The Annals of Probability
2015, Vol. 43, No. 6, 3216–3238
DOI: 10.1214/14-AOP958

PLANAR LATTICES DO NOT RECOVER FROM FOREST FIRES

BY DEMETER KISS[1], IOAN MANOLESCU[2] AND VLADAS SIDORAVICIUS

University of Cambridge and AIMR Tohoku University, Université de Genève and IMPA

Self-destructive percolation with parameters p, δ is obtained by taking a site percolation configuration with parameter p, closing all sites belonging to infinite clusters, then opening every closed site with probability δ, independently of the rest. Call $\theta(p, \delta)$ the probability that the origin is in an infinite cluster in the configuration thus obtained.

For two-dimensional lattices, we show the existence of $\delta > 0$ such that, for any $p > p_c$, $\theta(p, \delta) = 0$. This proves the conjecture of van den Berg and Brouwer [*Random Structures Algorithms* **24** (2004) 480–501], who introduced the model. Our results combined with those of van den Berg and Brouwer [*Random Structures Algorithms* **24** (2004) 480–501] imply the nonexistence of the infinite parameter forest-fire model. The methods herein apply to site and bond percolation on any two-dimensional planar lattice with sufficient symmetry.

1. Introduction. Self-destructive percolation was introduced in 2004 by van den Berg and Brouwer [17]. It may be formulated for both bond and site percolation; we choose to consider the latter. Fix some infinite connected graph G.

For $\delta, p \geq 0$ consider a regular site percolation configuration with intensity p. Close all sites contained in the, possibly many, infinite clusters; we say infinite clusters are "burned." Finally, open every closed site in the above configuration with probability δ, independently of all previous choices. Call $\mathbb{P}_{p,\delta}$ the measure governing the configuration thus obtained and $\theta(p, \delta)$ the $\mathbb{P}_{p,\delta}$-probability that a given site (called the origin) is in an infinite cluster. Formal and extended definitions will be given in Section 2.

Let $\delta_c(p) = \inf\{\delta : \theta(p, \delta) > 0\}$, and let $p_c = p_c(G)$ denote the critical point for regular site percolation. For $p < p_c$ and $\delta \geq 0$, $\mathbb{P}_{p,\delta}$ is just a regular percolation measure with parameter $p + (1 - p)\delta$. In particular $\delta_c(p) = \frac{p_c - p}{1-p}$ when $p < p_c$. Consequently, self-destructive percolation is only interesting for $p \geq p_c$. In their original paper [17], van den Berg and Brouwer conjectured that, for planar lattices, δ_c is uniformly bounded away from 0 when $p > p_c$.

Received January 2014; revised July 2014.

[1]Supported in part by NWO Grant 613.000.804.

[2]Supported by the ERC Grant AG CONFRA as well as the FNS.

MSC2010 subject classifications. 60K35, 82B43.

Key words and phrases. Self-destructive percolation, planar percolation, critical percolation, near-critical percolation, forest fires.

The conjecture is somewhat surprising. Recall that on planar lattices there is no infinite cluster at $p = p_c$. Hence, when p is only slightly larger than p_c, the infinite percolation cluster is very thin, and it may be expected that, after burning it, opening only few sites suffices to obtain a new infinite cluster.

Recently Ahlberg, Sidoravicius and Tykesson [2] proved that, for nonamenable graphs G, the conclusion of the conjecture is false, that is, $\delta_c(p) \to 0$ as $p \to p_c$. The same has been shown by Ahlberg et al. [1] for high-dimensional lattices (more precisely for bond percolation on $\mathbb{Z}^d$ with d large enough).

In two dimensions it has been proved in [17], Proposition 3.1, that $\delta_c(p) > 0$ for any given $p > p_c$. This was later strengthened by van den Berg and de Lima [20] to the linear lower bound $\delta_c(p) \geq (p - p_c)/p$, but a bound which is nonzero and uniform in p could not be obtained. In the present paper we prove the aforementioned conjecture. For illustration we will consider site percolation on the two-dimensional lattice $\mathbb{Z}^2$; see Section 2 for precise definitions. Fix $G = \mathbb{Z}^2$ from now on.

THEOREM 1. *There exists $\delta > 0$ such that, for all $p > p_c$, $\theta(p, \delta) = 0$.*

REMARK 2. Theorem 1 also holds for site and bond self-destructive percolation on any planar graph which is invariant under a translation (by some $u \in \mathbb{R}^2 \setminus \{0\}$), a rotation [of an angle $\varphi \in (0, \pi)$] and reflection with respect to some line. These symmetry conditions are needed for the RSW result of Proposition 3. Indeed Proposition 3 may be adapted to lattices with the symmetry conditions above.

In particular, an analogue of Theorem 1 also holds for site percolation on the triangular lattice and bond percolation on the square lattice.

Let us discuss some implications of Theorem 1. Let δ_c be the limit of $\delta_c(p)$ as $p \searrow p_c$. Theorem 1 together with the results in [19] shows that the function $(p, \delta) \to \theta(p, \delta)$ is continuous on the set $[0, 1]^2 \setminus \{p_c\} \times (0, \delta_c]$, while it is discontinuous on $\{p_c\} \times (0, \delta_c]$.

Our result has important consequences for forest fires, a class of model introduced in [5]. Intuitively, an infinite-parameter forest fire is a process indexed by $t \geq 0$ defined as follows. At time $t = 0$, all sites are closed. As t increases, sites open independently at rate 1. When an infinite cluster appears it is immediately burned (i.e., all its sites are closed). Then sites become open again at rate 1, etcetera.

It is not clear whether such a model actually exists. We show in Section 3 that our results combined with those in [17] imply that infinite-parameter forest fires cannot be defined on two-dimensional lattices.

To avoid the problems of definition, one can investigate the N-parameter forest fire models with $N < \infty$. That is, we modify the dynamics above by burning clusters as soon as their "size" reaches N. Our results with those of [18] provide some insight to the behavior of these processes. We find a behavior which is quite different compared to that of a mean field version of the forest fire model; cf. [13]. See Section 3 for a more detailed discussion.

Organization of the paper. In Section 2 we introduce the formal definitions and notation used throughout the paper. Once the notation is in place, in Section 3 we state a result on certain box-crossing probabilities (Theorem 4) and show how Theorem 1 can be deduced from it. We also discuss in more detail its implications for forest fire models. Theorem 4 is our main contribution.

Section 4 contains a review of the notion of arm events essential, to the proofs of the next section. In Section 5 we provide a delicate counting argument which proves Theorem 4.

2. Definitions and notation.

2.1. *The model.* Let $\mathbb{Z}^2$ denote the square lattice with vertices $V(\mathbb{Z}^2)$ (also called *sites*) and edges $E(\mathbb{Z}^2)$. For sites $x, y \in V(\mathbb{Z}^2)$ we write $x \sim y$, alternatively $(x, y) \in E(\mathbb{Z}^2)$, when $\|x - y\|_2 = 1$. Set $\Omega = \{0, 1\}^{V(\mathbb{Z}^2)}$. We call an element $\omega \in \Omega$ a *configuration* and write $\{\omega(x) : x \in V(\mathbb{Z}^2)\}$ for its coordinates. A site x with $\omega(x) = 1$ is called *open* (or ω-open when the configuration needs to be specified), while one with $\omega(x) = 0$ is called *closed*.

A path on $\mathbb{Z}^2$ is a sequence of sites $\gamma = (u_0, \dots, u_n)$ with $u_i \sim u_{i+1}$ for $i = 0, \dots, n-1$. Moreover we ask all paths to be self-avoiding, that is, for the vertices $u_0, \dots, u_n$ to be pairwise distinct. A path is called ω-open (resp., ω-closed) for a configuration ω if all its vertices are ω-open (resp., ω-closed).

For a configuration ω and $x, y \in V(\mathbb{Z}^2)$, we say x is *connected* to y in ω, and write $x \overset{\omega}{\leftrightarrow} y$, if there exists an ω-open path with endpoints x and y. We write $x \overset{\omega}{\leftrightarrow} \infty$ and say that x is *connected to infinity* if there exists an infinite ω-open path starting at x. Finally we write $x \overset{\omega}{\nleftrightarrow} y$ and $x \overset{\omega}{\nleftrightarrow} \infty$ for the negations of the above events. A *cluster* is a connected component of the graph induced by the open sites of $\mathbb{Z}^2$.

For $p \in [0, 1]$, let $\mathbb{P}_p$ be the site percolation measure on $\mathbb{Z}^2$ with intensity p. That is, $\mathbb{P}_p$ is the product measure on Ω with $\mathbb{P}_p(\omega(x) = 1) = p$ for all $x \in V(\mathbb{Z}^2)$. Finally let $p_c = \sup\{p \geq 0 : \mathbb{P}_p(0 \overset{\omega}{\leftrightarrow} \infty) = 0\}$. For $p > p_c$ it is well known that there exists $\mathbb{P}_p$-a.s. a unique infinite cluster. For this and further details on percolation we direct the reader to [6].

Let $p \in [0, 1]$, and consider a configuration ω chosen according to $\mathbb{P}_p$. We define a modification of ω, called $\overline{\omega}$, as follows. For $x \in V(\mathbb{Z}^2)$,

$$\overline{\omega}(x) = \begin{cases} 1, & \text{if } \omega(x) = 1 \text{ and } x \overset{\omega}{\nleftrightarrow} \infty, \\ 0, & \text{otherwise.} \end{cases}$$

Let $\delta \geq 0$ and σ be a configuration chosen according to $\mathbb{P}_\delta$, independently of ω. The enhancement of $\overline{\omega}$ with intensity δ is $\overline{\omega}^\sigma(x) = \overline{\omega}(x) \vee \sigma(x)$.

