

MOMENTS OF TRACES OF CIRCULAR BETA-ENSEMBLES

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Let $\theta_1, \dots, \theta_n$ be random variables from Dyson's circular β -ensemble with probability density function $\text{Const} \cdot \prod_{1 \leq j < k \leq n} |e^{i\theta_j} - e^{i\theta_k}|^\beta$. For each $n \geq 2$ and $\beta > 0$, we obtain some inequalities on $\mathbb{E}[p_\mu(Z_n)\overline{p_\nu(Z_n)}]$, where $Z_n = (e^{i\theta_1}, \dots, e^{i\theta_n})$ and p_μ is the power-sum symmetric function for partition μ . When $\beta = 2$, our inequalities recover an identity by Diaconis and Evans for Haar-invariant unitary matrices. Further, we have the following: $\lim_{n \rightarrow \infty} \mathbb{E}[p_\mu(Z_n)\overline{p_\nu(Z_n)}] = \delta_{\mu\nu} (\frac{2}{\beta})^{l(\mu)} z_\mu$ for any $\beta > 0$ and partitions μ, ν ; $\lim_{m \rightarrow \infty} \mathbb{E}[|p_m(Z_n)|^2] = n$ for any $\beta > 0$ and $n \geq 2$, where $l(\mu)$ is the length of μ and z_μ is explicit on μ . These results apply to the three important ensembles: COE ($\beta = 1$), CUE ($\beta = 2$) and CSE ($\beta = 4$). We further examine the nonasymptotic behavior of $\mathbb{E}[|p_m(Z_n)|^2]$ for $\beta = 1, 4$. The central limit theorems of $\sum_{j=1}^n g(e^{i\theta_j})$ are obtained when (i) $g(z)$ is a polynomial and $\beta > 0$ is arbitrary, or (ii) $g(z)$ has a Fourier expansion and $\beta = 1, 4$. The main tool is the Jack function.

1. Introduction. Let M_n be an $n \times n$ Haar-invariant unitary matrix, that is, the entries of unitary matrix M_n are random variables satisfying that the probability distribution of the entries of M_n is the same as that of UM_n and that of M_nU for any $n \times n$ unitary matrix U . Diaconis and Evans (Theorem 2.1 from [4]) proved that

(a) Consider $a = (a_1, \dots, a_k)$ and $b = (b_1, \dots, b_k)$ with $a_j, b_j \in \{0, 1, 2, \dots\}$. Then for $n \geq \sum_{j=1}^k ja_j \vee \sum_{j=1}^k jb_j$,

$$(1.1) \quad \mathbb{E} \left[\prod_{j=1}^k (\text{Tr}(M_n^j))^{a_j} \overline{(\text{Tr}(M_n^j))^{b_j}} \right] = \delta_{ab} \prod_{j=1}^k j^{a_j} a_j!$$

where δ_{ab} is Kronecker's delta.

(b) For any positive integers j and k ,

$$(1.2) \quad \mathbb{E}[\text{Tr}(M_n^j)\overline{\text{Tr}(M_n^k)}] = \delta_{jk} \cdot j \wedge n.$$

The idea of the proof is based on the group representation theory of unitary group $U(n)$. Some other derivations for (1.1) and (1.2) are given in [5, 23–25].

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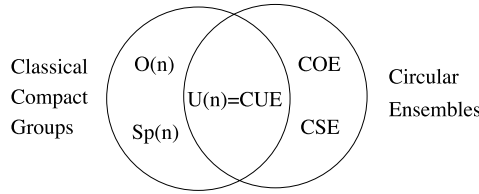


FIG. 1. *Circular Ensembles and Haar-invariant matrices from classical compact groups.*

The right-hand side of (1.1) is evidently equal to $\mathbb{E}[\prod_{j=1}^k \xi_j^{a_j} \bar{\xi}_j^{b_j}]$ where ξ_j 's are independent complex-normal random variables with $\xi_j \sim \mathbb{C}N(0, j)$ for each j .

Notice an $n \times n$ Haar-invariant unitary matrix is also called a CUE, which belongs to the Circular Ensembles of three members: the Circular Orthogonal Ensemble (COE), the Circular Unitary Ensemble (CUE) and the Circular Symplectic Ensemble (CSE); see Figure 1 for the relationship, where the left circle consists of matrices which induce the Haar probability measure on the orthogonal group $O(n)$, Haar probability measure on the unitary group $U(n)$ and Haar probability measure on the real symplectic group $Sp(n)$, respectively.

Let $e^{i\theta_1}, \dots, e^{i\theta_n}$ be the eigenvalues of an $n \times n$ Haar-invariant unitary matrix, or equivalently, an $n \times n$ CUE, it is known (see, e.g., [12, 22]) that the density function of $\theta_1, \dots, \theta_n$ is $f(\theta_1, \dots, \theta_n|\beta)$ with $\beta = 2$, where

$$(1.3) \quad f(\theta_1, \dots, \theta_n|\beta) = (2\pi)^{-n} \cdot \frac{\Gamma(1 + \beta/2)^n}{\Gamma(1 + \beta n/2)} \prod_{1 \leq j < k \leq n} |e^{i\theta_j} - e^{i\theta_k}|^\beta$$

with $\beta > 0$ and $\theta_i \in [0, 2\pi)$ for $1 \leq i \leq n$. The density function of $\theta_1, \dots, \theta_n$ for the COE is $f(\theta_1, \dots, \theta_n|\beta)$ with $\beta = 1$, and that for the CSE is $f(\theta_1, \dots, \theta_n|\beta)$ with $\beta = 4$.

The purpose of this paper is to study the analogues of (1.1) and (1.2) for the circular β -ensembles with density function $f(\theta_1, \dots, \theta_n|\beta)$ in (1.3) for any $\beta > 0$. Further, we develop the central limit theorems for functions of $(e^{i\theta_1}, \dots, e^{i\theta_n})$. Before stating the main results, we next introduce some background about the circular β -ensembles.

The circular ensembles were first introduced by physicist Dyson [8–10] for the study of nuclear scattering data. In fact, as studied in [8], Dyson shows that the consideration of time reversal symmetry leading to the three Gaussian ensembles behaves equally well to unitary matrices. A time reversal symmetry requires that $U = U^T$, no time reversal symmetry has no constraint, and a time reversal symmetry for a system with an odd number of spin 1/2 particles requires $U = U^D$, where D denotes the quaternion dual. Choosing such matrices with a uniform probability then gives COE, CUE and CSE, respectively (see, e.g., [11, 22]). The entries of COE and CUE are asymptotically complex normal random variables when the sizes of the matrices are large [14, 16, 17].

Let U be an $n \times n$ Haar-invariant unitary matrix. As mentioned earlier, U is also a CUE; the matrix $U^T U$ gives a COE. Furthermore, the matrix $U^D U$ gives a CSE when n is even; see Chapter 9 from [22]. For the relations among the zonal polynomials, the Schur functions, the Gelfand pairs and the three circular ensembles; see, for example, Chapter VII in [20] or Section 2.7 in [1] for reference.

Now we consider the moments in (1.1) and (1.2) for the circular β -ensembles. Taking $\beta = 1$ in (1.3), that is, choosing W_n such that it is an $n \times n$ COE, by an elementary check in Lemma A.1, we have

$$(1.4) \quad \mathbb{E}[|\mathrm{Tr}(W_n)|^2] = \frac{2n}{n+1}$$

for all $n \geq 2$. This suggests that, unlike the right-hand sides of (1.1) or (1.2) that are free of n , the moments for the general circular β -ensemble may depend on n for $\beta \neq 2$. In fact, by using the Jack functions, we will soon see from (2.4) below that the second moment in (1.4) does depend on n except $\beta = 2$, in which case W_n is an $n \times n$ CUE.

In this paper, we will first prove some inequalities on the moments in (1.1) and (1.2) for the circular β -ensembles with arbitrary $\beta > 0$. In particular, some of our inequalities for $\beta = 2$ recover the equality in (1.1) by Diaconis and Evans [4]. Further, we evaluate the limiting behavior by letting $n \rightarrow \infty$ for the left-hand side in (1.1) and $k \rightarrow \infty$ for the left-hand side in (1.2), respectively. Their limits exist and look quite similar to the right-hand sides of (1.1) and (1.2). Finally, we spend much effort to study the central limit theorems of $\sum_{j=1}^n g(e^{i\theta_j})$ for two situations: (a) $g(x)$ is a polynomial and $\beta > 0$ is arbitrary; (b) $g(x)$ has a Fourier expansion and $\beta = 1, 4$. The key to obtain (b) is the nonasymptotic behavior of $\mathbb{E}|\sum_{j=1}^n e^{im\theta_j}|^2$ for any n and m , which are analyzed in detail.

The method of the proof is the Jack functions. The main results are obtained by using their orthogonal properties and combinatorial structures.

From the studies in this paper, it is obvious to see the importance of understanding the circular β -ensembles through the Jack functions. Realizing that the Jack functions are a special class of the Macdonald polynomials, we have obtained the analogue of the results in this paper in the setting of the Macdonald polynomials. These will be published elsewhere in the future.

The organization of the rest of the paper is as follows. We present the moment inequalities in Section 2 and their proofs are given in Section 4; the nonasymptotic behavior of $\mathbb{E}|\sum_{j=1}^n e^{im\theta_j}|^2$ and the central limit theorems are stated in Section 3 and their proofs are arranged in Section 5. In the Appendix, we prove (1.4) by two ways different from the method of the Jack functions. Some other explicit formulas of moments are also given in the same section.

2. Moment inequalities for circular beta-ensembles. Let $\lambda = (\lambda_1, \lambda_2, \dots)$ be a partition, that is, the sequence is in nonincreasing order and only finite of λ_i 's are nonzero. The weight of λ is $|\lambda| = \lambda_1 + \lambda_2 + \dots$. Denote by $m_i(\lambda)$ the

multiplicity of i in $(\lambda_1, \lambda_2, \dots)$ for each i , and $l(\lambda)$ the length of λ : $l(\lambda) = m_1(\lambda) + m_2(\lambda) + \dots$. Recall the convention $0! = 1$. Set

$$(2.1) \quad z_\lambda = \prod_{i \geq 1} i^{m_i(\lambda)} m_i(\lambda)!$$

Let $\rho = (\rho_1, \rho_2, \dots)$ be a partition, and

$$(2.2) \quad p_\rho = \prod_{i=1}^{l(\rho)} p_{\rho_i} \quad \text{where } p_k(x_1, x_2, \dots) = x_1^k + x_2^k + \dots$$

for integer $k \geq 1$ and indeterminates x_i 's. The function p_ρ is called the power-sum symmetric function. For real number $\alpha > 0$, integers $K \geq 1$ and $n \geq 1$, define two constants $A = A(n, K, \alpha)$ and $B = B(n, K, \alpha)$ by

$$(2.3) \quad \begin{aligned} A &= \left(1 - \frac{|\alpha - 1|}{n - K + \alpha} \delta(\alpha \geq 1)\right)^K \quad \text{and} \\ B &= \left(1 + \frac{|\alpha - 1|}{n - K + \alpha} \delta(\alpha < 1)\right)^K, \end{aligned}$$

where $\delta(\alpha \geq 1) = 1 - \delta(\alpha < 1)$ is 1 if $\alpha \geq 1$, or 0 otherwise. With these notation, we have one of main results as follows.

THEOREM 1. *Let $\beta > 0$ and $\theta_1, \dots, \theta_n$ have density $f(\theta_1, \dots, \theta_n | \beta)$ as in (1.3). Set $Z_n = (e^{i\theta_1}, \dots, e^{i\theta_n})$ and $\alpha = 2/\beta$. For partitions μ and ν , the following hold:*

(a) *If $n \geq K = |\mu|$, then*

$$A \leq \frac{\mathbb{E}[|p_\mu(Z_n)|^2]}{\alpha^{l(\mu)} z_\mu} \leq B.$$

(b) *If $|\mu| \neq |\nu|$, then $\mathbb{E}[p_\mu(Z_n) \overline{p_\nu(Z_n)}] = 0$. If $\mu \neq \nu$ and $n \geq K = |\mu| \vee |\nu|$, then*

$$|\mathbb{E}[p_\mu(Z_n) \overline{p_\nu(Z_n)}]| \leq \max\{|A - 1|, |B - 1|\} \cdot \alpha^{(l(\mu)+l(\nu))/2} (z_\mu z_\nu)^{1/2}.$$

(c) *There exists a constant C depending only on β such that for any $m \geq 1$ and $n \geq 2$, we have*

$$|\mathbb{E}[|p_m(Z_n)|^2] - n| \leq C \frac{n^3 2^{n\beta}}{m^{1 \wedge \beta}}.$$

Take $\beta = 2$ in (a) and (b) of Theorem 1, then $A = 1$ and $B = 1$. The two results recover the result of Diaconis and Evans in (1.1). Further, letting $n \rightarrow \infty$ in (a) and (b) of Theorem 1, we see that A and B (depending on n) converge to 1; letting $m \rightarrow \infty$ in (c) of the theorem, then the last term in (c) goes to 0. So we obviously have the following results.

COROLLARY 1. *Let the conditions be as in Theorem 1. Then, for any $\beta > 0$,*

- (a) $\lim_{n \rightarrow \infty} \mathbb{E}[p_\mu(Z_n) \overline{p_\nu(Z_n)}] = \delta_{\mu\nu} \left(\frac{2}{\beta}\right)^{l(\mu)} z_\mu;$
- (b) $\lim_{m \rightarrow \infty} \mathbb{E}[|p_m(Z_n)|^2] = n \quad \text{for any } n \geq 2.$

Part (b) of the above corollary says that, as $m \rightarrow \infty$, the limit of $\mathbb{E}[|p_m(Z_n)|^2]$ does not depend on parameter β , which is consistent with (1.2). We further take a careful examination on $\mathbb{E}[|p_m(Z_n)|^2]$ as $\beta = 1$ and 4. Some upper bounds of $\mathbb{E}[|p_m(Z_n)|^2]$ are given in Propositions 1 and 2. By studying A and B in (2.3), we have the following corollary from Theorem 1.

COROLLARY 2. *Let $\beta > 0$ and $f(\theta_1, \dots, \theta_n | \beta)$ be as in (1.3). Set $\alpha = 2/\beta$ and $Z_n = (e^{i\theta_1}, \dots, e^{i\theta_n})$. Let μ and ν be partitions with $\mu \neq \nu$ and $K = |\mu| \vee |\nu|$. If $n \geq 2K$, then*

- (a) $\left| \frac{\mathbb{E}[|p_\mu(Z_n)|^2]}{\alpha^{l(\mu)} z_\mu} - 1 \right| \leq \frac{6|1 - \alpha|K}{n};$
- (b) $|\mathbb{E}[p_\mu(Z_n) \overline{p_\nu(Z_n)}]| \leq \frac{6|1 - \alpha|K}{n} \cdot \alpha^{(l(\mu)+l(\nu))/2} (z_\mu z_\nu)^{1/2}.$

The above results are in the forms of inequalities or limits. We actually derive an exact formula in Proposition 3 to compute $\mathbb{E}[|p_\mu(Z_n)|^2]$ for every partition μ . In general, it is not easy to evaluate this quantity for arbitrary μ , however, we are able to do so when μ is special. For instance, by using the exact formula we calculate the moment in (1.4) for any $\beta > 0$ as follows.

EXAMPLE. For any $n \geq 1$,

$$(2.4) \quad \mathbb{E}[|p_1(Z_n)|^2] = \frac{2}{\beta} \frac{n}{n - 1 + 2\beta^{-1}} = \begin{cases} \frac{2n}{n + 1}, & \text{if } \beta = 1; \\ 1, & \text{if } \beta = 2; \\ \frac{n}{2n - 1}, & \text{if } \beta = 4. \end{cases}$$

The verification of this formula through Proposition 3 is provided in the Appendix. We also give $\mathbb{E}[|p_1(Z_n)|^4]$, $\mathbb{E}[|p_2(Z_n)|^2]$ and $\mathbb{E}[p_2(Z_n) \overline{p_1(Z_n)}^2]$ in closed forms in the Appendix.

The main tool used in our proofs is the Jack functions. Diaconis and Evans [4] and Diaconis and Shahshahani [5] use the group representation theory to study (1.1) and (1.2) because $U(n)$ is a compact Lie group. The situations for the Circular Orthogonal Ensembles ($\beta = 1$) and the Circular Symplectic Ensembles ($\beta = 4$) are different. In fact, the two ensembles are not groups.

The proofs of (1.1) and (1.2) involve with the Schur functions. The connection is that the irreducible characters of the unitary groups, when seen as symmetric functions in the eigenvalues, are given by Schur functions. Looking at Figure 1, an Haar-invariant unitary matrix is also a CUE. From the perspective of symmetric functions, the COE is connected to the zonal polynomials, and the CSE to symplectic zonal polynomials. The three functions are special cases of the Jack polynomial $J_\lambda^{(\alpha)}$ with $\alpha = 1, 2$ and $1/2$, respectively, where λ is a partition. See Section 4.1 for this or [20] for general properties of the Jack polynomials. By using the Jack functions, we are able to prove (a) and (b) in Theorem 1. Part (c) in the theorem is proved by evaluating the expectation/integral with respect to $f(\theta_1, \dots, \theta_n|\beta)$ in (1.3) directly.

Treating n as a variable, the bound $n^3 2^{n\beta} m^{-(1 \wedge \beta)}$ in (c) of Theorem 1 seems quite large. It is possibly to be improved. However, as $\beta = 4$, we show in Proposition 2 in the next section that $\mathbb{E}[|p_m(Z_n)|^2]$ has the scale of $m \log m$ when n and m are not far from each other. This partially explains why the bound is large.

3. Central limit theorems for circular beta-ensembles. For the sake of precision, we replace Z_n appeared earlier with Z_n^α . Specifically, let $Z_n^\alpha = (e^{i\theta_1}, \dots, e^{i\theta_n})$ follow the β -circular ensemble with $\alpha = \frac{2}{\beta}$ and the density function $f(\theta_1, \dots, \theta_n|\beta)$ as in (1.3). According to our notation in previous sections, $p_m(Z_n^\alpha) = \sum_{j=1}^n e^{im\theta_j}$ for any integer $m \geq 0$. In the paper, the symbol $\mathbb{CN}(0, \sigma^2)$ stands for the complex normal distribution generated by $\sigma \cdot (\xi_1 + i\xi_2)/\sqrt{2}$, where ξ_1 and ξ_2 are i.i.d. real random variables with the standard normal distribution $N(0, 1)$. The first result is a CLT for general circular β -ensemble.

THEOREM 2 (CLT for any β -circular ensemble). *Let $Z_n^\alpha = (e^{i\theta_1}, \dots, e^{i\theta_n})$ follow the β -circular ensemble. Then, for fixed $m \geq 1$, the random vector $(p_1(Z_n^\alpha), p_2(Z_n^\alpha), \dots, p_m(Z_n^\alpha))$ converges weakly to (ξ_1, \dots, ξ_m) as $n \rightarrow \infty$, where ξ_j 's are independent random variables with $\xi_j \sim \mathbb{CN}(0, \frac{2j}{\beta})$ for each j .*

An immediate consequence of the theorem is as follows.

COROLLARY 3. *Let $(e^{i\theta_1}, \dots, e^{i\theta_n})$ follow the β -circular ensemble. Let $g(z) = \sum_{k=0}^m c_k z^k$ with fixed $m \geq 1$ and $c_k \in \mathbb{C}$ for all k . Set $X_n = \sum_{j=1}^n g(e^{i\theta_j})$. Then $X_n - \mu_n$ converges weakly to $\mathbb{CN}(0, \sigma^2)$ as $n \rightarrow \infty$, where*

$$\mu_n = nc_0 \quad \text{and} \quad \sigma^2 = \frac{2}{\beta} \sum_{k=1}^m k|c_k|^2.$$

We next study the central limit theorem when the function $g(z)$ is not a polynomial. To avoid a lengthier paper, we only focus on the cases $\beta = 1$ and $\beta = 4$. A discussion on the general case will be given later in this section. We first need to understand the variance of $p_m(Z_n^\alpha)$.

PROPOSITION 1 (Bound of variance on COE). *For all $m \geq 1, n \geq 2$ and $\beta = 1$, there exists a universal constant $K > 0$ such that*

$$\mathbb{E}[|p_m(Z_n^2)|^2] \leq \begin{cases} 2m, & \text{if } 1 \leq m \leq n; \\ Kn, & \text{if } m > n. \end{cases}$$

PROPOSITION 2 (Bound of variance on CSE). *Let $\beta = 4$. Then there exists a universal constant $K > 0$ such that the following hold:*

- (i) $\mathbb{E}[|p_m(Z_n^{1/2})|^2] \leq K\delta^{-1}n$ for all $m \geq (1 + \delta)n$ and $\delta \in (0, 1]$.
- (ii) $\mathbb{E}[|p_m(Z_n^{1/2})|^2] \leq Km \log(m + 1)$ for all $m \geq 1$ and $n \geq 2$.
- (iii) $\mathbb{E}[|p_m(Z_n^{1/2})|^2] \geq K(w + 1)^{-2}m \log m$ for all $12 \leq n \leq m \leq 2n$ where $w = m - n \geq 0$.

From (ii) and (iii), we see that $\mathbb{E}[|p_m(Z_n^{1/2})|^2]$ is of the scale “ $m \log m$ ” when m and n are not far from each other. It is known from (1.2) and Proposition 1 that $\mathbb{E}|p_m(Z_n^\alpha)|^2 \leq Kn$ for any $n \geq 2, m \geq 1$ and $\beta = 1, 2$, where K is a universal constant. This together with (b) of Corollary 1 seems to suggest that the second moment for $\beta = 4$ is always bounded by Kn . Proposition 2 tells us a different story. However, (b) of Corollary 1 is indeed consistent with (i).

The proofs of Propositions 1 and 2 are very involved. We use the combinatorial structure (5.1) to understand the second moments. Major effort is devoted to analyzing (5.1) through (5.2) and (5.10).

Another way to calculate above variance is through the covariance of $e^{im\theta_1}$ and $e^{im\theta_2}$ by symmetry [see (4.31)], which again can be computed by using the two-point correlation function $\rho_{(2)}(\theta_1, \theta_2)$. The explicit form of $\rho_{(2)}(\theta_1, \theta_2)$ is given in Proposition 13.2.2 from [11]. It seems very hard to estimate the variance by using the proposition. But it is possible in principle.

THEOREM 3 (CLT for COE). *Let $(e^{i\theta_1}, \dots, e^{i\theta_n})$ follow the circular orthogonal ensemble ($\beta = 1$). Let $\{a_j, b_j \in \mathbb{C}; j = 1, 2, \dots\}$ satisfy $\sum_{j=1}^\infty j(|a_j|^2 + |b_j|^2) = \sigma^2 \in (0, \infty)$. Then, $\sum_{j=1}^\infty (a_j p_j(Z_n^2) + b_j \overline{p_j(Z_n^2)})$ converges weakly to the law of $U + iV$ as $n \rightarrow \infty$, where $(U, V) \in \mathbb{R}^2$ has the law $N_2(\mathbf{0}, \Sigma)$ with*

$$\Sigma = \begin{pmatrix} \sum_{j=1}^\infty j|a_j + \bar{b}_j|^2 & 2 \cdot \text{Im} \left(\sum_{j=1}^\infty ja_j b_j \right) \\ 2 \cdot \text{Im} \left(\sum_{j=1}^\infty ja_j b_j \right) & \sum_{j=1}^\infty j|a_j - \bar{b}_j|^2 \end{pmatrix}.$$

Obviously, if $b_j = 0$ for all j , then $\Sigma = \sigma^2 \mathbf{I}_2$ with $\sigma^2 = \sum_{j=1}^\infty j|a_j|^2$, and hence $U + iV \sim \mathbb{CN}(0, 2\sigma^2)$.

