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EXIT LAWS FROM LARGE BALLS OF (AN)ISOTROPIC RANDOM WALKS IN RANDOM ENVIRONMENT[1]

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We study exit laws from large balls in $\mathbb{Z}^d$, $d \geq 3$, of random walks in an i.i.d. random environment that is a small perturbation of the environment corresponding to simple random walk. Under a centering condition on the measure governing the environment, we prove that the exit laws are close to those of a symmetric random walk, which we identify as a perturbed simple random walk. We obtain bounds on total variation distances as well as local results comparing exit probabilities on boundary segments. As an application, we prove transience of the random walks in random environment.

Our work includes the results on isotropic random walks in random environment of Bolthausen and Zeitouni [*Probab. Theory Related Fields* **138** (2007) 581–645]. Since several proofs in Bolthausen and Zeitouni (2007) were incomplete, a somewhat different approach was given in the first author's thesis [Long-time behavior of random walks in random environment (2013) Zürich Univ.]. Here, we extend this approach to certain anisotropic walks and provide a further step towards a fully perturbative theory of random walks in random environment.

CONTENTS

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1. Introduction and main results.

1.1. *The model and main results.*

1.1.1. *Our model of random walks in random environment.* Consider the integer lattice $\mathbb{Z}^d$ with unit vectors e_i, whose ith component equals 1. We let $\mathcal{P}$ be the set of probability distributions on $\{\pm e_i : i = 1, \dots, d\}$. Given a probability measure μ on $\mathcal{P}$, we equip $\Omega = \mathcal{P}^{\mathbb{Z}^d}$ with its natural product σ-field $\mathcal{F}$ and the product measure $\mathbb{P} = \mu^{\otimes \mathbb{Z}^d}$. Each element $\omega \in \Omega$ yields transition probabilities of a nearest neighbor Markov chain on $\mathbb{Z}^d$, the *random walk in random environment* (RWRE for short), via

$$p_\omega(x, x + e) = \omega_x(e), \qquad e \in \{\pm e_i : i = 1, \dots, d\}.$$

We write $\mathrm{P}_{x,\omega}$ for the "quenched" law of the canonical Markov chain $(X_n)_{n\geq 0}$ with these transition probabilities, starting at $x \in \mathbb{Z}^d$.

We study asymptotic properties of the RWRE in dimension $d \geq 3$ when the underlying environments are small perturbations of the fixed environment $\omega_x(\pm e_i) = 1/(2d)$ corresponding to simple random walk.

- Let $0 < \varepsilon < 1/(2d)$. We say that $\mathbf{A0}(\varepsilon)$ holds if $\mu(\mathcal{P}_\varepsilon) = 1$, where

$$\mathcal{P}_\varepsilon = \{q \in \mathcal{P} : |q(\pm e_i) - 1/(2d)| \leq \varepsilon \text{ for all } i = 1, \dots, d\}.$$

The perturbative behavior concerns the behavior of the RWRE when $\mathbf{A0}(\varepsilon)$ holds for small ε. However, even for arbitrarily small ε, such walks can behave in very different manners. This motivates a further "centering" restriction on μ.

- We say that **A1** holds if μ is invariant under reflection in the coordinate hyperplanes, that is, under all d reflections $O_i : \mathbb{R}^d \to \mathbb{R}^d$ with $O_i e_i = -e_i$ and $O_i e_j = e_j$ for $j \neq i$.

Condition **A1** is weaker than the isotropy condition introduced by Bricmont and Kupiainen [9], which requires that μ is invariant under all orthogonal transformations $O : \mathbb{R}^d \to \mathbb{R}^d$ fixing the lattice $\mathbb{Z}^d$. This stronger condition was also assumed in Bolthausen and Zeitouni [8], in the first author's thesis [1] and in a similar form in Sznitman and Zeitouni [23], who consider isotropic diffusions. Weaker than **A1** is the requirement that μ is invariant under $(\omega_0(e))_{|e|=1} \to (\omega_0(-e))_{|e|=1}$, which is used in Bolthausen, Sznitman and Zeitouni [7], (2.1).

1.1.2. *Our main results.* Write $V_L = \{y \in \mathbb{Z}^d : |y| \le L\}$ for the discrete ball of radius L. Given $\omega \in \Omega$, denote by $\Pi_L = \Pi_L(\omega)$ the exit distribution from V_L of the random walk with law $\mathrm{P}_{x,\omega}$, that is,

$$\Pi_L(x, z) = \mathrm{P}_{x,\omega}(X_{\tau_L} = z),$$

where $\tau_L = \inf\{n \ge 0 : X_n \notin V_L\}$. For probability measures ν_1 and ν_2, we let $\|\nu_1 - \nu_2\|_1$ be the total variation distance between ν_1 and ν_2. Denote by $\mathbb{E}$ the expectation with respect to $\mathbb{P}$, and let $p_o(\pm e_i) = p_o(x, x \pm e_i) = 1/(2d)$ be the transition kernel of simple random walk.

PROPOSITION 1.1. *Assume* **A1**. *There is $\varepsilon_0 > 0$ such that for $\varepsilon \le \varepsilon_0$, under* **A0**($\varepsilon$) *the limit*

$$2p_\infty(\pm e_i) = \lim_{L\to\infty} \sum_{y\in\mathbb{Z}^d} \mathbb{E}[\Pi_L(0, y)] \frac{y_i^2}{|y|^2}$$

exists for $i = 1, \dots, d$. Moreover, $\|p_\infty - p_o\|_1 \to 0$ as $\varepsilon \downarrow 0$.

From now on, p_∞ is always given by the limit above. The proposition suggests that for large radii L, the RWRE exit measure should be close to that of a symmetric random walk with transition kernel p_∞. Write $\pi_L^{(p)}(x, \cdot)$ for the exit distribution from V_L of a random walk with homogeneous nearest neighbor kernel p, started at $x \in \mathbb{Z}^d$. Recall that we assume $d \ge 3$.

THEOREM 1.1. *Assume* **A1**. *For $\delta > 0$ small enough, there exists $\varepsilon_0 = \varepsilon_0(\delta) > 0$ such that if* **A0**(ε) *is satisfied for some $\varepsilon \le \varepsilon_0$, then*

$$\mathbb{P}\Big(\sup_{x\in V_{L/5}} \big\|(\Pi_L - \pi_L^{(p_\infty)})(x, \cdot)\big\|_1 > \delta\Big) \le \exp(-(\log L)^2).$$

The difference in total variation of the exit laws of the RWRE and the random walk with kernel p_∞ does not tend to zero as $L \to \infty$, due to localized perturbations near the boundary. However, with an additional smoothing, convergence occurs. Let ρ be a random variable that is independent of the environment and has a smooth density compactly supported in $(1, 2)$. For $m > 0$ and $y \in \mathbb{Z}^d$, write

$$\Sigma_m^{(p)}(y, \cdot) = E\big[\pi_{\rho\cdot m}^{(p)}(0, \cdot - y)\big]$$

for the averaged exit distribution from balls $y + V_t$, $t \in (m, 2m)$, of a random walk with kernel p, where E is the expectation with respect to ρ.

THEOREM 1.2. *Assume* **A1**. *There exists $\varepsilon_0 > 0$ such that if* **A0**(ε) *is satisfied for some $\varepsilon \le \varepsilon_0$, then for any $\eta > 0$, we can find L_η and a smoothing radius m_η such that for $m \ge m_\eta$, $L \ge L_\eta$,*

$$\mathbb{P}\Big(\sup_{x\in V_{L/5}} \big\|\big(\Pi_L - \pi_L^{(p_\infty)}\big)\Sigma_m^{(p_\infty)}(x, \cdot)\big\|_1 > \eta\Big) \le \exp\big(-(\log L)^2\big).$$

REMARK 1.1. (i) As an easy consequence of the last theorem, if one increases the smoothing scale with L, that is, if $m = m_L \uparrow \infty$ (arbitrary slowly) as $L \to \infty$, then

$$\sup_{x\in V_{L/5}} \big\|\big(\Pi_L - \pi_L^{(p_\infty)}\big)\Sigma_m^{(p_\infty)}(x, \cdot)\big\|_1 \to 0 \qquad \mathbb{P}\text{-almost surely.}$$

(ii) The averaging over the radius ensures that the smoothing kernel is smooth enough: we have, uniformly in y, y', z and for some constant C depending only on the dimension,

$$\Sigma_m^{(p_\infty)}(y, z) \le C m^{-d},$$
$$\big|\Sigma_m^{(p_\infty)}(y, z) - \Sigma_m^{(p_\infty)}\big(y', z\big)\big| \le C\big|y - y'\big| m^{-(d+1)} \log m;$$

see Lemma A.2 (there, $\hat{\pi}_\psi^{(p)}$ with $\psi \equiv m$ takes the role of $\Sigma_m^{(p)}$). Theorem 1.2 does still hold if $\Sigma_m^{(p_\infty)}$ is replaced by another probability kernel sharing these properties. However, our particular choice of the smoothing kernel simplifies the presentation of the proof.

Our methods enable us to compare the exit measures in a more local way. Denote by $\partial V_L = \{y \in \mathbb{Z}^d : \mathrm{d}(y, V_L) = 1\}$ the outer boundary of V_L. For positive t and $z \in \partial V_L$ let $W_t(z) = V_t(z) \cap \partial V_L$, where $V_t(z) = z + V_t$. Then $|W_t(z)|$ is of order t^{d-1}. We obtain the following:

THEOREM 1.3. *Assume* **A1**. *There exist $\varepsilon_0 > 0$ and $L_0 > 0$ such that if* **A0**(ε) *is satisfied for some $\varepsilon \le \varepsilon_0$, then for $L \ge L_0$, there exists an event $A_L \in \mathcal{F}$ with $\mathbb{P}(A_L^c) \le \exp(-(1/2)(\log L)^2)$ such that on A_L, the following holds true. If $0 < \eta < 1$ and $x \in V_{\eta L}$, then for all $z \in \partial V_L$:*

(i) *For $t \geq L/(\log L)^{15}$, there exists $C = C(\eta)$ with*

$$\Pi_L(x, W_t(z)) \leq C\pi_L^{(p_o)}(x, W_t(z)).$$

(ii) *There exists a homogeneous symmetric nearest neighbor kernel p_L such that for $t \geq L/(\log L)^6$,*

$$\Pi_L(x, W_t(z)) = \pi_L^{(p_L)}(x, W_t(z))\big(1 + O\big((\log L)^{-5/2}\big)\big).$$

Here, the constant in the O-notation depends only on d and η.

We give one possible choice for the kernel p_L in (2.8). Our results can also be used to deduce transience of the RWRE.

COROLLARY 1.1. *Assume* **A1**. *There exist ε_0 such that if* **A0**(ε) *is satisfied for some $\varepsilon \leq \varepsilon_0$, then on $\mathbb{P}$-almost all $\omega \in \Omega$ the RWRE $(X_n)_{n\geq 0}$ is transient.*

REMARK 1.2. Let us mention the simplest nontrivial example of a RWRE under conditions **A0**(ε) and **A1**, with no isotropy. Fix a coordinate direction, say e_1, and define two symmetric kernels $q, q' \in \mathcal{P}_\varepsilon$ by setting

$$q(e) = \frac{1}{2d} + \begin{cases} \varepsilon, & \text{for } e = e_1, \\ -\varepsilon, & \text{for } e = -e_1, \\ 0, & \text{for } e \neq \pm e_1, \end{cases}$$

$$q'(e) = \frac{1}{2d} + \begin{cases} -\varepsilon, & \text{for } e = e_1, \\ \varepsilon, & \text{for } e = -e_1, \\ 0, & \text{for } e \neq \pm e_1. \end{cases}$$

Then the law μ on $\mathcal{P}_\varepsilon$ with $\mu(q) = \mu(q') = 1/2$ satisfies **A0**(ε) and **A1**. With this choice of μ, Corollary 1.1 settles the generalization of Problem 4 in Kalikow [12] to dimensions $d \geq 3$ (for small disorder).

1.2. *Discussion of this work.* This paper is inspired by the work of Bolthausen and Zeitouni [8]. There, Theorems 1.1 and 1.2 appeared in a similar form for the case of isotropic RWRE in dimension $d \geq 3$. A corrected and extended version of [8] forms part of the first author's thesis [1]. Our work should in turn be understood as an extension of [1] to the case of certain anisotropic random walks in random environment.

Here, the main difficulty stems from the fact that the kernel p_∞ is not explicitly computable and depends in a complicated way on μ. We will *not* first prove the existence of p_∞ and then deduce our results about the exit measures—in fact, it will be a side effect of our multiscale analysis of exit laws that p_∞ exists and is the right object of comparison. The idea of its construction starts with the observation that if the statements of Theorems 1.1 and 1.2 are true for *some* kernel p_∞, then the averaged exit distribution on a global scale will be the same as the exit distribution of the random walk with kernel p_∞, when $L \to \infty$. Therefore, it is natural to choose for any scale L a symmetric transition kernel p_L which has the property that the covariance matrix of the averaged exit distribution from V_L, scaled down by L^2, is the covariance matrix of p_L [in fact, we will choose p_L in a slightly different way; see (2.8) for the precise definition]. The difficult task is then to show that $p_\infty = \lim_{L\to\infty} p_L$ exists. In the isotropic case, this problem is absent since one can choose $p_L = p_o$ for every L.

The thesis [1] develops a somewhat new approach to the isotropic case covered in [8], which is, as we hope, easier to understand. Since we follow here the same strategy, let us explain the main changes compared to [8] and point at some of the problems which appeared there.

Our focus lies on (coarse grained) Green's function estimates on a large class of environments, so-called *goodified* environments. These concepts are developed in Section 5. In contrast with [8], we state our core Green's function estimates (Lemma 5.2, often used in the version of Lemma 5.5) in terms of an appropriate notion of domination of kernels, and also employ basic operations on kernels; see, for example, Propositions 5.3 and 5.4. While Lemma 5.2 requires some effort to be set up, it then yields in a relatively straight forward and systematic way controls on both smoothed and nonsmoothed estimates, see, for example, Lemma 6.5. In contrast, the goodified Green's function estimates in [8], namely, (4.24) and (4.25), are weaker than our Lemma 5.2. (We note in passing that fleshing out the missing details in the proof of [8], (4.24), without a version of Lemma 5.2 seems challenging; see, e.g., the end of Section 4.3, page 606 there.) Likewise, the lack in [8] of a statement like Lemma 5.2 makes the derivation of (4.46) there incomplete. The same issue arises, for dimensions $d = 3, 4$, in the derivation of (4.48) and (4.49) in [8].

With our Green's function estimates and the concept of goodified environments, we give proofs of the main results in our Sections 6.2 and 6.3, which differ even in the mere isotropic case in many details from the derivation in [8]. We believe that our proofs are more transparent. The reader who is primarily interested in the isotropic case is, however, advised to consult the thesis [1] first.

Finally, our Appendix includes the results in [8] on simple random walk and standard Brownian motion as special cases. We include the proofs both because our statements are more general, and also because the proofs of different cases are only sketched or altogether omitted in [8], for example, in the proof of Lemma 3.4 there; we also provide a lower bound on exit probabilities [Lemma 4.2(iii)] which is implicitly used in [8], but not proved there.

For a better reading, a rough overview over this paper is given in Section 2.4.

1.3. *Some relevant literature.* Let us comment on some further literature which is relevant for our study. For a detailed survey on RWRE, we refer to the lecture notes of Sznitman [20, 22] and Zeitouni [25, 26], and also to the overview article of Bogachev [5].

Assuming $\mathbf{A0}(\varepsilon)$ for small ε and the stronger isotropy condition that was mentioned at the beginning, Bricmont and Kupiainen [9] prove a (quenched) invariance principle, showing that in dimensions $d \geq 3$, the RWRE is asymptotically Gaussian, on $\mathbb{P}$-almost all environments. A continuous counterpart, isotropic diffusions in a random environment which are small perturbations of Brownian motion, has been investigated by Sznitman and Zeitouni in [23]. They prove transience and a full quenched invariance principle in dimensions $d \geq 3$.

Our centering condition **A1** excludes so-called ballistic behavior, that is, the regime where the limit velocity $v = \lim_{n\to\infty} X_n/n$ is an almost sure constant vector different from zero. Ballistic behavior has been studied extensively, for example, by Kalikow [12], Sznitman [18, 19, 21], Bolthausen and Sznitman [6], or more recently by Berger [2] and Berger, Drewitz and Ramírez [4].

In the perturbative regime when $d \geq 3$, Sznitman [21] shows that some strength of the mean local drift $m = \mathbb{E}[\sum_{|e|=1} e\omega_0(e)]$ is enough to deduce ballisticity. However, as examples in Bolthausen, Sznitman and Zeitouni [7] for dimensions $d \geq 7$ demonstrate, ballisticity can also occur with $m = 0$, and one can even construct examples exhibiting ballistic behavior with $v = -cm$ and $c > 0$. Note that our condition **A1** implies $m = v = 0$.

The work of Bolthausen, Sznitman and Zeitouni [7] provides also examples and results for nonballistic behavior. They consider the special class of multidimensional RWRE for which the projection onto at least $d_1 \geq 5$ components behaves as a standard random walk. In particular, if $d_1 \geq 7$ and the law of the environment is invariant under the antipodal transformation ([7], (2.1)), mentionned at the beginning, a quenched invariance principle is proved.