Let $\mathbb{P}_{p,\delta}$ denote the probability measure governing ω, σ and thus $\overline{\omega}$ and $\overline{\omega}^\sigma$. To avoid confusion, when working with $\mathbb{P}_{p,\delta}$, we will usually state to which configuration we refer. When writing simply $\mathbb{P}_{p,\delta}(A)$ we mean $\mathbb{P}_{p,\delta}(\overline{\omega}^\sigma \in A)$. Let

$$\theta(p,\delta) = \mathbb{P}_{p,\delta}\big(0 \stackrel{\overline{\omega}^\sigma}{\longleftrightarrow} \infty\big).$$

Note that $\mathbb{P}_{p,\delta}$ is increasing in δ, hence so is θ.

2.2. *Further notation.* Let $\mathrm{dist}(\cdot,\cdot)$ denote the L^∞ distance on $\mathbb{Z}^2$. That is,

$$\mathrm{dist}(x,y) = \max\big(|x_1 - y_1|, |x_2 - y_2|\big) \qquad \text{for } x = (x_1,x_2), y = (y_1,y_2) \in \mathbb{Z}^2.$$

For $u \in V(\mathbb{Z}^2)$ and $n \geq 0$, denote by $\Lambda_n(u)$ the ball of radius n around u for the L^∞ distance. Hence $\Lambda_n(u) = ([-n,n]^2 + u) \cap \mathbb{Z}^2$. When u is omitted, it is assumed equal to the origin. We will usually identify regions of the plane with the set of vertices they contain.

For $A \subset V(\mathbb{Z}^2)$, we call the (*outer*) *boundary* of A the set

$$\partial A = \big\{y \in V\big(\mathbb{Z}^2\big) \setminus A : y \sim x \text{ for some } x \in A\big\};$$

the *internal boundary* of A is the set $\partial_i A = \partial(A^c)$. The diameter of the set A is $\mathrm{diam}(A) = \sup\{\mathrm{dist}(x,y) : x, y \in A\}$.

For a configuration ω and $x, y \in A \subset V(\mathbb{Z}^2)$, we say x is ω-connected to y in A, and write $x \stackrel{\omega,A}{\longleftrightarrow} y$, if there exists an ω-open path with endpoints x and y, fully contained in A.

The matching graph of $\mathbb{Z}^2$, written $(\mathbb{Z}^2)^*$, has the same vertex set as $\mathbb{Z}^2$ and an edge between any two vertices of the same face of $\mathbb{Z}^2$. We say that x and y are *-*connected*, and write $x \stackrel{\omega}{\leftrightarrow}^* y$, if there exists ω-closed path in $(\mathbb{Z}^2)^*$ with endpoints x and y. The notion of matching graph is proper to site percolation, so when working with bond percolation it should be replaced by the dual graph. For more details on matching and dual graphs consult [6].

For $m, n \in \mathbb{N}$, we define the rectangular box $B(m,n) = [0,m] \times [0,n]$. The sides of $B(m,n)$ are the sets $[0,m] \times \{0\}$, $[0,m] \times \{n\}$, $\{0\} \times [0,n]$ and $\{m\} \times [0,n]$, and they are called the bottom, top, left-hand and right-hand side, respectively. Given a configuration ω, we say $B(m,n)$ is crossed horizontally if there exists an ω-open path γ contained in $B(m,n)$, with one endpoint on the left-hand side and one on the right-hand side of $B(m,n)$. We say it is crossed vertically if an ω-open path contained in $B(m,n)$ connects the top and the bottom. We write $\mathcal{C}_h(m,n)$ and $\mathcal{C}_v(m,n)$ for the events that $B(m,n)$ is crossed horizontally, respectively, vertically. If R is a translate of the box $B(m,n)$, we write $\mathcal{C}_h(R)$ and $\mathcal{C}_v(R)$ for the appropriate translations of $\mathcal{C}_h(m,n)$ and $\mathcal{C}_v(m,n)$.

Finally, we mention a well-known result for standard percolation that is essential to our analysis. This type of result was initially proved separately by Russo [15] and Seymour and Welsh [16], hence the name of Russo–Seymour–Welsh (RSW) result. For reference we direct the reader to [6], Theorem 11.70. Extensions to percolation models on general graphs with the symmetries mentioned in Remark 2 are discussed in detail in [7], Section 6.

PROPOSITION 3 (RSW). *There exists a constant* $\alpha > 0$ *such that, for all* $n \geq 1$,

$$\mathbb{P}_{p_c}\big(\mathcal{C}_h(2n, n)\big) \geq \alpha. \tag{1}$$

*The analogue holds for *-crossings on the matching graph.*

3. Box-crossing estimates and consequences for forest fires.

3.1. *Crossing boxes after the burn.* The proof of Theorem 1 is based on a crossing-probability estimate. Some additional notation is needed.

Let $R_n = [-2n, 2n] \times [0, n]$ and $S_n = [-3n, 3n] \times [0, n]$. For a configuration ω let χ be the set of sites $x \in S_n$ which are connected to both the left-hand and right-hand sides of S_n by open paths contained in S_n. Define a configuration $\widetilde{\omega}$ by setting, for $x \in S_n$,

$$\widetilde{\omega}(x) = \begin{cases} 0, & \text{if } x \in \chi \cup \partial\chi, \\ 1, & \text{otherwise.} \end{cases} \tag{2}$$

In other words, the ω-open clusters containing horizontal crossings of S_n are declared closed in $\widetilde{\omega}$, as are their boundaries. All other sites are opened. The value of $\widetilde{\omega}$ outside of S_n is irrelevant for our purposes; for concreteness we take $\widetilde{\omega} = 0$ there. Finally we enhance the vertices inside R_n by setting

$$\widetilde{\omega}^\sigma(x) = \begin{cases} \widetilde{\omega}(x) \vee \sigma(x), & \text{if } x \in R_n, \\ \widetilde{\omega}(x), & \text{otherwise.} \end{cases} \tag{3}$$

THEOREM 4. *There exist constants* $\delta, \lambda, c > 0$ *such that, for all* $n \geq 1$,

$$\mathbb{P}_{p_c,\delta}\big[\omega \in \mathcal{C}_h(S_n) \text{ and } \widetilde{\omega}^\sigma \in \mathcal{C}_v(R_n)\big] \leq cn^{-\lambda}. \tag{4}$$

Similar statements to (4) have been shown to imply Theorem 1, but none has been proved. See, for instance, [17], Conjecture 3.2, and [18], Conjecture 2.1. Our criterion was inspired by the previous; the slightly different formulation is particularly adapted to our proof.

Theorem 4 will be proved in Section 5. For completeness we give a proof of Theorem 1 from Theorem 4 that follows the steps of [17]. We start with a corollary which requires some additional notation.

Recall the definition of Λ_n from Section 2.2. Consider some $n \in \mathbb{N}$, and define the annulus $\mathrm{A}(n, 2n) = \Lambda_{2n} \setminus \Lambda_{n-1}$. A circuit in $\mathrm{A}(n, 2n)$ is a path contained in $\mathrm{A}(n, 2n)$ that separates the origin from infinity. For a configuration ω, define a new modification $\check{\omega}$ of ω, by closing all sites that are connected by an ω-open path in $\mathrm{A}(n, 2n)$ to an ω-open circuit in $\mathrm{A}(n, 2n)$. As above, for a second configuration σ, set $\check{\omega}^\sigma = \check{\omega} \vee \sigma$.

COROLLARY 5. *There exists a constant $\rho > 0$ such that, with δ as in Theorem 4*,

$$\mathbb{P}_{p_c,\delta}\big(\partial \Lambda_{n-1} \xleftrightarrow{\check{\omega}^\sigma} \partial \Lambda_{2n}\big) \leq 1 - \rho$$

for all $n \geq 1$.

Before we dive in the proofs of Corollary 5 from Theorem 4 and of Theorem 1 from Corollary 5, let us turn to some other implications of Theorem 4 and Corollary 5.

3.2. *Consequences for forest fires.* The following was stated as a conditional result in [17]. Our results imply it.

THEOREM 6 (Theorem 4.1 of [17]). *The infinite-parameter forest fire process does not exist on $\mathbb{Z}^2$.*

In [17] the above was stated conditionally on [17], Conjecture 3.2. While the latter is not obviously implied by our results, its main consequence, [17], Lemma 3.4, is equivalent to Corollary 5 above. The proof of the theorem in [17] is based solely on [17], Lemma 3.4.

The intuition behind Theorem 6 is the following. Suppose an infinite-parameter forest fire process is defined, and let t_c (defined by $1 - e^{-t_c} = p_c$) be the time when fires start to appear. No fires ignite on $[0, t_c]$ since no infinite cluster is produced. But for any $t > t_c$ at least one infinite cluster was produced and burned before t. Thus an infinity of burning times have to accumulate after t_c. But Theorem 1 suggests that there exists a universal $\tau > 0$ such that, after one fire, the process needs at least time τ to recover and recreate a new infinite cluster. This leads to a contradiction, hence the nonexistence of the process.

In [18] van den Berg and Brouwer stated several results for finite-parameter forest fires conditionally on [18], Conjecture 2.1. Our Theorem 4 implies this conjecture, and hence their results. We will state two of them. In the following $\eta^{[N]}$ denotes the N-parameter forest fire process. We say $\eta^{[N]}$ has a fire in Λ_m when a cluster intersecting Λ_m reaches size N and is burned.

THEOREM 7 (Theorem 4.2 and Proposition 4.3 of [18]). *There exists $t > t_c$ such that for all $m \geq 0$*,

$$\liminf_{N\to\infty} \mathbb{P}\big(\eta^{[N]} \textit{ has a fire in } \Lambda_m \textit{ before time } t\big) \leq 1/2,$$

$$\lim_{N\to\infty} \mathbb{P}\big(\eta^{[N]} \textit{ has at least } 2 \textit{ fires in } \Lambda_m \textit{ before time } t\big) = 0.$$

The interested reader is referred to [17] and [18] for precise definitions of forest fires and more details. We conclude the section with the proofs of Corollary 5 and Theorem 1 given Theorem 4.