THEOREM 4 (CLT for CSE). *Let $(e^{i\theta_1}, \dots, e^{i\theta_n})$ follow the circular symplectic ensemble ($\beta = 4$). Let $\{a_j, b_j \in \mathbb{C}; j = 1, 2, \dots\}$ satisfy $\sum_{j=1}^\infty (j \log j)(|a_j|^2 + |b_j|^2) \in (0, \infty)$. Set $\sigma^2 = \sum_{j=1}^\infty j(|a_j|^2 + |b_j|^2)$. Then $\sum_{j=1}^\infty (a_j p_j(Z_n^{1/2}) + \overline{b_j p_j(Z_n^{1/2})})$ converges weakly to the law of $U + iV$ as $n \rightarrow \infty$, where $(U, V) \in \mathbb{R}^2$ has the law $N_2(\mathbf{0}, \Sigma)$ with*

$$\Sigma = \frac{1}{4} \begin{pmatrix} \sum_{j=1}^\infty j|a_j + \bar{b}_j|^2 & 2 \cdot \text{Im} \left(\sum_{j=1}^\infty j a_j b_j \right) \\ 2 \cdot \text{Im} \left(\sum_{j=1}^\infty j a_j b_j \right) & \sum_{j=1}^\infty j|a_j - \bar{b}_j|^2 \end{pmatrix}.$$

Similar to the comment below Theorem 3, if $b_j = 0$ for all j , then $\Sigma = \sigma^2 \mathbf{I}_2$ with $\sigma^2 = \frac{1}{4} \sum_{j=1}^\infty j|a_j|^2$, and hence $U + iV \sim \mathcal{CN}(0, 2\sigma^2)$.

Though Proposition 2 says that $\mathbb{E}[|p_m(Z_n^{1/2})|^2]$ is of scale “ $m \log m$ ” when m and n are not far from each other, the variance of the limiting distribution in Theorem 4 is not affected by this fact. The variance is similar to those in the circular orthogonal and unitary ensemble ($\beta = 1, 4$).

Diaconis and Evans [4] obtains the CLTs for the orthogonal groups, the unitary groups and the symplectic groups. Their tool is the identities in (1.1) and (1.2). Reviewing Corollary 2, we no longer have identities for any $\beta \neq 2$; this increases much difficulty to get the corresponding CLTs. It is understandable because after all the three members in the classical compact groups have group structures in addition to their combinatorial ones. So the group representation theory can be possibly used in the paper by Diaconis and Evans. The general circular β -ensemble loses the former property and has only the combinatorial structure.

Johansson in [18] further explores the convergence speed of $\text{Tr}(M_n^m)$ to a normal distribution, where m is fixed and M_n is an Haar-invariant orthogonal, unitary or symplectic random matrix. He shows that the convergence rate is exponentially fast.

By Proposition 2, the condition “ $\sum_{j=1}^\infty (j \log j)(|a_j|^2 + |b_j|^2) < \infty$ ” in Theorem 4 can be slightly relaxed. For simplicity, we just leave it as it is. Also, the conditions “ $\sum_{j=1}^\infty j(|a_j|^2 + |b_j|^2) < \infty$ ” and “ $\sum_{j=1}^\infty (j \log j)(|a_j|^2 + |b_j|^2) < \infty$ ” can be easily satisfied. For instance, the first condition is satisfied if a_j and b_j are of the order $\frac{1}{j(\log j)^{(1/2)+\delta}}$ for some $\delta > 0$, and the second one is satisfied if a_j and b_j are of the order $\frac{1}{j(\log j)^{1+\delta}}$ for some $\delta > 0$.

To study the number of eigenvalues falling in an arc of the unit circle in the complex plane, namely, $\sum_{j=1}^n I(e^{i\theta_j} \in A)$ with A being a subset of $S^1 = \{z \in \mathbb{C}; |z| = 1\}$, one needs to handle the Fourier expansion of the indicator function $I_{[a,b]}(x)$ with $[a, b] \subset [0, 2\pi]$. It is known from [4] that the coefficients a_j and b_j in the contexts of Theorems 3 and 4 are of scale $\frac{1}{j}$. Our theorems do not cover this

special case. By using a construction of the circular β -ensemble, Killip [19] specifically considers this situation and obtains a CLT. The author does not investigate the general CLTs as treated in our Theorems 2, 3 or 4.

Finally, we provide some examples which satisfy the condition

$$\sum_{j=1}^{\infty} (j \log j)(|a_j|^2 + |b_j|^2) < \infty.$$

They are the solutions of some classical partial differential equations. We leave readers for the trivial calculations of the means and the variances of the limiting normal distributions.

EXAMPLE. Let $u = u(x, y)$ be defined on \mathbb{R}^2 and satisfy the Laplace equation

$$\begin{cases} \Delta u = 0, & x^2 + y^2 < a^2; \\ u = h(\theta), & x^2 + y^2 = a^2, \end{cases}$$

where $h(\theta)$ is a known function and $a > 0$ is given. Let $(x, y) = (r \cos \theta, r \sin \theta)$. The solution has a Poisson’s formula. It can also be expressed in the following Fourier series:

$$(3.1) \quad u(r, \theta) = \frac{1}{2}A_0 + \sum_{j=1}^{\infty} r^j (A_j \cos j\theta + B_j \sin j\theta)$$

for $r \in (0, a)$ and $\theta \in [0, 2\pi]$, where A_j ’s and B_j ’s are obtained from the Fourier series of $h(\theta)$ so that

$$A_j = \frac{1}{\pi a^j} \int_0^{2\pi} h(\phi) \cos j\phi d\phi \quad \text{and} \quad B_j = \frac{1}{\pi a^j} \int_0^{2\pi} h(\phi) \sin j\phi d\phi.$$

See, for example, more details on page 160 from [26]. Clearly, if $C := \sup_{\phi \in [0, 2\pi]} |h(\phi)| < \infty$, then $|A_j| \leq \frac{2C}{a^j}$ and $|B_j| \leq \frac{2C}{a^j}$. And the coefficients $|r^j A_j|$ and $|r^j B_j|$ in (3.1) are bounded by $2C(\frac{r}{a})^j$ for $0 < r < a$. Then use the formulas $\cos j\theta = \frac{e^{ij\theta} + e^{-ij\theta}}{2}$ and $\sin j\theta = \frac{e^{ij\theta} - e^{-ij\theta}}{2i}$ to transfer $u(r, \theta)$ in (3.1) to the form of $a_0 + \sum_{j=1}^{\infty} (a_j e^{ij\theta} + b_j e^{-ij\theta})$, where a_j ’s and b_j ’s are complex numbers. Fix $r < a$. It is easy to see that $|a_j| = O((ra^{-1})^j)$ and $b_j = O((ra^{-1})^j)$ as $j \rightarrow \infty$. Theorems 3 and 4 can then be applied to get the CLT for $a_0 + \sum_{j=1}^{\infty} (a_j p_j(Z_n^\alpha) + b_j \overline{p_j(Z_n^\alpha)})$ for $\alpha = 2$ and $\alpha = \frac{1}{2}$, respectively.

EXAMPLE. Let $u(x, t)$ be a function defined on $[0, \pi] \times [0, \infty)$. Consider the following heat equation with boundary conditions defined by

$$(3.2) \quad \begin{cases} u_t = k u_{xx}, & x \in (0, \pi), t > 0; \\ u(0, t) = u(\pi, t) = 0; \\ u(x, 0) = \phi(x), \end{cases}$$

where $k > 0$ is a constant. Suppose $\phi(x) = \sum_{j=1}^{\infty} A_j \sin jx$ for all $x \in [0, \pi]$. Then the solution of (3.2) is given by

$$u(x, t) = \sum_{j=1}^{\infty} A_j e^{-j^2 kt} \sin jx.$$

See, for example, page 85 from [26]. If $\sup_{j \geq 0} |A_j| < \infty$, then $A_j e^{-j^2 kt} = O(e^{-j^2 kt})$ as $j \rightarrow \infty$. Similar to the previous example, we can write $u(x, t)$ in the form of $a_0 + \sum_{j=1}^{\infty} (a_j e^{ij\theta} + b_j e^{-ij\theta})$, where a_j 's and b_j 's are complex numbers with $|a_j| \vee |b_j| = O(e^{-j^2 kt})$ as $j \rightarrow \infty$. Theorems 3 and 4 can then be applied to obtain the CLT for $a_0 + \sum_{j=1}^{\infty} (a_j p_j(Z_n^\alpha) + b_j \overline{p_j(Z_n^\alpha)})$ with $\alpha = 2$ and $\alpha = \frac{1}{2}$, respectively.

To get the analogues of Theorems 3 and 4 for any $\beta \neq 1, 2, 4$, one needs to get upper bounds for $\mathbb{E}[|p_m(Z_n^\alpha)|^2]$ as in Propositions 1 and 2. It will be even more involved because of the lack of classifications of partitions as in (5.4) for general $\beta > 0$, particularly for irrational $\beta > 0$. However, by using our method, it is possible to get upper bounds for any $\beta = \dots, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1, 2, 3, 4, \dots$

4. Proofs of moment inequalities in Section 2. This section is divided into two parts. In Section 4.1, the necessary background of the Jack functions including their orthogonal properties and combinatorial structures are given. With this preparation, we prove parts (a) and (b) of Theorem 1 and Corollary 2. In Section 4.2, we prove part (c) of Theorem 1 by analysis.

4.1. *Proofs of (a) and (b) of Theorem 1 and Corollary 2.* For a partition λ , the notation $\lambda' = (\lambda'_1, \lambda'_2, \dots)$ represents the conjugate partition of λ , whose Young diagram is obtained by transposing the Young diagram of λ .

Let us review Jack symmetric functions briefly. We do not need the exact definition of Jack functions. In fact, their orthogonal properties are actively used here. For any real number $\alpha > 0$ and each integer $k \geq 1$, we denote by $\Lambda^k(\alpha)$ the algebra of symmetric functions of degree k over the field $\mathbb{Q}(\alpha)$. Recall power-sum symmetric function p_ρ in (2.2). The family of p_ρ over partitions ρ of k forms a basis on $\Lambda^k(\alpha)$. A scalar product on $\Lambda^k(\alpha)$ is defined by

$$(4.1) \quad \langle p_\lambda, p_\mu \rangle_\alpha = \delta_{\lambda\mu} \alpha^{l(\lambda)} z_\lambda$$

for any partitions λ and μ of k , where z_λ is as in (2.1). Set

$$(4.2) \quad C_\lambda(\alpha) = \prod_{(i,j) \in \lambda} \{(\alpha(\lambda_i - j) + \lambda'_j - i + 1)(\alpha(\lambda_i - j) + \lambda'_j - i + \alpha)\},$$

where (i, j) runs over all cells of the Young diagram of λ . By definition, Jack functions $\{J_\lambda^{(\alpha)}\}$ form an orthogonal basis on $\Lambda^k(\alpha)$ and satisfy

$$(4.3) \quad \langle J_\lambda^{(\alpha)}, J_\mu^{(\alpha)} \rangle_\alpha = \delta_{\lambda\mu} C_\lambda(\alpha);$$

see, for example, Chapter VI from [20] or [11].

Since both power-sum symmetric functions and Jack functions form a basis of $\Lambda^k(\alpha)$, they can be mutually expanded. Let $\theta_\rho^\lambda(\alpha)$ denote the coefficient of p_ρ in $J_\lambda^{(\alpha)}$, that is,

$$(4.4) \quad J_\lambda^{(\alpha)} = \sum_{\rho:|\rho|=|\lambda|} \theta_\rho^\lambda(\alpha) p_\rho.$$

The $\theta_\rho^\lambda(\alpha)$'s are real numbers. Inversely, let $\Theta_\rho^\lambda(\alpha)$ be the coefficient of $J_\lambda^{(\alpha)}$ in p_ρ , that is,

$$(4.5) \quad p_\rho = \sum_{\lambda:|\lambda|=|\rho|} \Theta_\rho^\lambda(\alpha) J_\lambda^{(\alpha)}.$$

LEMMA 4.1. *Recalling $\theta_\rho^\lambda(\alpha)$ in (4.4) and $\Theta_\rho^\lambda(\alpha)$ in (4.5). Then, for any partitions λ and ρ with $|\lambda| = |\rho|$, we have*

$$(4.6) \quad \Theta_\rho^\lambda(\alpha) = \frac{\alpha^{l(\rho)} z_\rho}{C_\lambda(\alpha)} \theta_\rho^\lambda(\alpha).$$

PROOF. It follows from (4.4) and (4.1) that

$$\langle J_\lambda^{(\alpha)}, p_\rho \rangle_\alpha = \left\langle \sum_v \theta_v^\lambda(\alpha) p_v, p_\rho \right\rangle_\alpha = \sum_v \theta_v^\lambda(\alpha) \langle p_v, p_\rho \rangle_\alpha = \theta_\rho^\lambda(\alpha) \alpha^{l(\rho)} z_\rho.$$

Similarly, by (4.5) and (4.3),

$$\langle J_\lambda^{(\alpha)}, p_\rho \rangle_\alpha = \left\langle J_\lambda^{(\alpha)}, \sum_v \Theta_\rho^v(\alpha) J_v^{(\alpha)} \right\rangle_\alpha = \sum_v \Theta_\rho^v(\alpha) \langle J_\lambda^{(\alpha)}, J_v^{(\alpha)} \rangle_\alpha = \Theta_\rho^\lambda(\alpha) C_\lambda(\alpha).$$

These two equalities lead to (4.6). \square

The coefficients θ_ρ^λ 's satisfy the following orthogonality relations ((10.31) and (10.32) from [20]):

$$(4.7) \quad \begin{aligned} \sum_\rho z_\rho \alpha^{l(\rho)} \theta_\rho^\lambda(\alpha) \theta_\rho^\mu(\alpha) &= \delta_{\lambda\mu} C_\lambda(\alpha); \\ \sum_\lambda \frac{1}{C_\lambda(\alpha)} \theta_\rho^\lambda(\alpha) \theta_\sigma^\lambda(\alpha) &= \delta_{\rho\sigma} z_\rho^{-1} \alpha^{-l(\rho)}. \end{aligned}$$

In other words, if $a_{\lambda\rho} := (z_\rho \alpha^{l(\rho)} / C_\lambda(\alpha))^{1/2} \theta_\rho^\lambda(\alpha)$, then $A_m = (a_{\lambda\rho})_{|\lambda|=|\rho|=m}$ is an orthogonal matrix of size $p(m)$ for $m \geq 1$. Here, $p(m)$ is the number of partitions of m . The following are some special cases of the Jack polynomials.

In other words, if $a_{\lambda\rho} := (z_\rho \alpha^{l(\rho)} / C_\lambda(\alpha))^{1/2} \theta_\rho^\lambda(\alpha)$, then $A_m = (a_{\lambda\rho})_{|\lambda|=|\rho|=m}$ is an orthogonal matrix of size $p(m)$ for $m \geq 1$. Here, $p(m)$ is the number of partitions of m . The following are some special cases of the Jack polynomials.

EXAMPLE. Let $\alpha = 1$, s_λ be the Schur polynomial and χ_μ^λ the character value for the irreducible representation of the symmetric groups. It is well known that $J_\lambda^{(1)} = h(\lambda)s_\lambda$ with $h(\lambda) = \sqrt{C_\lambda(1)}$ as the hook-length product. Further, by (7.8) from Chapter VI of [20] and (4.5) that

$$\theta_\mu^\lambda(1) = \frac{h(\lambda)\chi_\mu^\lambda}{z_\mu} \quad \text{and} \quad \Theta_\mu^\lambda(1) = \frac{\chi_\mu^\lambda}{h(\lambda)}.$$

EXAMPLE. Let $\alpha = 2$. Then $J_\lambda^{(2)}$ coincides with the zonal polynomial Z_λ . By (2.13) and (2.16) from Chapter VII of [20], we have

$$\theta_\mu^\lambda(2) = \frac{2^k k!}{2^{l(\mu)} z_\mu} \omega_\mu^\lambda \quad \text{and} \quad \Theta_\mu^\lambda(2) = \frac{2^k k!}{h(2\lambda)} \omega_\mu^\lambda,$$

with $k = |\lambda| = |\mu|$, where $h(2\lambda) = C_\lambda(2)$ is the hook-length product of $2\lambda = (2\lambda_1, 2\lambda_2, \dots)$ and ω_μ^λ is the value of the zonal spherical function of a Gelfand pair $(\mathfrak{S}_{2k}, \mathfrak{B}_k)$. Here, \mathfrak{S}_{2k} is the symmetric group and \mathfrak{B}_k is the hyperoctahedral group in \mathfrak{S}_{2k} .

EXAMPLE (Example 1(a) on page 383 from [20]). For each partition ρ of k , we have

$$(4.8) \quad \theta_\rho^{(k)}(\alpha) = \frac{k!}{z_\rho} \alpha^{k-l(\rho)} \quad \text{and} \quad \theta_\rho^{(1^k)}(\alpha) = \frac{k!}{z_\rho} (-1)^{k-l(\rho)}.$$

For each partition λ with $l(\lambda) \leq n$, we define

$$\mathcal{N}_\lambda^\alpha(n) = \prod_{(i,j) \in \lambda} \frac{n + (j - 1)\alpha - (i - 1)}{n + j\alpha - i},$$

which is a positive real number. As we saw in (4.3), Jack functions are orthogonal with respect to the scalar product $\langle \cdot, \cdot \rangle_\alpha$. We next need the second orthogonal property for them.

LEMMA 4.2. *Let λ and μ be two partitions. Let $\alpha > 0$ and $n \geq 1$. Then*

$$\begin{aligned} & \frac{1}{(2\pi)^n} \int_{[0, 2\pi]^n} J_\lambda^{(\alpha)}(e^{i\theta_1}, \dots, e^{i\theta_n}) J_\mu^{(\alpha)}(e^{-i\theta_1}, \dots, e^{-i\theta_n}) \\ & \quad \times \prod_{1 \leq p < q \leq n} |e^{i\theta_p} - e^{i\theta_q}|^{2/\alpha} d\theta_1 \cdots d\theta_n \\ & = \delta_{\lambda\mu} \cdot \delta(l(\lambda) \leq n) \cdot \frac{\Gamma(n/\alpha + 1)}{\Gamma(1 + 1/\alpha)^n} C_\lambda(\alpha) \mathcal{N}_\lambda^\alpha(n). \end{aligned}$$

PROOF. Since $J_\lambda^{(\alpha)}(x_1, \dots, x_n) = 0$ if $l(\lambda) > n$, we assume $l(\lambda) \leq n$ in the following discussion. It is known (e.g., Theorem 12.1.1 from [22]) that

$$(4.9) \quad \frac{1}{(2\pi)^n} \int_{[0,2\pi]^n} \prod_{1 \leq p < q \leq n} |e^{i\theta_p} - e^{i\theta_q}|^{2/\alpha} d\theta_1 \cdots d\theta_n = \frac{\Gamma(n/\alpha + 1)}{\Gamma(1 + 1/\alpha)^n}.$$

From (10.22), (10.35) and (10.37) in [20], we see that

$$\begin{aligned} & \frac{1}{(2\pi)^n n! C_\lambda(\alpha)} \int_{[0,2\pi]^n} J_\lambda^{(\alpha)}(e^{i\theta_1}, \dots, e^{i\theta_n}) J_\mu^{(\alpha)}(e^{-i\theta_1}, \dots, e^{-i\theta_n}) \\ & \quad \times \prod_{1 \leq p < q \leq n} |e^{i\theta_p} - e^{i\theta_q}|^{2/\alpha} d\theta_1 \cdots d\theta_n \\ & = \delta_{\lambda\mu} \cdot c_n \mathcal{N}_\lambda^\alpha(n), \end{aligned}$$

where

$$c_n := \frac{1}{(2\pi)^n n!} \int_{[0,2\pi]^n} \prod_{1 \leq p < q \leq n} |e^{i\theta_p} - e^{i\theta_q}|^{2/\alpha} d\theta_1 \cdots d\theta_n = \frac{1}{n!} \cdot \frac{\Gamma(n/\alpha + 1)}{\Gamma(1 + 1/\alpha)^n}$$

by (4.9). Hence, the desired conclusion follows. \square

PROPOSITION 3. Let $\beta > 0$ be a constant. Suppose $\theta_1, \dots, \theta_n$ have a joint density as in (1.3). Let $Z_n = (e^{i\theta_1}, \dots, e^{i\theta_n})$. Given partitions μ and ν of weight K , then

$$\mathbb{E}[p_\mu(Z_n) \overline{p_\nu(Z_n)}] = \alpha^{l(\mu)+l(\nu)} z_\mu z_\nu \sum_{\lambda \vdash K: l(\lambda) \leq n} \frac{\theta_\mu^\lambda(\alpha) \theta_\nu^\lambda(\alpha)}{C_\lambda(\alpha)} \mathcal{N}_\lambda^\alpha(n).$$

PROOF. Reviewing (1.3), by (4.5) and Lemma 4.2, we have

$$\mathbb{E}[p_\mu(Z_n) \overline{p_\nu(Z_n)}] = \sum_{\lambda \vdash K: l(\lambda) \leq n} \Theta_\mu^\lambda(\alpha) \Theta_\nu^\lambda(\alpha) C_\lambda(\alpha) \mathcal{N}_\lambda^\alpha(n),$$

where $\alpha = 2/\beta$. By Lemma 4.1, the above is identical to

$$\alpha^{l(\mu)+l(\nu)} z_\mu z_\nu \sum_{\lambda \vdash K: l(\lambda) \leq n} \frac{\theta_\mu^\lambda(\alpha) \theta_\nu^\lambda(\alpha)}{C_\lambda(\alpha)} \mathcal{N}_\lambda^\alpha(n).$$

The proof is completed. \square

For positive integers n and K and real number $\alpha > 0$. Define

$$(4.10) \quad \begin{aligned} \Gamma_{n,K}^\alpha &= \max_{\lambda \vdash K: l(\lambda) \leq n} \mathcal{N}_\lambda^\alpha(n) \\ &= \max_{\lambda \vdash K: l(\lambda) \leq n} \prod_{(i,j) \in \lambda} \frac{n + (j-1)\alpha - (i-1)}{n + j\alpha - i}; \end{aligned}$$

$$\begin{aligned}
 \gamma_{n,K}^\alpha &= \min_{\lambda \vdash K: l(\lambda) \leq n} \mathcal{N}_\lambda^\alpha(n) \\
 (4.11) \qquad &= \min_{\lambda \vdash K: l(\lambda) \leq n} \prod_{(i,j) \in \lambda} \frac{n + (j-1)\alpha - (i-1)}{n + j\alpha - i}.
 \end{aligned}$$

LEMMA 4.3. *Let $\alpha > 0, K \geq 1$ and $\Gamma_{n,K}^\alpha$ be as in (4.10) and $\gamma_{n,K}^\alpha$ be as in (4.11). If $n \geq K$, then $A \leq \gamma_{n,K}^\alpha \leq \Gamma_{n,K}^\alpha \leq B$ where A and B are as in (2.3). Further, if $n \geq K$, then*

$$(4.12) \qquad \max_{\lambda \vdash K} |\mathcal{N}_\lambda^\alpha(n) - 1| \leq \max\{|A - 1|, |B - 1|\}.$$

PROOF. For $\lambda \vdash K$ such that $l(\lambda) \leq n$ and $(i, j) \in \lambda$, it is easy to check that

$$(4.13) \qquad 1 \leq i \leq \min\{n, K\} \quad \text{and} \quad 1 \leq j \leq K.$$

Thus, $n + (j - 1)\alpha - (i - 1) \geq n - i + 1 > 0$ and $n + j\alpha - i \geq j\alpha > 0$. It follows that

$$(4.14) \qquad b_{i,j}(\alpha) := \frac{n + (j - 1)\alpha - (i - 1)}{n + j\alpha - i} > 0.$$

Write

$$(4.15) \qquad b_{i,j}(\alpha) = 1 + \frac{1 - \alpha}{n + j\alpha - i}.$$

Case 1: $\alpha \geq 1$. By (4.14) and (4.15), we see that $b_{i,j}(\alpha) \in [0, 1]$ for all $\lambda \vdash K$ such that $l(\lambda) \leq n$ and $(i, j) \in \lambda$, which concludes $\Gamma_{n,K}^\alpha \leq 1$.