Much is also known for the class of *balanced* RWRE when $\mathbb{P}(\omega_0(e_i) = \omega_0(-e_i)) = 1$ for all $i = 1, \dots, d$. Employing the method of environment viewed from the particle, Lawler proves in [15] that for $\mathbb{P}$-almost all ω, $X_{\lfloor n\cdot\rfloor}/\sqrt{n}$ converges in $\mathrm{P}_{0,\omega}$-distribution to a nondegenerate Brownian motion with diagonal covariance matrix, even in the nonperturbative regime. Moreover, the RWRE is recurrent in dimension $d = 2$ and transient when $d \geq 3$; see [25]. Recently, within the i.i.d. setting, diffusive behavior has been shown in the mere elliptic case by Guo and Zeitouni [11] and in the nonelliptic case by Berger and Deuschel [3].

2. Basic notation and main techniques.

2.1. *Basic notation.* Our purpose here is to cover the most relevant notation which will be used throughout this text. Further notation will be introduced later on when needed.

2.1.1. *Sets and distances.* We let $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ and $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$. For a set A, its complement is denoted by A^c. If $A \subset \mathbb{R}^d$ is measurable and nondiscrete, we write $|A|$ for its d-dimensional Lebesgue measure. Sometimes,

$|A|$ denotes the surface measure instead, but this will be clear from the context. If $A \subset \mathbb{Z}^d$, then $|A|$ denotes its cardinality.

For $x \in \mathbb{R}^d$, $|x|$ is the Euclidean norm. If $A, B \subset \mathbb{R}^d$, we set $\mathrm{d}(A, B) = \inf\{|x - y| : x \in A, y \in B\}$ and $\mathrm{diam}(A) = \sup\{|x - y| : x, y \in A\}$. Given $L > 0$, let $V_L = \{x \in \mathbb{Z}^d : |x| \leq L\}$ and for $x \in \mathbb{Z}^d$, $V_L(x) = x + V_L$. For Euclidean balls in $\mathbb{R}^d$ we write $C_L = \{x \in \mathbb{R}^d : |x| < L\}$ and for $x \in \mathbb{R}^d$, $C_L(x) = x + C_L$.

If $V \subset \mathbb{Z}^d$, then $\partial V = \{x \in V^c \cap \mathbb{Z}^d : \mathrm{d}(\{x\}, V) = 1\}$ is the outer boundary, while in the case of a nondiscrete set $V \subset \mathbb{R}^d$, ∂V stands for the usual topological boundary of V and $\overline{V}$ for its closure. For $x \in \overline{C}_L$, we set $\mathrm{d}_L(x) = L - |x|$. Finally, for $0 \leq a < b \leq L$, the "shell" is defined by

$$\mathrm{Sh}_L(a, b) = \{x \in V_L : a \leq \mathrm{d}_L(x) < b\}, \qquad \mathrm{Sh}_L(b) = \mathrm{Sh}_L(0, b).$$

2.1.2. *Functions*. If a, b are two real numbers, we set $a \wedge b = \min\{a, b\}$, $a \vee b = \max\{a, b\}$. The largest integer not greater than a is denoted by $\lfloor a \rfloor$. As usual, set $1/0 = \infty$. For us, log is the logarithm to the base e, and $\log_a$ is then the logarithm to the base a. For $x, z \in \mathbb{R}^d$, the Delta function $\delta_x(z)$ is defined to be equal to one for $z = x$ and zero otherwise.

Given two functions $F, G : \mathbb{Z}^d \times \mathbb{Z}^d \to \mathbb{R}$, we write FG for the (matrix) product $FG(x, y) = \sum_{u \in \mathbb{Z}^d} F(x, u)G(u, y)$, provided the right-hand side is absolutely summable. F^k is the kth power defined in this way, and $F^0(x, y) = \delta_x(y)$. F can also operate on functions $f : \mathbb{Z}^d \to \mathbb{R}$ from the left via $Ff(x) = \sum_{y \in \mathbb{Z}^d} F(x, y) f(y)$.

We use the symbol 1_W for the indicator function of the set W. By an abuse of notation, 1_W will also denote the kernel $(x, y) \mapsto 1_W(x)\delta_x(y)$. If $f : \mathbb{Z}^d \to \mathbb{R}$, $\|f\|_1 = \sum_{x \in \mathbb{Z}^d} |f(x)| \in [0, \infty]$ is its L^1-norm. When $\nu : \mathbb{Z}^d \to \mathbb{R}$ is a (signed) measure, $\|\nu\|_1$ is its total variation norm.

Let $U \subset \mathbb{R}^d$ be a bounded open set, and let $k \in \mathbb{N}$. For a real-valued function f with $f|_U \in C^k(U)$, that is, f is k-times continuously differentiable in U, we define for $i = 0, 1, \ldots, k$,

$$\|D^i f\|_U = \sup_{|\beta| = i} \sup_U \left| \frac{\partial^i}{\partial x_1^{\beta_1} \cdots \partial x_d^{\beta_d}} f \right|,$$

where the first supremum is over all multi-indices $\beta = (\beta_1, \ldots, \beta_d)$, $\beta_j \in \mathbb{N}$, with $|\beta| = \sum_{j=1}^d \beta_j$. We also write ∇ for the gradient of a function, and in some proofs, Δ denotes the Laplace operator.

Let $L > 0$, and put $\mathcal{U}_L = \{x \in \mathbb{R}^d : L/2 < |x| < 2L\}$. We denote by $\mathcal{M}_L$ the set of functions ψ, whose restrictions to $\mathcal{U}_L$ satisfy the following properties:

- $\psi|_{\mathcal{U}_L} : \mathcal{U}_L \to (L/10, 5L)$,
- $\psi|_{\mathcal{U}_L} \in C^4(\mathcal{U}_L)$ with $\|D^i \psi|_{\mathcal{U}_L}\|_{\mathcal{U}_L} \leq 10$ for $i = 1, 2, 3, 4$.

Functions in $\mathcal{M}_L$ will be used to define smoothing kernels with good smoothing properties.

2.1.3. *Transition probabilities and exit distributions.* Given (not necessarily nearest neighbor) transition probabilities $p = (p(x, y))_{x,y\in\mathbb{Z}^d}$, we write $\mathrm{P}_{x,p}$ for the law of the canonical Markov chain $(X_n)_{n\geq 0}$ on $((\mathbb{Z}^d)^{\mathbb{N}}, \mathcal{G})$, $\mathcal{G}$ the σ-algebra generated by cylinder functions, with transition probabilities p and starting point $X_0 = x$ $\mathrm{P}_{x,p}$-a.s. The expectation with respect to $\mathrm{P}_{x,p}$ is denoted by $\mathrm{E}_{x,p}$. The simple random walk kernel $p_o(x, x \pm e_i) = 1/(2d)$ will play a prominent role. Clearly, every $p \in \mathcal{P}$ gives rise to a homogeneous nearest neighbor kernel, which by a small abuse of notation we again denote by p.

If $V \subset \mathbb{Z}^d$, we denote by $\tau_V = \inf\{n \geq 0 : X_n \notin V\}$ the first exit time from V, with $\inf \varnothing = \infty$, whereas $T_V = \tau_{V^c}$ is the first hitting time of V. Given $x, z \in \mathbb{Z}^d$ and p, V as above, we define

$$\mathrm{ex}_V(x, z; p) = \mathrm{P}_{x,p}(X_{\tau_V} = z).$$

Notice that for $x \in V^c$, $\mathrm{ex}_V(x, z; p) = \delta_x(z)$.

For $p \in \mathcal{P}$, we write

$$\pi_V^{(p)}(x, z) = \mathrm{ex}_V(x, z; p),$$

and for $\omega \in \Omega$, we set

$$\Pi_{V,\omega}(x, z) = \mathrm{ex}_V(x, z; p_\omega).$$

We usually suppress ω from the notation and simply write Π_V. Mostly, we shall interpret Π_V as a *random* exit distribution. However, sometimes we work with a fixed environment $\omega \in \Omega$, and then we still write Π_V instead of $\Pi_{V,\omega}$.

Recall the definitions of the sets $\mathcal{P}$ and $\mathcal{P}_\varepsilon$ from the Introduction. For $0 < \kappa < 1/(2d)$, let

$$\mathcal{P}_\kappa^{\mathrm{s}} = \{p \in \mathcal{P}_\kappa : p(e_i) = p(-e_i), i = 1, \dots, d\},$$

that is, $\mathcal{P}_\kappa^{\mathrm{s}}$ is the subset of $\mathcal{P}_\kappa$ which contains all symmetric probability distributions on $\{\pm e_i : i = 1, \dots, d\}$. At various places, the parameter κ bounds the range of the symmetric transition kernels we work with.

2.1.4. *Coarse grained transition kernels.* Fix once for all a probability density $\varphi \in C^\infty(\mathbb{R}_+, \mathbb{R}_+)$ with compact support in $(1, 2)$. Given a transition kernel $p \in \mathcal{P}$ and a strictly positive function $\psi = (m_x)_{x\in W}$, where $W \subset \mathbb{R}^d$ with $W \cap \mathbb{Z}^d \neq \varnothing$, we define the coarse grained transition kernels on $W \cap \mathbb{Z}^d$ associated to (ψ, p),

$$\hat{\pi}_\psi^{(p)}(x, \cdot) = \frac{1}{m_x}\int_{\mathbb{R}_+} \varphi\left(\frac{t}{m_x}\right) \mathrm{ex}_{V_t(x)}(x, \cdot; p)\,\mathrm{d}t, \qquad x \in W \cap \mathbb{Z}^d. \tag{2.1}$$

Mostly, we will take $\psi \in \mathcal{M}_L$, and then (2.1) yields a collection of transition kernels on at least $\mathcal{U}_L \cap \mathbb{Z}^d$. Often, we consider for $m > 0$ the constant function $\psi \equiv m$ (sometimes denoted ψ_m), and then (2.1) gives coarse grained transition kernels on the whole grid $\mathbb{Z}^d$.

2.1.5. *Coarse graining schemes in the ball.* Similarly to (2.1), we will now introduce coarse grained transition kernels for the motion inside the ball V_L, for both symmetric random walk and RWRE.

We use a particular function ψ. Once for all, let

$$s_L = \frac{L}{(\log L)^3} \quad \text{and} \quad r_L = \frac{L}{(\log L)^{15}}.$$

Our coarse graining schemes in the ball are indexed by a parameter r, which can either be a constant ≥ 100, but much smaller than r_L, or, in most of the cases, $r = r_L$. We fix a smooth function $h : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$h(x) = \begin{cases} x, & \text{for } x \leq 1/2, \\ 1, & \text{for } x \geq 2, \end{cases}$$

such that h is concave and increasing on $(1/2, 2)$. Define $h_{L,r} : \overline{C}_L \to \mathbb{R}_+$ by

$$h_{L,r}(x) = \frac{1}{20} \max\left\{ s_L h\left(\frac{\mathrm{d}_L(x)}{s_L}\right), r \right\}. \tag{2.2}$$

Since we mostly work with $r = r_L$, we use the abbreviation $h_L = h_{L,r_L}$. We write $\hat{\Pi}_{L,r} (= \hat{\Pi}_{L,r,\omega})$ for the coarse grained RWRE transition kernel inside V_L associated to $(\psi = (h_{L,r}(x))_{x \in V_L}, p_\omega)$,

$$\hat{\Pi}_{L,r}(x, \cdot) = \frac{1}{h_{L,r}(x)} \int_{\mathbb{R}_+} \varphi\left(\frac{t}{h_{L,r}(x)}\right) \Pi_{V_t(x) \cap V_L}(x, \cdot)\, \mathrm{d}t,$$

and $\hat{\pi}_{L,r}^{(p)}$ for the coarse grained kernel coming from symmetric random walk with transition kernel $p \in \mathcal{P}$, where in the definition Π is replaced by $\pi^{(p)}$. Most of the time we view $\hat{\Pi}_{L,r}$ as a random transition kernel, but we shall also write $\hat{\Pi}_{L,r}$ if the underlying environment ω is fixed. For convenience, we set $\hat{\Pi}_{L,r}(x, \cdot) = \hat{\pi}_{L,r}^{(p)}(x, \cdot) = \delta_x(\cdot)$ for $x \in \mathbb{Z}^d \setminus V_L$. By the strong Markov property, the exit measures from the ball V_L remain unchanged under these transition kernels, that is,

$$\mathrm{ex}_{V_L}(x, \cdot; \hat{\Pi}_{L,r}) = \Pi_L(x, \cdot) \quad \text{and} \quad \mathrm{ex}_{V_L}\big(x, \cdot; \hat{\pi}_{L,r}^{(p)}\big) = \pi_L^{(p)}(x, \cdot). \tag{2.3}$$

See Figure 1 for a visualization of the coarse graining scheme.

REMARK 2.1. (i) Later on, we will also work with slightly modified transition kernels $\breve{\Pi}$ and $\breve{\pi}^{(p)}$, which depend on the environment. We elaborate on this in Section 5.3.

(ii) Due to the lack of the last smoothing step outside V_L, we need to zoom in near the boundary in order to handle nonsmoothed exit distributions in Section 6.3. The parameter r allows us to adjust the step size in the boundary region.

(iii) For every choice of r,

$$h_{L,r}(x) = \begin{cases} \mathrm{d}_L(x)/20, & \text{for } x \in V_L \text{ with } r_L \leq \mathrm{d}_L(x) \leq s_L/2, \\ s_L/20, & \text{for } x \in V_L \text{ with } \mathrm{d}_L(x) \geq 2s_L. \end{cases}$$

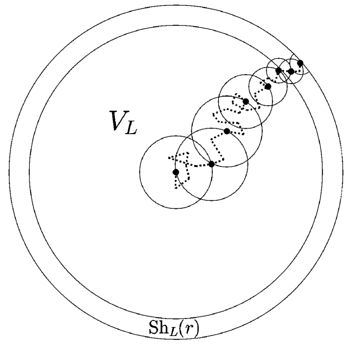

FIG. 1. *The coarse graining scheme in V_L. In the bulk $\{x \in V_L : \mathrm{d}_L(x) \geq 2s_L\}$, the exit distributions are taken from balls of radii between $(1/20)s_L$ and $(1/10)s_L$. When entering $\mathrm{Sh}_L(2s_L)$, the coarse graining radii start to shrink, up to the boundary layer $\mathrm{Sh}_L(r)$, where the exit distributions are taken from intersected balls $V_t(x) \cap V_L$, $t \in [(1/20)r, (1/10)r]$. The dotted lines indicate a corresponding random walk sample path.*

2.1.6. *Abbreviations.* If it is clear from the context which transition kernel p we are working with, we often drop the sub- or superscript p from notation. Then, for example, we write π_V for $\pi_V^{(p)}$, P_x instead of $\mathrm{P}_{x,p}$ or E_x for $\mathrm{E}_{x,p}$. Given transition probabilities p_ω coming from an environment ω, we use the notation $\mathrm{P}_{x,\omega}$, $\mathrm{E}_{x,\omega}$.

If $V = V_L$ is the ball around zero of radius L, we usually write π_L instead of π_V, Π_L for Π_V and τ_L for τ_V.

Many of our quantities, for example, the transition kernels $\hat{\Pi}_{L,r}$, $\hat{\pi}_{L,r}$ or the kernel $\Gamma_{L,r}$ which is introduced in Section 5, are indexed by both L and r. While we always keep the indices in the statements, we normally drop both of them in the proofs.

Finally, we will often use the abbreviations $\mathrm{d}(y, B)$ for $\mathrm{d}(\{y\}, B)$, T_x for $T_{\{x\}}$ and $\mathbb{P}(A; B)$ for $\mathbb{P}(A \cap B)$.

2.1.7. *Some words about constants, O-notation and large L behavior.* All our constants are positive. They only depend on the dimension $d \geq 3$ unless stated otherwise. In particular, constants do *not* depend on L, on δ, on ω or on any point $x \in \mathbb{Z}^d$, and they are also independent of the parameter r.

At some places, one might have the impression that constants depend on the transition kernel p. However, we only work with $p \in \mathcal{P}_\kappa^{\mathrm{s}}$, and κ can be chosen (arbitrarily) small. Such kernels p are therefore small perturbations of the simple random walk kernel p_o, and since all dependencies emerge in a continuous way (in p), we may always assume that constants are uniform in p.

We use C and c for generic positive constants whose values can change in different expressions, even in the same line. In the proofs, we often use other constants like K, C_1, c_1; their values are fixed throughout the proofs. Lower-case constants usually indicate small (positive) values.

Given two functions f, g defined on some subset of $\mathbb{R}$, we write $f(t) = O(g(t))$ if there exists a positive $C > 0$ and a real number t_0 such that $|f(t)| \le C|g(t)|$ for $t \ge t_0$.

If a statement holds for "L large (enough)," this means that there exists $L_0 > 0$ depending only on the dimension such that the statement is true for all $L \ge L_0$. This applies analogously to expressions like "δ (or ε, or κ) small (enough)."

One should always keep in mind that we are interested in asymptotics when $L \to \infty$ and the perturbation parameter ε is arbitrarily small, but fixed. Even though some of our statements are valid only for large L and ε (or δ, or κ) sufficiently small, we do not mention this every time.