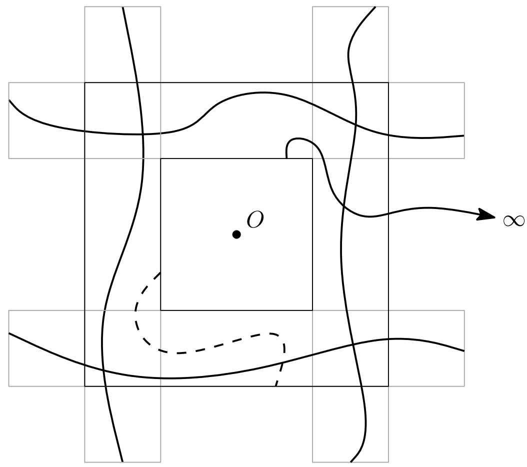

FIG. 1. *A situation with $\omega \in C_n$ and $\partial \Lambda_{n-1} \overset{\omega}{\leftrightarrow} \infty$ (the bold paths are open in ω) and such that there exists an $\overline{\varpi}^{\sigma}$-open crossing of the annulus* $\mathrm{A}(n, 2n)$ *(dashed path). Then $\overline{\varpi}^{\sigma} \leq \check{\omega}^{\sigma}$, and there exists an $\check{\omega}^{\sigma}$-open crossing in the easy direction of one of the four rectangles forming* $\mathrm{A}(n, 2n)$. *Note that any site of a ω-open horizontal crossing of a gray rectangle is closed in $\check{\omega}$.*

3.3. *Proofs*. Figure 1 sums up the proofs of both Corollary 5 and Theorem 1.

PROOF OF COROLLARY 5 FROM THEOREM 4. For $n \geq 1$ denote the four $6n \times n$ rectangles surrounding Λ_n by

$$S_B = [-3n, 3n] \times [-2n, -n], \qquad S_T = [-3n, 3n] \times [n, 2n],$$
$$S_L = [-2n, -n] \times [-3n, 3n], \qquad S_R = [n, 2n] \times [-3n, 3n].$$

Let $R_B = [-2n, 2n] \times [-2n, -n]$, and define similarly R_T, R_L and R_R. Also let $C_n = \mathcal{C}_h(S_B) \cap \mathcal{C}_h(S_T) \cap \mathcal{C}_v(S_L) \cap \mathcal{C}_v(S_R)$ and note that, if $\omega \in C_n$, then ω contains an open circuit in $\mathrm{A}(n, 2n)$. By Proposition 3 and the FKG inequality for regular percolation, there exists $\rho > 0$ such that $\mathbb{P}_{p_c}(C_n) \geq 2\rho$ for all $n \geq 1$.

Fix δ as in Theorem 4. Let $\omega \in C_n$ and σ be such that $\check{\omega}^{\sigma}$ contains an open path γ between $\partial \Lambda_{n-1}$ and $\partial_i \Lambda_{2n}$. Then it is easy to see that γ contains a crossing in the easy direction of one of the rectangles R_B, R_T, R_L and R_R. In other words,

$$\check{\omega}^{\sigma} \in \mathcal{C}_v(R_B) \cup \mathcal{C}_v(R_T) \cup \mathcal{C}_h(R_L) \cup \mathcal{C}_h(R_R). \tag{5}$$

Suppose, for instance, that $\check{\omega}^{\sigma} \in \mathcal{C}_v(R_B)$. Since $\omega \in C_n$, all sites connected to a horizontal crossings of S_B are closed in $\check{\omega}$, and (4) implies that

$$\mathbb{P}_{p_c,\delta}[\omega \in C_n \text{ and } \check{\omega}^{\sigma} \in \mathcal{C}_v(R_B)] \leq cn^{-\alpha}.$$

This combined with by (5), and the bound $\mathbb{P}_{p_c}(C_n) \geq 2\rho$ gives

$$\begin{aligned}
&\mathbb{P}_{p_c,\delta}\big(\partial\Lambda_{n-1} \overset{\check{\omega}^\sigma}{\longleftrightarrow} \partial\Lambda_{2n}\big) \\
&\quad \leq \mathbb{P}_{p_c,\delta}\big(\omega \in C_n \text{ and } \partial\Lambda_{n-1} \overset{\check{\omega}^\sigma}{\longleftrightarrow} \partial\Lambda_{2n}\big) + \mathbb{P}_{p_c,\delta}(\omega \notin C_n) \\
&\quad \leq 4cn^{-\alpha} + 1 - 2\rho.
\end{aligned}$$

Taking n large enough in the above completes the proof of Corollary 5. $\square$

PROOF OF THEOREM 1 FROM COROLLARY 5. Corollary 5 gives crossing probability estimates for measures $\mathbb{P}_{p_c,\delta}$. We start by extending these to measures $\mathbb{P}_{p,\delta'}$, with $p > p_c$. Let $\delta > 0$ be given by Theorem 4. Fix some $p > p_c$, and let $\delta' > 0$ be such that $p + (1-p)\delta' \leq p_c + (1-p_c)\delta$.

It is easy to check (see, e.g., [17], Corollary 2.4) that the configuration $\check{\omega}^\sigma$ obtained from $\mathbb{P}_{p_c,\delta}$ stochastically dominates that obtained from $\mathbb{P}_{p,\delta'}$. In particular,

$$\mathbb{P}_{p_c,\delta}\big(\partial\Lambda_{n-1} \overset{\check{\omega}^\sigma}{\longleftrightarrow} \partial\Lambda_{2n}\big) \geq \mathbb{P}_{p,\delta'}\big(\partial\Lambda_{n-1} \overset{\check{\omega}^\sigma}{\longleftrightarrow} \partial\Lambda_{2n}\big).$$

This together with Corollary 5 implies that

$$\mathbb{P}_{p,\delta/2}\big(\partial\Lambda_{n-1} \overset{\check{\omega}^\sigma}{\longleftrightarrow} \partial\Lambda_{2n}\big) \leq 1 - \rho$$

for all p sufficiently close to p_c and all sufficiently large n. We claim that the above yields $\theta(p, \delta/2) = 0$.

Since $p > p_c$, the probability that $\partial\Lambda_{n-1}$ does not have an ω-open path to ∞ is at most $\rho/2$ for n sufficiently large. Moreover, if $\partial\Lambda_{n-1} \overset{\omega}{\leftrightarrow} \infty$, then $\overline{\omega} \leq \check{\omega}$. Hence, for all sufficiently large n,

$$\begin{aligned}
\mathbb{P}_{p,\delta/2}\big(\partial\Lambda_{n-1} \overset{\overline{\omega}^\sigma}{\longleftrightarrow} \infty\big) &\leq \mathbb{P}_{p,\delta/2}\big(\partial\Lambda_{n-1} \overset{\check{\omega}^\sigma}{\longleftrightarrow} \Lambda_{2n}\big) + \mathbb{P}_{p,\delta/2}\big(\partial\Lambda_{n-1} \overset{\omega}{\not\leftrightarrow} \infty\big) \\
&\leq 1 - \rho + \rho/2.
\end{aligned}$$

The event $\{\overline{\omega}^\sigma$ contains an infinite cluster$\}$ is translation invariant; thus its probability is either 0 or 1. The above excludes the latter, hence $\theta(p, \delta/2) = 0$. $\square$

4. Arm events. A *color sequence* of length k is a sequence $\varsigma \in \{0, 1\}^k$. Fix such a color sequence ς, a vertex $u \in \mathbb{Z}^2$ and integers $n \leq N$. We write $\mathcal{A}_\varsigma(u; n, N)$ for the event that there exist k pairwise disjoint paths $\gamma_1, \dots, \gamma_k$ such that, for $j = 1, \dots, k$:

- if $\varsigma_j = 1$, then γ_j is a open path on $\mathbb{Z}^2$, and if $\varsigma_j = 0$, then γ_j is a closed path on $(\mathbb{Z}^2)^*$;
- $\gamma_j \subset \Lambda_N(u) \setminus \Lambda_n(u)$ and has one endpoint in $\partial\Lambda_n(u)$ and the other in $\partial_i\Lambda_N(u)$;
- the endpoints of $\gamma_1, \dots, \gamma_k$ are placed in counter-clockwise order on $\partial\Lambda_n(u)$.

The paths γ_j are called *arms*; and the event $\mathcal{A}_\varsigma(u; n, N)$ is called an *arm event*. When u is omitted, it is assumed to be the origin. The probabilities of arm events are denoted by $\pi_\varsigma(n, N) = \mathbb{P}_{p_c}(\mathcal{A}_\varsigma(n, N))$.

For very small values of n, $\mathcal{A}_\varsigma(n, N)$ could be empty because of geometric constraints. It will be convenient to redefine $\mathcal{A}_\varsigma(n, N)$ as $\mathcal{A}_\varsigma(|\varsigma|, N)$ when $n \le |\varsigma|$. Let $\mathcal{A}_\varsigma(n) = \mathcal{A}_\varsigma(0, n)$ and $\pi_\varsigma(n) = \pi_\varsigma(0, n)$.

A related notion is that of *half-plane arm events*. Let $\mathbb{H} = \mathbb{R} \times [0, \infty)$ be the upper half-plane. Define $\mathcal{A}_\varsigma^{hp}(n, N)$ as the event $\mathcal{A}_\varsigma(n, N)$, with the additional restriction that the arms $\gamma_1, \dots, \gamma_k$ are all contained in $\mathbb{H}$ and that γ_1 is the right-most arm.

The notation for arm events extends to half-plane arm events, thus $\pi_\varsigma^{hp}(n, N) = \mathbb{P}_{p_c}(\mathcal{A}_\varsigma^{hp}(n, N))$, $\mathcal{A}_\varsigma^{hp}(n) = \mathcal{A}_\varsigma^{hp}(0, n)$ and $\pi_\varsigma^{hp}(n) = \pi_\varsigma^{hp}(0, n)$.