Further, by (4.13), $n + j\alpha - i \geq n - K + \alpha > 0$ for all $\lambda \vdash K$ such that $l(\lambda) \leq n$ and $(i, j) \in \lambda$. Thus, noticing $1 - \alpha \leq 0$, we get

$$b_{i,j}(\alpha) \geq 1 + \frac{1 - \alpha}{n - K + \alpha} = 1 - \frac{|1 - \alpha|}{n - K + \alpha} > 0$$

for all $n \geq K$. This yields

$$\gamma_{n,K}^\alpha \geq \left(1 - \frac{|1 - \alpha|}{n - K + \alpha}\right)^K.$$

The above two conclusions lead to that

$$(4.16) \qquad \left(1 - \frac{|1 - \alpha|}{n - K + \alpha}\right)^K \leq \gamma_{n,K}^\alpha \leq \Gamma_{n,K}^\alpha \leq 1$$

for all $n \geq K$ and $\alpha \geq 1$.

Case 2: $\alpha \in (0, 1]$. By (4.15), $b_{i,j}(\alpha) \geq 1$ for all $\lambda \vdash K$ such that $l(\lambda) \leq n$ and $(i, j) \in \lambda$, which shows $\gamma_{n,K}^\alpha \geq 1$.

Moreover, by (4.13) again, $n + j\alpha - i \geq n - K + \alpha$ for all $\lambda \vdash K$ such that $l(\lambda) \leq n$ and $(i, j) \in \lambda$. Thus, with $1 - \alpha > 0$, we have from (4.15) that

$$b_{i,j}(\alpha) \leq 1 + \frac{1 - \alpha}{n - K + \alpha}.$$

By the definition of $\Gamma_{n,K}^\alpha$ and the earlier conclusion, we get

$$1 \leq \gamma_{n,K}^\alpha \leq \Gamma_{n,K}^\alpha \leq \left(1 + \frac{1 - \alpha}{n - K + \alpha}\right)^K$$

for all $n \geq K$ and $\alpha \in (0, 1]$. This and (4.16) prove the first part of the lemma.

Finally, by the definitions in (4.10) and (4.11),

$$\gamma_{n,K}^\alpha \leq \mathcal{N}_\lambda^\alpha(n) = \prod_{(i,j) \in \lambda} b_{i,j}(\alpha) \leq \Gamma_{n,K}^\alpha$$

for all $\lambda \vdash K$ since $l(\lambda) \leq n$ holds automatically if $n \geq K$. By the proved conclusion,

$$A - 1 \leq \mathcal{N}_\lambda^\alpha(n) - 1 \leq B - 1$$

for all $\lambda \vdash K$. This implies (4.12). \square

PROOF OF (A) AND (B) OF THEOREM 1. (a) By Proposition 3, take $\mu = \nu$ with weight K to have

$$\mathbb{E}[|p_\mu(Z_n)|^2] = \alpha^{2l(\mu)} z_\mu^2 \sum_{\lambda \vdash K: l(\lambda) \leq n} \frac{\theta_\mu^\lambda(\alpha)^2}{C_\lambda(\alpha)} \mathcal{N}_\lambda^\alpha(n).$$

Lemma 4.3 says that $\Gamma_{n,K}^\alpha > 0$ and $\gamma_{n,K}^\alpha > 0$ for all $n \geq K$. By the definitions of $\Gamma_{n,K}^\alpha$ in (4.10) and $\gamma_{n,K}^\alpha$ in (4.11), since $C_\lambda(\alpha) > 0$ for any partition λ and $\alpha > 0$,

$$\begin{aligned} \gamma_{n,K}^\alpha \cdot \alpha^{2l(\mu)} z_\mu^2 \sum_{\lambda \vdash K: l(\lambda) \leq n} \frac{\theta_\mu^\lambda(\alpha)^2}{C_\lambda(\alpha)} &\leq \mathbb{E}[|p_\mu(Z_n)|^2] \\ (4.17) \qquad \qquad \qquad &\leq \Gamma_{n,K}^\alpha \cdot \alpha^{2l(\mu)} z_\mu^2 \sum_{\lambda \vdash K: l(\lambda) \leq n} \frac{\theta_\mu^\lambda(\alpha)^2}{C_\lambda(\alpha)}. \end{aligned}$$

From assumption $n \geq K$, if $\lambda \vdash K$, we know $l(\lambda) \leq n$ automatically. Therefore, from (4.7) the two sums in (4.17) are both equal to $z_\mu^{-1} \alpha^{-l(\mu)}$. Consequently,

$$\gamma_{n,K}^\alpha \leq \frac{\mathbb{E}[|p_\mu(Z_n)|^2]}{\alpha^{l(\mu)} z_\mu} \leq \Gamma_{n,K}^\alpha.$$

The conclusion (a) then follows from the first part of Lemma 4.3.

(b) First, assume $|\mu| \neq |\nu|$. Notice

$$\begin{aligned} & \mathbb{E}[p_\mu(Z_n)\overline{p_\nu(Z_n)}] \\ &= \text{Const} \cdot \int_0^{2\pi} \cdots \int_0^{2\pi} p_\mu(e^{i\theta_1}, \dots, e^{i\theta_n}) \overline{p_\nu(e^{i\theta_1}, \dots, e^{i\theta_n})} \\ & \quad \times \prod_{1 \leq j < k \leq n} |e^{i\theta_j} - e^{i\theta_k}|^\beta d\theta_1 \cdots d\theta_n. \end{aligned}$$

For an integrable function $h(x)$, we know $\int_0^{2\pi} h(e^{ix}) dx = \int_b^{b+2\pi} h(e^{ix}) dx$ for any $b \in \mathbb{R}$. Using the induction and the Fubini theorem, we see that

$$\begin{aligned} & \mathbb{E}[p_\mu(Z_n)\overline{p_\nu(Z_n)}] \\ &= \text{Const} \cdot \int_b^{b+2\pi} \cdots \int_b^{b+2\pi} p_\mu(e^{i\theta_1}, \dots, e^{i\theta_n}) \overline{p_\nu(e^{i\theta_1}, \dots, e^{i\theta_n})} \\ & \quad \times \prod_{1 \leq j < k \leq n} |e^{i\theta_j} - e^{i\theta_k}|^\beta d\theta_1 \cdots d\theta_n. \end{aligned}$$

Making transform $\eta_j = \theta_j - b$ for $1 \leq j \leq n$, noting that $p_\mu(e^{ib+i\eta_1}, \dots, e^{ib+i\eta_n}) = e^{ib|\mu|} p_\mu(e^{i\eta_1}, \dots, e^{i\eta_n})$ for any $b \in \mathbb{R}$, we obtain that

$$\mathbb{E}[p_\mu(Z_n)\overline{p_\nu(Z_n)}] = e^{ib(|\mu|-|\nu|)} \mathbb{E}[p_\mu(Z_n)\overline{p_\nu(Z_n)}]$$

for any $b \in \mathbb{R}$. If $|\mu| \neq |\nu|$, since b is arbitrary, we then conclude

$$\mathbb{E}[p_\mu(Z_n)\overline{p_\nu(Z_n)}] = 0$$

for all $n \geq 2$.

To prove the second part of (b), by the first part, it suffices to prove the conclusion for $n \geq |\mu| = |\nu| = K$. Observe that $l(\lambda) \leq n$ if $\lambda \vdash K$. Thus, it follows from Proposition 3 that

$$\begin{aligned} & \mathbb{E}[p_\mu(Z_n)\overline{p_\nu(Z_n)}] \\ &= \alpha^{l(\mu)+l(\nu)} z_\mu z_\nu \sum_{\lambda \vdash K} \frac{\theta_\mu^\lambda(\alpha)\theta_\nu^\lambda(\alpha)}{C_\lambda(\alpha)} \mathcal{N}_\lambda^\alpha(n) \\ &= \alpha^{l(\mu)+l(\nu)} z_\mu z_\nu \left[\sum_{\lambda \vdash K} \frac{\theta_\mu^\lambda(\alpha)\theta_\nu^\lambda(\alpha)}{C_\lambda(\alpha)} + \sum_{\lambda \vdash K} \frac{\theta_\mu^\lambda(\alpha)\theta_\nu^\lambda(\alpha)}{C_\lambda(\alpha)} (\mathcal{N}_\lambda^\alpha(n) - 1) \right] \\ &= \alpha^{l(\mu)+l(\nu)} z_\mu z_\nu \sum_{\lambda \vdash K} \frac{\theta_\mu^\lambda(\alpha)\theta_\nu^\lambda(\alpha)}{C_\lambda(\alpha)} (\mathcal{N}_\lambda^\alpha(n) - 1), \end{aligned}$$

where the last identity comes from the orthogonal property in (4.7). Therefore,

$$\begin{aligned} & |\mathbb{E}[p_\mu(Z_n)\overline{p_\nu(Z_n)}]| \\ & \leq \max_{\lambda \vdash K} |\mathcal{N}_\lambda^\alpha(n) - 1| \cdot \alpha^{l(\mu)+l(\nu)} z_\mu z_\nu \sum_{\lambda \vdash K} \frac{|\theta_\mu^\lambda(\alpha)| \cdot |\theta_\nu^\lambda(\alpha)|}{C_\lambda(\alpha)}. \end{aligned}$$

Now, by the Cauchy–Schwarz inequality the sum above is bounded by

$$\left(\sum_{\lambda \vdash K} \frac{|\theta_\mu^\lambda(\alpha)|^2}{C_\lambda(\alpha)}\right)^{1/2} \cdot \left(\sum_{\lambda \vdash K} \frac{|\theta_\nu^\lambda(\alpha)|^2}{C_\lambda(\alpha)}\right)^{1/2} = z_\mu^{-1/2} z_\nu^{-1/2} \alpha^{-l(\mu)+l(\nu)/2}$$

by (4.7). The above two inequalities imply

$$\begin{aligned} |\mathbb{E}[p_\mu(Z_n) \overline{p_\nu(Z_n)}]| &\leq \max_{\lambda \vdash K} |\mathcal{N}_\lambda^\alpha(n) - 1| \cdot \alpha^{(l(\mu)+l(\nu))/2} (z_\mu z_\nu)^{1/2} \\ &\leq \max\{|A - 1|, |B - 1|\} \cdot \alpha^{(l(\mu)+l(\nu))/2} (z_\mu z_\nu)^{1/2} \end{aligned}$$

by (4.12). \square

LEMMA 4.4. *Let A and B be as in (2.3) with $\beta > 0$. Set $\alpha = 2/\beta$. If $n \geq 2K$, then*

$$\max\{|A - 1|, |B - 1|\} \leq \frac{6|1 - \alpha|K}{n}.$$

PROOF. By the definitions of A and B , it suffices to show that, as $n \geq 2K$,

$$(4.18) \quad 1 - \left(1 - \frac{\alpha - 1}{n - K + \alpha}\right)^K \leq \frac{6|1 - \alpha|K}{n} \quad \text{for } \alpha \geq 1;$$

$$(4.19) \quad \left(1 + \frac{1 - \alpha}{n - K + \alpha}\right)^K - 1 \leq \frac{6|1 - \alpha|K}{n} \quad \text{for } \alpha \in (0, 1).$$

First, if $\alpha \geq 1$, then $(\alpha - 1)/(n - K + \alpha) \in [0, 1)$. Notice $(1 + x)^K \geq 1 + Kx$ for all $x \geq -1$ (see, e.g., Theorem 42 on page 40 from [13]), we have

$$1 - \left(1 - \frac{\alpha - 1}{n - K + \alpha}\right)^K \leq \frac{K(\alpha - 1)}{n - K + \alpha} \leq \frac{2K|1 - \alpha|}{n}$$

since $n - K + \alpha \geq n/2$ as $n \geq 2K$. This proves (4.18).

Second, for $\alpha \in (0, 1)$, it is easy to verify that $(1 - \alpha)/(n - K + \alpha) \leq 1/K$ provided $n \geq 2K$. By the fact that $(1 + x)^K \leq 1 + 3Kx$ for all $0 \leq x \leq 1/K$, we obtain

$$\left(1 + \frac{1 - \alpha}{n - K + \alpha}\right)^K - 1 \leq \frac{3(1 - \alpha)K}{n - K + \alpha} \leq \frac{6|1 - \alpha|K}{n}$$

since $n - K + \alpha \geq n/2$ if $n \geq 2K$ as used earlier. This concludes (4.19). \square

PROOF OF COROLLARY 2. (a) By Theorem 1

$$A - 1 \leq \frac{\mathbb{E}[|p_\mu(Z_n)|^2]}{\alpha^{l(\mu)} z_\mu} - 1 \leq B - 1.$$

Thus,

$$\left| \frac{\mathbb{E}[|p_\mu(Z_n)|^2]}{\alpha^{l(\mu)} z_\mu} - 1 \right| \leq \max\{|A - 1|, |B - 1|\}.$$

The conclusion (a) then follows from Lemma 4.4.

(b) The conclusion obviously holds if $|\mu| \neq |\nu|$ by (b) of Theorem 1. If $|\mu| = |\nu| = K$, by (b) of Theorem 1 and Lemma 4.4, we get the desired result. \square

4.2. *Proof of (c) of Theorem 1.* We start the proof through a series of lemmas.

LEMMA 4.5. *Let $\beta > 0$. For positive integers m and k and real numbers a_1, \dots, a_k , define*

$$D = \int_0^\pi \cos(2mt) \left| \prod_{i=1}^k \sin(t + a_i) \right|^\beta dt.$$

Then $|D| \leq 6(1 + \beta) \left(\frac{k}{m}\right)^{1 \wedge \beta}$.

PROOF. First, since $|D| \leq \int_0^\pi 1 dt = \pi$, the conclusion obviously holds for $m = 1$. Now we assume $m \geq 2$. Set $s = mt$. Then

$$\begin{aligned} D &= \frac{1}{m} \int_0^{m\pi} \cos(2s) \left| \prod_{i=1}^k \sin\left(\frac{s}{m} + a_i\right) \right|^\beta ds \\ &= \frac{1}{m} \sum_{j=0}^{m-1} \int_{j\pi}^{(j+1)\pi} \cos(2s) \left| \prod_{i=1}^k \sin\left(\frac{s}{m} + a_i\right) \right|^\beta ds \\ (4.20) \quad &= \frac{1}{m} \sum_{j=0}^{m-1} \int_0^\pi \cos(2s) \left| \prod_{i=1}^k \sin\left(\frac{s + j\pi}{m} + a_i\right) \right|^\beta ds \\ &= \int_0^\pi L_m(s) \cos(2s) ds, \end{aligned}$$

where we make a transform: $s \rightarrow s - j\pi$ in the second identity to get the third one, and

$$L_m(s) = \frac{1}{m} \sum_{j=0}^{m-1} \left| \prod_{i=1}^k \sin\left(b_{ij} + \frac{s}{m}\right) \right|^\beta$$

for $0 \leq s \leq \pi$ and $b_{ij} = a_i + \frac{j\pi}{m}$. Since $(a + b)^\beta \leq a^\beta + b^\beta$ for any $a \geq 0, b \geq 0, \beta \in (0, 1]$, and $|c^\beta - d^\beta| \leq \beta|c - d|$ for any $c, d \in [-1, 1], \beta > 1$, it is not

difficult to see that $||x|^\beta - |y|^\beta| \leq (1 + \beta)||x| - |y||^{1 \wedge \beta} \leq (1 + \beta)|x - y|^{1 \wedge \beta}$ for any $\beta > 0$ and $x, y \in [-1, 1]$. Therefore,

$$\begin{aligned}
 & \left| L_m(s) - \frac{1}{m} \sum_{j=0}^{m-1} \left| \prod_{i=1}^k \sin b_{ij} \right|^\beta \right| \\
 (4.21) \quad & \leq \frac{1}{m} \sum_{j=0}^{m-1} \left| \left| \prod_{i=1}^k \sin \left(b_{ij} + \frac{s}{m} \right) \right|^\beta - \left| \prod_{i=1}^k \sin b_{ij} \right|^\beta \right| \\
 & \leq \frac{1 + \beta}{m} \sum_{j=0}^{m-1} \left| \prod_{i=1}^k \sin \left(b_{ij} + \frac{s}{m} \right) - \prod_{i=1}^k \sin b_{ij} \right|^{\beta \wedge 1}.
 \end{aligned}$$

Now, by the product rule, $(\prod_{i=1}^k \sin(b_{ij} + t))' = \sum_{l=1}^k \cos(b_{lj} + t) \times \prod_{1 \leq i \leq k, i \neq l} \sin(b_{ij} + t)$ for any $t \in \mathbb{R}$. Thus, the absolute value of the derivative is bounded by k for ant $t \in \mathbb{R}$. By the mean-value theorem,

$$\left| \prod_{i=1}^k \sin \left(b_{ij} + \frac{s}{m} \right) - \prod_{i=1}^k \sin b_{ij} \right| \leq \frac{ks}{m}.$$

This implies that the last term in (4.21) is controlled by

$$\frac{1 + \beta}{m} \sum_{j=0}^{m-1} \left(\frac{ks}{m} \right)^{1 \wedge \beta} = (1 + \beta) \left(\frac{ks}{m} \right)^{1 \wedge \beta}.$$

It follows from (4.21) that

$$\left| L_m(s) - \frac{1}{m} \sum_{j=0}^{m-1} \left| \prod_{i=1}^k \sin b_{ij} \right|^\beta \right| \leq (1 + \beta) \left(\frac{ks}{m} \right)^{1 \wedge \beta}.$$

Set $C = \frac{1}{m} \sum_{j=0}^{m-1} \left| \prod_{i=1}^k \sin b_{ij} \right|^\beta$. Notice $\int_0^\pi \cos(2s) ds = 0$. From the above, we use the simple fact that $|\cos(2s)| \leq 1$ to have

$$\begin{aligned}
 \left| \int_0^\pi L_m(s) \cos(2s) ds \right| &= \left| \int_0^\pi C \cos(2s) ds + \int_0^\pi (L_m(s) - C) \cos(2s) ds \right| \\
 &\leq (1 + \beta) \left(\frac{k}{m} \right)^{1 \wedge \beta} \int_0^\pi s^{1 \wedge \beta} ds.
 \end{aligned}$$

Now the last integral above is bounded by $\int_0^1 1 ds + \int_1^\pi s ds = (\pi^2 + 1)/2 \leq 6$. The proof is completed by using (4.20). \square

LEMMA 4.6. For $\beta > 0$, let $f(\theta_1, \dots, \theta_n | \beta)$ be as in (1.3). Define

$$\begin{aligned}
 I(m, n) &= \int_0^{2\pi} \cdots \int_0^{2\pi} \cos(m(\theta_2 - \theta_1)) \\
 &\quad \times f(\theta_1, \dots, \theta_n | \beta) d\theta_1 \cdots d\theta_n \quad (m \geq 0, n \geq 2).
 \end{aligned}$$

Then, for some constant $K = K(\beta)$, we have $|I(m, n)| \leq (Kn2^{n\beta})m^{-(1 \wedge \beta)}$ for all $m \geq 1$ and $n \geq 2$.

PROOF. Evidently, since $f(\theta_1, \dots, \theta_n|\beta)$ is a probability density function, we know

$$(4.22) \quad I(0, n) = 1$$

for all $n \geq 2$. Since $|e^{ix} - e^{iy}|^2 = |1 - e^{i(x-y)}|^2 = (1 - \cos(x-y))^2 + \sin^2(x-y) = 2(1 - \cos(x-y)) = 4 \sin^2((x-y)/2)$ for any $x, y \in \mathbb{R}$, the probability density function in (1.3) becomes

$$f(\theta_1, \dots, \theta_n|\beta) = C_n \prod_{1 \leq j < k \leq n} \left| \sin\left(\frac{\theta_j - \theta_k}{2}\right) \right|^\beta,$$

where $\theta_1, \dots, \theta_n \in [0, 2\pi]$ and

$$C_n = 2^{n(n-1)\beta/2} (2\pi)^{-n} \cdot \frac{\Gamma(1 + \beta/2)^n}{\Gamma(1 + \beta n/2)}.$$

Now,

$$\begin{aligned} I(m, n) &= \int_0^{2\pi} \cdots \int_0^{2\pi} \cos(m(\theta_2 - \theta_1)) f(\theta_1, \dots, \theta_n|\beta) d\theta_1 \cdots d\theta_n \\ &= C_n \int_0^{2\pi} \cdots \int_0^{2\pi} \cos(m(\theta_2 - \theta_1)) \\ &\quad \times \prod_{1 \leq j < k \leq n} \left| \sin\left(\frac{\theta_j - \theta_k}{2}\right) \right|^\beta d\theta_2 \cdots d\theta_n d\theta_1. \end{aligned}$$

Making transforms $x_i = \theta_i - \theta_1$ for $i = 2, 3, \dots, n$, we obtain that

$$I(m, n) = C_n \int_0^{2\pi} \int_{-\theta_1}^{2\pi - \theta_1} \cdots \int_{-\theta_1}^{2\pi - \theta_1} \cos(mx_2) \cdot G_n(x) dx_2 \cdots dx_n d\theta_1$$

with

$$G_n(x) = \prod_{i=2}^n \left| \sin\left(\frac{x_i}{2}\right) \right|^\beta \cdot \prod_{2 \leq j < k \leq n} \left| \sin\left(\frac{x_j - x_k}{2}\right) \right|^\beta,$$

where the second product is understood to be 1 if $n = 2$. For a periodic and integrable function $h(x)$ with period 2π , we know that $\int_b^{b+2\pi} h(x) dx = \int_0^{2\pi} h(x) dx$. By induction and the Fubini theorem, we have

$$\begin{aligned} (4.23) \quad I(m, n) &= C_n \int_0^{2\pi} \cdots \int_0^{2\pi} \cos(mx_2) \cdot G_n(x) dx_2 \cdots dx_n d\theta_1 \\ &= (2\pi)C_n \int_0^{2\pi} \cdots \int_0^{2\pi} \cos(mx_2) \cdot G_n(x) dx_2 \cdots dx_n \end{aligned}$$

$$(4.24) \quad = (2\pi)C_n \int_0^{2\pi} \cdots \int_0^{2\pi} \cos(mx_2) J_n(x) H_n(x) dx_2 \cdots dx_n,$$

where $G_n(x) = J_n(x)H_n(x)$ and

$$H_n(x) = \begin{cases} \prod_{i=3}^n \left| \sin\left(\frac{x_i}{2}\right) \right|^\beta \cdot \prod_{3 \leq j < k \leq n} \left| \sin\left(\frac{x_j - x_k}{2}\right) \right|^\beta, & \text{if } n \geq 4; \\ \left| \sin\left(\frac{x_3}{2}\right) \right|^\beta, & \text{if } n = 3; \\ 1, & \text{if } n = 2, \end{cases}$$

and

$$J_n(x) = \begin{cases} \left| \sin\left(\frac{x_2}{2}\right) \right|^\beta \prod_{i=3}^n \left| \sin\left(\frac{x_2 - x_i}{2}\right) \right|^\beta, & \text{if } n \geq 3; \\ \left| \sin\left(\frac{x_2}{2}\right) \right|^\beta, & \text{if } n = 2. \end{cases}$$

