1\le i\le N$ and a natural number N. Then the pair $(\sum_i J^i\bullet U^i,\sum_i K^i\bullet V^i)$ of finite sums of stochastic integrals is said to have a *normal covariation* if

$$\left\langle\sum_i J^i\bullet U^i(\phi_1),\sum_i K^i\bullet V^i(\phi_2)\right\rangle=H\bullet\lambda(\phi_1\phi_2)\qquad\text{a.s. }\forall\phi_1,\phi_2\in\mathscr{C}_c^\infty(\mathbb{R})\tag{2.50}$$

for some $H\in L^2_{\mathrm{loc}}(U^i)$, where the measure λ in (2.50) is defined by

$$d\lambda(x,s)\triangleq dx\,ds.\tag{2.51}$$

In this case, we write $\langle\!\langle\sum_i J^i\bullet U^i,\sum_i K^i\bullet V^i\rangle\!\rangle$ for $H\bullet\lambda$. If $Z=Z(x,t)$ and $Z'=Z'(x,t)$ are solutions to SPDEs with stochastic integral terms characterized by $\sum_i J^i\bullet U^i$ and $\sum_i K^i\bullet V^i$, respectively, then the pair (Z,Z') is also said to have a normal covariation and we write $\langle\!\langle Z,Z'\rangle\!\rangle$ for $\langle\!\langle\sum_i J^i\bullet U^i,\sum_i K^i\bullet V^i\rangle\!\rangle$.

PROPOSITION 2.8. *Fix $\varepsilon\in(0,1]$ and an interlacing pair (X,Y) of ε-approximating solutions. Let $\{X^i\}$ and $\{Y^i\}$ be immigrant processes as in Theorem 2.6 and let $\{W^{X^i}\}$ and $\{W^{Y^i}\}$ be the auxiliary space–time white noises chosen before Remark 2.7. Then for all $i,j\in\mathbb{N}$, (X^i,X) and (X,Y^i) have normal covariations, and we have*

$$\begin{aligned}&\langle\!\langle X^i,Y\rangle\!\rangle=\mathbb{1}_{(X>0,Y>0)}(X^i)^{1/2}\left(\frac{X^i}{X}\right)^{1/2}Y^{1/2}\bullet\lambda\quad\textit{and}\\&\langle\!\langle X,Y^j\rangle\!\rangle=\mathbb{1}_{(X>0,Y>0)}X^{1/2}(Y^j)^{1/2}\left(\frac{Y^j}{Y}\right)^{1/2}\bullet\lambda,\end{aligned}\tag{2.52}$$

where the measure λ is defined by (2.51).

PROOF. We only show that (X^i,Y) has a normal covariation and compute $\langle\!\langle X^i,Y\rangle\!\rangle$, as the covariations of (X,Y^j) can be handled similarly. For $\phi_1,\phi_2\in$

$\mathscr{C}_c^\infty(\mathbb{R})$, write

$$\begin{aligned}&\langle (X^i)^{1/2} \bullet W^{X^i}(\phi_1), Y^{1/2} \bullet W(\phi_2)\rangle \\ &\quad = \langle (X^i)^{1/2} \bullet W^{X^i}(\phi_1), \mathbb{1}_{(X>0)} Y^{1/2} \bullet W(\phi_2)\rangle \\ &\qquad + \langle (X^i)^{1/2} \bullet W^{X^i}(\phi_1), \mathbb{1}_{(X=0)} Y^{1/2} \bullet W(\phi_2)\rangle,\end{aligned} \tag{2.53}$$

and consider the two terms on the right-hand side separately. For the first one, we turn the space–time white noise W into functionals of $\{(X^i, W^{X^i}); i \in \mathbb{N}\}$ over the space–time subset $(X > 0)$ by writing

$$\mathbb{1}_{(X>0)} \bullet W = \mathbb{1}_{(X>0)} \frac{X^{1/2}}{X^{1/2}} \bullet W = \sum_{j=1}^{\infty} \mathbb{1}_{(X>0)} \left(\frac{X^j}{X}\right)^{1/2} \bullet W^{X^j},$$

where the last equality follows from the compatibility condition $X^{1/2} \bullet W = \sum_{j=1}^\infty (X^j)^{1/2} \bullet W^{X^j}$ (compare the stochastic integral terms of the SPDEs for X and $\sum_{j=1}^\infty X^j$). Note that the infinite series in the foregoing display is well defined since there are only finite many immigration events in a bounded time interval. The independence of the noises W^{X^j}, $j \in \mathbb{N}$, and the foregoing equality imply that

$$\begin{aligned}&\langle (X^i)^{1/2} \bullet W^{X^i}(\phi_1), \mathbb{1}_{(X>0)} Y^{1/2} \bullet W(\phi_2)\rangle \\ &\quad = \mathbb{1}_{(X>0)} \frac{X^i}{X^{1/2}} Y^{1/2} \bullet \lambda(\phi_1\phi_2).\end{aligned} \tag{2.54}$$

Next, we consider the second term on the right-hand side of (2.53). A standard property of one-parameter stochastic integrals implies that

$$\langle (X^i)^{1/2} \bullet W^{X^i}(\phi_1), \mathbb{1}_{[s_j,s_{j+1})} \mathbb{1}_{(X=0)} \bullet W(\phi_2)\rangle \equiv 0, \qquad 0 \le j < i,$$

since X^i does not arrive before time s_{j+1} for these pairs of indices (i, j). For $j \in \mathbb{N}$ with $j \ge i$, the corresponding substitution identity in (2.11) [recall (vi) of Theorem 2.6] gives $\mathbb{1}_{[s_j,s_{j+1})}\mathbb{1}_{(X=0)} \bullet W = \mathbb{1}_{[s_j,s_{j+1})}\mathbb{1}_{(X=0)} \bullet V^{X,j}$. Hence, for all t,

$$\begin{aligned}&\langle (X^i)^{1/2} \bullet W^{X^i}(\phi_1), \mathbb{1}_{(X=0)} Y^{1/2} \bullet W(\phi_2)\rangle_t \\ &\quad = \sum_{j=i}^{\infty} \langle (X^i)^{1/2} \bullet W^{X^i}(\phi_1), Y^{1/2} \mathbb{1}_{[s_j,s_{j+1})}\mathbb{1}_{(X=0)} \bullet V^{X,j}(\phi_2)\rangle_t \\ &\quad = \sum_{j=i}^{\infty} \langle (X^i)^{1/2} \bullet W^{X^i}(\phi_1), Y^{1/2} \mathbb{1}_{[s_j,s_{j+1})}\mathbb{1}_{(X=0)} \bullet V^{X,j}(\phi_2)\rangle_{t\wedge(s_{j+1})} \\ &\quad = 0,\end{aligned} \tag{2.55}$$

where the last equality follows from the independence (2.47). Applying (2.54) and (2.55) to the right-hand side of (2.53), we see that (X^i, Y) has a normal covariation and get the first equality of (2.52). The proof is complete. □

2.4. *Immigrant processes obeying a system of SPDEs.* Equations (2.52) in Proposition 2.8 give partial information for the covariations between X^i and Y^j. The symmetry of these equations in X and Y suggests that if we consider the case in which all of the pairs (X^i, Y^j) have normal covariations, then one possibility for $\{\langle\langle X^i, Y^j\rangle\rangle; i, j \in \mathbb{N}\}$ should be that the *coexistence condition* is satisfied:

$$\begin{aligned}\langle\langle X^i, Y^j\rangle\rangle &= \mathbb{1}_{(X>0,Y>0)}(X^i)^{1/2}(Y^j)^{1/2}\left(\frac{X^i}{X}\right)^{1/2}\left(\frac{Y^j}{Y}\right)^{1/2}\bullet\lambda\\ &= \mathbb{1}_{(X>0,Y>0)}\frac{X^iY^j}{X^{1/2}Y^{1/2}}\bullet\lambda, \qquad i, j \in \mathbb{N}.\end{aligned} \tag{2.56}$$

Equations such as (2.56), if valid, would complement the fact that all (X^i, X^j) and (Y^i, Y^j) have normal covariations and

$$\langle\langle X^i, X^j\rangle\rangle = \delta_{ij}X^i\bullet\lambda \quad \text{and} \quad \langle\langle Y^i, Y^j\rangle\rangle = \delta_{ij}Y^i\bullet\lambda, \tag{2.57}$$

where δ_{ij} denote Kronecker's deltas [recall that $\{X^i\}$ and $\{Y^i\}$ are families of independent super-Brownian motions obeying the SPDE (1.1)]. In terms of stochastic calculus, (2.56) and (2.57) would completely characterize the immigrant processes.

We do not pursue the question whether every interlacing pair of ε-approximating solutions admits immigrant processes subject to the coexistence condition (2.56). For our purpose to study pathwise nonuniqueness in the SPDE (1.2), it is enough to turn to the converse point of view and study whether there exist such immigrant processes so that they define an interlacing pair of ε-approximating solutions (subject to the same white noise) as in (2.1) and (2.3). More precisely, our plan is to construct, for every $\varepsilon \in (0, 1]$, immigrant processes $\{X^i\}$ and $\{Y^i\}$ satisfying conditions (i)–(v) of Theorem 2.6 [we do not require (vi)], and in addition, (2.56) so that they are nonnegative solutions to a system of SPDEs of the form

$$\begin{cases} X_t^i(\phi) = \psi(\mathbb{1})J_\varepsilon^{x_i}(\phi)\mathbb{1}_{t\geq s_i} + \displaystyle\int_{s_i}^{s_i\vee t} X_s^i\left(\frac{\Delta\phi}{2}\right)ds \\ \qquad\qquad + \displaystyle\sum_{j=1}^{\infty}\int_{s_i}^{s_i\vee t}\int_{\mathbb{R}}\sigma_{2i-1,j}(X^1, Y^1, X^2, Y^2, \dots, s)\phi(x)\,dW^j(x,s), \\ Y_t^i(\phi) = \psi(\mathbb{1})J_\varepsilon^{y_i}(\phi)\mathbb{1}_{t\geq t_i} + \displaystyle\int_{t_i}^{t_i\vee t} Y_s^i\left(\frac{\Delta\phi}{2}\right)ds \\ \qquad\qquad + \displaystyle\sum_{j=1}^{\infty}\int_{t_i}^{t_i\vee t}\int_{\mathbb{R}}\sigma_{2i,j}(X^1, Y^1, X^2, Y^2, \dots, s)\phi(x)\,dW^j(x,s), \end{cases} \tag{2.58}$$

for $\phi \in \mathscr{C}_c^\infty(\mathbb{R})$ and some infinite-dimensional deterministic diffusion coefficient matrix $\sigma(x^1, y^1, x^2, y^2, \dots, s)$ depending on space variables $x^1, y^1, x^2, y^2, \dots$ (in contrast, recall that x_i and y_i denote the landing targets of X^i and Y^i, resp.) and time variable s. In (2.58), (1) x_i and y_i are i.i.d. with distribution (1.15) as before,

(2) $(\sigma_{i,j}(\cdot,\cdot,s))_{s\in[0,t]}$ are zero for all but finitely many j for every fixed i and finite t so that the infinite series in (2.58) reduce to finite sums, and (3) $\{W^j\}$ is a family of i.i.d. space–time white noises. We remark that the various restrictions on t in the equations for X^i and Y^i in (2.58) (namely, $\mathbb{1}_{t\geq s_i}$, $s_i \vee t$, $\mathbb{1}_{t\geq t_i}$ and $t_i \vee t$) imply that X^i and Y^i are nonzero only in $[s_i,\infty)$ and $[t_i,\infty)$, respectively, and in writing $x^1, y^1, x^2, y^2, \dots$ for the arguments of σ, we keep track of the order in which immigrants land. Finding an appropriate grand coefficient matrix σ is the major task of this section. Below we make a series of observations, and the conclusion will be stated in Theorem 2.12 by the end of this section.

PROPOSITION 2.9. *Fix $\varepsilon \in (0,1]$. Let $\{X^i\}$ and $\{Y^i\}$ be adapted super-Brownian motions defined on a filtered probability space $(\Omega,\mathscr{F},(\mathscr{G}_t),\mathbb{P})$, so that $\{X^i\}$ is subject to* (i) *and* (iii) *of Theorem* 2.6 *and $\{Y^i\}$ is subject to the analogous conditions. As before, the landing targets x_i and y_i here are i.i.d. with distribution* (1.15) *and satisfy* (2.16). *Suppose that all pairs (X^i,Y^j) have normal covariations and the coexistence condition* (2.56) *is satisfied. Then $X=\sum_i X^i$ and $Y=\sum_i Y^i$ define an interlacing pair of ε-approximating solutions.*

PROOF. As explained in Section 2.3, we may assume the existence of two families of independent space–time white noises $\{W^{X^i}\}$ and $\{W^{Y^i}\}$ so that (X^i, W^{X^i}) and (Y^i, W^{Y^i}) solve the SPDE (1.1) of super-Brownian motion.

We have to show that X and Y are subject to the SPDEs (2.1) and (2.3), respectively, both with respect to the same $(\mathscr{G}_t)$-space–time white noise W. We start with the definition of W. Let V be a $(\mathscr{G}_t)$-space–time white noise independent of $\{(X^i, W^{X^i})\}$ and $\{(Y^i, W^{Y^i})\}$. By our assumptions and Lévy's theorem (cf. Theorem IV.3.6 of [23]), we deduce that

$$
\begin{aligned}
W \triangleq{}& \sum_{i=1}^{\infty}\mathbb{1}_{(Y>0)}\left(\frac{Y^i}{Y}\right)^{1/2}\bullet W^{Y^i}+\sum_{i=1}^{\infty}\mathbb{1}_{(X>0,Y=0)}\left(\frac{X^i}{X}\right)^{1/2}\bullet W^{X^i}\\
&+\mathbb{1}_{(X=0,Y=0)}\bullet V
\end{aligned}
\tag{2.59}
$$

defines a $(\mathscr{G}_t)$-space–time white noise. Then Y is subject to the SPDE (2.3) with respect to W since the compatibility condition $Y^{1/2}\bullet W=\sum_{i=1}^{\infty}(Y^i)^{1/2}\bullet W^{Y^i}$ holds. Indeed, we have

$$
\begin{aligned}
Y^{1/2}\bullet W = Y^{1/2}\mathbb{1}_{(Y>0)}\bullet W &= \sum_{i=1}^{\infty}\mathbb{1}_{(Y>0)}(Y^i)^{1/2}\bullet W^{Y^i}\\
&=\sum_{i=1}^{\infty}(Y^i)^{1/2}\bullet W^{Y^i},
\end{aligned}
\tag{2.60}
$$

where the last equality follows from the nonnegativity of Y^i's.

To prove that X is also subject to W, one may wish that the roles of $(\{X^i\}, X)$ and $(\{Y^i\}, Y)$ on the right-hand side of (2.59) can be exchanged, that is W can also be rewritten as

$$
\begin{aligned}
W = &\sum_{i=1}^{\infty} \mathbb{1}_{(X>0)} \left(\frac{X^i}{X}\right)^{1/2} \bullet W^{X^i} + \sum_{i=1}^{\infty} \mathbb{1}_{(Y>0, X=0)} \left(\frac{Y^i}{Y}\right)^{1/2} \bullet W^{Y^i} \\
&+ \mathbb{1}_{(X=0, Y=0)} \bullet V,
\end{aligned}
\tag{2.61}
$$

and so the argument in (2.60) applies to X. In this direction, it is enough to claim that

$$
\sum_{i=1}^{\infty} \mathbb{1}_{(X>0, Y>0)} \left(\frac{X^i}{X}\right)^{1/2} \bullet W^{X^i} - \sum_{i=1}^{\infty} \mathbb{1}_{(X>0, Y>0)} \left(\frac{Y^i}{Y}\right)^{1/2} \bullet W^{Y^i} = 0.
\tag{2.62}
$$

Recall the measure λ defined in (2.51). Since all of the pairs (X^i, Y^j) have normal covariations, the left-hand side of the foregoing equality also has a normal covariation and we have

$$
\begin{aligned}
&\Bigg\langle\!\!\Bigg\langle \sum_{i=1}^{\infty} \mathbb{1}_{(X>0, Y>0)} \left(\frac{X^i}{X}\right)^{1/2} \bullet W^{X^i} - \sum_{i=1}^{\infty} \mathbb{1}_{(X>0, Y>0)} \left(\frac{Y^i}{Y}\right)^{1/2} \bullet W^{Y^i}, \\
&\quad \sum_{j=1}^{\infty} \mathbb{1}_{(X>0, Y>0)} \left(\frac{X^j}{X}\right)^{1/2} \bullet W^{X^j} - \sum_{j=1}^{\infty} \mathbb{1}_{(X>0, Y>0)} \left(\frac{Y^j}{Y}\right)^{1/2} \bullet W^{Y^j} \Bigg\rangle\!\!\Bigg\rangle \\
&\quad = \sum_{i=1}^{\infty} \mathbb{1}_{(X>0, Y>0)} \frac{X^i}{X} \bullet \lambda - 2 \sum_{i,j=1}^{\infty} \mathbb{1}_{(X>0, Y>0)} \frac{1}{X^{1/2} Y^{1/2}} \frac{X^i Y^j}{X^{1/2} Y^{1/2}} \bullet \lambda \\
&\qquad + \sum_{i=1}^{\infty} \mathbb{1}_{(X>0, Y>0)} \frac{Y^i}{Y} \bullet \lambda \\
&\quad = \mathbb{1}_{(X>0, Y>0)} \bullet \lambda - 2\mathbb{1}_{(X>0, Y>0)} \bullet \lambda + \mathbb{1}_{(X>0, Y>0)} \bullet \lambda = 0,
\end{aligned}
$$

where the second equality follows from (2.56) and (2.57) [the stochastic integral terms of the SPDEs for X^i and Y^j are characterized by $(X^i)^{1/2} \bullet W^{X^i}$ and $(Y^j)^{1/2} \bullet W^{Y^j}$]. We deduce our claim (2.62) from the last equality and the fact that the action of the left-hand side of (2.62) on every function in $\mathscr{C}_c^{\infty}(\mathbb{R})$ induces a continuous martingale. We have proved the alternative expression (2.61) of W, and the proof is complete. $\square$

LEMMA 2.10. *Suppose that for $N \in \mathbb{N}$, $\xi^1, \ldots, \xi^N$ are continuous nonnegative $\mathscr{C}_{\mathrm{rap}}(\mathbb{R})$-valued solutions to the SPDE (1.1) with respect to the same filtration and independent initial conditions, and all pairs (ξ^i, ξ^j) have normal covariations with $\langle\!\langle \xi^i, \xi^j \rangle\!\rangle = \delta_{ij} \xi^i \bullet \lambda$ for λ given by (2.51). Then $\xi^1, \ldots, \xi^N$ are independent super-Brownian motions.*

SKETCH OF PROOF. The proof is to generalize the exponential duality argument for super-Brownian motion. For each i, let ϕ^i be a nonnegative $\mathscr{C}_c^\infty(\mathbb{R})$-function and u^i be the unique nonnegative solution of the PDE

$$\partial_r u_r^i = \frac{\Delta u_r^i}{2} - \frac{1}{2}(u_r^i)^2 \text{ in } \mathbb{R} \times (0, \infty) \qquad \text{with } u_0^i = \phi^i$$

(cf. Lemma 4 in Section II.2 of [16] or pages 167–169 of [21]). Then for every fixed $t \in (0, \infty)$, the continuous semimartingale $\exp\{-\sum_{i=1}^N \xi_s^i(u_{t-s}^i)\}$, $0 \le s \le t$, has zero finite variation by Itô's lemma and the assumption that $\langle\langle \xi^i, \xi^j \rangle\rangle = \delta_{ij}\xi^i \bullet \lambda$ (cf. Proposition II.5.7 of [21]), and hence has constant mean. It follows that one-dimensional marginals of $(\xi^1, \dots, \xi^N)$ are uniquely determined as those of independent super-Brownian motions. A standard argument for martingale problems (cf. Section 4.4 in [10]) implies the desired result. $\square$

Thanks to Lemma 2.10, the main assumptions of Proposition 2.9 are reduced to the covariation equations (2.56) and (2.57) for $\{X^i\}$ and $\{Y^i\}$, as well as other minor conditions. Then as in the standard construction of solutions to systems of stochastic differential equations, the issue is whether these covariation equations (2.56) and (2.57) are induced by the nonnegative definite matrix $\sigma\sigma^\top$ for some diffusion coefficient matrix σ as in (2.58).

Below we write $\mathbf{x} = (x^1, x^2, \dots)$, $\mathbf{y} = (y^1, y^2, \dots)$, and $\mathbf{0} = (0, 0, \dots)$ for which the dimensions may vary from line to line but will be clear from the context.

LEMMA 2.11. *Fix $n, m \in \mathbb{N}$, and consider the matrix-valued function*

$$(\mathbf{x}, \mathbf{y}) \longmapsto a^{(n,m)}(\mathbf{x}, \mathbf{y}) = \left[a_{k,\ell}^{(n,m)}(\mathbf{x}, \mathbf{y})\right]_{1 \le k, \ell \le m+n} \tag{2.63}$$

defined on $(\mathbb{R}^n \setminus \{\mathbf{0}\}) \times (\mathbb{R}^m \setminus \{\mathbf{0}\})$ as follows. For $\mathbf{x} = (x^1, x^2, \dots, x^n)$ and $\mathbf{y} = (y^1, y^2, \dots, y^m)$ with $x^i, y^j \ge 0$ and $\sum_{i'} x^{i'}, \sum_{j'} y^{j'} > 0$, we set

$$\begin{cases} a_{i,j}^{(n,m)}(\mathbf{x}, \mathbf{y}) = x^i \delta_{ij}, & 1 \le i, j \le n, \\ a_{n+i,n+j}^{(n,m)}(\mathbf{x}, \mathbf{y}) = y^j \delta_{ij}, & 1 \le i, j \le m, \\ a_{i,n+j}^{(n,m)}(\mathbf{x}, \mathbf{y}) = a_{n+j,i}^{(n,m)}(\mathbf{x}, \mathbf{y}) & \\ \qquad = (x^i)^{1/2}(y^j)^{1/2}\left(\dfrac{x^i}{\sum_{i'} x^{i'}}\right)^{1/2}\left(\dfrac{y^j}{\sum_{j'} y^{j'}}\right)^{1/2}, & \\ & 1 \le i \le n, 1 \le j \le m. \end{cases}$$

For other $(\mathbf{x}, \mathbf{y}) \in (\mathbb{R}^n \setminus \{\mathbf{0}\}) \times (\mathbb{R}^m \setminus \{\mathbf{0}\})$, we set

$$a^{(n,m)}(\mathbf{x}, \mathbf{y}) = a^{(n,m)}(|x^1|, |x^2|, \dots, |x^n|, |y^1|, |y^2|, \dots, |y^m|). \tag{2.64}$$

Then $a^{(n,m)}$ extends continuously to the entire space $\mathbb{R}^n \times \mathbb{R}^m$, and the extension, still denoted by $a^{(n,m)}$, takes values in $(m+n)$-by-$(m+n)$ nonnegative definite matrices.

PROOF. Our assertion that $a^{(n,m)}$ extends continuously to $\mathbb{R}^n \times \mathbb{R}^m$ follows plainly from the fact that

$$\frac{|x^i|}{\sum_{i'} |x^{i'}|}, \frac{|y^j|}{\sum_{j'} |y^{j'}|} \in [0,1] \qquad \forall \mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}, \mathbf{y} \in \mathbb{R}^m \setminus \{\mathbf{0}\}.$$

We turn to the nonnegative definiteness of $a^{(n,m)}$. By continuity and (2.64), we only need to show that $a^{(n+m)}(\mathbf{x}, \mathbf{y})$ is nonnegative definite for $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^m$ satisfying $x_i, y_j > 0$ for all i, j. Write

$$a^{(n+m)}(\mathbf{x}, \mathbf{y}) = \begin{bmatrix} D^X & A \\ A^\top & D^Y \end{bmatrix}$$

for an n-by-n diagonal matrix D^X and an m-by-m diagonal matrix D^Y. For any $(u, v) \in \mathbb{R}^n \times \mathbb{R}^m$, we regard u and v as column vectors and compute

$$\begin{aligned}
&\begin{bmatrix} u^\top & v^\top \end{bmatrix} a^{(n+m)}(\mathbf{x}, \mathbf{y}) \begin{bmatrix} u \\ v \end{bmatrix} \\
&\quad = \begin{bmatrix} u^\top & v^\top \end{bmatrix} \begin{bmatrix} D^X & A \\ A^\top & D^Y \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \\
&\quad = u^\top D^X u + 2u^\top A v + v^\top D^Y v \qquad (2.65) \\
&\quad = \sum_i (u^i)^2 x^i + 2 \sum_{i,j} u^i (x^i)^{1/2} \left(\frac{x^i}{\sum_{i'} x^{i'}} \right)^{1/2} v^j (y^j)^{1/2} \left(\frac{y^j}{\sum_{j'} y^{j'}} \right)^{1/2} \\
&\qquad + \sum_j (v^j)^2 y^j
\end{aligned}$$

by the definition of $a^{(n,m)}$. Notice that for all $\alpha^1, \dots, \alpha^n, \beta^1, \dots, \beta^m \in \mathbb{R}$,

$$\begin{aligned}
2 \sum_{i,j} \alpha^i \beta^j = \left(\sum_i \alpha^i + \sum_j \beta^j \right)^2 - \sum_i (\alpha^i)^2 - 2 \sum_{i_1 < i_2} \alpha^{i_1} \alpha^{i_2} \\
- \sum_j (\beta^j)^2 - 2 \sum_{j_1 < j_2} \beta^{j_1} \beta^{j_2}.
\end{aligned}$$

Applying the foregoing equality to the second term on the right-hand side of (2.65) with the choice

$$\alpha^i = u^i (x^i)^{1/2} \left(\frac{x^i}{\sum_{i'} x^{i'}} \right)^{1/2}$$

and

$$\beta^j = v^j (y^j)^{1/2} \left(\frac{y^j}{\sum_{j'} y^{j'}} \right)^{1/2},$$

we obtain

$$
\begin{aligned}
&[u^\top \quad v^\top]a^{(n+m)}(\mathbf{x},\mathbf{y})\begin{bmatrix} u \\ v \end{bmatrix} \\
&\quad = \sum_i (u^i)^2 x^i \\
&\qquad + \left[\sum_i u^i (x^i)^{1/2}\left(\frac{x^i}{\sum_{i'} x^{i'}}\right)^{1/2} + \sum_j v^j (y^j)^{1/2}\left(\frac{y^j}{\sum_{j'} y^{j'}}\right)^{1/2}\right]^2 \\
&\qquad - \sum_i (u^i)^2 \frac{(x^i)^2}{\sum_{i'} x^{i'}} - 2\sum_{i_1<i_2} u^{i_1}u^{i_2}\frac{x^{i_1}}{(\sum_{i'} x^{i'})^{1/2}}\frac{x^{i_2}}{(\sum_{i'} x^{i'})^{1/2}} \\
&\qquad - \sum_j (v^j)^2 \frac{(y^j)^2}{\sum_{j'} y^{j'}} - 2\sum_{j_1<j_2} v^{j_1}v^{j_2}\frac{y^{j_1}}{(\sum_{j'} y^{j'})^{1/2}}\frac{y^{j_2}}{(\sum_{j'} y^{j'})^{1/2}} \\
&\qquad + \sum_j (v^j)^2 y^j.
\end{aligned} \tag{2.66}
$$

The first, third and fourth terms on the right-hand side of the above equality (with their signs taken into account as well) sum to

$$
\begin{aligned}
&\sum_i (u^i)^2 x^i - \sum_i (u^i)^2 \frac{(x^i)^2}{\sum_{i'} x^{i'}} - 2\sum_{i_1<i_2} u^{i_1}u^{i_2}\frac{x^{i_1}}{(\sum_{i'} x^{i'})^{1/2}}\frac{x^{i_2}}{(\sum_{i'} x^{i'})^{1/2}} \\
&\quad = \sum_i (u^i)^2 x^i - \left[\sum_i u^i \frac{x^i}{(\sum_{i'} x^{i'})^{1/2}}\right]^2 \geq 0,
\end{aligned} \tag{2.67}
$$

since the Cauchy–Schwarz inequality implies that

$$
\begin{aligned}
\left[\sum_i u^i \frac{x^i}{(\sum_{i'} x^{i'})^{1/2}}\right]^2 &= \left[\sum_i u^i (x^i)^{1/2} \times \left(\frac{x^i}{\sum_{i'} x^{i'}}\right)^{1/2}\right]^2 \\
&\leq \left(\sum_i (u^i)^2 x^i\right)\left(\sum_i \frac{x^i}{\sum_{i'} x^{i'}}\right) \\
&= \sum_i (u^i)^2 x^i.
\end{aligned}
$$

Similarly, the last three terms on the right-hand side of (2.66) sum to

$$
\begin{aligned}
&-\sum_j (v^j)^2 \frac{(y^j)^2}{\sum_{j'} y^{j'}} - 2\sum_{j_1<j_2} v^{j_1}v^{j_2}\frac{y^j}{(\sum_{j'} y^{j'})^{1/2}}\frac{y^j}{(\sum_{j'} y^{j'})^{1/2}} + \sum_j (v^j)^2 y^j \\
&\quad = \sum_j (v^j)^2 y^j - \left[\sum_j v^j \frac{y^j}{(\sum_{j'} y^{j'})^{1/2}}\right]^2 \geq 0.
\end{aligned} \tag{2.68}
$$

Apply (2.67) and (2.68) to the right-hand side of (2.66), and we obtain

$$\begin{aligned}&\begin{bmatrix} u^\top & v^\top \end{bmatrix} a^{(n+m)}(\mathbf{x},\mathbf{y})\begin{bmatrix} u \\ v \end{bmatrix}\\ &\quad\geq \left[\sum_i u^i (x^i)^{1/2}\left(\frac{x^i}{\sum_{i'} x^{i'}}\right)^{1/2} + \sum_j v^j (y^j)^{1/2}\left(\frac{y^j}{\sum_{j'} y^{j'}}\right)^{1/2}\right]^2 \geq 0\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall (u,v)\in\mathbb{R}^n\times\mathbb{R}^m,\end{aligned}$$

that is, $a^{(n+m)}(\mathbf{x},\mathbf{y})$ is nonnegative definite. The proof is complete. □

We are now ready to define the sought-after diffusion coefficient matrix σ. For convenience, we reorder the arguments and entries of the matrix-valued function $a^{(n,m)}(\mathbf{x},\mathbf{y})$ in Lemma 2.11 in accordance with the order in which immigrants land, for (n,m) equal to (n,n) or $(n,n-1)$. This results in the matrix-valued functions $A^{(n,n)}$ and $A^{(n,n-1)}$ defined by

$$\begin{aligned}&A^{(n,n)}(x^1,y^1,x^2,y^2,\dots,x^n,y^n) \triangleq \Pi_{n,n} a^{(n,n)}(\mathbf{x},\mathbf{y})\Pi_{n,n}^\top, \qquad n\geq 1,\\ &A^{(n,n-1)}(x^1,y^1,x^2,y^2,\dots,x^{n-1},y^{n-1},x^n)\\ &\qquad \triangleq \Pi_{n,n-1} a^{(n,n-1)}(\mathbf{x},\mathbf{y})\Pi_{n,n-1}^\top, \qquad n\geq 2.\end{aligned} \tag{2.69}$$

Here, $\Pi_{n,n}$ and $\Pi_{n,n-1}$ are the permutation matrices defined by

$$\begin{aligned}&\Pi_{n,n}[x^1,x^2,\dots,x^n,y^1,y^2,\dots,y^n]^\top\\ &\qquad = [x^1,y^1,x^2,y^2,\dots,x^n,y^n]^\top,\\ &\Pi_{n,n-1}[x^1,x^2,\dots,x^n,y^1,y^2,\dots,y^{n-1}]^\top\\ &\qquad = [x^1,y^1,x^2,y^2,\dots,x^{n-1},y^{n-1},x^n]^\top.\end{aligned} \tag{2.70}$$

Then we choose a continuous square root, denoted by $\sigma^{(n,m)}$, of the square matrix $A^{(n,m)}$ (cf. Theorem 1.1 of [5] or [11] for its existence) for $(n,m)=(n,n)$ or $(n,n-1)$, and so $\sigma^{(n,m)}$ satisfies

$$\sigma^{(n,m)}[\sigma^{(n,m)}]^\top \equiv A^{(n,m)}. \tag{2.71}$$

Let $\sigma^{(1,0)}(x^1)$ be the 1-by-1 matrix $[|x^1|^{1/2}]$. Then we define σ as follows. We set $\sigma\equiv 0$ on $[0,s_1)$, and for all $n,i,j\in\mathbb{N}$,

$$\begin{aligned}&\sigma_{i,j}(x^1,y^1,x^2,y^2,\dots,s)\\ &\quad\equiv\begin{cases}\sigma_{i,j}^{(n,n-1)}(x^1,y^1,x^2,y^2,\dots,x^{n-1},y^{n-1},x^n), & \\ \qquad\text{if } i,j\leq 2n-1 \text{ and } s\in[s_n,t_n), & \\ \sigma_{i,j}^{(n,n)}(x^1,y^1,x^2,y^2,\dots,x^n,y^n), & \\ \qquad\text{if } i,j\leq 2n \text{ and } s\in[t_n,s_{n+1}), & \\ 0, \quad \text{otherwise}. & \end{cases}\end{aligned} \tag{2.72}$$

In other words, solutions to the system (2.58) of SPDE's subject to the above choice of diffusion coefficient matrix σ, if any, can be described as follows: over $[s_n, t_n)$ for $n \in \mathbb{N}$,

$$\left(X^1, Y^1, X^2, Y^2, \dots, X^{n-1}, Y^{n-1}, X^n\right)$$

is subject to the diffusion coefficient $\sigma^{(n,n-1)}$ and the independent noises $W^1, \dots, W^{2n-1}$, and over $[t_n, s_{n+1})$ for $n \in \mathbb{N}$,

$$\left(X^1, Y^1, X^2, Y^2, \dots, X^n, Y^n\right)$$

is subject to the diffusion coefficient $\sigma^{(n,n)}$ and the independent noises $W^1, \dots, W^{2n}$. Note that σ depends only on space variables between two consecutive immigration times.

THEOREM 2.12. *Fix an immigration function $\psi \in \mathscr{C}_c^+(\mathbb{R}) \setminus \{0\}$. For any $\varepsilon \in (0, 1]$, we can construct a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{G}_t), \mathbb{P}_\varepsilon)$, with $(\mathscr{G}_t)$ satisfying the usual conditions, on which there exist random elements $\{x_i\}$, $\{y_i\}$, $\{X^i\}$, $\{Y^i\}$, $\{W^i\}$ and W with the following properties*:

(i) *x_i and y_i are i.i.d. with law (1.15) and take values in the topological support of ψ.*

(ii) *W^i and W are $(\mathscr{G}_t)$-space–time white noises, and the noises $\{W^i\}$ are independent.*

(iii) *For each $i \in \mathbb{N}$, $(X_t^i)_{t\in[s_i,\infty)}$ and $(Y_t^i)_{t\in[t_i,\infty)}$ are nonnegative processes with sample paths in $C([s_i, \infty), \mathscr{C}_{\mathrm{rap}}(\mathbb{R}))$ and $C([t_i, \infty), \mathscr{C}_{\mathrm{rap}}(\mathbb{R}))$, respectively.*

(iv) *$\{X^i\}$ and $\{Y^i\}$ obey the system of SPDE's (2.58) with respect to $\{W^i\}$ for σ defined by (2.72).*

(v) *The independent landing property (2.16) for immigrants holds.*

(vi) *The sums $X = \sum_i X^i$ and $Y = \sum_i Y^i$ define an interlacing pair of ε-approximating solutions with respect to W (see Definition 2.1).*

In particular, $\{X^i\}$ and $\{Y^i\}$ are two families of independent super-Brownian motions for which the covariation equations (2.56) hold.

The proof of Theorem 2.12 follows similarly as the existence of interlacing pairs of ε-approximating solutions (see Definition 2.1). We introduce i.i.d. landing targets x_i and y_i which are independent of a family of i.i.d. noises $\{W^i\}$, and then solve (2.58) over $[s_1, t_1)$, $[t_1, s_2)$, $[s_2, t_2), \dots$ sequentially (see [2] for a similar construction). More precisely, over any of these intervals, (2.58) reduces to a finite-dimensional system of SPDEs to which the classical Peano approximation argument as in the proof of Theorem 2.6 of [25] applies (cf. Section 6 of [17] as well). Indeed, by comparing diagonal entries on both sides of (2.71), we deduce that every entry $\sigma_{i,j}^{(n,m)}$ is bouded by $z \longmapsto |z|^{1/2}$, where $z = x^k$ if $i = 2k - 1$, or $z = y^k$ if $i = 2k$. We omit other details.

As an immediate consequence of (2.56), we obtain the following (see Remark 2.7).

PROPOSITION 2.13. *Let $\{X^i\}$ and $\{Y^i\}$ be as in Theorem 2.12. Then for any $i, j_1, \dots, j_n \in \mathbb{N}$ for $n \in \mathbb{N}$ with $j_1 < j_2 < \cdots < j_n$, except outside a null event, the inequality*

$$
\begin{aligned}
&\left| \int_0^t H_s \, d\left\langle X^i(\mathbb{1}), \sum_{\ell=1}^n Y^{j_\ell}(\mathbb{1}) \right\rangle_s \right| \\
&\quad \le \int_{s_i \vee t_{j_1}}^t |H_s| \int_{\mathbb{R}} \left(X^i(x,s) \cdot \sum_{\ell=1}^n Y^{j_\ell}(x,s) \right)^{1/2} dx\, ds \qquad (2.73) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall t \in [s_i \vee t_{j_1}, \infty),
\end{aligned}
$$

holds for any locally bounded Borel measurable function H on $\mathbb{R}_+$.

CHOICE OF APPROXIMATING SOLUTIONS. From now on, we *only* work with $\{X^i\}$ and $\{Y^i\}$ as in Theorem 2.12, and the corresponding interlacing pairs of ε-approximating solutions $X = \sum_i X^i$ and $Y = \sum_i Y^i$, except in Section 5.

3. Conditional separation of approximating solutions.

3.1. *Basic results.* The theme of Section 3 is conditional separation of the approximating solutions defined by the immigrant processes $\{X^i; i \in \mathbb{N}\}$ and $\{Y^i; i \in \mathbb{N}\}$ chosen in Theorem 2.12. For any $\varepsilon \in (0, [8\psi(\mathbb{1})]^{-1} \wedge 1]$, we condition on the event that the total mass of a generic cluster X^i hits 1, and then the conditional separation refers to the separation of the approximating solutions under

$$
\mathbb{Q}^i_\varepsilon(A) \equiv \mathbb{P}_\varepsilon\big(A \mid T_1^{X^i} < \infty\big). \tag{3.1}
$$

Here, the restriction $[8\psi(\mathbb{1})]^{-1}$ for ε is just to make sure that $X^i(\mathbb{1})$ stays in $(0,1)$ initially, and we set

$$
T_x^H \triangleq \inf\{t \ge 0; H_t(\mathbb{1}) = x\} \tag{3.2}
$$

for any nonnegative two-parameter process $H = (H(x,t); (x,t) \in \mathbb{R} \times \mathbb{R}_+)$. Our specific goal is to study the differences in the growth rates of local masses of X and Y over the "initial part" of the space–time support of X^i. In the following, we prove a few basic results concerning $\mathbb{Q}^i_\varepsilon$.