Here are two well-known properties of arm events.

PROPOSITION 8. *Fix a color sequence ς. There exists a constant $c = c(\varsigma) > 0$ such that, for $0 \le n \le m \le N$,*

$$c\pi_\varsigma(n, m)\pi_\varsigma(m, N) \le \pi_\varsigma(n, N) \le \pi_\varsigma(n, m)\pi_\varsigma(m, N), \tag{6}$$

$$\pi_\varsigma(n, 2n) \ge c. \tag{7}$$

The above also holds for half-plane arm events.

The proposition is not specific to site percolation on $\mathbb{Z}^2$; the only thing needed for the proof is the crossing estimate (1). The bound (6) first appeared in [9], combination of Lemmas 4 and 6, while (7) is a simple consequence of (1). For a modern treatment of Proposition 8 and for other proofs in this section, we refer the reader to the survey [12].

We also need to introduce the notion of *arms with defects*. Let $\mathcal{A}_\varsigma^*(n, N)$ be the set of configurations ω such that there exists a point u and a configuration ω' equal to ω outside $\Lambda_3(u)$ with $\omega' \in \mathcal{A}_\varsigma(n, N)$. All the notation defined above extends to arm events with defects, with the attached asterisk.

PROPOSITION 9 (Proposition 18 of [12]). *Fix a color sequence ς. There exists a positive constant $C = C(\varsigma)$ such that, for all $n \le N$,*

$$\begin{aligned} \mathbb{P}_{p_c}\big(\mathcal{A}_\varsigma^*(n, N)\big) &\le C\big(1 + \log(N/n)\big)\pi_\varsigma(n, N) \quad \textit{and} \\ \mathbb{P}_{p_c}\big(\mathcal{A}_\varsigma^{hp*}(n, N)\big) &\le C\big(1 + \log(N/n)\big)\pi_\varsigma^{hp}(n, N). \end{aligned} \tag{8}$$

REMARK 10. In [12] an arm event with a defect is defined as a modification of the event $\mathcal{A}_\varsigma(n, N)$ where the arms are allowed to have at most one vertex of the opposing color. Our definition is slightly different; nevertheless, Nolin's proof readily extends to our case.

In the rest of the paper, the following types of arm events will play a special role. Call $\mathcal{A}_1$, $\mathcal{A}_5$ and $\mathcal{A}_6$ the event $\mathcal{A}_\varsigma$ with $\varsigma = (1)$, $\varsigma = (1, 0, 0, 1, 0)$ and $\varsigma = (0, 1, 0, 0, 1, 0)$, respectively. In addition, write $\mathcal{A}_3^{hp}$ and $\mathcal{A}_4^{hp}$ for the event $\mathcal{A}_\varsigma^{hp}$ with $\varsigma = (1, 0, 1)$ and $\varsigma = (1, 0, 0, 1)$, respectively. The same notation applies to π.

The following is a well-known consequence of (1). See [12], Theorem 24, and [6], Theorem 11.89.

PROPOSITION 11. *There exist constants $\lambda, c, C > 0$ such that for all $n \le N$,*

$$\pi_1(n, N) \le (N/n)^{-\lambda}, \tag{9}$$

$$c(N/n)^{-2} \le \pi_5(n, N) \le C(N/n)^{-2}, \tag{10}$$

$$c(N/n)^{-2} \le \pi_3^{hp}(n, N) \le C(N/n)^{-2}. \tag{11}$$

As a consequence of the above, we have the following estimates for the probabilities of arm events of interest to us. The proof is a simple application of Reimer's inequality [14].

COROLLARY 12. *There exist constants $c, \lambda > 0$ such that, for all $n \le N$,*

$$\pi_6(n, N) \le c(N/n)^{-(2+\lambda)} \quad \textit{and} \quad \pi_4^{hp}(n, N) \le c(N/n)^{-(2+\lambda)}.$$

Among the results of this section, only the following corollary will be used explicitly in the rest of the paper.

COROLLARY 13. *There exist constants $c \ge 1$ and $\lambda > 0$ so that, for all $n \le N$,*

$$\begin{aligned} \mathbb{P}_{p_c}\big(\mathcal{A}_6^*(n, N)\big) &\le c(N/n)^{-(2+\lambda)} \quad \textit{and} \\ \mathbb{P}_{p_c}\big(\mathcal{A}_4^{hp*}(n, N)\big) &\le c(N/n)^{-(2+\lambda)}. \end{aligned} \tag{12}$$

PROOF. The statement above follows directly from Proposition 9 and Corollary 12. $\square$

5. Proof of Theorem 4.

5.1. *Plan of proof.* The proof of Theorem 4 is quite intricate; we start with some notation and a brief description of the strategy.

Fix some $\delta > 0$ and $n \in \mathbb{N}$. Consider a pair of configurations ω, σ, and recall the definition of χ, $\widetilde{\omega}$ and $\widetilde{\omega}^\sigma$ from the lines above (2), (2) and (3), respectively. Call a point x is called *enhanced* if $\widetilde{\omega}(x) = 0$ but $\widetilde{\omega}^\sigma(x) = 1$. We will bound

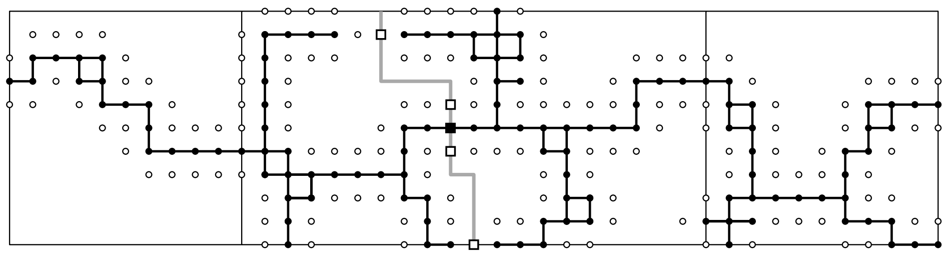

FIG. 2. *The set χ in black, surrounded by $\partial\chi$. The gray path γ needs to use at least one passage point to cross χ (black square). Even if a site which is pivotal for $\{\omega \in \mathcal{C}_h(S_n)\}$ is enhanced, γ generally needs additional passage points to cross $\partial\chi$ (see the empty squares).*

the probability $\mathbb{P}_{p_c,\delta}(\omega \in \mathcal{C}_h(S_n)$ and $\widetilde{\omega}^\sigma \in \mathcal{C}_v(S_n))$, which is obviously larger than $\mathbb{P}_{p_c,\delta}(\omega \in \mathcal{C}_h(S_n)$ and $\widetilde{\omega}^\sigma \in \mathcal{C}_v(R_n))$.

If $\omega \in \mathcal{C}_h(S_n)$, then χ contains a horizontal crossing of S_n. If in addition there exists a $\widetilde{\omega}^\sigma$-open vertical crossing of S_n, then it must cross χ, and hence it contains at least one enhanced point; see Figure 2.

For $\omega \in \mathcal{C}_h(S_n)$ and σ such that $\widetilde{\omega}^\sigma \in \mathcal{C}_v(S_n)$, let γ be the left-most $\widetilde{\omega}^\sigma$-open vertical crossing of S_n containing the minimal number of enhanced points. (We only take γ to be left-most for it to be uniquely defined.) Call the enhanced points of γ *passage points*, and let $\mathscr{X}$ be the set of passage points. If $\widetilde{\omega}^\sigma \notin \mathcal{C}_v(S_n)$ or $\omega \notin \mathcal{C}_h(S_n)$, then let $\mathscr{X} = \varnothing$.

Recall from the definition of $\widetilde{\omega}^\sigma$ that all enhanced points are contained in R_n. Thus, under $\mathbb{P}_{p_c,\delta}$, $\mathscr{X}$ is a random set of vertices of R_n, nonempty when $\omega \in \mathcal{C}_h(S_n)$ and $\widetilde{\omega}^\sigma \in \mathcal{C}_v(S_n)$.

We will prove (4) by estimating the probability for $\mathscr{X}$ to take specific values. More precisely we will use the equality

$$\mathbb{P}_{p_c,\delta}\big(\omega \in \mathcal{C}_h(S_n) \text{ and } \widetilde{\omega}^\sigma \in \mathcal{C}_v(S_n)\big) = \sum_{X \neq \varnothing} \mathbb{P}_{p_c,\delta}(\mathscr{X} = X). \tag{13}$$

The computation used to estimate $\mathbb{P}_{p_c,\delta}(\mathscr{X} = X)$ is quite delicate. Here are the main ideas; the actual proof is given in the following sections.

Fix a nonempty set of vertices X with $|X| = k+1$, and let ω, σ be configurations such that $\mathscr{X} = X$. Since the passage points act as passages between the clusters of $\widetilde{\omega}$, they have, in ω, a (local) six arm structure around them (see Figure 3), and we may control the probabilities of such configurations by π_6.

Imagine the following dynamics. Around each point $x \in X$ we grow a ball at unit speed, $\Lambda_t(x) : t \geq 0$. For a given time t, we call *blobs* the connected components of $\bigcup_x \Lambda_t(x)$.