In particular,

$$(4.25) \quad I(m, 2) = 2\pi C_2 \int_0^{2\pi} \cos(mx_2) J_2(x) dx_2.$$

Taking $m = 0$ in (4.23), we know from (4.22) that

$$\int_0^{2\pi} \cdots \int_0^{2\pi} \prod_{i=2}^n \left| \sin\left(\frac{x_i}{2}\right) \right|^\beta \cdot \prod_{2 \leq j < k \leq n} \left| \sin\left(\frac{x_j - x_k}{2}\right) \right|^\beta dx_2 dx_3 \cdots dx_n = \frac{1}{2\pi C_n}$$

for all $n \geq 2$, where the second product above is understood to be 1 if $n = 2$. This implies

$$(4.26) \quad \int_0^{2\pi} \cdots \int_0^{2\pi} H_n(x) dx_3 \cdots dx_n = \frac{1}{2\pi C_{n-1}}$$

for all $n \geq 3$. Now, recalling the definition of $J_n(x)$, let $t = x_2/2$, we have

$$\int_0^{2\pi} \cos(mx_2) J_n(x) dx_2 = 2 \int_0^\pi \cos(2mt) \left| \prod_{i=1}^{n-1} \sin(t + a_i) \right|^\beta dt$$

for all $n \geq 2$, where $a_1 = 0, a_i = -x_{i+1}/2$ for $i = 2, \dots, n - 1$. By Lemma 4.5,

$$(4.27) \quad \left| \int_0^{2\pi} \cos(mx_2) J_n(x) dx_2 \right| \leq 12(1 + \beta) \left(\frac{n}{m}\right)^{1 \wedge \beta}$$

for all $n \geq 2$. Therefore, this and (4.25) imply that for some constant $K_1 = K_1(\beta)$,

$$(4.28) \quad |I(m, 2)| \leq \frac{K_1}{m^{(1 \wedge \beta)}}.$$

Now assume $n \geq 3$. By (4.24) and (4.27), and then (4.26), we obtain

$$\begin{aligned}
 (4.29) \quad |I(m, n)| &\leq 24\pi(1 + \beta)C_n \left(\frac{n}{m}\right)^{1 \wedge \beta} \int_0^{2\pi} \cdots \int_0^{2\pi} H_n(x) dx_3 \cdots dx_n \\
 &= 12(1 + \beta) \left(\frac{n}{m}\right)^{1 \wedge \beta} \frac{C_n}{C_{n-1}}
 \end{aligned}$$

for all $n \geq 3$. Now,

$$(4.30) \quad \frac{C_n}{C_{n-1}} = \frac{\Gamma(1 + \beta/2)}{2\pi} \cdot \frac{\Gamma(1 + \beta n/2 - \beta/2)}{\Gamma(1 + \beta n/2)} \cdot 2^{(n-1)\beta}$$

for all $n \geq 3$. By Lemma 2.4 from [6], there exists a constant $K_2 = K_2(\beta)$ such that

$$\frac{\Gamma(1 + \beta n/2 - \beta/2)}{\Gamma(1 + \beta n/2)} \leq \frac{K_2}{n^{\beta/2}}$$

for all $n \geq 1$. This, (4.29) and (4.30) imply that there exists a constant $K = K(\beta)$ such that

$$|I(m, n)| \leq K \cdot \left(\frac{n}{m}\right)^{1 \wedge \beta} \cdot \frac{1}{n^{\beta/2}} \cdot 2^{n\beta} = K n^{(1 \wedge \beta) - \beta/2} \frac{2^{n\beta}}{m^{1 \wedge \beta}} \leq K \frac{n 2^{n\beta}}{m^{1 \wedge \beta}}$$

for all $n \geq 3$. This together with (4.28) proves the lemma. \square

PROOF OF (C) OF THEOREM 1. Observe that, for any real numbers x_1, \dots, x_n ,

$$\begin{aligned}
 \left| \sum_{j=1}^n e^{ix_j} \right|^2 &= \sum_{j=1}^n e^{ix_j} \cdot \sum_{j=1}^n e^{-ix_j} \\
 &= n + \sum_{j \neq k} e^{i(x_j - x_k)} = n + \sum_{1 \leq j < k \leq n} (e^{i(x_j - x_k)} + e^{-i(x_j - x_k)}) \\
 &= n + 2 \sum_{1 \leq j < k \leq n} \cos(x_j - x_k).
 \end{aligned}$$

Thus, by the symmetry of $f(\theta_1, \dots, \theta_n | \beta)$,

$$\begin{aligned}
 (4.31) \quad \mathbb{E}[|p_m(Z_n)|^2] &= \mathbb{E} \left[\left| \sum_{j=1}^n e^{im\theta_j} \right|^2 \right] \\
 &= n + n(n - 1) \cdot \mathbb{E}[\cos\{m(\theta_1 - \theta_2)\}].
 \end{aligned}$$

The conclusion then follows from Lemma 4.6. \square

5. Proofs of central limit theorems in Section 3. Before proving the central limit theorems, we will spend a lot efforts in studying the second moments, which enable us to reduce the infinite Fourier series in Theorems 3 and 4 to finite sums, and hence we can apply the moment inequalities stated in Section 2. We will prove Proposition 1 in Section 5.1, and Proposition 2 in Section 5.2. All of the central limit theorems will be proved in Section 5.3. We start with the combinatorial structure of the second moment.

Review that $Z_n^\alpha = (e^{i\theta_1}, \dots, e^{i\theta_n})$ follow the β -circular ensemble with $\alpha = \frac{2}{\beta}$. Its probability density function is given in (1.3). Following our notation, $p_m(Z_n^\alpha) = \sum_{j=1}^n e^{im\theta_j}$ for any integer $m \geq 0$. We know from Proposition 3 that

$$(5.1) \quad \mathbb{E}[|p_m(Z_n^\alpha)|^2] = \alpha^2 m^2 \sum_{\lambda \vdash m: l(\lambda) \leq n} \frac{\theta_{(m)}^\lambda(\alpha)^2}{C_\lambda(\alpha)} \mathcal{N}_\lambda^\alpha(n),$$

where

$$(5.2) \quad \begin{aligned} \mathcal{N}_\lambda^\alpha(n) &= \prod_{(i,j) \in \lambda} \frac{n + (j - 1)\alpha - (i - 1)}{n + j\alpha - i} \\ &= \prod_{(i,j) \in \lambda} \left(1 + \frac{1 - \alpha}{n + j\alpha - i} \right). \end{aligned}$$

We also know the following formula (page 383 from [20]): For each $\lambda \vdash m$,

$$(5.3) \quad \theta_{(m)}^\lambda(\alpha) = \prod_{\substack{(i,j) \in \lambda \\ (i,j) \neq (1,1)}} (\alpha(j - 1) - (i - 1)),$$

where the product runs over all boxes of Young diagram λ , except the $(1, 1)$ -box.

5.1. *Proof of Proposition 1.* Let us first evaluate $\theta_{(m)}^\lambda(2)$ and $C_\lambda(2)$. Suppose $\alpha = 2$. The $(3, 2)$ th box in the Young diagram λ gives $\alpha(j - 1) - (i - 1) = 2 \cdot (2 - 1) - (3 - 1) = 0$, and hence $\theta_{(m)}^\lambda(2)$ vanishes if λ has the $(3, 2)$ -box. In other words, $\theta_{(m)}^\lambda(2)$ vanishes unless $\lambda_3 \leq 1$. Denote by $\mathcal{P}_m^{(2)}(n)$ the set of such partitions of m with lengths $\leq n$:

$$(5.4) \quad \mathcal{P}_m^{(2)}(n) = \{ \lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \vdash m; \lambda_3 \leq 1 \}.$$

The elements in $\mathcal{P}_m^{(2)}(n)$ can be classified into the following three categories.

1. The one-row partition (m) ;
2. A two-row partition $(m - r, r)$ with $r = 1, 2, \dots, \lfloor \frac{m}{2} \rfloor$;
3. $\lambda = (r, s, 1^{m-r-s})$ with $r \geq s \geq 1$ and $3 \leq l(\lambda) = m - r - s + 2 \leq n$.

For each case, the quantity $\theta_{(m)}^\lambda(2) = \prod_{\substack{(i,j) \in \lambda \\ (i,j) \neq (1,1)}} (2j - i - 1)$ is computed as follows:

$$(5.5) \quad \theta_{(m)}^{(m)}(2) = 2 \cdot 4 \cdots (2m - 2) = 2^{m-1} \cdot (m - 1)!;$$

$$(5.6) \quad \theta_{(m)}^{(m-r,r)}(2) = (-1)^{m-2r} (m - r - 1)! \cdot \frac{(2r - 2)!}{(r - 1)!};$$

$$(5.7) \quad \begin{aligned} \theta_{(m)}^{(r,s,1^{m-r-s})}(2) &= (-1)^{m-r-s+1} \cdot 2^{r-s} \cdot (r - 1)! \\ &\quad \times \frac{(2s - 2)!}{(s - 1)!} \cdot (m - r - s + 1)!. \end{aligned}$$

Now we study $C_\lambda(2)$. Note that $C_\lambda(2)$ coincides with the hook-length product of $2\lambda = (2\lambda_1, 2\lambda_2, \dots)$. The hook-length product of λ is computed in Section 6 from [3]:

1. $C_{(m)}(2) = (2m)!;$
2. $C_{(m-r,r)}(2) = \frac{(2r)!(2m-2r+1)!}{2^{m-4r+1}};$
3. $C_{(r,s,1^{m-r-s})}(2) = (m + r - s + 1)(m + r - s)(m - r + s)(m - r + s - 1) \cdot (m - r - s + 1)!(m - r - s)! \frac{(2r-1)!(2s-2)!}{2^{r-2s+1}}.$

Hence, the term $[(\alpha^{l(\mu)} z_\mu)^2 \frac{\theta_\mu^\lambda(\alpha)^2}{C_\lambda(\alpha)}]_{\mu=(m), \alpha=2} = 4m^2 \frac{\theta_{(m)}^\lambda(2)^2}{C_\lambda(2)}$ is given below.

$$(5.8) \quad 4m^2 \frac{\theta_{(m)}^{(m)}(2)^2}{C_{(m)}(2)} = \frac{2^{2m} (m!)^2}{(2m)!};$$

$$(5.9) \quad 4m^2 \frac{\theta_{(m)}^{(m-r,r)}(2)^2}{C_{(m-r,r)}(2)} = \frac{\binom{2r}{m-r}}{\binom{2(m-r)}{m-r}} \cdot \frac{2^{2m-4r} m^2 (2m - 4r + 1)}{(m - r)^2 (2r - 1)^2 (2m - 2r + 1)};$$

$$(5.10) \quad \begin{aligned} &4m^2 \frac{\theta_{(m)}^{(r,s,1^{m-r-s})}(2)^2}{C_{(r,s,1^{m-r-s})}(2)} \\ &= \frac{4m^2 (m - r - s + 1)}{(m + r - s + 1)(m + r - s)(m - r + s)(m - r + s - 1)} \\ &\quad \cdot \frac{2^{2r-2s} [(r - 1)!]^2 (2r - 2s + 1) \cdot (2s - 2)!}{[(s - 1)!]^2 (2r - 1)!}. \end{aligned}$$

Note: through the rest of the paper, C stands for a generic constant which may change from line to line.

LEMMA 5.1. Recall $N_\lambda^2(n)$ as in (5.2). Then there exists a universal constant $K \in (0, \infty)$ such that $N_\lambda^2(n) \leq K \sqrt{\frac{n}{m}}$ uniformly for all m, n and all λ satisfying:

- (i) $\lambda = (m)$ and $m \geq n \geq 1$,
- (ii) $\lambda = (m - r, r)$ with $1 \leq r \leq m/2$ and $m \geq n \geq 2$ or
- (iii) $\lambda = (r, s, 1^{m-r-s})$ with $r \geq s \geq 1, 3 \leq m - r - s + 2 \leq n$ and $m \geq n$.

PROOF. The following basic estimate will be used several times.

$$(5.11) \quad \log \frac{l}{k} \leq \sum_{j=k}^l \frac{1}{j} \leq 1 + \log \frac{l}{k}$$

for all $1 \leq k \leq l$. It is obviously true if $k = l$. Now, for $1 \leq k < l$,

$$\sum_{j=k}^l \frac{1}{j} \leq 1 + \sum_{j=k+1}^l \int_{j-1}^j \frac{1}{x} dx = 1 + \int_k^l \frac{1}{x} dx = 1 + \log \frac{l}{k}.$$

Similarly,

$$\sum_{j=k}^l \frac{1}{j} \geq \sum_{j=k}^l \int_j^{j+1} \frac{1}{x} dx = \int_k^{l+1} \frac{1}{x} dx \geq \log \frac{l}{k}.$$

(i) Since $\lambda = (m)$, we have from (5.2) and the fact $1 - x \leq e^{-x}$ for all $x \in \mathbb{R}$ that

$$(5.12) \quad \begin{aligned} \mathcal{N}_\lambda^2(n) &= \prod_{(i,j) \in \lambda} \left(1 - \frac{1}{n + 2j - i}\right) = \prod_{j=1}^m \left(1 - \frac{1}{n + 2j - 1}\right) \\ &\leq \exp\left(-\frac{1}{2} \sum_{j=1}^m \frac{1}{n - 1 + j}\right) \end{aligned}$$

since $n + 2j - 1 \leq 2(n - 1 + j)$. From (5.11), we get that

$$\sum_{j=1}^m \frac{1}{n - 1 + j} = \sum_{j=n}^{n+m-1} \frac{1}{j} \geq \log \frac{n + m - 1}{n} \geq \log \frac{m}{n}$$

for all $m \geq n \geq 1$. This gives that $\mathcal{N}_\lambda^2(n) \leq \sqrt{\frac{n}{m}}$ for any $m \geq n \geq 1$.

(ii) Now, $\lambda = (m - r, r)$ with $1 \leq r \leq m/2$ and $n \geq 2$. Recall (5.12). We have

$$\begin{aligned} \mathcal{N}_\lambda^2(n) &= \prod_{(i,j) \in \lambda} \left(1 - \frac{1}{n + 2j - i}\right) \\ &= \prod_{j=1}^{m-r} \left(1 - \frac{1}{n + 2j - 1}\right) \cdot \prod_{j=1}^r \left(1 - \frac{1}{n + 2j - 2}\right) \\ &\leq \exp\left(-\frac{1}{2} \sum_{j=1}^{m-r} \frac{1}{n - 1 + j} - \frac{1}{2} \sum_{j=1}^r \frac{1}{n - 2 + j}\right) \end{aligned}$$

by the inequality $n + 2j - i \leq 2(n - i + j)$ for $i = 1, 2$. Hence,

$$\begin{aligned} -2 \log \mathcal{N}_\lambda^2(n) &\geq \sum_{j=n}^{m+n-r-1} \frac{1}{j} + \sum_{j=n-1}^{n+r-2} \frac{1}{j} \\ &\geq \log \left(\frac{m+n-r-1}{n} \cdot \frac{n+r-2}{n-1} \right) \end{aligned}$$

for any $1 \leq r \leq m/2$ by (5.11). Notice $\frac{m+n-r-1}{n} \geq \frac{m}{2n}$ and $\frac{n+r-2}{n-1} \geq 1$ since $1 \leq r \leq m/2$. We then have

$$\mathcal{N}_\lambda^2(n) \leq 2\sqrt{\frac{n}{m}}.$$

(iii) In this case, $\lambda = (r, s, 1^{m-r-s})$ with $r \geq s \geq 1$ and $3 \leq l(\lambda) = m - r - s + 2 \leq n$ and $m \geq n$. First, these restrictions imply

$$(5.13) \quad r \geq \frac{m-n}{2} + 1, \quad m-r \geq 2 \text{ and } n \geq 3.$$

Now,

$$\begin{aligned} \mathcal{N}_\lambda^2(n) &= \prod_{(i,j) \in \lambda} \left(1 - \frac{1}{n+2j-i} \right) \\ &= \prod_{j=1}^r \left(1 - \frac{1}{n+2j-1} \right) \cdot \prod_{j=1}^s \left(1 - \frac{1}{n+2j-2} \right) \\ &\quad \times \prod_{i=3}^{m-r-s+2} \left(1 - \frac{1}{n+2-i} \right) \\ &\leq \exp \left(-\frac{1}{2} \sum_{j=1}^r \frac{1}{n-1+j} - \frac{1}{2} \sum_{j=1}^s \frac{1}{n-2+j} - \frac{1}{2} \sum_{i=3}^{m-r-s+2} \frac{1}{n-i+1} \right) \end{aligned}$$

by the inequality $n + 2j - i \leq 2(n + j - i)$ for all $j \geq 1$ and $i \leq m - r - s + 2 \leq n$. Rearranging the indices in the sums and using (5.11), we obtain that

$$\begin{aligned} -2 \log \mathcal{N}_\lambda^2(n) &\geq \sum_{j=n}^{n+r-1} \frac{1}{j} + \sum_{j=n-1}^{n+s-2} \frac{1}{j} + \sum_{j=n+r+s-m-1}^{n-2} \frac{1}{j} \\ &\geq \log \frac{n+r-1}{n} \cdot \frac{n+s-2}{n-1} \cdot \frac{n-2}{n+r+s-m-1} \\ &\geq \log \frac{(n+r-1)(n+s-2)}{2n(n+r+s-m)} \end{aligned}$$

since $\frac{n-2}{n-1} \geq \frac{1}{2}$ by (5.13). Equivalently,

$$\begin{aligned} \mathcal{N}_\lambda^2(n) &\leq \sqrt{\frac{2n(n+r+s-m)}{(n+r-1)(n+s-2)}} \\ &\leq 2\sqrt{\frac{n}{m}} \cdot \sqrt{\frac{n+r+s-m}{n+s-2}} \\ &\leq 2\sqrt{\frac{n}{m}} \end{aligned}$$

since $n+r-1 \geq n+(m-n)/2 \geq m/2$ and $\frac{n+s-(m-r)}{n+s-2} \leq 1$ by (5.13). \square

LEMMA 5.2. *Let m, r, s be positive integers such that $r \geq s \geq 1$ and $m > r + s$. Set $\mu = (m)$ and $\lambda = (r, s, 1^{m-r-s})$. Then there exists a universal constant $K > 0$ such that*

$$m^2 \frac{\theta_\mu^\lambda(2)^2}{C_\lambda(2)} \leq K \cdot \frac{1}{m-r+s} \cdot \sqrt{\frac{r}{s}}.$$

Further, if $r \geq 2s$ then

$$m^2 \frac{\theta_\mu^\lambda(2)^2}{C_\lambda(2)} \geq \frac{1}{K} \cdot \frac{m-r-s}{(m-r+s)^2} \cdot \sqrt{\frac{r}{s}}.$$

PROOF. From (5.10), we see that

$$\begin{aligned} (5.14) \quad & m^2 \frac{\theta_\mu^\lambda(2)^2}{C_\lambda(2)} \\ &= \frac{m^2(m-r-s+1)}{(m+r-s+1)(m+r-s)(m-r+s)(m-r+s-1)} \\ &\quad \times \frac{2^{2r-2s} [(r-1)!]^2 (2s-2)! (2r-2s+1)}{[(s-1)!]^2 (2r-1)!} \end{aligned}$$

$$(5.15) \quad \leq \frac{(m-r-s+1)}{(m-r+s-1)^2} \cdot \frac{2^{2r-2s} [(r-1)!]^2 (2s-2)! (2r-2s+1)}{[(s-1)!]^2 (2r-1)!}$$

since $m+r-s+1 \geq m$ and $m+r-s \geq m$. Now, write

$$\begin{aligned} (5.16) \quad & \frac{2^{2r-2s} [(r-1)!]^2 (2s-2)! (2r-2s+1)}{[(s-1)!]^2 (2r-1)!} \\ &= \frac{2^{2r-2s} (2s-2)! [r!]^2 \cdot 2(2r-2s+1)}{[(s-1)!]^2 (2r)!} \cdot \frac{2(2r-2s+1)}{r} \\ &\leq 4 \cdot \frac{2^{2r-2s} \binom{2s-2}{s-1}}{\binom{2r}{r}} \end{aligned}$$

due to the fact that $\frac{2(2r-2s+1)}{r} \leq 4$. We regard $\binom{0}{0} = 1$. The Stirling formula says that

$$(5.17) \quad 1 < \frac{k!}{\sqrt{2\pi k} k^k e^{-k}} < 2$$

for all $k \geq 1$. It is easy to check from (5.17) that there exists a universal constant $K > 0$ such that

$$(5.18) \quad \frac{1}{K} \cdot \frac{2^{2k}}{\sqrt{k}} \leq \binom{2k}{k} \leq K \cdot \frac{2^{2k}}{\sqrt{k}}$$

for all $k \geq 1$. We claim that

$$(5.19) \quad \frac{2^{2r-2s} \binom{2s-2}{s-1}}{\binom{2r}{r}} \leq C \sqrt{\frac{r}{s}}$$

for all $r \geq s \geq 1$. In fact, if $s = 1$,

$$\frac{2^{2r-2s} \binom{2s-2}{s-1}}{\binom{2r}{r}} = \frac{2^{2r-2}}{\binom{2r}{r}} \leq C \sqrt{r} = C \sqrt{\frac{r}{s}}$$

by (5.18). If $r \geq s \geq 2$, by (5.18) again,

$$\frac{2^{2r-2s} \binom{2s-2}{s-1}}{\binom{2r}{r}} \leq C \sqrt{\frac{r}{s-1}} \leq 2C \sqrt{\frac{r}{s}}.$$

So (5.19) holds. Hence, this and (5.15) imply that

$$\begin{aligned} m^2 \frac{\theta_\mu^\lambda(2)^2}{C_\lambda(2)} &\leq C \cdot \frac{m-r-s+1}{(m-r+s-1)^2} \cdot \sqrt{\frac{r}{s}} \\ &\leq C \cdot \frac{1}{m-r+s} \cdot \sqrt{\frac{r}{s}} \end{aligned}$$

since $m-r-s+1 \leq m-r+s-1$ and $m-r+s-1 \geq \frac{1}{2}(m-r+s)$.

Now we prove the lower bound. By the fact $r \leq m$ it is seen that $m+r-s+1 \leq 2m$. Therefore, by (5.14) and (5.16),

$$\begin{aligned} m^2 \frac{\theta_\mu^\lambda(2)^2}{C_\lambda(2)} &\geq \frac{1}{4} \cdot \frac{m-r-s}{(m-r+s)^2} \cdot \frac{2^{2r-2s} [(r-1)!]^2 (2s-2)! (2r-2s+1)}{[(s-1)!]^2 (2r-1)!} \\ &= \frac{1}{4} \cdot \frac{m-r-s}{(m-r+s)^2} \cdot \frac{2^{2r-2s} \binom{2s-2}{s-1}}{\binom{2r}{r}} \cdot \frac{2(2r-2s+1)}{r}. \end{aligned}$$

The condition $r \geq 2s$ implies that $\frac{2(2r-2s+1)}{r} \geq 2$. By (5.18) again,

$$\frac{2^{2r-2s} \binom{2s-2}{s-1}}{\binom{2r}{r}} \geq C \sqrt{\frac{r}{s}}.$$

We complete the proof. \square

PROOF OF PROPOSITION 1. Look at (a) of Theorem 1, $B = 1$ since $\alpha = 2$. It follows that $\mathbb{E}[|p_\mu(Z_n)|^2] \leq 2m$ for $1 \leq m \leq n$. So, in the rest of the paper, we only need to study the case for $m > n \geq 2$.

Review (5.1),

$$(5.20) \quad \mathbb{E}[|p_m(Z_n^2)|^2] = 4m^2 \sum_{\lambda \vdash m: l(\lambda) \leq n} \frac{\theta_{(m)}^\lambda(2)^2}{C_\lambda(2)} \mathcal{N}_\lambda^2(n).$$

To study this quantity, we will differentiate the three cases for λ in the sum as appeared earlier.

Case 1: $\lambda = (m)$. By (5.8) and (5.17),

$$(5.21) \quad 4m^2 \frac{\theta_{(m)}^\lambda(2)^2}{C_\lambda(2)} = \frac{2^{2m} (m!)^2}{(2m)!} < \frac{2^{2m} (2\sqrt{2\pi m} m^m e^{-m})^2}{\sqrt{4\pi m} (2m)^{2m} e^{-2m}} < C\sqrt{m}.$$

Hence, by (i) of Lemma 5.1,

$$(5.22) \quad 4m^2 \frac{\theta_{(m)}^\lambda(2)^2}{C_\lambda(2)} \mathcal{N}_\lambda^2(n) \leq C\sqrt{n}$$

for any $m \geq n \geq 1$ and $\lambda = (m)$.