First, let us represent the Radon–Nikodym derivative process of $\mathbb{Q}^i_\varepsilon$ relative to $\mathbb{P}_\varepsilon$ (cf. Section VIII.1 of [23] for its role in Girsanov's theorem).

LEMMA 3.1. *For any $i \in \mathbb{N}$ and $\varepsilon \in (0, [8\psi(\mathbb{1})]^{-1} \wedge 1]$,*

$$
\mathbb{P}_\varepsilon\big(T_1^{X^i} < T_0^{X^i}\big) = \psi(\mathbb{1})\varepsilon, \tag{3.3}
$$

and the Radon–Nikodym derivative process $\mathbb{E}^{\mathbb{P}^{\varepsilon}}[d\mathbb{Q}^i_\varepsilon/d\mathbb{P}_\varepsilon|\mathscr{G}_t]$, $t\in[s_i,\infty)$, *of* $\mathbb{Q}^i_\varepsilon$ *relative to* $\mathbb{P}_\varepsilon$ *is given by the stopped* $((\mathscr{G}_t)_{t\geq s_i},\mathbb{P}_\varepsilon)$*-martingale* $X^i(\mathbb{1})^{T_1^{X^i}}/\psi(\mathbb{1})\varepsilon$, *that is*,

$$\mathbb{Q}^i_\varepsilon(A)=\int_A\frac{X^i_t(\mathbb{1})^{T_1^{X^i}}}{\psi(\mathbb{1})\varepsilon}\,d\mathbb{P}_\varepsilon\qquad\forall A\in\mathscr{G}_t\text{ with }t\in[s_i,\infty).\tag{3.4}$$

Here, $X^i(\mathbb{1})^{T_1^{X^i}}$ *denotes the total mass process* $X^i(\mathbb{1})$ *stopped at* $T_1^{X^i}$ [*see* (3.2) *for* $T_1^{X^i}$].

PROOF. The proof is a standard application of Doob's h-transforms (cf. Section VII.3 of [23]). Recall that $X^i(\mathbb{1})$ under $\mathbb{P}_\varepsilon$ is a Feller diffusion with initial condition $\psi(\mathbb{1})\varepsilon$ and plainly the scale function of Feller diffusion is given by $x\longmapsto x$. Hence, (3.3) follows from Proposition VII.3.2 of [23]. To see the second assertion, we recall the definition (3.1) of $\mathbb{Q}^i_\varepsilon$, and then apply (3.3), Proposition VII.3.2 in [23] again and the Markov property of $X^i(\mathbb{1})$. $\square$

Some basic properties of the total mass processes $X^i(\mathbb{1})$ and $Y^j(\mathbb{1})$ for $t_j>s_i$ under $\mathbb{Q}^i_\varepsilon$ are stated in the following lemma.

LEMMA 3.2. *Fix* $i\in\mathbb{N}$ *and* $\varepsilon\in(0,[8\psi(\mathbb{1})]^{-1}\wedge 1]$. *Then we have the following.*

(1) $X^i(\mathbb{1})^{T_1^{X^i}}$ *under* $\mathbb{Q}^i_\varepsilon$ *is a copy of* $\frac{1}{4}\mathrm{BES}Q^4(4\psi(\mathbb{1})\varepsilon)$ *started at* s_i *and stopped upon hitting* 1.

(2) *For any* $j\in\mathbb{N}$ *with* $t_j>s_i$, *the process* $(Y^j(\mathbb{1})_t)_{t\geq t_j}$ *is a continuous* $(\mathscr{G}_t)_{t\geq t_j}$*-semimartingale under* $\mathbb{Q}^i_\varepsilon$ *with canonical decomposition*

$$Y^j_t(\mathbb{1})=\psi(\mathbb{1})\varepsilon+I^j_t+M^j_t,\qquad t\in[t_j,\infty),\tag{3.5}$$

where the finite variation process I^j *satisfies*

$$I^j_t=\int_{t_j}^t\frac{1}{X^i_s(\mathbb{1})^{T_1^{X^i}}}\,d\big\langle X^i(\mathbb{1})^{T_1^{X^i}},Y^j(\mathbb{1})\big\rangle_s,\tag{3.6}$$

$$0\leq I^j_t\leq\int_{t_j}^t\mathbb{1}_{[0,T_1^{X^i}]}(s)\frac{1}{X^i_s(\mathbb{1})}\int_{\mathbb{R}}X^i(x,s)^{1/2}Y^j(x,s)^{1/2}\,dx\,ds,\tag{3.7}$$

for $t\in[t_j,\infty)$, *and* M^j *is a true* $(\mathscr{G}_t)_{t\geq t_j}$*-martingale under* $\mathbb{Q}^i_\varepsilon$.

(3) *For any* $j\in\mathbb{N}$ *with* $t_j>s_i$,

$$x_i,X^i(\mathbb{1})\restriction[s_i,t_j],y_j\quad\text{and}\quad Y^j(\mathbb{1})\restriction[t_j,\infty)\qquad\text{are }\mathbb{P}_\varepsilon\text{-independent.}\tag{3.8}$$

(4) *For any* $j \in \mathbb{N}$,

$$\mathbb{Q}^i_\varepsilon(|y_j - x_i| \in dx) = \mathbb{P}_\varepsilon(|y_j - x_i| \in dx), \qquad x \in \mathbb{R},$$
$$\mathbb{P}_\varepsilon(y_j \in dx) \le \frac{\|\psi\|_\infty}{\psi(\mathbb{1})}\, dx, \qquad x \in \mathbb{R}. \tag{3.9}$$

PROOF. (1) The proof is omitted since it is a straightforward application of Girsanov's theorem and Lemma 3.1, and can be found in the proof of Lemma 4.1 of [17].

(2) Under $\mathbb{P}_\varepsilon$, the total mass process $(Y^j_t(\mathbb{1}))_{t \ge t_j}$ for any $j \in \mathbb{N}$ with $t_j > s_i$ is a $(\mathscr{G}_t)_{t\ge t_j}$-Feller diffusion and hence a $(\mathscr{G}_t)_{t\ge t_j}$-martingale. By Lemma 3.1 and Girsanov's theorem (cf. Theorem VIII.1.4 of [23]), $(Y^j_t(\mathbb{1}))_{t \ge t_j}$ for any $j \in \mathbb{N}$ with $t_j > s_i$ is a continuous $(\mathscr{G}_t)_{t\ge t_j}$-semimartingale under $\mathbb{Q}^i_\varepsilon$ with canonical decomposition given by (3.5). Here, $(M^j_t)_{t\ge t_j}$ is a continuous $(\mathscr{G}_t)_{t\ge t_j}$-local martingale under $\mathbb{Q}^i_\varepsilon$ with quadratic variation

$$\langle M^j\rangle_t = \int_{t_j}^t Y^j_s(\mathbb{1})\, ds, \qquad t \in [t_j, \infty), \tag{3.10}$$

and by Lemma 3.1 the finite variation process $(I^j_t)_{t\ge t_j}$ is given by (3.6). Applying (2.56) and (2.73) to (3.6), we obtain (3.7) at once.

For the martingale property of M^j under $\mathbb{Q}^i_\varepsilon$, we note that the one-dimensional marginals of $Y^j(\mathbb{1})$ have pth moments which are locally bounded on compacts, for any $p \in (0, \infty)$. [$Y^j(\mathbb{1})$ under $\mathbb{P}_\varepsilon$ is a Feller diffusion.] Applying this to (3.10) shows that $\mathbb{E}^{\mathbb{Q}^i_\varepsilon}[\langle M^j\rangle_t] < \infty$ for every $t \in [t_j, \infty)$, and hence M^j is a true martingale under $\mathbb{Q}^i_\varepsilon$.

(3) The assertion (3.8) is an immediate consequence of the independent landing property (2.16) and the Markov properties of $X^i(\mathbb{1})$ and $Y^j(\mathbb{1})$ (cf. Theorem 2.12).

(4) We consider (3.9). Recall that $x_i \in \mathscr{G}_{s_i}$ and $y_j \in \mathscr{G}_{t_j}$ by (2.4). If $t_j > s_i$, then we obtain from (3.4) that

$$\begin{aligned} \mathbb{Q}^i_\varepsilon(|y_j - x_i| \in dx) &= \frac{1}{\psi(\mathbb{1})\varepsilon} \mathbb{E}^{\mathbb{P}_\varepsilon}\big[X^i_{t_j}(\mathbb{1})^{T_1^{X^i}}; |y_j - x_i| \in dx\big] \\ &= \mathbb{P}_\varepsilon(|y_j - x_i| \in dx), \end{aligned} \tag{3.11}$$

where the last equality follows from (3.8). If $t_j < s_i$, then a similar argument applies [without using (3.4)] since $X^i_{s_i}(\mathbb{1}) = \psi(\mathbb{1})\varepsilon$. Hence, the equality in (3.9) holds. The inequality in (3.9) is obvious. The proof is complete. $\square$

3.2. *Setup*. In order to state precisely our quantifications of the local growth rates of X and Y, we need several preliminary results which have similar counterparts in [17]. First, we choose in Proposition 3.3 below a $(\mathscr{G}_t)$-stopping time τ^i satisfying $\tau^i > s_i$, so that within $[s_i, \tau^i]$ we can explicitly bound from below the growth rate of $X^i(\mathbb{1})$. Since $X \geq X^i$, this gives a lower bound for the size of X over the initial part of the space–time support of X^i. Our objective is to study the local growth rate of Y within this part.

PROPOSITION 3.3. *For any* $\varepsilon \in (0, [8\psi(\mathbb{1})]^{-1} \wedge 1]$, *parameter vector* $(\eta, \alpha, L) \in (1, \infty) \times (0, \frac{1}{2}) \times (0, \infty)$, *and* $i \in \mathbb{N}$, *we define four* $(\mathscr{G}_t)$*-stopping times by*

$$\tau^{i,(1)} \triangleq \inf\left\{t \geq s_i; X_t^i(\mathbb{1})^{T_1^{X^i}} < \frac{(t-s_i)^\eta}{4}\right\} \wedge T_1^{X^i},$$

$$\tau^{i,(2)} \triangleq \inf\left\{t \geq s_i; \left|X_t^i(\mathbb{1})^{T_1^{X^i}} - \psi(\mathbb{1})\varepsilon - (t-s_i)\right| > L\left(\int_{s_i}^t X_s^i(\mathbb{1})^{T_1^{X^i}}\, ds\right)^\alpha\right\} \wedge T_1^{X^i},$$

$$\tau^{i,(3)} \triangleq \inf\left\{t \geq s_i; \sum_{j: s_i < t_j \leq t} Y_t^j(\mathbb{1}) > 1\right\},$$

$$\tau^i \triangleq \tau^{i,(1)} \wedge \tau^{i,(2)} \wedge \tau^{i,(3)} \wedge (s_i + 1).$$

Then

$$\forall \rho > 0\ \exists \delta > 0 \text{ such that} \quad \sup\left\{\mathbb{Q}_\varepsilon^i(\tau^i \leq s_i + \delta); i \in \mathbb{N}, \varepsilon \in \left(0, \frac{1}{8\psi(\mathbb{1})} \wedge 1\right]\right\} \leq \rho. \tag{3.12}$$

See Section 6 for the proof of Proposition 3.3.

Let us explain the meanings of the parameters η, α, L in this proposition. Since $X^i(\mathbb{1})$ is a Feller diffusion under $\mathbb{P}_\varepsilon$, a straightforward application of Girsanov's theorem (cf. Theorem VIII.1.4 of [23]) shows that $X^i(\mathbb{1})^{T_1^{X^i}}$ under $\mathbb{Q}_\varepsilon^i$ is a $\frac{1}{4}\mathrm{BES}Q^4(4\psi(\mathbb{1})\varepsilon)$ stopped upon hitting 1; see Lemma 4.1 of [17] for details. As a result, by the lower escape rate of $\mathrm{BES}Q^4$ (cf. Theorem 5.4.6 of [14]), the time $\tau^{i,(1)}$ is strictly positive $\mathbb{Q}_\varepsilon^i$-a.s. for any $\eta \in (1, \infty)$. In particular, we may take the parameter η close to 1.

The definition of $\tau^{i,(2)}$ involves the notion of improved modulus of continuity. We will take the parameter α in the definition of $\tau^{i,(2)}$ close to $\frac{1}{2}$ and consider the local Hölder exponent of the martingale part of $\mathrm{BES}Q^4$ in terms of its quadratic variation. The parameter L bounds the associated local Hölder coefficient. Hence, we have the integral inequality

$$\left|X_t^i(\mathbb{1})^{T_1^{X^i}} - \psi(\mathbb{1})\varepsilon\right| \leq (t-s_i) + L\left(\int_{s_i}^t X_s^i(\mathbb{1})^{T_1^{X^i}}\, ds\right)^\alpha \quad \forall t \in [s_i, \tau^i], \mathbb{Q}_\varepsilon^i\text{-a.s.}, \forall i \in \mathbb{N}, \varepsilon \in (0, [8\psi(\mathbb{1})]^{-1} \wedge 1], \tag{3.13}$$

by the choice of $\tau^{i,(2)}$ in Proposition 3.3. The integral inequality (3.13) is reminiscent of the integral inequalities to which Gronwall's lemma applies, and hence suggests an iteration argument if we wish to bound more explicitly the difference $|X_t^i(\mathbb{1})^{T_1^{X^i}} - \psi(\mathbb{1})\varepsilon|$. A general result for this is given by Corollary 8.2. Applying Corollary 8.2 to the random function

$$t \longmapsto X_t^i(\mathbb{1})^{T_1^{X^i}} : [s_i, \tau^i] \longrightarrow \mathbb{R},$$

we obtain from (3.13) that whenever $\xi \in (0,1)$ and $N_0 \in \mathbb{N}$ satisfies

$$\sum_{j=1}^{N_0} \alpha^j \le \xi < \sum_{j=1}^{N_0+1} \alpha^j, \tag{3.14}$$

the following inequality holds:

$$\begin{aligned} &\left|X_t^i(\mathbb{1})^{T_1^{X^i}} - \psi(\mathbb{1})\varepsilon\right| \le K_1^X\left[\psi(\mathbb{1})\varepsilon\right]^{\alpha^{N_0}}(t-s_i)^\alpha + K_2^X(t-s_i)^\xi \\ &\qquad \forall t \in [s_i, \tau^i], \mathbb{Q}_\varepsilon^i\text{-a.s.}, \forall i \in \mathbb{N}, \varepsilon \in \left(0, \left[8\psi(\mathbb{1})\right]^{-1} \wedge 1\right], \end{aligned} \tag{3.15}$$

where the constants $K_1^X, K_2^X \ge 1$ depend only on (α, L, ξ, N_0). Moreover, since α is close to $\frac{1}{2}$, we can choose N_0 large in (3.14) to make ξ close to 1, as is our intention in the sequel. Informally, we can interpret the foregoing inequality as the statement:

$$t \longmapsto X_t^i(\mathbb{1})^{T_1^{X^i}} \qquad \text{is Hölder-1 continuous at } s_i \text{ from the right.}$$

A similar derivation of the improved modulus of continuity of $Y^j(\mathbb{1})$ will appear in the proof of Lemma 3.12 below.

To use the support of X^i within which we study the local growth rate of Y, we take a parameter $\beta \in (0, \frac{1}{2})$, which is now close to $\frac{1}{2}$. We use this parameter to get a better control of the supports of X^i and Y^j, and this means we use the parabola

$$\mathcal{P}_\beta^{X^i}(t) \triangleq \left\{(x,s) \in \mathbb{R} \times [s_i, t]; |x - x_i| \le \left(\varepsilon^{1/2} + (s-s_i)^\beta\right)\right\} \tag{3.16}$$

to *envelop* the space–time support of $X^i \restriction [s_i, t]$, for $t \in (s_i, \infty)$, with a similar practice applied to other clusters Y^j. (See the speed of support propagation of super-Brownian motions in Theorem III.1.3 of [21].) More precisely, we can use the $(\mathscr{G}_t)$-stopping time

$$\begin{aligned} \sigma_\beta^{X^i} \triangleq \inf\{&s \ge s_i; \\ &\operatorname{supp}(X_s^i) \nsubseteq [x_i - \varepsilon^{1/2} - (s-s_i)^\beta, x_i + \varepsilon^{1/2} + (s-s_i)^\beta]\} \end{aligned} \tag{3.17}$$

as well as the analogous stopping times $\sigma_\beta^{Y^j}$ for Y^j to identify the duration of the foregoing enveloping.

We now specify the clusters Y^j selected for computing the local growth rate of Y. Suppose that at time t with $t > s_i$, we can still envelop the support of X^i by $\mathcal{P}_\beta^{X^i}(t)$ and the analogous enveloping for the support of Y^j holds for any $j \in \mathbb{N}$

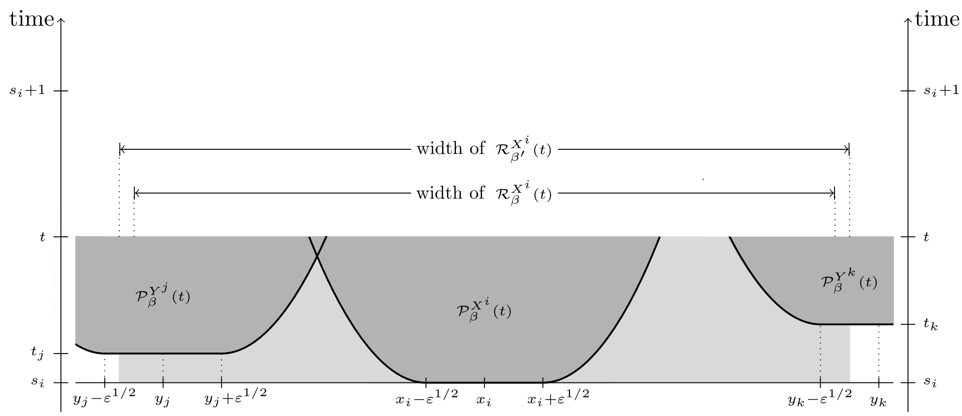

FIG. 2. *Parabolas* $\mathcal{P}_\beta^{X^i}(t), \mathcal{P}_\beta^{Y_j}(t), \mathcal{P}_\beta^{Y_k}(t)$ *and rectangles* $\mathcal{R}_\beta^{X^i}(t)$ *and* $\mathcal{R}_{\beta'}^{X^i}(t)$, *for* $0 < \beta' < \beta$ *and* $t \in [s_i, s_i + 1)$.

satisfying $t_j \in (s_i, t]$. Informally, we can ignore the clusters Y^j landing before X^i, because the probability that they can invade the initial part of the support of X^i is small for small t (cf. Lemma 7.3). Under such circumstances, simple geometric arguments show that only the Y^j clusters landing inside the space–time rectangle

$$\mathcal{R}_\beta^{X^i}(t) \triangleq \left[x_i - 2\left(\varepsilon^{1/2} + (t - s_i)^\beta\right), x_i + 2\left(\varepsilon^{1/2} + (t - s_i)^\beta\right)\right] \times [s_i, t] \tag{3.18}$$

can invade the initial part of the support of X^i by time t (see Lemma 7.3). We remark that this choice of clusters Y^j for $(y_j, t_j) \in \mathcal{R}_\beta^{X^i}(t)$ is also used in [17].

For technical reasons (cf. Section 3.5 below), however, we will consider the super-Brownian motions Y^j landing inside the slightly larger rectangle $\mathcal{R}_{\beta'}^{X^i}(t)$ for $t \in (s_i, s_i + 1]$, where β' is another value close to $\frac{1}{2}$, has the same meaning as β, and satisfies $\beta' < \beta$. See Figure 2 for these rectangles as well as an example for three parabolas $\mathcal{P}_\beta^{X^i}(t)$, $\mathcal{P}_\beta^{Y^j}(t)$, and $\mathcal{P}_\beta^{Y^k}(t)$ where $(y_j, t_j) \in \mathcal{R}_\beta^{X^i}(t)$ and $(y_k, t_k) \notin \mathcal{R}_\beta^{X^i}(t)$. The labels $j \in \mathbb{N}$ of the clusters Y^j landing inside $\mathcal{R}_{\beta'}^{X^i}(t)$ constitute the random index set

$$\mathcal{J}_{\beta'}^i(t) \equiv \mathcal{J}_{\beta'}^i(t, t), \tag{3.19}$$

where

$$\begin{aligned}\mathcal{J}_{\beta'}^i(t, t') \triangleq \left\{j \in \mathbb{N}; |y_j - x_i| \le 2\left(\varepsilon^{1/2} + (t - s_i)^{\beta'}\right), s_i < t_j \le t'\right\} \\ \forall t, t' \in (s_i, \infty).\end{aligned} \tag{3.20}$$

ASSUMPTION 3.4 (Choice of auxiliary parameters). Throughout the remainder of this section and Section 4, we fix a parameter vector

$$\begin{aligned}&(\eta, \alpha, L, \beta, \beta', \xi, N_0) \\ &\quad \in (1, \infty) \times \left(0, \tfrac{1}{2}\right) \times (0, \infty) \times \left[\tfrac{1}{3}, \tfrac{1}{2}\right) \times \left[\tfrac{1}{3}, \tfrac{1}{2}\right) \times (0, 1) \times \mathbb{N}\end{aligned} \tag{3.21}$$

satisfying

$$
\begin{cases}
\text{(a)} \quad \displaystyle\sum_{j=1}^{N_0} \alpha^j \leq \xi < \sum_{j=1}^{N_0+1} \alpha^j, \\
\text{(b)} \quad \alpha < \dfrac{\beta'}{\beta} < 1, \\
\text{(c)} \quad \beta' - \dfrac{\eta}{2} + \dfrac{3}{2}\alpha > 0, \\
\text{(d)} \quad (\beta' + 1) \wedge \left(\beta' - \dfrac{\eta}{2} + \dfrac{3\xi}{2}\right) > \eta.
\end{cases}
\tag{3.22}
$$

[Note that we restate (3.14) in (a).] We insist that the parameter vector in (3.21) is chosen to be *independent* of $i \in \mathbb{N}$ and $\varepsilon \in (0, [8\psi(\mathbb{1})]^{-1} \wedge 1]$. For example, we can choose these parameters in the following order: first choose $\eta, \alpha, \beta', \xi$ according to (c) and (d), choose β according to (b), and finally choose N_0 according to (a) by enlarging ξ if necessary; the parameter L, however, can be chosen arbitrarily.

The following theorem gives our quantification of the local growth rates of Y under $\mathbb{Q}_\varepsilon^i$.

THEOREM 3.5. *Under Assumption 3.4, set three strictly positive constants by*

$$
\begin{aligned}
&\kappa_1 = (\beta' + 1) \wedge \left(\beta' - \frac{\eta}{2} + \frac{3\xi}{2}\right), \qquad \kappa_2 = \frac{\alpha^{N_0}}{4}, \\
&\kappa_3 = \beta' - \frac{\eta}{2} + \frac{3\alpha}{2}.
\end{aligned}
\tag{3.23}
$$

Then there exists a constant $K^* \in (0, \infty)$, depending only on the parameter vector in (3.21) and the immigration function ψ, such that for any $\delta \in (0, \kappa_1 \wedge \kappa_3)$, the following uniform bound holds:

$$
\begin{aligned}
&\mathbb{Q}_\varepsilon^i \Biggl(\exists s \in (s_i, t], \sum_{j \in \mathcal{J}_{\beta'}^i (s \wedge \tau^i \wedge \sigma_\beta^{X^i})} Y_s^j(\mathbb{1})^{\tau^i \wedge \sigma_\beta^{X^i} \wedge \sigma_\beta^{Y^j}} \\
&\qquad > K^* \bigl[(s - s_i)^{\kappa_1 - \delta} + \varepsilon^{\kappa_2} \cdot (s - s_i)^{\kappa_3 - \delta} \bigr] \Biggr) \\
&\quad \leq \frac{2 \cdot 2^{\kappa_1 \vee \kappa_3}}{2^{(N+1)\delta}(1 - 2^{-\delta})} \\
&\quad \forall t \in \bigl[s_i + 2^{-(N+1)}, s_i + 2^{-N} \bigr], N \in \mathbb{Z}_+, i \in \mathbb{N}, \varepsilon \in \left(0, \frac{1}{8\psi(\mathbb{1})} \wedge 1\right],
\end{aligned}
\tag{3.24}
$$

where the $(\mathscr{G}_t)$-stopping times τ^i are defined in Proposition 3.3.

REMARK 3.6. If we follow the aforementioned interpretation of the parameter vector in (3.21) that (η, β', ξ) is close to $(1, \frac{1}{2}, 1)$, then κ_1 in (3.23) is close to $\frac{3}{2}$. Informally, if we regard the stopping times τ^i, $\sigma_\beta^{X^i}$, and $\sigma_\beta^{Y^j}$ as being *bounded away* from s_i, then by the above reason for choosing the random index sets $\mathcal{J}_{\beta'}^i(\cdot)$ in (3.19), we can regard Theorem 3.5 as a formalization of the statement in (1.19).

In fact, the proof of Theorem 3.5 is reduced to a study of some nonnegative $(\mathscr{G}_t)_{t\geq s_i}$-submartingale dominating the process

$$\sum_{j\in\mathcal{J}_{\beta'}^i(t\wedge\tau^i\wedge\sigma_\beta^{X^i})} Y_t^j(\mathbb{1})^{\tau^i\wedge\sigma_\beta^{X^i}\wedge\sigma_\beta^{Y^j}}, \qquad t\in[s_i,\infty), \tag{3.25}$$

in (3.24), and the main task will be to prove Theorem 3.8 below. We explain the reductions as follows.

We observe that by Lemma 3.2(2), the process in (3.25) is dominated by the nonnegative process

$$\begin{aligned}\sum_{j\in\mathcal{J}_{\beta'}^i(t,t\wedge\tau^i\wedge\sigma_\beta^{X^i})}&\Bigg(\psi(\mathbb{1})\varepsilon\\ &+\int_{t_j}^{t\wedge\tau^i\wedge\sigma_\beta^{X^i}\wedge\sigma_\beta^{Y^j}}\frac{1}{X_s^i(\mathbb{1})}\int_{\mathbb{R}}X^i(x,s)^{1/2}Y^j(x,s)^{1/2}\,dx\,ds\\ &+M^j_{t\wedge\tau^i\wedge\sigma_\beta^{X^i}\wedge\sigma_\beta^{Y^j}}\Bigg),\\ &\qquad t\in[s_i,\infty),\end{aligned} \tag{3.26}$$

under $\mathbb{Q}_\varepsilon^i$ for any $i\in\mathbb{N}$ and $\varepsilon\in(0,[8\psi(\mathbb{1})]^{-1}\wedge 1]$. The process in (3.26) is in fact a nonnegative $(\mathscr{G}_t)_{t\geq s_i}$-submartingale under $\mathbb{Q}_\varepsilon^i$, since for any $j\in\mathbb{N}$ with $s_i<t_j$, $j\in\mathcal{J}_{\beta'}^i(t,t\wedge\tau^i\wedge\sigma_\beta^{X^i})$ if and only if the following $\mathscr{G}_{t_j}$-event occurs:

$$\{|y_j-x_i|\leq 2(\varepsilon^{1/2}+(t-s_i)^{\beta'})\text{ and }t_j\leq t\wedge\tau^i\wedge\sigma_\beta^{X^i}\}.$$

(Recall that $y_j\in\mathscr{G}_{t_j}$ and $x_i\in\mathscr{G}_{s_i}$ by Theorem 2.12.) It suffices to prove the bound (3.24) of Theorem 3.5 with the involved process in (3.25) replaced by the nonnegative submartingale in (3.26). To further reduce the problem, we resort to the following simple corollary of Doob's maximal inequality.

LEMMA 3.7. *Let F be a nonnegative function on $[0,1]$ such that $F\restriction(0,1]>0$ and $\sup_{s,t:\,1\leq t/s\leq 2}\frac{F(t)}{F(s)}<\infty$. In addition, assume that*

$$\textit{for some } \delta>0,\qquad t\longmapsto\frac{F(t)}{t^\delta}\textit{ is increasing}. \tag{3.27}$$

Suppose that Z is a nonnegative submartingale with càdlàg sample paths such that $\mathbb{E}[Z_t] \leq F(t)$ for any $t \in [0,1]$. Then for every $N \in \mathbb{Z}_+$,

$$\begin{aligned}\sup_{t\in[2^{-(N+1)},2^{-N}]} \mathbb{P}\left(\exists s \in (0,t], Z_s > \frac{F(s)}{s^\delta}\right)\\ \leq \left(\sup_{s,t\,:\,1\leq t/s\leq 2} \frac{F(t)}{F(s)}\right) \times \frac{1}{2^{(N+1)\delta}(1-2^{-\delta})}.\end{aligned}\tag{3.28}$$

PROOF. For each $m \in \mathbb{Z}_+$,

$$\begin{aligned}&\mathbb{P}\left(\exists s \in [2^{-(m+1)}, 2^{-m}], Z_s \geq \frac{F(s)}{s^\delta}\right)\\ &\quad\leq \mathbb{P}\left(\sup_{2^{-(m+1)}\leq s\leq 2^{-m}} Z_s \geq F\left(\frac{1}{2^{(m+1)}}\right)\Big/\frac{1}{2^{(m+1)\delta}}\right)\\ &\quad\leq \frac{\mathbb{E}[Z_{1/2^m}]}{F(1/2^{(m+1)})/(1/2^{(m+1)\delta})}\\ &\quad\leq \frac{F(1/2^m)}{F(1/2^{(m+1)})/(1/2^{(m+1)\delta})}\\ &\quad= \sup_{s,t\,:\,1\leq t/s\leq 2} \frac{F(t)}{F(s)} \times \frac{1}{2^{(m+1)\delta}},\end{aligned}$$

where the first inequality follows from (3.27) and the second inequality follows from Doob's maximal inequality. Hence, whenever $t \in [2^{-(N+1)}, 2^{-N}]$ for $N \in \mathbb{Z}_+$, the last inequality gives

$$\begin{aligned}&\mathbb{P}\left(\exists s \in (0,t], Z_s > \frac{F(s)}{s^\delta}\right)\\ &\quad\leq \sum_{m=N}^{\infty} \mathbb{P}\left(\exists s \in [2^{-(m+1)}, 2^{-m}], Z_s \geq \frac{F(s)}{s^\delta}\right)\\ &\quad\leq \left(\sup_{s,t\,:\,1\leq t/s\leq 2} \frac{F(t)}{F(s)}\right) \sum_{m=N}^{\infty} \frac{1}{2^{(m+1)\delta}}\\ &\quad= \left(\sup_{s,t\,:\,1\leq t/s\leq 2} \frac{F(t)}{F(s)}\right) \times \frac{1}{2^{(N+1)\delta}(1-2^{-\delta})}.\end{aligned}$$

This completes the proof. □

THEOREM 3.8. *Under Assumption 3.4, take the same constants κ_j as in Theorem 3.5. Then we can choose a constant $K^* \in (0,\infty)$ as stated in Theorem 3.5,* *such that the following uniform bound holds*:

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}^i_\varepsilon}\Bigg[& \sum_{j\in\mathcal{J}^i_{\beta'}(t,t\wedge\tau^i\wedge\sigma^{X^i}_\beta)} \Bigg(\psi(\mathbb{1})\varepsilon \\
& \qquad + \int_{t_j}^{t\wedge\tau^i\wedge\sigma^{X^i}_\beta\wedge\sigma^{Y^j}_\beta} \frac{1}{X^i_s(\mathbb{1})}\int_{\mathbb{R}} X^i(x,s)^{1/2}Y^j(x,s)^{1/2}\,dx\,ds\Bigg)\Bigg] \\
& \leq K^*\big[(t-s_i)^{\kappa_1}+\varepsilon^{\kappa_2}\cdot(t-s_i)^{\kappa_3}\big] \\
& \qquad \forall t\in(s_i,s_i+1], i\in\mathbb{N}, \varepsilon\in\left(0,\frac{1}{8\psi(\mathbb{1})}\wedge 1\right].
\end{aligned}
$$

Now, we prove the main result of this section, that is Theorem 3.5, assuming Theorem 3.8.

PROOF OF THEOREM 3.5. In, and only in, this proof, we denote by $Z^{(0)}$ the submartingale defined in (3.26).

Since $[j\in\mathcal{J}^i_{\beta'}(t,t\wedge\tau^i\wedge\sigma^{X^i}_\beta)]\in\mathscr{G}_{t_j}$, we obtain immediately from Lemma 3.2(2) that the part

$$
\sum_{j\in\mathcal{J}^i_{\beta'}(t,t\wedge\tau^i\wedge\sigma^{X^i}_\beta)} M^j_{t\wedge\tau^i\wedge\sigma^{X^i}_\beta\wedge\sigma^{Y^j}_\beta}, \qquad t\in[s_i,\infty),
$$

in the definition of $Z^{(0)}$ is a true $\mathbb{Q}^i_\varepsilon$-martingale with mean zero, for any $i\in\mathbb{N}$ and $\varepsilon\in(0,[8\psi(\mathbb{1})]^{-1}\wedge 1]$. Hence, setting

$$
F^{(0)}(s)=K^*\big(s^{\kappa_1}+\varepsilon^{\kappa_2}\cdot s^{\kappa_3}\big), \qquad s\in[0,1],
$$

we see from Theorem 3.8 that

$$
\mathbb{E}_\varepsilon\big[Z^{(0)}_t\big]\leq F^{(0)}(t-s_i)
$$

for any $t\in(s_i,s_i+1]$, $i\in\mathbb{N}$ and $\varepsilon\in(0,[8\psi(\mathbb{1})]^{-1}\wedge 1]$. Note that

$$
\sup_{s,t:\,1\leq t/s\leq 2}\frac{F^{(0)}(t)}{F^{(0)}(s)}\leq \sup_{s,t:\,1\leq t/s\leq 2}\left(\frac{t^{\kappa_1}}{s^{\kappa_1}}+\frac{t^{\kappa_3}}{s^{\kappa_3}}\right)\leq 2\cdot 2^{\kappa_1\vee\kappa_3}.
$$

Hence, applying Lemma 3.7 with (Z,F) taken to be $(Z^{(0)},F^{(0)})$, we see that (3.24) with the involved process in (3.25) replaced by $Z^{(0)}$ holds. The proof is complete. □

The remainder of this section is to prove Theorem 3.8. For this purpose, we need to classify the clusters Y^j for $j\in\mathcal{J}^i_{\beta'}(t,t\wedge\tau^i\wedge\sigma^{X^i}_\beta)$. Set

$$
\begin{aligned}
\mathcal{C}^i_{\beta'}(t) &\triangleq \big\{j\in\mathbb{N}; |y_j-x_i|<2\big(\varepsilon^{1/2}+(t_j-s_i)^{\beta'}\big), s_i<t_j\leq t\big\}, \\
\mathcal{L}^i_{\beta'}(t,t') &\triangleq \big\{j\in\mathbb{N}; 2\big(\varepsilon^{1/2}+(t_j-s_i)^{\beta'}\big)\leq|y_j-x_i|\leq 2\big(\varepsilon^{1/2}+(t-s_i)^{\beta'}\big), \\
&\qquad\qquad s_i<t_j\leq t'\big\},
\end{aligned}
$$

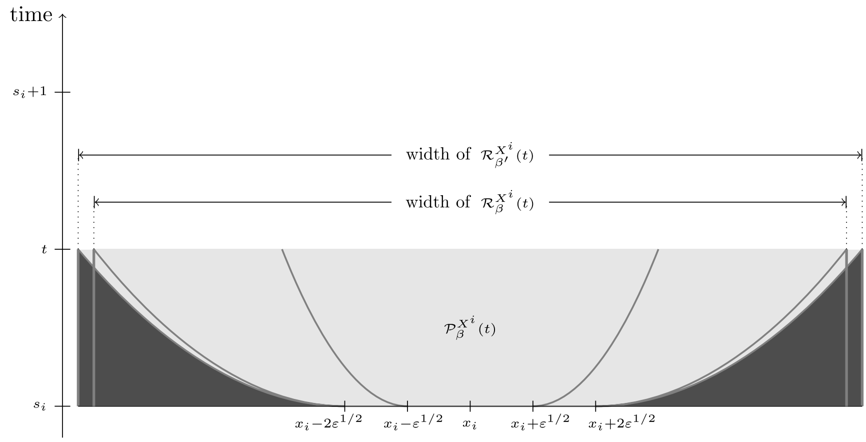

FIG. 3. $\mathcal{P}_{\beta}^{X^i}(t)$, $\mathcal{R}_{\beta}^{X^i}(t)$, *and* $\mathcal{R}_{\beta'}^{X^i}(t)$ *for* $0<\beta'<\beta$ *and* $t\in[s_i,s_i+1]$.

for $t',t\in(s_i,\infty)$ with $t\geq t'$. Hence, as far as the clusters Y^j landing inside the rectangle $\mathcal{R}_{\beta'}^{X^i}(t)$ are concerned, the clusters Y^j, $j\in\mathcal{C}^i_{\beta'}(t)$, are those landing inside the double parabola

$$\big\{(x,s)\in\mathbb{R}\times[s_i,t];\,|x-x_i|<2\big(\varepsilon^{1/2}+(s-s_i)^{\beta'}\big)\big\}$$

(the light grey area in Figure 3), and the clusters Y^j, $j\in\mathcal{L}^i_{\beta'}(t,t)$, are those landing outside (the dark grey area in Figure 3). For any $i\in\mathbb{N}$, we say a cluster Y^j is a *critical cluster* if $j\in\mathcal{C}^i_{\beta'}(t)$ and a *lateral cluster* if $j\in\mathcal{L}^i(t,t')$ for some t,t'.

Since $\{\mathcal{C}^i_{\beta'}(t),\mathcal{L}^i_{\beta'}(t,t')\}$ is a cover of $\mathcal{J}^i_{\beta'}(t,t')$ by disjoint sets, Theorem 3.8 can be obtained by the following two lemmas.