For small times, the blobs are just balls centered at the points of X. As time increases two blobs may *merge* to create a bigger blob. For a point $x \in X$ set $t(x)$ to be the first time of merger for the blob containing x. Thus $t(x) = \frac{1}{2}\inf\{\text{dist}(x, y) : y \in X, y \neq x\}$. Then ω contains six arms from x to $\partial\Lambda_{t(x)}(x)$, an

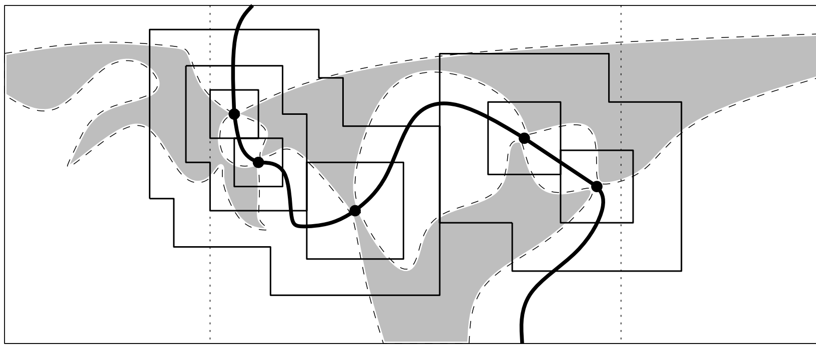

FIG. 3. *The crossing γ is drawn in bold and the passage points are marked. The set χ of sites open in ω but closed in $\tilde{\omega}$ is drawn in gray. Its boundary is closed in ω. The blobs at the times of merger are outlined. Observe the six-arm structure between the boundaries of the blobs.*

event which has probability bounded by $\pi_6(t(x))$. Moreover the regions $\Lambda_{t(x)}(x)$ for $x \in X$ are disjoint. Finally, in order to be a passage point, x has to be enhanced. This happens with probability δ, independently of ω, thus

$$\mathbb{P}_{p_c,\delta}(\mathscr{X} = X) \leq \prod_{x \in X} \pi_6\big(t(x)\big)\delta. \tag{14}$$

Unfortunately, this bound is not sufficient to obtain Theorem 4. If points are grouped in small bunches, then all values $t(x)$ are small, and the right-hand side of (14) is not significantly smaller than δ^{k+1}.

In order to improve (14), we will also study the blobs after their first mergers. Consider a blob at the time of formation (e.g., by the merger of two smaller blobs), and the same blob at the first time it merges with another blob. Let B_1 denote the blob at the initial time, and B_2 at the latter time. Then we also observe six arms between ∂B_1 and ∂B_2. This will add terms to the bound in (14), thus improving it.

If we denote by d_i the times of merger of blobs (counted with multiplicity when more than two blobs merge at the same time), then we obtain a bound on $\mathbb{P}_{p_c,\delta}(\mathscr{X} = X)$ as a function of $d_1, \dots, d_k$; see Proposition 14.

In order to compute (13), we also need to estimate the number of sets X that yield a given set of merger times $d_1, \dots, d_k$. This is done in Proposition 15.

In the above analysis we have omitted certain technical complications. One is the influence of the boundary of S_n. As blobs expand, they may touch the top and bottom of S_n, and special situations arise. Another has to do with defects in arm events around passage points.

Before diving into the actual proof, we mention that a simplified one-arm version of this argument already appeared in [3, 8] for the study of the moments of the volume of the largest critical percolation cluster in Λ_n. It was shown there that the kth moment of this quantity is bounded above by $k!(Cn^2\pi_1(n))^k$ for a constant $C > 0$. Contrary to the argument presented here, in [3, 8] blobs only need to be studied up to their first time of merger, and the resulting bound (14) (with π_6 replaced by π_1) suffices. The fundamental reason for which (14) suffices in that case is that the one-arm exponent is smaller than 2, hence $\sum_{k=1}^{n} k\pi_1(k) = O(n^2\pi_1(n))$. The six-arm exponent, however, is larger than 2, and the series $\sum_{k=1}^{\infty} k\pi_6(k)$ converges, thus requiring a more sophisticated analysis.

In [10], the first author applies the refined counting arguments presented here to the one-arm case in order to derive an improved upper bound of $(Cn^2\pi_1(n/\sqrt{k}))^k$ for the kth moment of the volume of the largest critical cluster in Λ_n. These arguments lead to large deviation bounds for the volumes of large critical percolation clusters.

5.2. *Two propositions.* Fix some nonempty set $X \subset R_n$. We associate to X a tree $\mathcal{T} = \mathcal{T}(X)$ as described below. Although this is not important for our proof, let us mention that $\mathcal{T}$ is a minimal spanning tree of X and that the algorithm by which we construct it is Kruskal's algorithm [11].

The vertices of $\mathcal{T}$ are the points of X, and the edges are added successively as follows.

Let $\mathcal{T}_0$ be the graph with no edges and vertex-set X. For $j \in \mathbb{N}$ define $\mathcal{T}_j$ by adding to $\mathcal{T}_{j-1}$ a maximal set of edges (x, y) with $\operatorname{dist}(x, y) = j$, which does not create cycles in $\mathcal{T}_j$. Since $\operatorname{diam}(X) \le 4n$, $\mathcal{T}_j = \mathcal{T}_{j+1}$ for $j \ge 4n$, and we define $\mathcal{T} = \mathcal{T}_{4n}$. The graph $\mathcal{T}$ thus obtained is indeed a tree: by construction it does not contain cycles and it is easy to check that it is connected.

Note that there is some ambiguity in the definition of $\mathcal{T}$ since there may be multiple choices for the set of edges added to $\mathcal{T}_{j-1}$ to create $\mathcal{T}_j$. To settle this, when multiple choices are available, we choose the minimal one with respect to the lexicographical order of $\mathbb{Z}^2 \times \mathbb{Z}^2$. Let the root of $\mathcal{T}$ be the smallest element of $V(\mathcal{T}) = X$ for the lexicographical order of $\mathbb{Z}^2$.

Let $E(\mathcal{T})$ denote the edge-set of $\mathcal{T}$. Then $\#E(\mathcal{T}) = k$. For $e = (x, y) \in E(\mathcal{T})$, let $\mathrm{d}_e = \lfloor \frac{1}{2} \operatorname{dist}(x, y) \rfloor + 1$. The multiset $\mathcal{D}(X) = [\mathrm{d}_e : e \in \mathcal{T}]$ is called the set of *merger times* of X.

The terminology of merger times is inspired by the dynamics described in Section 5.1. Indeed, each edge e of $\mathcal{T}$ corresponds to the merger of two blobs and d_e is their (approximate) time of merger.

PROPOSITION 14. *There exist constants $c, \lambda > 0$ such that, for all $\delta > 0$, $n \in \mathbb{N}$, and $X \subset R_n$ with $|X| = k + 1$, we have*

$$\mathbb{P}_{p_c,\delta}(\mathscr{X} = X) \le (c\delta)^{k+1} n^{-(2+\lambda)} \prod_{e \in E(\mathcal{T})} \mathrm{d}_e^{-(2+\lambda)}. \tag{15}$$

Since the above offers a bound on $\mathbb{P}_{p_c,\delta}(\mathscr{X} = X)$ as a function of the set $\mathcal{D}(X)$, it is natural to group the sum in (13) by the value of $\mathcal{D}(X)$.

PROPOSITION 15. *There exists a constant $K > 0$ such that, for any given multiset of values $D = [d_1, \dots, d_k]$, the number of sets X with $\mathcal{D}(X) = D$ is bounded as follows*:

$$\#\{X \subset R_n : \mathcal{D}(X) = D\} \le \mathcal{Q}(D) K^{k+1} n^2 \prod_{i=1}^{k} d_i, \tag{16}$$

where $\mathcal{Q}(D)$ is the number of different ways of ordering $d_1, \dots, d_k$.

Theorem 4 follows easily from the two propositions.

PROOF OF THEOREM 4 FROM PROPOSITIONS 14 AND 15. Let c, λ and K be the constants provided by Propositions 14 and 15. Choose $\delta > 0$ small enough to have $cK\delta \sum_{d \ge 1} d^{-(1+\lambda)} \le 1/2$. It is essential here that $\lambda > 0$, so that the sum above converges. Then, by Propositions 14 and 15, for $n \in \mathbb{N}$,

$$\begin{aligned}
&\mathbb{P}_{p_c,\delta}\big(\omega \in \mathcal{C}_h(S_n) \text{ and } \widetilde{\omega}^\sigma \in \mathcal{C}_v(S_n)\big) \\
&\quad = \sum_{k \ge 0} \sum_{d_k \ge \cdots \ge d_1 \ge 1} \sum_{\substack{X \subseteq R_n \\ \mathcal{D}(X) = [d_1, \dots, d_k]}} \mathbb{P}_{p_c,\delta}(\mathscr{X} = X) \\
&\quad \le \sum_{k \ge 0} (cK\delta)^{k+1} n^{-\lambda} \sum_{d_k \ge \cdots \ge d_1 \ge 1} \mathcal{Q}\big([d_1, \dots, d_k]\big) \prod_{i=1}^{k} d_i^{-(1+\lambda)} \\
&\quad = cK\delta n^{-\lambda} \sum_{k \ge 0} \Big(cK\delta \sum_{d \ge 1} d^{-(1+\lambda)}\Big)^k \\
&\quad \le 2cK\delta n^{-\lambda}.
\end{aligned}$$

□

5.3. *Proof of Proposition* 14. The following lemma formalizes the fact that passage points have six arms around them, possibly with a defect.

LEMMA 16. *Fix $n \in \mathbb{N}$*:

(i) *Let $u \in S_n$ and $r \le R$ such that $\Lambda_R(u) \subset S_n$. If ω and σ are configurations such that, when $\widetilde{\omega}^\sigma$ is defined as in* (3):
  (a) *$\Lambda_r(u)$ contains at least one passage point*,
  (b) *$\Lambda_R(u) \setminus \Lambda_r(u)$ contains no passage points.*
  Then $\omega \in \mathcal{A}_6^*(u; r, R)$.