Case 2: $\lambda = (m - r, r)$ with $1 \leq r \leq m/2$. First, by (5.9),

$$(5.23) \quad 4m^2 \frac{\theta_{(m)}^\lambda(2)^2}{C_\lambda(2)} = \frac{\binom{2r}{m-r}}{\binom{2r}{m-r}} \cdot \frac{2^{2m-4r} m^2 (2m - 4r + 1)}{(m - r)^2 (2r - 1)^2 (2m - 2r + 1)}.$$

By using the fact $1 \leq r \leq m/2$, we have that $(m - r)^2 r^2 (2m - 2r + 1) \geq m^3 r^2 / 4$ and $2^{2m-4r} m^2 (2m - 4r + 1) \leq 2 \cdot 2^{2m-4r} m^3$. It follows that the last ratio in (5.23) is dominated by $8 \cdot 2^{2m-4r} / r^2$. Thus, by (5.18),

$$(5.24) \quad \begin{aligned} 4m^2 \frac{\theta_{(m)}^\lambda(2)^2}{C_\lambda(2)} &\leq C \frac{\sqrt{m-r}}{2^{2m-2r}} \cdot \frac{2^{2r}}{\sqrt{r}} \cdot \frac{2^{2m-4r}}{r^2} \\ &= C \frac{\sqrt{m-r}}{r^{5/2}} \leq \frac{C}{r^{5/2}} \sqrt{m} \end{aligned}$$

for all $1 \leq r \leq m/2$. It follows from (ii) of Lemma 5.1 that

$$(5.25) \quad \begin{aligned} &4m^2 \sum_{\lambda=(m-r,r), 1 \leq r \leq m/2} \frac{\theta_{(m)}^\lambda(2)^2}{C_\lambda(2)} \mathcal{N}_\lambda^2(n) \\ &\leq C \cdot \sum_{1 \leq r \leq m/2} \frac{1}{r^{5/2}} \sqrt{m} \cdot \sqrt{\frac{n}{m}} \\ &\leq C \cdot \left(\sum_{r=1}^{\infty} \frac{1}{r^{5/2}} \right) \sqrt{n}. \end{aligned}$$

Case 3: $\lambda = (r, s, 1^{m-r-s})$ with $r \geq s \geq 1$ and $3 \leq l(\lambda) = m - r - s + 2 \leq n$. From (iii) of Lemma 5.1 and the first assertion of Lemma 5.2, we get that

$$(5.26) \quad 4m^2 \sum_{\lambda=(r,s,1^{m-r-s})} \frac{\theta_{(m)}^\lambda(2)^2}{C_\lambda(2)} \mathcal{N}_\lambda^2(n) \leq C\sqrt{n} \sum_{r,s} \frac{1}{m-r+s} \cdot \frac{1}{\sqrt{s}},$$

where both sums are taken over all possible $r \geq s \geq 1$ with $3 \leq l(\lambda) = m - r - s + 2 \leq n$. These restrictions imply that $s + 1 \leq m - r \leq s + n$ and hence $2s + 1 \leq m - r + s \leq 2s + n$. It follows that the last sum in (5.26) is bounded by

$$(5.27) \quad \sum_{s=1}^m \sum_{j=2s+1}^{2s+n} \frac{1}{j} \cdot \frac{1}{\sqrt{s}} = \sum_{s=1}^m \frac{1}{\sqrt{s}} \sum_{j=2s+1}^{2s+n} \frac{1}{j}$$

for all $n \geq 2$. Now,

$$\sum_{j=2s+1}^{2s+n} \frac{1}{j} \leq \sum_{j=2s+1}^{2s+n} \int_{j-1}^j \frac{1}{x} dx = \int_{2s}^{2s+n} \frac{1}{x} dx = \log\left(1 + \frac{n}{2s}\right)$$

for all $s \geq 1$. This implies that (5.27) is controlled by

$$\sum_{s=1}^m \frac{1}{\sqrt{s}} \log\left(1 + \frac{n}{s}\right) \leq \sum_{s=1}^m \int_{s-1}^s \frac{1}{\sqrt{y}} \log\left(1 + \frac{n}{y}\right) dy = \int_0^m \frac{1}{\sqrt{y}} \log\left(1 + \frac{n}{y}\right) dy.$$

Set $u = y/n$. Then the last integral is equal to

$$\int_0^{m/n} \frac{1}{\sqrt{nu}} \log\left(1 + \frac{1}{u}\right) \cdot n du \leq \sqrt{n} \int_0^\infty \frac{1}{\sqrt{u}} \log\left(1 + \frac{1}{u}\right) du.$$

Trivially, $\frac{1}{\sqrt{u}} \log(1 + \frac{1}{u}) \sim \frac{1}{u^{3/2}}$ as $u \rightarrow +\infty$ and $\frac{1}{\sqrt{u}} \log(1 + \frac{1}{u}) \sim -\frac{\log u}{\sqrt{u}}$ as $u \rightarrow 0^+$. It follows that $0 < \int_0^\infty \frac{1}{\sqrt{u}} \log(1 + \frac{1}{u}) du < \infty$. Therefore, by (5.26),

$$4m^2 \sum_{\lambda=(r,s,1^{m-r-s})} \frac{\theta_{(m)}^\lambda(2)^2}{C_\lambda(2)} \mathcal{N}_\lambda^2(n) \leq Cn$$

for all $m \geq n \geq 2$, where the sum is taken over all possible $r \geq s \geq 1$ and $3 \leq l(\lambda) = m - r - s + 2 \leq n$. Combining this, (5.20), (5.22) and (5.25), we arrive at

$$\mathbb{E}[|p_m(Z_n)|^2] \leq Kn$$

for all $m \geq n \geq 2$, where K is a universal constant. \square

5.2. *Proof of Proposition 2.* The following result allows us to express the variance for the circular symplectic ensembles ($\beta = 4$) in terms of some familiar quantities treated earlier in the case of the circular orthogonal ensembles and a new quantity $\mathcal{N}_\lambda^2(-2n)$.

LEMMA 5.3 (Duality lemma). *Recall (5.1). For any $m \geq 1$ and $n \geq 2$, the following holds:*

$$(5.28) \quad \mathbb{E}[|p_m(Z_n^{1/2})|^2] = m^2 \sum_{\lambda \vdash m: \lambda_1 \leq n} \frac{\theta_{(m)}^\lambda(2)^2}{C_\lambda(2)} \mathcal{N}_\lambda^2(-2n),$$

where $\lambda = (\lambda_1, \lambda_2, \dots)$ and

$$(5.29) \quad \mathcal{N}_\lambda^2(-2n) = \prod_{(i,j) \in \lambda} \left(1 + \frac{1}{2n - 2j + i}\right).$$

PROOF. The quantity $\theta_\mu^\lambda(\alpha)$ has the following duality (see (10.30) from [20]): for partitions λ, μ of m ,

$$\theta_\mu^\lambda(\alpha) = (-\alpha)^{m-l(\mu)} \theta_\mu^{\lambda'}(1/\alpha),$$

where λ' is the partition of m corresponding to the Young diagram of the transpose of λ . From (4.2), it is easy to see the duality

$$C_\lambda(\alpha) = \prod_{(i,j) \in \lambda} (\alpha(\lambda_i - j) + \lambda'_j - i + 1)(\alpha(\lambda_i - j) + \lambda'_j - i + \alpha) = \alpha^{2m} C_{\lambda'}(1/\alpha).$$

We furthermore have

$$\begin{aligned} \mathcal{N}_\lambda^\alpha(n) &= \prod_{(i,j) \in \lambda} \frac{n + (j - 1)\alpha - (i - 1)}{n + j\alpha - i} \\ &= \prod_{(i,j) \in \lambda'} \frac{n + (i - 1)\alpha - (j - 1)}{n + i\alpha - j} \\ &= \prod_{(i,j) \in \lambda'} \frac{-n/\alpha - (i - 1) + (j - 1)/\alpha}{-n/\alpha - i + j/\alpha} \\ &= \mathcal{N}_{\lambda'}^{1/\alpha}(-n/\alpha), \end{aligned}$$

where

$$(5.30) \quad \mathcal{N}_\mu^\gamma(x) := \prod_{(i,j) \in \mu} \frac{x - (i - 1) + \gamma(j - 1)}{x - i + \gamma j}$$

for any partition μ , $\gamma > 0$ and $x \in \mathbb{R}$ satisfying that the denominators in the product are not equal to zero. It follows from dualities given above and (5.1) that

$$\begin{aligned} \mathbb{E}[|p_m(Z_n^\alpha)|^2] &= \alpha^2 m^2 \sum_{\lambda \vdash m: l(\lambda) \leq n} \frac{\alpha^{2m-2} \theta_{(m)}^{\lambda'} (1/\alpha)^2}{\alpha^{2m} C_{\lambda'}(1/\alpha)} \mathcal{N}_{\lambda'}^{1/\alpha}(-n/\alpha) \\ &= m^2 \sum_{\lambda \vdash m: \lambda_1 \leq n} \frac{\theta_{(m)}^\lambda (1/\alpha)^2}{C_\lambda(1/\alpha)} \mathcal{N}_\lambda^{1/\alpha}(-n/\alpha), \end{aligned}$$

where $\lambda = (\lambda_1, \lambda_2, \dots)$. Plugging $\alpha = 1/2$ into this identity,

$$\mathbb{E}[|p_m(Z_n^{1/2})|^2] = m^2 \sum_{\lambda \vdash m: \lambda_1 \leq n} \frac{\theta_{(m)}^\lambda (2)^2}{C_\lambda(2)} \mathcal{N}_\lambda^2(-2n).$$

Finally, from (5.30),

$$\mathcal{N}_\lambda^2(-2n) = \prod_{(i,j) \in \lambda} \frac{-2n + 2j - i - 1}{-2n + 2j - i} = \prod_{(i,j) \in \lambda} \left(1 + \frac{1}{2n - 2j + i} \right).$$

The proof is completed. \square

LEMMA 5.4. *Let $m \geq n \geq 1$ and $\lambda = (\lambda_1, \lambda_2, \dots) \vdash m$ with $\lambda_1 \leq n$. Let $\mathcal{N}_\lambda^2(-2n)$ be as in (5.29). Then there exists a universal constant $K > 0$ such that:*

- (i) $\mathcal{N}_\lambda^2(-2n) \leq K\sqrt{n}$ if $m = n$ and $\lambda = (n)$;
- (ii) $\mathcal{N}_\lambda^2(-2n) \leq K \frac{n}{\sqrt{(n-r+1)(n-s+1)}}$ if $\lambda = (r, s)$ with $1 \leq s \leq r \leq n$ and $r + s = m$.

PROOF. Let $C := \max_{\lambda, n \leq 2} \mathcal{N}_\lambda^2(-2n)$, where λ goes over all partitions as in (i) and (ii) with $\lambda_1 \leq 2$. Since $2n - 2j + i \geq i \geq 1$ for all $(i, j) \in \lambda$ with $\lambda_1 \leq n$, we know $\mathcal{N}_\lambda^2(-2n) > 1$, and hence $C > 1$. Also, since $m = r + s \leq 2n \leq 4$, these partitions are only of finitely many. Thus, $1 < C < \infty$. Then (i) and (ii) hold by taking $K = C$. From now on, we assume, without loss of generality, that $n \geq 3$.

(i) In this case,

$$\mathcal{N}_\lambda^2(-2n) = \prod_{j=1}^n \left(1 + \frac{1}{2n - 2j + 1} \right) = \prod_{k=1}^n \left(1 + \frac{1}{2k - 1} \right) \leq \exp \left(\sum_{k=1}^n \frac{1}{2k - 1} \right).$$

Now,

$$\sum_{k=1}^n \frac{1}{2k - 1} \leq 1 + \sum_{k=2}^n \int_{k-1}^k \frac{1}{2x - 1} dx = 1 + \int_1^n \frac{1}{2x - 1} dx = 1 + \frac{1}{2} \log(2n - 1).$$

The desired result then follows.

(ii) By the same argument as in the proof of (i),

$$\begin{aligned}
 \log N_\lambda^2(-2n) &\leq \sum_{j=1}^r \frac{1}{2n-2j+1} + \sum_{j=1}^s \frac{1}{2n-2j+2} \\
 &\leq \sum_{k=n-r+1}^n \frac{1}{2k-1} + \sum_{k=n-s+1}^n \frac{1}{2k-1}.
 \end{aligned}
 \tag{5.31}$$

Similar to (i),

$$\begin{aligned}
 \sum_{k=n-r+1}^n \frac{1}{2k-1} &\leq 1 + \int_{n-r+1}^n \frac{1}{2x-1} dx = 1 + \frac{1}{2} \log \frac{2n-1}{2(n-r)+1} \\
 &\leq 1 + \frac{1}{2} \log \frac{2n}{n-r+1}.
 \end{aligned}$$

A similar inequality also holds true for the last sum in (5.31). Thus,

$$\log N_\lambda^2(-2n) \leq C + \frac{1}{2} \log \frac{n^2}{(n-r+1)(n-s+1)}.$$

This implies (ii). \square

LEMMA 5.5. *Let $N_\lambda^2(-2n)$ be as in (5.29). Let $\lambda = (r, s, 1^{m-r-s})$ with $1 \leq s \leq r \leq n$, $m > r + s$ and $m \geq n$. Then there exists a universal constant $K > 0$ such that*

$$\frac{1}{K} \cdot \frac{m}{\sqrt{(n-r+1)(n-s+1)}} \leq N_\lambda^2(-2n) \leq K \cdot \frac{m}{\sqrt{(n-r+1)(n-s+1)}}.$$

PROOF. We prove the upper bound and lower bound in two steps.

Step 1: *Upper bound.* First,

$$\begin{aligned}
 \log N_\lambda^2(-2n) &= \sum_{j=1}^r \log \left(1 + \frac{1}{2n-2j+1} \right) + \sum_{j=1}^s \log \left(1 + \frac{1}{2n-2j+2} \right) \\
 &\quad + \sum_{i=3}^{m-r-s+2} \log \left(1 + \frac{1}{2n+i-2} \right) \\
 &\leq \sum_{j=1}^r \frac{1}{2n-2j+1} \\
 &\quad + \sum_{j=1}^s \frac{1}{2n-2j+2} + \sum_{i=3}^{m-r-s+2} \frac{1}{2n+i-2}
 \end{aligned}
 \tag{5.32}$$

(5.33)

by the inequality $\log(1 + x) \leq x$ for all $x > -1$. Easily, if $r > 1$ then

$$\begin{aligned} \sum_{j=1}^r \frac{1}{2n - 2j + 1} &\leq 1 + \sum_{j=1}^{r-1} \int_j^{j+1} \frac{1}{2n - 2x + 1} dx \\ &= 1 + \int_1^r \frac{1}{2n - 2x + 1} dx \\ &= 1 + \frac{1}{2} \log \frac{2n - 1}{2n - 2r + 1} \end{aligned}$$

and this assertion is evidently true for $r = 1$. So the above inequality holds for all $r \geq 1$. Thus,

$$\sum_{j=1}^s \frac{1}{2n - 2j + 2} \leq \sum_{j=1}^s \frac{1}{2n - 2j + 1} \leq 1 + \frac{1}{2} \log \frac{2n - 1}{2n - 2s + 1}.$$

Likewise,

$$\begin{aligned} \sum_{i=3}^{m-r-s+2} \frac{1}{2n + i - 2} &\leq \sum_{i=3}^{m-r-s+2} \int_{i-1}^i \frac{1}{2n + x - 2} dx \\ &= \int_2^{m-r-s+2} \frac{1}{2n + x - 2} dx = \log \frac{m + 2n - r - s}{2n}. \end{aligned}$$

Combining the three inequalities with (5.33), we get

$$\begin{aligned} \log \mathcal{N}_\lambda^2(-2n) &\leq 2 + \frac{1}{2} \log \frac{(2n - 1)^2(m + 2n - r - s)^2}{(2n)^2(2n - 2r + 1)(2n - 2s + 1)} \\ (5.34) \qquad &\leq 2 + \log 3 + \frac{1}{2} \log \frac{m^2}{(2n - 2r + 1)(2n - 2s + 1)} \\ &\leq (2 + \log 3) + \log \frac{m}{\sqrt{(n - r + 1)(n - s + 1)}}, \end{aligned}$$

where the fact $\frac{2n-1}{2n} \leq 1$ and the fact $m + 2n - r - s \leq m + 2n \leq 3m$ are used in the second inequality; the facts $2n - 2r + 1 \geq n - r + 1$ and $2n - 2s + 1 \geq n - s + 1$ are used in the last inequality.

Step 2: Lower bound. Review (5.32). Use the inequality that $\log(1 + x) \geq x - x^2$ for all $x \geq 0$ to have

$$\begin{aligned} \log \mathcal{N}_\lambda^2(-2n) &\geq \sum_{j=1}^r \frac{1}{2n - 2j + 1} + \sum_{j=1}^s \frac{1}{2n - 2j + 2} + \sum_{i=3}^{m-r-s+2} \frac{1}{2n + i - 2} \\ &\quad - \sum_{j=1}^r \frac{1}{(2n - 2j + 1)^2} - \sum_{j=1}^s \frac{1}{(2n - 2j + 2)^2} - \sum_{i=3}^{m-r-s+2} \frac{1}{(2n + i - 2)^2}. \end{aligned}$$

Observe that each term in the last three sums is strictly monotone in its corresponding index. From the fact $\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}$, we know that the sum of the last three sums is bounded by $\frac{\pi^2}{2}$. By the same arguments as before,

$$\begin{aligned} \sum_{j=1}^r \frac{1}{2n - 2j + 1} &\geq \sum_{j=1}^r \int_{j-1}^j \frac{1}{2n - 2x + 1} dx \\ &= \int_0^r \frac{1}{2n - 2x + 1} dx \\ &= \frac{1}{2} \log \frac{2n + 1}{2n + 1 - 2r}. \end{aligned}$$

And

$$\sum_{j=1}^s \frac{1}{2n - 2j + 2} \geq \sum_{j=1}^s \frac{1}{2(n + 1) - 2j + 1} \geq \frac{1}{2} \log \frac{2n + 3}{2n + 3 - 2s}.$$

Now,

$$\begin{aligned} \sum_{i=3}^{m-r-s+2} \frac{1}{2n + i - 2} &\geq \sum_{i=3}^{m-r-s+2} \int_i^{i+1} \frac{1}{2n + x - 2} dx \\ &= \int_3^{m-r-s+3} \frac{1}{2n + x - 2} dx = \log \frac{m + 2n - r - s + 1}{2n + 1}. \end{aligned}$$

In summary,

$$\begin{aligned} \log \mathcal{N}_\lambda^2(-2n) &\geq -\frac{\pi^2}{2} + \frac{1}{2} \log \frac{(2n + 3)(m + 2n - r - s + 1)^2}{(2n + 1)(2n - 2r + 1)(2n - 2s + 3)} \\ &\geq -\frac{\pi^2}{2} + \frac{1}{2} \log \frac{m^2}{(2n - 2r + 1)(2n - 2s + 3)} \\ &\geq \left(-\frac{\pi^2}{2} - \frac{1}{2} \log 6\right) + \log \frac{m}{\sqrt{(n - r + 1)(n - s + 1)}}, \end{aligned}$$

where we use the fact that $r + s \leq 2n$ in the second inequality, and the facts that $2n - 2r + 1 \leq 2(n - r + 1)$ and $2n - 2s + 3 \leq 3(n - s + 1)$ in the last inequality. □

LEMMA 5.6. *Let $\mathcal{N}_\lambda^2(-2n)$ be as in (5.29). Then there exists a universal constant $K > 0$ such that*

- (i) $N_\lambda^2(-2n) \leq K \sqrt{\frac{n}{n - m + 1}}$ if $\lambda = (m)$ and $1 \leq m \leq n$;
- (ii) $N_\lambda^2(-2n) \leq K \frac{n}{\sqrt{(n - r + 1)(n - s + 1)}}$

if $\lambda = (r, s)$ with $1 \leq s \leq r$ and $r + s = m \leq n$, or $\lambda = (r, s, 1^{m-r-s})$ with $1 \leq s \leq r$ and $n \geq m > r + s$.

PROOF. (i) Look at (i) in the proof of Lemma 5.4, replace “ $\prod_{j=1}^n$ ” with “ $\prod_{j=1}^m$ ” to have

$$\log N_\lambda^2(-2n) \leq \sum_{j=1}^m \frac{1}{2n - 2j + 1} = \sum_{k=n-m+1}^n \frac{1}{2k - 1} \leq 1 + \int_{n-m+1}^n \frac{1}{2x - 1} dx$$

since $\frac{1}{2k-1} \leq \int_{k-1}^k \frac{1}{2x-1} dx$ for all $k \geq 2$. Thus,

$$\log N_\lambda^2(-2n) \leq 1 + \frac{1}{2} \log \frac{2n - 1}{2n - 2m + 1} \leq \left(1 + \frac{1}{2} \log 2\right) + \frac{1}{2} \log \frac{n}{n - m + 1}$$

since $2n - 1 < 2n$ and $2n - 2m + 1 \geq n - m + 1$. This gives (i).

(ii) We consider the two aforementioned cases separately.

Case (a): $\lambda = (r, s)$ with $1 \leq s \leq r$ and $r + s = m \leq n$. Review the proof of (ii) of Lemma 5.4. The first paragraph is still true. The only occurrence of “ m ”, which is in “ $r + s = m$ ”, does not show up in the proof. So we obtain the same inequality.

Case (b): $\lambda = (r, s, 1^{m-r-s})$ with $1 \leq s \leq r$ and $n \geq m > r + s$. Review step 1 in the proof of Lemma 5.5, no restriction on the relationship between m and n is used from the beginning to (5.34). So, by (5.34), we have

$$\begin{aligned} \log N_\lambda^2(-2n) &\leq 2 + \frac{1}{2} \log \frac{(2n - 1)^2(m + 2n - r - s)^2}{(2n)^2(2n - 2r + 1)(2n - 2s + 1)} \\ &\leq 2 + \frac{1}{2} \log \frac{9n^2}{(2n - 2r + 1)(2n - 2s + 1)} \\ &\leq (2 + \log 3) + \frac{1}{2} \log \frac{n^2}{(n - r + 1)(n - s + 1)} \end{aligned}$$

since $m + 2n - r - s \leq 3n$. This gives the conclusion. \square

LEMMA 5.7. *There exists a universal constant $K > 0$ such that*

- (i) $\sqrt{\frac{mn}{n - m}} \leq Km$ for all $1 \leq m < n$;
- (ii) $\log\left(1 + \sqrt{\frac{m}{n - m}}\right) \leq K\sqrt{\frac{m}{n}} \log(m + 1)$ for all $1 \leq m < n$;
- (iii) $\sup_{m > n \geq 1} \left\{ \frac{m}{n} \cdot \frac{1}{\sqrt{m - n}} \tan^{-1} \sqrt{\frac{n}{m - n}} \right\} \leq K$.

PROOF. (i) If $1 \leq m \leq \frac{n}{2}$, then $n - m \geq \frac{n}{2}$ and hence $\sqrt{\frac{mn}{n - m}} \leq \sqrt{2m}$. If $\frac{n}{2} \leq m \leq n - 1$, then $\sqrt{\frac{mn}{n - m}} \leq \sqrt{mn} \leq \sqrt{2m}$.

(ii) If $1 \leq m \leq \frac{n}{2}$ then:

$$\log\left(1 + \sqrt{\frac{m}{n-m}}\right) \leq \sqrt{\frac{m}{n-m}} \leq 2\sqrt{\frac{m}{n}} \leq \frac{2}{\log 2} \cdot \sqrt{\frac{m}{n}} \cdot \log(m+1).$$

If $\frac{n}{2} < m < n$, then $\frac{m}{n-m} \geq 1$ and $2\sqrt{\frac{m}{n}} \geq 1$. It follows that

$$\begin{aligned} \log\left(1 + \sqrt{\frac{m}{n-m}}\right) &\leq \log\left(2\sqrt{\frac{m+1}{n-m}}\right) \\ &\leq \log 2 + \frac{1}{2} \log(m+1) \\ &\leq 2\log(m+1) \leq 4\sqrt{\frac{m}{n}} \log(m+1). \end{aligned}$$

(iii) Define

$$A_{m,n} = \frac{m}{n} \cdot \frac{1}{\sqrt{m-n}} \tan^{-1} \sqrt{\frac{n}{m-n}}.$$

Obviously,

$$\sup_{m>n \geq 1} A_{m,n} \leq \sup_{n < m \leq 5n} A_{m,n} + \sup_{m > 5n} A_{m,n} \leq \frac{5\pi}{2} + \sup_{m > 5n} A_{m,n}.$$

Note that $\tan^{-1} x < x$ for all $x > 0$. It follows that

$$\begin{aligned} \sup_{m > 5n} A_{m,n} &\leq \sup_{m > 5n} \left\{ \frac{m}{n} \cdot \frac{1}{\sqrt{m-n}} \sqrt{\frac{n}{m-n}} \right\} \\ &\leq \sup_{m > 5n} \left\{ \frac{1}{\sqrt{n}} \cdot \frac{1}{1-n/m} \right\} \leq \frac{5}{4}. \end{aligned}$$

Then (iii) follows. \square

LEMMA 5.8. *Recall (5.28). Let $m \geq n \geq 2$. Define*

$$E_{m,n} = m^2 \sum_{\lambda} \frac{\theta_{(m)}^{\lambda}(2)^2}{C_{\lambda}(2)} \mathcal{N}_{\lambda}^2(-2n),$$

where the sum is taken over all $\lambda = (r, s, 1^{m-r-s})$ with $1 \leq s \leq r \leq n$ and $m > r + s$. Then there exists a universal constant $K > 0$ such that the following hold

- (i) $E_{m,n} \leq K\delta^{-1}n$ for all $m \geq (1 + \delta)n$ and $\delta \in (0, 1]$.
- (ii) $E_{m,n} \leq Kn \log n$ for all $m \geq n \geq 2$.
- (iii) Let $w = m - n \geq 0$. Then $E_{m,n} \geq K(w + 1)^{-2}n \log n$ for all $n \geq 12$.