LEMMA 3.9. *Let* κ_j *be as in Theorem* 3.5. *We can choose a constant* $K^*\in(0,\infty)$ *as in Theorem* 3.5 *such that the following uniform bound holds*:

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}^i_\varepsilon}\Bigg[\sum_{j\in\mathcal{C}^i_{\beta'}(t\wedge\tau^i\wedge\sigma_\beta^{X^i})}\Bigg(\psi(\mathbb{1})\varepsilon\\
&\qquad\qquad+\int_{t_j}^{t\wedge\tau^i\wedge\sigma_\beta^{X^i}\wedge\sigma_\beta^{Y^j}}\frac{1}{X^i_s(\mathbb{1})}\int_{\mathbb{R}}X^i(x,s)^{1/2}Y^j(x,s)^{1/2}\,dx\,ds\Bigg)\Bigg]\\
&\quad\leq\frac{K^*}{2}\big[(t-s_i)^{\kappa_1}+\varepsilon^{\kappa_2}\cdot(t-s_i)^{\kappa_3}\big]\\
&\qquad\qquad\forall t\in(s_i,s_i+1],i\in\mathbb{N},\varepsilon\in\Big(0,\frac{1}{8\psi(\mathbb{1})}\wedge1\Big].
\end{aligned}\tag{3.29}$$

LEMMA 3.10. *Let κ_j be as in Theorem 3.5. By enlarging the constant K^* in Lemma 3.9 if necessary, the following uniform bound holds*:

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}^i_\varepsilon}\Bigg[\sum_{j\in\mathcal{L}^i_{\beta'}(t,t\wedge\tau^i\wedge\sigma^{X^i}_\beta)}\Bigg(\psi(\mathbb{1})\varepsilon \\
&\qquad + \int_{t_j}^{t\wedge\tau^i\wedge\sigma^{X^i}_\beta\wedge\sigma^{Y^j}_\beta}\frac{1}{X^i_s(\mathbb{1})}\int_{\mathbb{R}}X^i(x,s)^{1/2} \\
&\qquad\qquad \times Y^j(x,s)^{1/2}\,dx\,ds\Bigg)\Bigg] \qquad (3.30)\\
&\quad\le\frac{K^*}{2}\big[(t-s_i)^{\kappa_1}+\varepsilon^{\kappa_2}\cdot(t-s_i)^{\kappa_3}\big] \\
&\qquad\qquad \forall t\in(s_i,s_i+1], i\in\mathbb{N}, \varepsilon\in\left(0,\frac{1}{8\psi(\mathbb{1})}\wedge 1\right].
\end{aligned}$$

Despite some technical details, the methods of proof for Lemmas 3.9 and 3.10 are very similar. For clarity, they are given in Sections 3.4 and 3.5 separately, with some preliminaries set in Section 3.3 below.

3.3. *Auxiliary results and notation*. For each $z,\delta\in\mathbb{R}_+$, let $(Z,\mathbf{P}^\delta_z)$ denote a copy of $\frac{1}{4}\mathrm{BES}Q^{4\delta}(4z)$. We assume that $(Z,\mathbf{P}^\delta_z)$ is defined by a $(\mathscr{H}_t)$-Brownian motion B, where $(\mathscr{H}_t)$ satisfies the usual conditions. This means that

$$Z_t = z+\delta t+\int_0^t\sqrt{Z_s}\,dB_s,\qquad \mathbf{P}^\delta_z\text{-a.s.}$$

(Cf. Section XI.1 of [23] for Bessel squared processes.) As we will often investigate Z before it hits a constant level, we set the following notation similar to (3.2): for any real-valued process $H=(H_t)$

$$T^H_x=\inf\{t\ge 0; H_t=x\},\qquad x\in\mathbb{R}.$$

For $\delta=0$, $(Z,\mathbf{P}^0_z)$ gives a Feller diffusion and its marginals are characterized by

$$\mathbb{E}^{\mathbf{P}^0_z}\big[\exp(-\lambda Z_t)\big]=\exp\left(\frac{-2\lambda z}{2+\lambda t}\right),\qquad \lambda,t\in\mathbb{R}_+.$$

In particular, the survival probability of $(Z,\mathbf{P}^0_z)$ is given by

$$\begin{aligned}
\mathbf{P}^0_z(Z_t>0)&=\lim_{\lambda\to\infty}\big(1-\mathbb{E}^{\mathbf{P}^0_z}\big[\exp(-\lambda Z_t)\big]\big)=1-\exp\left(-\frac{2z}{t}\right), \qquad (3.31)\\
&\qquad\qquad z,t\in(0,\infty).
\end{aligned}$$

Using the elementary inequality $1-e^{-x}\le x$ for $x\in\mathbb{R}_+$, we obtain from the last inequality that

$$\mathbf{P}^0_z(Z_t>0)\le\frac{2z}{t},\qquad z,t\in(0,\infty). \tag{3.32}$$

To save notation in the following Sections 3.4 and 3.5, we write $A \lesssim B$ if $A \leq CB$ for some constant $C \in (0,\infty)$ which may vary from line to line but depends only on ψ *and* the parameter vector chosen in Assumption 3.4.

3.4. *Proof of Lemma* 3.9. Fix $i \in \mathbb{N}$ and $\varepsilon \in (0, [8\psi(\mathbb{1})]^{-1} \wedge 1]$, and henceforth we drop the subscripts ε of $\mathbb{P}_\varepsilon$ and $\mathbb{Q}^i_\varepsilon$. In addition, we may only consider $t \in [s_i + \frac{\varepsilon}{2}, s_i + 1]$ as there are no immigrants for Y arriving in $[s_i, s_i + \frac{\varepsilon}{2})$. Our analysis proceeds with the following steps.

Step 1. We start with the simplification:

$$
\begin{aligned}
&\sum_{j \in \mathcal{C}^i_{\beta'}(t \wedge \tau^i \wedge \sigma^{X^i}_\beta)} \Bigg(\psi(\mathbb{1})\varepsilon \\
&\qquad\qquad + \int_{t_j}^{t \wedge \tau^i \wedge \sigma^{X^i}_\beta \wedge \sigma^{Y^j}_\beta} \frac{1}{X^i_s(\mathbb{1})} \int_{\mathbb{R}} X^i(x,s)^{1/2} Y^j(x,s)^{1/2}\, dx\, ds \Bigg) \\
&\quad \leq \sum_{j \in \mathcal{C}^i_{\beta'}(t \wedge \tau^i)} \Bigg(\psi(\mathbb{1})\varepsilon + \int_{t_j}^{t \wedge \tau^i} \frac{1}{[X^i_s(\mathbb{1})]^{1/2}} \big[Y^j_s(\mathbb{1})\big]^{1/2} ds \Bigg) \\
&\quad \leq \sum_{j \in \mathcal{C}^i_{\beta'}(t \wedge \tau^i)} \Bigg(\psi(\mathbb{1})\varepsilon + \int_{t_j}^{t \wedge \tau^i} \frac{2}{(s-s_i)^{\eta/2}} \big[Y^j_s(\mathbb{1})\big]^{1/2} ds \Bigg),
\end{aligned}
\tag{3.33}
$$

where the first inequality follows from the Cauchy–Schwarz inequality and the second one follows by using the component $\tau^{i,(1)}$ of τ^i in Proposition 3.3.

We claim that

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}^i}\Bigg[\sum_{j \in \mathcal{C}^i_{\beta'}(t \wedge \tau^i \wedge \sigma^{X^i}_\beta)} \Bigg(\psi(\mathbb{1})\varepsilon \\
&\qquad\qquad + \int_{t_j}^{t \wedge \tau^i \wedge \sigma^{X^i}_\beta \wedge \sigma^{Y^j}_\beta} \frac{1}{X^i_s(\mathbb{1})} \int_{\mathbb{R}} X^i(x,s)^{1/2} \\
&\qquad\qquad\qquad\qquad \times Y^j(x,s)^{1/2}\, dx\, ds \Bigg) \Bigg] \\
&\quad \lesssim \sum_{j : s_i < t_j \leq t} (t_j - s_i)^{\beta'} \varepsilon \\
&\qquad + \sum_{j : s_i < t_j \leq t} \int_{t_j}^{t} ds \frac{1}{(s-s_i)^{\eta/2}} \mathbb{E}^{\mathbb{Q}^i}\big[\big[Y^j_s(\mathbb{1})\big]^{1/2}; s < \tau^i, \\
&\qquad\qquad\qquad |y_j - x_i| < 2\big(\varepsilon^{1/2} + (t_j - s_i)^{\beta'}\big)\big].
\end{aligned}
\tag{3.34}
$$

Note that

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}^i}\left[\psi(\mathbb{1})\varepsilon\#\mathcal{C}^i_{\beta'}\left(t\wedge\tau^i\right)\right]\\
&\quad\precsim \varepsilon\mathbb{E}^{\mathbb{Q}^i}\left[\#\mathcal{C}^i_{\beta'}(t)\right]\\
&\quad=\varepsilon\sum_{j\,:\,s_i<t_j\leq t}\mathbb{Q}^i\left(|y_j-x_i|<2\left(\varepsilon^{1/2}+(t_j-s_i)^{\beta'}\right)\right)\\
&\quad\precsim\sum_{j\,:\,s_i<t_j\leq t}4\left(\varepsilon^{1/2}+(t_j-s_i)^{\beta'}\right)\varepsilon\\
&\quad\precsim\sum_{j\,:\,s_i<t_j\leq t}(t_j-s_i)^{\beta'}\varepsilon,
\end{aligned}
\tag{3.35}
$$

where the second $\precsim$-inequality follows from Lemma 3.2(4), and the last $\precsim$-inequality follows since

$$
\varepsilon^{1/2}\leq\varepsilon^{\beta'}\leq 2^{\beta'}(t_j-s_i)^{\beta'}\qquad\forall j\in\mathbb{N}\text{ with }t_j>s_i.
\tag{3.36}
$$

Our claim (3.34) follows from (3.33) and (3.35).

From the display (3.34), we see the necessity to obtain the order of

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}^i}\left[\left[Y^j_s(\mathbb{1})\right]^{1/2};s<\tau^i,|y_j-x_i|<2\left(\varepsilon^{1/2}+(t_j-s_i)^{\beta'}\right)\right],\\
&\qquad\qquad\qquad\qquad\qquad\qquad s\in(t_j,t],s_i<t_j<t,
\end{aligned}
\tag{3.37}
$$

in s_i,t_j,s,t.

We subdivide our analysis of a generic term in (3.37) into the following steps 2-1–2-4, with a summary given in step 2-5.

Step 2-1. We convert the $\mathbb{Q}^i$-expectations in (3.37) to $\mathbb{P}$-expectations. Recalling that $x_i,y_j\in\mathscr{G}_{t_j}$ by (2.4), we can use Lemma 3.1 to get

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}^i}\left[\left[Y^j_s(\mathbb{1})\right]^{1/2};s<\tau^i,|y_j-x_i|<2\left(\varepsilon^{1/2}+(t_j-s_i)^{\beta'}\right)\right]\\
&\quad=\frac{1}{\psi(\mathbb{1})\varepsilon}\mathbb{E}^{\mathbb{P}}\Big[X^i_s(\mathbb{1})^{T_1^{X^i}}\left[Y^j_s(\mathbb{1})\right]^{1/2};s<\tau^i,\\
&\qquad\qquad\qquad|y_j-x_i|<2\left(\varepsilon^{1/2}+(t_j-s_i)^{\beta'}\right)\Big].
\end{aligned}
\tag{3.38}
$$

We break the $\mathbb{P}$-expectation in (3.38) into finer pieces by considering the following. For $s>t_j$, $X^i(\mathbb{1})_s^{T_1^{X^i}}$ is nonzero on the union of the two disjoint events:

$$
\left\{X^i_s(\mathbb{1})^{T_1^{X^i}}>0,T_0^{X^i}\leq t_j\right\}=\left\{T_1^{X^i}<T_0^{X^i}\leq t_j\right\}
\tag{3.39}
$$

and

$$
\left\{X^i_s(\mathbb{1})^{T_1^{X^i}}>0,t_j<T_0^{X^i}\right\}.
\tag{3.40}
$$

Here, the equality in (3.39) holds $\mathbb{P}$-a.s. since 0 is an absorbing state of $X^i(\mathbb{1})$ under $\mathbb{P}$. In fact, $X^i(\mathbb{1})_s^{T_1^{X^i}} = 1$ on the event in (3.39). To invoke the additional order provided by the improved modulus of continuity of $X^i(\mathbb{1})$ at its starting point s_i, we use the trivial inequality

$$X_s^i(\mathbb{1})^{T_1^{X^i}} \leq \left|X_s^i(\mathbb{1})^{T_1^{X^i}} - \psi(\mathbb{1})\varepsilon\right| + \psi(\mathbb{1})\varepsilon$$

on the event (3.40).

Putting things together, we see from (3.38) that

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}^i}\big[\big[Y_s^j(\mathbb{1})\big]^{1/2}; s < \tau^i, |y_j - x_i| < 2\big(\varepsilon^{1/2} + (t_j - s_i)^{\beta'}\big)\big] \\
&\quad \leq \frac{1}{\psi(\mathbb{1})\varepsilon}\mathbb{E}^{\mathbb{P}}\big[\big[Y_s^j(\mathbb{1})\big]^{1/2}; s \leq T_1^{Y^j}, \\
&\qquad\qquad |y_j - x_i| < 2\big(\varepsilon^{1/2} + (t_j - s_i)^{\beta'}\big), T_1^{X^i} < T_0^{X^i} \leq t_j\big] \\
&\qquad + \frac{1}{\psi(\mathbb{1})\varepsilon}\mathbb{E}^{\mathbb{P}}\big[\big|X_s^i(\mathbb{1})^{T_1^{X^i}} - \psi(\mathbb{1})\varepsilon\big|\big[Y_s^j(\mathbb{1})\big]^{1/2}; s < \tau^i, \\
&\qquad\qquad |y_j - x_i| < 2\big(\varepsilon^{1/2} + (t_j - s_i)^{\beta'}\big), X_s^i(\mathbb{1})^{T_1^{X^i}} > 0, t_j < T_0^{X^i}\big] \\
&\qquad + \frac{1}{\psi(\mathbb{1})\varepsilon} \cdot \psi(\mathbb{1})\varepsilon\mathbb{E}^{\mathbb{P}}\big[\big[Y_s^j(\mathbb{1})\big]^{1/2}; s \leq T_1^{Y^j}, \\
&\qquad\qquad |y_j - x_i| < 2\big(\varepsilon^{1/2} + (t_j - s_i)^{\beta'}\big), t_j < T_0^{X^i}\big] \\
&\qquad\qquad\qquad \forall s \in (t_j, t], s_i < t_j < t,
\end{aligned} \tag{3.41}$$

where for the first and the third terms on the right-hand side, it is legitimate to replace the event $\{s < \tau^i\}$ by the larger one $\{s \leq T_1^{Y^j}\}$ since, in Proposition 3.3, $\tau^{i,(3)}$ is a component of τ^i, and for the third term we replace the event in (3.40) by the larger one $\{t_j < T_0^{X^i}\}$.

In steps 2-2–2-4 below, we derive a bound for each of the three terms in (3.41) which involves only Feller's diffusion. We use the notation in Section 3.3.

Step 2-2. Consider the first term on the right-hand side of (3.41), and recall the notation in Section 3.3. It follows from (3.8) and (3.9) that

$$\begin{aligned}
&\frac{1}{\psi(\mathbb{1})\varepsilon}\mathbb{E}^{\mathbb{P}}\big[\big[Y_s^j(\mathbb{1})\big]^{1/2}; s \leq T_1^{Y^j}, \\
&\qquad |y_j - x_i| < 2\big(\varepsilon^{1/2} + (t_j - s_i)^{\beta'}\big), T_1^{X^i} < T_0^{X^i} \leq t_j\big] \\
&\quad \precsim \frac{1}{\varepsilon}\mathbb{P}\big(T_1^{X^i} < T_0^{X^i} \leq t_j\big)\big(\varepsilon^{1/2} + (t_j - s_i)^{\beta'}\big) \\
&\qquad \times \mathbb{E}^{\mathbf{P}^0_{\psi(\mathbb{1})\varepsilon}}\big[(Z_{s-t_j})^{1/2}; s - t_j \leq T_1^Z\big]
\end{aligned} \tag{3.42}$$

$$
\begin{aligned}
&\le \frac{1}{\varepsilon}\mathbb{P}\big(T_1^{X^i} < T_0^{X^i}\big)\big(\varepsilon^{1/2} + (t_j - s_i)^{\beta'}\big) \\
&\quad \times \mathbb{E}^{\mathbf{P}^0_{\psi(\mathbb{1})\varepsilon}}\big[(Z_{s-t_j})^{1/2}; s - t_j \le T_1^Z\big] \\
&\precsim (t_j - s_i)^{\beta'}\mathbb{E}^{\mathbf{P}^0_{\psi(\mathbb{1})\varepsilon}}\big[(Z_{s-t_j})^{1/2}; s - t_j \le T_1^Z\big] \\
&\qquad\qquad\qquad\qquad \forall s \in (t_j, t], s_i < t_j < t,
\end{aligned}
$$

where the last inequality follows from (3.36) and Lemma 3.1.

Step 2-3. Let us deal with the second term in (3.41). We claim that

$$
\begin{aligned}
&\frac{1}{\psi(\mathbb{1})\varepsilon}\mathbb{E}^{\mathbb{P}}\big[\big|X_s^i(\mathbb{1})^{T_1^{X^i}} - \psi(\mathbb{1})\varepsilon\big|\big[Y_s^j(\mathbb{1})\big]^{1/2}; s < \tau^i, \\
&\qquad |y_j - x_i| < 2\big(\varepsilon^{1/2} + (t_j - s_i)^{\beta'}\big), X_s^i(\mathbb{1})^{T_1^{X^i}} > 0, t_j < T_0^{X^i}\big] \\
&\quad \precsim \big(\varepsilon^{\alpha^{N_0}}(s - s_i)^{\alpha} + (s - s_i)^{\xi}\big)(t_j - s_i)^{\beta' - 1} \\
&\qquad \times \mathbb{E}^{\mathbf{P}^0_{\psi(\mathbb{1})\varepsilon}}\big[(Z_{s-t_j})^{1/2}; s - t_j \le T_1^Z\big] \qquad \forall s \in (t_j, t], s_i < t_j < t.
\end{aligned} \tag{3.43}
$$

Fix such s throughout step 2-3.

First, let us transfer the improved modulus of $X^i(\mathbb{1})$ under $\mathbb{Q}^i$ to one under $\mathbb{P}$. It follows from (3.15) that on $\{s < \tau^i, X^i(\mathbb{1})_s^{T_1^{X^i}} > 0\} \in \mathscr{G}_s$, we have

$$
\big|X_s^i(\mathbb{1})^{T_1^{X^i}} - \psi(\mathbb{1})\varepsilon\big| \le K_1^X\big[\psi(\mathbb{1})\varepsilon\big]^{\alpha^{N_0}}(s - s_i)^{\alpha} + K_2^X(s - s_i)^{\xi} \qquad \mathbb{Q}^i\text{-a.s.}
$$

and hence

$$
\begin{aligned}
0 &= \mathbb{Q}^i\big(\big|X_s^i(\mathbb{1})^{T_1^{X^i}} - \psi(\mathbb{1})\varepsilon\big| > K_1^X\big[\psi(\mathbb{1})\varepsilon\big]^{\alpha^{N_0}}(s - s_i)^{\alpha} \\
&\qquad\qquad + K_2^X(s - s_i)^{\xi}, s < \tau^i, X_s^i(\mathbb{1})^{T_1^{X^i}} > 0\big) \\
&= \frac{1}{\psi(\mathbb{1})\varepsilon}\mathbb{E}^{\mathbb{P}}\big[X_s^i(\mathbb{1})^{T_1^{X^i}}; \big|X_s^i(\mathbb{1})^{T_1^{X^i}} - \psi(\mathbb{1})\varepsilon\big| > K_1^X\big[\psi(\mathbb{1})\varepsilon\big]^{\alpha^{N_0}}(s - s_i)^{\alpha} \\
&\qquad\qquad + K_2^X(s - s_i)^{\xi}, s < \tau^i, X_s^i(\mathbb{1})^{T_1^{X^i}} > 0\big],
\end{aligned} \tag{3.44}
$$

where the last equality follows from Lemma 3.1 since the event evaluated under $\mathbb{Q}^i$ is a $\mathscr{G}_s$-event. Using the restriction $X_s^i(\mathbb{1})^{T_1^{X^i}} > 0$, we see that the equality (3.44) implies

$$
\begin{aligned}
&\big|X_s^i(\mathbb{1})^{T_1^{X^i}} - \psi(\mathbb{1})\varepsilon\big| \le K_1^X\big[\psi(\mathbb{1})\varepsilon\big]^{\alpha^{N_0}}(s - s_i)^{\alpha} + K_2^X(s - s_i)^{\xi} \\
&\qquad\qquad \mathbb{P}\text{-a.s. on } \big[s < \tau^i, X_s^i(\mathbb{1})^{T_1^{X^i}} > 0\big].
\end{aligned} \tag{3.45}
$$

Using (3.45), we obtain

$$
\begin{aligned}
&\frac{1}{\psi(\mathbb{1})\varepsilon}\mathbb{E}^{\mathbb{P}}\big[\big|X_s^i(\mathbb{1})^{T_1^{X^i}} - \psi(\mathbb{1})\varepsilon\big|\big[Y_s^j(\mathbb{1})\big]^{1/2}; s < \tau^i, \\
&\qquad |y_j - x_i| < 2\big(\varepsilon^{1/2} + (t_j - s_i)^{\beta'}\big), X_s^i(\mathbb{1})^{T_1^{X^i}} > 0, t_j < T_0^{X^i}\big] \\
&\quad \lesssim \frac{\varepsilon^{\alpha^{N_0}}(s-s_i)^{\alpha} + (s-s_i)^{\xi}}{\varepsilon} \\
&\qquad \times \mathbb{E}^{\mathbb{P}}\big[\big[Y_s^j(\mathbb{1})\big]^{1/2}; s \le T_1^{Y^j}, |y_j - x_i| < 2\big(\varepsilon^{1/2} + (t_j - s_i)^{\beta'}\big), t_j < T_0^{X^i}\big],
\end{aligned}
\tag{3.46}
$$

where in the last inequality we use the component $\tau^{i,(3)}$ of τ^i in Proposition 3.3 and discard the event $\{X_s^i(\mathbb{1})^{T_1^{X^i}} > 0\}$. Applying (3.8) and (3.9) to (3.46) gives

$$
\begin{aligned}
&\frac{1}{\psi(\mathbb{1})\varepsilon}\mathbb{E}^{\mathbb{P}}\big[\big|X_s^i(\mathbb{1})^{T_1^{X^i}} - \psi(\mathbb{1})\varepsilon\big|\big[Y_s^j(\mathbb{1})\big]^{1/2}; s < \tau^i, \\
&\qquad |y_j - x_i| \le 2\big(\varepsilon^{1/2} + (t_j - s_i)^{\beta'}\big), X_s^i(\mathbb{1})^{T_1^{X^i}} > 0, t_j < T_0^{X^i}\big] \\
&\quad \lesssim \frac{\varepsilon^{\alpha^{N_0}}(s-s_i)^{\alpha} + (s-s_i)^{\xi}}{\varepsilon} \cdot \big(\varepsilon^{1/2} + (t_j - s_i)^{\beta'}\big)\mathbb{P}\big(t_j < T_0^{X^i}\big) \\
&\qquad \times \mathbb{E}^{\mathbf{P}^0_{\psi(\mathbb{1})\varepsilon}}\big[(Z_{s-t_j})^{1/2}; s - t_j \le T_1^Z\big].
\end{aligned}
\tag{3.47}
$$

We have

$$
\mathbb{P}\big(t_j < T_0^{X^i}\big) \le \frac{2\psi(\mathbb{1})\varepsilon}{t_j - s_i}
\tag{3.48}
$$

by (3.32). Applying the last display and (3.36) to the right-hand side of (3.47) then gives the desired inequality (3.43).

Step 2-4. For the third term in (3.41), the arguments step 2-3 [cf. (3.46) and (3.47)] readily give

$$
\begin{aligned}
&\frac{1}{\psi(\mathbb{1})\varepsilon} \cdot \psi(\mathbb{1})\varepsilon\mathbb{E}^{\mathbb{P}}\big[\big[Y_s^j(\mathbb{1})\big]^{1/2}; s \le T_1^{Y_j}, \\
&\qquad |y_j - x_i| \le 2\big(\varepsilon^{1/2} + (t_j - s_i)^{\beta'}\big), t_j < T_0^{X^i}\big] \\
&\quad \lesssim (t_j - s_i)^{\beta'-1}\varepsilon\mathbb{E}^{\mathbf{P}^0_{\psi(\mathbb{1})\varepsilon}}\big[(Z_{s-t_j})^{1/2}; s - t_j \le T_1^Z\big] \\
&\qquad \forall s \in (t_j, t], s_i < t_j < t.
\end{aligned}
\tag{3.49}
$$

Step 2-5. We note that in (3.42), (3.43) and (3.49), there is a common fractional moment, or more precisely

$$
\mathbb{E}^{\mathbf{P}^0_{\psi(\mathbb{1})\varepsilon}}\big[(Z_{s-t_j})^{1/2}; s - t_j \le T_1^Z\big],
\tag{3.50}
$$

left to be estimated, as will be done in this step.

Recall the filtration $(\mathscr{H}_t)$ defined in Section 3.3.

LEMMA 3.11. *Fix $z, T \in (0, \infty)$. Under the conditional probability measure $\mathbf{P}_z^{(T)}$ defined by*

$$\mathbf{P}_z^{(T)}(A) \triangleq \mathbf{P}_z^0(A|Z_T > 0), \qquad A \in \mathscr{H}_T, \tag{3.51}$$

the process $(Z_t)_{0\leq t\leq T}$ is a continuous $(\mathscr{H}_t)$-semimartingale with canonical decomposition

$$Z_t = z + \int_0^t F\left(\frac{2Z_s}{T-s}\right) ds + M_t, \qquad 0 \leq t \leq T. \tag{3.52}$$

Here, $F : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ defined by

$$F(x) \triangleq \begin{cases} \dfrac{e^{-x}x}{1-e^{-x}}, & x > 0, \\ 1, & x = 0, \end{cases} \tag{3.53}$$

is continuous and decreasing, and M is a continuous $(\mathscr{H}_t)$-martingale under $\mathbf{P}_z^{(T)}$ with quadratic variation $\langle M \rangle_t \equiv \int_0^t Z_s\, ds$.

PROOF. The proof of this lemma is a standard application of Girsanov's theorem (cf. Theorem VIII.1.4 of [23]), and we proceed as follows.

First, let $(D_t)_{0\leq t\leq T}$ denote the $(\mathscr{H}_t, \mathbf{P}_z^0)$-martingale associated with the Radon–Nikodym derivative of $\mathbf{P}_z^{(T)}$ with respect to $\mathbf{P}_z^0$, that is,

$$D_t \equiv \frac{\mathbf{P}_z^0(Z_T > 0|\mathscr{H}_t)}{\mathbf{P}_z^0(Z_T > 0)}, \qquad 0 \leq t \leq T. \tag{3.54}$$

To obtain the explicit form of D under $\mathbf{P}_z^0$, we first note that the $(\mathscr{H}_t, \mathbf{P}_z^0)$-Markov property of Z and (3.31) imply

$$\mathbf{P}_z^0(Z_T > 0|\mathscr{H}_t) = \mathbf{P}_{Z_t}^0(Z_{T-t} > 0) = 1 - \exp\left(-\frac{2Z_t}{T-t}\right), \qquad 0 \leq t < T. \tag{3.55}$$

Hence, it follows from Itô's formula and the foregoing display that, under $\mathbf{P}_z^0$,

$$\begin{aligned} D_t ={}& \frac{1}{\mathbf{P}_z^0(Z_T > 0)}\left[1 - \exp\left(-\frac{2z}{T}\right)\right] \\ &+ \frac{1}{\mathbf{P}_z^0(Z_T > 0)} \int_0^t \exp\left(-\frac{2Z_s}{T-s}\right) \cdot \left(\frac{2}{T-s}\right)\sqrt{Z_s}\, dB_s, \\ &\qquad 0 \leq t < T. \end{aligned} \tag{3.56}$$

We now apply Girsanov's theorem and verify that the components of the canonical decomposition of $(Z_t)_{0\leq t\leq T}$ under $\mathbf{P}_z^{(T)}$ satisfy the asserted properties. Under $\mathbf{P}_z^{(T)}$, we have

$$Z_t = z + \int_0^t D_s^{-1}\, d\langle D, Z\rangle_s + M_t, \qquad 0 \leq t \leq T.$$

Here,

$$M_t = \int_0^t \sqrt{Z_s}\, dB_s - \int_0^t D_s^{-1}\, d\langle D, Z\rangle_s, \qquad 0 \leq t \leq T$$

is a continuous $(\mathscr{H}_t, \mathbf{P}_z^{(T)})$-local martingale with the asserted quadratic variation $\langle M_t\rangle \equiv \int_0^t Z_s\, ds$, which implies that M is a true martingale under $\mathbf{P}_z^{(T)}$. In addition, it follows from (3.55) and (3.56) that the finite variation process of Z under $\mathbf{P}_z^{(T)}$ is given by

$$\begin{aligned}\int_0^t D_s^{-1}\, d\langle D, Z\rangle_s &= \int_0^t \frac{1}{\mathbf{P}_z^0(Z_T > 0|\mathscr{H}_s)}\, d\big\langle \mathbf{P}_z^0(Z_T > 0) D, Z\big\rangle_s \\ &= \int_0^t \frac{\exp(-2Z_s/(T-s))2Z_s/(T-s)}{1-\exp(-2Z_s/(T-s))}\, ds \\ &= \int_0^t F\left(\frac{2Z_s}{T-s}\right) ds, \qquad 0 \leq t \leq T,\end{aligned}$$

where F is given by (3.53). The proof is complete. $\square$

LEMMA 3.12. *For any $p \in (0,\infty)$, there exists a constant $K_p \in (0,\infty)$ depending only on p and (α, ξ, N_0) such that*

$$\begin{aligned}&\mathbb{E}^{\mathbf{P}_z^0}\big[(Z_T)^p; T \leq T_1^Z\big] \\ &\quad \leq K_p\big[\big(z^{p\alpha^{N_0}} T^{p\alpha} + z^p\big)\mathbf{P}_z^0(Z_T > 0) + zT^{p\xi - 1}\big], \qquad \forall z, T \in (0,1].\end{aligned} \tag{3.57}$$

PROOF. Recall the conditional probability measure $\mathbf{P}_z^{(T)}$ defined in (3.51) and write

$$\begin{aligned}\mathbb{E}^{\mathbf{P}_z^0}\big[(Z_T)^p; T \leq T_1^Z\big] &\leq \mathbf{P}_z^0(Z_T > 0)\mathbb{E}^{\mathbf{P}_z^0}\big[(Z_{T\wedge T_1^Z})^p | Z_T > 0\big] \\ &= \mathbf{P}_z^0(Z_T > 0)\mathbb{E}^{\mathbf{P}_z^{(T)}}\big[(Z_{T\wedge T_1^Z})^p\big].\end{aligned} \tag{3.58}$$

Henceforth, we work under the conditional probability measure $\mathbf{P}_z^{(T)}$.

We turn to the improved modulus of continuity of Z at its starting time 0 under $\mathbf{P}_z^{(T)}$ in order to bound the right-hand side of (3.58). We first claim that, by enlarging the underlying probability space if necessary,

$$|Z_t - z| \leq t + C_\alpha^Z \left(\int_0^t Z_s\, ds\right)^\alpha \qquad \forall t \in \big[0, T \wedge T_1^Z\big] \text{ under } \mathbf{P}_z^{(T)}, \tag{3.59}$$

where the random variable C_α^Z under $\mathbf{P}_z^{(T)}$ has distribution depending only on α and finite $\mathbf{P}_z^{(T)}$-moment of any finite order. We show how to obtain (3.59) by using the canonical decomposition of the continuous $(\mathscr{H}_t, \mathbf{P}_z^{(T)})$-semimartingale $(Z_t)_{0\le t\le T}$ in (3.52). First, since its martingale part M has quadratic variation $\int_0^\cdot Z_s\,ds$, the Dambis–Dubins–Schwarz theorem (cf. Theorem V.1.6 of [23]) implies that, by enlarging of the underlying probability space if necessary,

$$M_t = \widetilde{B}\left(\int_0^t Z_s\,ds\right), \qquad t \in [0, T \wedge T_1^Z],$$

for some standard Brownian motion $\widetilde{B}$ under $\mathbf{P}_z^{(T)}$. Here, the random clock $\int_0^t Z_s\,ds$, $t \in [0, T \wedge T_1^Z]$, for $\widetilde{B}$ is bounded by 1 by the assumption that $z, T \le 1$. On the other hand, recall that the chosen parameter α lies in $(0, \frac{1}{2})$ and the uniform Hölder-α modulus of continuity of standard Brownian motion on compacts has moments of any finite order. (See, e.g., the discussion preceding Theorem I.2.2 of [23] and its proof.) Hence,

$$\left|\widetilde{B}\left(\int_0^t Z_s\,ds\right)\right| \le C_\alpha^Z \left(\int_0^t Z_s\,ds\right)^\alpha, \qquad t \in [0, T \wedge T_1^Z],$$

where the random variable C_α^Z is as in (3.59). Second, Lemma 3.11 also states that the finite variation process of Z under $\mathbf{P}_z^{(T)}$ given by (3.52) is a time integral with integrand uniformly bounded by 1. This and the last two displays are enough to obtain our claim (3.59).

With the integral inequality (3.59) and the distributional properties of C_α^Z, we obtain the following improved modulus of continuity of Z (cf. Corollary 8.2):

$$|Z_{T\wedge T_1^Z} - z| \le K_1^Z z^{\alpha^{N_0}} T^\alpha + K_2^Z T^\xi \tag{3.60}$$

for some random variables $K_1^Z, K_2^Z \in \bigcap_{q\in(0,\infty)} L^q(\mathbf{P}_z^{(T)})$ obeying a joint law under $\mathbf{P}_z^{(T)}$ depending only on (α, ξ, N_0) by the analogous property of C_α^Z and Corollary 8.2.

We return to the calculation in (3.58). Applying (3.60), we get

$$\begin{aligned}
&\mathbb{E}^{\mathbf{P}_z^0}\left[(Z_T)^p; T \le T_1^Z\right] \\
&\quad \le \mathbf{P}_z^0(Z_T > 0)\mathbb{E}^{\mathbf{P}_z^{(T)}}\left[(Z_{T\wedge T_1^Z})^p\right] \\
&\quad \le \mathbf{P}_z^0(Z_T > 0)(2^{p-1} \vee 1)\mathbb{E}^{\mathbf{P}_z^{(T)}}\left[|Z_{T\wedge T_1^Z} - z|^p + z^p\right] \\
&\quad \le \mathbf{P}_z^0(Z_T > 0)K_p'\left(z^{p\alpha^{N_0}} T^{p\alpha} + T^{p\xi} + z^p\right)
\end{aligned} \tag{3.61}$$

for some constant K_p' depending only on p and (α, ξ, N_0) by (3.60) and the distributional properties of K_j^Z, where the second inequality follows from the elementary inequality

$$(x+y)^p \le \bigl(2^{p-1} \vee 1\bigr) \cdot \bigl(x^p + y^p\bigr) \qquad \forall x, y \in \mathbb{R}_+.$$

The desired result follows by applying (3.32) to (3.61). The proof is complete. □

Step 2-6. At this step, we summarize our results in steps 2-1–2-4, using Lemma 3.12. We apply (3.42), (3.43) and (3.49) to (3.41). This gives

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}^i}\bigl[[Y_s^j(\mathbb{1})]^{1/2}; s < \tau^i, |y_j - x_i| < 2\bigl(\varepsilon^{1/2} + (t_j - s_i)^{\beta'}\bigr)\bigr] \\
&\quad \lesssim \bigl[(t_j - s_i)^{\beta'} + (t_j - s_i)^{\beta'-1}\bigl(\varepsilon^{\alpha^{N_0}}(s - s_i)^{\alpha} + (s - s_i)^{\xi} + \varepsilon\bigr)\bigr] \\
&\qquad \times \mathbb{E}^{\mathbf{P}^0_{\psi(\mathbb{1})\varepsilon}}\bigl[(Z_{s-t_j})^{1/2}; s - t_j \le T_1^Z\bigr] \\
&\quad \lesssim \bigl[(t_j - s_i)^{\beta'} + (t_j - s_i)^{\beta'-1}\bigl(\varepsilon^{\alpha^{N_0}}(s - s_i)^{\alpha} + (s - s_i)^{\xi} + \varepsilon\bigr)\bigr] \\
&\qquad \times \bigl[\bigl(\varepsilon^{\alpha^{N_0}/2}(s - t_j)^{\alpha/2} + \varepsilon^{1/2}\bigr)\mathbf{P}^0_{\psi(\mathbb{1})\varepsilon}(Z_{s-t_j} > 0) + \varepsilon(s - t_j)^{\xi/2-1}\bigr] \\
&\quad \lesssim (t_j - s_i)^{\beta'-1} \times \bigl((t_j - s_i) + \varepsilon^{\alpha^{N_0}}(s - s_i)^{\alpha} + (s - s_i)^{\xi} + \varepsilon\bigr) \\
&\qquad \times \varepsilon^{\alpha^{N_0}/2}(s - t_j)^{\alpha/2}\mathbf{P}^0_{\psi(\mathbb{1})\varepsilon}(Z_{s-t_j} > 0) \\
&\qquad + (t_j - s_i)^{\beta'-1} \times \bigl((t_j - s_i) + \varepsilon^{\alpha^{N_0}}(s - s_i)^{\alpha} + (s - s_i)^{\xi} + \varepsilon\bigr) \\
&\qquad\quad \times \varepsilon^{1/2}\mathbf{P}^0_{\psi(\mathbb{1})\varepsilon}(Z_{s-t_j} > 0) \\
&\qquad + (t_j - s_i)^{\beta'} \times \varepsilon(s - t_j)^{\xi/2-1} \\
&\qquad + (t_j - s_i)^{\beta'-1} \times \bigl(\varepsilon^{\alpha^{N_0}}(s - s_i)^{\alpha} + \varepsilon\bigr) \times \varepsilon(s - t_j)^{\xi/2-1} \\
&\qquad + (t_j - s_i)^{\beta'-1} \times (s - s_i)^{\xi} \times \varepsilon(s - t_j)^{\xi/2-1} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall s \in (t_j, t], s_i < t_j < t,
\end{aligned} \tag{3.62}$$

where the last $\lesssim$-inequality follows by some algebra.