2.2. *Perturbation expansion for Green's functions.* Our approach of comparing the RWRE exit distribution with that of an appropriate symmetric random walk is based on a perturbation argument. Namely, the resolvent equation allows us to express Green's functions of the RWRE in terms of Green's functions of homogeneous random walks. More generally, let $p = (p(x, y))_{x,y\in\mathbb{Z}^d}$ be a family of finite range transition probabilities on $\mathbb{Z}^d$, and let $V \subset \mathbb{Z}^d$ be a finite set. The corresponding Green's kernel or Green's function for V is defined by

$$g_V(p)(x, y) = \sum_{k=0}^{\infty} (1_V p)^k(x, y).$$

The connection with the exit measure is given by the fact that for $z \notin V$, we have

$$g_V(p)(\cdot, z) = \mathrm{ex}_V(\cdot, z; p). \tag{2.4}$$

Now write g for $g_V(p)$, and let P be another transition kernel with corresponding Green's function G for V. With $\Delta = 1_V(P - p)$, we have by the resolvent equation

$$G - g = g\Delta G = G\Delta g. \tag{2.5}$$

In order to get rid of G on the right-hand side, we iterate (2.5) and obtain

$$G - g = \sum_{k=1}^{\infty} (g\Delta)^k g, \tag{2.6}$$

provided the infinite series converges, which is always the case in our setting. Writing (2.6) as

$$G = g \sum_{k=0}^{\infty} (\Delta g)^k,$$

replacing the rightmost g by $g(x, \cdot) = \delta_x(\cdot) + 1_V pg(x, \cdot)$ and reordering terms, we get

$$G = g \sum_{m=0}^{\infty} (Rg)^m \sum_{k=0}^{\infty} \Delta^k, \tag{2.7}$$

where $R = \sum_{k=1}^{\infty} \Delta^k p$.

Two Green's functions for the ball V_L will play a particular role: the (coarse grained) RWRE Green's function $\hat{G}_{L,r}$ corresponding to $\hat{\Pi}_{L,r}$, and the Green's function $\hat{g}_{L,r}$ corresponding to $\hat{\pi}_{L,r}$,

$$\hat{G}_{L,r}(x,y) = \sum_{k=0}^{\infty} (1_{V_L} \hat{\Pi}_{L,r})^k(x,y), \qquad \hat{g}_{L,r}(x,y) = \sum_{k=0}^{\infty} (1_{V_L} \hat{\pi}_{L,r})^k(x,y).$$

A "goodified" version of $\hat{G}_{L,r}$ will be introduced in Section 3.

2.3. *Main technical statement*. We will deduce our main results from Proposition 2.1 below. The latter involves a technical condition, which we will propagate from one level to the next. This condition depends on the deviation δ (cf. Theorem 1.1) and on a parameter $L_0 \geq 3$ which will finally be chosen sufficiently large.

Recall the coarse graining schemes on V_L. Even though we use the "final" kernel p_∞ in the formulation of our main theorems, we will work in the proofs with intermediate kernels p_L depending on the radius of the ball. More precisely, we assign to each $L > 0$ the symmetric transition kernel ($i = 1, \dots, d$)

$$p_L(\pm e_i) = \begin{cases} 1/(2d), & \text{for } 0 < L \leq L_0, \\ \dfrac{1}{2} \displaystyle\sum_{y \in \mathbb{Z}^d} \mathbb{E}[\hat{\Pi}_{L,r}(0,y)] \frac{y_i^2}{|y|^2}, & \text{for } L > L_0. \end{cases} \tag{2.8}$$

Since $h_{L,r}(0) = s_L/20$, the definition of p_L does not depend on the parameter r. In words, for radii $0 < L \leq L_0$, p_L agrees with the simple random walk kernel p_o, while for $L > L_0$ the kernel p_L is defined as an average of variances of normalized mean exit distributions from balls of radii $t \in [(1/20)s_L, (1/10)s_L]$.

For $\psi \in \mathcal{M}_t$ and $p, q \in \mathcal{P}$, define

$$D^*_{t,p,\psi,q} = \sup_{x \in V_{t/5}} \big\| (\Pi_{V_t} - \pi^{(p)}_{V_t}) \hat{\pi}^{(q)}_{\psi}(x, \cdot) \big\|_1,$$

$$D^*_{t,p} = \sup_{x \in V_{t/5}} \big\| (\Pi_{V_t} - \pi^{(p)}_{V_t})(x, \cdot) \big\|_1.$$

With $\delta > 0$, we set for $i = 1, 2, 3$

$$\begin{aligned} &b_i(L, p, \psi, q, \delta) \\ &\quad = \mathbb{P}\big(\big\{(\log L)^{-9+9(i-1)/4} < D^*_{L,p,\psi,q} \leq (\log L)^{-9+9i/4}\big\} \cap \big\{D^*_{L,p} \leq \delta\big\}\big) \end{aligned}$$

and

$$b_4(L, p, \psi, q, \delta) = \mathbb{P}\big(\big\{D^*_{L,p,\psi,q} > (\log L)^{-3+3/4}\big\} \cup \big\{D^*_{L,p} > \delta\big\}\big).$$

Put $\iota = (\log L_0)^{-7}$, and let us now formulate the following:

2.3.1. *Condition* **Cond**. Let $\delta > 0$ and $L_1 \geq L_0 \geq 3$. We say that $\mathbf{Cond}(\delta, L_0, L_1)$ holds if:

- For all $3 \leq L \leq 2L_0$, all $\psi \in \mathcal{M}_L$ and all $q \in \mathcal{P}_\iota^{\mathrm{s}}$,

$$\mathbb{P}(\{D^*_{L,p_o,\psi,q} > (\log L)^{-9}\} \cup \{D^*_{L,p_o} > \delta\}) \leq \exp(-(\log(2L_0))^2).$$

- For all $L_0 < L \leq L_1$, $L' \in [L, 2L]$, $\psi \in \mathcal{M}_{L'}$ and $q \in \mathcal{P}_\iota^{\mathrm{s}}$,

$$b_i(L', p_L, \psi, q, \delta) \leq \tfrac{1}{4}\exp(-((3+i)/4)(\log L')^2) \qquad \text{for } i = 1, 2, 3, 4.$$

Let us summarize this condition in words.

The first point controls the total variation distance of the RWRE exit measure to the exit measure of simple random walk on balls of radii $3 \leq L \leq 2L_0$. Note that the bound on the probability is given in terms of L_0, for all such L.

The second point concerns radii $L_0 < L \leq L_1$ and gives control over the deviation of the RWRE exit measure from that of a symmetric random walk with kernel p_L. It also includes a continuity property of RWRE exit measures when L' varies (note that we use p_L on the left-hand side, not $p_{L'}$), which will be crucial to compare the distance between two kernels for different radii; see Lemma 3.2.

The main technical statement of this paper is the following:

PROPOSITION 2.1. *Assume* **A1**. *For $\delta > 0$ small enough, there exists $\varepsilon_0 = \varepsilon_0(\delta) > 0$ with the following property: if $\varepsilon \leq \varepsilon_0$ and* $\mathbf{A0}(\varepsilon)$ *holds, then:*

(i) *There exists $L_0 = L_0(\delta)$ such that for $L_1 \geq L_0$,*

$$\mathbf{Cond}(\delta, L_0, L_1) \Rightarrow \mathbf{Cond}(\delta, L_0, L_1(\log L_1)^2).$$

(ii) *There exist $L_0 = L_0(\delta)$ and sequences ℓ_n, $m_n \to \infty$ with the following property: if $L_1 \geq \ell_n$ and $L_1 \leq L \leq L_1(\log L_1)^2$, then for $q \in \mathcal{P}_\iota^{\mathrm{s}}$ and $m \geq m_n$, with $\psi \equiv m$,*

$$\mathbf{Cond}(\delta, L_0, L_1) \Rightarrow (\mathbb{P}(D^*_{L,p_L,\psi,q} > 1/n) \leq \exp(-(\log L)^2)).$$

REMARK 2.2. (i) It is important to realize that for every choice of δ and L_0, we can make sure that $\mathbf{Cond}(\delta, L_0, L_0)$ is fulfilled, simply by choosing the perturbation ε small enough. This observation provides us with the base step of the induction in Proposition 2.1(i): once we know that **Cond** propagates for properly chosen δ and L_0, we can choose ε so small such that $\mathbf{Cond}(\delta, L_0, L)$ holds for *all* $L \geq L_0$.

(ii) One should note that under $\mathbf{Cond}(\delta, L_0, L_1)$, if $L \leq L_1(\log L_1)^2$, then $h_{L,r}(x) \leq s_L \leq L_1/2$, so that $\mathbf{Cond}(\delta, L_0, L_1)$ can be used to control the exit distributions of the coarse grained walks inside V_L.

(iii) The number ι defined above the condition bounds the range of symmetric transition kernels q from which smoothing kernels $\hat{\pi}_\psi^{(q)}$ are built. In Lemma 3.2

we will see that under $\mathbf{Cond}(\delta, L_0, L_1)$, for $L \leq L_1(\log L_1)^2$, the kernels p_L are elements of $\mathcal{P}_\iota^{\mathrm{s}}$.

(iv) If $\mathbf{Cond}(\delta, L_0, L_1)$ is satisfied, then for any $3 \leq L \leq L_1$ and for all $L' \in [L, 2L]$, all $\psi \in \mathcal{M}_{L'}$ and all $q \in \mathcal{P}_\iota^{\mathrm{s}}$,

$$\mathbb{P}\big(\big\{D^*_{L',p_L,\psi,q} > (\log L')^{-9}\big\} \cup \big\{D^*_{L',p_L} > \delta\big\}\big) \leq \exp\big(-(\log L')^2\big).$$

For the rest of this paper, if we write "assume $\mathbf{Cond}(\delta, L_0, L_1)$," this means that we assume $\mathbf{Cond}(\delta, L_0, L_1)$ for some $\delta > 0$ and some $L_1 \geq L_0$, where δ can be chosen arbitrarily small and L_0 arbitrarily large.

2.4. *A short reading guide.* The key idea behind our proofs is to compare exit measures by means of the expansion

$$\Pi_L - \pi_L = \hat{G}_{L,r} 1_{V_L} (\hat{\Pi}_{L,r} - \hat{\pi}_{L,r}) \pi_L, \tag{2.9}$$

which results from (2.3), (2.4) and (2.5). Our coarse grained transition kernels are given by exit distributions from smaller balls inside V_L, and we obtain our results by transferring inductively information on smaller scales to the scale L. The notion of good and bad points, introduced in Section 3, allows us to classify the exit behavior on smaller scales. If inside V_L all points are good, then the estimates on smaller balls can be transferred to a (globally smoothed) estimate on the larger ball V_L (Lemma 6.4). But *bad* points can appear, and in fact we have to distinguish four different levels of badness (Section 3.3). When bad points are present, it is convenient to "goodify" the environment, that is, to replace bad points by good ones. This important concept is first explained in Section 3 and then further developed in Section 5.

However, for the globally smoothed estimate, we only have to deal with the case where all bad points are enclosed in a comparably small region; two or more such regions are too unlikely (Lemma 3.3). Some special care is required for the worst class of bad points in the interior of the ball. For environments containing such points, we slightly modify the coarse graining scheme inside V_L, as described in Section 5.3.

In Lemma 6.5, we prove the smoothed estimates on environments with bad points and show that the degree of badness decreases by one from one scale to the next.

For exit measures where no or only a local last smoothing step is added (Section 6.3, Lemmata 6.6 and 6.7, resp.), bad points near the boundary of V_L are much more delicate to handle, since we have to take into account several possibly bad regions. However, they do not occur too frequently (Lemma 3.4) and can be controlled by capacity arguments.

All these estimates require precise bounds on coarse grained Green's functions, which are developed in Section 5. Roughly speaking, we show that on environments with no bad points, the coarse grained RWRE Green's function for the ball is dominated from above by the analogous quantity coming from simple random walk (or some symmetric perturbation).

In Section 4, we present various bounds on hitting probabilities for both symmetric random walk and Brownian motion, and difference estimates of smoothed exit measures. One main difficulty is that we have to work with a whole family (p_L) of nearest neighbor transition kernels. For example, we have to control the total variation distance of exit measures corresponding to two different kernels. Here, the crucial statement is Lemma 4.4, which is formulated in terms of Brownian motion and then transferred to random walks via coupling arguments.

The statements from Section 6 are finally used in Section 7 to prove the main results. In the Appendix we prove the main statements from Section 4, as well as a local central limit theorem for the coarse grained symmetric random walk.

3. Transition kernels and notion of badness. Here, we look closer at the family of kernels defined in (2.8) and introduce the concept of "good" and "bad" points. Furthermore, we define "goodified" transition kernels and prove two estimates ensuring that we do not have to consider environments with bad points that are widely spread out in the ball or densely packed in the boundary region.

3.1. *Some properties of the kernels p_L.* The first general statement exemplifies how to extract information about a symmetric kernel $p \in \mathcal{P}_\kappa^{\mathrm{s}}$, $0 < \kappa < 1/(2d)$, out of the corresponding exit measure on ∂V_L.

LEMMA 3.1. *For $i = 1, \dots, d$,*

$$p(e_i) = \frac{1}{2} \sum_{y \in \partial V_L} \pi_L^{(p)}(0, y) \left(\frac{y_i}{L}\right)^2 + O(L^{-1}).$$

PROOF. Recall that under $\mathrm{P}_{0,p}$, $(X_n)_{n \geq 0}$ denotes the canonical random walk on $\mathbb{Z}^d$ with transition kernel p starting at the origin. Write $\mathcal{G}_m = \sigma(X_1, \dots, X_m)$ for the filtration up to time m, and denote by $X_{n,i}$ the ith component of X_n. Due to the symmetry of p, the process $X_{n,i}^2 - 2p(e_i)n$, $n \geq 0$, is a martingale with respect to $\mathcal{G}_n$. By the optional stopping theorem,

$$\mathrm{E}_{0,p}[X_{\tau_L,i}^2] = 2p(e_i)\mathrm{E}_{0,p}[\tau_L].$$

Since $X_{\tau_L,1}^2 + \cdots + X_{\tau_L,d}^2 = (L + O(1))^2$, it follows that $\mathrm{E}_{0,p}[\tau_L] = (L + O(1))^2$, and the claim is proved. $\square$

Now let us turn to the kernel p_L.

LEMMA 3.2. *Assume* $\mathbf{Cond}(\delta, L_0, L_1)$. *There exists a constant $C > 0$ such that*:

(i) *For* $3 \le L \le L_1(\log L_1)^2$,

$$\|p_{s_L/20} - p_L\|_1 \le C(\log L)^{-9}.$$

In particular, if L_0 *is sufficiently large, we have* $p_L \in \mathcal{P}^{\mathrm{s}}_{\iota}$.

(ii) *Let* $3 \le L \le L_1$ *and* $L' \in [L/2, L]$. *Then*

$$\|p_{L'} - p_L\|_1 \le C(\log L)^{-9}.$$

PROOF. (i) For $L \le L_0$, there is nothing to show since $p_L = p_o$. Now assume $L_0 < L \le L_1(\log L_1)^2$. We apply Lemma 3.1 to $t \in [s_L/20, s_L/10]$ in place of L. Writing p for $p_{s_L/20}$ and $\hat{\pi}$ for $\hat{\pi}_{L,r}$, we then obtain for each $i = 1, \dots, d$ after an integration

$$2p(e_i) = \sum_{y \in V_L} \hat{\pi}^{(p)}(0, y)\frac{y_i^2}{|y|^2} + O(s_L^{-1}). \tag{3.1}$$

Therefore, by the definition of p_L,

$$2|p(e_i) - p_L(e_i)| = \sum_y (\hat{\pi}^{(p)} - \mathbb{E}[\hat{\Pi}])(0, y)\frac{y_i^2}{|y|^2} + O(s_L^{-1}).$$

We now use the elementary fact that for centered random variables Y, Y', Z with Y, Y' independent of Z, we have $E[Y^2] - E[Y'^2] = E[(Y + Z)^2] - E[(Y' + Z)^2]$. Moreover, $\hat{\pi}^{(p)}(0, \cdot)$ and $\mathbb{E}[\hat{\Pi}](0, \cdot)$ have support in $V_{s_L/10}$, where the kernel $\hat{\pi}^{(p_o)}$ is homogeneous. We can therefore write

$$\sum_y (\hat{\pi}^{(p)} - \mathbb{E}[\hat{\Pi}])(0, y)\frac{y_i^2}{|y|^2} = \sum_y (\hat{\pi}^{(p)} - \mathbb{E}[\hat{\Pi}])\hat{\pi}^{(p_o)}(0, y)\frac{y_i^2}{|y|^2}.$$

We next note that by definition of the coarse-grained transition kernels, we have

$$\|(\mathbb{E}[\hat{\Pi}] - \hat{\pi}^{(p)})\hat{\pi}^{(p_o)}(0, \cdot)\|_1 \le \sup_{t \in [s_L/20, s_L/10]} \mathbb{E}[\|(\Pi_{V_t} - \pi^{(p)}_{V_t})\hat{\pi}^{(p_o)}(0, \cdot)\|_1].$$

We apply condition **Cond**(δ, L_0, L_1) in order to bound the right-hand side. Clearly, $s_L/10 \le L_1$. Moreover, the function $h_{L,r}$ defined in (2.2) lies in $\mathcal{M}_t$ for each $t \in [s_L/20, s_L/10]$. Recalling the last point of Remark 2.2, we obtain under **Cond**(δ, L_0, L_1) (with t in place of L' and $p = p_{s_L/20}$ in place of p_L in this remark)

$$\mathbb{E}[\|(\Pi_{V_t} - \pi^{(p)}_{V_t})\hat{\pi}^{(p_o)}(0, \cdot)\|_1] \le C(\log L)^{-9}$$

for some constant C which is uniform in $t \in [s_L/20, s_L/10]$. Putting the pieces together, we have shown that for each i, $|p(e_i) - p_L(e_i)| \le C(\log L)^{-9}$ and the first part of (i) follows.