(ii) *Let $u \in \mathbb{Z} \times \{0, n\}$ and $r \le R \le n/2$. If ω, σ are configurations with the properties* (a) *and* (b) *above, then $\omega \in \mathcal{A}_4^{hp*}(u; r, R)$.*

REMARK 17. In point (ii) above, when $u \in \mathbb{Z} \times \{n\}$, we write $\mathcal{A}_4^{hp*}(u; r, R)$ for the event that there exist four arms from $\partial\Lambda_r(u)$ to $\partial\Lambda_R(u)$ in the half plane

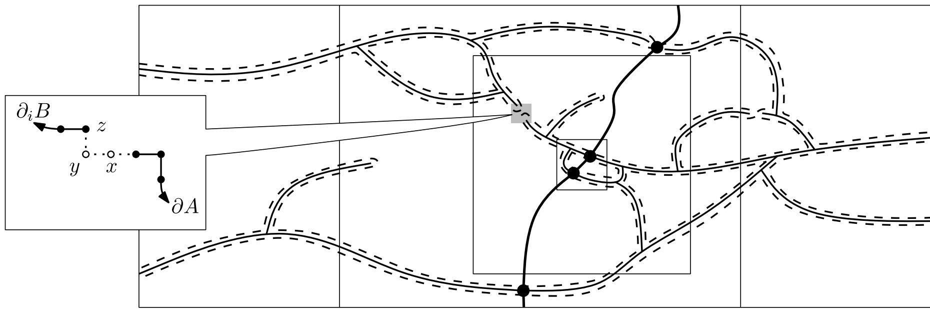

FIG. 4. *Two concentric balls $A \subset B$ with two passage points in A but no passage points in $B \setminus A$. Note the three arms on either side of γ. The gray square marks the defect on one of the open arms.*

below $\mathbb{R} \times \{n\}$. This is a slight abuse of notation that we will ask the reader to accept.

PROOF OF LEMMA 16. We start by giving a full proof of point (i); we will then sketch the proof of (ii), marking the differences with the previous point.

Let u, r, R be as in (i). For simplicity of notation we will write $A = \Lambda_r(u)$, $B = \Lambda_R(u)$ and $H = B \setminus A$. Then $A \subset B \subset S_n$ and A contains passage points, but H does not.

Since A contains passage points, γ intersects A. Thus we may find two disjoint sub-paths, γ_1 and γ_2, of γ, both contained in H, each connecting ∂A to $\partial_i B$ and such that γ contains at least one passage point between γ_1 and γ_2. Let $\overline{\gamma} = \gamma_1 \cup \gamma_2$. Then $\overline{\gamma}$ splits H into two disjoint regions, H^L and H^R; see Figure 4.

Since A contains passage points, there exists an ω-open path contained in χ, connecting ∂A to the left-hand side of S_n. This must contain a sub-path $\tau_1 \subset H$, connecting ∂A to $\partial_i B$. Since τ_1 is ω-open and $\widetilde{\omega}$-closed, it can only intersect γ at passage points. But, as part of H, τ_1 does not contain passage points, thus is fully contained in either H^L or H^R.

Assume $\tau_1 \subset H^R$. Then τ_1 separates H^R into two regions H^{R+} and H^{R-}. Let $\chi_R = \{x \in H : x \xleftrightarrow{\omega,H} \tau_1\}$ be the ω-open cluster of τ_1 in H. The points of χ_R and those of $\partial\chi_R \cap H$ are closed in $\widetilde{\omega}$ and are not passage points. Thus they are not part of γ. Hence $\partial\chi_R \cap H$ provides two paths τ_0 and τ_2 in $(\mathbb{Z}^2)^*$, contained in H^{R-} and H^{R+}, respectively, both closed in ω and connecting ∂A to $\partial_i B$.

We have found up to now three arms τ_0, τ_1, τ_2 in H^R, with states closed, open and closed, respectively, in ω. It is natural to expect the same structure in H^L. Some complications may arise though, hence the defect in the arm event.

Let $\chi_L = \{x \in \chi \cap H^L : x \xleftrightarrow{\omega,H} \partial A\}$. Also denote the *-cluster of ∂A in H^L in the configuration $\widetilde{\omega}$ by $\Delta = \{x \in H^L : x \overset{\widetilde{\omega};H^L}{\longleftrightarrow}{}^{*} \partial A\}$. Then $\chi_L \subset \Delta$. First we claim that Δ must intersect $\partial_i B$.

Indeed, if it does not, consider the set $\partial(\Delta \cup A) \cap H^L$. All sites of this set are $\widetilde{\omega}$-open. Moreover, this set contains an $\widetilde{\omega}^\sigma$-open path γ' joining γ_1 with γ_2. Then γ' contains no passage points, and this contradicts the choice of γ as having minimal number of passage points.

Let ρ be a path in Δ, connecting ∂A to $\partial_i B$. Let x be the last point of ρ (when going from ∂A toward $\partial_i B$) contained in $\chi_L \cup \partial\chi_L$. If $x \in \partial_i B$, then there exists a path τ_4 connecting ∂A to $\partial_i B$, contained in χ (hence ω-open), except possibly for its endpoint x. Two additional ω-closed arms τ_3 and τ_5 may be found in H^L as previously done in H^R.

Suppose $x \notin \partial_i B$. Then $x \in \partial\chi_L$, and let y be the next point visited by ρ. Since $y \in \Delta$, there exists $z \sim y$ (or $z = y$) which is open in ω but closed in $\widetilde{\omega}$. In particular, z is connected to the left side of S_n by a ω-open path τ (part of χ). By choice of $x, z \notin \chi_L$, thus τ does not intersect A. It does not intersect $\overline{\gamma}$ either, since the latter contains no passage points. Thus τ contains a sub-path in H^L, linking z to $\partial_i B$.

In conclusion there exists a path τ_4 linking ∂A to $\partial_i B$, contained in H^L and open in ω, with the possible exception of the sites x and y. With a possible modification of the configuration in $\Lambda_3(x)$, two ω-closed *-arms τ_3 and τ_5 may be found by inspecting the boundary of the ω-open cluster of τ_4 in H^L.

Since the arms τ_0, τ_1, τ_2 are contained in H^R and τ_3, τ_4, τ_5 are contained in H^L, they are necessarily disjoint. This completes the proof of (i).

For (ii) consider $u \in [-3n, 3n] \times \{0\}$ and $r \leq R \leq n/2$ such that $\Lambda_R(u) \setminus \Lambda_r(u)$ does not contain passage points, but $\Lambda_r(u)$ contains at least one. In particular $\Lambda_R(u)$ intersects R_n, and since $R \leq n/2$, $\Lambda_R(u) \cap \mathbb{H} \subset S_n$.

As before, write $A = \Lambda_r(u)$, $B = \Lambda_R(u)$ and $H = (B \setminus A) \cap \mathbb{H}$. In this case there exists a single sub-path $\overline{\gamma}$ of γ connecting ∂A to $\partial_i B$. Still $\overline{\gamma}$ splits H into disjoint regions H^L and H^R.

We may proceed as before in defining τ_1 and χ_R. The key difference with part (i) is that only one part of the boundary of χ_R is guaranteed to contain an ω-closed arm. Indeed, the part of the boundary above τ_1 contains a path τ_2, contained in H^R, closed in ω, and connecting ∂A to $\partial_i B$. The part below τ_1, however, can intersect the bottom of S_n very close to ∂A. The same type of phenomenon takes place in H^L. In conclusion we obtain four arms in the half plane with one possible defect. See also Figure 5. $\square$

We now turn to a consequence of Lemma 16 that will be used in the proof of Proposition 15. To state it we need some additional notation. Let $n \in \mathbb{N}$ and $A \subset B$ be two sets intersecting R_n. Let $r = \lceil \mathrm{diam}(A)/2 \rceil$. Then there exists a vertex u such that $A \subset \Lambda_r(u)$. If several such vertices exist, let u be the minimal one for the lexicographical order of $\mathbb{Z}^2$. If $\mathrm{dist}(u, \mathbb{R} \times \{0\}) \leq n/2$, let v be the projection of u onto $\mathbb{R} \times \{0\}$. Otherwise let v be the projection of u onto $\mathbb{R} \times \{n\}$.

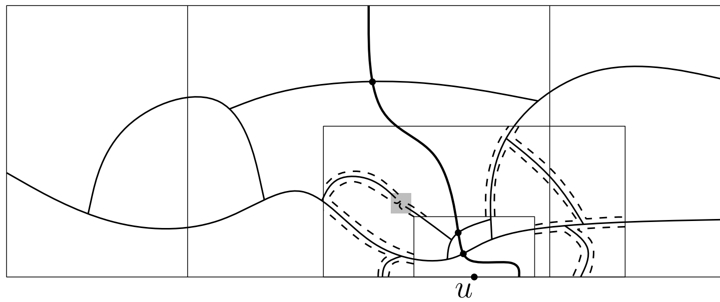

FIG. 5. *The intersection of $\Lambda_r(u)$ with $\mathbb{H}$ contains passage points, but $\Lambda_R(u) \setminus \Lambda_r(u)$ does not. Then there are two open arms on either sides of γ between $\partial\Lambda_r(u)$ and $\partial_i\Lambda_R(u)$. Above each open arm (but not necessarily also below) there is a closed arm.*

We define the following additional quantities:

$$R = \sup\{s \in \mathbb{N} : \Lambda_s(u) \subset B \cap S_n\} \vee r,$$

$$r' = \inf\{s \in \mathbb{N} : \Lambda_R(u) \subset \Lambda_s(v)\} \wedge n/2,$$

$$R' = (\sup\{s \in \mathbb{N} : \Lambda_s(v) \subset B\} \wedge n/2) \vee r'.$$

See Figure 6 for the meaning of u, v, r, R, r' and R'. Define the event

$$\mathcal{E}(A, B) = \mathcal{A}_6^*(u; r, R) \cap \mathcal{A}_4^{hp*}(v; r', R').$$

When $v \in \mathbb{R} \times \{n\}$, we use the notation $\mathcal{A}_4^{hp*}(v; r', R')$ as described in Remark 17.