We make some simplifications for the right-hand side of (3.62) before going further. Some orders in ε and other variables will be discarded here. We bound the survival probability in (3.62) by

$$\mathbf{P}^0_{\psi(\mathbb{1})\varepsilon}(Z_{s-t_j} > 0) \le \left(\frac{2\psi(\mathbb{1})\varepsilon}{s - t_j}\right)^{1-\alpha^{N_0}/4}, \tag{3.63}$$

as follows from the elementary inequalities $x \le x^{\gamma}$ for any $x \in [0,1]$ and $\gamma \in (0,1]$, and then (3.32). Assuming $s \in (t_j, t]$ for $s_i < t_j < t$, we have the inequalities

$$1 \ge s - s_i \ge t_j - s_i \ge \frac{\varepsilon}{2}, \qquad s - t_j \le 1 \quad \text{and} \quad 0 < \alpha + \alpha^{N_0} < \xi < 1$$

[cf. (3.22)(a) for the third inequality]. These and (3.63) imply that the first term of (3.62) satisfies

$$
\begin{aligned}
&(t_j - s_i)^{\beta'-1} \times \big((t_j - s_i) + \varepsilon^{\alpha^{N_0}}(s - s_i)^{\alpha} + (s - s_i)^{\xi} + \varepsilon\big) \\
&\qquad \times \varepsilon^{\alpha^{N_0}/2}(s - t_j)^{\alpha/2}\mathbf{P}^0_{\psi(\mathbb{1})\varepsilon}(Z_{s-t_j} > 0) \\
&\qquad \precsim (t_j - s_i)^{\beta'-1}(s - s_i)^{\alpha}(s - t_j)^{\alpha/2+\alpha^{N_0}/4-1}\varepsilon^{1+\alpha^{N_0}/4},
\end{aligned}
\tag{3.64}
$$

the second term of (3.62) satisfies

$$
\begin{aligned}
&(t_j - s_i)^{\beta'-1} \times \big((t_j - s_i) + \varepsilon^{\alpha^{N_0}}(s - s_i)^{\alpha} + (s - s_i)^{\xi} + \varepsilon\big) \\
&\quad \times \varepsilon^{1/2}\mathbf{P}^0_{\psi(\mathbb{1})\varepsilon}(Z_{s-t_j} > 0) \\
&\quad \precsim (t_j - s_i)^{\beta'-1}(s - s_i)^{\alpha}(s - t_j)^{\alpha^{N_0}/4-1}\varepsilon^{3/2-\alpha^{N_0}/4} \\
&\quad \precsim (t_j - s_i)^{\beta'+(1/2-\alpha^{N_0}/2)-1}(s - s_i)^{\alpha}(s - t_j)^{\alpha^{N_0}/4-1}\varepsilon^{1+\alpha^{N_0}/4} \\
&\quad \precsim (t_j - s_i)^{\beta'+\alpha/2-1}(s - s_i)^{\alpha}(s - t_j)^{\alpha^{N_0}/4-1}\varepsilon^{1+\alpha^{N_0}/4},
\end{aligned}
\tag{3.65}
$$

and, finally, the fourth term of (3.62) satisfies

$$
\begin{aligned}
&(t_j - s_i)^{\beta'-1} \times \big(\varepsilon^{\alpha^{N_0}}(s - s_i)^{\alpha} + \varepsilon\big) \times \varepsilon(s - t_j)^{\xi/2-1} \\
&\qquad \precsim (t_j - s_i)^{\beta'-1}(s - s_i)^{\alpha}(s - t_j)^{\xi/2-1}\varepsilon^{1+\alpha^{N_0}} \\
&\qquad \precsim (t_j - s_i)^{\beta'-1}(s - s_i)^{\alpha}(s - t_j)^{\alpha/2+\alpha^{N_0}/4-1}\varepsilon^{1+\alpha^{N_0}/4}.
\end{aligned}
\tag{3.66}
$$

Note that the bounds in (3.64) and (3.66) coincide. Using (3.64)–(3.66) in (3.62), we obtain

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}^i}\big[[Y^j_s(\mathbb{1})]^{1/2}; s < \tau^i, |y_j - x_i| \le 2(\varepsilon^{1/2} + (t_j - s_i)^{\beta'})\big] \\
&\quad \precsim (t_j - s_i)^{\beta'-1}(s - s_i)^{\alpha}(s - t_j)^{\alpha/2+\alpha^{N_0}/4-1}\varepsilon^{1+\alpha^{N_0}/4} \\
&\qquad + (t_j - s_i)^{\beta'+\alpha/2-1}(s - s_i)^{\alpha}(s - t_j)^{\alpha^{N_0}/4-1}\varepsilon^{1+\alpha^{N_0}/4} \\
&\qquad + (t_j - s_i)^{\beta'}(s - t_j)^{\xi/2-1}\varepsilon \\
&\qquad + (t_j - s_i)^{\beta'-1}(s - s_i)^{\xi}(s - t_j)^{\xi/2-1}\varepsilon \qquad \forall s \in (t_j, t], s_i < t_j < t.
\end{aligned}
\tag{3.67}
$$

Step 3. We digress to a *conceptual* discussion for some elementary integrals which will play an important role in the forthcoming calculations in step 4. First, for $a, b, c \in \mathbb{R}$ and $T \in (0, \infty)$, a straightforward application of Fubini's theorem and changes of variables shows that

$$
\begin{aligned}
&I(a, b, c)_T \triangleq \int_0^T dr\, r^a \int_r^T ds\, s^b (s - r)^c < \infty \\
&\quad \Longleftrightarrow \quad a, c \in (-1, \infty) \quad \text{and} \quad a + b + c > -2.
\end{aligned}
\tag{3.68}
$$

Furthermore, when $I(a,b,c)_T$ is finite, it can be expressed as

$$I(a,b,c)_T = \left(\int_0^1 dr\, r^a(1-r)^c\right) \cdot \frac{T^{a+b+c+2}}{a+b+c+2}.$$

Given $a+b+c>-2$ with $a,c\in(-1,\infty)$, we consider alternative ways to show that the integral $I(a,b,c)_T$ is finite while preserving the same order $T^{a+b+c+2}$ in T, according to $b\geq 0$ and $b<0$. If $b\geq 0$, then

$$\begin{aligned} I(a,b,c)_T &\leq \int_0^T dr\, r^a \times T^b \times \int_0^T ds\, s^c \\ &= \frac{1}{a+1}\frac{1}{c+1}T^{a+b+c+2}, \end{aligned} \tag{3.69}$$

where the first inequality follows since $s^b \leq T^b$ for any $s\in[r,T]$. For the case that $b<0$, we decompose the function $s\longmapsto s^b$ in the following way. For $b_1,b_2<0$ such that $b_1+b_2=b$, we have

$$\begin{aligned} I(a,b,c)_T &\leq \int_0^T dr\, r^{a+b_1}\int_r^T ds(s-r)^{b_2+c} \\ &\leq \int_0^T dr\, r^{a+b_1} \times \int_0^T ds\, s^{b_2+c}, \end{aligned} \tag{3.70}$$

where the first inequality follows since for $s>r$, $s^{b_1}\leq r^{b_1}$ and $s^{b_2}\leq (s-r)^{b_2}$. Using the following elementary lemma, we obtain from (3.70) that

$$I(a,b,c)_T \leq \frac{1}{a+b_1+1}\frac{1}{b_2+c+1}T^{a+b+c+2}.$$

LEMMA 3.13. *For any reals $a,c>-1$ and $b<0$ such that $a+b+c>-2$, there exists a pair $(b_1,b_2)\in(-\infty,0)\times(-\infty,0)$ such that $b=b_1+b_2$ and $a+b_1>-1$ and $b_2+c>-1$.*

The two simple *concepts* for the inequalities (3.69) and (3.70) will be applied later on in step 4 to bound *Riemann sums* by integrals of the type $I(a,b,c)_T$.

Step 4. We complete the proof of Lemma 3.9 in this step. Apply the bound (3.67) to the right-hand side of the inequality (3.34). We have

$$\begin{aligned} &\mathbb{E}^{\mathbb{Q}^i}\Bigg[\sum_{j\in\mathcal{C}^i_{\beta'}(t\wedge\tau^i\wedge\sigma_\beta^{X^i})}\Bigg(\psi(\mathbb{1})\varepsilon \\ &\qquad + \int_{t_j}^{t\wedge\tau^i\wedge\sigma_\beta^{X^i}\wedge\sigma_\beta^{Y^j}} \frac{1}{X^i_s(\mathbb{1})}\int_{\mathbb{R}} X^i(x,s)^{1/2}Y^j(x,s)^{1/2}\,dx\,ds\Bigg)\Bigg] \end{aligned} \tag{3.71}$$

$$
\begin{aligned}
&\lesssim \sum_{j:s_i<t_j\le t}(t_j-s_i)^{\beta'}\varepsilon \\
&\quad+\sum_{j:s_i<t_j\le t}(t_j-s_i)^{\beta'-1}\int_{t_j}^{t}(s-s_i)^{-\eta/2+\alpha}(s-t_j)^{\alpha/2+\alpha^{N_0}/4-1}\,ds \\
&\quad\times\varepsilon^{1+\alpha^{N_0}/4} \\
&\quad+\sum_{j:s_i<t_j\le t}(t_j-s_i)^{\beta'+\alpha/2-1}\int_{t_j}^{t}(s-s_i)^{-\eta/2+\alpha}(s-t_j)^{\alpha^{N_0}/4-1}\,ds \\
&\quad\times\varepsilon^{1+\alpha^{N_0}/4} \\
&\quad+\sum_{j:s_i<t_j\le t}(t_j-s_i)^{\beta'}\int_{t_j}^{t}(s-s_i)^{-\eta/2}(s-t_j)^{\xi/2-1}\,ds\cdot\varepsilon \\
&\quad+\sum_{j:s_i<t_j\le t}(t_j-s_i)^{\beta'-1}\int_{t_j}^{t}(s-s_i)^{-\eta/2+\xi}(s-t_j)^{\xi/2-1}\,ds\cdot\varepsilon.
\end{aligned}
$$

Recall the notation $I(a,b,c)$ in (3.68). It should be clear that, up to a translation of time by s_i, the first, the fourth, and the fifth sums are Riemann sums of

$$
I(\beta',0,0)_{t-s_i},\qquad I\biggl(\beta',-\frac{\eta}{2},\frac{\xi}{2}-1\biggr)_{t-s_i},\qquad I\biggl(\beta'-1,-\frac{\eta}{2}+\xi,\frac{\xi}{2}-1\biggr)_{t-s_i},
$$

respectively, and so are the second and the third sums after a division by $\varepsilon^{\alpha^{N_0}/4}$ with the corresponding integrals equal to

$$
\begin{aligned}
&I\biggl(\beta'-1,-\frac{\eta}{2}+\alpha,\frac{\alpha}{2}+\frac{\alpha^{N_0}}{4}-1\biggr)_{t-s_i},\\
&I\biggl(\beta'+\frac{\alpha}{2}-1,-\frac{\eta}{2}+\alpha,\frac{\alpha^{N_0}}{4}-1\biggr)_{t-s_i},
\end{aligned}
$$

respectively. It follows from (3.22)(c) and (d) and (3.68) that all of the integrals in the last two displays are finite.

We now aim to bound each of the five sums in (3.71) by suitable powers of ε and t, using integral comparisons. Observe that, whenever $\gamma\in(-1,\infty)$, the monotonicity of $r\longmapsto(r-s_i)^{\gamma}$ over (s_i,∞) implies

$$
\begin{aligned}
\sum_{j:s_i<t_j\le t}(t_j-s_i)^{\gamma}\cdot\varepsilon &\le 2\int_{s_i}^{t+\varepsilon}(r-s_i)^{\gamma}\,dr \\
&=\frac{2}{\gamma+1}(t+\varepsilon-s_i)^{\gamma+1} \qquad (3.72)\\
&\le\frac{2\cdot 3^{\gamma+1}}{\gamma+1}(t-s_i)^{\gamma+1}
\end{aligned}
$$

since $t \geq s_i + \frac{\varepsilon}{2}$. (The constant 2 is used to accommodate the case that $\gamma < 0$.) Hence, the first sum in (3.71) can be bounded as

$$\sum_{j\,:\,s_i<t_j\leq t} (t_j - s_i)^{\beta'} \varepsilon \precsim (t - s_i)^{\beta'+1}. \tag{3.73}$$

Consider the other sums in (3.71). Recall our discussion of some alternative ways to bound $I(a, b, c)$ for given $a + b + c > -2$ and $a, c, \in (-1, \infty)$ according to $b \geq 0$ or $b < 0$; see (3.69) and (3.70). We use Lemma 3.13 in the following whenever necessary. The second sum in (3.71) can be bounded as

$$\begin{aligned}
&\sum_{j\,:\,s_i<t_j\leq t} (t_j - s_i)^{\beta'-1} \int_{t_j}^{t} (s - s_i)^{-\eta/2+\alpha} (s - t_j)^{\alpha/2+\alpha^{N_0}/4-1}\, ds \cdot \varepsilon^{1+\alpha^{N_0}/4} \\
&\quad = \sum_{j\,:\,s_i<t_j\leq t} (t_j - s_i)^{\beta'-1} \int_{t_j-s_i}^{t-s_i} s^{-\eta/2+\alpha} \big[s - (t_j - s_i)\big]^{\alpha/2+\alpha^{N_0}/4-1}\, ds \\
&\qquad \times \varepsilon^{1+\alpha^{N_0}/4} \\
&\quad \precsim (t - s_i)^{\beta'-\eta/2+(3\alpha)/2+\alpha^{N_0}/4} \cdot \varepsilon^{\alpha^{N_0}/4} \\
&\quad \precsim (t - s_i)^{\beta'-\eta/2+(3\alpha)/2} \cdot \varepsilon^{\alpha^{N_0}/4}.
\end{aligned} \tag{3.74}$$

Here, in the foregoing $\precsim$-inequality, we use the integral comparison discussed in step 3 (with Lemma 3.13 to algebraically allocate the exponent $-\frac{\eta}{2} + \frac{\alpha}{2}$ if necessary) and the Riemman-sum bound (3.72). The other sums on the right-hand side of (3.71) can be bounded similarly as follows. The third sum satisfies

$$\begin{aligned}
&\sum_{j\,:\,s_i<t_j\leq t} (t_j - s_i)^{\beta'+\alpha/2-1} \int_{t_j}^{t} (s - s_i)^{-\eta/2+\alpha} (s - t_j)^{\alpha^{N_0}/4-1}\, ds \cdot \varepsilon^{1+\alpha^{N_0}/4} \\
&\quad \precsim (t - s_i)^{\beta'-\eta/2+(3\alpha)/2} \cdot \varepsilon^{\alpha^{N_0}/4}.
\end{aligned} \tag{3.75}$$

The fourth sum satisfies

$$\begin{aligned}
&\sum_{j\,:\,s_i<t_j\leq t} (t_j - s_i)^{\beta'} \int_{t_j}^{t} (s - s_i)^{-\eta/2} (s - t_j)^{\xi/2-1}\, ds \cdot \varepsilon \\
&\quad \precsim (t - s_i)^{\beta'-\eta/2+\xi/2+1} \precsim (t - s_i)^{\beta'-\eta/2+(3\xi)/2},
\end{aligned} \tag{3.76}$$

where the last inequality applies since $\xi \in (0, 1)$. The last sum satisfies

$$\begin{aligned}
&\sum_{j\,:\,s_i<t_j\leq t} (t_j - s_i)^{\beta'-1} \int_{t_j}^{t} (s - s_i)^{-\eta/2+\xi} (s - t_j)^{\xi/2-1}\, ds \cdot \varepsilon \\
&\quad \precsim (t - s_i)^{\beta'-\eta/2+(3\xi)/2}.
\end{aligned} \tag{3.77}$$

The proof of Lemma 3.9 is complete upon applying (3.73)–(3.77) to the right-hand side of (3.71).

3.5. *Proof of Lemma* 3.10. As in Section 3.4, we fix $t \in [s_i + \frac{\varepsilon}{2}, s_i + 1]$, $i \in \mathbb{N}$, and $\varepsilon \in (0, [8\psi(\mathbb{1})]^{-1} \wedge 1]$ and drop the subscripts of $\mathbb{P}_\varepsilon$ and $\mathbb{Q}^i_\varepsilon$. For the proof of Lemma 3.10, the arguments in Section 3.4 work essentially. Now, we begin to use the condition (3.22)(b) in Assumption 3.4 and the upper limit $\sigma_\beta^{X^i} \wedge \sigma_\beta^{Y^j}$ in the time integral in (3.30), which are neglected when we prove Lemma 3.9.

To motivate our adaptation of the arguments for critical clusters in Section 3.4, we discuss some parts of Section 3.4. First, it is straightforward to modify the proof of (3.35) and obtain

$$\mathbb{Q}^i\big(2\big(\varepsilon^{1/2} + (t_j - s_i)^{\beta'}\big) \le |y_j - x_i| \le 2\big(\varepsilon^{1/2} + (t - s_i)^{\beta'}\big)\big) \precsim (t - s_i)^{\beta'}. \tag{3.78}$$

If we proceed as in (3.34) and use (3.78) in the obvious way, then this leads to

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}^i}\Bigg[\sum_{j \in \mathcal{L}^i_{\beta'}(t, t\wedge\tau^i\wedge\sigma_\beta^{X^i})}\Bigg(\psi(\mathbb{1})\varepsilon \\
&\qquad\qquad + \int_{t_j}^{t\wedge\tau^i\wedge\sigma_\beta^{X^i}\wedge\sigma_\beta^{Y^j}} \frac{1}{X^i_s(\mathbb{1})}\int_{\mathbb{R}} X^i(x,s)^{1/2} Y^j(x,s)^{1/2}\,dx\,ds\Bigg)\Bigg] \\
&\quad\precsim \sum_{j : s_i < t_j \le t} (t - s_i)^{\beta'}\varepsilon \\
&\qquad + \sum_{j : s_i < t_j \le t} \int_{t_j}^{t} ds \frac{1}{(s - s_i)^{\eta/2}} \mathbb{E}^{\mathbb{Q}^i}\big[[Y^j_s(\mathbb{1})]^{1/2}; s < \tau^i, 2\big(\varepsilon^{1/2} + (t_j - s_i)^{\beta'}\big) \\
&\qquad\qquad \le |y_j - x_i| \le 2\big(\varepsilon^{1/2} + (t - s_i)^{\beta'}\big)\big].
\end{aligned}$$

[Compare this with (3.34) for critical clusters.] If we argue by using (3.78) repeatedly in the steps analogous to steps 2-2–2-4 of Section 3.4, then we obtain the following $\precsim$-inequality similar to (3.71):

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}^i}\Bigg[\sum_{j \in \mathcal{L}^i_{\beta'}(t, t\wedge\tau^i\wedge\sigma_\beta^{X^i})}\Bigg(\psi(\mathbb{1})\varepsilon \\
&\qquad\qquad + \int_{t_j}^{t\wedge\tau^i\wedge\sigma_\beta^{X^i}\wedge\sigma_\beta^{Y^j}} \frac{1}{X^i_s(\mathbb{1})}\int_{\mathbb{R}} X^i(x,s)^{1/2} Y^j(x,s)^{1/2}\,dx\,ds\Bigg)\Bigg] \\
&\quad\precsim \sum_{j : s_i < t_j \le t} (t - s_i)^{\beta'}\varepsilon \\
&\qquad + \sum_{j : s_i < t_j \le t} (t - s_i)^{\beta'} (t_j - s_i)^{-1} \int_{t_j}^{t} (s - s_i)^{-\eta/2+\alpha} (s - t_j)^{\alpha/2 + \alpha^{N_0}/4 - 1}\,ds \\
&\qquad \times \varepsilon^{1+\alpha^{N_0}/4}
\end{aligned} \tag{3.79}$$

$$+\sum_{j:s_i<t_j\le t}(t-s_i)^{\beta'}(t_j-s_i)^{\alpha/2-1}\int_{t_j}^{t}(s-s_i)^{-\eta/2+\alpha}(s-t_j)^{\alpha^{N_0}/4-1}\,ds$$
$$\times\,\varepsilon^{1+\alpha^{N_0}/4}$$
$$+\sum_{j:s_i<t_j\le t}(t-s_i)^{\beta'}\int_{t_j}^{t}(s-s_i)^{-\eta/2}(s-t_j)^{\xi/2-1}\,ds\cdot\varepsilon$$
$$+\sum_{j:s_i<t_j\le t}(t-s_i)^{\beta'}(t_j-s_i)^{-1}\int_{t_j}^{t}(s-s_i)^{-\eta/2+\xi}(s-t_j)^{\xi/2-1}\,ds\cdot\varepsilon,$$

taking into account some simplifications similar to (3.64)–(3.66) where some orders are discarded. (We omit the derivation of the foregoing display, as it will not be used for the proof of Lemma 3.10.) In other words, replacing the factor $(t_j-s_i)^{\beta'}$ for each of the sums in (3.71) by $(t-s_i)^{\beta'}$ gives the bound in the foregoing display. Applying integral domination to the second and the last sums of the foregoing display as in step 4 of Section 3.5 results in bounds which are *divergent* integrals.

Examining the arguments in steps 2-2–2-4 of Section 3.4 shows that the problematic factor

$$(t_j-s_i)^{-1} \tag{3.80}$$

in (3.79) results from using the bound (3.48) for the survival probability $\mathbb{P}(t_j<T_0^{X^i})$. The exponent -1 in the foregoing display, however, is critical, and any decrease in this value will lead to convergent integrals. Also, we recall that (3.9) is used repeatedly in steps 2-2–2-4 of Section 3.4, while (3.9) is a consequence of (3.8) and the proof of (3.8) uses in particular the Markov property of $Y^j(\mathbb{1})$ at t_j. These observations suggest that we should modify the arguments in Section 3.4 by replacing t_j with a "larger" value, subject to the condition that certain $\mathbb{P}$-independence, similar to (3.8) with t_j replaced by the resulting value, still holds.

First, let us identify the value to replace t_j. The idea comes from the following observation.

OBSERVATION. It takes a positive amount of time before the support process of a lateral cluster Y^j intersects the support process of X^i, as leads to a time t_j^{c} *larger* than the landing time t_j of Y^j. Prior to t_j^{c}, the supports of X^i and Y^j are disjoint. See Figure 2.

We formalize the definition of this time t_j^{c} as follows. Let $j\in\mathbb{N}$ with $t_j\in(s_i,s_i+1]$. Recall that the range for the possible values y of y_j associated with a lateral cluster is

$$2\big(\varepsilon^{1/2}+(t_j-s_i)^{\beta'}\big)\le|y-x_i|\le 2\big(\varepsilon^{1/2}+(t-s_i)^{\beta'}\big), \tag{3.81}$$

and we use $\mathcal{P}_\beta^{X^i}(\cdot)$ and $\mathcal{P}_\beta^{Y^j}(\cdot)$ to envelop the support processes of X^i and Y^j, respectively. Let the processes of parabolas $\{\mathcal{P}_\beta^{X^i}(t); t \in [s_i, \infty)\}$ and $\{\mathcal{P}_\beta^{Y^j}(t); t \in [t_j, \infty)\}$ evolve in the deterministic way, and consider the *support contact time* $t_j^{\mathrm{c}}(y_j)$, that is, the first time t when $\mathcal{P}_\beta^{X^i}(t)$ and $\mathcal{P}_\beta^{Y^j}(t)$ intersect. Here, for any y satisfying (3.81), $t_j^{\mathrm{c}}(y) \in (t_j, \infty)$ solves

$$\begin{cases} x_i + \varepsilon^{1/2} + \big(t_j^{\mathrm{c}}(y) - s_i\big)^\beta = y - \varepsilon^{1/2} - \big(t_j^{\mathrm{c}}(y) - t_j\big)^\beta, & \text{if } y > x_i, \\ x_i - \varepsilon^{1/2} - \big(t_j^{\mathrm{c}}(y) - s_i\big)^\beta = y + \varepsilon^{1/2} + \big(t_j^{\mathrm{c}}(y) - t_j\big)^\beta, & \text{if } y < x_i. \end{cases} \tag{3.82}$$

By simple arithmetic, we see that the minimum of $t_j^{\mathrm{c}}(y)$ for y satisfying (3.81) is attained at the boundary cases where y satisfies $2(\varepsilon^{1/2} + (t_j - s_i)^{\beta'}) = |y - x_i|$. Let us consider the *worst* case of the support contact time as

$$t_j^\star \triangleq \min\big\{t_j^{\mathrm{c}}(y); y \text{ satisfies (3.81)}\big\}. \tag{3.83}$$

Recall that $\beta' < \beta$ by (3.22)(b).

LEMMA 3.14. *Let $j \in \mathbb{N}$ with $t_j \in (s_i, s_i + 1]$.*

(1) *The number $t_j^\star$ defined by* (3.83) *satisfies*

$$t_j^\star = s_i + A(t_j - s_i) \cdot (t_j - s_i)^{\beta'/\beta}, \tag{3.84}$$

where $A(r)$ is the unique number in $(r^{1-\beta'/\beta}, \infty)$ solving

$$A(r)^\beta + \big[A(r) - r^{1-\beta'/\beta}\big]^\beta = 2, \qquad r \in (0, 1]. \tag{3.85}$$

(2) *The function $A(\cdot)$ defined by* (3.85) *satisfies*

$$1 \le A(r) \le 1 + r^{1-\beta'/\beta} \qquad \forall r \in (0, 1]. \tag{3.86}$$

PROOF. Without loss of generality, we may assume that $t_j^\star = t_j^{\mathrm{c}}(y)$ for y satisfying

$$x_i - y = 2\big(\varepsilon^{1/2} + (t_j - s_i)^{\beta'}\big).$$

Using this particular value y of y_j in (3.82), we see that $t_j^\star$ solves the equation

$$\begin{aligned} x_i - \varepsilon^{1/2} - \big(t_j^\star - s_i\big)^\beta &= y + \varepsilon^{1/2} + \big(t_j^\star - t_j\big)^\beta \\ &= x_i - \varepsilon^{1/2} - 2(t_j - s_i)^{\beta'} + \big(t_j^\star - t_j\big)^\beta. \end{aligned}$$

Taking $t_j^\star = s_i + A \cdot (t_j - s_i)^{\beta'/\beta}$ for some constant $A \in (0, \infty)$ left to be determined, we obtain from the foregoing equality that

$$\begin{aligned} 2(t_j - s_i)^{\beta'} &= A^\beta \cdot (t_j - s_i)^{\beta'} + \big[A \cdot (t_j - s_i)^{\beta'/\beta} - (t_j - s_i)\big]^\beta \\ &= A^\beta \cdot (t_j - s_i)^{\beta'} + \big[A - (t_j - s_i)^{1-\beta'/\beta}\big]^\beta \cdot (t_j - s_i)^{\beta'}, \end{aligned}$$

which shows that $A = A(t_j - s_i)$ for $A(\cdot)$ defined by (3.85) upon cancelling $(t_j - s_i)^{\beta'}$ on both sides. We have obtained (1).

From the definition (3.85) of $A(\cdot)$, we obtain

$$2A(r)^{\beta} \geq A(r)^{\beta} + \big[A(r) - r^{1-\beta'/\beta}\big]^{\beta} = 2,$$
$$2\big[A(r) - r^{1-\beta'/\beta}\big]^{\beta} \leq A(r)^{\beta} + \big[A(r) - r^{1-\beta'/\beta}\big]^{\beta} = 2,$$

and both inequalities in (3.86) follow. The proof is complete. □

As a result of Lemma 3.14, we have

$$\mathbb{P}\big(t_j^{\star} < T_0^{X^i}\big) \lesssim \varepsilon (t_j - s_i)^{-\beta'/\beta}, \tag{3.87}$$

where the exponent $-\frac{\beta'}{\beta}$ is an improvement in terms of our preceding discussion about the factor (3.80). The value $t_j^{\star}$ will serve as the desired replacement of t_j.

Let us show how $t_j^{\star}$ still allows some independence similar to (3.8).

LEMMA 3.15 (Orthogonal continuation). *Let $(\mathscr{H}_t)$ be a filtration satisfying the usual conditions, and U and V be two $(\mathscr{H}_t)$-Feller diffusions such that $U_0 \perp\!\!\!\perp V_0$ and, for some $(\mathscr{H}_t)$-stopping $\sigma^{\perp}$, $\langle U, V\rangle^{\sigma^{\perp}} \equiv 0$. Then by enlarging the underlying filtered probability space if necessary and writing again $(\mathscr{H}_t)$ for the resulting filtration with a slight abuse of notation in this case, we can find a $(\mathscr{H}_t)$-Feller diffusion $\widehat{U}$ such that $\widehat{U} \perp\!\!\!\perp V$ and $\widehat{U} = U$ over* $[0, \sigma^{\perp}]$.

PROOF. We only give a sketch of the proof here, and leave the details, calling for standard arguments, to the readers. Using Lévy's theorem, we can define a Brownian motion $\widehat{B}$ by

$$\widehat{B}_t = \int_0^{T_0^U \wedge \sigma^{\perp} \wedge t} \frac{1}{\sqrt{U_s}}\, dU_s + \int_0^t \mathbb{1}_{\{T_0^U \wedge \sigma^{\perp} < s\}}\, dB_s,$$

for some Brownian motion B independent of (U, V). We can use $\widehat{B}$ to solve for a Feller diffusion $\widehat{U}$ with initial value U_0. Then the proof of pathwise uniqueness for Feller diffusions (cf. [27]) gives $\widehat{U} = U$ on $[0, \sigma^{\perp}]$. Note that $\langle \widehat{U}, V\rangle \equiv 0$, and consider the martingale problem associated with a two-dimensional independent Feller diffusions with initial values U_0 and V_0. By its uniqueness, $\widehat{U} \perp\!\!\!\perp V$. Hence, $\widehat{U}$ is the desired continuation of U beyond $\sigma^{\perp}$. □

We apply Lemma 3.15 to the total mass processes $X^i(\mathbb{1})$ and $Y^j(\mathbb{1})$ under $\mathbb{P}$ and prove the following analogue of (3.8).

PROPOSITION 3.16. *Let $i, j \in \mathbb{N}$ be given so that $s_i < t_j$. Suppose that $\sigma^{\perp}$ is a $(\mathscr{G}_t)$-stopping time such that $\sigma^{\perp} \geq t_j$ and $\langle X^i(\mathbb{1}), Y^j(\mathbb{1})\rangle^{\sigma^{\perp}} \equiv 0$. Then for*

$r_2 > r_1 \geq t_j$ and nonnegative Borel measurable functions H_1, H_2 and h,

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\big[H_1\big(Y_r^j(\mathbb{1}); r \in [t_j, r_2]\big) H_2\big(X_r^i(\mathbb{1}); r \in [s_i, r_1]\big) h(y_j, x_i); r_1 \leq \sigma^{\perp}\big] \\
&\qquad \leq \mathbb{E}^{\mathbb{P}}\big[H_1\big(Y_r^j(\mathbb{1}); r \in [t_j, r_2]\big)\big] \times \mathbb{E}^{\mathbb{P}}\big[H_2\big(X_r^i(\mathbb{1}); r \in [s_i, r_1]\big)\big] \\
&\qquad\quad \times \mathbb{E}^{\mathbb{P}}\big[h(y_j, x_i)\big].
\end{aligned}
\tag{3.88}
$$

PROOF. By the monotone class theorem, we may only consider the case that

$$
\begin{aligned}
H_1\big(Y_r^j(\mathbb{1}); r \in [t_j, r_2]\big) &= H_{1,1}\big(Y_r^j(\mathbb{1}); r \in [t_j, r_1]\big) H_{1,2}\big(Y_r^j(\mathbb{1}); r \in [r_1, r_2]\big), \\
H_2\big(X_r^i(\mathbb{1}); r \in [s_i, r_1]\big) &= H_{2,1}\big(X_r^i(\mathbb{1}); r \in [s_i, t_j]\big) H_{2,2}\big(X_r^i(\mathbb{1}); r \in [t_j, r_1]\big),
\end{aligned}
$$

for nonnegative Borel measurable functions $H_{k,\ell}$.

As the first step, we condition on $\mathscr{G}_{r_1}$ and obtain

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\big[H_1\big(Y_r^j(\mathbb{1}); r \in [t_j, r_2]\big) H_2\big(X_r^i(\mathbb{1}); r \in [s_i, r_1]\big) h(y_j, x_i); r_1 \leq \sigma^{\perp}\big] \\
&\qquad = \mathbb{E}^{\mathbb{P}}\big[H_{1,1}\big(Y_r^j(\mathbb{1}); r \in [t_j, r_1]\big) \mathbb{E}^{\mathbb{P}}\big[H_{1,2}\big(Y_r^j(\mathbb{1}); r \in [r_1, r_2]\big) | \mathscr{G}_{r_1}\big] \\
&\qquad\qquad \times H_2\big(X_r^i(\mathbb{1}); r \in [s_i, r_1]\big) h(y_j, x_i); r_1 \leq \sigma^{\perp}\big].
\end{aligned}
\tag{3.89}
$$

Since $Y^j(\mathbb{1})$ is a $(\mathscr{G}_t)$-Feller diffusion, we know that

$$
\mathbb{E}^{\mathbb{P}}\big[H_{1,2}\big(Y_r^j(\mathbb{1}); r \in [r_1, r_2]\big) | \mathscr{G}_{r_1}\big] = \widehat{H}_{1,2}\big(Y_{r_1}^j(\mathbb{1})\big)
\tag{3.90}
$$

for some nonnegative Borel measurable function $\widehat{H}_{1,2}$. Hence, from (3.89), we get

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\big[H_1\big(Y_r^j(\mathbb{1}); r \in [t_j, r_2]\big) H_2\big(X_r^i(\mathbb{1}); r \in [s_i, r_1]\big) h(y_j, x_i); r_1 \leq \sigma^{\perp}\big] \\
&\qquad = \mathbb{E}^{\mathbb{P}}\big[H_{1,1}\big(Y_r^j(\mathbb{1}); r \in [t_j, r_1]\big) \widehat{H}_{1,2}\big(Y_{r_1}^j(\mathbb{1})\big) \\
&\qquad\qquad \times H_2\big(X_r^i(\mathbb{1}); r \in [s_i, r_1]\big) h(y_j, x_i); r_1 \leq \sigma^{\perp}\big].
\end{aligned}
\tag{3.91}
$$

Next, since $Y_{t_j}^j(\mathbb{1}) \equiv \psi(\mathbb{1})\varepsilon$ is obviously $\mathbb{P}$-independent of $X_{t_j}^i(\mathbb{1})$ and $\sigma^{\perp} \geq t_j$ by assumption, we can do an orthogonal continuation of $X^i(\mathbb{1})$ over $[\sigma^{\perp}, \infty)$ by Lemma 3.15. This gives a Feller diffusion $\widehat{X}^i$ such that $\widehat{X}^i \perp\!\!\!\perp Y^j(\mathbb{1})$ under $\mathbb{P}$ and $\widehat{X}^{i,\sigma^{\perp}} = X^i(\mathbb{1})^{\sigma^{\perp}}$. Hence,

$$
X^i(\mathbb{1}) = \widehat{X}^i \qquad \text{over } [s_i, r_1] \text{ on } \{r_1 \leq \sigma^{\perp}\}
$$

and from (3.91) we get

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\big[H_1\big(Y_r^j(\mathbb{1}); r \in [t_j, r_2]\big) H_2\big(X_r^i(\mathbb{1}); r \in [s_i, r_1]\big) h(y_j, x_i); r_1 \leq \sigma^{\perp}\big] \\
&\qquad = \mathbb{E}^{\mathbb{P}}\big[H_{1,1}\big(Y_r^j(\mathbb{1}); r \in [t_j, r_1]\big) \widehat{H}_{1,2}\big(Y_{r_1}^j(\mathbb{1})\big) \\
&\qquad\qquad \times H_2\big(\widehat{X}_r^i; r \in [s_i, r_1]\big) h(y_j, x_i); r_1 \leq \sigma^{\perp}\big] \\
&\qquad \leq \mathbb{E}^{\mathbb{P}}\big[H_{1,1}\big(Y_r^j(\mathbb{1}); r \in [t_j, r_1]\big) \widehat{H}_{1,2}\big(Y_{r_1}^j(\mathbb{1})\big) \\
&\qquad\qquad \times H_2\big(\widehat{X}_r^i; r \in [s_i, r_1]\big) h(y_j, x_i)\big],
\end{aligned}
\tag{3.92}
$$

where the last inequality follows from the nonnegativity of $\widehat{H}_{1,2}$, $H_{k,\ell}$ and h.

Next, we condition on $\mathscr{G}_{t_j}$. From (3.92), we get

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\big[H_1\big(Y^j_r(\mathbb{1}); r\in[t_j,r_2]\big)H_2\big(X^i_r(\mathbb{1}); r\in[s_i,r_1]\big)h(y_j,x_i); r_1\leq\sigma^{\perp}\big]\\
&\quad\leq \mathbb{E}^{\mathbb{P}}\big[\mathbb{E}^{\mathbb{P}}\big[H_{1,1}\big(Y^j_r(\mathbb{1}); r\in[t_j,r_1]\big)\widehat{H}_{1,2}\big(Y^j_{r_1}(\mathbb{1})\big)\\
&\qquad\qquad\qquad\qquad \times H_{2,2}\big(\widehat{X}^i_r; r\in[t_j,r_1]\big)|\mathscr{G}_{t_j}\big]\\
&\qquad\qquad\qquad \times H_{2,1}\big(\widehat{X}^i_r; r\in[s_i,t_j]\big)h(y_j,x_i)\big].
\end{aligned}\tag{3.93}$$

To evaluate the conditional expectation in the last term, we use the independence between $\widehat{X}^i$ and $Y^j(\mathbb{1})$ and deduce from the martingale problem formulation and Theorem 4.4.2 of [10] that the two-dimensional process $(\widehat{X}^i, Y^j(\mathbb{1}))\restriction[t_j,\infty)$ is $(\mathscr{G}_t)_{t\geq t_j}$-Markov with joint law

$$\mathscr{L}\big(\widehat{X}^i\restriction[t_j,\infty)\big)\otimes\mathscr{L}\big(Y^j(\mathbb{1})\restriction[t_j,\infty)\big).$$

Hence,

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\big[H_{1,1}\big(Y^j_r(\mathbb{1}); r\in[t_j,r_1]\big)\widehat{H}_{1,2}\big(Y^j_{r_1}(\mathbb{1})\big)H_{2,2}\big(\widehat{X}^i_r; r\in[t_j,r_1]\big)|\mathscr{G}_{t_j}\big]\\
&\quad=\mathbb{E}^{\mathbb{P}}\big[H_{1,1}\big(Y^j_r(\mathbb{1}); r\in[t_j,r_1]\big)\widehat{H}_{1,2}\big(Y^j_{r_1}(\mathbb{1})\big)\big]\\
&\qquad\times\mathbb{E}^{\mathbf{P}^0_{\widehat{X}^i_{t_j}}}\big[H_{2,2}\big(Z_r; r\in[0,r_1-t_j]\big)\big],
\end{aligned}$$

where we recall that $(Z,\mathbf{P}^0_z)$ denotes a copy of $\frac{1}{4}\mathrm{BES}Q^0(4z)$. [The value of $Y^j(\mathbb{1})$ at t_j is $\psi(\mathbb{1})\varepsilon$.] Applying the foregoing equality to (3.93) and using (3.90), we obtain

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\big[H_1\big(Y^j_r(\mathbb{1}); r\in[t_j,r_2]\big)H_2\big(X^i_r(\mathbb{1}); r\in[s_i,r_1]\big)h(y_j,x_i); r_1\leq\sigma^{\perp}\big]\\
&\quad\leq\mathbb{E}^{\mathbb{P}}\big[H_1\big(Y^j_r(\mathbb{1}); r\in[t_j,r_2]\big)\big]\\
&\qquad\times\mathbb{E}^{\mathbb{P}}\big[\mathbb{E}^{\mathbf{P}^0_{\widehat{X}^i_{t_j}}}\big[H_{2,2}\big(Z_r; r\in[0,r_1-t_j]\big)\big]H_{2,1}\big(\widehat{X}^i_r; r\in[s_i,t_j]\big)h(y_j,x_i)\big]\\
&\quad=\mathbb{E}^{\mathbb{P}}\big[H_1\big(Y^j_r(\mathbb{1}); r\in[t_j,r_2]\big)\big]\\
&\qquad\times\mathbb{E}^{\mathbb{P}}\big[\mathbb{E}^{\mathbf{P}^0_{X^i(\mathbb{1})_{t_j}}}\big[H_{2,2}\big(Z_r; r\in[0,r_1-t_j]\big)\big]\\
&\qquad\qquad\times H_{2,1}\big(X^i_r(\mathbb{1}); r\in[s_i,t_j]\big)h(y_j,x_i)\big],
\end{aligned}\tag{3.94}$$

where the last equality follows since we only redefine $X^i(\mathbb{1})_t$ for $t\geq\sigma^{\perp}$ to obtain $\widehat{X}^i$, whereas $\sigma^{\perp}\geq t_j$. The rest is easy to obtain. Using (3.8), we see that (3.94) gives

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\big[H_1\big(Y^j_r(\mathbb{1}); r\in[t_j,r_2]\big)H_2\big(X^i_r(\mathbb{1}); r\in[s_i,r_1]\big)h(y_j,x_i); r_1\leq\sigma^{\perp}\big]\\
&\quad\leq\mathbb{E}^{\mathbb{P}}\big[H_1\big(Y^j_r(\mathbb{1}); r\in[t_j,r_2]\big)\big]
\end{aligned}$$

$$
\begin{aligned}
&\times \mathbb{E}^{\mathbb{P}}\big[\mathbb{E}^{\mathbf{P}^0_{X^i(\mathbb{1})_{t_j}}}\big[H_{2,2}\big(Z_r; r\in[0,r_1-t_j]\big)\big]H_{2,1}\big(X^i_r(\mathbb{1}); r\in[s_i,t_j]\big)\big]\\
&\times \mathbb{E}^{\mathbb{P}}\big[h(y_j,x_i)\big]\\
=\,&\mathbb{E}^{\mathbb{P}}\big[H_1\big(Y^j_r(\mathbb{1}); r\in[t_j,r_2]\big)\big]\mathbb{E}^{\mathbb{P}}\big[H_2\big(X^i_r(\mathbb{1}); r\in[s_i,r_1]\big)\big]\mathbb{E}^{\mathbb{P}}\big[h(y_j,x_i)\big].
\end{aligned}
$$

We have obtained the desired inequality, and the proof is complete. □

We are ready to prove Lemma 3.10 with arguments similar to those in Section 3.4. The following steps are labelled in the same way as their counterparts in Section 3.4, except that steps 2-5 and 3 below correspond to steps 2-6 and 4 in Section 3.4, respectively. Due to the similarity, we will only point out the key changes, leaving other details to readers.