In order to see that $p_L \in \mathcal{P}_\iota^{\mathrm{s}}$, we put $\ell_1 = L$, $\ell_{k+1} = s_{\ell_k}/20$ and then apply the above bound repeatedly to the differences $\|p_{\ell_k} - p_{\ell_{k+1}}\|_1$, until $\ell_{k+1} \le L_0$ and hence $p_{\ell_{k+1}} = p_o$. With $K = \lfloor \log_2(L/L_0) \rfloor$, we obtain the bound

$$\|p_L - p_o\|_1 \le C \sum_{i=1}^{K} \bigl(\log(2^{-i}L)\bigr)^{-9} \le C(\log L_0)^{-8},$$

which implies the second part of (i).

(ii) Let $\psi \equiv L \in \mathcal{M}_L$. By Lemma 3.1 and the same variance additivity property as in the proof of (i),

$$\begin{aligned}
\|p_{L'} - p_L\|_1 &= \frac{1}{L^2}\left| \sum_{y \in \partial V_L} \bigl(\pi_L^{(p_{L'})} - \pi_L^{(p_L)}\bigr)(0, y) y_i^2 \right| + O(L^{-1}) \\
&= \frac{1}{L^2}\left| \sum_{y \in \mathbb{Z}^d} \bigl(\pi_L^{(p_{L'})} - \pi_L^{(p_L)}\bigr)\hat{\pi}_\psi^{(p_o)}(0, y) y_i^2 \right| + O(L^{-1}) \\
&\le C \bigl\| \bigl(\pi_L^{(p_{L'})} - \pi_L^{(p_L)}\bigr)\hat{\pi}_\psi^{(p_o)}(0, \cdot)\bigr\|_1 + O(L^{-1}).
\end{aligned}$$

Since under $\mathbf{Cond}(\delta, L_0, L_1)$,

$$\begin{aligned}
\bigl\| \bigl(\pi_L^{(p_{L'})} - \pi_L^{(p_L)}\bigr)\hat{\pi}_\psi^{(p_o)}(0, \cdot)\bigr\|_1 &\le C\bigl(\mathbb{E}\bigl[D^*_{L,p_L,\psi,p_o}\bigr] + \mathbb{E}\bigl[D^*_{L,p_{L'},\psi,p_o}\bigr]\bigr) \\
&\le C(\log L)^{-9},
\end{aligned}$$

the second claim is proved. $\square$

3.2. *Good and bad points*. We shall partition the grid points inside V_L according to their influence on the exit behavior. Recall assignment (2.8), and fix an environment $\omega \in \Omega$. We say that a point $x \in V_L$ is *good* (with respect to ω, L, $\delta > 0$ and r, $100 \le r \le r_L$) if:

- For all $t \in [h_{L,r}(x), 2h_{L,r}(x)]$, with $q = p_{h_{L,r}(x)}$,
$$\bigl\| \bigl(\Pi_{V_t(x)} - \pi_{V_t(x)}^{(q)}\bigr)(x, \cdot)\bigr\|_1 \le \delta.$$
- If $\mathrm{d}_L(x) > 2r$, then additionally
$$\bigl\| \bigl(\hat{\Pi}_{L,r} - \hat{\pi}_{L,r}^{(q)}\bigr)\hat{\pi}_{L,r}^{(q)}(x, \cdot)\bigr\|_1 \le \bigl(\log h_{L,r}(x)\bigr)^{-9}.$$

A point $x \in V_L$ which is not good is called *bad*. We denote by $\mathcal{B}_{L,r} = \mathcal{B}_{L,r}(\omega)$ the set of all bad points inside V_L and write $\mathcal{B}_L = \mathcal{B}_{L,r_L}$ for short. Furthermore, set $\mathcal{B}_{L,r}^{\partial} = \mathcal{B}_{L,r} \cap \mathrm{Sh}_L(r_L)$ and $\mathcal{B}_{L,r}^{\star} = \mathcal{B}_{L,r} \cup \mathcal{B}_L = \mathcal{B}_{L,r}^{\partial} \cup \mathcal{B}_L$. Of course, the set of bad points depends also on δ, but we do not indicate this.

REMARK 3.1. (i) For the coarse graining scheme associated to $r = r_L$, we have by definition $\mathcal{B}_{L,r_L}^{\star} = \mathcal{B}_L$. When performing the nonsmoothed estimates in

Section 6.3, we work with constant r. In this case, $\mathcal{B}^{\star}_{L,r}$ can contain more points than $\mathcal{B}_L$.

(ii) Assume L large. If $x \in V_L$ with $\mathrm{d}_L(x) > 2r$, then the function $h_{L,r}(x + \cdot)$ lies in $\mathcal{M}_t$ for each $t \in [h_{L,r}(x), 2h_{L,r}(x)]$. Thus, for all $x \in V_L$, we can use **Cond**(δ, L_0, L_1) to control the event $\{x \in \mathcal{B}_{L,r}\}$, provided $2h_{L,r}(x) \le L_1$.

We shall replace the RWRE transition kernels at bad points by those of a symmetric random walk. Write p for $p_{s_L/20}$. For all environments, we introduce the "goodified" transition kernels as follows:

$$\hat{\Pi}^g_{L,r}(x, \cdot) = \begin{cases} \hat{\Pi}_{L,r}(x, \cdot), & \text{for } x \in V_L \setminus \mathcal{B}^{\star}_{L,r}, \\ \hat{\pi}^{(p)}_{L,r}(x, \cdot), & \text{for } x \in \mathcal{B}^{\star}_{L,r}. \end{cases} \tag{3.2}$$

We write $\hat{G}^g_{L,r}$ for the corresponding (random) Green's function. Note that the transition kernel q used in the definition of a good point $x \in V_L$ does depend on the location of x inside the ball, whereas the goodifying-procedure uses the same transition kernel p for all points [which agrees with q for $x \in V_L$ with $\mathrm{d}_L(x) \ge 2s_L$, since in this region $h_{L,r} \equiv (1/20)s_L$]. Goodified transition kernels and Green's functions will play a major role from Section 5 onwards.

3.3. *Bad regions in the case $r = r_L$.* The next lemma shows that with high probability, all bad points with respect to $r = r_L$ are contained in a ball of radius $4h_L(x)$. Let

$$\mathcal{D}_L = \{V_{4h_L(x)}(x) : x \in V_L\}.$$

We will look at the events $\mathrm{OneBad}_L = \{\mathcal{B}_L \subset D \text{ for some } D \in \mathcal{D}_L\}$, see Figure 2, and $\mathrm{ManyBad}_L = (\mathrm{OneBad}_L)^c$. It is also useful to define the set of *good* environments, $\mathrm{Good}_L = \{\mathcal{B}_L = \varnothing\} \subset \mathrm{OneBad}_L$.

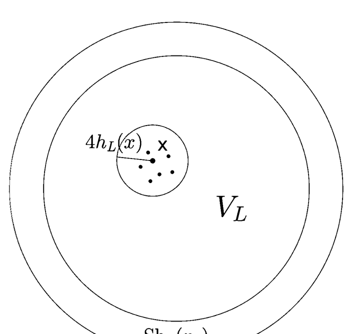

FIG. 2. *On environments $\omega \in \mathrm{OneBad}_L$, all bad points are enclosed in a ball $V_{4h_L(x)}(x)$.*

LEMMA 3.3. *Assume* $\mathbf{Cond}(\delta, L_0, L_1)$. *Then for* $L_1 \le L \le L_1(\log L_1)^2$,

$$\mathbb{P}(\mathrm{ManyBad}_L) \le \exp\bigl(-\tfrac{19}{10}(\log L)^2\bigr).$$

PROOF. Let $x \in V_L$ with $\mathrm{d}_L(x) > 2r_L$. Set $q = p_{h_L(x)}$ and $\Delta = 1_{V_L}(\hat{\Pi}_{L,r_L} - \hat{\pi}^{(q)}_{L,r_L})$. Put $D_{t,q,h_L,q}(x) = \|(\Pi_{V_t(x)} - \pi^{(q)}_{V_t(x)})\hat{\pi}^{(q)}_{h_L}(x,\cdot)\|_1$, $D_{t,q}(x) = \|(\Pi_{V_t(x)} - \pi^{(q)}_{V_t(x)})(x,\cdot)\|_1$. Using $r_L/20 \le h_L(x) \le s_L \le L_1/2$ and the second point of Remark 3.1,

$$\begin{aligned}\mathbb{P}(x \in \mathcal{B}_L) &\le \mathbb{P}\Bigl(\bigcup_{t\in[h_L(x),2h_L(x)]} \{D_{t,q,h_L,q}(x) > (\log h_L(x))^{-9}\} \cup \{D_{t,q}(x) > \delta\}\Bigr)\\ &\le C s_L^d \exp\bigl(-(\log(r_L/20))^2\bigr),\end{aligned}$$

and a similar estimate holds when $\mathrm{d}_L(x) \le 2r_L$. On the event $\mathrm{ManyBad}_L$, there exist $x, y \in \mathcal{B}_L$ with $|x - y| > 2h_L(x) + 2h_L(y)$. But for such x, y, the events $\{x \in \mathcal{B}_L\}$ and $\{y \in \mathcal{B}_L\}$ are independent, whence for L large

$$\begin{aligned}\mathbb{P}(\mathrm{ManyBad}_L) &\le C L^{2d} s_L^{2d}\bigl[\exp\bigl(-(\log(r_L/20))^2\bigr)\bigr]^2\\ &\le \exp\bigl(-(19/10)(\log L)^2\bigr).\end{aligned}$$

□

The estimate is good enough for our inductive procedure, so we only have to deal with the case where all possibly bad points are enclosed in a ball $D \in \mathcal{D}_L$. However, inside D we need to look closer at the degree of badness.

We say that $\omega \in \mathrm{OneBad}_L$ is *bad on level* i, $i = 1, 2, 3$, if the following holds:

- For all $x \in V_L$, for all $t \in [h_L(x), 2h_L(x)]$, with $q = p_{h_L(x)}$,
$$\bigl\|(\Pi_{V_t(x)} - \pi^{(q)}_{V_t(x)})(x,\cdot)\bigr\|_1 \le \delta.$$
- For all $x \in V_L$ with $\mathrm{d}_L(x) > 2r_L$, additionally
$$\bigl\|(\hat{\Pi}_{L,r_L} - \hat{\pi}^{(q)}_{L,r_L})\hat{\pi}^{(q)}_{L,r_L}(x,\cdot)\bigr\|_1 \le (\log h_L(x))^{-9+9i/4}.$$
- There exists $x \in \mathcal{B}_L(\omega)$ with $\mathrm{d}_L(x) > 2r_L$ such that
$$\bigl\|(\hat{\Pi}_{L,r_L} - \hat{\pi}^{(q)}_{L,r_L})\hat{\pi}^{(q)}_{L,r_L}(x,\cdot)\bigr\|_1 > (\log h_L(x))^{-9+9(i-1)/4}.$$

If $\omega \in \mathrm{OneBad}_L$ is neither bad on level $i = 1, 2, 3$ nor good, we call ω *bad on level* 4. In this case, $\mathcal{B}_L(\omega)$ contains "really bad" points. We write $\mathrm{OneBad}_L^{(i)} \subset \mathrm{OneBad}_L$ for the subset of all those ω which are bad on level $i = 1, 2, 3, 4$. Observe that we have the partition

$$\mathrm{OneBad}_L = \mathrm{Good}_L \cup \bigl(\mathrm{OneBad}_L^{(1)} \cup \cdots \cup \mathrm{OneBad}_L^{(4)}\bigr).$$

On Good_L, $\hat{\Pi}^g_{L,r_L} = \hat{\Pi}_{L,r_L}$ and therefore $\hat{G}^g_{L,r_L} = \hat{G}_{L,r_L}$.

3.4. *Bad regions when r is a constant.* When estimating nonsmoothed exit measures, we cannot stop the refinement of the coarse graining in the boundary region $\mathrm{Sh}_L(r_L)$. Instead, we will choose r as a (large) constant. However, now it is no longer true that essentially all bad points are contained in one single region $D \in \mathcal{D}_L$. For example, if $x \in V_L$ is such that $\mathrm{d}_L(x)$ is of order $\log L$, we only have a bound of the form

$$\mathbb{P}(x \in \mathcal{B}_{L,r}) \le \exp\bigl(-c(\log\log L)^2\bigr),$$

which is clearly not enough to get an estimate as in Lemma 3.3. We therefore choose a different strategy to handle bad points within $\mathrm{Sh}_L(r_L)$. We split the boundary region into layers of an appropriate size and use independence to show that with high probability, bad regions are rather sparse within those layers. Then the Green's function estimates of Corollary 5.1 will ensure that on such environments, there is a high chance to never hit points in $\mathcal{B}^{\partial}_{L,r}$ before leaving the ball.

To begin with the first part, fix r with $r \ge r_0 \ge 100$, where $r_0 = r_0(d)$ is a constant that will be chosen below. Let L be large enough such that $r < r_L$, and set $J_1 = J_1(L) = \lfloor \log_2(r_L/r) \rfloor + 1$. We define layers $\Lambda_0 = \mathrm{Sh}_L(2r)$ and $\Lambda_j = \mathrm{Sh}_L(r2^j, r2^{j+1})$ for integers $1 \le j \le J_1$. Then

$$\mathrm{Sh}_L(2r_L) \subset \bigcup_{0 \le j \le J_1} \Lambda_j \subset \mathrm{Sh}_L(4r_L).$$

Let $1 \le j \le J_1$. For $k \in \mathbb{Z}$, consider the interval $I_k^{(j)} = (kr2^j, (k+1)r2^j] \cap \mathbb{Z}$. We divide Λ_j into subsets by setting $D_{\mathbf{k}}^{(j)} = \Lambda_j \cap (I_{k_1} \times \cdots \times I_{k_d})$, where $\mathbf{k} = (k_1, \dots, k_d) \in \mathbb{Z}^d$, cf. Figure 3. Denote by $\mathcal{Q}_{j,r}$ the set of those subsets which are not empty. Setting $N_{j,r} = |\mathcal{Q}_{j,r}|$, it follows that

$$\frac{1}{C}\left(\frac{L}{r2^j}\right)^{d-1} \le N_{j,r} \le C\left(\frac{L}{r2^j}\right)^{d-1}.$$

We say that a set $D \in \mathcal{Q}_{j,r}$ is *bad* if $\mathcal{B}^{\partial}_{L,r} \cap D \ne \varnothing$. As we want to make use of independence, we partition $\mathcal{Q}_{j,r}$ into disjoint sets $\mathcal{Q}^{(1)}_{j,r}, \dots, \mathcal{Q}^{(R)}_{j,r}$, such that for each $1 \le m \le R$, we have:

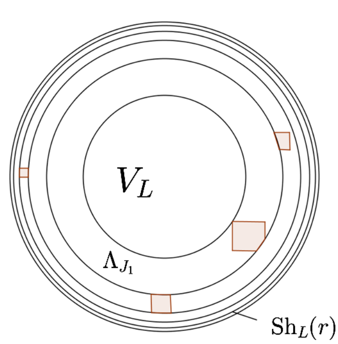

FIG. 3. *The layers Λ_j, $0 \le j \le J_1$, with $\Lambda_0 = \mathrm{Sh}_L(2r)$. Subsets $D_{\mathbf{k}}^{(j)} \subset \Lambda_j$ containing bad points are shaded.*

- $\mathrm{d}(D, D') > 4 \max_{x\in\Lambda_j} h_{L,r}(x)$ for all $D \neq D' \in \mathcal{Q}_{j,r}^{(m)}$,
- $N_{j,r}^{(m)} = |\mathcal{Q}_{j,r}^{(m)}| \geq \frac{N_{j,r}}{2R}$.