REMARK 18. Henceforth we will write, for $n \leq N$, $\pi(n, N) = \pi(N/n) = c(N/n)^{-(2+\lambda)}$, where c and λ are given by Corollary 13. This is to emphasize that the computations may be carried through with different types of arm events with

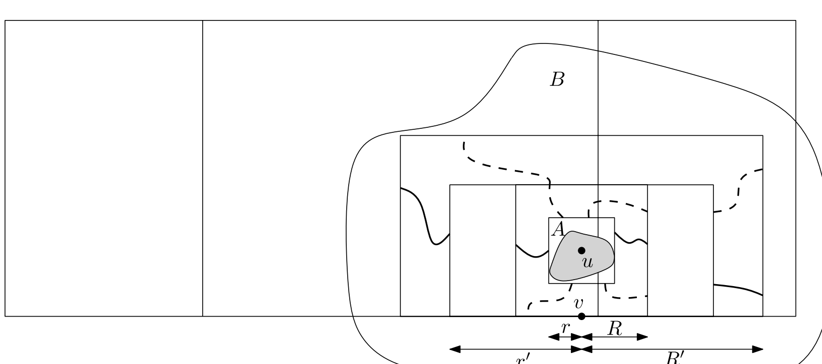

FIG. 6. *Two sets $A \subset B$ intersecting R_n. The six arms between $\partial\Lambda_r(u)$ and $\partial_i\Lambda_R(u)$ and the four arms in $\mathbb{H}$ between $\partial\Lambda_{r'}(u)$ and $\partial_i\Lambda_{R'}(u)$ ensure that $\mathcal{E}(A, B)$ occurs.*

power-law behavior. The quasi-multiplicativity property of probabilities of such events is essential. For π it states that there exist constants $c_1, c_2 > 0$ such that, for all $n \leq m \leq N$,

$$\pi(n, N) \leq \pi(n, m)\pi(m, N) \leq c_1\pi(n, N), \tag{17}$$

$$\pi(n, N) \leq c_2\pi(n, 2N). \tag{18}$$

LEMMA 19. (i) *Let $A \subset B$ be two sets of vertices of $\mathbb{Z}^2$. If ω, σ are such that A contains at least one passage point and $B \setminus A$ contains none, then $\omega \in \mathcal{E}(A, B)$.*

(ii) *There exists a constant $c > 0$ such that, for all $n \in \mathbb{N}$ and all sets $A \subset B$ intersecting R_n with* $\mathrm{diam}(B) \leq 6n$,

$$\mathbb{P}_{p_c}\big(\mathcal{E}(A, B)\big) \leq c\pi\big(\mathrm{diam}(A), \mathrm{diam}(A) + \mathrm{dist}\big(A, B^c\big)\big), \tag{19}$$

with $\pi(\cdot, \cdot)$ as in Remark 18.

PROOF. (i) Let $A \subset B$ and ω, σ be as in the lemma. With the notation in the definition of $\mathcal{E}(A, B)$, if $r < R$, then $A \subset \Lambda_r(u) \subset \Lambda_R(u) \subset B \cap S_n$. By Lemma 16(i), $\omega \in \mathcal{A}_6^*(u; r, R)$. If $r = R$, then $\mathcal{A}_6^*(u; r, R)$ is trivial.

As in the previous paragraph, if $r' = R'$, then $\mathcal{A}_4^{hp*}(v; r', R')$ is trivial. Suppose that $r' < R'$. Without loss of generality we may assume $v \in \mathbb{Z} \times \{0\}$. Then

$$A \cap \mathbb{H} \subset \Lambda_{r'}(v) \cap \mathbb{H} \subset \Lambda_{R'}(v) \cap \mathbb{H} \subset B \cap S_n.$$

By Lemma 16(ii), $\omega \in \mathcal{A}_4^{hp*}(v; r', R')$. In conclusion $\omega \in \mathcal{E}(A, B)$.

(ii) If $R \geq n/4$, we have $\mathrm{diam}(B) \leq 6n \leq 24R$. Then the first inequality of Corollary 13, the fact that $\mathcal{A}_6^*(u; r, R) \subset \mathcal{E}(A, B)$ and (17) yield (19) after some simple arithmetic manipulations. Thus we may restrict ourselves to $R < n/4$.

We distinguish two cases. First consider that $\Lambda_{R+1}(u)$ intersects B^c. Then $R = \mathrm{dist}(u, B^c) - 1$, and the first inequality of Corollary 13 yields (19) as above.

Suppose now that $\Lambda_{R+1}(u)$ does not intersect B^c. Then $\Lambda_R(u)$ necessarily intersects $\mathbb{R} \times \{0, n\}$. It follows that $r' = 2R < n/2$ and $v \in B$. By considering the cases $R' < n/2$ and $R' \geq n/2$ separately, we find

$$R' \geq \frac{1}{12}\mathrm{dist}\big(v, B^c\big) \geq \frac{1}{12}\big(\mathrm{dist}\big(A, B^c\big) - r'\big).$$

The second inequality of Corollary 13, equations (17) and (18) and the above imply

$$\mathbb{P}_{p_c}\big(\mathcal{A}_4^{hp*}(v; r', R')\big) \leq c'\pi\big(R, \mathrm{dist}\big(A, B^c\big)\big),$$

for some $c' > 0$. In addition, by the first inequality of Corollary 13, we have

$$\mathbb{P}_{p_c}\big(\mathcal{A}_6^*(u; r, R)\big) \leq c''\pi\big(\mathrm{diam}(A), R\big),$$

for some $c'' > 0$. Finally note that $\mathcal{A}_6^*(u; r, R)$ and $\mathcal{A}_4^{hp*}(v; r', R')$ depend on disjoint regions of the plane, hence

$$\begin{aligned}\mathbb{P}_{p_c}(\mathcal{E}(A, B)) &\le c'c''\pi(\mathrm{diam}(A), R)\pi(R, \mathrm{dist}(A, B^c)) \\ &\le c\pi(\mathrm{diam}(A), \mathrm{diam}(A) + \mathrm{dist}(A, B^c)),\end{aligned}$$

where $c > 0$ is obtained using again equations (17) and (18). □

Finally we are ready for the proof of Proposition 14.

PROOF OF PROPOSITION 14. Fix some nonempty set $X \subset R_n$ with $\#X = k+1$. Let $e_1, \dots, e_k$ be an ordering of the edges of $\mathcal{T}$ such that the sequence d_{e_i} is increasing.

For an edge e_i of $\mathcal{T}$, let C_i be the set of vertices of $\mathcal{T}$ connected to e_i via edges e_j with $j \le i$. Let $\mathcal{C} = \{C_i : i = 1, \dots, k\}$ and $\overline{\mathcal{C}} := \mathcal{C} \cup \{\{x\} : x \in X\}$. Inclusion provides a natural partial order of the elements of $\overline{\mathcal{C}}$. The singletons are the lowest elements; the maximal element is X.

For each $i = 1, \dots, k$, C_i is the union of two smaller disjoint elements of $\overline{\mathcal{C}}$, which we will call the offspring of C_i. If we write $e_i = (x, y)$, the offspring of C_i are the connected components of x and y, respectively, in the graph with vertices X and edges $\{e_1, \dots, e_{i-1}\}$.

Thus the elements of $\overline{\mathcal{C}}$ form a binary tree with the singletons of X as leaves. We will sometimes refer to $\overline{\mathcal{C}}$ itself as a tree. In the vision given in Section 5.1, $\overline{\mathcal{C}}$ is the coalescence tree of the blobs (at least when blobs merge only two at a time). Indeed, at time d_{e_i} two blobs merge and form a larger one, that contains the vertices of C_i. The two offspring of C_i correspond to the two merging blobs. If more than two blobs merge at the same time, we split this into sequential pairwise mergers.

For $U \in \mathcal{C}$ let

$$\begin{aligned}\mathrm{d}_U &= \left\lfloor \frac{1}{2} \max\{\mathrm{dist}(x, y) : x, y \in U \text{ and } (x, y) \in E(\mathcal{T})\} \right\rfloor, \\ \Delta_U &= \left\lfloor \frac{1}{2} \mathrm{diam}(U) \right\rfloor, \\ \Lambda_r(U) &= \bigcup_{u \in U} \Lambda_r(u),\end{aligned}$$

for $r \ge 0$. For $U = \{x\}$ a singleton, set $\mathrm{d}_U = \Delta_U = 0$ and $\Lambda_r(U) = \Lambda_r(x)$.

Consider ω and σ such that $\mathscr{X} = X$. For $U \in \mathcal{C}$, let $V = V(U)$ and $W = W(U)$ denote its offspring. The two regions $\Lambda_{\mathrm{d}_U}(V) \setminus \Lambda_{\mathrm{d}_V}(V)$ and $\Lambda_{\mathrm{d}_U}(W) \setminus \Lambda_{\mathrm{d}_W}(W)$ are disjoint and do not contain passage points. On the other hand both $\Lambda_{\mathrm{d}_V}(V)$ and $\Lambda_{\mathrm{d}_W}(W)$ contain passage points. Thus, by Lemma 19(i), the event

$$\mathcal{E}_U = \mathcal{E}(\Lambda_{\mathrm{d}_V}(V), \Lambda_{\mathrm{d}_U}(V)) \cap \mathcal{E}(\Lambda_{\mathrm{d}_W}(W), \Lambda_{\mathrm{d}_U}(W)),$$

must occur in ω. By Lemma 19(ii) there exists some constant $c > 0$ such that

$$\mathbb{P}_p(\mathcal{E}_U) \le c\pi(\Delta_V + \mathrm{d}_V, \Delta_V + \mathrm{d}_U)\pi(\Delta_W + \mathrm{d}_W, \Delta_W + \mathrm{d}_U). \tag{20}$$

To extend the definition of $\mathcal{E}_U$ to $U \in \overline{\mathcal{C}}$, define it as the full event (i.e., equal to Ω) when U is a singleton.