Recall that we fix $t\in[s_i+\frac{\varepsilon}{2}, s_i+1]$, $i\in\mathbb{N}$ and $\varepsilon\in(0,[8\psi(\mathbb{1})]^{-1}\wedge 1]$.

Step 1. We begin with a simple observation for the integral term

$$
\int_{t_j}^{t\wedge\tau^i\wedge\sigma_\beta^{X^i}\wedge\sigma_\beta^{Y^j}} \frac{1}{X^i_s(\mathbb{1})}\int_{\mathbb{R}} X^i(x,s)^{1/2}Y^j(x,s)^{1/2}\,dx\,ds
$$

in (3.30), for $y_j=y$ satisfying (3.81) and $j\in\mathbb{N}$ with $t_j\in(s_i,s_i+1]$. For $s\in[t_j, t\wedge\tau^i\wedge\sigma_\beta^{X^i}\wedge\sigma_\beta^{Y^j}]$ with $s<t_j^\star$, the support processes of X^i and Y^j can be enveloped by $\mathcal{P}_\beta^{X^i}(\cdot)$ and $\mathcal{P}_\beta^{Y^j}(\cdot)$ up to time s, respectively, and $\mathcal{P}_\beta^{X^i}(s)\cap\mathcal{P}_\beta^{Y^j}(s)=\varnothing$ by the definition of $t_j^\star$ in (3.83). Hence, for such s,

$$
\int_{\mathbb{R}} X^i(x,s)^{1/2}Y^j(x,s)^{1/2}\,dx=0.
$$

Using the bound (3.78), we obtain as for (3.34) that

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}^i}\Bigg[\sum_{j\in\mathcal{L}^i_{\beta'}(t,t\wedge\tau^i\wedge\sigma_\beta^{X^i})}\Bigg(\psi(\mathbb{1})\varepsilon\\
&\qquad\qquad + \int_{t_j}^{t\wedge\tau^i\wedge\sigma_\beta^{X^i}\wedge\sigma_\beta^{Y^j}}\frac{1}{X^i_s(\mathbb{1})}\int_{\mathbb{R}}X^i(s,x)^{1/2}Y^j(s,x)^{1/2}\,dx\,ds\Bigg)\Bigg]\\
&\quad\precsim \sum_{j:\,s_i<t_j\le t}(t-s_i)^{\beta'}\varepsilon\\
&\qquad + \sum_{j:\,s_i<t_j\le t}\int_{t_j}^{t} ds\,\mathbb{1}_{t_j^\star<s}\frac{1}{(s-s_i)^{\eta/2}}\mathbb{E}^{\mathbb{Q}^i}\big[\big[Y^j_s(\mathbb{1})\big]^{1/2}; s<\tau^i\wedge\sigma_\beta^{X^i}\wedge\sigma_\beta^{Y^j},\\
&\qquad\qquad 2\big(\varepsilon^{1/2}+(t_j-s_i)^{\beta'}\big)\le|y_j-x_i|\\
&\qquad\qquad \le 2\big(\varepsilon^{1/2}+(t-s_i)^{\beta'}\big)\big].
\end{aligned}
\tag{3.95}
$$

Hence, for lateral clusters, we consider

$$\mathbb{E}^{\mathbb{Q}^i}\big[\big[Y_s^j(\mathbb{1})\big]^{1/2}; s < \tau^i \wedge \sigma_\beta^{X^i} \wedge \sigma_\beta^{Y^j}, 2\big(\varepsilon^{1/2} + (t_j - s_i)^{\beta'}\big) \leq |y_j - x_i| \leq 2\big(\varepsilon^{1/2} + (t - s_i)^{\beta'}\big)\big], \quad s \in \big(t_j^\star, t\big], s_i < t_j < t, t_j^\star < t.$$

Step 2-1. We partition the event $\{X_s^i(\mathbb{1})^{T_1^{X^i}} > 0\}$ into the two events in (3.39) and (3.40) with t_j replaced by $t_j^\star$. Then as in (3.41), we write

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}^i}\big[\big[Y_s^j(\mathbb{1})\big]^{1/2}; s < \tau^i \wedge \sigma_\beta^{X^i} \wedge \sigma_\beta^{Y^j}, 2\big(\varepsilon^{1/2} + (t_j - s_i)^{\beta'}\big) \\
&\qquad \leq |y_j - x_i| \leq 2\big(\varepsilon^{1/2} + (t - s_i)^{\beta'}\big)\big] \\
&\leq \frac{1}{\psi(\mathbb{1})\varepsilon}\mathbb{E}^{\mathbb{P}}\big[\big[Y_s^j(\mathbb{1})\big]^{1/2}; s < \tau^i \wedge \sigma_\beta^{X^i} \wedge \sigma_\beta^{Y^j}, 2\big(\varepsilon^{1/2} + (t_j - s_i)^{\beta'}\big) \\
&\qquad \leq |y_j - x_i| \leq 2\big(\varepsilon^{1/2} + (t - s_i)^{\beta'}\big), T_1^{X^i} < T_0^{X^i} \leq t_j^\star\big] \\
&\quad + \frac{1}{\psi(\mathbb{1})\varepsilon}\mathbb{E}^{\mathbb{P}}\big[\big|X^i(\mathbb{1})_s^{T_1^{X^i}} - \psi(\mathbb{1})\varepsilon\big|\big[Y_s^j(\mathbb{1})\big]^{1/2}; s < \tau^i \wedge \sigma_\beta^{X^i} \wedge \sigma_\beta^{Y^j}, \\
&\qquad 2\big(\varepsilon^{1/2} + (t_j - s_i)^{\beta'}\big) \leq |y_j - x_i| \\
&\qquad \leq 2\big(\varepsilon^{1/2} + (t - s_i)^{\beta'}\big), X_s^i(\mathbb{1})^{T_1^{X^i}} > 0, t_j^\star < T_0^{X^i}\big] \\
&\quad + \frac{1}{\psi(\mathbb{1})\varepsilon} \cdot \psi(\mathbb{1})\varepsilon\mathbb{E}^{\mathbb{P}}\big[\big[Y_s^j(\mathbb{1})\big]^{1/2}; s < \tau^i \wedge \sigma_\beta^{X^i} \wedge \sigma_\beta^{Y^j}, \\
&\qquad 2\big(\varepsilon^{1/2} + (t_j - s_i)^{\beta'}\big) \leq |y_j - x_i| \\
&\qquad \leq 2\big(\varepsilon^{1/2} + (t - s_i)^{\beta'}\big), t_j^\star < T_0^{X^i}\big] \\
&\qquad \forall s \in \big(t_j^\star, t\big], s_i < t_j < t, t_j^\star < t,
\end{aligned} \tag{3.96}$$

where we replace the event $\{X_s^i(\mathbb{1})^{T_1^{X^i}} > 0, t_j^\star < T_0^{X^i}\}$ by the larger one $\{t_j^\star < T_0^{X^i}\}$ for the third term.

Step 2-2. Consider the first term on the right-hand side of (3.96). We have

$$\begin{aligned}
&\frac{1}{\psi(\mathbb{1})\varepsilon}\mathbb{E}^{\mathbb{P}}\big[\big[Y_s^j(\mathbb{1})\big]^{1/2}; s < \tau^i \wedge \sigma_\beta^{X^i} \wedge \sigma_\beta^{Y^j}, \\
&\qquad 2\big(\varepsilon^{1/2} + (t_j - s_i)^{\beta'}\big) \leq |y_j - x_i| \\
&\qquad \leq 2\big(\varepsilon^{1/2} + (t - s_i)^{\beta'}\big), T_1^{X^i} < T_0^{X^i} \leq t_j^\star\big] \\
&\leq \frac{1}{\psi(\mathbb{1})\varepsilon}\mathbb{E}^{\mathbb{P}}\big[\big[Y_s^j(\mathbb{1})\big]^{1/2}; s \leq T_1^{Y^j}, t_j^\star \leq \sigma_\beta^{X^i} \wedge \sigma_\beta^{Y^j},
\end{aligned} \tag{3.97}$$

$$
\begin{aligned}
&2\big(\varepsilon^{1/2}+(t_j-s_i)^{\beta'}\big)\le |y_j-x_i|\\
&\le 2\big(\varepsilon^{1/2}+(t-s_i)^{\beta'}\big), T_1^{X^i}<T_0^{X^i}\le t_j^\star\big]\\
&\qquad \forall s\in (t_j^\star,t], s_i<t_j<t, t_j^\star<t.
\end{aligned}
$$

We then apply Proposition 3.16, taking

$$
\sigma^\perp=\big(\sigma_\beta^{X^i}\wedge\sigma_\beta^{Y^j}\wedge t_j^\star\big)\vee t_j,\qquad r_1=t_j^\star,\qquad r_2=s. \tag{3.98}
$$

Hence, from (3.78) and (3.97), we obtain

$$
\begin{aligned}
&\frac{1}{\psi(\mathbb{1})\varepsilon}\mathbb{E}^{\mathbb{P}}\big[\big[Y_s^j(\mathbb{1})\big]^{1/2}; s<\tau^i\wedge\sigma_\beta^{X^i}\wedge\sigma_\beta^{Y^j}, 2\big(\varepsilon^{1/2}+(t_j-s_i)^{\beta'}\big)\\
&\qquad\le |y_j-x_i|\le 2\big(\varepsilon^{1/2}+(t-s_i)^{\beta'}\big), T_1^{X^i}<T_0^{X^i}\le t_j^\star\big]\\
&\quad\precsim \frac{1}{\varepsilon}\mathbb{P}\big(T_1^{X^i}<T_0^{X^i}\le t_j^\star\big)(t-s_i)^{\beta'}\mathbb{E}^{\mathbf{P}^0_{\psi(\mathbb{1})\varepsilon}}\big[(Z_{s-t_j})^{1/2}; s-t_j\le T_1^Z\big] \qquad (3.99)\\
&\quad\precsim (t-s_i)^{\beta'}\cdot\mathbb{E}^{\mathbf{P}^0_{\psi(\mathbb{1})\varepsilon}}\big[(Z_{s-t_j})^{1/2}; s-t_j\le T_1^Z\big]\\
&\qquad \forall s\in (t_j^\star,t], s_i<t_j<t, t_j^\star<t.
\end{aligned}
$$

Step 2-3. Let us consider the second term in (3.96). As before, using (3.45) gives

$$
\begin{aligned}
&\frac{1}{\psi(\mathbb{1})\varepsilon}\mathbb{E}^{\mathbb{P}}\big[\big|X_s^i(\mathbb{1})^{T_1^{X^i}}-\psi(\mathbb{1})\varepsilon\big|\big[Y_s^j(\mathbb{1})\big]^{1/2}; s<\tau^i\wedge\sigma_\beta^{X^i}\wedge\sigma_\beta^{Y^j},\\
&\qquad 2\big(\varepsilon^{1/2}+(t_j-s_i)^{\beta'}\big)\le |y_j-x_i|\\
&\qquad\quad \le 2\big(\varepsilon^{1/2}+(t-s_i)^{\beta'}\big), X_s^i(\mathbb{1})^{T_1^{X^i}}>0, t_j^\star<T_0^{X^i}\big]\\
&\quad\precsim \frac{\varepsilon^{\alpha^{N_0}}(s-s_i)^\alpha+(s-s_i)^\xi}{\varepsilon}\\
&\qquad\times\mathbb{E}^{\mathbb{P}}\big[\big[Y_s^j(\mathbb{1})\big]^{1/2}; s\le T_1^{Y^j}, t_j^\star\le\sigma_\beta^{X^i}\wedge\sigma_\beta^{Y^j}, \qquad (3.100)\\
&\qquad\quad 2\big(\varepsilon^{1/2}+(t_j-s_i)^{\beta'}\big)\le |y_j-x_i|\\
&\qquad\quad \le 2\big(\varepsilon^{1/2}+(t-s_i)^{\beta'}\big), t_j^\star<T_0^{X^i}\big]\\
&\qquad \forall s\in (t_j^\star,t], s_i<t_j<t, t_j^\star<t.
\end{aligned}
$$

Taking the choice (3.98) again, we obtain from Proposition 3.16, (3.78) and the last display that

$$
\begin{aligned}
&\frac{1}{\psi(\mathbb{1})\varepsilon}\mathbb{E}^{\mathbb{P}}\big[\big|X_s^i(\mathbb{1})^{T_1^{X^i}}-\psi(\mathbb{1})\varepsilon\big|\big[Y_s^j(\mathbb{1})\big]^{1/2}; s<\tau^i\wedge\sigma_\beta^{X^i}\wedge\sigma_\beta^{Y^j},\\
&\qquad 2\big(\varepsilon^{1/2}+(t_j-s_i)^{\beta'}\big)\le |y_j-x_i|
\end{aligned}
$$

$$\leq 2\big(\varepsilon^{1/2} + (t-s_i)^{\beta'}\big), X_s^i(\mathbb{1})^{T_1^{X^i}} > 0, t_j^\star < T_0^{X^i}\big]$$
$$\precsim \frac{\varepsilon^{\alpha^{N_0}}(s-s_i)^\alpha + (s-s_i)^\xi}{\varepsilon}(t-s_i)^{\beta'}\mathbb{P}\big(t_j^\star < T_0^{X^i}\big)$$
$$\times \mathbb{E}^{\mathbf{P}^0_{\psi(\mathbb{1})\varepsilon}}\big[(Z_{s-t_j})^{1/2}; s-t_j \leq T_1^Z\big] \qquad \forall s \in \big(t_j^\star, t\big], s_i < t_j < t, t_j^\star < t.$$

Hence, by a computation similar to (3.47) and Lemma 3.14, the foregoing display gives

$$\begin{aligned}
&\frac{1}{\psi(\mathbb{1})\varepsilon}\mathbb{E}^{\mathbb{P}}\big[\big|X_s^i(\mathbb{1})^{T_1^{X^i}} - \psi(\mathbb{1})\varepsilon\big|\big[Y_s^j(\mathbb{1})\big]^{1/2}; s < \tau^i \wedge \sigma_\beta^{X^i} \wedge \sigma_\beta^{Y^j},\\
&\qquad 2\big(\varepsilon^{1/2} + (t_j-s_i)^{\beta'}\big) \leq |y_j - x_i|\\
&\qquad\qquad \leq 2\big(\varepsilon^{1/2} + (t-s_i)^{\beta'}\big), X_s^i(\mathbb{1})^{T_1^{X^i}} > 0, t_j^\star < T_0^{X^i}\big]\\
&\quad\precsim \big(\varepsilon^{\alpha^{N_0}}(s-s_i)^\alpha + (s-s_i)^\xi\big)(t-s_i)^{\beta'}(t_j-s_i)^{-\beta'/\beta}\\
&\qquad \times \mathbb{E}^{\mathbf{P}^0_{\psi(\mathbb{1})\varepsilon}}\big[(Z_{s-t_j})^{1/2}; s-t_j \leq T_1^Z\big]\\
&\qquad\qquad \forall s \in \big(t_j^\star, t\big], s_i < t_j < t, t_j^\star < t.
\end{aligned}\tag{3.101}$$

Step 2-4. For the third term in (3.96), the calculation in the foregoing step 2-3 readily shows

$$\begin{aligned}
&\frac{1}{\psi(\mathbb{1})\varepsilon}\cdot\psi(\mathbb{1})\varepsilon\mathbb{E}^{\mathbb{P}}\big[\big[Y_s^j(\mathbb{1})\big]^{1/2}; s < \tau^i \wedge \sigma_\beta^{X^i} \wedge \sigma_\beta^{Y^j}, 2\big(\varepsilon^{1/2} + (t_j-s_i)^{\beta'}\big)\\
&\qquad \leq |y_j - x_i| \leq 2\big(\varepsilon^{1/2} + (t-s_i)^{\beta'}\big), t_j^\star < T_0^{X^i}\big]\\
&\quad\precsim (t-s_i)^{\beta'}(t_j-s_i)^{-\beta'/\beta}\cdot\varepsilon\cdot\mathbb{E}^{\mathbf{P}^0_{\psi(\mathbb{1})\varepsilon}}\big[(Z_{s-t_j})^{1/2}; s-t_j \leq T_1^Z\big]\\
&\qquad\qquad \forall s \in \big(t_j^\star, t\big], s_i < t_j < t, t_j^\star < t.
\end{aligned}\tag{3.102}$$

Step 2-5. At this step, we apply (3.99), (3.101) and (3.102) to (3.96) and give a summary as follows:

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}^i}\big[\big[Y_s^j(\mathbb{1})\big]^{1/2}; s < \tau^i \wedge \sigma_\beta^{X^i} \wedge \sigma_\beta^{Y^j}, 2\big(\varepsilon^{1/2} + (t_j-s_i)^{\beta'}\big) \leq |y_j - x_i|\\
&\qquad \leq 2\big(\varepsilon^{1/2} + (t-s_i)^{\beta'}\big)\big]\\
&\quad\precsim \big[(t-s_i)^{\beta'} + (t-s_i)^{\beta'}(t_j-s_i)^{-\beta'/\beta}\big(\varepsilon^{\alpha^{N_0}}(s-s_i)^\alpha + (s-s_i)^\xi + \varepsilon\big)\big]\\
&\qquad \times \mathbb{E}^{\mathbf{P}^0_{\psi(\mathbb{1})\varepsilon}}\big[(Z_{s-t_j})^{1/2}; s-t_j \leq T_1^Z\big]\\
&\quad\precsim (t-s_i)^{\beta'}(t_j-s_i)^{-\beta'/\beta}\big((t_j-s_i)^{\beta'/\beta} + \varepsilon^{\alpha^{N_0}}(s-s_i)^\alpha + (s-s_i)^\xi + \varepsilon\big)\\
&\qquad \times \varepsilon^{\alpha^{N_0}/2}(s-t_j)^{\alpha/2}\bigg(\frac{\varepsilon}{s-t_j}\bigg)^{1-\alpha^{N_0}/4}
\end{aligned}\tag{3.103}$$

$$
\begin{aligned}
&+ (t-s_i)^{\beta'}(t_j-s_i)^{-\beta'/\beta}\big((t_j-s_i)^{\beta'/\beta}+\varepsilon^{\alpha^{N_0}}(s-s_i)^{\alpha}+(s-s_i)^{\xi}+\varepsilon\big)\\
&\quad\times \varepsilon^{1/2}\bigg(\frac{\varepsilon}{s-t_j}\bigg)^{1-\alpha^{N_0}/4}+(t-s_i)^{\beta'}\varepsilon(s-t_j)^{\xi/2-1}\\
&+ (t-s_i)^{\beta'}(t_j-s_i)^{-\beta'/\beta}\big(\varepsilon^{\alpha^{N_0}}(s-s_i)^{\alpha}+\varepsilon\big)\varepsilon(s-t_j)^{\xi/2-1}\\
&+ (t-s_i)^{\beta'}(t_j-s_i)^{-\beta'/\beta}(s-s_i)^{\xi}\varepsilon(s-t_j)^{\xi/2-1}\\
&\hspace{18em}\forall s\in\big(t_j^\star,t\big],\, s_i<t_j<t,\, t_j^\star<t,
\end{aligned}
$$

where as in step 2-6 of Section 3.4, the last "$\lesssim$"-inequality follows again from Lemma 3.12, some arithmetic, and an application of (3.63).

For any $s\in(t_j^\star,t]$ with $s_i<t_j<t$ and $t_j^\star<t$, we have

$$
(t_j-s_i)^{\beta'/\beta}+\varepsilon^{\alpha^{N_0}}(s-s_i)^{\alpha}+(s-s_i)^{\xi}+\varepsilon\lesssim(s-s_i)^{\alpha},
$$

which results from (3.22)(a), (3.22)(b), Lemma 3.14 and $t_j-s_i\geq\frac{\varepsilon}{2}$. Hence, with some simplifications similar to (3.64)–(3.66), we obtain

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}^i}\big[\big[Y_s^j(\mathbb{1})\big]^{1/2}; s<\tau^i\wedge\sigma_\beta^{X^i}\wedge\sigma_\beta^{Y^j},\\
&\qquad 2\big(\varepsilon^{1/2}+(t_j-s_i)^{\beta'}\big)\leq|y_j-x_i|\leq 2\big(\varepsilon^{1/2}+(t-s_i)^{\beta'}\big)\big]\\
&\qquad\lesssim (t-s_i)^{\beta'}(t_j-s_i)^{-\beta'/\beta}(s-s_i)^{\alpha}(s-t_j)^{\alpha/2+\alpha^{N_0}/4-1}\varepsilon^{1+\alpha^{N_0}/4}\\
&\qquad\quad+(t-s_i)^{\beta'}(t_j-s_i)^{\alpha/2-\beta'/\beta}(s-s_i)^{\alpha}(s-t_j)^{\alpha^{N_0}/4-1}\varepsilon^{1+\alpha^{N_0}/4}\\
&\qquad\quad+(t-s_i)^{\beta'}(s-t_j)^{\xi/2-1}\varepsilon\\
&\qquad\quad+(t-s_i)^{\beta'}(t_j-s_i)^{-\beta'/\beta}(s-s_i)^{\xi}(s-t_j)^{\xi/2-1}\varepsilon\\
&\hspace{18em}\forall s\in\big(t_j^\star,t\big],\, s_i<t_j<t,\, t_j^\star<t.
\end{aligned}
\tag{3.104}
$$

Step 3. We complete the proof of Lemma 3.10 in this step. Apply the bound (3.104) to the right-hand side of the inequality (3.95). We have

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}^i}\bigg[\sum_{j\in\mathcal{L}^i_{\beta'}(t,t\wedge\tau^i\wedge\sigma_\beta^{X^i})}\bigg(\psi(\mathbb{1})\varepsilon\\
&\qquad\qquad+\int_{t_j}^{t\wedge\tau^i\wedge\sigma_\beta^{X^i}\wedge\sigma_\beta^{Y^j}}\frac{1}{X_s^i(\mathbb{1})}\int_{\mathbb{R}}X^i(x,s)^{1/2}Y^j(x,s)^{1/2}\,dx\,ds\bigg)\bigg]\\
&\quad\lesssim(t-s_i)^{\beta'}\sum_{j:\,s_i<t_j\leq t}\varepsilon\\
&\qquad+(t-s_i)^{\beta'}\sum_{j:\,s_i<t_j\leq t}(t_j-s_i)^{-\beta'/\beta}\int_{t_j}^{t}(s-s_i)^{-\eta/2+\alpha}
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\qquad\qquad\qquad\qquad \times (s-t_j)^{\alpha/2+\alpha^{N_0}/4-1}\,ds\cdot\varepsilon^{1+\alpha^{N_0}/4}\\
&+(t-s_i)^{\beta'}\sum_{j:s_i<t_j\le t}(t_j-s_i)^{\alpha/2-\beta'/\beta}\int_{t_j}^t (s-s_i)^{-\eta/2+\alpha}\\
&\qquad\qquad\qquad\qquad\qquad \times (s-t_j)^{\alpha^{N_0}/4-1}\,ds\cdot\varepsilon^{1+\alpha^{N_0}/4}\\
&+(t-s_i)^{\beta'}\sum_{j:s_i<t_j\le t}\int_{t_j}^t (s-s_i)^{-\eta/2}(s-t_j)^{\xi/2-1}\,ds\cdot\varepsilon\\
&+(t-s_i)^{\beta'}\sum_{j:s_i<t_j\le t}(t_j-s_i)^{-\beta'/\beta}\int_{t_j}^t (s-s_i)^{-\eta/2+\xi}(s-t_j)^{\xi/2-1}\,ds\cdot\varepsilon.
\end{aligned}
$$

Thanks to the second inequality in (3.22)(b), the integral domination outlined in step 3 of Section 3.4 can be applied to each term on the right-hand side of the foregoing $\lesssim$-inequality, giving bounds which are convergent integrals. As in step 4 of Section 3.4, we obtain

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}^i}\Bigg[\sum_{j\in\mathcal{L}^i_{\beta'}(t,t\wedge\tau^i\wedge\sigma_\beta^{X^i})}\Bigg(\psi(\mathbb{1})\varepsilon\\
&\qquad\qquad +\int_{t_j}^{t\wedge\tau^i\wedge\sigma_\beta^{X^i}\wedge\sigma_\beta^{Y^j}}\frac{1}{X_s^i(\mathbb{1})}\int_{\mathbb{R}}X^i(x,s)^{1/2}Y^j(x,s)^{1/2}\,dx\,ds\Bigg)\Bigg]\\
&\quad\lesssim (t-s_i)^{\beta'+1}+(t-s_i)^{\beta'-\eta/2+(3\alpha/2)}\cdot\varepsilon^{\alpha^{N_0}/4}+(t-s_i)^{\beta'-\eta/2+(3\xi/2)},
\end{aligned}
$$

which proves Lemma 3.10.

4. Uniform separation of approximating solutions. In this section, we prove the main result of the present paper that there is pathwise nonuniqueness in the SPDE (1.2). The result is summarized in Theorem 4.4. We continue to suppress the dependence on ε of the approximation solutions and emphasize it only through $\mathbb{P}_\varepsilon$, unless otherwise mentioned.

Our program to obtain uniform separation of the approximating solutions is sketched as follows (cf. the discussion in Section 1). For small $r,\varepsilon\in(0,1]$, we will define an event $S(r)=S_\varepsilon(r)$ which keeps track of certain separation of the ε-approximating solutions X and Y over the territory of a "large" immigrant process X^i. The immigrant processes range over those large and arriving approximately by time r. The definition of $S(r)$ is based on the earlier results for conditional separation of the approximating solutions. The effect is that these particular events $S(r)$ imply the required uniform separation: for some $\Delta(r)\in(0,\infty)$ depending only on the parameter vector in Assumption 3.4 and r, we have

$$
S(r)\subseteq\Big\{\sup_{0\le s\le 2r}\|X_s-Y_s\|_{\mathrm{rap}}\ge\Delta(r)\Big\}\tag{4.1}
$$

[recall the definition of $\|\cdot\|_{\mathrm{rap}}$ in (1.6)] and, for fixed r,

$$\liminf_{\varepsilon\searrow 0}\mathbb{P}_\varepsilon\big(S(r)\big)>0. \tag{4.2}$$

Let us give the precise definition of the events $S(r)$ and discuss the ingredients. First, recall the parameter vector chosen in Assumption 3.4 as well as the constants κ_j defined in Theorem 3.5. We need to use small portions of the constants κ_1 and κ_3, and by (3.22)(d) we can find a constant $\wp$ satisfying

$$\wp\in(0,\kappa_1\wedge\kappa_3)\qquad\text{such that }\kappa_1-\wp>\eta.$$

We insist that $\wp$ depends only on the parameter vector in (3.21). For any $i\in\mathbb{N}$, $\varepsilon\in(0,[8\psi(\mathbb{1})]^{-1}\wedge 1]$, and random time $T\geq s_i$, let $G^i(T)=G^i_\varepsilon(T)$ be the growth event defined by

$$G^i(T)=\left\{\begin{array}{l} X^i_s\big(\big[x_i-\varepsilon^{1/2}-(s-s_i)^\beta,x_i+\varepsilon^{1/2}+(s-s_i)^\beta\big]\big)\\ \qquad\geq\dfrac{(s-s_i)^\eta}{4}\quad\text{and}\\ Y_s\big(\big[x_i-\varepsilon^{1/2}-(s-s_i)^\beta,x_i+\varepsilon^{1/2}+(s-s_i)^\beta\big]\big)\\ \qquad\leq K^*\big[(s-s_i)^{\kappa_1-\wp}+\varepsilon^{\kappa_2}(s-s_i)^{\kappa_3-\wp}\big]\\ \hfill\forall s\in[s_i,T]\end{array}\right\}, \tag{4.3}$$

where the constant $K^*\in(0,\infty)$ is as in Theorem 3.5. Note that $G^i(\cdot)$ is *decreasing* in the sense that, for any random times T_1,T_2 with $T_1\leq T_2$, $G^i(T_1)\supseteq G^i(T_2)$. Later on when taking into consideration of the support propagation of the immigrant processes, we will explain how the description in (4.3) for X is related to the stopping time $\tau^{i,(1)}$ underlying (3.12), and how the description in the same display for Y is related to the event in (3.24) for partial sums of the total mass processes $Y^j(\mathbb{1})$.

Next, we set

$$S(r)=S_\varepsilon(r)\triangleq\bigcup_{i=1}^{\lfloor r\varepsilon^{-1}\rfloor}G^i_\varepsilon(s_i+r),\qquad r\in(0,\infty),\varepsilon\in\left(0,\frac{1}{8\psi(\mathbb{1})}\wedge 1\right]. \tag{4.4}$$

Whereas the event $S(r)$ depends on all immigrant processes X^i arriving approximately by time r, it is intended to keep track of separation over the territories of the "large" ones, as we have planned above. This idea underlies the arguments below, and will become explicit in the proof of Lemma 4.3 where inclusion–exclusion and conditioning come into play in applying the result of Theorem 3.5 immigrant-by-immigrant with respect to X^i's.

By the following lemma, (4.1) is a simple consequence of the events $G^i(\cdot)$.

LEMMA 4.1. *For some $r_0\in(0,1]$, we can find $\varepsilon_0(r)\in(0,r\wedge[8\psi(\mathbb{1})]^{-1}\wedge 1]$ and $\Delta(r)\in(0,\infty)$ for any $r\in(0,r_0]$ so that the inclusion* (4.1) *holds almost surely for any $\varepsilon\in(0,\varepsilon_0(r)]$. The constant $\Delta(r)$ depends only on r and the parameter vector chosen in Assumption* 3.4.

PROOF. First, we specify the strictly positive numbers r_0, $\varepsilon_0(r)$, and $\Delta(r)$. Since the small portion $\wp$ taken away from κ_1 and κ_3 satisfies $\kappa_1 - \wp > \eta$, we can choose $r_0 \in (0, 1]$ such that

$$\frac{r^\eta}{4} - 2K^* r^{\kappa_1 - \wp} > 0 \qquad \forall r \in (0, r_0]. \tag{4.5}$$

Then we choose, for every $r \in (0, r_0]$, a number $\varepsilon_0(r) \in (0, r \wedge [8\psi(\mathbb{1})]^{-1} \wedge 1]$ such that

$$0 < \varepsilon_0(r)^{\kappa_2} \le r^{\kappa_1 - \kappa_3}. \tag{4.6}$$

Finally, we set

$$\Delta(r) \triangleq \frac{1}{2}\left[\left(\frac{r^\eta/4 - 2K^* r^{\kappa_1 - \kappa}}{2 + 2r^\beta}\right) \wedge 1\right] > 0, \qquad r \in (0, r_0]. \tag{4.7}$$

We check that the foregoing choices give (4.1). Fix $r \in (0, r_0]$, $\varepsilon \in (0, \varepsilon_0(r)]$, and $1 \le i \le \lfloor r\varepsilon^{-1} \rfloor$ (note that $\lfloor r\varepsilon^{-1} \rfloor \ge 1$ since $\varepsilon \le r$). In this paragraph, we assume that the event $G^i(s_i + r)$ occurs. Then by definition,

$$\begin{aligned} &Y_s\big([x_i - \varepsilon^{1/2} - (s - s_i)^\beta, x_i + \varepsilon^{1/2} + (s - s_i)^\beta]\big) \\ &\quad \le K^*\big[(s - s_i)^{\kappa_1 - \wp} + \varepsilon^{\kappa_2}(s - s_i)^{\kappa_3 - \wp}\big] \qquad \forall s \in [s_i, s_i + r]. \end{aligned} \tag{4.8}$$

In particular, (4.6) and (4.8) imply that

$$Y_{s_i + r}\big([x_i - \varepsilon^{1/2} - r^\beta, x_i + \varepsilon^{1/2} + r^\beta]\big) \le 2K^* r^{\kappa_1 - \wp}.$$

Since $X \ge X^i$, the last inequality and the definition of $G^i(s_i + r)$ imply

$$\begin{aligned} &X_{s_i + r}\big([x_i - \varepsilon^{1/2} - r^\beta, x_i + \varepsilon^{1/2} + r^\beta]\big) - Y_{s_i + r}\big([x_i - \varepsilon^{1/2} - r^\beta, x_i + \varepsilon^{1/2} + r^\beta]\big) \\ &\quad \ge \frac{r^\eta}{4} - 2K^* r^{\kappa_1 - \wp}, \end{aligned}$$

where the lower bound is strictly positive by (4.5). To carry this to the $\mathscr{C}_{\mathrm{rap}}(\mathbb{R})$-norm of $X_{s_i + r} - Y_{s_i + r}$, we make an elementary observation: if f is Borel measurable, integrable on a finite interval I, and satisfies $\int_I f > A$, then there must exist some $x \in I$ such that $f(x) > A/\ell(I)$, where $\ell(I)$ is the length of I. Using this, we obtain from the last inequality that, for some $x \in [x_i - \varepsilon^{1/2} - r^\beta, x_i + \varepsilon^{1/2} + r^\beta]$,

$$X(x, s_i + r) - Y(x, s_i + r) \ge \frac{r^\eta/4 - 2K^* r^{\kappa_1 - \wp}}{2\varepsilon^{1/2} + 2r^\beta} \ge \frac{r^\eta/4 - 2K^* r^{\kappa_1 - \wp}}{2 + 2r^\beta},$$

so the definition of $\|\cdot\|_{\mathrm{rap}}$ [in (1.6)] and the definition (4.7) of $\Delta(r)$ entail

$$\Delta(r) \le \|X_{s_i + r} - Y_{s_i + r}\|_{\mathrm{rap}} \le \sup_{0 \le s \le 2r} \|X_s - Y_s\|_{\mathrm{rap}},$$

where the second inequality follows since $s_i = \frac{(2i-1)}{2}\varepsilon$ and $1 \le i \le \lfloor r\varepsilon^{-1} \rfloor$.

In summary, we have shown that (4.1) holds because each component $G^i(s_i+r)$ of $S(r)$ satisfies the analogous inclusion. The proof is complete. $\square$

We proceed to the proof of (4.2) for small enough $r>0$. From now on, we take into account the support propagation of the immigrant processes. The major argument will be in Lemma 4.3 below. As the preliminary step to use Theorem 3.5, we bring the involved stopping times into the events $G^i(\cdot)$ and then translate the descriptions about Y in (4.3) into ones about its immigrant processes (see Lemma 4.2 below).

Recall Figure 2, and define the event $\Gamma^i(r)=\Gamma^i_\varepsilon(r)$ by

$$\begin{aligned}\Gamma^i(r) &\triangleq \Bigl\{\mathcal{P}^{X^i}_\beta(s_i+r)\cap\Bigl(\bigcup_{j:\,t_j\leq s_i}\operatorname{supp}(Y^j)\Bigr)=\varnothing\Bigr\}\\ &\quad\cap\bigcap_{j:\,t_j\leq s_i+r}\bigl\{\sigma^{Y^j}_\beta>t_j+3r\bigr\}\cap\bigl\{\sigma^{X^i}_\beta>s_i+2r\bigr\},\\ &\qquad r\in(0,1],i\in\mathbb{N},\varepsilon\in\Bigl(0,\frac{1}{8\psi(\mathbb{1})}\wedge 1\Bigr],\end{aligned}\tag{4.9}$$

where $\operatorname{supp}(Y^j)$ denotes the topological support of the two-parameter (random) function $(x,s)\longmapsto Y^j(x,s)$, and the time durations r, $3r$ and $2r$ on the right-hand side in restricting $\mathcal{P}^{X^i}_\beta$, $\sigma^{Y^j}_\beta$ and $\sigma^{X^i}_\beta$, respectively, are chosen only for technical convenience and can be replaced by suitably large constant multiples of r. Through $\Gamma^i(r)$, we confine the ranges of the supports of Y^j, for $j\in\mathbb{N}$ satisfying $t_j\leq s_i+r$, and X^i. It will become clear in passing that one of the reasons for considering this event is to make precise the informal argument of choosing $\mathcal{J}^i_{\beta'}(\cdot)$, as discussed in Section 3.2.