Note that since $h_{L,r}$ is proportional to d_L on Λ_j for $1 \leq j \leq J_1$, the number $R \in \mathbb{N}$ can be chosen to depend on the dimension only. Then the events $\{D \text{ is bad}\}$, $D \in \mathcal{Q}_{j,r}^{(m)}$, are independent. Furthermore, if $L_1 \leq L \leq L_1(\log L_1)^2$, it follows that under $\mathbf{Cond}(\delta, L_0, L_1)$,

$$\begin{aligned}\mathbb{P}(D \text{ is bad}) &\leq C\big(r2^j\big)^{2d} \exp\big(-\big(\log\big(r2^j/20\big)\big)^2\big)\\ &\leq \exp\big(-(\log r + j)^{5/3}\big) = p_{j,r},\end{aligned}$$

for all $r \geq r_0$ and $j \in \mathbb{N}$, if r_0 is big enough. Let $Y_{j,r}$ and $Y_{j,r}^{(m)}$ be the number of bad sets in $\mathcal{Q}_{j,r}$ and $\mathcal{Q}_{j,r}^{(m)}$, respectively. For $r \geq 5$, we have $p_{j,r} \leq (\log r + j)^{-3/2} \leq 1/2$. A standard large deviation estimate for Bernoulli random variables yields

$$\mathbb{P}\big(Y_{j,r}^{(m)} \geq (\log r + j)^{-3/2} N_{j,r}^{(m)}\big) \leq \exp\big(-N_{j,r}^{(m)} I\big((\log r + j)^{-3/2} | p_{j,r}\big)\big),$$

with $I(x|p) = x\log(x/p) + (1-x)\log((1-x)/(1-p))$. By enlarging r_0 if necessary, we get $I((\log r + j)^{-3/2}|p_{j,r}) \geq 2R(\log r + j)^{1/7}$ for $r \geq r_0$, whence

$$\begin{aligned}&\mathbb{P}\big(Y_{j,r} \geq (\log r + j)^{-3/2} N_{j,r}\big)\\ &\quad\leq R \max_{m=1,\dots,R} \mathbb{P}\big(Y_{j,r}^{(m)} \geq (\log r + j)^{-3/2} N_{j,r}^{(m)}\big)\\ &\quad\leq R \exp\big(-(\log r + j)^{1/7} N_{j,r}\big)\\ &\quad\leq R \exp\left(-\frac{1}{C}(\log r + j)^{1/7}\left(\frac{L}{r2^j}\right)^{d-1}\right)\\ &\quad\leq \exp\big(-(\log r + j)^{1/7}(\log L)^{29}\big),\end{aligned}$$

for $r_0 \leq r < r_L$, $0 \leq j \leq J_1(L)$ and L large enough. In particular,

$$\sum_{0\leq j\leq J_1(L)} \mathbb{P}\big(Y_{j,r} \geq (\log r + j)^{-3/2} N_{j,r}\big) \leq \exp\big(-(\log L)^{28}\big).$$

Therefore, introducing the set of environments with plenty of bad points in the boundary region,

$$\mathrm{BdBad}_{L,r} = \bigcup_{0\leq j\leq J_1(L)} \big\{Y_{j,r} \geq (\log r + j)^{-3/2} N_{j,r}\big\},$$

we have proved the following:

LEMMA 3.4. *There exists a constant $r_0 > 0$ such that if $r \geq r_0$, then* $\mathbf{Cond}(\delta, L_0, L_1)$ *implies that for $L_1 \leq L \leq L_1(\log L_1)^2$,*

$$\mathbb{P}(\mathrm{BdBad}_{L,r}) \leq \exp\big(-(\log L)^{28}\big).$$

4. Some important estimates. In this section, we collect estimates on symmetric random walks with kernel $p \in \mathcal{P}_\kappa^{\mathrm{s}}$ and on d-dimensional Brownian motion with (diagonal) covariance matrix given by

$$\Lambda_p = \big(2dp(e_i)\delta_{i,j}\big)_{i,j=1}^d. \tag{4.1}$$

We can safely use the same letter as for the layers defined in the foregoing section, since it will always be clear from the context what is meant. The following statements hold for small κ, meaning that there exists $0 < \kappa_0 < 1/(2d)$ such that for $0 < \kappa \le \kappa_0$, the statements hold true. All constants are then uniform in $p \in \mathcal{P}_\kappa^{\mathrm{s}}$.

4.1. *Hitting probabilities.* The first two lemmata concern symmetric random walk. The proofs are provided in the Appendix.

LEMMA 4.1. *Let $p \in \mathcal{P}_\kappa^{\mathrm{s}}$, and let $0 < \eta < 1$.*

(i) *There exists $C = C(\eta) > 0$ such that for all $x \in V_{\eta L}$, $z \in \partial V_L$,*

$$C^{-1}L^{-d+1} \le \pi_L^{(p)}(x,z) \le CL^{-d+1}.$$

(ii) *There exists $C = C(\eta) > 0$ such that for all $x, x' \in V_{\eta L}$, $z \in \partial V_L$,*

$$\big|\pi_L^{(p)}(x,z) - \pi_L^{(p)}(x',z)\big| \le C|x - x'|L^{-d}.$$

A good control over hitting probabilities is given by the following:

LEMMA 4.2. *Let $a \ge 1$ and $x, y \in \mathbb{Z}^d$ with $x \notin V_a(y)$. There exists a constant $C > 0$ such that for $p \in \mathcal{P}_\kappa^{\mathrm{s}}$,*

(i)

$$\mathrm{P}_{x,p}(T_{V_a(y)} < \infty) \le C\left(\frac{a}{|x-y|}\right)^{d-2}.$$

(ii) *There exists $C > 0$, independent of a, such that when $|x - y| > 7a$,*

$$\mathrm{P}_{x,p}(T_{V_a(y)} < \tau_L) \le C\frac{a^{d-2}\max\{a, \mathrm{d}_L(y)\}\max\{1, \mathrm{d}_L(x)\}}{|x-y|^d}.$$

(iii) *There exists $C > 0$ such that for all $x \in V_L$, $z \in \partial V_L$,*

$$C^{-1}\frac{\mathrm{d}_L(x)}{|x-z|^d} \le \pi_L^{(p)}(x,z) \le C\frac{\max\{1, \mathrm{d}_L(x)\}}{|x-z|^d}.$$

We need to compare exit laws of random walks with different kernels $p \in \mathcal{P}_\kappa^{\mathrm{s}}$, and we need difference estimates on smoothed exit measures. In this direction, it is easier to work with Brownian motion and then transfer the results back to the discrete setting. Let us first introduce some additional notation. Let $p \in \mathcal{P}_\kappa^{\mathrm{s}}$. For a domain $U \subset \mathbb{R}^d$ with smooth boundary and $x \in U$, denote by $\pi_U^{\mathrm{B}(p)}(x, dz)$ the exit measure from U of a d-dimensional Brownian motion W_t started at x, with diffusion (or covariance) matrix Λ_p defined in (4.1), that is, $\mathbb{E}[(W_1 - x)^2] = \Lambda_p$. In the case $U = C_L$, we simply write $\pi_L^{\mathrm{B}(p)}(x, dz)$. By a small abuse of notation, we also write $\pi_U^{\mathrm{B}(p)}(x, z)$ [or $\pi_L^{\mathrm{B}(p)}(x, z)$ if $U = C_L$] for the (continuous version of the) density with respect to surface measure on U.

In particular, $\pi_U^{\mathrm{B}(p_o)}$ is the exit measure from U of standard d-dimensional Brownian motion with covariance matrix I_d. Its density $\pi_L^{\mathrm{B}(p_o)}(x, z)$ is given by the Poisson kernel

$$\pi_L^{\mathrm{B}(p_o)}(x, z) = \frac{1}{d\alpha(d)L} \frac{L^2 - |x|^2}{|x - z|^d}, \tag{4.2}$$

where $\alpha(d)$ is the volume of the unit ball. For general $p \in \mathcal{P}_\kappa^{\mathrm{s}}$, there is no explicit expression for the kernel $\pi_L^{\mathrm{B}(p)}(x, z)$. However, we have the following:

LEMMA 4.3. *There exists $C > 0$ such that for $p \in \mathcal{P}_\kappa^{\mathrm{s}}$ and all $x \in C_L$, $z \in \partial C_L$,*

(i)

$$C^{-1} \frac{\mathrm{d}_L(x)}{|x - z|^d} \le \pi_L^{\mathrm{B}(p)}(x, z) \le C \frac{\mathrm{d}_L(x)}{|x - z|^d}.$$

(ii) *For $k \in \mathbb{N}$,*

$$C^{-1} \frac{\mathrm{d}_L(x)}{|x - z|^{d+k}} \le \nabla_x^k \pi_L^{\mathrm{B}(p)}(x, z) \le C \frac{\mathrm{d}_L(x)}{|x - z|^{d+k}}.$$

This lemma gives us immediately the statements corresponding to Lemma 4.1 for Brownian motion with covariance matrix Λ_p, $p \in \mathcal{P}_\kappa^{\mathrm{s}}$. Clearly, also Lemma 4.2 has a direct analog. In fact, part (iii) is reformulated for Brownian motion in the last lemma. For the results corresponding to (i) and (ii), one can follow the proof of Lemma 4.2 in the Appendix, replacing the random walk estimates by those for Brownian motion. These analogous results will be used in the Appendix.

The following important lemma controls the difference of two Brownian exit densities on ∂C_L, when the corresponding diffusion matrices are close together.

LEMMA 4.4. *There exists $C > 0$ such that for $p, q \in \mathcal{P}_\kappa^{\mathrm{s}}$, for all $x \in C_{(2/3)L}$, $z \in \partial C_L$,*

$$\left|\left(\pi_L^{\mathrm{B}(p)} - \pi_L^{\mathrm{B}(q)}\right)(x, z)\right| \le C \|q - p\|_1 L^{-(d-1)}.$$

The proof involves techniques from the theory of elliptic PDEs and is given in the Appendix, as well as the proof of the foregoing lemma. It should be pretty clear that differences of exit probabilities of symmetric random walks can be bounded in the same way, that is,

$$\left|\left(\pi_L^{(p)} - \pi_L^{(q)}\right)(x, z)\right| \leq C\|q - p\|_1 L^{-(d-1)}.$$

However, it seems more difficult to prove this, and in any case, we will only need a weaker form, which can be readily deduced from the last lemma and a coupling argument given in the Appendix.

LEMMA 4.5. *There exists $C > 0$ such that for $p, q \in \mathcal{P}_\kappa^{\mathrm{s}}$, for large L, $\psi \in \mathcal{M}_L$, any $x \in \mathcal{U}_L \cap \mathbb{Z}^d$ and any $z \in \mathbb{Z}^d$,*

$$\left|\left(\hat{\pi}_\psi^{(p)} - \hat{\pi}_\psi^{(q)}\right)(x, z)\right| \leq C\|q - p\|_1 L^{-d}.$$

Moreover, for $x \in V_L$ with $\mathrm{d}_L(x) > (1/10)r$,

$$\left\|\left(\hat{\pi}_{L,r}^{(p)} - \hat{\pi}_{L,r}^{(q)}\right)(x, \cdot)\right\|_1 \leq C \max\left\{h_{L,r}(x)^{-1/4}, \|q - p\|_1\right\}.$$

PROOF. By comparing $\hat{\pi}_\psi^{(p)}$ to the analogous Brownian quantity $\hat{\pi}_\psi^{\mathrm{B}(p)}$ defined in (A.7), the first claim follows from Lemma 4.4, Lemma A.2(vii) from the Appendix and the triangle inequality. The second statement is proved in the same way, with the choice $\psi(x) = h_{L,r}(x)$. The restriction to x with $\mathrm{d}_L(x) > (1/10)r$ ensures that all exit distributions are taken from balls that lie completely inside V_L. $\square$

Let us finish this part by proving the following useful estimate.

LEMMA 4.6. *Let $a > 0$, $\ell, m \geq 1$ and $x \in \mathbb{Z}^d$. Set $R_\ell = V_\ell \setminus V_{\ell-1}$, $\alpha = \max\{||x| - \ell|, a\}$. Then for some constant $C = C(m) > 0$*

$$\sum_{y \in R_\ell} \frac{1}{(a + |x - y|)^m} \leq C \begin{cases} \ell^{d-(m+1)}, & \text{for } 1 \leq m < d - 1, \\ \max\{\log(\ell/\alpha), 1\}, & \text{for } m = d - 1, \\ \alpha^{d-(m+1)}, & \text{for } m \geq d. \end{cases}$$

PROOF. If $\alpha > \ell$, then the left-hand side is bounded by

$$C\ell^{d-1}\alpha^{-m} \leq C \max\left\{\alpha^{d-(m+1)}, \ell^{d-(m+1)}\right\}.$$

If $\alpha \leq \ell$, we set $A_k = \{y \in R_\ell : |x - y| \in [(k-1)\alpha, k\alpha)\}$. Then, for all $k \geq 1$,

$$\max_{y \in A_k} \frac{1}{(a + |x - y|)^m} \leq 2^m k^{-m} \alpha^{-m}.$$

Since for $k\alpha \le \ell/10$, we have $|A_k| \le C\alpha(k\alpha)^{d-2}$, the claim then follows from

$$\sum_{y\in R_\ell} \frac{1}{(a+|x-y|)^m}$$
$$\le C\biggl(\sum_{1\le k\le \lfloor \ell/(10\alpha)\rfloor} \frac{\alpha(k\alpha)^{d-2}}{(k\alpha)^m}\biggr) + C\ell^{d-1}\ell^{-m}$$
$$\le C\alpha^{d-(m+1)} \sum_{1\le k\le \lfloor \ell/(10\alpha)\rfloor} k^{d-(m+2)} + C\ell^{d-(m+1)}. \qquad \square$$

4.2. *Smoothed exit measures*. In order to obtain difference estimates for smoothed exit distributions of a symmetric random walk, we will compare them to the corresponding quantities of Brownian motion.

Let $p, q \in \mathcal{P}_\kappa^{\mathrm{s}}$, and let $\psi = (m_x) \in \mathcal{M}_L$. The smoothed exit distribution from V_L of the random walk (with respect to p, q, ψ) is defined as

$$\phi_{L,p,\psi,q}(x,z) = \pi_L^{(p)}\hat{\pi}_\psi^{(q)}(x,z)$$
$$= \sum_{y\in\partial V_L} \pi_L^{(p)}(x,y)\frac{1}{m_y}\int_{\mathbb{R}_+} \varphi\biggl(\frac{t}{m_y}\biggr)\pi_{V_t(y)}^{(q)}(y,z)\,\mathrm{d}t.$$

For Brownian motion, the smoothing step is defined analogously to (2.1), namely,

$$\hat{\pi}_\psi^{\mathrm{B}(q)}(x,dz) = \frac{1}{m_x}\int_{\mathbb{R}_+} \varphi\biggl(\frac{t}{m_x}\biggr)\pi_{C_t(x)}^{\mathrm{B}(q)}(x,dz)\,\mathrm{d}t.$$

The smoothed exit distribution from C_L is then given by

$$\phi_{L,p,\psi,q}^{\mathrm{B}}(x,dz) = \pi_L^{\mathrm{B}(p)}\hat{\pi}_\psi^{\mathrm{B}(q)}(x,dz)$$
$$= \int_{\partial C_L} \pi_L^{\mathrm{B}(p)}(x,dy)\frac{1}{m_y}\int_{\mathbb{R}_+} \varphi\biggl(\frac{t}{m_y}\biggr)\pi_{C_t(y)}^{\mathrm{B}(q)}(y,dz)\,\mathrm{d}t.$$

By $\phi_{L,p,\psi,q}^{\mathrm{B}}(x,z)$ we denote the density of $\phi_{L,p,\psi,q}^{\mathrm{B}}(x,dz)$ with respect to d-dimensional Lebesgue measure. For the proof of the next lemma, we refer to the Appendix.