PROOF. (i) From the first assertion of Lemmas 5.2 and 5.5, we know

$$E_{m,n} \leq Cm\sqrt{n} \sum_{r,s} \frac{1}{\sqrt{(n-r+1)(n-s+1)s(m-r+s)}},$$

where the sum runs over all possible r and s satisfying $1 \leq s \leq r \leq n$, $m-r-s \geq 1$. Obviously, $s \leq \frac{m}{2}$. Therefore,

$$(5.35) \quad E_{m,n} \leq Cm\sqrt{n} \sum_{s=1}^{m_n} \frac{1}{\sqrt{(n-s+1)s}} \sum_{r=s}^n \frac{1}{\sqrt{n-r+1(m-r+s)}},$$

where $m_n = n \wedge [m/2]$.

Step 1. First, we consider the term corresponding to $s = 1$ dividing by C , which is equal to

$$(5.36) \quad m \sum_{r=1}^n \frac{1}{\sqrt{(n-r+1)(m-r+1)}} := V_{m,n}^1.$$

Easily $V_{n,n}^1 = n \sum_{r=1}^n \frac{1}{(n-r+1)^{3/2}} < n \sum_{j=1}^\infty \frac{1}{j^{3/2}} = n\zeta(3/2)$, where $\zeta(z)$ is the Riemann zeta function. Assume now $m > n \geq 1$. Then

$$(5.37) \quad V_{m,n}^1 = m \sum_{j=1}^n \frac{1}{\sqrt{j}} \cdot \frac{1}{m-n+j}$$

by setting $j = n - r + 1$. Each term in the sum is bounded by $\int_{j-1}^j \frac{1}{\sqrt{x}} \cdot \frac{1}{m-n+x} dx$. Consequently,

$$\begin{aligned} V_{m,n}^1 &\leq m \int_0^n \frac{1}{\sqrt{x}} \cdot \frac{1}{m-n+x} dx \\ &= \frac{2m}{\sqrt{m-n}} \int_0^{\sqrt{n/(m-n)}} \frac{1}{1+y^2} dy \\ &= \frac{2m}{\sqrt{m-n}} \tan^{-1} \sqrt{\frac{n}{m-n}} \end{aligned}$$

by defining $y = \sqrt{\frac{x}{m-n}}$. From (iii) of Lemma 5.7, we obtain that

$$(5.38) \quad V_{m,n}^1 \leq (2n) \cdot \sup_{m>n \geq 1} \left\{ \frac{m}{n} \cdot \frac{1}{\sqrt{m-n}} \tan^{-1} \sqrt{\frac{n}{m-n}} \right\} \leq Cn$$

for any $m > n \geq 1$. Hence, to prove the conclusion, it suffices to show

$$(5.39) \quad W_{m,n} := \sum_{s=2}^{m_n} \frac{1}{\sqrt{(n-s+1)s}} \sum_{r=s}^n \frac{1}{\sqrt{n-r+1(m-r+s)}} \leq C\delta^{-1} \frac{\sqrt{n}}{m}.$$

Step 2. In this step, we prove (5.39) holds for all $m \geq (1 + \delta)n$. Set $j = n - r + 1$. Then, using the same argument as in estimating the term in (5.37), we have

$$\begin{aligned} \sum_{r=s}^n \frac{1}{\sqrt{(n-r+1)(m-r+s)}} &= \sum_{j=1}^{n-s+1} \frac{1}{(m-n+s-1)+j} \cdot \frac{1}{\sqrt{j}} \\ &\leq \sum_{j=1}^{n-s+1} \int_{j-1}^j \frac{1}{a+x} \cdot \frac{1}{\sqrt{x}} dx \\ &= \int_0^{n-s+1} \frac{1}{a+x} \cdot \frac{1}{\sqrt{x}} dx, \end{aligned}$$

where $a = m - n + s - 1 \geq 1$ for $s \geq 2$. Let $y = \sqrt{x/a}$. It follows that

$$\begin{aligned} \int_0^{n-s+1} \frac{1}{a+x} \cdot \frac{1}{\sqrt{x}} dx &= \frac{2}{\sqrt{a}} \int_0^{\sqrt{(n-s+1)/a}} \frac{1}{1+y^2} dy \\ &= \frac{2}{\sqrt{a}} \cdot \tan^{-1} \sqrt{\frac{n-s+1}{a}}. \end{aligned}$$

Thus,

$$(5.40) \quad \sum_{r=s}^n \frac{1}{\sqrt{(n-r+1)(m-r+s)}} \leq \frac{2}{\sqrt{a}} \cdot \tan^{-1} \sqrt{\frac{n-s+1}{a}}$$

for any $m \geq n \geq 1$ and $s \geq 2$ [we do not need the condition “ $m \geq (1 + \delta)n$ ” here]. Therefore, for any $n \geq 2$,

$$\begin{aligned} W_{m,n} &\leq 2 \sum_{s=2}^n \frac{1}{\sqrt{s(n-s+1)(m-n+s-1)}} \cdot \tan^{-1} \sqrt{\frac{n-s+1}{m-n+s-1}} \\ &\leq 2 \sum_{k=1}^{n-1} \frac{1}{\sqrt{k(m-k)(n-k)}} \cdot \tan^{-1} \sqrt{\frac{k}{m-k}} \end{aligned}$$

by letting $k = n - s + 1$. The term corresponding to $k = n - 1$ in the sum is equal to

$$\frac{1}{\sqrt{(n-1)(m-n+1)}} \tan^{-1} \sqrt{\frac{n-1}{m-n+1}}.$$

By the inequality $\tan^{-1} x < x$ for all $x > 0$, it is seen that the above quantity is controlled by $\frac{1}{m-n+1} \leq \frac{2}{\delta} \cdot \frac{\sqrt{n}}{m}$ due to the fact $\frac{m}{m-n} = (1 - \frac{n}{m})^{-1} \leq 1 + \delta^{-1} \leq 2\delta^{-1}$ from the assumption $m \geq (1 + \delta)n$. Consequently, to prove (5.39), it suffices to show

$$(5.41) \quad U_{m,n} := \sum_{k=1}^{n-2} \frac{1}{\sqrt{k(m-k)(n-k)}} \cdot \tan^{-1} \sqrt{\frac{k}{m-k}} \leq C\delta^{-1} \frac{\sqrt{n}}{m}$$

for all $m \geq (1 + \delta)n$ and $n \geq 3$. In fact, since $\tan^{-1} x < x$ for all $x > 0$,

$$\begin{aligned} U_{m,n} &\leq \sum_{k=1}^{n-2} \frac{1}{(m-k)\sqrt{n-k}} \\ &\leq \sum_{k=1}^{n-2} \int_k^{k+1} \frac{1}{(m-x)\sqrt{n-x}} dx \\ &= \int_1^{n-1} \frac{1}{(m-x)\sqrt{n-x}} dx \end{aligned}$$

by the obvious monotonicity. Now,

$$\begin{aligned} U_{m,n} &\leq \frac{1}{m-n+1} \int_1^{n-1} \frac{1}{\sqrt{n-x}} dx \\ &= \frac{2\sqrt{n-1}-2}{m-n+1} \\ &\leq \frac{2\sqrt{n}}{m-n}. \end{aligned}$$

By the inequality $\frac{m}{m-n} \leq 2\delta^{-1}$ again, $U_{m,n} \leq 4\delta^{-1} \frac{\sqrt{n}}{m}$. We get (5.41).

(ii) By taking $\delta = \frac{1}{2}$ in (i), we know $E_{m,n} \leq Cn$ for $m \geq \frac{3}{2}n$. So, to prove (ii), we assume, without loss of generality, that $n \leq m \leq \frac{3}{2}n$. Recall (5.35). We know $m_n = n \wedge [m/2] \leq \frac{3}{4}n$. Then $n - s + 1 \geq \frac{n}{4}$ for $1 \leq s \leq m_n$. It follows that

$$\begin{aligned} E_{m,n} &\leq CV_{m,n}^1 + Cm \sum_{s=2}^{m_n} \frac{1}{\sqrt{s}} \sum_{r=s}^n \frac{1}{\sqrt{n-r+1}(m-r+s)} \\ &\leq Cn + Cn \sum_{s=2}^n \frac{1}{\sqrt{s}} \sum_{r=s}^n \frac{1}{\sqrt{n-r+1}(n-r+s)} \end{aligned}$$

by (5.36) and (5.38) since $n \leq m \leq \frac{3}{2}n$. Thus, to complete the proof, we only need to show

$$(5.42) \quad H_n := \sum_{s=2}^n \frac{1}{\sqrt{s}} \sum_{r=s}^n \frac{1}{\sqrt{n-r+1}(n-r+s)} \leq C \log n$$

for all $n \geq 2$. In fact, apply (5.40) to the case $m = n$ so that $a = s - 1 \geq \frac{s}{2}$ for $s \geq 2$. We know that

$$\sum_{r=s}^n \frac{1}{\sqrt{n-r+1}(n-r+s)} \leq \frac{\pi}{\sqrt{a}} \leq \frac{2\pi}{\sqrt{s}}$$

for $s \geq 2$. It follows that

$$H_n \leq (2\pi) \cdot \sum_{s=2}^n \frac{1}{s} \leq (2\pi) \cdot \left(1 + \frac{\log n}{\log 2}\right) \leq C \log n$$

by (5.11) where $C = (4\pi)(\log 2)^{-1}$. This gives (5.42).

(iii) From Lemmas 5.2 and 5.5,

$$E_{m,n} \geq Cm \sum_{r,s} \frac{m-r-s}{(m-r+s)^2} \cdot \sqrt{\frac{r}{s}} \cdot \frac{1}{\sqrt{(n-r+1)(n-s+1)}},$$

where $2s \leq r \leq n$ and $m > r + s$. Since $n - s + 1 \leq n$ and $\sqrt{\frac{r}{s}} \geq \sqrt{\frac{n}{2s}}$ if $r \geq \frac{n}{2}$. Then

$$\begin{aligned} E_{m,n} &\geq Cn \sum_{(r,s) \in T_1} \frac{m-r-s}{(m-r+s)^2} \cdot \frac{1}{\sqrt{s}} \cdot \frac{1}{\sqrt{n-r+1}} \\ &= Cn \sum_{(s,t) \in T_2} \frac{w+t-s}{(w+t+s)^2} \cdot \frac{1}{\sqrt{s}} \cdot \frac{1}{\sqrt{t+1}}, \end{aligned}$$

where $w = m - n$ as defined in the statement of the lemma,

$$T_1 = \left\{ (r, s) \in \mathbb{N}^2; 2s \leq r \leq n, m > r + s \text{ and } r \geq \frac{n}{2} \right\},$$

$t = n - r$ and

$$T_2 = \left\{ (s, t) \in \mathbb{Z}^2; s \geq 1, t \geq 0, 2s + t \leq n, t \leq \frac{n}{2} \text{ and } w + t - s \geq 1 \right\},$$

where \mathbb{N} is the set of positive integers and \mathbb{Z} is the set of real integers. Easily,

$$T_2 \supset T_3 := \left\{ (s, t) \in \mathbb{N}^2; 1 \leq s \leq \frac{t}{2} \text{ and } 2 \leq t \leq \frac{n}{2} \right\}.$$

Consequently,

$$\begin{aligned} E_{m,n} &\geq Cn \sum_{2 \leq t \leq n/2} \sum_{1 \leq s \leq t/2} \frac{w+t-s}{(w+t+s)^2} \cdot \frac{1}{\sqrt{s}} \cdot \frac{1}{\sqrt{t+1}} \\ &\geq \frac{Cn}{(1+w)^2} \sum_{2 \leq t \leq n/2} \sum_{1 \leq s \leq t/2} \frac{t-s}{(t+s)^2} \cdot \frac{1}{\sqrt{s}} \cdot \frac{1}{\sqrt{t}} \end{aligned}$$

since $w + t + s \leq (w + 1)(t + s)$ and $t + 1 \leq 2t$. Note that $\frac{t-s}{(t+s)^2}$ is strictly decreasing in $s \in [1, \frac{t}{2}]$, it is bounded below by $\frac{2}{9} \cdot \frac{1}{t}$ for all $1 \leq s \leq \frac{t}{2}$. Thus,

$$\begin{aligned} E_{m,n} &\geq \frac{Cn}{(w+1)^2} \sum_{2 \leq t \leq n/2} \sum_{1 \leq s \leq t/2} \frac{1}{t^{3/2}} \cdot \frac{1}{\sqrt{s}} \\ &\geq \frac{Cn}{(w+1)^2} \sum_{2 \leq t \leq n/2} \frac{1}{t} \end{aligned}$$

because $\frac{1}{t^{3/2}} \cdot \frac{1}{\sqrt{s}} \geq \frac{1}{t^2}$ for $1 \leq s \leq t$. Finally, by (5.11),

$$\sum_{2 \leq t \leq n/2} \frac{1}{t} \geq \log\left(\frac{1}{2} \left\lfloor \frac{n}{2} \right\rfloor\right) \geq C \log n$$

for $n \geq 12$, where $C = \inf_{n \geq 12} \{(\log n)^{-1} \log(\frac{1}{2} \lfloor \frac{n}{2} \rfloor)\} \in (0, \infty)$. In summary,

$$E_{m,n} \geq C \cdot \frac{n \log n}{(w + 1)^2}$$

for all $n \geq 12$. \square

LEMMA 5.9. *Recall (5.28). There exists a universal constant $K > 0$ such that*

$$\mathbb{E}[|p_m(Z_n^{1/2})|^2] \leq Km \log(m + 1)$$

for all $1 \leq m < n$.

PROOF. By Lemma 5.3,

$$(5.43) \quad \mathbb{E}[|p_m(Z_n^{1/2})|^2] = m^2 \sum_{\lambda \vdash m: \lambda_1 \leq n} \frac{\theta_{(m)}^\lambda(2)^2}{C_\lambda(2)} \mathcal{N}_\lambda^2(-2n).$$

Since $m < n$, the restriction $\lambda_1 \leq n$ automatically holds. Review (5.4). Many of the terms in the sum are equal to zero except the following three types of partitions: (i) $\lambda = (m)$; (ii) $\lambda = (m - r, r)$ with $1 \leq r \leq \frac{m}{2}$; (iii) $\lambda = (r, s, 1^{m-r-s})$ with $1 \leq s \leq r$ and $m - r - s \geq 1$.

Now let us analyze the three sums separately.

Step 1: *Analysis of the sum corresponding to case (i).* By (5.21), (i) of Lemma 5.6 and (i) of Lemma 5.7,

$$(5.44) \quad m^2 \sum_{\lambda=(m)} \frac{\theta_{(m)}^\lambda(2)^2}{C_\lambda(2)} \mathcal{N}_\lambda^2(-2n) \leq C \sqrt{\frac{mn}{n - m + 1}} \leq C'm,$$

where both C and C' are universal constants.

Step 2: *Analysis of the sum corresponding to case (ii).* Review (5.24). Replace “ (r, s) ” in (ii) of Lemma 5.6 by “ $(m - r, r)$ ” to obtain

$$\begin{aligned} & m^2 \sum_{\lambda=(m-r,r)} \frac{\theta_{(m)}^\lambda(2)^2}{C_\lambda(2)} \mathcal{N}_\lambda^2(-2n) \\ & \leq Cn\sqrt{m} \sum_{\lambda=(m-r,r)} \frac{1}{r^{5/2}} \cdot \frac{1}{\sqrt{(n-r+1)(n-m+r+1)}}, \end{aligned}$$

where the sum runs over all possible $\lambda = (m - r, r)$ with $1 \leq r \leq \frac{m}{2}$ and $m < n$. Use the trivial estimate $n - r + 1 \geq \frac{n}{2}$ and $n - m + r + 1 \geq n - m$ to see that

$$(5.45) \quad m^2 \sum_{\lambda=(m-r,r)} \frac{\theta_{(m)}^\lambda(2)^2}{C_\lambda(2)} \mathcal{N}_\lambda^2(-2n) \leq C \sqrt{\frac{mn}{n - m}} \sum_{r=1}^\infty \frac{1}{r^{5/2}} = C \cdot \zeta\left(\frac{5}{2}\right) \cdot m$$

by (i) of Lemma 5.7, where $\zeta(z)$ is the Riemann zeta function.

Step 3: Analysis of the sum corresponding to case (iii). Consider

$$E_{m,n} := m^2 \sum_{\lambda=(r,s,1^{m-r-s})} \frac{\theta_{(m)}^\lambda(2)^2}{C_\lambda(2)} \mathcal{N}_\lambda^2(-2n),$$

where the sum is taken over all partition $\lambda = (r, s, 1^{m-r-s})$ with $1 \leq s \leq r$ and $n > m > r + s$. From the first assertion of Lemmas 5.2 and 5.6, we know

$$E_{m,n} \leq Cn\sqrt{m} \sum_{r,s} \frac{1}{\sqrt{(n-r+1)(n-s+1)s(m-r+s)}},$$

where the sum runs over all possible r and s satisfying $1 \leq s \leq r, m-r-s \geq 1$ and $n > m$. Clearly, $s \leq \frac{m}{2}$, hence $n-s+1 \geq \frac{n}{2}$. Further, the restriction “ $m-r-s \geq 1$ ” implies that $m \geq 3$. Therefore,

$$(5.46) \quad E_{m,n} \leq C\sqrt{mn} \cdot \sum_{r=1}^{m-2} \frac{1}{\sqrt{n-r}} \sum_{s=1}^r \frac{1}{\sqrt{s(m-r+s)}}.$$

Use the inequality $\frac{1}{\sqrt{s(m-r+s)}} \leq \int_{s-1}^s \frac{1}{\sqrt{x(m-r+x)}} dx$ to get

$$\begin{aligned} \sum_{s=1}^r \frac{1}{\sqrt{s(m-r+s)}} &\leq \int_0^r \frac{1}{\sqrt{x(m-r+x)}} dx \\ &= \frac{2}{\sqrt{m-r}} \int_0^{\sqrt{r/(m-r)}} \frac{1}{1+y^2} dy = \frac{2}{\sqrt{m-r}} \tan^{-1} \sqrt{\frac{r}{m-r}} \end{aligned}$$

by setting $y = \sqrt{\frac{x}{m-r}}$. Since $\tan^{-1} x \leq \min\{x, \frac{\pi}{2}\}$ for all $x > 0$, we have

$$(5.47) \quad \begin{aligned} &\sum_{r=1}^{m-2} \frac{1}{\sqrt{n-r}} \sum_{s=1}^r \frac{1}{\sqrt{s(m-r+s)}} \\ &\leq 2 \sum_{1 \leq r \leq m/2} \frac{\sqrt{r}}{\sqrt{n-r}(m-r)} + \pi \sum_{m/2 \leq r \leq m-2} \frac{1}{\sqrt{(n-r)(m-r)}}. \end{aligned}$$

Observe that $n-r \geq \frac{n}{2}$ and $m-r \geq \frac{m}{2}$ for $1 \leq r \leq \frac{m}{2}$. Then

$$(5.48) \quad \begin{aligned} \sum_{1 \leq r \leq m/2} \frac{\sqrt{r}}{\sqrt{n-r}(m-r)} &\leq \frac{4}{m\sqrt{n}} \sum_{1 \leq r \leq m/2} \sqrt{r} \\ &\leq \frac{4}{m\sqrt{n}} \sum_{1 \leq r \leq m/2} \int_r^{r+1} \sqrt{x} dx \\ &\leq \frac{4}{m\sqrt{n}} \int_1^m \sqrt{x} dx \leq 3\sqrt{\frac{m}{n}} \end{aligned}$$

since $\int_1^m \sqrt{x} \, dx = \frac{2}{3}(m^{3/2} - 1)$. On the other hand,

$$\begin{aligned} \sum_{m/2 \leq r \leq m-2} \frac{1}{\sqrt{(n-r)(m-r)}} &\leq \sum_{m/2 \leq r \leq m-2} \int_r^{r+1} \frac{1}{\sqrt{(n-x)(m-x)}} \, dx \\ &\leq \int_{m/2}^{m-1} \frac{1}{\sqrt{(n-x)(m-x)}} \, dx \\ &= \int_1^{m/2} \frac{1}{\sqrt{y(n-m+y)}} \, dy \end{aligned}$$

by taking $y = m - x$. Now, let $u = \sqrt{y/(n-m)}$, the above integral becomes

$$\begin{aligned} 2 \int_{1/\sqrt{n-m}}^{\sqrt{m/(2(n-m))}} \frac{1}{\sqrt{1+u^2}} \, du &\leq 4 \int_0^{\sqrt{m/(n-m)}} \frac{1}{u+1} \, du \\ &= 4 \log\left(1 + \sqrt{\frac{m}{n-m}}\right) \end{aligned}$$

by using the inequality $1 + u^2 \geq \frac{1}{2}(1 + u)^2$. By (ii) of Lemma 5.7,

$$\sum_{m/2 \leq r \leq m-2} \frac{1}{\sqrt{(n-r)(m-r)}} \leq C \sqrt{\frac{m}{n}} \log(m+1),$$

which together with (5.47) and (5.48) gives

$$\sum_{r=1}^{m-2} \frac{1}{\sqrt{n-r}} \sum_{s=1}^r \frac{1}{\sqrt{s(m-r+s)}} \leq C \cdot \left(\sqrt{\frac{m}{n}} + \sqrt{\frac{m}{n}} \log(m+1) \right).$$

This inequality and (5.46) conclude that

$$(5.49) \quad E_{m,n} \leq C \cdot m \log(m+1).$$

At last, according to (5.43) and its following paragraph, the desired result follows by considering (5.44), (5.45) and (5.49) together. \square

PROOF OF PROPOSITION 2. From Lemma 5.9, we know that we only need to prove the theorem for the case $m \geq n$. By (5.28),

$$(5.50) \quad \mathbb{E}[|p_m(Z_n^{1/2})|^2] = m^2 \sum_{\lambda} \frac{\theta_{(m)}^{\lambda}(2)^2}{C_{\lambda}(2)} \mathcal{N}_{\lambda}^2(-2n),$$

where the sum is taken over all $\lambda \vdash m : \lambda_1 \leq n$. Review (5.4). Many of the terms in the sum are equal to zero except the following three types of partitions: (i) $\lambda = (m)$; (ii) $\lambda = (m-r, r)$ with $1 \leq m-r \leq n$ and $1 \leq r \leq \frac{m}{2}$; (iii) $\lambda = (r, s, 1^{m-r-s})$ with $1 \leq s \leq r \leq n$ and $m-r-s \geq 1$.

Now let us analyze the three cases one by one.