LEMMA 4.2. *Fix $r\in(0,1]$, $i\in\mathbb{N}$ and $\varepsilon\in(0,[8\psi(\mathbb{1})]^{-1}\wedge 1]$. Then on the event $\Gamma^i(r)$ defined by (4.9), we have*

$$\begin{aligned}&Y_s\bigl([x_i-\varepsilon^{1/2}-(s-s_i)^\beta,x_i+\varepsilon^{1/2}+(s-s_i)^\beta]\bigr)\\ &\quad=\sum_{j\in\mathcal{J}^i_{\beta'}(s)}Y^j_s\bigl([x_i-\varepsilon^{1/2}-(s-s_i)^\beta,x_i+\varepsilon^{1/2}+(s-s_i)^\beta]\bigr)\\ &\qquad\forall s\in[s_i,s_i+r].\end{aligned}\tag{4.10}$$

In particular, on $\Gamma^i(r)$,

$$\begin{aligned}&Y_s\bigl([x_i-\varepsilon^{1/2}-(s-s_i)^\beta,x_i+\varepsilon^{1/2}+(s-s_i)^\beta]\bigr)\leq\sum_{j\in\mathcal{J}^i_{\beta'}(s)}Y^j_s(\mathbb{1})\\ &\qquad\forall s\in[s_i,s_i+r].\end{aligned}\tag{4.11}$$

PROOF. In this proof, we argue on the event $\Gamma^i(r)$ and call $\Theta_s \triangleq \{x; (x,s) \in \Theta\}$ the s-*section* of a subset Θ of $\mathbb{R} \times \mathbb{R}_+$ for any $s \in \mathbb{R}_+$.

Consider (4.10). Since the s-section $\mathrm{supp}(Y^j)_s$ contains the support of $Y^j_s(\cdot)$, it suffices to show that, for any $s \in [s_i, s_i + r]$ and $j \in \mathbb{N}$ with $t_j \le s$ and $j \notin \mathcal{J}^i_{\beta'}(s)$,

$$[x_i - \varepsilon^{1/2} - (s - s_i)^\beta, x_i + \varepsilon^{1/2} + (s - s_i)^\beta] \cap \mathrm{supp}(Y^j)_s = \varnothing. \tag{4.12}$$

If $j \in \mathbb{N}$ satisfies $t_j \le s_i$, then using the first item in the definition (4.9) of $\Gamma^i(r)$ gives

$$\mathcal{P}^{X^i}_\beta(s_i + r) \cap \mathrm{supp}(Y^j) = \varnothing.$$

Hence, taking the s-sections of both $\mathcal{P}^{X^i}_\beta(s_i + r)$ and $\mathrm{supp}(Y^j)$ shows that Y^j satisfies (4.12).

Next, suppose that $j \in \mathbb{N}$ satisfies $s_i < t_j \le s$ but $j \notin \mathcal{J}^i_{\beta'}(s)$. On one hand, this choice of j implies

$$|y_j - x_i| > 2(\varepsilon^{1/2} + (s - s_i)^{\beta'}) \ge 2(\varepsilon^{1/2} + (s - s_i)^\beta),$$

where the second inequality follows from the assumption $r \in (0, 1]$ and the choice $\beta' < \beta$ by (3.22)(b), so Lemma 7.3 entails

$$\mathcal{P}^{X^i}_\beta(s) \cap \mathcal{P}^{Y^j}_\beta(s) = \varnothing. \tag{4.13}$$

On the other hand, using the second item in the definition of $\Gamma^i(r)$, we deduce that

$$\mathrm{supp}(Y^j) \cap (\mathbb{R} \times [t_j, t_j + 3r]) \subseteq \mathcal{P}^{Y^j}_\beta(t_j + 3r).$$

Using $t_j + r > s_i + r \ge s$ and taking s-sections of $\mathrm{supp}(Y^j)$ and $\mathcal{P}^{Y^j}_\beta(t_j + 3r)$, we obtain from the foregoing inclusion that

$$\begin{aligned}\mathrm{supp}(Y^j)_s &\subseteq [y_j - \varepsilon^{1/2} - (s - t_j)^\beta, y_j + \varepsilon^{1/2} + (s - t_j)^\beta] \\ &= \mathcal{P}^{Y_j}_\beta(t_j + 3r)_s \\ &= \mathcal{P}^{Y_j}_\beta(s)_s.\end{aligned} \tag{4.14}$$

Since

$$\mathcal{P}^{X^i}_\beta(s)_s = [x_i - \varepsilon^{1/2} - (s - s_i)^\beta, x_i + \varepsilon^{1/2} + (s - s_i)^\beta],$$

(4.13) and (4.14) give our assertion (4.12) for $j \in \mathbb{N}$ satisfying $s_j < t_j \le s$ and $j \notin \mathcal{J}^i_{\beta'}(s)$. We have considered all cases for which $j \in \mathbb{N}$, $t_j \le s$, and $j \notin \mathcal{J}^i_{\beta'}(s)$. The proof is complete. □

Recall $r_0 \in (0, 1]$ and $\varepsilon_0(r) \in (0, r \wedge [8\psi(\mathbb{1})]^{-1} \wedge 1]$ chosen in Lemma 4.1 and the events $S(r)$ in (4.4). The following lemma completes the last step (4.2) to obtain uniform separation of the approximating solutions.

LEMMA 4.3. *For some $r_1 \in (0, r_0]$, we can find $\varepsilon_1(r) \in (0, \varepsilon_0(r)]$ for any $r \in (0, r_1]$ such that*

$$\inf_{\varepsilon \in (0, \varepsilon_1(r)]} \mathbb{P}_\varepsilon\big(S(r)\big) > 0. \tag{4.15}$$

PROOF. In this proof, we transfer the $\mathbb{Q}_\varepsilon^i$-probabilities of separation in Theorem 3.5 to $\mathbb{P}_\varepsilon$-probabilities of separation by conditioning and use inclusion–exclusion as in [17]. The latter makes the $\mathbb{P}_\varepsilon$-probabilities of separation stand out among others.

For any $i \in \mathbb{N}$, $\varepsilon \in (0, [8\psi(\mathbb{1})]^{-1} \wedge 1]$, and random time $T \geq s_i$, we define $\widehat{G}^i(\cdot) = \widehat{G}_\varepsilon^i(\cdot)$ by

$$\widehat{G}^i(T) = \left\{ \begin{array}{l} X_s^i(\mathbb{1}) \geq \dfrac{(s - s_i)^\eta}{4} \quad \text{and} \\ \displaystyle\sum_{j \in \mathcal{J}_{\beta'}^i(s)} Y_s^j(\mathbb{1}) \leq K^*\big[(s - s_i)^{\kappa_1 - \wp} + \varepsilon^{\kappa_2}(s - s_i)^{\kappa_3 - \wp}\big] \\ \hfill \forall s \in [s_i, T] \end{array} \right\}. \tag{4.16}$$

Note that $\widehat{G}^i(\cdot)$ is decreasing, and its definition about the masses of Y is the same as the event considered in Theorem 3.5 except for the restrictions from stopping times τ^i, $\sigma_\beta^{X^i}$ and $\sigma_\beta^{Y^j}$.

The connection between $\widehat{G}^i(\cdot)$ and $G^i(\cdot)$ is as follows. First, we note that by (4.11), the statement about the masses of Y in $\widehat{G}^i(r) \cap \Gamma^i(r)$ implies that in $G^i(r) \cap \Gamma^i(r)$. Also, the statements in $G^i(\cdot)$ and $\widehat{G}^i(\cdot)$ concerning the masses of X^i are linked by the obvious equality:

$$X_s^i(\mathbb{1}) = X_s^i\big([x_i - \varepsilon^{1/2} - (s - s_i)^\beta, x_i + \varepsilon^{1/2} + (s - s_i)^\beta]\big) \qquad \forall s \in \big[s_i, \sigma_\beta^{X^i}\big].$$

Since $\sigma_\beta^{X^i} > s_i + 2r$ on $\Gamma^i(r)$, we are led to the inclusion

$$\widehat{G}^i\big(\tau^i \wedge (s_i + r)\big) \cap \Gamma^i(r) \subseteq G^i\big(\tau^i \wedge (s_i + r)\big) \cap \Gamma^i(r) \tag{4.17}$$

for any $r \in (0, 1]$, $i \in \mathbb{N}$ and $\varepsilon \in (0, [8\psi(\mathbb{1})]^{-1} \wedge 1]$ (τ^i is defined in Proposition 3.3). We can also write (4.17) as

$$\widehat{G}^i\big(\widehat{\tau}^i(s_i + r) \wedge (s_i + r)\big) \cap \Gamma^i(r) \subseteq G^i\big(\widehat{\tau}^i(s_i + r) \wedge (s_i + r)\big) \cap \Gamma^i(r), \tag{4.18}$$

where

$$\widehat{\tau}^i(s_i + r) \triangleq \tau^i \wedge \sigma_\beta^{X^i} \wedge \bigwedge_{j : s_i < t_j \leq s_i + r} \sigma_\beta^{Y^j}. \tag{4.19}$$

Here, although the restriction $\sigma_\beta^{X^i} \wedge \bigwedge_{j : s_i < t_j \leq s_i + r} \sigma_\beta^{Y^j}$ is redundant in (4.18) [because $\sigma_\beta^{Y^j} > t_j + 3r > s_i + r$ for each $j \in \mathbb{N}$ with $s_i < t_j \leq s_i + r$ by the definition of $\Gamma^i(r)$], we emphasize its role by writing it out.

We start bounding $\mathbb{P}_\varepsilon(S(r))$. For any $r \in (0, r_0]$ and $\varepsilon \in (0, \varepsilon_0(r)]$, we have

$$\begin{aligned}\mathbb{P}_\varepsilon\big(S(r)\big) \geq \mathbb{P}_\varepsilon\Bigg(\bigcup_{i=1}^{\lfloor r\varepsilon^{-1}\rfloor} G^i\big(\widehat{\tau}^i(s_i+r)\wedge(s_i+r)\big)\cap\Gamma^i(r)\cap\big\{T_1^{X^i}<T_0^{X^i}\big\}\\ \cap\big\{\widehat{\tau}^i(s_i+r)\geq s_i+r\big\}\Bigg)\\ \geq \mathbb{P}_\varepsilon\Bigg(\bigcup_{i=1}^{\lfloor r\varepsilon^{-1}\rfloor} \widehat{G}^i\big(\widehat{\tau}^i(s_i+r)\wedge(s_i+r)\big)\cap\Gamma^i(r)\cap\big\{T_1^{X^i}<T_0^{X^i}\big\}\\ \cap\big\{\widehat{\tau}^i(s_i+r)\geq s_i+r\big\}\Bigg),\end{aligned}$$

where the last inequality follows from the inclusion (4.18). We make the restrictions $\{T_1^{X^i} < T_0^{X^i}\}$ in order to invoke $\mathbb{Q}^i$-probabilities later on. By considering separately $\widehat{\tau}^i(s_i+r)\geq s_i+r$ and $\widehat{\tau}^i(s_i+r)<s_i+r$, we obtain from the last inequality that

$$\begin{aligned}\mathbb{P}_\varepsilon\big(S(r)\big) \geq \mathbb{P}_\varepsilon\Bigg(\bigcup_{i=1}^{\lfloor r\varepsilon^{-1}\rfloor} \widehat{G}_i\big(\widehat{\tau}^i(s_i+r)\wedge(s_i+r)\big)\cap\Gamma^i(r)\cap\big\{T_1^{X^i}<T_0^{X^i}\big\}\Bigg)\\ -\mathbb{P}_\varepsilon\Bigg(\bigcup_{i=1}^{\lfloor r\varepsilon^{-1}\rfloor}\big[\widehat{\tau}^i(s_i+r)<s_i+r\big]\cap\big\{T_1^{X^i}<T_0^{X^i}\big\}\Bigg).\end{aligned}\tag{4.20}$$

Applying another inclusion–exclusion to the first term on the right-hand side of (4.20) gives the main inequality of this proof:

$$\begin{aligned}\mathbb{P}_\varepsilon\big(S(r)\big) \geq \mathbb{P}_\varepsilon\Bigg(\bigcup_{i=1}^{\lfloor r\varepsilon^{-1}\rfloor} \widehat{G}^i\big(\widehat{\tau}^i(s_i+r)\wedge(s_i+r)\big)\cap\big\{T_1^{X^i}<T_0^{X^i}\big\}\Bigg)\\ -\mathbb{P}_\varepsilon\Bigg(\bigcup_{i=1}^{\lfloor r\varepsilon^{-1}\rfloor}\Gamma^i(r)^{\complement}\cap\big\{T_1^{X^i}<T_0^{X^i}\big\}\Bigg)\\ -\mathbb{P}_\varepsilon\Bigg(\bigcup_{i=1}^{\lfloor r\varepsilon^{-1}\rfloor}\big\{\widehat{\tau}^i(s_i+r)<s_i+r\big\}\cap\big\{T_1^{X^i}<T_0^{X^i}\big\}\Bigg)\\ \forall r\in(0,r_0],\ \varepsilon\in\big(0,\varepsilon_0(r)\big].\end{aligned}\tag{4.21}$$

In the rest of this proof, we bound each of the three terms on the right-hand side of (4.21) and then choose according to these bounds the desired r_1 and $\varepsilon_1(r)$ for (4.15).

At this stage, we use Proposition 3.3 and Theorem 3.5 in the following way. For any $\rho \in (0, \frac{1}{2})$, we choose $\delta_1 \in (0, 1]$, independent of $i \in \mathbb{N}$ and $\varepsilon \in (0, [8\psi(\mathbb{1})]^{-1} \wedge 1]$, such that

$$\sup\left\{\mathbb{Q}_\varepsilon^i(\tau^i \le s_i + \delta_1); i \in \mathbb{N}, \varepsilon \in \left(0, \frac{1}{8\psi(\mathbb{1})} \wedge 1\right]\right\} \le \rho, \tag{4.22}$$

$$\begin{aligned}\sup\Bigg\{\mathbb{Q}_\varepsilon^i\Bigg(\exists s \in (s_i, s_i + \delta_1], &\sum_{j \in \mathcal{J}^i_{\beta'}(s \wedge \tau^i \wedge \sigma_\beta^{X^i})} Y_s^j(\mathbb{1})^{\tau^i \wedge \sigma_\beta^{X^i} \wedge \sigma_\beta^{Y^j}} \\ &> K^*\big[(s - s_i)^{\kappa_1 - \wp} + \varepsilon^{\kappa_2} \cdot (s - s_i)^{\kappa_3 - \wp}\big]\Bigg); \\ &i \in \mathbb{N}, \varepsilon \in \left(0, \frac{1}{8\psi(\mathbb{1})} \wedge 1\right]\Bigg\} \le \rho.\end{aligned} \tag{4.23}$$

Consider the first probability on the right-hand side of (4.21). We use the elementary inequality: for any events $A_1, \ldots, A_n$ for $n \in \mathbb{N}$,

$$\mathbb{P}\left(\bigcup_{j=1}^n A_j\right) \ge \sum_{j=1}^n \mathbb{P}(A_j) - \sum_{i=1}^n \sum_{\substack{j : j \ne i \\ 1 \le j \le n}} \mathbb{P}(A_i \cap A_j).$$

Then

$$\begin{aligned}\mathbb{P}_\varepsilon&\left(\bigcup_{i=1}^{\lfloor r\varepsilon^{-1}\rfloor} \widehat{G}^i\big(\widehat{\tau}^i(s_i + r) \wedge (s_i + r)\big) \cap \{T_1^{X^i} < T_0^{X^i}\}\right) \\ &\ge \sum_{i=1}^{\lfloor r\varepsilon^{-1}\rfloor} \mathbb{P}_\varepsilon\big(\widehat{G}^i\big(\widehat{\tau}^i(s_i + r) \wedge (s_i + r)\big) \cap \{T_1^{X^i} < T_0^{X^i}\}\big) \\ &\quad - \sum_{i=1}^{\lfloor r\varepsilon^{-1}\rfloor} \sum_{\substack{j : j \ne i \\ 1 \le j \le \lfloor r\varepsilon^{-1}\rfloor}} \mathbb{P}_\varepsilon\big(T_1^{X^i} < T_0^{X^i}, T_1^{X^j} < T_0^{X^j}\big) \\ &\qquad \forall r \in (0, r_0], \varepsilon \in \big(0, \varepsilon_0(r)\big].\end{aligned} \tag{4.24}$$

The first term on the right-hand side of (4.24) can be written as

$$\begin{aligned}&\sum_{i=1}^{\lfloor r\varepsilon^{-1}\rfloor} \mathbb{P}_\varepsilon\big(\widehat{G}^i\big(\widehat{\tau}^i(s_i + r) \wedge (s_i + r)\big) \cap \{T_1^{X^i} < T_0^{X^i}\}\big) \\ &\quad = \sum_{i=1}^{\lfloor r\varepsilon^{-1}\rfloor} \psi(\mathbb{1})\varepsilon \cdot \mathbb{Q}_\varepsilon^i\big(\widehat{G}^i\big(\widehat{\tau}^i(s_i + r) \wedge (s_i + r)\big)\big)\end{aligned} \tag{4.25}$$

by the definition of $\mathbb{Q}^i_\varepsilon$ in (3.1). By inclusion–exclusion, we have

$$\begin{aligned}
&\mathbb{Q}^i_\varepsilon\big(\widehat{G}^i\big(\widehat{\tau}^i(s_i+r)\wedge(s_i+r)\big)\big)\\
&\qquad \geq \mathbb{Q}^i_\varepsilon\biggl(X^i_s(\mathbb{1})\geq \frac{(s-s_i)^\eta}{4}, \forall s\in\big[s_i,\widehat{\tau}^i(s_i+r)\wedge(s_i+r)\big]\biggr)\\
&\qquad\quad - \mathbb{Q}^i_\varepsilon\biggl(\exists s\in\big(s_i,\widehat{\tau}^i(s_i+r)\wedge(s_i+r)\big], \sum_{j\in\mathcal{J}^i_{\beta'}(s)} Y^j(\mathbb{1})_s \\
&\qquad\qquad > K^*\big[(s-s_i)^{\kappa_1-\wp}+\varepsilon^{\kappa_2}\cdot(s-s_i)^{\kappa_3-\wp}\big]\biggr)\\
&\qquad\qquad\qquad \forall i\in\mathbb{N}, \varepsilon\in\biggl(0,\frac{1}{8\psi(\mathbb{1})}\wedge 1\biggr].
\end{aligned}\tag{4.26}$$

Recall that $\tau^{i,(1)}\leq\tau^i$ and $X^i_{s_i}(\mathbb{1})=\psi(\mathbb{1})\varepsilon>0$. Hence, by the definition of $\widehat{\tau}^i(s_i+r)$,

$$\begin{aligned}
&\mathbb{Q}^i_\varepsilon\biggl(X^i_s(\mathbb{1})\geq\frac{(s-s_i)^\eta}{4}, \forall s\in\big[s_i,\widehat{\tau}^i(s_i+r)\wedge(s_i+r)\big]\biggr)=1\\
&\qquad \forall i\in\mathbb{N}, \varepsilon\in\biggl(0,\frac{1}{8\psi(\mathbb{1})}\wedge 1\biggr].
\end{aligned}\tag{4.27}$$

For $r\in(0,\delta_1]$, $i\in\mathbb{N}$, and $\varepsilon\in(0,[8\psi(\mathbb{1})]^{-1}\wedge 1]$, the second probability in (4.26) can be bounded as

$$\begin{aligned}
&\mathbb{Q}^i_\varepsilon\biggl(\exists s\in\big(s_i,\widehat{\tau}^i(s_i+r)\wedge(s_i+r)\big], \sum_{j\in\mathcal{J}^i_{\beta'}(s)} Y^j_s(\mathbb{1})\\
&\qquad > K^*\big[(s-s_i)^{\kappa_1-\wp}+\varepsilon^{\kappa_2}\cdot(s-s_i)^{\kappa_3-\wp}\big]\biggr)\\
&\quad\leq \mathbb{Q}^i_\varepsilon\biggl(\exists s\in\big(s_i,\tau^i\wedge(s_i+r)\big], \sum_{j\in\mathcal{J}^i_{\beta'}(s\wedge\tau^i\wedge\sigma^{X^i}_\beta)} Y^j_s(\mathbb{1})^{\tau^i\wedge\sigma^{X^i}_\beta\wedge\sigma^{Y^j}_\beta}\\
&\qquad\qquad > K^*\big[(s-s_i)^{\kappa_1-\wp}+\varepsilon^{\kappa_2}\cdot(s-s_i)^{\kappa_3-\wp}\big]\biggr)\\
&\quad\leq \mathbb{Q}^i_\varepsilon\biggl(\exists s\in(s_i,s_i+\delta_1], \sum_{j\in\mathcal{J}^i_{\beta'}(s\wedge\tau^i\wedge\sigma^{X^i}_\beta)} Y^j_s(\mathbb{1})^{\tau^i\wedge\sigma^{X^i}_\beta\wedge\sigma^{Y^j}_\beta}\\
&\qquad\qquad > K^*\big[(s-s_i)^{\kappa_1-\wp}+\varepsilon^{\kappa_2}\cdot(s-s_i)^{\kappa_3-\wp}\big]\biggr)\\
&\qquad + \mathbb{Q}^i_\varepsilon\big(\tau^i\leq s_i+\delta_1\big)\leq 2\rho.
\end{aligned}\tag{4.28}$$

Here, the first inequality follows since for $s \in (s_i, \widehat{\tau}^i(s_i+r) \wedge (s_i+r)]$, we have $s \leq \tau^i \wedge \sigma_\beta^{X^i}$ and

$$
\begin{aligned}
j \in \mathcal{J}^i_{\beta'}(s) \quad &\Longrightarrow \quad j \in \mathcal{J}^i_{\beta'}(s_i+r) \quad \Longrightarrow \quad t_j \in (s_i, s_i+r] \\
&\Longrightarrow \quad \widehat{\tau}^i(s_i+r) \leq \sigma_\beta^{Y^j} \quad \Longrightarrow \quad s \leq \sigma_\beta^{Y^j},
\end{aligned}
$$

where the third implication follows from the definition of $\widehat{\tau}^i(s_i+r)$ in (4.19). The first term in the second inequality follows by considering the scenario $\tau^i > s_i + \delta_1$ and using $r \in (0, \delta_1]$, and the last inequality follows from (4.22) and (4.23). Applying (4.27) and (4.28) to (4.26), we get

$$
\begin{aligned}
&\mathbb{Q}^i_\varepsilon\big(\widehat{G}^i\big(\widehat{\tau}^i(s_i+r) \wedge (s_i+r)\big)\big) \geq 1 - 2\rho \\
&\qquad \forall r \in (0, \delta_1 \wedge r_0], i \in \mathbb{N}, \varepsilon \in \big(0, \varepsilon_0(r)\big].
\end{aligned}
\tag{4.29}
$$

From (4.25) and the last inequality, we have shown that

$$
\begin{aligned}
&\sum_{i=1}^{\lfloor r\varepsilon^{-1}\rfloor} \mathbb{P}_\varepsilon\big(\widehat{G}_i\big(\widehat{\tau}^i(s_i+r) \wedge (s_i+r)\big) \cap \big\{T_1^{X^i} < T_0^{X^i}\big\}\big) \geq \psi(\mathbb{1})(r-\varepsilon)(1-2\rho) \\
&\qquad \forall r \in (0, \delta_1 \wedge r_0], \varepsilon \in \big(0, \varepsilon_0(r)\big].
\end{aligned}
$$

[Recall that $\varepsilon_0(r) \leq r$.] The second term on the right-hand side of (4.24) is relatively easy to bound. Indeed, by using the independence between the clusters X^i and Lemma 3.1,

$$
\begin{aligned}
&\sum_{i=1}^{\lfloor r\varepsilon^{-1}\rfloor} \sum_{\substack{j:\, j\neq i \\ 1\leq j\leq \lfloor r\varepsilon^{-1}\rfloor}} \mathbb{P}_\varepsilon\big(T_1^{X^i} < T_0^{X^i}, T_1^{X^j} < T_0^{X^j}\big) \\
&\quad = \sum_{i=1}^{\lfloor r\varepsilon^{-1}\rfloor} \sum_{\substack{j:\, j\neq i \\ 1\leq j\leq \lfloor r\varepsilon^{-1}\rfloor}} \mathbb{P}_\varepsilon\big(T_1^{X^i} < T_0^{X^i}\big)\mathbb{P}_\varepsilon\big(T_1^{X^j} < T_0^{X^j}\big) \leq \psi(\mathbb{1})^2 r^2 \\
&\qquad \forall r \in (0,1], \varepsilon \in \left(0, \frac{1}{8\psi(\mathbb{1})} \wedge 1\right].
\end{aligned}
$$

Recalling (4.24) and using the last two displays, we have the following bound for the first term on the right-hand side of (4.21):

$$
\begin{aligned}
&\mathbb{P}_\varepsilon\left(\bigcup_{i=1}^{\lfloor r\varepsilon^{-1}\rfloor} \widehat{G}_i\big(\widehat{\tau}^i(s_i+t) \wedge (s_i+r)\big) \cap \big\{T_1^{X^i} < T_0^{X^i}\big\}\right) \\
&\quad \geq \psi(\mathbb{1})(r-\varepsilon)(1-2\rho) - \psi(\mathbb{1})^2 r^2 \\
&\qquad \forall r \in (0, \delta_1 \wedge r_0], \varepsilon \in \big(0, \varepsilon_0(r)\big].
\end{aligned}
\tag{4.30}
$$

Next, we consider the second probability on the right-hand side of (4.21). By the definition of $\Gamma^i(r)$ in (4.9) and the general inclusion $(A_1 \cap A_2 \cap A_3)^{\complement} \subseteq (A_1^{\complement} \cap A_2 \cap A_3) \cup A_2^{\complement} \cup A_3^{\complement}$, we have

$$
\begin{aligned}
\Gamma^i(r)^{\complement} \subseteq \Bigg(& \bigg\{ \mathcal{P}_\beta^{X^i}(s_i + r) \cap \Big(\bigcup_{j : t_j \le s_i} \operatorname{supp}(Y^j) \Big) \neq \varnothing \bigg\} \\
& \cap \bigcap_{j : t_j \le s_i} \{\sigma_\beta^{Y^j} > t_j + 3r\} \cap \{\sigma_\beta^{X^i} > s_i + 2r\} \Bigg) \\
& \cup \Big(\bigcup_{j : t_j \le s_i + r} \{\sigma_\beta^{Y^j} \le t_j + 3r\} \Big) \cup \{\sigma_\beta^{X^i} \le s_i + 2r\},
\end{aligned} \tag{4.31}
$$

where we note that the indices j in $\bigcap_{j : t_j \le s_i} \{\sigma_\beta^{Y^j} > t_j + 3r\}$ range only over $j \in \mathbb{N}$ with $t_j \le s_i$. Hence,

$$
\begin{aligned}
& \mathbb{P}_\varepsilon \Bigg(\bigcup_{i=1}^{\lfloor r\varepsilon^{-1} \rfloor} \Gamma^i(r)^{\complement} \cap \{T_1^{X^i} < T_0^{X^i}\} \Bigg) \\
& \quad \le \mathbb{P}_\varepsilon \Bigg(\bigcup_{i=1}^{\lfloor r\varepsilon^{-1} \rfloor} \Bigg(\bigg\{ \mathcal{P}_\beta^{X^i}(s_i + r) \cap \Big(\bigcup_{j : t_j \le s_i} \operatorname{supp}(Y^j) \Big) \neq \varnothing \bigg\} \\
& \qquad \cap \bigcap_{j : t_j \le s_i} \{\sigma_\beta^{Y^j} > t_j + 3r\} \cap \{\sigma_\beta^{X^i} > s_i + 2r\} \cap \{T_1^{X^i} < T_0^{X^i}\} \Bigg) \Bigg) \\
& \qquad + \mathbb{P}_\varepsilon \Bigg(\bigcup_{j=1}^{\lfloor 2r\varepsilon^{-1} \rfloor + 1} \{\sigma_\beta^{Y^j} \le t_j + 3r\} \Bigg) + \mathbb{P}_\varepsilon \Bigg(\bigcup_{i=1}^{\lfloor r\varepsilon^{-1} \rfloor} \{\sigma_\beta^{X^i} \le s_i + 2r\} \Bigg),
\end{aligned} \tag{4.32}
$$

where we have the second probability in the foregoing inequality since

$$
t_j \le s_{\lfloor r\varepsilon^{-1} \rfloor} + r \quad \Longrightarrow \quad t_j \le 2r \quad \Longrightarrow \quad j \le \lfloor 2r\varepsilon^{-1} \rfloor + 1.
$$

Resorting to the conditional probability measures $\mathbb{Q}_\varepsilon^i$, we see that the first probability in (4.32) can be bounded as

$$
\begin{aligned}
& \mathbb{P}_\varepsilon \Bigg(\bigcup_{i=1}^{\lfloor r\varepsilon^{-1} \rfloor} \Bigg(\bigg\{ \mathcal{P}_\beta^{X^i}(s_i + r) \cap \Big(\bigcup_{j : t_j \le s_i} \operatorname{supp}(Y^j) \Big) \neq \varnothing \bigg\} \\
& \qquad \cap \bigcap_{j : t_j \le s_i} \{\sigma_\beta^{Y^j} > t_j + 3r\} \cap \{\sigma_\beta^{X^i} > s_i + 2r\} \cap \{T_1^{X^i} < T_0^{X^i}\} \Bigg) \Bigg) \\
& \quad \le \sum_{i=1}^{\lfloor r\varepsilon^{-1} \rfloor} \psi(\mathbb{1}) \varepsilon \mathbb{Q}_\varepsilon^i \Bigg(\bigg\{ \mathcal{P}_\beta^{X^i}(s_i + r) \cap \Big(\bigcup_{j : t_j \le s_i} \operatorname{supp}(Y^j) \Big) \neq \varnothing \bigg\}
\end{aligned}
$$

$$\cap \bigcap_{j:t_j\leq s_i} \{\sigma_\beta^{Y^j} > t_j + 3r\} \cap \{\sigma_\beta^{X^i} > s_i + 2r\}\Big)$$

$$\leq \sum_{i=1}^{\lfloor r\varepsilon^{-1}\rfloor} \psi(\mathbb{1})\varepsilon C^1_{\mathrm{supp}} r^{1/6} \leq \psi(\mathbb{1}) C^1_{\mathrm{supp}} r^{7/6} \qquad \forall r \in (0, r_0], \varepsilon \in \big(0, \varepsilon_0(r)\big],$$

where the next to the last inequality follows from Proposition 7.2 and the constant $C^1_{\mathrm{supp}} \in (0,\infty)$ is independent of $r \in (0, r_0]$ and $\varepsilon \in (0, r_0]$. (Here, we use the choice $\beta \in [\frac{1}{3}, \frac{1}{2})$ to apply this proposition.) By Proposition 7.1, the second probability in (4.32) can be bounded as [recall $\varepsilon_0(r) \leq r$]

$$\begin{aligned} &\mathbb{P}_\varepsilon\Bigg(\bigcup_{j=1}^{\lfloor 2r\varepsilon^{-1}\rfloor+1} \{\sigma_\beta^{Y^j} \leq t_j + 3r\}\Bigg) \\ &\quad \leq C^0_{\mathrm{supp}}\big(2r\varepsilon^{-1}+1\big)\cdot 3\varepsilon r \\ &\quad \leq 9C^0_{\mathrm{supp}} r^2 \qquad \forall r \in (0, r_0], \varepsilon \in \big(0, \varepsilon_0(r)\big], \end{aligned} \tag{4.33}$$

where C^0_{supp} is a constant independent of $r \in (0, r_0]$ and $\varepsilon \in (0, \varepsilon_0(r)]$. Similarly,

$$\mathbb{P}_\varepsilon\Bigg(\bigcup_{i=1}^{\lfloor r\varepsilon^{-1}\rfloor} \{\sigma_\beta^{X^i} \leq s_i + 2r\}\Bigg) \leq 2C^0_{\mathrm{supp}} r^2 \qquad \forall r \in (0, r_0], \varepsilon \in \big(0, \varepsilon_0(r)\big]. \tag{4.34}$$

From (4.32) and the last three displays, we have shown that the second probability in (4.21) satisfies the bound

$$\begin{aligned} &\mathbb{P}_\varepsilon\Bigg(\bigcup_{i=1}^{\lfloor r\varepsilon^{-1}\rfloor} \Gamma^i(r)^\complement \cap \{T_1^{X^i} < T_0^{X^i}\}\Bigg) \leq 11C^0_{\mathrm{supp}} r^2 + \psi(\mathbb{1}) C^1_{\mathrm{supp}} r^{7/6} \\ &\qquad\qquad \forall r \in (0, r_0], \varepsilon \in \big(0, \varepsilon_0(r)\big]. \end{aligned} \tag{4.35}$$

It remains to bound the last probability on the right-hand side of (4.21). Recall the number δ_1 chosen for (4.22). Similar to the derivation of (4.32), we have

$$\begin{aligned} &\mathbb{P}_\varepsilon\Bigg(\bigcup_{i=1}^{\lfloor r\varepsilon^{-1}\rfloor} \{\widehat{\tau}^i(s_i + r) < s_i + r\} \cap \{T_1^{X^i} < T_0^{X^i}\}\Bigg) \\ &\quad \leq \mathbb{P}_\varepsilon\Bigg(\bigcup_{i=1}^{\lfloor r\varepsilon^{-1}\rfloor} \{\sigma_\beta^{X^i} \leq s_i + r\}\Bigg) + \mathbb{P}_\varepsilon\Bigg(\bigcup_{i=1}^{\lfloor 2r\varepsilon^{-1}\rfloor+1} \{\sigma_\beta^{Y^i} \leq t_i + r\}\Bigg) \\ &\qquad + \sum_{i=1}^{\lfloor r\varepsilon^{-1}\rfloor} \psi(\mathbb{1})\varepsilon \mathbb{Q}^i_\varepsilon\big(\tau^i < s_i + r\big) \\ &\quad \leq 11C^0_{\mathrm{supp}} r^2 + \psi(\mathbb{1}) r\rho \qquad \forall r \in (0, \delta_1 \wedge r_0], \varepsilon \in \big(0, \varepsilon_0(r)\big], \end{aligned} \tag{4.36}$$

where we use (4.33) and (4.34) in the last inequality.

We apply the three bounds (4.30), (4.35) and (4.36) to (4.21). This shows that for any $\rho \in (0, \frac{1}{2})$, there exist $\delta_1 > 0$ such that for any $r \in (0, \delta_1 \wedge r_0]$ and $\varepsilon \in (0, \varepsilon_0(r)]$ [note that $\varepsilon_0(r) \le r \wedge 1$],

$$\begin{aligned}\mathbb{P}_\varepsilon\big(S(r)\big) &\ge \big[\psi(\mathbb{1})(r-\varepsilon)(1-2\rho) - \psi(\mathbb{1})^2 r^2\big] - \big(11C^0_{\mathrm{supp}} r^2 + \psi(\mathbb{1})C^1_{\mathrm{supp}} r^{7/6}\big)\\ &\quad - \big(11C^0_{\mathrm{supp}} r^2 + \psi(\mathbb{1}) r\rho\big)\\ &= r\big[\psi(\mathbb{1})(1-3\rho) - \big(\psi(\mathbb{1})^2 + 22C^0_{\mathrm{supp}}\big)r - \psi(\mathbb{1})C^1_{\mathrm{supp}} r^{1/6}\big]\\ &\quad - \psi(\mathbb{1})\varepsilon(1-2\rho).\end{aligned}$$

Finally, to attain the uniform lower bound (4.15), we choose $\rho \in (0, \frac{1}{2})$ and $r_1 \in (0, \delta_1 \wedge r_0]$ such that

$$\psi(\mathbb{1})(1-3\rho) - \big(\psi(\mathbb{1})^2 + 22C^0_{\mathrm{supp}}\big)r - \psi(\mathbb{1})C^1_{\mathrm{supp}} r^{1/6} \ge \frac{\psi(\mathbb{1})}{2} \qquad \forall r \in (0, r_1],$$

and then $\varepsilon_1(r) \in (0, \varepsilon_0(r)]$ such that

$$\psi(\mathbb{1})\varepsilon_1(r)(1-2\rho) \le \frac{\psi(\mathbb{1})r}{4}.$$

By the last three displays, we obtain

$$\mathbb{P}_\varepsilon\big(S(r)\big) \ge \frac{\psi(\mathbb{1})r}{4} \qquad \forall \varepsilon \in \big(0, \varepsilon_1(r)\big], r \in (0, r_1],$$

and hence (4.15) follows. The proof is complete. $\square$

We use Lemma 4.3 to give the proof for a more precise version of our main theorem, namely Theorem 1, in Theorem 4.4 below.

THEOREM 4.4 (Separation of limiting solutions). *Let $(\varepsilon_n) \subseteq (0, [8\psi(\mathbb{1})]^{-1} \wedge 1]$ with $\varepsilon_n \searrow 0$ be such that the sequence of laws of $((X,Y), \mathbb{P}_{\varepsilon_n})$ converges to the law of $((X,Y), \mathbb{P}_0)$ of a pair of solutions to the SPDE (1.2) in the space of probability measures on the product space $D(\mathbb{R}_+, \mathscr{C}_{\mathrm{rap}}(\mathbb{R})) \times D(\mathbb{R}_+, \mathscr{C}_{\mathrm{rap}}(\mathbb{R}))$ (cf. Proposition 2.3). Then we have*

$$\mathbb{P}_0\Big(\sup_{0 \le s \le 2r_1} \|X - Y\|_{\mathrm{rap}} \ge \frac{\Delta(r_1)}{2}\Big) \ge \inf_{\varepsilon \in (0, \varepsilon_1(r_1)]} \mathbb{P}_\varepsilon\big(S(r_1)\big) > 0,$$

where $\Delta(r_1) > 0$ is chosen in Lemma 4.1 and $r_1, \varepsilon_1(r_1) \in (0,1]$ are chosen in Lemma 4.3.

PROOF. By Skorokhod's representation theorem, we may take $(X^{(\varepsilon_n)}, Y^{(\varepsilon_n)})$ to be copies of the ε_n-approximating solutions which live on the same probability space, and assume that $(X^{(\varepsilon_n)}, Y^{(\varepsilon_n)})$ converges almost surely to $(X^{(0)}, Y^{(0)})$ in the product (metric) space $D(\mathbb{R}_+, \mathscr{C}_{\mathrm{rap}}(\mathbb{R})) \times D(\mathbb{R}_+, \mathscr{C}_{\mathrm{rap}}(\mathbb{R}))$.