LEMMA 4.7. *There exists $C > 0$ such that for $p, q \in \mathcal{P}_\kappa^{\mathrm{s}}$ and $\psi \in \mathcal{M}_L$:*

(i)

$$\sup_{x\in V_L}\sup_{z\in\mathbb{Z}^d} \bigl|(\phi_{L,p,\psi,q} - \phi_{L,p,\psi,q}^{\mathrm{B}})(x,z)\bigr| \le CL^{-(d+1/4)},$$

(ii)

$$\sup_{z\in\mathbb{R}^d} \bigl\|D^i\phi_{L,p,\psi,q}^{\mathrm{B}}(\cdot,z)\bigr\|_{C_L} \le CL^{-(d+i)}, \qquad i = 0,1,2,3,$$

(iii)

$$\sup_{x,x'\in V_L\cup\partial V_L}\sup_{z\in\mathbb{Z}^d}\left|\phi_{L,p,\psi,q}(x,z)-\phi_{L,p,\psi,q}(x',z)\right|$$
$$\le C\left(L^{-(d+1/4)}+\left|x-x'\right|L^{-(d+1)}\right).$$

The next proposition will be applied at the end of the proof of Lemma 6.4. At this point, the symmetry condition **A1** comes into play. We give a general formulation in terms of a signed measure ν. Let us introduce the following notation. For $x=(x_1,\dots,x_d)\in\mathbb{Z}^d$, $i=1,\dots,d$, put

$$x^{(i)}=(x_1,\dots,x_{i-1},-x_i,x_{i+1},\dots,x_d).$$

PROPOSITION 4.1. *Let $p,q\in\mathcal{P}_\kappa^{\mathrm{s}}$ and $\ell>0$. Consider a measure ν on V_ℓ with total mass zero satisfying $\nu(x)=\nu(x^{(i)})$ for all x and all $i=1,\dots,d$. Then there is a constant $C>0$ such that for $y'\in V_L$ with $V_\ell(y')\subset V_L$ and all $z\in\mathbb{Z}^d$, $\psi\in\mathcal{M}_L$,*

$$\left|\sum_{y\in V_\ell(y')}\nu(y-y')\phi_{L,p,\psi,q}(y,z)\right|\le C\|\nu\|_1\left(L^{-(d+1/4)}+\left(\frac{\ell}{L}\right)^2L^{-d}\right).$$

PROOF. We simply write ϕ for $\phi_{L,p,\psi,q}$ and ϕ^{B} for $\phi^{\mathrm{B}}_{L,p,\psi,q}$. Since the proof is the same for all $y'\in V_L$ with $V_\ell(y')\subset V_L$, we can assume $y'=0$. By Lemma 4.7(i),

$$\left|\sum_y\nu(y)\phi(y,z)-\sum_y\nu(y)\phi^{\mathrm{B}}(y,z)\right|\le C\|\nu\|_1L^{-(d+1/4)}.$$

Taylor's expansion gives

$$\begin{aligned}&\sum_y\nu(y)\phi^{\mathrm{B}}(y,z)\\ &\quad=\sum_y\nu(y)\left[\phi^{\mathrm{B}}(y,z)-\phi^{\mathrm{B}}(0,z)\right]\\ &\quad=\sum_y\nu(y)\nabla_x\phi^{\mathrm{B}}(0,z)\cdot y+\frac{1}{2}\sum_y\nu(y)y\cdot H_x\phi^{\mathrm{B}}(0,z)y+R(\nu,0,z),\end{aligned}\tag{4.3}$$

where $\nabla_x\phi^{\mathrm{B}}$ is the gradient, $H_x\phi^{\mathrm{B}}$ the Hessian of ϕ^{B} with respect to the first variable and $R(\nu,0,z)$ is the remainder term. Due to the symmetry condition on ν, the first summand on the right-hand side of (4.3) vanishes, and for the second and third summand one can use Lemma 4.7(ii). □

REMARK 4.1. In [8] and [1], it is assumed that μ, the measure governing the environment, is isotropic. This leads us to consider a measure ν that is invariant not only under $x\mapsto x^{(i)}$, but also under $x\mapsto x^{\leftrightarrow(i,j)}$, where for $i<j$,

$$x^{\leftrightarrow(i,j)}=(x_1,\dots,x_{i-1},x_j,x_{i+1},\dots,x_{j-1},x_i,x_{j+1},\dots,x_d).$$

In this case, the choice $p = p_o$ results in the sharper bound

$$\left|\sum_{y\in V_\ell(y')} \nu(y-y')\phi_{L,p_o,\psi,q}(y,z)\right| \le C\|\nu\|_1\left(L^{-(d+1/4)} + \left(\frac{\ell}{L}\right)^3 L^{-d}\right);$$

see Proposition 3.1 in [1]. It is then clear from the proof of Lemma 6.4 that one can work with $p_L = p_o$ for *all* radii L, that is, the (isotropic) RWRE exit measure approaches that of simple random walk.

5. Green's functions for the ball. One main task of our approach aims at developing good estimates on Green's functions for the ball of both coarse grained (goodified) RWRE as well as coarse grained symmetric random walk in the perturbative regime. The main result is Lemma 5.2. For the coarse grained symmetric random walk, the estimates on hitting probabilities of the last section together with Proposition 5.2 yield the right control.

On a certain class of environments, we need to modify the transition kernels in order to ensure that bad points are not visited too often by the coarse grained random walks. This modification will be described in Section 5.3.

We work with the same convention concerning the parameter κ as in Section 4.

5.1. *A local central limit theorem.* Let $p \in \mathcal{P}_\kappa^{\mathrm{s}}$ and $m \ge 1$. Denote by $\hat{\pi}_{\psi_m} = \hat{\pi}_{\psi_m}^{(p)}$ the coarse grained transition probabilities on $\mathbb{Z}^d$ associated to the constant function $\psi_m \equiv m$; cf. (2.1). We constantly drop p from notation. Notice that $\hat{\pi}_{\psi_m}$ is centered, and the covariances satisfy

$$\sum_{y\in\mathbb{Z}^d}(y_i - x_i)(y_j - x_j)\hat{\pi}_{\psi_m}(x,y) = \lambda_{m,i}\delta_i(j),$$

where for large m, $C^{-1} < \lambda_{m,i}/m^2 < C$ for some $C > 0$. Define the matrix

$$\Lambda_m = \left(\lambda_{m,i}\delta_i(j)\right)_{i,j=1}^d,$$

and let for $x \in \mathbb{Z}^d$

$$\mathcal{J}_m(x) = \left|\Lambda_m^{-1/2}x\right|.$$

PROPOSITION 5.1 (Local central limit theorem). *Let $p \in \mathcal{P}_\kappa^{\mathrm{s}}$, and let $x, y \in \mathbb{Z}^d$. For $m \ge 1$ and all integers $n \ge 1$,*

$$(\hat{\pi}_{\psi_m})^n(x,y) = \frac{1}{(2\pi n)^{d/2}\det\Lambda_m^{1/2}}\exp\left(-\frac{\mathcal{J}_m^2(x-y)}{2n}\right) + O\left(m^{-d}n^{-(d+2)/2}\right).$$

For the corresponding Green's function $\hat{g}_{m,\mathbb{Z}^d}(x,y) = \sum_{n=0}^{\infty}(\hat{\pi}_{\psi_m})^n(x,y)$ we obtain the following:

PROPOSITION 5.2. *Let $p \in \mathcal{P}_\kappa^{\mathrm{s}}$. Let $x, y \in \mathbb{Z}^d$, and assume $m \geq m_0 > 0$ large enough.*

(i) *For $|x-y| < 3m$,*

$$\hat{g}_{m,\mathbb{Z}^d}(x,y) = \delta_x(y) + O(m^{-d}).$$

(ii) *For $|x-y| \geq 3m$, there exists a constant $c(d) > 0$ such that*

$$\hat{g}_{m,\mathbb{Z}^d}(x,y) = \frac{c(d)\det\Lambda_m^{-1/2}}{\mathcal{J}_m(x-y)^{d-2}} + O\left(\frac{1}{|x-y|^d}\left(\log\frac{|x-y|}{m}\right)^d\right).$$

Note that the constants in the O-notation are independent of n, m and $|x-y|$.

In our applications, m will be a function of L. Although these results look rather standard, we cannot directly refer to the literature because we have to keep track of the m-dependency. We give a proof of both statements in the Appendix. The last proposition will be used to estimate the Green's function for the ball V_L, $\hat{g}_{m,V_L}(x,y) = \sum_{n=0}^\infty (1_{V_L}\hat{\pi}_{\psi_m})^n(x,y)$. Clearly, $\hat{g}_{m,V_L}$ is bounded from above by $\hat{g}_{m,\mathbb{Z}^d}$.

5.2. *Estimates on coarse grained Green's functions*. As we will show, the perturbation expansion enables us to control the goodified Green's function $\hat{G}^g_{L,r}$ essentially in terms of $\hat{g}^{(p)}_{L,r}$, where p is the kernel corresponding to the radius $s_L/20$, stemming from assignment (2.8).

The first step in controlling the Green's function is provided by the following lemma.

LEMMA 5.1. *Assume* $\mathbf{Cond}(\delta, L_0, L_1)$, *let* $L_1 \leq L \leq L_1(\log L_1)^2$, *and put* $p = p_{s_L/20}$. *Then for all $x \in V_L \setminus \mathrm{Sh}_L(2r)$, with $H(x) = \max\{L_0, h_{L,r}(x)\}$,*

$$\begin{aligned}&\|(\hat{\Pi}^g_{L,r} - \hat{\pi}^{(p)}_{L,r})\hat{\pi}^{(p)}_{L,r}(x,\cdot)\|_1\\&\qquad\leq C\min\{\log(s_L/H(x))(\log H(x))^{-9}, (\log H(x))^{-8}\}\end{aligned}$$

and

$$\|(\hat{\Pi}^g_{L,r} - \hat{\pi}^{(p)}_{L,r})(x,\cdot)\|_1 \leq 2\delta.$$

PROOF. For $x \in \mathcal{B}^\star_{L,r}$, both left-hand sides are zero. Now let $x \in V_L \setminus (\mathrm{Sh}_L(2r) \cup \mathcal{B}^\star_{L,r})$ and set $q = p_{h_{L,r}(x)}$. By the triangle inequality,

$$\begin{aligned}&\|(\hat{\Pi}^g - \hat{\pi}^{(p)})\hat{\pi}^{(p)}(x,\cdot)\|_1\\&\quad\leq \|(\hat{\Pi}^g - \hat{\pi}^{(q)})\hat{\pi}^{(p)}(x,\cdot)\|_1 + \|(\hat{\pi}^{(p)} - \hat{\pi}^{(q)})(x,\cdot)\|_1\\&\quad\leq \|(\hat{\Pi}^g - \hat{\pi}^{(q)})\hat{\pi}^{(q)}(x,\cdot)\|_1 + 2\sup_{y\in V_L\setminus\mathrm{Sh}(r)}\|(\hat{\pi}^{(p)} - \hat{\pi}^{(q)})(y,\cdot)\|_1\\&\quad\leq C\left((\log H(x))^{-9} + \|p-q\|_1\right),\end{aligned}$$

where in the last line we used that x is good and Lemma 4.5. Now, with $K = \lfloor \log_2(s_L/H(x)) \rfloor$, Lemma 3.2 shows

$$\begin{aligned}\|p - q\|_1 &\le C \sum_{i=1}^{K} \left(\log(2^{-i} s_L)\right)^{-9} \\ &\le C \min\left\{K \left(\log H(x)\right)^{-9}, \left(\log H(x)\right)^{-8}\right\}.\end{aligned}$$

This proves the claim for the smoothed difference. For the nonsmoothed difference,

$$\left\|\left(\hat{\Pi}^g - \hat{\pi}^{(p)}\right)(x,\cdot)\right\|_1 \le \left\|\left(\hat{\Pi}^g - \hat{\pi}^{(q)}\right)(x,\cdot)\right\|_1 + \left\|\left(\hat{\pi}^{(p)} - \hat{\pi}^{(q)}\right)(x,\cdot)\right\|_1.$$

Since x is good, the first term is bounded by δ, and, by what we have just seen, the second term is bounded by δ as well if we choose L_0 (and so L) large enough. □

REMARK 5.1. Notice that the choice of the parameter r depends on δ. See also the preliminary remarks of Section 6.3.

Recall that in the goodifying-procedure introduced in Section 3, "bad" exit distributions inside V_L are replaced by such of a symmetric random walk with one-step distribution $p = p_{s_L/20}$. For this p and good points x within the boundary region $\mathrm{Sh}_L(2r)$, we would like to use at least an estimate of the form

$$\left\|\left(\hat{\Pi}_{L,r} - \hat{\pi}^{(p)}_{L,r}\right)(x,\cdot)\right\|_1 \le C\delta.$$

However, exit measures at points x inside $\mathrm{Sh}_L((1/10)r)$ are taken from intersected balls $V_t(x) \cap V_L$. We therefore work in this (and only in this) section with slightly modified transition kernels $\tilde{\Pi}_{L,r}$, $\tilde{\pi}_{L,r}$, $\tilde{\Pi}^g_{L,r}$ in the enlarged ball V_{L+r}, taking the exit measure in $\mathrm{Sh}_L(2r)$ from uncut balls $V_t(x) \subset V_{L+r}$, $t \in [h_{L,r}(x), 2h_{L,r}(x)]$.

Now, to make things precise, for $q \in \mathcal{P}^{\mathrm{s}}_{\kappa}$, we set $h_{L,r}(x) = (1/20)r$ for $x \notin C_L$, and let $\tilde{\pi}^{(q)}_{L,r}$ be the coarse grained symmetric random walk kernel in V_{L+r} associated to $\tilde{\psi} = (h_{L,r}(x))_{x \in V_{L+r}}$,

$$\tilde{\pi}^{(q)}_{L,r}(x,\cdot) = \frac{1}{h_{L,r}(x)} \int_{\mathbb{R}_+} \varphi\left(\frac{t}{h_{L,r}(x)}\right) \pi^{(q)}_{V_t(x)\cap V_{L+r}}(x,\cdot)\,\mathrm{d}t.$$

For the corresponding RWRE kernel, we forget about the environment on $V_{L+r} \setminus V_L$ and set

$$\tilde{\Pi}^{(q)}_{L,r}(x,\cdot) = \begin{cases} \dfrac{1}{h_{L,r}(x)} \displaystyle\int_{\mathbb{R}_+} \varphi\left(\frac{t}{h_{L,r}(x)}\right) \Pi_{V_t(x)}(x,\cdot)\,\mathrm{d}t, & \text{for } x \in V_L, \\ \tilde{\pi}^{(q)}_{L,r}(x,\cdot), & \text{for } x \in V_{L+r} \setminus V_L. \end{cases}$$

For $p = p_{s_L/20}$ and all good $x \in V_L$, we now have $\|(\tilde{\Pi}^{(p)}_{L,r} - \tilde{\pi}^{(p)}_{L,r})(x,\cdot)\|_1 \le \delta$ provided ε is small enough, while for $x \in V_{L+r} \setminus V_L$, the difference vanishes anyway. The goodified version of $\tilde{\Pi}^{(p)}_{L,r}$ is then obtained in an analogous way to (3.2),

$$\tilde{\Pi}^{g}_{L,r}(x,\cdot) = \begin{cases} \tilde{\Pi}^{(p)}_{L,r}(x,\cdot), & \text{for } x \notin \mathcal{B}^{\star}_{L,r}, \\ \tilde{\pi}^{(p)}_{L,r}(x,\cdot), & \text{for } x \in \mathcal{B}^{\star}_{L,r}. \end{cases}$$

Clearly, for $x \in V_L \setminus \mathrm{Sh}_L(2r)$, the first statement of Lemma 5.1 holds with the left-hand side there replaced by

$$\|(\tilde{\Pi}^{g}_{L,r} - \tilde{\pi}^{(p)}_{L,r})\tilde{\pi}^{(p)}_{L,r}(x,\cdot)\|_1.$$

But thanks to the modified transition kernels, we now have

$$\|(\tilde{\Pi}^{g}_{L,r} - \tilde{\pi}^{(p)}_{L,r})(x,\cdot)\|_1 \le 2\delta$$

for *all* $x \in V_L$. Indeed, one just has to notice that Lemma 5.1 can now also be applied to points $x \in \mathrm{Sh}_L(2r)$, with the same proof.

We write $\tilde{G}_{L,r}$, $\tilde{g}_{L,r}$ and $\tilde{G}^{g}_{L,r}$ for the Green's functions on V_{L+r} corresponding to $\tilde{\Pi}_{L,r}$, $\tilde{\pi}_{L,r}$ and $\tilde{\Pi}^{g}_{L,r}$. Note

$$\hat{G}_{L,r} \le \tilde{G}_{L,r}, \qquad \hat{g}_{L,r} \le \tilde{g}_{L,r}, \qquad \hat{G}^{g}_{L,r} \le \tilde{G}^{g}_{L,r} \quad \text{pointwise on } V_{L+r} \times (V_{L+r} \setminus \partial V_L). \tag{5.1}$$

Since we do not have exact expressions for $\tilde{g}_{L,r}$ or $\tilde{G}_{L,r}$, we construct a (deterministic) kernel $\Gamma_{L,r}$ that bounds the Green's functions from above. For $x \in V_{L+r}$, set

$$\tilde{\mathrm{d}}(x) = \max\left(\frac{\mathrm{d}_{L+r}(x)}{2}, 3r\right), \qquad a(x) = \min\bigl(\tilde{\mathrm{d}}(x), s_L\bigr).$$

Furthermore, for $x, y \in V_{L+r}$, let

$$\Gamma^{(1)}_{L,r}(x,y) = \frac{\tilde{\mathrm{d}}(x)\tilde{\mathrm{d}}(y)}{a(y)^2(a(y) + |x-y|)^d},$$

$$\Gamma^{(2)}_{L,r}(x,y) = \frac{1}{a(y)^2(a(y) + |x-y|)^{d-2}}.$$

The kernel $\Gamma_{L,r}$ is defined as the pointwise minimum

$$\Gamma_{L,r} = \min\bigl\{\Gamma^{(1)}_{L,r}, \Gamma^{(2)}_{L,r}\bigr\}. \tag{5.2}$$

We cannot derive pointwise estimates on the Green's functions in terms of $\Gamma_{L,r}$, but we can use this kernel to obtain upper bounds on neighborhoods $U(x) = V_{a(x)}(x) \cap V_{L+r}$. Call a function $F : V_{L+r} \times V_{L+r} \to \mathbb{R}_+$ a *positive kernel*. Given two positive kernels F and G, we write $F \preceq G$ if for all $x, y \in V_{L+r}$,

$$F\bigl(x, U(y)\bigr) \le G\bigl(x, U(y)\bigr),$$

where $F(x, U)$ stands for $\sum_{y\in U\cap\mathbb{Z}^d} F(x, y)$. We write $F \asymp 1$, if there is a constant $C > 0$ such that for all $x, y \in V_{L+r}$,

$$\frac{1}{C}F(x, y) \le F(\cdot,\cdot) \le CF(x, y) \qquad \text{on } U(x) \times U(y).$$

We adapt this notation to positive functions of one argument: for $f : V_{L+r} \to \mathbb{R}_+$, $f \asymp 1$ means that for some $C > 0$, $C^{-1}f(x) \le f(\cdot) \le Cf(x)$ on any $U(x) \subset V_{L+r}$. Finally, given $0 < \eta < 1$, we say that a positive kernel A on V_{L+r} is η-*smoothing*, if for all $x \in V_{L+r}$, $A(x, U(x)) \le \eta$, and $A(x, y) = 0$ whenever $y \notin U(x)$.