(a): *The estimate of the sum corresponding to case (i).* When $\lambda = (m)$ with $\lambda_1 = m \leq n$, it is seen that $m = n$, then from (5.21),

$$m^2 \frac{\theta_{(m)}^\lambda(2)^2}{C_\lambda(2)} = m^2 \frac{\theta_{(m)}^{(m)}(2)^2}{C_{(m)}(2)} \leq C\sqrt{n}.$$

By (i) of Lemma 5.4, we know

$$(5.51) \quad m^2 \frac{\theta_{(m)}^\lambda(2)^2}{C_\lambda(2)} N_\lambda^2(-2n) \leq Cn.$$

(b): *The estimate of the sum corresponding to case (ii).* If $\lambda = (m - r, r)$ with $1 \leq m - r \leq n$ and $1 \leq r \leq \frac{m}{2}$, then from (5.24) and (ii) of Lemma 5.4 [replace “(r, s)” by “(m - r, r)”],

$$\begin{aligned} m^2 \frac{\theta_{(m)}^\lambda(2)^2}{C_\lambda(2)} N_\lambda^2(-2n) &\leq \frac{C}{r^{5/2}} \sqrt{m} \cdot \frac{n}{\sqrt{(n-r+1)(n-m+r+1)}} \\ &\leq C \cdot \frac{n^{3/2}}{r^{5/2}} \cdot \frac{1}{\sqrt{n-r+1}} \end{aligned}$$

since the restrictions on r imply that $m \leq 2n$ and $1 \leq r \leq n$. Therefore,

$$\begin{aligned} m^2 \sum_\lambda \frac{\theta_{(m)}^\lambda(2)^2}{C_\lambda(2)} N_\lambda^2(-2n) \\ \leq Cn^{3/2} \sum_{1 \leq r \leq n/2} \frac{1}{r^{5/2}} \cdot \frac{1}{\sqrt{n-r+1}} + Cn^{3/2} \sum_{n/2 \leq r \leq n} \frac{1}{r^{5/2}} \cdot \frac{1}{\sqrt{n-r+1}}, \end{aligned}$$

where the sum is taken over all $\lambda = (m - r, r)$ with $1 \leq m - r \leq n$ and $1 \leq r \leq \frac{m}{2}$. The term in the first sum is controlled by $\frac{2}{r^{5/2}\sqrt{n}}$; each term in the second sum is dominated by $\frac{8}{n^{5/2}}$. Consequently,

$$(5.52) \quad \begin{aligned} m^2 \sum_\lambda \frac{\theta_{(m)}^\lambda(2)^2}{C_\lambda(2)} N_\lambda^2(-2n) \\ \leq C \cdot \left(2\zeta\left(\frac{5}{2}\right)n + 8 \right) \leq C \cdot \left(2\zeta\left(\frac{5}{2}\right) + 8 \right)n, \end{aligned}$$

where the sum is taken corresponding to case (ii) and $\zeta(z)$ is the Riemann zeta function.

(c): *The estimate of the sum corresponding to case (iii).* Let $m \geq n \geq 2$. Define

$$E_{m,n} = m^2 \sum_\lambda \frac{\theta_{(m)}^\lambda(2)^2}{C_\lambda(2)} N_\lambda^2(-2n),$$

where the sum is taken over all $\lambda = (r, s, 1^{m-r-s})$ with $1 \leq s \leq r \leq n$ and $m > r + s$. By Lemma 5.8, there exists a universal constant $K > 0$ such that the following hold:

- (A) $E_{m,n} \leq K\delta^{-1}n$ for all $m \geq (1 + \delta)n$ and $\delta \in (0, 1]$.
- (B) $E_{m,n} \leq Kn \log n$ for all $m \geq n \geq 2$.
- (C) Let $w = m - n \geq 0$. Then $E_{m,n} \geq K(w + 1)^{-2}n \log n$ for all $n \geq 12$.

If $12 \leq n \leq m \leq 2n$ then $n \log n \geq \frac{1}{4}m \log m$. It follows that $E_{m,n} \geq K(w + 1)^{-2}m \log m$ for all $12 \leq n \leq m \leq 2n$. These combined with (5.50), (5.51) and (5.52) imply that

- (A)' $\mathbb{E}[|p_m(Z_n^{1/2})|^2] \leq K\delta^{-1}n$ for all $m \geq (1 + \delta)n$ and $\delta \in (0, 1]$.
- (B)' $\mathbb{E}[|p_m(Z_n^{1/2})|^2] \leq Km \log m$ for all $m \geq n \geq 2$.
- (C)' $\mathbb{E}[|p_m(Z_n^{1/2})|^2] \geq E_{m,n} \geq K(w + 1)^{-2}m \log m$ for all $12 \leq n \leq m \leq 2n$.

Finally, (A)' and (C)' are identical to (i) and (iii) in the statement of Proposition 2, respectively. As mentioned at the beginning of the proof, (B)' and Lemma 5.9 implies (ii) of the proposition. \square

5.3. *Proofs of Theorems 2, 3 and 4.* With the preparations in Sections 5.1 and 5.2, we are now ready to prove the central limit theorems.

PROOF OF THEOREM 2. For any complex numbers c_k 's and d_k 's with $\sum_{k=1}^m (|c_k| + |d_k|) \neq 0$, define

$$X_n = \sum_{j=1}^m [c_j p_j(Z_n^\alpha) + d_j \overline{p_j(Z_n^\alpha)}] \quad \text{and} \quad X = \sum_{j=1}^m [c_j \xi_j + d_j \bar{\xi}_j].$$

We claim that, to prove the theorem, it is enough to show

$$(5.53) \quad \lim_{n \rightarrow \infty} \mathbb{E}(X_n^p \bar{X}_n^q) = \mathbb{E}(X^p \bar{X}^q)$$

for any integers $p \geq 0$ and $q \geq 0$ with $p + q \geq 1$. In fact, for a complex random vector $U = (U_1, \dots, U_m) \in \mathbb{C}^m$, we treat it as the real vector $\tilde{U} \in \mathbb{R}^{2m}$ by listing their real and imaginary parts in a column. Since the real and the complex parts of U_j are $\frac{U_j + \bar{U}_j}{2}$ and $\frac{U_j - \bar{U}_j}{2i}$, respectively, for each j , then $a' \tilde{U}$ for $a \in \mathbb{R}^{2m}$ is a linear combination of U_j 's and \bar{U}_j 's with complex coefficients. Thus, by the Cramér–Wold device (see, e.g., page 176 from [7]), to prove the theorem, it suffices to show X_n converges weakly to X as $n \rightarrow \infty$. Trivially, X has the same distribution as that of $a\eta_1 + b\eta_2$ where η_1, η_2 are i.i.d. with distribution $N(0, 1)$ and a, b are complex numbers, hence X is uniquely determined by its moments. By the moment method, we only need to check (5.53).

First, $\mathbb{E}[p_\mu(Z_n^\alpha) \overline{p_\nu(Z_n^\alpha)}] = 0$ unless the weights $|\mu|$ and $|\nu|$ are equal. This fact follows in a way similar to the proof of Proposition 3. The key is Lemma 4.2: Jack polynomials are orthogonal. We have

$$(5.54) \quad \mathbb{E}[p_\mu(Z_n^\alpha)] = 0 \quad \text{in particular} \quad \mathbb{E}[p_k(Z_n^\alpha)] = 0$$

for all $|\mu| \geq 1$ and $k \geq 1$.

Second, expand $X_n^p \bar{X}_n^q$ and $X^p \bar{X}^q$ as sums of M terms, where the number M does not depend on n . In the same way, it is seen that, to prove (5.53), we only need to show

$$(5.55) \quad \lim_{n \rightarrow \infty} \mathbb{E} \left(\prod_{j=1}^m p_j(Z_n^\alpha)^{l_j} \cdot \prod_{j=1}^m \overline{p_j(Z_n^\alpha)^{l'_j}} \right) = \mathbb{E} \left(\prod_{j=1}^m \xi_j^{l_j} \cdot \prod_{j=1}^m \bar{\xi}_j^{l'_j} \right)$$

for nonnegative integers l_j 's and l'_j 's with $\sum_{j=1}^m l_j \geq 1$ or $\sum_{j=1}^m l'_j \geq 1$. Set $\mu = (1^{l_1}, 2^{l_2}, \dots, m^{l_m})$ and $\nu = (1^{l'_1}, 2^{l'_2}, \dots, m^{l'_m})$. Then, according to (2.1),

$$l(\mu) = \sum_{j=1}^m l_j \quad \text{and} \quad z_\mu = \prod_{j=1}^s j^{l_j} l_j!$$

The quantities $l(\nu)$ and z_ν are defined similarly. Hence, by (a) of Corollary 1,

$$\begin{aligned} \text{The left-hand side of (5.55)} &= \lim_{n \rightarrow \infty} \mathbb{E}[p_\mu(Z_n^\alpha) \overline{p_\nu(Z_n^\alpha)}] \\ &= \delta_{\mu\nu} \left(\frac{2}{\beta}\right)^{l(\mu)} z_\mu. \end{aligned}$$

By independence and rotation-invariance, we know that the right-hand side of (5.55) is zero if $l_j \neq l'_j$ for some j , or equivalently, $\mu \neq \nu$. If $\mu = \nu$, then

$$(5.56) \quad \mathbb{E} \left(\prod_{j=1}^m \xi_j^{l_j} \cdot \prod_{j=1}^m \bar{\xi}_j^{l'_j} \right) = \prod_{j=1}^m \mathbb{E}(|\xi_j|^{2l_j}) = \left(\frac{2}{\beta}\right)^{l(\mu)} \prod_{j=1}^m j^{l_j} l_j!$$

since $|\xi_j|^2 \sim \frac{2j}{\beta} W$ where W is the exponential distribution with density $e^{-x} I(x \geq 0)$ and $\mathbb{E}W^l = l!$ for all integer $l \geq 1$. We then obtain (5.55). \square

PROOF OF COROLLARY 3. Let $\alpha = \frac{2}{\beta}$ and $Z_n^\alpha = (e^{i\theta_1}, \dots, e^{i\theta_n})$. Write

$$X_n = \sum_{j=1}^n \sum_{k=0}^m c_k e^{ik\theta_j} = \sum_{k=0}^m c_k \sum_{j=1}^n e^{ik\theta_j} = \mu_n + \sum_{k=1}^m c_k p_k(Z_n^\alpha)$$

with $p_k(z) = \sum_{j=1}^n z_j^k$ for $z = (z_1, \dots, z_n)$. By Theorem 2 and the continuous mapping theorem, $X_n - \mu_n$ converges weakly to $Z := \sum_{j=1}^m c_j \xi_j$ as $n \rightarrow \infty$, where ξ_j 's are independent random variables and $\xi_j \sim \mathbb{C}N(0, \frac{2j}{\beta})$ for each j . It is easy to check that $Z \sim \mathbb{C}N(0, \sigma^2)$. \square

LEMMA 5.10. Let $X \sim \mathbb{C}N(0, 1)$ and c, d be two complex numbers. Then $cX + d\bar{X} = U + iV$ where $(U, V)' \sim N_2(\mathbf{0}, \Sigma)$ with

$$\Sigma = \frac{1}{2} \begin{pmatrix} |c + \bar{d}|^2 & 2\text{Im}(cd) \\ 2\text{Im}(cd) & |c - \bar{d}|^2 \end{pmatrix}.$$

PROOF. Let ξ be a standard normal random variable and $a = a_1 + a_2i$ be a complex number, where $a_1 \in \mathbb{R}$ and $a_2 \in \mathbb{R}$. Then $a\xi$, as a 2-dimensional random vector, has the same distribution as that of $(a_1\xi, a_2\xi) \sim N_2(\mathbf{0}, \Sigma_1)$ where

$$\Sigma_1 = \begin{pmatrix} a_1^2 & a_1a_2 \\ a_1a_2 & a_2^2 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} (a + \bar{a})^2 & (\bar{a}^2 - a^2)i \\ (\bar{a}^2 - a^2)i & -(a - \bar{a})^2 \end{pmatrix}.$$

Let ξ_1, ξ_2 be i.i.d. with distribution $N(0, 1)$, and c, d be complex numbers. Then

$$c \frac{\xi_1 + i\xi_2}{\sqrt{2}} + d \frac{\xi_1 - i\xi_2}{\sqrt{2}} = \frac{c + d}{\sqrt{2}} \xi_1 + i \frac{c - d}{\sqrt{2}} \xi_2,$$

as a sum of independent (2-dimensional) normal random vectors, has distribution $N_2(\mathbf{0}, \Sigma_2)$ where

$$\Sigma_2 = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}.$$

Since the covariance matrix of the sum of two independent random variables is the sum of their individual covariance matrices, we have

$$\begin{aligned} \sigma_{11} &= \frac{1}{8}((c + d + \bar{c} + \bar{d})^2 - (c - d - \bar{c} + \bar{d})^2) \\ &= \frac{1}{8} \cdot 4(c + \bar{d})(\bar{c} + d) = \frac{1}{2}|c + \bar{d}|^2 \end{aligned}$$

by using the identity $x^2 - y^2 = (x + y)(x - y)$. And by the identity again,

$$\begin{aligned} \sigma_{22} &= -\frac{1}{8}((c + d - \bar{c} - \bar{d})^2 - (c + \bar{c} - d - \bar{d})^2) \\ &= -\frac{1}{8} \cdot 4(c - \bar{d})(d - \bar{c}) = \frac{1}{2}|c - \bar{d}|^2. \end{aligned}$$

Now,

$$\sigma_{12} = \frac{i}{8}((\bar{c} + \bar{d})^2 - (c + d)^2 - (\bar{c} - \bar{d})^2 + (c - d)^2) = \frac{i}{2}(\bar{c}\bar{d} - cd).$$

Thus, $c\xi + d\bar{\xi} = U + iV$ where $(U, V)' \sim N_2(\mathbf{0}, \Sigma_3)$ and

$$\Sigma_3 = \frac{1}{2} \begin{pmatrix} |c + \bar{d}|^2 & (\bar{c}\bar{d} - cd)i \\ (\bar{c}\bar{d} - cd)i & |c - \bar{d}|^2 \end{pmatrix}. \quad \square$$

LEMMA 5.11. *Let $\{X_n; 1 \leq n \leq \infty\}$ be complex normal random variables with mean zero for each n . Then, X_n converges to X_∞ weakly if and only if $\lim_{n \rightarrow \infty} \mathbb{E}(X_n^p \bar{X}_n^q) = \mathbb{E}(X_\infty^p \bar{X}_\infty^q)$ for any integers $p \geq 0$ and $q \geq 0$ with $p + q \geq 1$.*

PROOF. Write $X_n = U_n + iV_n$ for all $1 \leq n \leq \infty$, where U_n and V_n are real random variables. Then there exists a 2×2 nonnegative definite matrix Σ_n such that $(U_n, V_n) \sim N_2(\mathbf{0}, \Sigma_n)$ for each n . Since both U_n and V_n can be expressed by linear combinations of X_n and \bar{X}_n and vice versa. The lemma then can be interpreted as follows: (U_n, V_n) converges to (U_∞, V_∞) weakly if and only if $\lim_{n \rightarrow \infty} \mathbb{E}(U_n^p V_n^q) = \mathbb{E}(U_\infty^p V_\infty^q)$ for any integers $p \geq 0$ and $q \geq 0$ with $p + q \geq 1$.

The sufficiency is obtained by using the moment method and the Cramér–Wold device. We now show the necessity. By using characteristic functions, it is easily seen that (U_n, V_n) converges to (U_∞, V_∞) weakly if and only if $\lim_{n \rightarrow \infty} (\Sigma_n)_{ij} = (\Sigma_\infty)_{ij}$ for all $1 \leq i, j \leq 2$. Now, assuming (U_n, V_n) converges to (U_∞, V_∞) , then $U_n^p V_n^q$ converges weakly to $U_\infty^p V_\infty^q$ by the continuous mapping theorem. So we only need to show the uniform integrability. In fact, let $r = p + q + 1$, then by the Hölder inequality, $\mathbb{E}(U_n^{2p} V_n^{2q}) \leq [\mathbb{E}(U_n^{2r})]^{p/r} \cdot [\mathbb{E}(V_n^{2r})]^{q/r}$. We know $\mathbb{E}(U_n^{2r}) = (\Sigma_n)_{11}^{2r} \mathbb{E}(N(0, 1)^{2r}) \rightarrow (\Sigma_\infty)_{11}^{2r} \mathbb{E}(N(0, 1)^{2r}) = \mathbb{E}(U_\infty^{2r})$ as $n \rightarrow \infty$. This shows that $\sup_{n \geq 1} \mathbb{E}(U_n^{2p} V_n^{2q}) < \infty$. In particular, $\{U_n^p V_n^q; n \geq 1\}$ is uniformly integrable. \square

PROOF OF THEOREM 3. Set $m = m_n = [\log n] + 1$ for $n \geq 1$, where $[x]$ is the integer part of $x \geq 0$. Review (b) of Theorem 1 and (5.54). We know

$$\begin{aligned} \mathbb{E}[p_j(Z_n^2) \overline{p_k(Z_n^2)}] &= 0 && \text{for any } j \neq k \geq 1; \\ \mathbb{E}[p_j(Z_n^2) p_k(Z_n^2)] &= \mathbb{E} p_\mu(Z_n^2) = 0 && \text{for any } j \geq 1 \text{ and } k \geq 1, \end{aligned}$$

where $\mu := (j, k)$ is a partition. In particular,

$$\begin{aligned} \mathbb{E}[(a_j p_j(Z_n^2) + b_j \overline{p_j(Z_n^2)}) \cdot \overline{(a_k p_k(Z_n^2) + b_k \overline{p_k(Z_n^2)})}] \\ = \delta_{jk} \cdot (|a_j|^2 + |b_j|^2) \mathbb{E}|p_j(Z_n^2)|^2 \end{aligned}$$

for all $j \geq 1$ and $k \geq 1$. Set

$$Y_n := \sum_{j=1}^m (a_j \xi_j + b_j \bar{\xi}_j),$$

where ξ_j 's are i.i.d. random variables such that $\xi_j \sim \mathbb{C}N(0, 2j)$ for each $j \geq 1$. By the Minkowski inequality,

$$\begin{aligned} \mathbb{E} \left| \sum_{j=1}^\infty (a_j \xi_j + b_j \bar{\xi}_j) \right|^2 &\leq 2 \mathbb{E} \left| \sum_{j=1}^\infty a_j \xi_j \right|^2 + 2 \mathbb{E} \left| \sum_{j=1}^m b_j \bar{\xi}_j \right|^2 \\ &\leq 4 \sum_{j=1}^\infty j (|a_j|^2 + |b_j|^2) < \infty. \end{aligned}$$

Therefore, Y_n converges weakly to $Y := \sum_{j=1}^\infty (a_j \xi_j + b_j \bar{\xi}_j)$. Write $a_j \xi_j + b_j \bar{\xi}_j = U_j + iV_j$ for each j such that $(U_j, V_j) \in \mathbb{R}^2$ and $(U_j, V_j) \sim N_2(\mathbf{0}, \Sigma_j)$. Then, by Lemma 5.10,

$$\Sigma_j = \begin{pmatrix} j|a_j + \bar{b}_j|^2 & 2j \cdot \text{Im}(a_j b_j) \\ 2j \cdot \text{Im}(a_j b_j) & j|a_j - \bar{b}_j|^2 \end{pmatrix}$$

for each j . Thus, $\sum_{j=1}^{\infty} (a_j \xi_j + b_j \bar{\xi}_j)$ has the law of $U + iV$ where $(U, V)' \sim N_2(\mathbf{0}, \Sigma)$ with

$$\Sigma = \begin{pmatrix} \sum_{j=1}^{\infty} j |a_j + \bar{b}_j|^2 & 2 \cdot \text{Im} \left(\sum_{j=1}^{\infty} j a_j b_j \right) \\ 2 \cdot \text{Im} \left(\sum_{j=1}^{\infty} j a_j b_j \right) & \sum_{j=1}^{\infty} j |a_j - \bar{b}_j|^2 \end{pmatrix}$$

since the covariance matrix of the sum of independent random variables is the sum of their individual covariance matrices. By Lemma 5.11,

$$(5.57) \quad \lim_{n \rightarrow \infty} \mathbb{E}[Y_n^p \bar{Y}_n^q] = \mathbb{E}[Y^p \bar{Y}^q].$$

Proposition 1 tells us that $\mathbb{E}[|p_j(Z_n^2)|^2] \leq K j$ for all $j \geq 1$ and $n \geq 2$, where K is a universal constant. We then have

$$(5.58) \quad \begin{aligned} \mathbb{E} \left| \sum_{j>m}^{\infty} (a_j p_j(Z_n^2) + b_j \overline{p_j(Z_n^2)}) \right|^2 &= \sum_{j>m} (|a_j|^2 + |b_j|^2) \mathbb{E} |p_j(Z_n^2)|^2 \\ &\leq K \sum_{j>m}^{\infty} j (|a_j|^2 + |b_j|^2) \rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. This shows that $\sum_{j>m}^{\infty} (a_j p_j(Z_n^2) + b_j \overline{p_j(Z_n^2)})$ converges to zero in probability as $n \rightarrow \infty$. By the Slutsky lemma, to prove the theorem, we only need to show

$$(5.59) \quad X_n := \sum_{j=1}^m (a_j p_j(Z_n^2) + b_j \overline{p_j(Z_n^2)}) \rightarrow Y$$

weakly as $n \rightarrow \infty$. Thus from (5.57), similar to (5.53), to prove (5.59) it suffices to show that

$$(5.60) \quad \lim_{n \rightarrow \infty} (\mathbb{E}[X_n^p \bar{X}_n^q] - \mathbb{E}[Y^p \bar{Y}^q]) = 0.$$

Recall the multinomial formula,

$$(5.61) \quad (x_1 + \dots + x_k)^p = \sum_{l_1 + \dots + l_k = p} \binom{p}{l_1, l_2, \dots, l_k} x_1^{l_1} x_2^{l_2} \dots x_k^{l_k}$$

for any complex number x_i 's, positive integers $k \geq 2$ and $p \geq 1$, where l_i 's are nonnegative integers. Note that X_n is a sum of $2m$ terms. Expand $X_n^p \bar{X}_n^q$ to have

$$\begin{aligned} \mathbb{E}[X_n^p \bar{X}_n^q] &= \sum \binom{p}{l_1, \dots, l_{2m}} \cdot \binom{q}{l'_1, \dots, l'_{2m}} \cdot \prod_{j=1}^m (a_j^{l_j} b_j^{l_{m+j}}) \cdot \prod_{j=1}^m \bar{a}_j^{l'_j} \bar{b}_j^{l'_{m+j}} \\ &\quad \times \mathbb{E} \left[\prod_{j=1}^m (p_j(Z_n^2)^{l_j} \overline{p_j(Z_n^2)^{l_{m+j}}}) \cdot \prod_{j=1}^m (\overline{p_j(Z_n^2)^{l'_j}} p_j(Z_n^2)^{l'_{m+j}}) \right], \end{aligned}$$

where the sum runs over all possible nonnegative integers l_j 's and l'_j 's with $\sum_{j=1}^{2m} l_j = p$ and $\sum_{j=1}^{2m} l'_j = q$. Rearranging the products in the expectation, we get

$$\begin{aligned}
 \mathbb{E}[X_n^p \bar{X}_n^q] &= \sum \binom{p}{l_1, \dots, l_{2m}} \cdot \binom{q}{l'_1, \dots, l'_{2m}} \cdot \prod_{j=1}^m (a_j^{l_j} b_j^{l_{m+j}}) \cdot \prod_{j=1}^m \bar{a}_j^{l'_j} \bar{b}_j^{l'_{m+j}} \\
 (5.62) \quad &\times \mathbb{E} \left[\prod_{j=1}^m p_j(Z_n^2)^{l_j+l'_{m+j}} \cdot \prod_{j=1}^m \overline{p_j(Z_n^2)^{l'_j+l_{m+j}}} \right].
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 \mathbb{E}[Y_n^p \bar{Y}_n^q] &= \sum \binom{p}{l_1, \dots, l_{2m}} \cdot \binom{q}{l'_1, \dots, l'_{2m}} \cdot \prod_{j=1}^m (a_j^{l_j} b_j^{l_{m+j}}) \cdot \prod_{j=1}^m \bar{a}_j^{l'_j} \bar{b}_j^{l'_{m+j}} \\
 (5.63) \quad &\times \mathbb{E} \left[\prod_{j=1}^m \xi_j^{l_j+l'_{m+j}} \cdot \prod_{j=1}^m \bar{\xi}_j^{l'_j+l_{m+j}} \right].
 \end{aligned}$$

We claim that

$$\begin{aligned}
 &\left| \mathbb{E} \left[\prod_{j=1}^m p_j(Z_n^2)^{l_j+l'_{m+j}} \cdot \prod_{j=1}^m \overline{p_j(Z_n^2)^{l'_j+l_{m+j}}} \right] \right. \\
 (5.64) \quad &\quad \left. - \mathbb{E} \left[\prod_{j=1}^m \xi_j^{l_j+l'_{m+j}} \cdot \prod_{j=1}^m \bar{\xi}_j^{l'_j+l_{m+j}} \right] \right| \\
 &\leq C_{p,q} \cdot \frac{m}{n} \cdot \prod_{j=1}^m j^{(l_j+l_{m+j})/2} \cdot \prod_{j=1}^m j^{(l'_j+l'_{m+j})/2}
 \end{aligned}$$

uniformly for all possible l_j 's and l'_j 's in the two sums, where $C_{p,q}$ is constant depending on p and q only. In fact, let μ and ν be two partitions so that

$$\begin{aligned}
 \mu &= (1^{l_1+l'_{m+1}}, 2^{l_2+l'_{m+2}}, \dots, m^{l_m+l'_{2m}}); \\
 \nu &= (1^{l'_1+l_{m+1}}, 2^{l'_2+l_{m+2}}, \dots, m^{l'_m+l_{2m}}).
 \end{aligned}$$

Then $l(\mu) = \sum_{j=1}^m (l_j + l'_{m+j}) \leq p + q$ and similarly $l(\nu) \leq p + q$, and

$$K := |\mu| \vee |\nu| = \sum_{j=1}^m j(l_j + l'_{m+j}) \vee \sum_{j=1}^m j(l'_j + l_{m+j}) \leq m(p + q).$$

According to this notation,

$$(5.65) \quad \mathbb{E} \left[\prod_{j=1}^m p_j(Z_n^2)^{l_j+l'_{m+j}} \cdot \prod_{j=1}^m \overline{p_j(Z_n^2)^{l'_j+l_{m+j}}} \right] = \mathbb{E}[p_\mu(Z_n^2) \overline{p_\nu(Z_n^2)}].$$

By (5.56),

$$\begin{aligned}
 (5.66) \quad & \mathbb{E} \left[\prod_{j=1}^m \xi_j^{l_j+l'_{m+j}} \cdot \prod_{j=1}^m \bar{\xi}_j^{l'_j+l_{m+j}} \right] \\
 & = \delta_{\mu\nu} \left(\frac{2}{\beta} \right)^{l(\mu)} \prod_{j=1}^m j^{l_j+l'_{m+j}} (l_j + l_{m+j'})! = \delta_{\mu\nu} \alpha^{l(\mu)} z_\mu,
 \end{aligned}$$

where $\alpha = \frac{2}{\beta} = 2$ and z_μ is as in (2.1). Since $\sum_{j=1}^m (l_j + l_{m+j'}) \leq p + q$, then

$$\text{Card}\{1 \leq j \leq m; l_j + l_{m+j'} \geq 2\} \leq \frac{p+q}{2}.$$

Using $l_j + l_{m+j'} \leq p + q$ for all $1 \leq j \leq m$, we get

$$0 < \alpha^{l(\mu)} z_\mu \leq (2^{p+q} ((p+q)!)^{(p+q)/2}) \cdot \prod_{j=1}^m j^{l_j+l'_{m+j}}.$$

A similar inequality also holds for $\alpha^{l(\nu)} z_\nu$. From (a) and (b) of Corollary 2, we see that

$$|\mathbb{E}[p_\mu(Z_n^2) \overline{p_\nu(Z_n^2)}] - \delta_{\mu\nu} \alpha^{l(\mu)} z_\mu| \leq C_{p,q} \cdot \frac{m}{n} \cdot \prod_{j=1}^m j^{(l_j+l'_{m+j})/2} \cdot \prod_{j=1}^m j^{(l'_j+l_{m+j})/2},$$

where $C_{p,q}$ is a constant depending on p and q only. This together with (5.65) and (5.66) yields (5.64).