It follows from Lemmas 4.1 and 4.3 that

$$\inf_{n:\,\varepsilon_n\le\varepsilon_1(r_1)}\mathbb{P}\Big(\sup_{0\le s\le 2r_1}\|X_s^{(\varepsilon_n)}-Y_s^{(\varepsilon_n)}\|_{\mathrm{rap}}\ge\Delta(r_1)\Big)\ge\inf_{\varepsilon\in(0,\varepsilon_1(r_1)]}\mathbb{P}_\varepsilon\big(S(r_1)\big)>0.$$

Hence, by Fatou's lemma, we get

$$\begin{aligned}
0&<\inf_{\varepsilon\in(0,\varepsilon_1(r_1)]}\mathbb{P}_\varepsilon\big(S(r_1)\big)\\
&\le\limsup_{n\to\infty}\mathbb{P}\Big(\sup_{0\le s\le 2r_1}\|X_s^{(\varepsilon_n)}-Y_s^{(\varepsilon_n)}\|_{\mathrm{rap}}\ge\Delta(r_1)\Big)\\
&\le\mathbb{P}\Big(\limsup_{n\to\infty}\Big\{\sup_{0\le s\le 2r_1}\|X_s^{(\varepsilon_n)}-Y_s^{(\varepsilon_n)}\|_{\mathrm{rap}}\ge\Delta(r_1)\Big\}\Big)\\
&\le\mathbb{P}\Big(\sup_{0\le s\le 2r_1}\|X_s^{(0)}-Y_s^{(0)}\|_{\mathrm{rap}}\ge\frac{\Delta(r_1)}{2}\Big),
\end{aligned}\tag{4.37}$$

where the last inequality follows from the convergence

$$X^{(\varepsilon_n)}\xrightarrow[n\to\infty]{\text{a.s.}}X^{(0)}\quad\text{and}\quad Y^{(\varepsilon_n)}\xrightarrow[n\to\infty]{\text{a.s.}}Y^{(0)}$$

in the Skorokhod space $D(\mathbb{R}_+,\mathscr{C}_{\mathrm{rap}}(\mathbb{R}))$, the continuity of $X^{(0)}$ and $Y^{(0)}$, and Proposition 3.6.5(a) of [10]. The proof is complete. $\square$

5. Proof of Proposition 2.3. Many arguments in this section can be modified from the proofs in Section 6 in [17] because of the apparent similarity of the involved stochastic processes, and so we only give sketches whenever necessary. Readers interested in a complete proof of Proposition 2.3 may see Section 3.9 of [3]. Some connections between limit theorems for $\mathscr{C}_{\mathrm{rap}}(\mathbb{R})$-valued processes and limit theorems for processes taking values in the space of real-valued continuous functions over $\mathbb{R}$ can be found in Section 3.11 of [3].

Throughout this section, we fix a sequence $(\varepsilon_n)\subseteq(0,1]$ with $\varepsilon_n\searrow 0$ and assume that the ε_n-approximating solutions live on the same probability space. To save notation, we write $\{(X^{(n)},Y^{(n)});n\in\mathbb{N}\}$ for this approximating sequence and denote by $\mathbb{P}$ the underlying probability measure. We will begin with the C-tightness of the sequence of joint laws of $\{(X^{(n)},Y^{(n)})\}$ in $D(\mathbb{R}_+,\mathscr{C}_{\mathrm{rap}}(\mathbb{R}))\times D(\mathbb{R}_+,\mathscr{C}_{\mathrm{rap}}(\mathbb{R}))$, where $D(\mathbb{R}_+,\mathscr{C}_{\mathrm{rap}}(\mathbb{R}))$ is equipped with Skorokhod's J_1-topology. Here, C-tightness means not only tightness but also the property that the limiting object of any convergent subsequence is a continuous process. We will only discuss the C-tightness of the sequence of laws of $\{X^{(n)}\}$ in $D(\mathbb{R}_+,\mathscr{C}_{\mathrm{rap}}(\mathbb{R}))$, and the argument for $\{Y^{(n)}\}$ follows similarly. Later on in Lemma 5.4, we will prove that the limit of any convergent subsequence of laws of $\{(X^{(n)},Y^{(n)})\}$ is the law of a pair of solutions to the SPDE (1.2) with respect to the same space–time white noise.

Consider our first objective that the sequence of laws of $\{X^{(n)}\}$ is C-tight as probability measures on $D(\mathbb{R}_+, \mathscr{C}_{\mathrm{rap}}(\mathbb{R}))$. The proof uses the mild forms of $\{X^{(n)}\}$ stated below. Let

$$p_s(x)\,dx \equiv \begin{cases} \dfrac{1}{\sqrt{2\pi s}}\exp\left(-\dfrac{x^2}{2s}\right)dx, & s \in (0,\infty), \\ \delta_0(dx), & s = 0, \\ 0, & s \in (-\infty, 0). \end{cases}$$

Recall the random measure $A^{X^{(n)}}$ [cf. (2.2)] associated with $X^{(n)}$ which is contributed by the initial masses of its immigrants, and write

$$M_t^{X^{(n)}}(\phi) \equiv X_t^{(n)}(\phi) - \int_0^t X_s^{(n)}\left(\frac{\Delta}{2}\phi\right)ds - \int_{(0,t]}\int_{\mathbb{R}} \phi(y)\,dA^{X^{(n)}}(y,s), \qquad \phi \in \mathscr{C}_c^\infty(\mathbb{R}), \tag{5.1}$$

for the martingale measure of $X^{(n)}$.

By summing up the mild forms of the immigrant processes for $X^{(n)}$ which are solutions to the SPDE (1.1) and have initial conditions taking the form $\psi(\mathbb{1})J_\varepsilon^a(\cdot)$ for $J_\varepsilon^a(\cdot)$ defined by (1.13) (see Theorem 2.1 of [25]), we deduce that the mild form of $X^{(n)}$ is given by

$$X^{(n)}(x,t) = p \star A^{X^{(n)}}(x,t) + p \star M^{X^{(n)}}(x,t), \qquad (x,t) \in \mathbb{R}\times\mathbb{R}_+. \tag{5.2}$$

Here, the convolutions on the right-hand side are given by

$$\begin{aligned} p \star A^{X^{(n)}}(x,t) &= \int_{(0,t]}\int_{\mathbb{R}} p_{t-s}(x-y)\,dA^{X^{(n)}}(y,s) \\ &= \psi(\mathbb{1}) \sum_{i\,:\,0<s_i\le t} \int_{\mathbb{R}} p_{t-s_i}(x-y) J_{\varepsilon_n}^{x_i}(y)\,dy, \end{aligned} \tag{5.3}$$

$$\begin{aligned} p \star M^{X^{(n)}}(x,t) &= \int_0^t\int_{\mathbb{R}} p_{t-s}(x-y)\,dM^{X^{(n)}}(y,s) \\ &= \int_0^t\int_{\mathbb{R}} p_{t-s}(x-y) X^{(n)}(y,s)^{1/2}\,dW(y,s). \end{aligned} \tag{5.4}$$

More precisely, in $p \star A^{X^{(n)}}$, we read $p_0(x-y)\,dy = \delta_0(x-dy) = \delta_x(dy)$, and hence

$$\int_{\mathbb{R}} p_0(x-y) J_{\varepsilon_n}^{x_i}(y)\,dy \equiv J_{\varepsilon_n}^{x_i}(x). \tag{5.5}$$

The mild form (5.2) implies the C-tightness of the sequence of laws of $\{X^{(n)}\}$ in $D(\mathbb{R}_+, \mathscr{C}_{\mathrm{rap}}(\mathbb{R}))$, provided that the sequences of laws of $\{p \star A^{X^{(n)}}; n \in \mathbb{N}\}$ and $\{p \star M^{X^{(n)}}; n \in \mathbb{N}\}$ are both C-tight as probability measures on the same space.

LEMMA 5.1. *The sequences of laws of* $\{p \star A^{X^{(n)}}; n \in \mathbb{N}\}$ *is* C*-tight and converges in probability in* $D(\mathbb{R}_+, \mathscr{C}_{\mathrm{rap}}(\mathbb{R}))$ *to the deterministic process* $(\int_0^t \int_{\mathbb{R}} p_{t-s}(\cdot - y)\psi(y)\,dy\,ds)_{t\in\mathbb{R}_+}$.

LEMMA 5.2. *For any* $q \in [1, \infty)$ *and* $\lambda, T \in (0, \infty)$, *there exists a constant* $\check{C} \in (0, \infty)$ *depending only on* (ψ, q, λ, T) *such that*

$$\sup_{n\in\mathbb{N}} \sup_{0\le t\le T} \sup_{x\in\mathbb{R}} e^{\lambda|x|}\mathbb{E}\big[X^{(n)}(x,t)^q + Y^{(n)}(x,t)^q\big] \le \check{C}.$$

LEMMA 5.3. *For some universal constants* $q \in (0, \infty)$ *and* $\gamma \in (2, \infty)$, *the following inequality holds for any* $\lambda, T \in (0, \infty)$:

$$\begin{aligned} &\sup_{n\in\mathbb{N}} \mathbb{E}\big[\big|p \star M^{X^{(n)}}(x', t') - p \star M^{X^{(n)}}(x, t)\big|^q\big] \\ &\qquad \le \check{C}\big(\big|x' - x\big|^{2\gamma} + \big|t' - t\big|^{\gamma}\big)e^{-\lambda|x|} \\ &\qquad\qquad \forall t, t' \in [0, T], \big|x - x'\big| \le 1. \end{aligned} \tag{5.6}$$

Here, *the constants* $\check{C}$ *are as in Lemma* 5.2 *and are enlarged if necessary. Moreover, the sequence of laws of* $\{p \star M^{X^{(n)}}\}$ *is tight as probability measures on* $C(\mathbb{R}_+, \mathscr{C}_{\mathrm{rap}}(\mathbb{R}))$.

The proofs of Lemmas 5.1, 5.2 and 5.3 can be obtained by arguments similar to the proofs of Lemmas 6.6, 6.1 and 6.7 in [17], respectively. In this direction, the proofs of Lemmas 5.1 and 5.2 use the particular form of the distribution (1.15) of x_i and y_i which is dominated by a constant multiple of Lebesgue measure over a compact interval, as well as the fact that in our case, immigrants can land throughout time. The latter does not create additional difficulties since C-tightness of $\mathscr{C}_{\mathrm{rap}}(\mathbb{R})$-valued processes and the bound in Lemma 5.2 only concern distributional properties of the corresponding processes over compact intervals. In addition, for the proof of Lemma 5.3, we need the moment bound in Lemma 5.2 for its first assertion, and the second assertion follows from (5.6) and Lemma 6.4 of [17].

By Lemmas 5.1 and 5.3, the sequence of laws of $\{X^{(n)}\}$ is C-tight as probability measures on $D(\mathbb{R}_+, \mathscr{C}_{\mathrm{rap}}(\mathbb{R}))$, thanks to (5.2). By similar arguments, the same is true for the sequence of laws of $\{Y^{(n)}\}$.

LEMMA 5.4. *Suppose that*, *by taking a subsequence if necessary*, *we have*

$$\big(X^{(n)}, Y^{(n)}\big) \xrightarrow[n\to\infty]{(\mathrm{d})} \big(X^{(0)}, Y^{(0)}\big) \tag{5.7}$$

for some continuous $\mathscr{C}_{\mathrm{rap}}(\mathbb{R})$*-valued processes* $X^{(0)}$ *and* $Y^{(0)}$. *Then* $X^{(0)}$ *and* $Y^{(0)}$ *solve the SPDE* (1.2) *with respect to the same space–time white noise*.

SKETCH OF PROOF. The argument in the proof of Proposition 2.2 of [17] (in Section 6 there) still applies and implies that both $X^{(0)}$ and $Y^{(0)}$ are solutions to the SPDE (1.2). Here, the readers may use as supporting facts a reinforcement of the convergence in (2.6) to almost-sure convergence along (ε_n) (by the strong law of large numbers), and the moment bound in Lemma 5.2.

It remains to show that $X^{(0)}$ and $Y^{(0)}$ can be subject to the same space–time white noise, and moreover, all of these random objects obey their defining properties with respect to the same filtration satisfying the usual conditions. Observe that, by the moment bound in Lemma 5.2 and the fact that $X^{(n)}$ and $Y^{(n)}$ are subject to the same SPDE, the covariation of $X^{(0)}$ and $Y^{(0)}$ satisfies

$$\langle X^{(0)}(\phi_1), Y^{(0)}(\phi_2)\rangle_t = \int_0^t \int_{\mathbb{R}} X^{(0)}(x,s)^{1/2} Y^{(0)}(x,s)^{1/2} \phi_1(x)\phi_2(x)\,dx\,ds,$$
$$\phi_1, \phi_2 \in \mathscr{C}_c^\infty(\mathbb{R}).$$

By an enlargement of the underlying probability space, we may assume that for some filtration $(\mathscr{H}_t)$ satisfying the usual conditions, $X^{(0)}$ and $Y^{(0)}$ are adapted to $(\mathscr{H}_t)$ and there exists an $(\mathscr{H}_t)$-space–time white noise $\widetilde{W}$ independent of $(X^{(0)}, Y^{(0)})$. Let $M^{X^{(0)}}$ and $M^{Y^{(0)}}$ denote the martingale measures of $X^{(0)}$ and $Y^{(0)}$, respectively [cf. the definition of $M^{X^{(n)}}$ in (5.1)]. Then by the foregoing display, the required space–time white noise can be chosen to be

$$\begin{aligned} W_t(\phi) \triangleq{}& \int_0^t \int_{\mathbb{R}} \mathbb{1}_{(X^{(0)}>0)}(y,s) \frac{\phi(x)}{X^{(0)}(y,s)^{1/2}}\, dM^{X^{(0)}}(y,s) \\ &+ \int_0^t \int_{\mathbb{R}} \mathbb{1}_{(X^{(0)}=0, Y^{(0)}>0)}(y,s) \frac{\phi(x)}{Y^{(0)}(y,s)^{1/2}}\, dM^{Y^{(0)}}(y,s) \\ &+ \int_0^t \int_{\mathbb{R}} \mathbb{1}_{(X^{(0)}=0, Y^{(0)}=0)}(y,s)\phi(x)\, d\widetilde{W}(y,s), \qquad \phi \in \mathscr{C}_c^\infty(\mathbb{R}) \end{aligned}$$

[recall the notation $(Z \in \Gamma)$ in (2.7)]. The proof is complete. □

6. Proof of Proposition 3.3. In this section, we prove Proposition 3.3 by verifying all of the following analogues of (3.12):

$$\begin{aligned} &\forall \rho > 0\ \exists \delta > 0 \text{ such that} \\ &\sup\left\{\mathbb{Q}_\varepsilon^i\left(\tau^{i,(j)} \le s_i + \delta\right); i \in \mathbb{N}, \varepsilon \in \left(0, \frac{1}{8\psi(\mathbb{1})} \wedge 1\right]\right\} \le \rho, \end{aligned} \tag{6.1}$$

where $1 \le j \le 3$. The proofs rely on the basic fact that for any $i \in \mathbb{N}$ and $\varepsilon \in (0, [8\psi(\mathbb{1})]^{-1} \wedge 1]$, $X^i(\mathbb{1})^{T_1^{X^i}}$ under $\mathbb{Q}_\varepsilon^i$ is a $\frac{1}{4}\mathrm{BES}Q^4(4\psi(\mathbb{1})\varepsilon)$ started at s_i and stopped upon hitting 1 (see the discussion after Proposition 3.3), and we will work with various couplings of $\frac{1}{4}\mathrm{BES}Q^4(4z)$. We assume that the couplings are obtained from a $(\mathscr{H}_t)$-standard Brownian motion B for a filtration $(\mathscr{H}_t)$ satisfying the usual conditions, the constructions explained in detail later on.

Recall that we write $\mathbf{P}^1_z$ for the law of a copy Z of $\frac{1}{4}\mathrm{BES}Q^4(4z)$. Throughout this section, we do *not* impose the constraints in Assumption 3.4 on the auxiliary parameters.

LEMMA 6.1. *Fix $\eta \in (1,\infty)$, and let $\tau^{i,(1)}$ be the stopping times defined in Proposition 3.3. Then (6.1) holds for $j=1$.*

PROOF. The proof is an application of the lower escape rate of $\mathrm{BES}Q^4(0)$:

$$\mathbf{P}^1_0\big(\exists h>0 \text{ such that } 4Z_t \geq t^\eta, \forall t\in[0,h]\big)=1 \tag{6.2}$$

(cf. Theorem 5.4.6 of [14]).

We will need a monotonicity of $\mathrm{BES}Q^4$ in initial values. For this purpose, we construct all $\frac{1}{4}\mathrm{BES}Q^4(4z)$-processes Z^z with initial values $z\in\mathbb{R}_+$ from the $(\mathscr{H}_t)$-standard Brownian motion. This is implied by the pathwise uniqueness in their stochastic differential equations (cf. Theorems IX.1.7 and IX.3.5 of [23]), and we can characterize them by

$$Z^z_t \equiv z+t+\int_0^t \sqrt{Z^z_s}\,dB_s, \qquad z\in\mathbb{R}_+. \tag{6.3}$$

In view of the first components in $\tau^{i,(1)}_\varepsilon$ (cf. Proposition 3.3), we consider

$$\sigma_z \triangleq \inf\left\{t\geq 0;\ Z^z_{T_1^{Z^z}\wedge t} < \frac{t^\eta}{4}\right\}, \qquad z\in\left[0,\frac{1}{8}\right].$$

Let us bound the distribution function of $\sigma_z\wedge T_1^{Z^z}$. The comparison theorem of stochastic differential equations (cf. Theorem IX.3.7 of [23]) implies that $Z^{z_1}\leq Z^{z_2}$ whenever $0\leq z_1\leq z_2<\infty$. In particular, for any $z\in(0,\frac{1}{8}]$,

$$T_1^{Z^{1/8}} \leq T_1^{Z^z} \leq T_1^{Z^0} \quad\text{and}\quad \sigma_z\geq\sigma_0 \qquad \text{a.s.}, \tag{6.4}$$

where the second inequality follows since

$$Z^z_{t\wedge T_1^{Z^z}} \geq Z^0_{t\wedge T_1^{Z^0}} \geq \frac{t^\eta}{4} \qquad \forall t\in[0,\sigma_0].$$

Hence, by (6.4), we have

$$\begin{aligned}\sup_{z\in(0,1/8]}\mathbb{P}\big(\sigma_z\wedge T_1^{Z^z}\leq\delta\big) &\leq \sup_{z\in(0,1/8]}\mathbb{P}(\sigma_z\leq\delta)+\sup_{z\in(0,1/8]}\mathbb{P}\big(T_1^{Z^z}\leq\delta\big)\\ &\leq \mathbb{P}(\sigma_0\leq\delta)+\mathbb{P}\big(T_1^{Z^{1/8}}\leq\delta\big) \qquad \forall\delta\in(0,\infty).\end{aligned}$$

Applying the lower escape rate (6.2) to the right-hand side of the foregoing inequality shows that

$$\forall\rho>0\ \exists\delta>0 \qquad \text{such that } \sup_{z\in(0,1/8]}\mathbb{P}\big(\sigma_z\wedge T_1^{Z^z}\leq\delta\big)\leq\rho.$$

Using the foregoing display and the distributional property of $X^i(\mathbb{1})^{T_1^{X^i}}$ under $\mathbb{Q}^i_\varepsilon$, for $i\in\mathbb{N}$ and $\varepsilon\in(0,[8\psi(\mathbb{1})]^{-1}\wedge 1]$ mentioned above, we have proved our assertion (6.1) for $j=1$. $\square$

LEMMA 6.2. *Fix $L \in (0, \infty)$ and $\alpha \in (0, \frac{1}{2})$, and let $\tau^{i,(2)}$ be the stopping times defined in Proposition 3.3. Then (6.1) holds for $j = 2$.*

PROOF. As in the proof of Lemma 6.1, we need a grand coupling of all $\frac{1}{4}\mathrm{BES}Q^4(4z)$, $z \in \mathbb{R}_+$, on the same probability space. For the first component of $\tau^{i,(2)}$, we need to measure the modulus of continuity of the martingale part of a $\frac{1}{4}\mathrm{BES}Q^4$ in terms of its quadratic variation. Hence, it will be convenient to extract all of the $\frac{1}{4}\mathrm{BES}Q^4(4z)$'s, say Z^z, from a fixed copy Z of $\frac{1}{4}\mathrm{BES}Q^4(0)$, and we consider $Z^z_t \equiv Z_{T^Z_z + t}$ for $z \in \mathbb{R}_+$, where the stopping times T^Z_z are finite almost surely by the transience of $\mathrm{BES}Q^4$ (cf. page 442 of [23]). We may further assume that $Z = Z^0$ and is defined by (6.3). It follows that

$$Z^z_t = z + t + \int_{T^Z_z}^{T^Z_z + t} \sqrt{Z_s}\, dB_s. \tag{6.5}$$

In this case, the analogues of $\tau^{i,(2)}$ are given by, for $z \in (0, \frac{1}{8}]$,

$$\begin{aligned}\sigma_z &\triangleq \inf\left\{t \geq 0; \left|Z^z_{t \wedge T_1^{Z^z}} - z - t\right| > L\left(\int_0^t Z^z_{s \wedge T_1^{Z^z}}\, ds\right)^{\alpha}\right\} \wedge T_1^{Z^z} \\ &= \inf\left\{t \geq 0; \left|\int_{T^Z_z}^{(T^Z_z + t)\wedge T^Z_1} \sqrt{Z_s}\, dB_s + \left(t \wedge T_1^{Z^z} - t\right)\right|\right. \\ &\qquad \left. > L\left[\int_{T^Z_z}^{(T^Z_z + t)\wedge T^Z_1} Z_s\, ds + \left(t \wedge T_1^{Z^z} - t\right)\right]^{\alpha}\right\} \wedge T_1^{Z^z},\end{aligned} \tag{6.6}$$

where the last equality follows from (6.5) and the obvious equality $T^Z_z + T_1^{Z^z} = T^Z_1$.

Let us bound the distribution function of σ_z. By the Dambis–Dubins–Schwarz theorem (cf. Theorem V.1.6 of [23]), $\sqrt{Z} \bullet B = B'_{\langle Z \rangle}$ for some standard Brownian motion B', where $\langle Z \rangle = \int_0^\cdot Z_s\, ds$. Also,

$$0 < \int_{T^Z_z}^{(T^Z_z + t)\wedge T^Z_1} Z_s\, ds \leq t \qquad \text{if } t > 0,$$

where the first inequality follows since $\{0\}$ is polar for $\mathrm{BES}Q^4$ (cf. page 442 of [23]). Hence, from (6.6), we deduce that, for any $H, \delta \in (0, \infty)$,

$$\begin{aligned}&\sup_{z \in (0, 1/8]} \mathbb{P}(\sigma_z \leq \delta) \\ &\quad \leq \mathbb{P}\left(T^Z_1 > H\right) + \mathbb{P}\left(\sup_{\substack{0 < |t-s| \leq 2\delta \\ 0 \leq s < t \leq H}} \frac{|B'_t - B'_s|}{|t-s|^{\alpha}} > L\right) + \mathbb{P}\left(T_1^{Z^{1/8}} \leq \delta\right)\end{aligned} \tag{6.7}$$

[for the third probability, recall the inequalities for hitting times of 1 by $\mathrm{BES}Q^4$ in (6.4)].

Let us make the dependence on δ of the second probability of (6.7) explicit. For the fixed $\alpha \in (0, \frac{1}{2})$, we pick $\alpha' \in (0, \frac{1}{2})$ and $p > 1$ such that $\alpha < \alpha' < \frac{p-1}{2p}$. Then applying Chebyshev's inequality to the second term on the right-hand side of (6.7), we get

$$
\begin{aligned}
&\sup_{z\in(0,1/8]} \mathbb{P}(\sigma_z \le \delta)\\
&\qquad \le \mathbb{P}\big(T_1^Z > H\big) + \frac{(2\delta)^{2p(\alpha'-\alpha)}}{L^{2p}} \mathbb{E}\bigg[\bigg(\sup_{0\le s<t\le H} \frac{|B'_t - B'_s|}{|t-s|^{\alpha'}}\bigg)^{2p}\bigg] \qquad (6.8)\\
&\qquad\quad + \mathbb{P}\big(T_1^{Z^{1/8}} \le \delta\big) \qquad \forall H, \delta \in (0,\infty),
\end{aligned}
$$

where

$$
\mathbb{E}\bigg[\bigg(\sup_{0\le s<t\le H} \frac{|B'_t - B'_s|}{|t-s|^{\alpha'}}\bigg)^{2p}\bigg] < \infty \tag{6.9}
$$

(cf. the discussion preceding Theorem I.2.2 of [23] as well as its Theorem I.2.1).

By the transience of $\mathrm{BES}Q^4$, the first probability on the right-hand side of (6.8) can be made as small as possible by choosing sufficiently large H. Since $(\sigma_{\psi(\mathbb{1})\varepsilon}, \mathbb{P})$ and $(\tau^{i,(2)}, \mathbb{P}_\varepsilon)$ have the same distribution and $\psi(\mathbb{1})\varepsilon \le \frac{1}{8}$, (6.8) and (6.9) are enough to obtain (6.1) for $j = 2$. The proof is complete. $\square$

It remains to prove (6.1) for $j = 3$. We need a few preliminary results.

LEMMA 6.3. *Fix $i \in \mathbb{N}$ and $\varepsilon \in (0, [8\psi(\mathbb{1})]^{-1} \wedge 1]$. Then*

$$
\mathbb{E}^{\mathbb{Q}_\varepsilon^i}\bigg[\bigg(\sup_{r\in[0,R]} \sum_{j:\, s_i<t_j\le s_i+r} Y^j_{s_i+r}(\mathbb{1})\bigg)^p\bigg] < \infty \qquad \forall p, R \in (0,\infty). \tag{6.10}
$$

PROOF. Plainly, it suffices to consider $p > 1$. By Lemma 3.1, we have

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}_\varepsilon^i}\bigg[\bigg(\sup_{r\in[0,R]} \sum_{j:\, s_i<t_j\le s_i+r} Y^j_{s_i+r}(\mathbb{1})\bigg)^p\bigg]\\
&\qquad = \frac{1}{\psi(\mathbb{1})\varepsilon} \mathbb{E}^{\mathbb{P}_\varepsilon}\bigg[X^i_{s_i+R}(\mathbb{1})^{T_1^{X^i}} \bigg(\sup_{r\in[0,R]} \sum_{j:\, s_i<t_j\le s_i+r} Y^j_{s_i+r}(\mathbb{1})\bigg)^p\bigg]\\
&\qquad \le \frac{1}{\psi(\mathbb{1})\varepsilon} \mathbb{E}^{\mathbb{P}_\varepsilon}\bigg[\bigg(\sup_{r\in[0,R]} \sum_{j:\, s_i<t_j\le s_i+r} Y^j_{s_i+r}(\mathbb{1})\bigg)^p\bigg]\\
&\qquad \le \frac{1}{\psi(\mathbb{1})\varepsilon} \mathbb{E}^{\mathbb{P}_\varepsilon}\bigg[\bigg(\sum_{j:\, s_i<t_j\le s_i+R} \sup_{t\in[t_j, s_i+R]} Y^j_t(\mathbb{1})\bigg)^p\bigg] \qquad (6.11)
\end{aligned}
$$

$$\leq \frac{1}{\psi(\mathbb{1})\varepsilon}\#\{j; s_i < t_j \leq s_i + R\}^{p-1} \times \sum_{j : s_i < t_j \leq s_i + R} \mathbb{E}^{\mathbb{P}_\varepsilon}\Big[\Big(\sup_{t \in [t_j, s_i+R]} Y_t^j(\mathbb{1})\Big)^p\Big],$$

where the last inequality follows from Hölder's inequality. Since each $Y^j(\mathbb{1})$ under $\mathbb{P}_\varepsilon$ is a Feller diffusion with initial value $\psi(\mathbb{1})\varepsilon$ and started at t_j, the summands on the right-hand side of (6.11) are finite. This gives (6.10), and the proof is complete. □

Next, we recall the canonical decomposition of $Y^j(\mathbb{1})$ for $t_j > s_i$ under $\mathbb{Q}^i_\varepsilon$ in Lemma 3.2(2). Recall (2.73) and the explicit form (3.6) of the finite variation process I^j of $Y^j(\mathbb{1})$ under $\mathbb{Q}^i_\varepsilon$. Then by the Cauchy–Schwarz inequality, we deduce that

$$\begin{aligned} \sum_{j : s_i < t_j \leq t} \big(\psi(\mathbb{1})\varepsilon + I_t^j\big) &\leq \psi(\mathbb{1})\varepsilon\#\{j; s_i < t_j \leq t\} + \int_{s_i}^{t \wedge T_1^{X^i}} \bigg(\frac{\sum_{j : s_i < t_j \leq t} Y_s^j(\mathbb{1})}{X_s^i(\mathbb{1})}\bigg)^{1/2} ds \\ &\leq 2\psi(\mathbb{1})(t - s_i) + \int_{s_i}^{t \wedge T_1^{X^i}} \bigg(\frac{\sum_{j : s_i < t_j \leq s} Y_s^j(\mathbb{1})}{X_s^i(\mathbb{1})}\bigg)^{1/2} ds \\ &\qquad \forall t \in [s_i, \infty). \end{aligned} \tag{6.12}$$

Here, the last inequality follows since for $t \geq s_i + \frac{\varepsilon}{2}$, $s_i + \varepsilon(\#\{j; s_i < t_j \leq t\} - \frac{1}{2}) \leq t$ and the clusters Y^j with $s < t_j \leq t$ have no contributions to $\sum_{j : s_i < t_j \leq t} Y_s^j(\mathbb{1})$.

Also, recall that M^j denotes the martingale part of $Y^j(\mathbb{1})$ under $\mathbb{Q}^i_\varepsilon$, and the super-Brownian motions Y^j are $\mathbb{P}_\varepsilon$-independent by Theorem 2.12. Hence, we deduce from Girsanov's theorem (cf. Theorem VIII.1.4 of [23]) that

$$\Big\langle \sum_{j : s_i < t_j \leq \cdot} M^j \Big\rangle_t = \int_{s_i}^t \sum_{j : s_i < t_j \leq t} Y_s^j(\mathbb{1})\, ds = \int_{s_i}^t \sum_{j : s_i < t_j \leq s} Y_s^j(\mathbb{1})\, ds \qquad \forall t \in [s_i, \infty), \tag{6.13}$$

where the omission of the clusters Y^j for $s < t_j \leq t$ follows from the same reason as in (6.12).

LEMMA 6.4. *Fix $i \in \mathbb{N}$ and $\varepsilon \in (0, [8\psi(\mathbb{1})]^{-1} \wedge 1]$. Then*

$$\mathbb{E}^{\mathbb{Q}^i_\varepsilon}\bigg[\frac{1}{[X^i_{s_i+r}(\mathbb{1})]^a}; s_i + r \leq T_1^{X^i}\bigg] \leq \frac{1}{r^a}\mathbb{E}^{\mathbf{P}^1_0}\bigg[\frac{1}{(Z_1)^a}\bigg] \qquad \forall r, a \in (0, \infty), \tag{6.14}$$

where

$$\mathbb{E}^{\mathbf{P}^1_0}\left[\frac{1}{(Z_1)^a}\right] < \infty \quad \Longleftrightarrow \quad a \in (-\infty, 2). \tag{6.15}$$

PROOF. Recall the grand coupling of $\frac{1}{4}\mathrm{BES}Q^4(4z)$ in the proof of Lemma 6.1 under which $Z^{z_1} \leq Z^{z_2}$ whenever $0 \leq z_1 \leq z_2$. Then for every $r, a \in (0, \infty)$,

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}^i_\varepsilon}\left[\frac{1}{[X^i_{s_i+r}(\mathbb{1})]^a}; s_i + r \leq T_1^{X^i}\right] &\leq \mathbb{E}^{\mathbf{P}^1_{\psi(\mathbb{1})\varepsilon}}\left[\frac{1}{(Z_r)^a}\right] \\
&\leq \mathbb{E}^{\mathbf{P}^1_0}\left[\frac{1}{(Z_r)^a}\right] \\
&= \frac{1}{r^a}\mathbb{E}^{\mathbf{P}^1_0}\left[\frac{1}{(Z_1)^a}\right],
\end{aligned}$$

where the last equality follows from the scaling property of Bessel squared processes (cf. Proposition XI.1.6 of [23]). This gives the bound (6.14). In addition, notice that Z under $\mathbf{P}^1_0$ has the same distribution as the image of a 4-dimensional standard Brownian motion under $x \longmapsto \|x\|^2$ where $\|\cdot\|$ denotes the Euclidean norm, and so we deduce (6.15) by writing out the expectation on its left-hand side as an elementary integral in polar coordinates. The proof is complete. □

With Lemma 6.4, we have the following improvement of (6.10).

LEMMA 6.5. *Fix $i \in \mathbb{N}$ and $\varepsilon \in (0, [8\psi(\mathbb{1})]^{-1} \wedge 1]$. Then we have*

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}^i_\varepsilon}\left[\sum_{j:\, s_i < t_j \leq s_i + r} Y^j_{s_i+r}(\mathbb{1})\right] \\
&\quad \leq \left(2\psi(\mathbb{1})R + \mathbb{E}^{\mathbf{P}^1_0}\left[\frac{1}{Z_1}\right]^{1/2} 2R^{1/2}\right)\exp\left(2\mathbb{E}^{\mathbf{P}^1_0}\left[\frac{1}{Z_1}\right]^{1/2}\sqrt{r}\right) \\
&\qquad \forall r \in [0, R], R \in (0, \infty),
\end{aligned} \tag{6.16}$$

where $\mathbb{E}^{\mathbf{P}^1_0}[1/Z_1] < \infty$ by (6.15).

PROOF. Recall that the local martingale part of $Y^j(\mathbb{1})$ under $\mathbb{Q}^i_\varepsilon$ is a true martingale by Lemma 3.2(2). Hence, for any $r \in [0, R]$, we obtain from (6.12) that

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}^i_\varepsilon}\left[\sum_{j:\, s_i < t_j \leq s_i + r} Y^j_{s_i+r}(\mathbb{1})\right] \\
&\quad \leq 2\psi(\mathbb{1})r + \int_{s_i}^{s_i+r} \mathbb{E}^{\mathbb{Q}^i_\varepsilon}\left[\left(\frac{\sum_{j:\, s_i < t_j \leq s} Y^j_s(\mathbb{1})}{X^i_s(\mathbb{1})}\right)^{1/2}; s \leq T_1^{X^i}\right] ds
\end{aligned} \tag{6.17}$$

$$
\begin{aligned}
&\leq 2\psi(\mathbb{1})r \\
&\quad + \int_{s_i}^{s_i+r} \mathbb{E}^{\mathbb{Q}^i_\varepsilon}\left[\frac{1}{X^i_s(\mathbb{1})}; s \leq T_1^{X^i}\right]^{1/2} \mathbb{E}^{\mathbb{Q}^i_\varepsilon}\left[\sum_{j\,:\,s_i<t_j\leq s} Y^j_s(\mathbb{1})\right]^{1/2} ds \\
&\leq 2\psi(\mathbb{1})r \\
&\quad + \int_{s_i}^{s_i+r} \frac{1}{\sqrt{s-s_i}} \mathbb{E}^{\mathbf{P}^1_0}\left[\frac{1}{Z_1}\right]^{1/2} \left(1 + \mathbb{E}^{\mathbb{Q}^i_\varepsilon}\left[\sum_{j\,:\,s_i<t_j\leq s} Y^j_s(\mathbb{1})\right]\right) ds \\
&\leq \left(2\psi(\mathbb{1})R + \mathbb{E}^{\mathbf{P}^1_0}\left[\frac{1}{Z_1}\right]^{1/2} 2R^{1/2}\right) \\
&\quad + \mathbb{E}^{\mathbf{P}^1_0}\left[\frac{1}{Z_1}\right]^{1/2} \int_0^r \frac{1}{\sqrt{s}} \mathbb{E}^{\mathbb{Q}^i_\varepsilon}\left[\sum_{j\,:\,s_i<t_j\leq s_i+s} Y^j_{s_i+s}(\mathbb{1})\right] ds,
\end{aligned} \tag{6.18}
$$

where the third inequality follows from Lemma 6.4. With the change of variables $s' = \sqrt{s}$, the foregoing inequality with r replaced by r^2 and R by R^2 becomes

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}^i_\varepsilon}\left[\sum_{j\,:\,s_i<t_j\leq s_i+r^2} Y^j_{s_i+r^2}(\mathbb{1})\right] \\
&\quad \leq \left(2\psi(\mathbb{1})R^2 + \mathbb{E}^{\mathbf{P}^1_0}\left[\frac{1}{Z_1}\right]^{1/2} 2R\right) \\
&\quad\quad + 2\mathbb{E}^{\mathbf{P}^1_0}\left[\frac{1}{Z_1}\right]^{1/2} \int_0^r \mathbb{E}^{\mathbb{Q}^i_\varepsilon}\left[\sum_{j\,:\,s_i<t_j\leq s_i+(s')^2} Y^j_{s_i+(s')^2}(\mathbb{1})\right] ds' \\
&\hspace{25em} \forall r \in [0, R],
\end{aligned}
$$

so by Lemma 6.3 and Gronwall's lemma

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}^i_\varepsilon}\left[\sum_{j\,:\,s_i<t_j\leq s_i+r^2} Y^j_{s_i+r^2}(\mathbb{1})\right] \\
&\quad \leq \left(2\psi(\mathbb{1})R^2 + \mathbb{E}^{\mathbf{P}^1_0}\left[\frac{1}{Z_1}\right]^{1/2} 2R\right) \exp\left(2\mathbb{E}^{\mathbf{P}^1_0}\left[\frac{1}{Z_1}\right]^{1/2} r\right) \\
&\hspace{25em} \forall r \in [0, R].
\end{aligned}
$$

With another change of time scales by $r' = r^2$, the foregoing gives the desired inequality (6.16). The proof is complete. □

We are ready to prove (6.1) for $j = 3$.

LEMMA 6.6. *Let $\tau^{i,(3)}$ be the stopping times defined in Proposition 3.3. Then (6.1) holds for $j = 3$.*

PROOF. Fix $i \in \mathbb{N}$ and $\varepsilon \in (0, [8\psi(\mathbb{1})]^{-1} \wedge 1]$. It follows from (6.12) that, for any $R > 0$ with $\frac{1}{3} \geq 2\psi(\mathbb{1})R$, we have

$$\begin{aligned}
&\mathbb{Q}^i_\varepsilon\Big(\sup_{r\in[0,R]} \sum_{j:s_i<t_j\leq s_i+r} Y^j_{s_i+r}(\mathbb{1}) > 1\Big) \\
&\quad\leq \mathbb{Q}^i_\varepsilon\bigg(\int_{s_i}^{(s_i+R)\wedge T_1^{X^i}} \bigg(\frac{\sum_{j:s_i<t_j\leq s} Y^j_s(\mathbb{1})}{X^i_s(\mathbb{1})}\bigg)^{1/2} ds > \frac{1}{3}\bigg) \\
&\qquad + \mathbb{Q}^i_\varepsilon\bigg(\sup_{r\in[0,R]}\bigg|\sum_{j:s_i<t_j\leq s_i+r} M^j_{s_i+r}\bigg| > \frac{1}{3}\bigg) \qquad (6.19)\\
&\quad\leq 3\mathbb{E}^{\mathbb{Q}^i_\varepsilon}\bigg[\int_{s_i}^{(s_i+R)\wedge T_1^{X^i}} \bigg(\frac{\sum_{j:s_i<t_j\leq s} Y^j_s(\mathbb{1})}{X^i_s(\mathbb{1})}\bigg)^{1/2} ds\bigg] \\
&\qquad + 9\sup_{r\in[0,R]} \mathbb{E}^{\mathbb{Q}^i_\varepsilon}\bigg[\bigg(\sum_{j:s_i<t_j\leq s_i+r} M^j_{s_i+r}\bigg)^2\bigg],
\end{aligned}$$

where the first term of the last inequality follows from Chebyshev's inequality, and the second term follows by applying Doob's L^2-inequality to the $\mathbb{Q}^i_\varepsilon$-martingale $\sum_{j:s_i<t_j\leq\cdot} M^j$.