Since there are no passage points outside of $\Lambda_{\mathrm{d}_X}(X)$, we also have $\omega \in \mathcal{E}_{\text{out}} := \mathcal{E}(\Lambda_{\mathrm{d}_X}(X), \Lambda_{\mathrm{d}_X \vee n}(X))$. Finally all passage points need to be enhanced, hence $\sigma(x) = 1$ for all $x \in X$. Thus

$$\{\mathscr{X} = X\} \subset \left(\bigcap_{U \in \mathcal{C}} \{\omega \in \mathcal{E}_U\}\right) \cap \{\omega \in \mathcal{E}_{\text{out}}\} \cap \left(\bigcap_{x \in X} \{\sigma(x) = 1\}\right).$$

Note that the events $\mathcal{E}_U : U \in \mathcal{C}$ and the event $\mathcal{E}_{\text{out}}$ are defined on disjoint parts of the plane. Hence, by (20),

$$\begin{aligned}
\mathbb{P}_{p_c,\delta}(\mathscr{X} = X) &\le \left(\prod_{U \in \mathcal{C}} \mathbb{P}_{p_c,\delta}(\omega \in \mathcal{E}_U)\right) \times \mathbb{P}_{p_c,\delta}(\omega \in \mathcal{E}_{\text{out}}) \\
&\qquad \times \left(\prod_{x \in X} \mathbb{P}_{p_c,\delta}\big(\sigma(x) = 1\big)\right) \\
&\le (c\delta)^{k+1} \pi(\Delta_X + \mathrm{d}_X, \Delta_X + \mathrm{d}_X + n) \\
&\qquad \times \prod_{U \in \mathcal{C}} \pi(\Delta_V + \mathrm{d}_V, \Delta_V + \mathrm{d}_U)\pi(\Delta_W + \mathrm{d}_W, \Delta_W + \mathrm{d}_U).
\end{aligned} \tag{21}$$

In estimating the product above, we will use an induction on the binary tree $\overline{\mathcal{C}}$. For $Y \in \mathcal{C}$ let

$$\Phi(Y) = \prod_{U \in \mathcal{C}; U \subseteq Y} \pi(\Delta_V + \mathrm{d}_V, \Delta_V + \mathrm{d}_U)\pi(\Delta_W + \mathrm{d}_W, \Delta_W + \mathrm{d}_U),$$

and set $\Phi(Y) = 1$ when Y is a singleton.

Let us prove by induction on the tree that there exists $c_0 > 0$ such that, for all $Y \in \overline{\mathcal{C}}$,

$$\Phi(Y) \le \pi(\Delta_Y + \mathrm{d}_Y) \prod_{U \in \mathcal{C}; U \subseteq Y} c_0 \pi(\mathrm{d}_U). \tag{22}$$

When Y is a leaf of $\mathcal{C}$, that is, a singleton of X, then $\Phi(Y) = 1$, and (22) is trivially true for any $c_0 \ge 1$. Assume Y is an element of $\mathcal{C}$ with offspring Z_1, Z_2. We have

$$\Delta_Y \le \Delta_{Z_1} + \Delta_{Z_2} + \mathrm{d}_Y.$$

Thus for at least one $i \in \{1, 2\}$, $\Delta_Y + \mathrm{d}_Y \le 2(\Delta_{Z_i} + \mathrm{d}_Y)$. Assume it is the case for $i = 1$. Then

$$\begin{aligned}
\Phi(Y) &= \Phi(Z_1)\Phi(Z_2)\pi(\Delta_{Z_1} + \mathrm{d}_{Z_1}, \Delta_{Z_1} + \mathrm{d}_Y)\pi(\Delta_{Z_2} + \mathrm{d}_{Z_2}, \Delta_{Z_2} + \mathrm{d}_Y) \\
&\le \pi(\Delta_{Z_1} + \mathrm{d}_{Z_1})\pi(\Delta_{Z_1} + \mathrm{d}_{Z_1}, \Delta_{Z_1} + \mathrm{d}_Y)
\end{aligned}$$

$$\times \pi(\Delta_{Z_2} + \mathrm{d}_{Z_2})\pi(\Delta_{Z_2} + \mathrm{d}_{Z_2}, \Delta_{Z_2} + \mathrm{d}_Y) \prod_{U \in \mathcal{C}; U \subsetneq Y} c_0 \pi(\mathrm{d}_U)$$

$$\leq c_1^2 \pi(\Delta_{Z_1} + \mathrm{d}_Y)\pi(\Delta_{Z_2} + \mathrm{d}_Y) \prod_{U \in \mathcal{C}; U \subsetneq Y} c_0 \pi(\mathrm{d}_U) \tag{23}$$

$$\leq \frac{c_1^2 c_2}{c_0} \pi(\Delta_Y + \mathrm{d}_Y) \prod_{U \in \mathcal{C}; U \subseteq Y} c_0 \pi(\mathrm{d}_U). \tag{24}$$

In (23) we have used the quasi-multiplicativity property of π (17), hence the constant c_1. In (24) we have used that $\pi(\Delta_{Z_1} + \mathrm{d}_Y) \leq c_2 \pi(2(\Delta_{Z_1} + \mathrm{d}_Y)) \leq c_2 \pi(\Delta_Y + \mathrm{d}_Y)$, and $\pi(\Delta_{Z_2} + \mathrm{d}_Y) \leq \pi(\mathrm{d}_Y)$. The constant c_2 is given by (18). In conclusion, the recurrence holds, provided that $c_0 \geq c_1^2 c_2$.

Let us get back to bound (21). Using (22), we have

$$\begin{aligned}\mathbb{P}_{p_c,\delta}(\mathscr{X} = X) &\leq (c\delta)^{k+1} \pi(\Delta_X + \mathrm{d}_X, \Delta_X + \mathrm{d}_X + n)\pi(\Delta_X + \mathrm{d}_X) \prod_{U \in \mathcal{C}} c_0 \pi(\mathrm{d}_U) \\ &\leq c_1 (c c_0 \delta)^{k+1} \pi(n) \prod_{U \in \mathcal{C}} \pi(\mathrm{d}_U) \qquad \text{by (17).}\end{aligned}$$

This proves Proposition 14. □

5.4. *Proof of Proposition* 15. We begin with a lemma. The number of rooted trees with n vertices is less than that of rooted plane trees with n vertices (since these are rooted trees along with an ordering of the offspring of each vertex). Since the latter is well known to be the nth Catalan number (see, e.g., Theorem 3.2 of [4]), we find the following.

LEMMA 20. *The number of rooted trees on n vertices is less than*

$$c_n = \frac{1}{n+1}\binom{2n}{n} < 4^n,$$

where c_n is the nth Catalan number.

We turn to the proof of Proposition 15.

PROOF OF PROPOSITION 15. Fix $n \in \mathbb{N}$, $k \geq 0$, and let D be a multiset of k not necessarily distinct natural numbers.

Consider a rooted tree T with k edges. Let v_0 denote the root of T, and let $v_0, \dots, v_k$ denote a fixed depth-first ordering of the vertices of T when we start at v_0. For $i \geq 1$, let e_i be the edge linking v_i to $\{v_0, \dots, v_{i-1}\}$. In addition, associate to each edge e_i a number d_i such that $[d_1, \dots, d_k] = D$. Thus T is a rooted tree with decorated edges.

Let us bound the number of sets $X \subset R_n$ for which $\mathcal{T}(X)$ is isomorphic to T in the sense of rooted trees with decorated edges. [The decorations of $E(\mathcal{T}(X))$ are the merger times d_e defined in the beginning of Section 5.2.] We will do this by placing the points of X sequentially in R_n, and counting at every stage the number of possibilities.

Since $X \subset R_n$, there are at most $4n^2$ choices for the position of v_0, which we denote by x_0. Once x_0 is fixed, there are at most $8\mathrm{d}_1$ choices for x_1, the position of v_1. We continue in this fashion. For every choice of $x_0, \dots, x_{i-1}$, there are at most $8\mathrm{d}_i$ choices for x_i, the position of v_i. In conclusion there are at most $4n^2 \prod_{i=1}^{k} 8d_i$ sets of points $X \subset R_n$ with $\mathcal{T}(X)$ isomorphic to T in the sense of rooted decorated trees.

To compute the number of sets $X \subset R_n$ with $\mathcal{D}(X) = D$, we need to consider all possible values of T and all the different ways of assigning the decorations d_i to its edges. By Lemma 20 there are at most 4^k choices for T. The number of ways to assign the decorations is obviously bounded by $\mathcal{Q}(D)$. Proposition 15 follows with $K = 8 \cdot 4$. $\square$

Acknowledgments. The first two authors thank IMPA for its hospitality during August 2013, when this project took place. The project was initiated while the first author was working at Centrum Wiskunde & Informatica (CWI), Amsterdam. He is grateful to CWI for the hospitality and to NWO for the support in that period. He thanks ENS for its hospitality during a week long visit and Gábor Pete and Artem Sapozhnikov for fruitful discussions.

All three authors are grateful to Rob van den Berg for introducing them to the model and for comments on the earlier versions of the paper. They also thank Hugo Duminil-Copin for various comments and discussions.

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D. KISS
STATISTICAL LABORATORY
CENTRE FOR MATHEMATICAL SCIENCES
UNIVERSITY OF CAMBRIDGE
WILBERFORCE ROAD
CAMBRIDGE, CB3 0WB
UNITED KINGDOM
E-MAIL: d.kiss@statslab.cam.ac.uk

I. MANOLESCU
DÉPARTEMENT DE MATHÉMATIQUES
UNIVERSITÉ DE GENÈVE
2–4 RUE DU LIÈVRE, GENÈVE
SWITZERLAND
E-MAIL: ioan.manolescu@unige.ch

V. SIDORAVICIUS
IMPA
ESTRADA DONA CASTORINA 110
22460-320, RIO DE JANEIRO
BRAZIL
E-MAIL: vladas@impa.br