Now we are in position to formulate our main statement of this section. Recall our convention concerning constants: they only depend on the dimension unless stated otherwise.

LEMMA 5.2. (i) *There exists a constant* $C_1 > 0$ *such that for all* $q \in \mathcal{P}_\kappa^{\mathrm{s}}$,

$$\hat{g}_{L,r}^{(q)} \preceq C_1\Gamma_{L,r} \quad \textit{and} \quad \tilde{g}_{L,r}^{(q)} \preceq C_1\Gamma_{L,r}.$$

(ii) *Assume* **Cond**(δ, L_0, L_1), *and let* $L_1 \le L \le L_1(\log L_1)^2$. *There exists a constant* $C > 0$ *such that for* $\delta > 0$ *small*,

$$\hat{G}_{L,r}^{g} \preceq C\Gamma_{L,r} \quad \textit{and} \quad \tilde{G}_{L,r}^{g} \preceq C\Gamma_{L,r}.$$

REMARK 5.2. (i) Thanks to (5.1), it suffices to show the bounds for $\tilde{g}_{L,r}$ and $\tilde{G}_{L,r}^{g}$. For later use, we keep track of the constant in part (i) of the lemma.

(ii) We will later apply part (i) with $q = p_L$. From Lemma 3.2 we know that we can assume $p_L \in \mathcal{P}_\kappa^{\mathrm{s}}$ for every choice of $\kappa > 0$, if L_0 is large.

We first prove part (i), which will be a straightforward consequence of the estimates on hitting probabilities in Section 4 and the next lemma.

LEMMA 5.3. *There exists a constant* $C > 0$ *such that for all* $q \in \mathcal{P}_\kappa^{\mathrm{s}}$, *for all* $x \in V_{L+r}$ *and* $y \in V_L$ *with* $\mathrm{d}_L(y) \ge 4s_L$,

$$\tilde{g}_{L,r}^{(q)}(x, y) \le C \begin{cases} s_L^{-2}\max\{|x-y|, s_L\}^{-(d-2)}, & \textit{for } y \ne x,\\ 1, & \textit{for } y = x.\end{cases}$$

PROOF. The underlying one-step transition kernel is always given by $q \in \mathcal{P}_\kappa^{\mathrm{s}}$, which we therefore omit from notation. For example, $\tilde{g} = \tilde{g}^{(q)}$, $\hat{g}_{m,V_L} = \hat{g}_{m,V_L}^{(q)}$, $\mathrm{P}_x = \mathrm{P}_{x,q}$.

If $x = y$, then the claim follows from transience of simple random walk. Now assume $x \ne y$, and always $\mathrm{d}_L(y) \ge 4s_L$. Consider first the case $|x - y| \le s_L$. Recall that $\hat{g}_{m,V_L}$ denotes the Green's function for the ball V_L associated to $\hat{\pi}_{\psi_m}$, where $\psi_m \equiv m$. With $m = s_L/20$ we have

$$\tilde{g}(x,y) \le \hat{g}_{m,V_L}(x,y) + \sup_{v\in \mathrm{Sh}_L(2s_L)} \mathrm{P}_v(T_{V_{s_L}(y)} < \tau_{V_{L+r}}) \sup_{\substack{w:w\neq y,\\ |w-y|\le s_L}} \tilde{g}(w,y).$$

Since

$$\sup_{v\in \mathrm{Sh}_L(2s_L)} \mathrm{P}_v(T_{V_{s_L}(y)} < \tau_{V_{L+r}}) < 1$$

uniformly in L, it follows from Proposition 5.2 that

$$\tilde{g}(x,y) \le C \sup_{\substack{w:w\neq y,\\ |w-y|\le s_L}} \hat{g}_{m,V_L}(w,y) \le C \sup_{\substack{w:w\neq y,\\ |w-y|\le s_L}} \hat{g}_{m,\mathbb{Z}^d}(w,y) \le \frac{C}{s_L^d}.$$

If $|x-y| > s_L$ we use Lemma 4.2(i) and the first case to get

$$\tilde{g}(x,y) \le \mathrm{P}_x(T_{V_{s_L}(y)} < \infty) \sup_{\substack{w:w\neq y,\\ |w-y|\le s_L}} \tilde{g}(w,y) \le \frac{C}{s_L^2|x-y|^{d-2}}.$$

□

PROOF OF LEMMA 5.2(i). It suffices to prove the bound for $\tilde{g}$. First we show that there exists a constant $C > 0$ such that for all $y \in V_{L+r}$,

$$\sup_{x\in V_{L+r}} \tilde{g}\big(x, U(y)\big) \le C. \tag{5.3}$$

At first let $\mathrm{d}_{L+r}(y) \le 6r$. Then $U(y) \subset \mathrm{Sh}_{L+r}(10r)$. We claim that even

$$\sup_{x\in V_{L+r}} \tilde{g}\big(x, \mathrm{Sh}_{L+r}(10r)\big) \le C \tag{5.4}$$

for some $C > 0$. Indeed, if $z \in \mathrm{Sh}_{L+r}(10r)$, then $\tilde{\pi}(z,\cdot)$ is an (averaging) exit distribution from balls $V_\ell(z) \cap V_{L+r}$, where $\ell \ge r/20$. Using Lemma 4.1(i), we find a constant $k_1 = k_1(d)$ such that starting at any $z \in \mathrm{Sh}_{L+r}(10r)$, V_{L+r} is left after k_1 steps with probability > 0, uniformly in z. This together with the strong Markov property implies (5.4). Next assume $6r < \mathrm{d}_{L+r}(y) \le 6s_L$. Then $U(y) \subset S(y) = \mathrm{Sh}_{L+r}(\frac{1}{2}\mathrm{d}_{L+r}(y), 2\mathrm{d}_{L+r}(y))$. We claim that

$$\sup_{x\in V_{L+r}} \tilde{g}\big(x, S(y)\big) \le C. \tag{5.5}$$

For $z \in S(y)$, $\tilde{\pi}(z,\cdot)$ is an averaging exit distribution from balls $V_l(z)$, where $l \ge \mathrm{d}_{L+r}(y)/240$. By Lemma 4.1(i), we find some small $0 < c < 1$ and a constant $k_2(c,d)$ such that after k_2 steps, the walk has probability > 0 to be in $\mathrm{Sh}_{L+r}(\frac{1-c}{2}\mathrm{d}_{L+r}(y))$, uniformly in z and y. But starting in $\mathrm{Sh}_{L+r}(\frac{1-c}{2}\mathrm{d}_{L+r}(y))$, an iterative application of Lemma 4.1(i) shows that with probability > 0, the ball

V_{L+r} is left before $S(y)$ is visited again. Therefore (5.5) and hence (5.3) hold in this case. At last, let $\mathrm{d}_{L+r}(y) > 6s_L$. Then $\mathrm{d}_L(w) \geq 4s_L$ for $w \in U(y)$. Estimating

$$\tilde{g}(x,w) \leq 1 + \sup_{v\,:\,v\neq w} \tilde{g}(v,w),$$

we get with part (i) that

$$\sup_{w\in U(y)} \tilde{g}(x,w) \leq 1 + \frac{C}{s_L^d}.$$

Summing over $w \in U(y)$, (5.3) follows. Finally, note that for any $x \in V_{L+r}$,

$$\tilde{g}\big(x, U(y)\big) \leq \mathrm{P}_x(T_{U(y)} < \tau_{V_{L+r}}) \sup_{w\in U(y)} \tilde{g}\big(w, U(y)\big).$$

Now $\tilde{g} \preceq C\Gamma$ follows from (5.3) and the hitting estimates of Lemma 4.2. $\square$

Let us now explain our strategy for proving part (ii). By version (2.7) of the perturbation expansion, we can express $\tilde{G}^g_{L,r}$ in a series involving $\tilde{g}_{L,r}$ and differences of exit measures. The Green's function $\tilde{g}_{L,r}$ is already controlled by means of $\Gamma_{L,r}$. Looking at (2.7), we thus have to understand what happens if $\Gamma_{L,r}$ is concatenated with certain smoothing kernels. This will be the content of Proposition 5.3.

We start with collecting some important properties of $\Gamma_{L,r}$, which will be used throughout this text. Define for $j \in \mathbb{N}$,

$$\mathcal{L}_j = \big\{y \in V_L : j \leq \mathrm{d}_L(y) < j+1\big\},$$
$$\mathcal{E}_j = \big\{y \in V_{L+r} : \tilde{\mathrm{d}}(y) \leq 3jr\big\}.$$

LEMMA 5.4 (Properties of $\Gamma_{L,r}$). (i) *Both* $\tilde{\mathrm{d}}$ *and a are Lipschitz with constant* $1/2$. *Moreover, for* $x, y \in V_{L+r}$,

$$a(y) + |x-y| \leq a(x) + \tfrac{3}{2}|x-y|.$$

(ii)

$$\Gamma_{L,r} \asymp 1.$$

(iii) *For* $0 \leq j \leq 2s_L$, $x \in V_{L+r}$,

$$\sum_{y\in\mathcal{L}_j} \left(\max\left\{1, \frac{\tilde{\mathrm{d}}(x)}{a(y)}\right\} \frac{1}{(a(y)+|x-y|)^d}\right) \leq C\frac{1}{j\vee r}.$$

(iv) *For* $1 \leq j \leq \frac{1}{3r}s_L$,

$$\sup_{x\in V_{L+r}} \Gamma_{L,r}(x,\mathcal{E}_j) \leq C\log(j+1),$$

and for $0 \le \alpha < 3$,

$$\sup_{x \in V_{L+r}} \Gamma_{L,r}\big(x, \mathrm{Sh}_L\big(s_L, L/(\log L)^\alpha\big)\big) \le C(\log\log L)(\log L)^{6-2\alpha}.$$

(v) *For* $x \in V_{L+r}$, *in the case of constant* r,

$$\Gamma_{L,r}(x, V_L) \le C \max\left\{\frac{\tilde{\mathrm{d}}(x)}{L}(\log L)^6, \left(\frac{\tilde{\mathrm{d}}(x)}{r} \wedge \log L\right)\right\}.$$

In the case $r = r_L$,

$$\Gamma_{L,r_L}(x, V_L) \le C \max\left\{\frac{\tilde{\mathrm{d}}(x)}{L}(\log L)^6, \left(\frac{\tilde{\mathrm{d}}(x)}{r_L} \wedge \log\log L\right)\right\}.$$

PROOF. (i) The second statement is a direct consequence of the Lipschitz property, which in turn follows immediately from the definitions of $\tilde{\mathrm{d}}$ and a.

(ii) As for $y' \in U(y)$, $\frac{1}{2}a(y) \le a(y') \le \frac{3}{2}a(y)$ and similarly with a replaced by $\tilde{\mathrm{d}}$, it suffices to show that for $x' \in U(x)$, $y' \in U(y)$,

$$\frac{1}{C}\big(a(y) + |x-y|\big) \le a\big(y'\big) + \big|x'-y'\big| \le C\big(a(y) + |x-y|\big). \tag{5.6}$$

First consider the case $|x-y| \ge 4\max\{a(x), a(y)\}$. Then

$$a(y) + |x-y| \le 2a\big(y'\big) + 2\big(|x-y| - a(x) - a(y)\big) \le 2\big(a\big(y'\big) + \big|x'-y'\big|\big).$$

If $|x-y| \le 4a(y)$, then

$$a(y) + |x-y| \le 5a(y) \le 5a(y) + \big|x'-y'\big| \le 10\big(a\big(y'\big) + \big|x'-y'\big|\big),$$

while for $|x-y| \le 4a(x)$, using part (i) in the first inequality,

$$a(y) + |x-y| \le a(x) + \tfrac{3}{2}|x-y| \le 7a(x) \le 14\big(a\big(y'\big) + \big|x'-y'\big|\big).$$

This proves the first inequality in (5.6). The second one follows from

$$a\big(y'\big) + \big|x'-y'\big| \le \tfrac{5}{2}a(y) + a(x) + |x-y| \le \tfrac{7}{2}\big(a(y) + |x-y|\big).$$

(iii) If $j \le 2s_L$ and $y \in \mathcal{L}_j$, then $a(y)$ is of order $j \vee r$. By Lemma 4.6 we have

$$\sum_{y \in \mathcal{L}_j} \frac{1}{(j \vee r + |x-y|)^d} \le C \min\left\{\frac{1}{j \vee r}, \frac{1}{|\mathrm{d}_{L+r}(x) - (j+r)|}\right\}.$$

It remains to show that

$$\max\left\{1, \frac{\tilde{\mathrm{d}}(x)}{j \vee r}\right\} \min\left\{\frac{1}{j \vee r}, \frac{1}{|\mathrm{d}_{L+r}(x) - (j+r)|}\right\} \le C\frac{1}{j \vee r}. \tag{5.7}$$

If $\tilde{\mathrm{d}}(x) \le (j \vee 3r)$, this is clear. If $\tilde{\mathrm{d}}(x) > (j \vee 3r)$, (5.7) follows from $|\mathrm{d}_{L+r}(x) - (j+r)| \ge \tilde{\mathrm{d}}(x)/2$.

(iv) We follow our convention and write Γ instead of $\Gamma_{L,r}$. If $\tilde{\mathrm{d}}(y) \leq 3jr$, then $\mathrm{d}_L(y) \leq 6jr$. Estimating Γ by $\Gamma^{(1)}$, we get

$$\Gamma(x, \mathcal{E}_j) \leq C \sum_{i=0}^{6jr} \sum_{y \in \mathcal{L}_i} \frac{\tilde{\mathrm{d}}(x)}{a(y)} \frac{1}{(a(y) + |x-y|)^d}.$$

Now the first assertion of (iv) follows from (iii). The second is proved similarly, so we omit the details.

(v) Set $B = \{y \in V_L : \tilde{\mathrm{d}}(y) \leq s_L \vee 2\tilde{\mathrm{d}}(x)\}$. For $y \in V_L \setminus B$, it holds that $a(y) = s_L$ and $|x - y| \geq \tilde{\mathrm{d}}(y)$. Therefore,

$$\Gamma(x, V_L \setminus B) \leq \Gamma^{(1)}(x, V_L \setminus B) \leq \frac{\tilde{\mathrm{d}}(x)}{s_L^2} \sum_{y \in V_{2L}} \frac{1}{(s_L + |y|)^{d-1}} \leq C \frac{\tilde{\mathrm{d}}(x)}{L} (\log L)^6.$$

Furthermore,

$$\Gamma(x, B) \leq \sum_{i=0}^{2s_L} \sum_{y \in \mathcal{L}_i} \frac{\tilde{\mathrm{d}}(x)}{a(y)} \frac{1}{(a(y) + |x-y|)^d} + \frac{1}{s_L^2} \sum_{\substack{y \in V_L : \\ s_L \leq \tilde{\mathrm{d}}(y) \leq 2\tilde{\mathrm{d}}(x)}} \frac{1}{(s_L + |x-y|)^{d-2}}.$$

Lemma 4.6 bounds the second term by $C(\tilde{\mathrm{d}}(x)/L)(\log L)^6$. For the first term, we use twice part (iii) and once Lemma 4.6 to get

$$\sum_{i=0}^{2s_L} \sum_{y \in \mathcal{L}_i} \frac{\tilde{\mathrm{d}}(x)}{a(y)} \frac{1}{(a(y) + |x-y|)^d} \leq C \sum_{i=0}^{5r} \frac{1}{i \vee r} + C \min\left\{\tilde{\mathrm{d}}(x) \sum_{i=5r}^{2s_L} \frac{1}{i^2}, \sum_{i=5r}^{2s_L} \frac{1}{i}\right\}.$$

This proves (v). $\square$

PROPOSITION 5.3. *Let F, G be positive kernels with $F \preceq G$.*

(i) *If A is η-smoothing and $G \asymp 1$, then for some constant $C = C(d, G) > 0$,*

$$FA \preceq C\eta G.$$

(ii) *If Φ is a positive function on V_{L+r} with $\Phi \asymp 1$, then for some $C = C(d, \Phi) > 0$,*

$$F\Phi \leq CG\Phi.$$

PROOF. (i) Let $y \in V_{L+r}$. As a is Lipschitz with constant $1/2$, we can choose $K = K(d)$ points y_k out of the set $M = \{y' \in V_{L+r} : U(y') \cap U(y) \neq \varnothing\}$ such that M is covered by the union of the $U(y_k)$, $k = 1, \ldots, K$. Since $A(y', U(y)) \neq 0$ implies $y' \in M$, we then have

$$\begin{aligned} FA(x, U(y)) &= \sum_{y' \in M} F(x, y') \sum_{y'' \in U(y)} A(y', y'') \leq \eta \sum_{k=1}^{K} F(x, U(y_k)) \\ &\leq \eta \sum_{k=1}^{K} G(x, U(y_k)). \end{aligned}$$

Using $G \asymp 1$, we get $G(x, U(y_k)) \le C|U(y_k)|G(x, y)$. Clearly $|U(y_k)| \le C|U(y)|$, so that

$$FA(x, U(y)) \le CK\eta|U(y)|G(x, y).$$

A second application of $G \asymp 1$ yields the claim.