We claim that the right-hand side of (6.19) converges to zero uniformly in $i \in \mathbb{N}$ and $\varepsilon \in (0, [8\psi(\mathbb{1})\varepsilon]^{-1} \wedge 1]$ as $R \longrightarrow 0+$. Inspecting the arguments from (6.17) to (6.18) shows that the first term in (6.19) satisfies

$$\begin{aligned}
&3\mathbb{E}^{\mathbb{Q}^i_\varepsilon}\bigg[\int_{s_i}^{(s_i+R)\wedge T_1^{X^i}} \bigg(\frac{\sum_{j:s_i<t_j\leq s} Y^j_s(\mathbb{1})}{X^i_s(\mathbb{1})}\bigg)^{1/2} ds\bigg] \\
&\quad\leq 3\bigg(2\psi(\mathbb{1})R + \mathbb{E}^{\mathbf{P}^1_0}\bigg[\frac{1}{Z_1}\bigg]^{1/2} 2R^{1/2}\bigg) \\
&\qquad + 3\mathbb{E}^{\mathbf{P}^1_0}\bigg[\frac{1}{Z_1}\bigg]^{1/2} \int_0^R \frac{1}{\sqrt{s}} \mathbb{E}^{\mathbb{Q}^i_\varepsilon}\bigg[\sum_{j:s_i<t_j\leq s_i+s} Y^j_{s_i+s}(\mathbb{1})\bigg] ds.
\end{aligned}$$

For the second term on the right-hand side of (6.19), we use (6.13) and obtain

$$\begin{aligned}
&9\sup_{r\in[0,R]} \mathbb{E}^{\mathbb{Q}^i_\varepsilon}\bigg[\bigg(\sum_{j:s_i<t_j\leq s_i+r} M^j_{s_i+r}\bigg)^2\bigg] \\
&\quad\leq \int_0^R \mathbb{E}^{\mathbb{Q}^i_\varepsilon}\bigg[\sum_{j:s_i<t_j\leq s_i+s} Y^j_{s_i+s}(\mathbb{1})\bigg] ds.
\end{aligned}$$

Applying the uniform bound (6.16) to the right-hand sides of the last two displays shows the existence of a constant $C \in (0, \infty)$ depending only on ψ such that

$$\mathbb{Q}_\varepsilon^i\Bigg(\sup_{r\in[0,R]} \sum_{j\,:\,s_i<t_j\le s_i+r} Y^j_{s_i+r}(\mathbb{1}) > 1\Bigg) \le C R^{1/2}$$
$$\forall R \in \left(0, \frac{1}{6\psi(\mathbb{1})}\right], i \in \mathbb{N}, \varepsilon \in \left(0, \frac{1}{8\psi(\mathbb{1})} \wedge 1\right],$$

where the restriction on R follows since $\frac{1}{3} \ge 2\psi(\mathbb{1})R$. The foregoing inequality proves our claim and is enough for our assertion of the present lemma. $\square$

7. Some properties of support processes. We study the supports of the immigrant processes X^i, Y^j in this section. Recall that Assumption 3.4 does not apply to the present section.

PROPOSITION 7.1. *There is a constant $C^0_{\mathrm{supp}} \in (0, \infty)$ depending only on the immigration function ψ and the parameter $\beta \in [\frac{1}{4}, \frac{1}{2})$ such that*

$$\mathbb{P}_\varepsilon\big(\sigma_\beta^{X^i} - s_i \le r\big) + \mathbb{P}_\varepsilon\big(\sigma_\beta^{Y^i} - t_i \le r\big) \le C^0_{\mathrm{supp}}\varepsilon(r \vee \varepsilon) \tag{7.1}$$
$$\forall \varepsilon, r \in (0, 1], i \in \mathbb{N}.$$

PROOF. The immigrant processes satisfy the SPDE (1.1) with initial condition taking the form $\psi(\mathbb{1})J^a_\varepsilon$ [recall that J^a_ε is defined by (1.13)]. Hence, Corollary 7.2 of [17] applies to the normalized processes $X^i/\psi(\mathbb{1})$ and $Y^i/\psi(\mathbb{1})$ with the parameter a in equation (7.1) set to be $\psi(\mathbb{1})^{-1/2}$. Our assertion follows. $\square$

In the remainder of this section, we consider, under $\mathbb{Q}^i_\varepsilon$, the supports of the immigrant processes Y^j landing by time $s_i + r \in (s_i, \infty)$ and with space–time locations (y_j, t_j) lying *outside* the rectangle $\mathcal{R}^{X^i}_\beta(s_i + r)$ defined by (3.18). We start with the immigrants Y^j landing before time s_i.

PROPOSITION 7.2. *There exists a constant $C^1_{\mathrm{supp}} \in (0, \infty)$ depending only on the immigration function ψ such that whenever $\beta \in [\frac{1}{3}, \frac{1}{2})$,*

$$\mathbb{Q}^i_\varepsilon\Bigg(\mathcal{P}^{X^i}_\beta(s_i + r) \cap \Bigg(\bigcup_{j\,:\,t_j\le s_i} \mathrm{supp}(Y^j)\Bigg) \ne \varnothing,$$
$$\min_{j\,:\,t_j\le s_i} \big(\sigma^{Y^j}_\beta - t_j\big) > 3r, \sigma^{X^i}_\beta - s_i > 2r\Bigg)$$
$$\le C^1_{\mathrm{supp}} r^{1/6} \qquad \forall i \in \mathbb{N} \textit{ with } s_i \le 1, r \in [s_i, 1], \varepsilon \in (0, r].$$

The proof of Proposition 7.2 is similar to the proof of Lemma 8.4 in [17] for $\gamma = 1/2$ (note that our notation β is denoted by α there instead), except that in [17] the immigrant processes are subject to i.i.d. space–time white noises, but in our case they are not. For this reason, we need a slightly different argument whenever covariations between the involved immigrants may be nonzero. Roughly speaking, we will handle the Y^j-immigrants which land a bit "far away" from the support of X^i in both space and time. Since these immigrants do not interfere with X^i immediately, we can apply orthogonal continuation (Lemma 3.15) to $X^i(\mathbb{1})$ and then argue as in [17] accordingly.

SKETCH OF PROOF OF PROPOSITION 7.2. We give the details to handle the Y^j-immigrants mentioned above and sketch the rest of the proof. A complete proof can be found in Section 3.12 of [3].

Fix $(\beta, i, r, \varepsilon)$ as described in the statement of Proposition 7.2. We will argue throughout this proof on the event that

$$\min_{j\,:\,t_j \le s_i} \big(\sigma_\beta^{Y^j} - t_j\big) > 3r \quad \text{and} \quad \sigma_\beta^{X^i} - s_i > 2r. \tag{7.2}$$

Let n_0 and n_1 be nonnegative integers chosen as equation (8.5) in [17], that is,

$$2^{-n_0-1} < r \le 2^{-n_0} \quad \text{and} \quad 2^{-n_1-1} < \varepsilon \le 2^{-n_1}. \tag{7.3}$$

Then as in the proof of Lemma 8.4 in [17] [cf. equation (8.8) there], we have

$$\begin{aligned} &\big\{(y_j, t_j); t_j \in (0, s_i), \mathcal{P}_\beta^{X^i}(s_i + r) \cap \operatorname{supp}(Y^j) \ne \varnothing\big\} \\ &\quad \subseteq \big[x_i - 7 \cdot 2^{-n_0\beta}, x_i + 7 \cdot 2^{-n_0\beta}\big] \times [0, s_i). \end{aligned} \tag{7.4}$$

The inclusion in (7.4) rules out a number of clusters Y^j landing before s_i whose space–time supports can intersect $\mathcal{P}_\beta^{X^i}(s_i + r)$ by time $s_i + r$. In the following, we handle the remaining immigrant processes Y^j for $j \in \mathbb{N}$ with $t_j < s_i$.

As in the proof of Lemma 8.4 of [17], we classify the clusters Y^j for $j \in \mathbb{N}$ satisfying $t_j \in (0, s_i)$ and $y_j \notin [x_i - 7 \cdot 2^{-n_0\beta}, x_i + 7 \cdot 2^{-n_0\beta}]$ according to the space–time landing locations (y_j, t_j). Define the following *random* rectangles

$$\begin{aligned} \mathcal{R}_n^0 &= \big[x_i - 7 \cdot 2^{-n\beta}, x_i + 7 \cdot 2^{-n\beta}\big] \times \big[s_i - 2^{-n+1}, s_i - 2^{-n}\big], \\ \mathcal{R}_n^{\mathtt{L}} &= \big[x_i - 7 \cdot 2^{-n\beta}, x_i - 7 \cdot 2^{-(n+1)\beta}\big] \times \big[s_i - 2^{-n}, s_i\big], \\ \mathcal{R}_n^{\mathtt{R}} &= \big[x_i + 7 \cdot 2^{-(n+1)\beta}, x_i + 7 \cdot 2^{-n\beta}\big] \times \big[s_i - 2^{-n}, s_i\big], \end{aligned}$$

for nonnegative integers $n \ge n_0$, and we group the clusters Y^j according to these rectangles by

$$Y^{(n),q} \triangleq \sum_{j\,:\,t_j \le s_i} \mathbb{1}_{\mathcal{R}_n^q}(y_j, t_j) Y^j, \qquad q = \mathtt{L}, \mathtt{0}, \mathtt{R}, n \ge n_0.$$

Then as in equation (8.11) of [17], the probability under consideration can be bounded as

$$
\begin{aligned}
&\mathbb{Q}^i_\varepsilon\Big(\mathcal{P}^{X^i}_\beta(s_i+r)\cap\Big(\bigcup_{j:t_j\le s_i}\operatorname{supp}(Y^j)\Big)\neq\varnothing,\\
&\qquad\min_{j:t_j\le s_i}\big(\sigma^{Y^j}_\beta-t_j\big)>3r,\sigma^{X^i}_\beta-s_i>2r\Big)\\
&\quad\le\mathbb{Q}^i_\varepsilon\Big(\bigcup_{n=n_1+1}^{\infty}\bigcup_{q=\mathtt{L},0,\mathtt{R}}\big\{\mathcal{P}^{X^i}_\beta(s_i+r)\cap\operatorname{supp}\big(Y^{(n),q}\big)\neq\varnothing\big\}\Big)\\
&\qquad+\sum_{n=n_0}^{n_1}\sum_{q=\mathtt{L},0,\mathtt{R}}\mathbb{Q}^i_\varepsilon\Big(\mathcal{P}^{X^i}_\beta(s_i+r)\cap\operatorname{supp}\big(Y^{(n),q}\big)\neq\varnothing,\\
&\qquad\qquad\min_{j:t_j\le s_i}\big(\sigma^{Y^j}_\beta-t_j\big)>3r,\sigma^{X^i}_\beta-s_i>2r\Big).
\end{aligned}\tag{7.5}
$$

Recall that the landing locations x_i and y_i have distributions given by (1.15). Then following the arguments between equations (8.11) and (8.22) in [17], we deduce that

$$
\mathbb{Q}^i_\varepsilon\Big(\bigcup_{n=n_1+1}^{\infty}\bigcup_{q=\mathtt{L},0,\mathtt{R}}\big\{\mathcal{P}^{X^i}_\beta(s_i+t)\cap\operatorname{supp}\big(Y^{(n),q}\big)\neq\varnothing\big\}\Big)\le C_\psi\varepsilon^\beta,\tag{7.6}
$$

$$
\begin{aligned}
&\mathbb{Q}^i_\varepsilon\Big(\mathcal{P}^{X^i}_\beta(s_i+r)\cap\operatorname{supp}\big(Y^{(n),0}\big)\neq\varnothing,\\
&\qquad\min_{j:t_j\le s_i}\big(\sigma^{Y^j}_\beta-t_j\big)>3r,\sigma^{X^i}_\beta-s_i>2r\Big)\\
&\quad\le C_\psi 2^{-n/6}
\end{aligned}\tag{7.7}
$$

for some constant C_ψ depending only on the immigration function ψ.

It remains to deal with the summands on the right-hand side of (7.5) associated with $Y^{(n),\mathtt{R}}$ for $n_0\le n\le n_1$ (the probability bounds for $Y^{(n),\mathtt{L}}$ follow similarly). In this case, the Y^j summands in $Y^{(n),\mathtt{R}}$ can arrive up to $s_i-\frac{\varepsilon}{2}$, and hence, $Y^{(n),\mathtt{R}}$ can survive beyond s_i when the covariation between $Y^{(n),\mathtt{R}}$ and X^i may become nonzero.

Fix n such that $n_0\le n\le n_1$. Following the argument from equation (8.24) to equation (8.26) in [17], we deduce that

$$
\begin{aligned}
&\mathbb{Q}^i_\varepsilon\Big(\mathcal{P}^{X^i}_\beta(s_i+r)\cap\operatorname{supp}\big(Y^{(n),\mathtt{R}}\big)\neq\varnothing,\\
&\qquad\min_{j:t_j\le s_i}\big(\sigma^{Y^j}_\beta-t_j\big)>3r,\sigma^{X^i}_\beta-s_i>2r\Big)\\
&\quad\le\mathbb{Q}^i_\varepsilon\Big(Y^{(n),\mathtt{R}}_{s_i+2^{-n}}(\mathbb{1})>0,\ \min_{j:t_j\le s_i}\big(\sigma^{Y^j}_\beta-t_j\big)>3r,\sigma^{X^i}_\beta-s_i>2r\Big).
\end{aligned}\tag{7.8}
$$

We can use a calculation of Feller diffusions to bound the right-hand side of (7.8). Let us start with the inequality:

$$\mathbb{Q}^i_\varepsilon\Big(Y^{(n),\mathrm{R}}_{s_i+2^{-n}}(\mathbb{1})>0,\ \min_{j\,:\,t_j\le s_i}\big(\sigma^{Y^j}_\beta-t_j\big)>3r,\sigma^{X^i}_\beta-s_i>2r\Big)$$

$$\tag{7.9}\begin{aligned}&\le\frac{1}{\psi(\mathbb{1})\varepsilon}\mathbb{E}^{\mathbb{P}_\varepsilon}\big[X^i_{s_i+2^{-n}}(\mathbb{1})^{T_1^{X^i}};Y^{(n),\mathrm{R}}_{s_i+2^{-n}}(\mathbb{1})>0,\sigma^{X^i}_\beta-s_i>2^{-n},\\&\qquad\qquad\sigma^{Y^j}_\beta-s_i>2^{-n},\forall(y_j,t_j)\in\mathcal{R}^{\mathrm{R}}_n\big],\end{aligned}$$

where the restriction for $\sigma^{Y^j}_\beta$ applies since $r\ge s_i$ and $2r\ge 2^{-n_0}\ge 2^{-n}$.

To evaluate the right-hand side of (7.9), we apply orthogonal continuation in the following way. First, note that under $\mathbb{P}_\varepsilon$, $X^i(\mathbb{1})\restriction[s_i,\infty)$ and $Y^{(n),\mathrm{R}}(\mathbb{1})\restriction[s_i,\infty)$ are $(\mathscr{G}_s)_{s\ge s_i}$-Feller diffusions with independent starting values by the independent landing property (2.16). Define a $(\mathscr{G}_s)_{s\ge s_i}$-stopping time $\sigma^\perp$ by

$$\sigma^\perp=\Big(\sigma^{X^i}_\beta\wedge\bigwedge_{j\,:\,t_j\le s_i}\widehat{\sigma}^{Y^j}_\beta\wedge\big(s_i+2^{-n}\big)\Big)\vee s_i,$$

where the $(\mathscr{G}_s)_{s\ge s_i}$-stopping times $\widehat{\sigma}^{Y^j}_\beta$ are given by

$$\widehat{\sigma}^{Y^j}_\beta=\begin{cases}\sigma^{Y^j}_\beta, & (y_j,t_j)\in\mathcal{R}^{\mathrm{R}}_n,\\ \infty, & \text{otherwise.}\end{cases}$$

Through $\sigma^\perp$, we control the support propagation of X^i and Y^j for $(y_j,t_j)\in\mathcal{R}^{\mathrm{R}}_n$. Note that

$$\tag{7.10}\mathcal{P}^{X^i}_\beta\big(s_i+2^{-n}\big)\cap\mathcal{P}^{Y^j}_\beta\big(s_i+2^{-n}\big)=\varnothing$$

for any $j\in\mathbb{N}$ with $(y_j,t_j)\in\mathcal{R}^{\mathrm{R}}_n$, since the distance between $\mathcal{P}^{X^i}(s_i+2^{-n})$ and $\mathcal{P}^{Y^j}_\beta(s_i+2^{-n})$ is given by

$$\begin{aligned}&\big(y_j-\big(s_i+2^{-n}-t_j\big)^\beta-\varepsilon^{1/2}\big)-\big(x_i-2^{-n\beta}-\varepsilon^{1/2}\big)\\&\quad\ge 7\cdot 2^{-(n+1)\beta}-\big[s_i+2^{-n}-\big(s_i-2^{-n}\big)\big]^\beta-2^{-n\beta}-2\cdot 2^{-n\beta}\\&\quad\ge\big(7\cdot 2^{-\beta}-2^\beta-3\big)\cdot 2^{-n\beta}>0,\end{aligned}$$

where for the first inequality, we recall (7.3) and $\beta\in[\frac{1}{3},\frac{1}{2})$. The equality (7.10) implies $\langle X^i(\mathbb{1}),Y^{(n),\mathrm{R}}(\mathbb{1})\rangle^{\sigma^\perp}=0$. This allows for orthogonal continuation of $X^i(\mathbb{1})$ beyond $\sigma^\perp$ (cf. Lemma 3.15), and thereby we get a $(\mathscr{G}_s)_{s\ge s_i}$-Feller diffusion $\widehat{X}^i$ independent of $Y^{(n),\mathrm{R}}(\mathbb{1})\restriction[s_i,\infty)$ and satisfying $\widehat{X}^i=X^i(\mathbb{1})$ over $[s_i,\sigma^\perp]$, under $\mathbb{P}_\varepsilon$.

We use $\widehat{X}^i$ to compute the right-hand side of (7.9) and get

$$
\begin{aligned}
&\mathbb{Q}_{\varepsilon}^{i}\Big(Y_{s_i+2^{-n}}^{(n),\mathtt{R}}(\mathbb{1})>0,\ \min_{j:t_j\le s_i}\big(\sigma_\beta^{Y^j}-t_j\big)>3r,\sigma_\beta^{X^i}-s_i>2r\Big)\\
&\quad\le \frac{1}{\psi(\mathbb{1})\varepsilon}\mathbb{E}^{\mathbb{P}_\varepsilon}\big[X_{s_i+2^{-n}}^{i}(\mathbb{1})^{T_1^{X^i}};Y_{s_i+2^{-n}}^{(n),\mathtt{R}}(\mathbb{1})>0,\sigma^{\perp}=s_i+2^{-n}\big]\\
&\quad= \frac{1}{\psi(\mathbb{1})\varepsilon}\mathbb{E}^{\mathbb{P}_\varepsilon}\big[\big(\widehat{X}_{s_i+2^{-n}}^{i}\big)^{T_1^{\widehat{X}^i}};Y_{s_i+2^{-n}}^{(n),\mathtt{R}}(\mathbb{1})>0,\sigma^{\perp}=s_i+2^{-n}\big]\\
&\quad\le \frac{1}{\psi(\mathbb{1})\varepsilon}\mathbb{E}^{\mathbb{P}_\varepsilon}\big[\big(\widehat{X}_{s_i+2^{-n}}^{i}\big)^{T_1^{\widehat{X}^i}};Y_{s_i+2^{-n}}^{(n),\mathtt{R}}(\mathbb{1})>0\big]\\
&\quad= \mathbb{P}_\varepsilon\big(Y_{s_i+2^{-n}}^{(n),\mathtt{R}}(\mathbb{1})>0\big),
\end{aligned}\tag{7.11}
$$

where the last quantity follows from the independence of $\widehat{X}^i$ and $Y^{(n),\mathtt{R}}$ and the martingale property of $\widehat{X}^i$, both under $\mathbb{P}_\varepsilon$. With an argument similar to equation (8.22) in [17], we have

$$
\begin{aligned}
&\mathbb{P}_\varepsilon\big(Y_{s_i+2^{-n}}^{(n),\mathtt{R}}(\mathbb{1})>0\big)\\
&\quad\le \mathbb{P}_\varepsilon\big(Y_{s_i}^{(n),\mathtt{R}}\ge 2^{-n(1+\beta-1/6)}\big)+2\cdot 2^n\cdot 2^{-n(1+\beta-1/6)}\\
&\quad\le \psi(\mathbb{1})\varepsilon 2^{n(1+\beta-1/6)}\#\{j\in\mathbb{N};s_i-2^{-n}\le t_j<s_i\}\cdot\frac{\|\psi\|_\infty}{\psi(\mathbb{1})}14\cdot 2^{-n\beta}\\
&\qquad+2\cdot 2^{-n(\beta-1/6)}\\
&\quad\le (28\|\psi\|_\infty+2)2^{-n/6},
\end{aligned}\tag{7.12}
$$

where the last inequality follows since $\beta\ge\frac{1}{3}$ and $\#\{j\in\mathbb{N};s_i-2^{-n}\le t_j<s_i\}\le\varepsilon^{-1}2^{-n}+1$. Then we apply (7.11) and (7.12) to bound the probability on the right-hand side of (7.8). By symmetry, the resulting bound also holds when $Y^{(n),\mathtt{R}}$ is replaced by $Y^{(n),\mathtt{L}}$. We have shown that, by enlarging the constant C_ψ for (7.6) and (7.7) if necessary,

$$
\begin{aligned}
&\mathbb{Q}_\varepsilon^i\Big(\mathcal{P}_\beta^{X^i}(s_i+r)\cap\operatorname{supp}\big(Y^{(n),q}\big)\neq\varnothing,\\
&\qquad\min_{j:t_j\le s_i}\big(\sigma_\beta^{Y^j}-t_j\big)>3r,\sigma_\beta^{X^i}-s_i>2r\Big)\\
&\qquad\le C_\psi 2^{-n/6},\qquad q=\mathtt{L},\mathtt{R},\forall n_0\le n\le n_1.
\end{aligned}\tag{7.13}
$$

We apply (7.6), (7.7) and (7.13) to (7.5). This gives the conclusion that

$$
\begin{aligned}
&\mathbb{Q}_\varepsilon^i\bigg(\mathcal{P}_\beta^{X^i}(s_i+r)\cap\bigg(\bigcup_{j:t_j\le s_i}\operatorname{supp}(Y^j)\bigg)\neq\varnothing,\\
&\qquad\min_{j:t_j\le s_i}\big(\sigma_\beta^{Y^j}-t_j\big)>3r,\sigma_\beta^{X^i}-s_i>2r\bigg)
\end{aligned}
$$

$$\leq C_\psi \varepsilon^\beta + \sum_{n=n_0}^{n_1} (3C_\psi)2^{-n/6}$$

$$\leq C_\psi \varepsilon^\beta + \left[\left(\sum_{n=0}^{\infty}(3C_\psi)2^{-n/6}\right)\cdot 2^{1/6}\right]\cdot 2^{(-n_0-1)/6}.$$

Since $2^{-n_0-1} \leq r$ by (7.3) an $\varepsilon^\beta \leq r^\beta \leq r^{1/6}$, our assertion follows from the last inequality. $\square$

Finally, we deal with the simple case where the clusters land after the landing time s_i of X^i but outside the rectangle $\mathcal{R}_\beta^{X^i}(s_i + r)$ defined by (3.18).

LEMMA 7.3. *Let $r \in (0,\infty)$. Then for any $j \in \mathbb{N}$ with $t_j \in (s_i, s_i + r]$ and $|y_j - x_i| > 2(\varepsilon^{1/2} + r^\beta)$, $\mathcal{P}_\beta^{X^i}(s_i + r) \cap \mathcal{P}_\beta^{Y^j}(s_i + r) = \varnothing$.*

PROOF. We only consider the case that $x_i < y_j$, as the other case follows by symmetry. Note that the distance between $\mathcal{P}_\beta^{X^i}(s_i + r)$ and $\mathcal{P}_\beta^{Y^j}(s_i + r)$ is strictly positive since

$$\begin{aligned}
&\big(y_j - (s_i + r - t_j)^\beta - \varepsilon^{1/2}\big) - \big(x_i + r^\beta + \varepsilon^{1/2}\big) \\
&\quad > 2\varepsilon^{1/2} + 2r^\beta - (s_i + r - t_j)^\beta - r^\beta - 2\varepsilon^{1/2} \\
&\quad \geq 2r^\beta - r^\beta - r^\beta = 0.
\end{aligned}$$

It follows that $\mathcal{P}_\beta^{X^i}(s_i + r)$ and $\mathcal{P}_\beta^{Y^j}(s_i + r)$ are disjoint. $\square$

8. Improved modulus of continuity. In this section, we study pointwise modulus of continuity for bounded Borel-measurable functions satisfying certain Gronwall-type integral inequalities.

THEOREM 8.1. *Let $T \in (0,\infty)$. Suppose that $(f_t)_{t\in[0,T]}$ is a real-valued bounded Borel-measurable function such that for some $b, a \in (0,\infty)$ and $C, B, A \in \mathbb{R}_+$ which are all independent of $t \in [0,T]$, we have*

$$|f_t - f_0| \leq C + Bt^b + A\left(\int_0^t |f_s|\,ds\right)^a \qquad \forall t \in [0,T]. \tag{8.1}$$

Set $\|f\|_\infty \triangleq \sup_{s\in[0,T]}|f_t|$ and $D_a \triangleq 2^{a-1} \vee 1$. Then for any $n \in \mathbb{N}$,

$$\begin{aligned}
&|f_t - f_0| \leq C + Bt^b \\
&\quad + (D_a)^{2n} \sum_{j=1}^{n}\left[\frac{\prod_{k=1}^{j}(A)^{a^{k-1}}\cdot\prod_{k=1}^{j-1}(D_a)^{2(n-k)a^k}\cdot(|f_0|+C)^{a^j}}{\prod_{k=1}^{j-1}(a_k+1)^{a^{j-k}}}\right]t^{a_j}
\end{aligned} \tag{8.2}$$

$$+ (D_a)^{2n} \sum_{j=1}^{n} \left[\frac{\prod_{k=1}^{j} (A)^{a^{k-1}} \cdot \prod_{k=1}^{j-1} (D_a)^{2(n-k)a^k} \cdot (B/(b+1))^{a^j}}{\prod_{k=1}^{j-1} (b_k+1)^{a^{j-k}}} \right] t^{b_j}$$

$$+ (D_a)^{2n} \left[\frac{(A)^{c_n} \cdot \prod_{k=1}^{n} (D_a)^{2(n-k)a^k} \cdot \|f\|_\infty^{a^{n+1}}}{\prod_{k=1}^{n} (a_k+1)^{a^{n-k+1}}} \right] t^{a_{n+1}} \qquad \forall t \in [0,T],$$

with the convention that $\prod_{k=1}^{0} \equiv 1$, *where the sequences* $\{a_k\}$, $\{b_k\}$, *and* $\{c_k\}$ *are given by*

$$a_k = \sum_{j=1}^{k} a^j, \qquad b_k = \sum_{j=1}^{k-1} a^j + (b+1)a^k \quad \textit{and} \quad c_k = \sum_{j=0}^{k} a^j \tag{8.3}$$

with the convention that $\sum_{j=1}^{0} \equiv 0$.

PROOF. By (8.3), we can characterize the sequences $\{a_k\}$, $\{b_k\}$ and $\{c_k\}$ alternatively by the equations:

$$\begin{aligned} &a_1 = a, \qquad a_{k+1} = a(a_k+1), \\ &b_1 = (b+1)a, \qquad b_{k+1} = a(b_k+1), \\ &c_1 = a+1, \qquad c_{k+1} = ac_k + 1. \end{aligned} \tag{8.4}$$

We use these identifications in the following argument.

We prove the theorem by an induction on $n \in \mathbb{N}$. We will need the following elementary inequality: for any $n \in \mathbb{N}$ with $n \geq 2$,

$$\left(\sum_{j=1}^{n} x_j \right)^a \leq (D_a)^{n-1} \left(\sum_{j=1}^{n} x_j^a \right) \qquad \forall x_1, \dots, x_n \in \mathbb{R}_+. \tag{8.5}$$

Consider (8.2) for $n = 1$. Note that (8.1) implies

$$|f_t - f_0| \leq C + Bt^b + A\|f\|_\infty^a t^a \qquad \forall t \in [0,T]. \tag{8.6}$$

Apply (8.6) to (8.1), and we obtain

$$\begin{aligned} &|f_t - f_0| \\ &\quad \leq C + Bt^b + A\left(|f_0|t + \int_0^t (C + Bs^b + A\|f\|_\infty^a s^a)\, ds \right)^a \\ &\quad = C + Bt^b + A\left((|f_0| + C)t + \frac{B}{b+1} t^{b+1} + \frac{A\|f\|_\infty^a}{a+1} t^{a+1} \right)^a \\ &\quad \leq C + Bt^b \\ &\qquad + A \cdot (D_a)^2 \left[(|f_0| + C)^a t^a + \left(\frac{B}{b+1} \right)^a t^{(b+1)a} + \left(\frac{A\|f\|_\infty^a}{a+1} \right)^a t^{(a+1)a} \right] \end{aligned}$$

$$
\begin{aligned}
&= C + Bt^b \\
&\quad + (D_a)^2\Big[A\big(|f_0| + C\big)^a t^{a_1} + A\Big(\frac{B}{b+1}\Big)^a t^{b_1} + \frac{(A)^{a+1}\|f\|_\infty^{a^2}}{(a+1)^a} t^{a_2}\Big],
\end{aligned}
$$

where the second inequality follows from (8.5). This proves (8.2) for $n = 1$.

Suppose that (8.2) holds for some $n \in \mathbb{N}$. Then for any $t \in [0, T]$, we have

$$
\begin{aligned}
&\int_0^t |f_s|\,ds \\
&\quad \le |f_0|t + \int_0^t |f_s - f_0|\,ds \\
&\quad \le \big(|f_0| + C\big)t + \frac{B}{b+1}t^{b+1} \\
&\qquad + (D_a)^{2n} \\
&\qquad \times \sum_{j=1}^n \Big[\frac{\prod_{k=1}^j (A)^{a^{k-1}} \cdot \prod_{k=1}^{j-1}(D_a)^{2(n-k)a^k} \cdot (|f_0| + C)^{a^j}}{\prod_{k=1}^{j-1}(a_k+1)^{a^{j-k}}}\Big]\frac{1}{(a_j+1)}t^{a_j+1} \\
&\qquad + (D_a)^{2n} \\
&\qquad \times \sum_{j=1}^n \Big[\frac{\prod_{k=1}^j (A)^{a^{k-1}} \cdot \prod_{k=1}^{j-1}(D_a)^{2(n-k)a^k} \cdot (B/(b+1))^{a^j}}{\prod_{k=1}^{j-1}(b_k+1)^{a^{j-k}}}\Big]\frac{1}{b_j+1}t^{b_j+1} \\
&\qquad + (D_a)^{2n}\Big[\frac{(A)^{c_n} \cdot \prod_{k=1}^n (D_a)^{2(n-k)a^k} \cdot \|f\|_\infty^{a^{n+1}}}{\prod_{k=1}^n (a_k+1)^{a^{n-k+1}}}\Big]\frac{1}{a_{n+1}+1}t^{a_{n+1}+1},
\end{aligned}
$$

where the right-hand side is a sum of $2n+3$ many terms (there are $2n$ terms in total under the two summation signs). Recall the recursive equations in (8.4). Applying (8.1) and (8.5) for n replaced by $2n+3$ to the foregoing inequality, we obtain, for every $t \in [0, T]$,

$$
\begin{aligned}
&|f_t - f_0| \\
&\quad \le C + Bt^b + A\cdot(D_a)^{2n+2}\big(|f_0| + C\big)^a t^a + A\cdot(D_a)^{2n+2}\Big(\frac{B}{b+1}\Big)^a t^{(b+1)a} \\
&\qquad + A\cdot(D_a)^{2n+2}\cdot(D_a)^{2na} \\
&\qquad \times \sum_{j=1}^n \Big[\frac{\prod_{k=2}^{j+1}(A)^{a^{k-1}} \cdot \prod_{k=2}^j (D_a)^{2[(n+1)-k]a^k} \cdot (|f_0| + C)^{a^{j+1}}}{\prod_{k=1}^{j-1}(a_k+1)^{a^{(j+1)-k}}}\Big] \\
&\qquad \times \frac{1}{(a_j+1)^a}t^{a_{j+1}} \\
&\qquad + A\cdot(D_a)^{2n+2}\cdot(D_a)^{2na}
\end{aligned}
$$

$$
\begin{aligned}
&\times \sum_{j=1}^{n}\left[\frac{\prod_{k=2}^{j+1}(A)^{a^{k-1}} \cdot \prod_{k=2}^{j}(D_a)^{2[(n+1)-k]a^k} \cdot (B/(b+1))^{a^{j+1}}}{\prod_{k=1}^{j-1}(b_k+1)^{a^{(j+1)-k}}}\right]\\
&\times \frac{1}{(b_j+1)^a} t^{b_{j+1}}\\
&+ A \cdot (D_a)^{2n+2} \cdot (D_a)^{2na}\left[\frac{(A)^{c_n a} \cdot \prod_{k=2}^{n+1}(D_a)^{2[(n+1)-k]a^k} \cdot \|f\|_\infty^{a^{n+2}}}{\prod_{k=1}^{n}(a_k+1)^{a^{(n+1)-k+1}}}\right]\\
&\times \frac{1}{(a_{n+1}+1)^a} t^{a_{n+2}}.
\end{aligned}
$$

Then the rest follows by writing the right-hand side of the foregoing inequality into the desired form:

$$
\begin{aligned}
&|f_t - f_0|\\
&\quad \le C + Bt^b + A \cdot (D_a)^{2n+2}(|f_0| + C)^a t^a + A \cdot (D_a)^{2n+2}\left(\frac{B}{b+1}\right)^a t^{(b+1)a}\\
&\qquad + (D_a)^{2n+2}\\
&\qquad \times \sum_{j=1}^{n}\left[\frac{\prod_{k=1}^{j+1}(A)^{a^{k-1}} \cdot \prod_{k=1}^{j}(D_a)^{2[(n+1)-k]a^k} \cdot (|f_0| + C)^{a^{j+1}}}{\prod_{k=1}^{j}(a_k+1)^{a^{(j+1)-k}}}\right] t^{a_{j+1}}\\
&\qquad + (D_a)^{2n+2}\\
&\qquad \times \sum_{j=1}^{n}\left[\frac{\prod_{k=1}^{j+1}(A)^{a^{k-1}} \cdot \prod_{k=1}^{j}(D_a)^{2[(n+1)-k]a^k} \cdot (B/(b+1))^{a^{j+1}}}{\prod_{k=1}^{j}(b_k+1)^{a^{(j+1)-k}}}\right] t^{b_{j+1}}\\
&\qquad + (D_a)^{2n+2}\left[\frac{(A)^{c_{n+1}} \cdot \prod_{k=1}^{n+1}(D_a)^{2[(n+1)-k]a^k} \cdot \|f\|_\infty^{a^{n+2}}}{\prod_{k=1}^{n+1}(a_k+1)^{a^{(n+1)-k+1}}}\right] t^{a_{n+2}}\\
&\quad = C + Bt^b\\
&\qquad + (D_a)^{2n+2}\\
&\qquad \times \sum_{j=1}^{n+1}\left[\frac{\prod_{k=1}^{j}(A)^{a^{k-1}} \cdot \prod_{k=1}^{j-1}(D_a)^{2[(n+1)-k]a^k} \cdot (|f_0| + C)^{a^j}}{\prod_{k=1}^{j-1}(a_k+1)^{a^{j-k}}}\right] t^{a_j}\\
&\qquad + (D_a)^{2n+2}\\
&\qquad \times \sum_{j=1}^{n+1}\left[\frac{\prod_{k=1}^{j}(A)^{a^{k-1}} \cdot \prod_{k=1}^{j-1}(D_a)^{2[(n+1)-k]a^k} \cdot (B/(b+1))^{a^j}}{\prod_{k=1}^{j-1}(b_k+1)^{a^{j-k}}}\right] t^{b_j}\\
&\qquad + (D_a)^{2n+2}\left[\frac{(A)^{c_{n+1}} \cdot \prod_{k=1}^{n+1}(D_a)^{2[(n+1)-k]a^k} \cdot \|f\|_\infty^{a^{n+2}}}{\prod_{k=1}^{n+1}(a_k+1)^{a^{(n+1)-k+1}}}\right] t^{a_{n+2}}.
\end{aligned}
$$

This proves our assertion for (8.2) when n is replaced by $n+1$, and the proof is complete by mathematical induction. □

COROLLARY 8.2 (Improved modulus of continuity). *Let $T \in (0,1]$, $a \in (0,\frac{1}{2})$, and $B, A \in \mathbb{R}_+$. If $(f_t)_{t\in[0,T]}$ is a real-valued Borel-measurable function uniformly bounded by 1 and satisfies (8.1) with $C = 0$ and $b = 1$, then for $\xi' \in (0,1)$ and $N' \in \mathbb{N}$ satisfying $\sum_{j=1}^{N'} a^j \le \xi' < \sum_{j=1}^{N'+1} a^j$, we have*

$$\begin{aligned} |f_t - f_0| \le {}& \Bigg[\big(A^{1/(1-a)}+1\big)\sum_{j=1}^{N'} |f_0|^{a^j}\Bigg] t^a \\ & + \Bigg[B + \big(A^{1/(1-a)}+1\big)\sum_{j=1}^{N'}\left(\frac{B}{2}\right)^{a^j} + A^{1/(1-a)} + 1\Bigg] t^{\xi'} \qquad (8.7)\\ & \forall t \in [0,T]. \end{aligned}$$

PROOF. We simplify the right-hand side of (8.2) with elementary algebra, using the present assumptions. First, $D_a = 1$ since $a \in (0,\frac{1}{2})$. Next, since $\sum_{k=1}^{j} a^{k-1} \le \frac{1}{1-a}$ for all $j \in \mathbb{N}$, we have

$$\prod_{k=1}^{j} A^{a^{k-1}} \le A^{1/(1-a)} + 1, \qquad 1 \le j \le n \quad \text{and} \quad A^{c_n} \le A^{1/(1-a)} + 1. \tag{8.8}$$

Finally, let us handle the exponents b_j in the second sum in (8.2). Using $b = 1$ and the definition of $\{b_k\}$ in (8.3), we obtain

$$\begin{aligned} b_k &= \frac{a(1-a^{k-1})}{1-a} + 2a^k = \frac{a - a^k + 2a^k - 2a^{k+1}}{1-a} \\ &= \frac{a + a^k(1-2a)}{1-a} \searrow \frac{a}{1-a} = \sum_{j=1}^{\infty} a^j \end{aligned}$$

as k tends to infinity since $a \in (0,\frac{1}{2})$. The inequality (8.7) follows by applying the above observations to (8.2). The proof is complete. □

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CENTER OF MATHEMATICAL SCIENCES AND APPLICATIONS
HARVARD UNIVERSITY
CAMBRIDGE, MASSACHUSETTS 02138
USA
E-MAIL: yuting_chen@cmsa.fas.harvard.edu