Now, combining (5.62), (5.63) and (5.64), we arrive at

$$\begin{aligned}
 & |\mathbb{E}[X_n^p \bar{X}_n^q] - \mathbb{E}[Y_n^p \bar{Y}_n^q]| \\
 & \leq C_{p,q} \cdot \frac{m}{n} \cdot \sum \binom{p}{l_1, \dots, l_{2m}} \cdot \binom{q}{l'_1, \dots, l'_{2m}} \cdot \prod_{j=1}^m (\sqrt{j}|a_j|)^{l_j} (\sqrt{j}|b_j|)^{l_{m+j}} \\
 & \quad \times \prod_{j=1}^m (\sqrt{j}|a_j|)^{l'_j} (\sqrt{j}|b_j|)^{l'_{m+j}} \\
 & = C_{p,q} \cdot \frac{m}{n} \cdot \left(\sum_{j=1}^m (\sqrt{j}|a_j| + \sqrt{j}|b_j|) \right)^p \cdot \left(\sum_{j=1}^m (\sqrt{j}|a_j| + \sqrt{j}|b_j|) \right)^q \\
 & = C_{p,q} \cdot \frac{m}{n} \cdot \left(\sum_{j=1}^m (\sqrt{j}|a_j| + \sqrt{j}|b_j|) \right)^{p+q},
 \end{aligned}$$

where (5.61) is used in the first identity. From the inequality $(x_1 + \dots + x_{2m})^2 \leq 2m(x_1^2 + \dots + x_{2m}^2)$ for any real number x_i 's we see that

$$|\mathbb{E}[X_n^p \bar{X}_n^q] - \mathbb{E}[Y_n^p \bar{Y}_n^q]| \leq (C'_{p,q} \sigma^{p+q}) \frac{1}{n} m^{1+(p+q)/2} \rightarrow 0$$

as $n \rightarrow \infty$ since $m = [\log n]$ for $n \geq 3$, where $C'_{p,q}$ is a constant depending on p and q only. This confirms (5.60). \square

PROOF OF THEOREM 4. From the assumption that $\sum_{j=1}^\infty (j \log j)(|a_j|^2 + |b_j|^2) \in (0, \infty)$, we know $\sigma^2 \in (0, \infty)$. Take $m = m_n = [\log n]$ for $n \geq 3$. By (ii) of Proposition 2, the assumption $\sum_{j=1}^\infty (j \log j)(|a_j|^2 + |b_j|^2) \in (0, \infty)$ and the same argument as the derivation of (5.58), to prove the theorem, it is enough to show that

$$\sum_{j=1}^m (a_j p_j(Z_n^{1/2}) + \overline{b_j p_j(Z_n^{1/2})}) \rightarrow \sum_{j=1}^\infty (a_j \xi_j + b_j \bar{\xi}_j)$$

weakly as $n \rightarrow \infty$, where $\{\xi_j; j \geq 1\}$ are independent random variables with $\xi_j \sim CN(0, \frac{1}{2}j)$ for each j . Write $a_j \xi_j + b_j \bar{\xi}_j = U_j + iV_j$ for each j with $(U_j, V_j) \in \mathbb{R}^2$. By Lemma 5.10, (U_j, V_j) has the distribution $N_2(\mathbf{0}, \Sigma_j)$ where

$$\Sigma_j = \frac{1}{4} \begin{pmatrix} j|a_j + \bar{b}_j|^2 & 2j \cdot \text{Im}(a_j b_j) \\ 2j \cdot \text{Im}(a_j b_j) & j|a_j - \bar{b}_j|^2 \end{pmatrix}.$$

It follows from the independence that $\sum_{j=1}^\infty (a_j \xi_j + b_j \bar{\xi}_j)$ has the law of $U + iV$ where $(U, V)' \sim N_2(\mathbf{0}, \Sigma)$ with

$$\Sigma = \frac{1}{4} \begin{pmatrix} \sum_{j=1}^\infty j|a_j + \bar{b}_j|^2 & 2 \cdot \text{Im}\left(\sum_{j=1}^\infty j a_j b_j\right) \\ 2 \cdot \text{Im}\left(\sum_{j=1}^\infty j a_j b_j\right) & \sum_{j=1}^\infty j|a_j - \bar{b}_j|^2 \end{pmatrix}.$$

Then the rest proof will be completed by following the same arguments as in the corresponding parts in the proof of Theorem 3. \square

APPENDIX

In this section, we calculate some moments for the circular β -ensembles. The first result below is an independent check of the second moment of the trace of a COE given in (1.4). The derivation does not depend on the Jack function as used in Section 4.1. It only uses the distribution of the entries of the COE.

LEMMA A.1. *Let W_n be an $n \times n$ circular orthogonal ensemble (COE), that is, $W_n = U_n^T U_n$ for some Haar-invariant unitary matrix U_n . Then $\mathbb{E}[|\text{Tr}(W_n)|^2] = 2n/(n + 1)$ for all $n \geq 2$.*

FIRST PROOF OF LEMMA A.1. We prove the lemma in three steps.

Step 1. Write $U_n = (u_{rs})$. First, we claim that

$$(A.1) \quad \mathbb{E}[u_{rs}^2 \bar{u}_{pq}^2] = 0$$

if $r \neq p$ or $s \neq q$. In fact, since U_n is Haar-invariant unitary, the distributions of UU_n and U_nU are the same as that of U_n for any unitary matrix U . In particular, take $U = \text{diag}(e^{i\theta_k})_{1 \leq k \leq n}$ to obtain that

$$(A.2) \quad \begin{aligned} \mathcal{L}((e^{i\theta_r} u_{rs})_{1 \leq r, s \leq n}) &= \mathcal{L}((e^{i\theta_s} u_{rs})_{1 \leq r, s \leq n}) \\ &= \mathcal{L}((u_{rs})_{1 \leq r, s \leq n}) \end{aligned}$$

for any $\theta_1, \dots, \theta_n \in \mathbb{R}$, where $\mathcal{L}(X)$ is the joint distribution of the entries of random matrix X . If $r \neq p$, taking $\theta_r - \theta_p = \pi/2$, then by (A.2), we have that

$$\mathbb{E}[u_{rs}^2 \bar{u}_{pq}^2] = e^{2i(\theta_r - \theta_p)} \mathbb{E}[u_{rs}^2 \bar{u}_{pq}^2] = -\mathbb{E}[u_{rs}^2 \bar{u}_{pq}^2],$$

which means (A.1). The case for $s = q$ can be proved similarly.

Step 2. Recall notation $(2m - 1)!! = (2m - 1)(2m - 3) \cdots 3 \cdot 1$ for any integer $m \geq 1$, and $(-1)!! = 1$ by convention. We have the following fact (Lemma 2.4 from [15]):

$$(A.3) \quad \mathbb{E}[\xi_1^{a_1} \xi_2^{a_2} \cdots \xi_n^{a_n}] = \frac{\prod_{i=1}^n (2a_i - 1)!!}{\prod_{i=1}^n (n + 2i - 2)},$$

where a_1, \dots, a_n are nonnegative integers with $a = \sum_{i=1}^n a_i$, $\xi_i = X_i^2 / (X_1^2 + \cdots + X_n^2)$ and X_1, \dots, X_n are i.i.d. random variables with $X_1 \sim N(0, 1)$.

Step 3. Evidently, $\text{Tr}(W_n) = \sum_{1 \leq i, j \leq n} u_{ij}^2$. Notice, from the invariant property, by exchanging some rows and some columns of U_n , we see that the distributions of u_{rs} and u_{11} are identical for any $1 \leq r, s \leq n$. By (A.1),

$$(A.4) \quad \begin{aligned} \mathbb{E}[|\text{Tr}(W_n)|^2] &= \mathbb{E}\left[\left(\sum_{r,s} u_{rs}^2\right)\left(\sum_{p,q} \bar{u}_{p,q}^2\right)\right] \\ &= \mathbb{E}\left[\sum_{r,s} |u_{rs}|^4\right] = n^2 E[|u_{11}|^4]. \end{aligned}$$

It is known (e.g., Lemma 2.1 in [15, 17]) that the probability distribution of $|u_{11}|^2$ is the same as that of $(X_1^2 + X_2^2) / \sum_{i=1}^{2n} X_i^2$. By (A.3),

$$\mathbb{E}[\xi_1^2] = \frac{3}{2n(2n + 2)} \quad \text{and} \quad \mathbb{E}[\xi_1 \xi_2] = \frac{1}{2n(2n + 2)}.$$

Then

$$\mathbb{E}[|u_{11}|^4] = \mathbb{E}[(\xi_1 + \xi_2)^2] = 2\mathbb{E}[\xi_1^2] + 2\mathbb{E}[\xi_1 \xi_2] = \frac{2}{n(n + 1)}.$$

Substitute this into (A.4) to see that $\mathbb{E}[|\text{Tr}(W_n)|^2] = 2n/(n + 1)$. \square

SECOND PROOF OF LEMMA A.1. We use the following formula due to Collins [2] (see also [21]): let $(u_{ij})_{1 \leq i, j \leq n}$ be an $n \times n$ CUE matrix (or equivalently, an Haar-distributed unitary matrix) and let $i_1, \dots, i_k, j_1, \dots, j_k, i'_1, \dots, i'_k, j'_1, \dots, j'_k$ be elements in $\{1, 2, \dots, n\}$. Then

$$(A.5) \quad \begin{aligned} & \mathbb{E}[u_{i_1 j_1} \cdots u_{i_k j_k} \overline{u_{i'_1 j'_1}} \cdots \overline{u_{i'_k j'_k}}] \\ &= \sum_{\sigma, \tau \in \mathfrak{S}_k} \text{Wg}_{n,k}(\sigma^{-1} \tau) \left(\prod_{p=1}^k \delta_{i_p, i'_{\sigma(p)}} \right) \left(\prod_{q=1}^k \delta_{j_q, j'_{\tau(q)}} \right). \end{aligned}$$

Here, \mathfrak{S}_k is the symmetric group and $\text{Wg}_{n,k}$ is a class function on \mathfrak{S}_k , called the Weingarten function for the unitary group. For our purpose, we do not need the explicit definition of $\text{Wg}_{n,k}$ but use the case for $k = 2$. In fact, for $n \geq 2$, we know (see (5.2) of [2])

$$(A.6) \quad \begin{aligned} \text{Wg}_{n,2}(\text{id}_2) &= \frac{1}{n^2 - 1} \quad \text{and} \\ \text{Wg}_{n,2}((1\ 2)) &= -\frac{1}{n(n^2 - 1)}, \end{aligned}$$

where id_2 and $(1\ 2)$ are the identity permutation and the transposition on $\{1, 2\}$, respectively.

We have $|\text{Tr}(W_n)|^2 = \sum_{r,s,p,q} u_{rs}^2 \overline{u_{pq}}^2$. By (A.5), $\mathbb{E}[u_{rs}^2 \overline{u_{pq}}^2]$ is zero unless $r = p$ and $s = q$. Moreover, $\mathbb{E}[u_{rs}^2 \overline{u_{rs}}^2] = \mathbb{E}[|u_{11}|^4]$ for all $1 \leq r, s \leq n$. Therefore, using (A.5) and (A.6), we obtain

$$\mathbb{E}[|\text{Tr}(W_n)|^2] = n^2 \mathbb{E}[|u_{11}|^4] = 2n^2 \{ \text{Wg}_{n,2}(\text{id}_2) + \text{Wg}_{n,2}((1\ 2)) \} = \frac{2n}{n+1}. \quad \square$$

Lemma A.1 corresponds to the conclusion for $\beta = 1$ in (2.4), which is derived through Proposition 3 by the Jack functions. Now we apply the same proposition to derive some other moments for the circular β -ensembles. Let p_k and Z_n be as in Theorem 1.

EXAMPLE. Assume $\alpha = 2/\beta > 0$. For $n \geq 2$,

$$(A.7) \quad \begin{aligned} \mathbb{E}[|p_1(Z_n)|^4] &= \frac{2n\alpha^2(n^2 + 2(\alpha - 1)n - \alpha)}{(n + \alpha - 1)(n + \alpha - 2)(n + 2\alpha - 1)} \\ &= \begin{cases} \frac{8(n^2 + 2n - 2)}{(n + 1)(n + 3)}, & \text{if } \beta = 1; \\ 2, & \text{if } \beta = 2; \\ \frac{2n^2 - 2n - 1}{(2n - 1)(2n - 3)}, & \text{if } \beta = 4. \end{cases} \end{aligned}$$

EXAMPLE. Assume $\alpha = 2/\beta > 0$. For $n \geq 2$,

$$(A.8) \quad \mathbb{E}[|p_2(Z_n)|^2] = \frac{2\alpha n(n^2 + 2(\alpha - 1)n + \alpha^2 - 3\alpha + 1)}{(n + \alpha - 1)(n + 2\alpha - 1)(n + \alpha - 2)}$$

$$= \begin{cases} \frac{4(n^2 + 2n - 1)}{(n + 1)(n + 3)}, & \text{if } \beta = 1; \\ 2, & \text{if } \beta = 2; \\ \frac{4n^2 - 4n - 1}{(2n - 1)(2n - 3)}, & \text{if } \beta = 4. \end{cases}$$

EXAMPLE. Assume $\alpha = 2/\beta > 0$. For $n \geq 2$,

$$(A.9) \quad \mathbb{E}[p_2(Z_n)\overline{p_1(Z_n)^2}] = \mathbb{E}[\overline{p_2(Z_n)}p_1(Z_n)^2]$$

$$= \frac{2\alpha^2(\alpha - 1)n}{(n + \alpha - 1)(n + 2\alpha - 1)(n + \alpha - 2)}$$

$$= \begin{cases} \frac{8}{(n + 1)(n + 3)}, & \text{if } \beta = 1; \\ 0, & \text{if } \beta = 2; \\ \frac{-1}{(2n - 1)(2n - 3)}, & \text{if } \beta = 4. \end{cases}$$

In particular, if $\beta \neq 2$, as $n \rightarrow \infty$,

$$(A.10) \quad \mathbb{E}[p_2(Z_n)\overline{p_1(Z_n)^2}] \sim 2\alpha^2(\alpha - 1)n^{-2}.$$

PROOFS OF (2.4), (A.7), (A.8) AND (A.9). Let $n \geq 2$, μ and ν be partitions of 2. Set $\alpha = 2/\beta$. By Proposition 3 and (4.8), we have

$$(A.11) \quad \mathbb{E}[p_\mu(Z_n)\overline{p_\nu(Z_n)}]$$

$$= \alpha^{l(\mu)+l(\nu)} \left(\frac{4\alpha^{2-l(\mu)}\alpha^{2-l(\nu)}}{2\alpha^2(\alpha + 1)} \frac{n(n + \alpha)}{(n + \alpha - 1)(n + 2\alpha - 1)} \right.$$

$$\left. + \frac{4(-1)^{2-l(\mu)}(-1)^{2-l(\nu)}}{2\alpha(\alpha + 1)} \frac{n(n - 1)}{(n + \alpha - 1)(n + \alpha - 2)} \right)$$

$$= \frac{2\alpha^{l(\mu)+l(\nu)}n}{\alpha^2(\alpha + 1)(n + \alpha - 1)} \left(\frac{\alpha^{4-l(\mu)-l(\nu)}(n + \alpha)}{n + 2\alpha - 1} + \frac{(-1)^{l(\mu)+l(\nu)}\alpha(n - 1)}{n + \alpha - 2} \right).$$

(i) Take $\mu = \nu = (1)$ in Proposition 3. Since $\theta_{(1)}^{(1)}(\alpha) = 1$, $C_{(1)}(\alpha) = \alpha$ for any $\alpha > 0$, we obtain (2.4).

(ii) Taking $\mu = \nu = (1, 1)$ in (A.11), (A.7) follows.

(iii) Taking $\mu = \nu = (2)$ in (A.11), (A.8) follows.

(iv) Taking $\mu = (2)$ and $\nu = (1, 1)$ in (A.11), we get the identity for the first expectation in (A.9). Since the value of the expectation is real, the identity for the second expectation follows. With the earlier conclusion, (A.10) is obvious. \square

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REFERENCES

- [1] BLOWER, G. (2009). *Random Matrices: High Dimensional Phenomena. London Mathematical Society Lecture Note Series* **367**. Cambridge Univ. Press, Cambridge. [MR2566878](#)
- [2] COLLINS, B. (2003). Moments and cumulants of polynomial random variables on unitary groups, the Itzykson–Zuber integral, and free probability. *Int. Math. Res. Not. IMRN* **17** 953–982. [MR1959915](#)
- [3] COLLINS, B. and MATSUMOTO, S. (2009). On some properties of orthogonal Weingarten functions. *J. Math. Phys.* **50** 113516, 14. [MR2567222](#)
- [4] DIACONIS, P. and EVANS, S. N. (2001). Linear functionals of eigenvalues of random matrices. *Trans. Amer. Math. Soc.* **353** 2615–2633. [MR1828463](#)
- [5] DIACONIS, P. and SHAHSHAHANI, M. (1994). On the eigenvalues of random matrices. *J. Appl. Probab.* **31A** 49–62. [MR1274717](#)
- [6] DONG, Z., JIANG, T. and LI, D. (2012). Circular law and arc law for truncation of random unitary matrix. *J. Math. Phys.* **53** 013301, 14. [MR2919538](#)
- [7] DURRETT, R. (2010). *Probability: Theory and Examples*, 4th ed. Cambridge Univ. Press, Cambridge. [MR2722836](#)
- [8] DYSON, F. J. (1962). Statistical theory of the energy levels of complex systems. I. *J. Math. Phys.* **3** 140–156. [MR0143556](#)
- [9] DYSON, F. J. (1962). Statistical theory of the energy levels of complex systems. II. *J. Math. Phys.* **3** 166–175. [MR0143558](#)
- [10] DYSON, F. J. (1962). Statistical theory of the energy levels of complex systems. III. *J. Math. Phys.* **3** 1191–1198.
- [11] FORRESTER, P. J. (2010). *Log-Gases and Random Matrices. London Mathematical Society Monographs Series* **34**. Princeton Univ. Press, Princeton, NJ. [MR2641363](#)
- [12] FORRESTER, P. J. and WARNAAR, S. O. (2008). The importance of the Selberg integral. *Bull. Amer. Math. Soc. (N.S.)* **45** 489–534. [MR2434345](#)
- [13] HARDY, G. H., LITTLEWOOD, J. E. and PÓLYA, G. (1988). *Inequalities*. Cambridge Univ. Press, Cambridge. [MR0944909](#)
- [14] JIANG, T. (2006). How many entries of a typical orthogonal matrix can be approximated by independent normals? *Ann. Probab.* **34** 1497–1529. [MR2257653](#)
- [15] JIANG, T. (2009). A variance formula related to quantum conductance. *Phys. Lett. A* **373** 2117–2121.
- [16] JIANG, T. (2009). The entries of circular orthogonal ensembles. *J. Math. Phys.* **50** 063302, 13. [MR2536107](#)
- [17] JIANG, T. (2010). The entries of Haar-invariant matrices from the classical compact groups. *J. Theoret. Probab.* **23** 1227–1243. [MR2735744](#)
- [18] JOHANSSON, K. (1997). On random matrices from the compact classical groups. *Ann. of Math. (2)* **145** 519–545. [MR1454702](#)

- [19] KILLIP, R. (2008). Gaussian fluctuations for β ensembles. *Int. Math. Res. Not. IMRN* 8 Art. ID rnn007, 19. [MR2428142](#)
- [20] MACDONALD, I. G. (1995). *Symmetric Functions and Hall Polynomials*, 2nd ed. The Clarendon Press, New York. [MR1354144](#)
- [21] MATSUMOTO, S. and NOVAK, J. (2013). Jucys–Murphy elements and unitary matrix integrals. *Int. Math. Res. Not. IMRN* 2 362–397. [MR3010693](#)
- [22] MEHTA, M. L. (2004). *Random Matrices*, 3rd ed. *Pure and Applied Mathematics (Amsterdam)* **142**. Elsevier/Academic Press, Amsterdam. [MR2129906](#)
- [23] PASTUR, L. and VASILCHUK, V. (2004). On the moments of traces of matrices of classical groups. *Comm. Math. Phys.* **252** 149–166. [MR2104877](#)
- [24] RAINS, E. M. (1997). High powers of random elements of compact Lie groups. *Probab. Theory Related Fields* **107** 219–241. [MR1431220](#)
- [25] STOLZ, M. (2005). On the Diaconis–Shahshahani method in random matrix theory. *J. Algebraic Combin.* **22** 471–491. [MR2191648](#)
- [26] STRAUSS, W. A. (1992). *Partial Differential Equations: An Introduction*. Wiley, New York. [MR1159712](#)

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