(ii) We can find a constant $K = K(d)$ and a covering of V_{L+r} by neighborhoods $U(y_k)$, $y_k \in V_{L+r}$, such that every $y \in V_{L+r}$ is contained in at most K many of the sets $U(y_k)$. Using $\Phi \asymp 1$, it follows that for $x \in V_{L+r}$,

$$\begin{aligned} F\Phi(x) &= \sum_{y \in V_{L+r}} F(x, y)\Phi(y) \le C \sum_{k=1}^{\infty} F(x, U(y_k))\Phi(y_k) \\ &\le C \sum_{k=1}^{\infty} G(x, U(y_k))\Phi(y_k) \le C \sum_{k=1}^{\infty} \sum_{y \in U(y_k)} G(x, y)\Phi(y) \\ &\le CK \sum_{y \in V_{L+r}} G(x, y)\Phi(y). \end{aligned}$$

□

In terms of our specific kernel $\Gamma_{L,r}$, we obtain the following:

PROPOSITION 5.4. *Let A be η-smoothing, and let F be a positive kernel satisfying $F \preceq \Gamma_{L,r}$.*

(i) *There exists a constant $C_2 > 0$ not depending on F such that*

$$FA \preceq C_2\eta\Gamma_{L,r}.$$

(ii) *If additionally $A(x, y) = 0$ for $x \notin V_L$ and $A(x, U(x)) \le (\log a(x))^{-15/2}$ for $x \in V_L \setminus \mathcal{E}_1$, then there exists a constant $C_3 > 0$ not depending on F such that for all $x, z \in V_{L+r}$,*

$$FA\Gamma_{L,r}(x, z) \le C_3\eta^{1/2}\Gamma_{L,r}(x, z).$$

PROOF. (i) This is Proposition 5.3(i) with $G = \Gamma$.

(ii) We set $B = V_L \setminus \mathcal{E}_1$ and split into

$$FA\Gamma = F1_{\mathcal{E}_1}A\Gamma + F1_B A\Gamma. \tag{5.8}$$

Let $x, z \in V_{L+r}$ be fixed, and consider first $F1_{\mathcal{E}_1}A\Gamma(x, z)$. Using $\Gamma \asymp 1$, $A\Gamma(y, z) \le C\eta\Gamma(y, z)$. As $\Gamma(\cdot, z) \asymp 1$ and $F1_{\mathcal{E}_1} \preceq \Gamma 1_{\mathcal{E}_2}$, we get by Proposition 5.3(ii),

$$F1_{\mathcal{E}_1}A\Gamma(x, z) \le C\eta\Gamma 1_{\mathcal{E}_2}\Gamma(x, z).$$

Setting $\mathcal{E}_2^1 = \{y \in \mathcal{E}_2 : |y - z| \ge |x - z|/2\}$, $\mathcal{E}_2^2 = \mathcal{E}_2 \setminus \mathcal{E}_2^1$, we split further into

$$\Gamma 1_{\mathcal{E}_2}\Gamma = \Gamma 1_{\mathcal{E}_2^1}\Gamma + \Gamma 1_{\mathcal{E}_2^2}\Gamma.$$

If $y \in \mathcal{E}_2^1$, then $\Gamma(y,z) \le C\Gamma(x,z)$. By Lemma 5.4(iv), $\Gamma(x,\mathcal{E}_2) \le C$. Together we obtain

$$\Gamma 1_{\mathcal{E}_2^1}\Gamma(x,z) \le C\Gamma(x,z).$$

If $y \in \mathcal{E}_2^2$, then $\Gamma(x,y) \le C\frac{a(z)^2}{r^2}\Gamma(x,z)$ and $\Gamma^{(1)}(y,z) \le C\frac{r^2}{a(z)^2}\Gamma^{(1)}(z,y)$, whence

$$\Gamma 1_{\mathcal{E}_2\setminus\mathcal{E}_2^2}\Gamma(x,z) \le C\Gamma(x,z)\Gamma^{(1)}(z,\mathcal{E}_2) \le C\Gamma(x,z).$$

We therefore have shown that

$$F 1_{\mathcal{E}_1} A\Gamma(x,z) \le C\eta\Gamma(x,z).$$

To handle the second summand of (5.8), set $\sigma(y) = \min\{\eta, (\log a(y))^{-15/2}\}$, $y \in V_{L+r}$. Clearly, $1_B A\Gamma(y,z) \le C\sigma(y)\Gamma(y,z)$ and $F1_B \preceq \Gamma 1_{V_L}$. Furthermore, $\sigma(\cdot)\Gamma(\cdot,z) \asymp 1$, so that by Proposition 5.3(ii)

$$F 1_B A\Gamma(x,z) \le C\Gamma 1_{V_L}\sigma\Gamma(x,z).$$

Consider $D^1 = \{y \in V_L : |y-z| \ge |x-z|/2\}$, $D^2 = V_L \setminus D^1$ and split into

$$\Gamma 1_{V_L}\sigma\Gamma = \Gamma 1_{D^1}\sigma\Gamma + \Gamma 1_{D^2}\sigma\Gamma.$$

If $y \in D^1$, then $\Gamma(y,z) \le C\max\{1, \frac{\tilde{\mathrm{d}}(y)}{\tilde{\mathrm{d}}(x)}\}\Gamma(x,z)$, implying $\Gamma 1_{D^1}\sigma\Gamma(x,z) \le C\eta^{1/2}\Gamma(x,z)$ if we prove

$$\sum_{y\in V_L}\max\left\{1, \frac{\tilde{\mathrm{d}}(y)}{\tilde{\mathrm{d}}(x)}\right\}\Gamma(x,y)\sigma(y) \le C\eta^{1/2}. \tag{5.9}$$

To this end, we treat the summation over $S^1 = \{y \in V_L : \mathrm{d}_L(y) \le 2s_L\}$ and $S^2 = V_L \setminus S_1$ separately. If $y \in S^2$, then $a(y) = s_L$. Estimating Γ by $\Gamma^{(1)}$ and $\tilde{\mathrm{d}}(y)$, $\tilde{\mathrm{d}}(x)$ simply by L, we get

$$\begin{aligned}
&\sum_{y\in S^2}\max\left\{1, \frac{\tilde{\mathrm{d}}(y)}{\tilde{\mathrm{d}}(x)}\right\}\Gamma(x,y)\sigma(y) \\
&\quad \le \frac{C}{(\log L)^{3/2}}\sum_{y\in V_{2L}}\frac{1}{(s_L+|y|)^d} \qquad (5.10)\\
&\quad \le \frac{C\log\log L}{(\log L)^{3/2}}.
\end{aligned}$$

If $y \in S^1$, we estimate Γ again by $\Gamma^{(1)}$ and split the summation into the layers $\mathcal{L}_j$, $j = 0,\dots,2s_L$. On $\mathcal{L}_j$, $\sigma(y) \le C\min\{\eta, (\log(j+1))^{-15/2}\}$. Thus, by

Lemma 5.4(iii),

$$\sum_{y\in S^1}\max\left\{1,\frac{\tilde{\mathrm{d}}(y)}{\tilde{\mathrm{d}}(x)}\right\}\Gamma(x,y)\sigma(y)$$
$$\le C\sum_{j=0}^{2s_L}\sum_{y\in\mathcal{L}_j}\max\left\{1,\frac{\tilde{\mathrm{d}}(x)}{a(y)}\right\}\frac{\min\{\eta,(\log(j+1))^{-15/2}\}}{(a(y)+|x-y|)^d}$$
$$\le C\sum_{j=0}^{2s_L}\frac{\min\{\eta,(\log(j+1))^{-15/2}\}}{j\vee r}\le C\eta^{1/2}.$$

Together with (5.10), we have proved (5.9). It remains to bound the term $\Gamma 1_{D^2}\sigma\Gamma(x,z)$. But if $y\in D^2$, then

$$a(y)+|x-y|\ge a(y)+\tfrac{1}{2}|x-z|\ge a(z)-\tfrac{1}{2}|y-z|+\tfrac{1}{2}|x-z|$$
$$\ge\tfrac{1}{4}\bigl(a(z)+|x-z|\bigr),$$

whence $\Gamma(x,y)\le C\frac{a(z)^2}{a(y)^2}\max\{1,\frac{\tilde{\mathrm{d}}(y)}{\tilde{\mathrm{d}}(z)}\}\Gamma(x,z)$. Using Lemma 5.4(i), we have

$$\frac{a(z)^2}{a(y)^2}\Gamma(y,z)\le C\Gamma(z,y),$$

so that $\Gamma 1_{D^2}\sigma\Gamma(x,z)\le C\eta^{1/2}\Gamma(x,z)$ again follows from (5.9). $\square$

Now we have collected all ingredients to finally prove part (ii) of our main Lemma 5.2.

PROOF OF LEMMA 5.2(ii). As already remarked, we only have to prove the statement involving $\tilde{G}^g$. We work with the kernel $p=p_{s_L/20}$, but suppress it from notation, that is, $\tilde{\pi}=\tilde{\pi}^{(p)}$, $\tilde{g}=\tilde{g}^{(p)}$. The perturbation expansion (2.7) yields

$$\tilde{G}^g=\tilde{g}\sum_{m=0}^{\infty}(R\tilde{g})^m\sum_{k=0}^{\infty}\Delta^k,$$

where $\Delta=1_{V_{L+r}}(\tilde{\Pi}^g-\tilde{\pi})$, $R=\sum_{k=1}^{\infty}\Delta^k\tilde{\pi}$. With the constants C_1 of Lemma 5.2(i) and C_2,C_3 of Proposition 5.4 we choose

$$\delta\le\frac{1}{32}\left(\frac{1}{C_2\vee C_1^2C_3^2}\right).$$

Recall the properties of $\tilde{\Pi}^g$ and $\tilde{\pi}$ mentioned after Remark 5.1. From Lemma 5.2(i) and Proposition 5.4(i) with $A=|\Delta|$, $\eta=2\delta$ we then deduce that $\tilde{g}|\Delta|\preceq(C_1/2)\Gamma$, and, by iterating,

$$\sum_{k=1}^{\infty}\tilde{g}|\Delta|^{k-1}\preceq 2C_1\Gamma.$$

Furthermore, by part (ii) of Proposition 5.4 with $A = |\Delta\tilde{\pi}|$ and Lemma 5.2(i),

$$\sum_{k=1}^{\infty} \tilde{g}|\Delta|^{k-1}|\Delta\tilde{\pi}|\tilde{g} \preceq (C_1/2)\Gamma.$$

Iterating this procedure shows that for $m \in \mathbb{N}$,

$$\tilde{g}(|R|\tilde{g})^m \preceq C_1 2^{-m}\Gamma.$$

Finally, by a further application of Proposition 5.4(i),

$$\tilde{g}\sum_{m=0}^{\infty}(|R|\tilde{g})^m \sum_{k=0}^{\infty}|\Delta|^k \preceq 4C_1\Gamma.$$

This proves the lemma. □

5.3. *Modified transitions on environments bad on level* 4. We shall now describe an environment-depending second version of the coarse graining scheme, which leads to modified transition kernels $\breve{\Pi}_{L,r}$, $\breve{\Pi}^g_{L,r}$, $\breve{\pi}_{L,r}$ on "really bad" environments.

We assume that $\mathbf{Cond}(\delta, L_0, L_1)$ holds, and take $L_1 \le L \le L_1(\log L_1)^2$, so that Lemma 5.2 can be applied.

Assume $\omega \in \mathrm{OneBad}_L$ is bad on level 4, with $\mathcal{B}_L(\omega) \subset V_{L/2}$. Then there exists $D = V_{4h_L(z)}(z) \in \mathcal{D}_L$ with $\mathcal{B}_L(\omega) \subset D$, $z \in V_{L/2}$. On D, $cr_L \le h_{L,r}(\cdot) \le Cr_L$. By Lemma 5.2 and the definition of $\Gamma_{L,r}$, it follows easily that we can find a constant $K_1 \ge 2$, depending only on d, such that whenever $|x - y| \ge K_1 h_{L,r}(y)$ for some $y \in \mathcal{B}_L$, we have

$$\hat{G}^g_{L,r}(x, \mathcal{B}_L) \le C\Gamma_{L,r}(x, D) \le \tfrac{1}{10}. \tag{5.11}$$

On such ω, we let $t(x) = K_1 h_{L,r}(x)$, and define on V_L,

$$\breve{\Pi}_{L,r}(x,\cdot) = \begin{cases} \mathrm{ex}_{V_{t(x)}(x)}(x,\cdot;\hat{\Pi}_{L,r}), & \text{for } x \in \mathcal{B}_L, \\ \hat{\Pi}_{L,r}(x,\cdot), & \text{otherwise.} \end{cases}$$

By replacing $\hat{\Pi}$ by $\hat{\pi}^{(q)}$ on the right-hand side, we define $\breve{\pi}^{(q)}_{L,r}(x,\cdot)$ in an analogous way, for all $q \in \mathcal{P}^{\mathrm{s}}_{\kappa}$. More precisely,

$$\breve{\pi}^{(q)}_{L,r}(x,\cdot) = \begin{cases} \mathrm{ex}_{V_{t(x)}(x)}(x,\cdot;\hat{\pi}^{(q)}_{L,r}), & \text{for } x \in \mathcal{B}_L, \\ \hat{\pi}^{(q)}_{L,r}(x,\cdot), & \text{otherwise.} \end{cases}$$

Note that $\breve{\pi}^{(q)}_{L,r_L}$ depends on the environment. See Figure 4 for a visualization of the modified transitions. We work again with a goodified version of $\breve{\Pi}_{L,r}$,

$$\breve{\Pi}^g_{L,r}(x,\cdot) = \breve{\Pi}^g_{L,r}(x,\cdot) \begin{cases} \mathrm{ex}_{V_{t(x)}(x)}(x,\cdot;\hat{\Pi}^g_{L,r}), & \text{for } x \in \mathcal{B}_L, \\ \hat{\Pi}^g_{L,r}(x,\cdot), & \text{otherwise.} \end{cases}$$

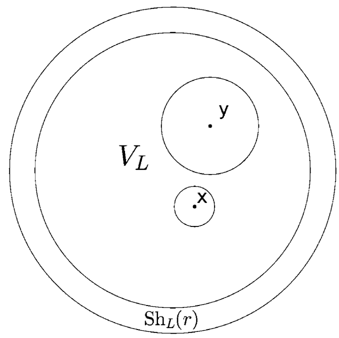

FIG. 4. *$\omega \in \mathrm{OneBad}_L$ bad on level 4, with $\mathcal{B}_L \subset V_{L/2}$. The point x is "good," so the coarse graining radii do not change at x. The point y is "bad." Therefore, at y, the exit distribution is taken from the larger set $V_{t(y)}(y)$, where $t(y) = K_1 h_{L,r}(y)$.*

For all other environments falling not into the above class, we change nothing and put $\breve{\Pi}_{L,r} = \hat{\Pi}_{L,r}$, $\breve{\Pi}^g_{L,r} = \hat{\Pi}^g_{L,r}$, $\breve{\pi}_{L,r} = \hat{\pi}_{L,r}$. This defines $\breve{\Pi}_{L,r}$, $\breve{\Pi}^g_{L,r}$ and $\breve{\pi}_{L,r}$ on all environments. We write $\breve{G}_{L,r}$, $\breve{G}^g_{L,r}$, $\breve{g}_{L,r}$ for the Green's functions corresponding to $\breve{\Pi}_{L,r}$, $\breve{\Pi}^g_{L,r}$ and $\breve{\pi}_{L,r}$.

5.3.1. *Some properties of the new transition kernels.* The following observations follow from the definition and will be tacitly used below:

- On environments which are good or bad on level at most 3, the new kernels agree with the old ones, and so do their Green's functions, that is, $\hat{G}_{L,r} = \breve{G}_{L,r}$ and $\hat{G}^g_{L,r} = \breve{G}^g_{L,r}$. On Good_L with the choice $r = r_L$, we have equality of all four Green's functions.
- If ω is not bad on level 4 with $\mathcal{B}_L \subset V_{L/2}$, then, with $p = p_{s_L/20}$,
$$1_{V_L}(\breve{\Pi}_{L,r} - \breve{\Pi}^g_{L,r}) = 1_{V_L}(\hat{\Pi}_{L,r} - \hat{\Pi}^g_{L,r}) = 1_{\mathcal{B}^\star_{L,r}}(\hat{\Pi} - \hat{\pi}^{(p)}).$$
This will be used in Section 6.
- In contrast to $\hat{\pi}_{L,r}$, the kernel $\breve{\pi}_{L,r}$ depends on the environment, too. However, $\breve{\Pi}_{L,r}$, $\breve{\Pi}^g_{L,r}$ and $\breve{\pi}_{L,r}$ do not change the exit measure from V_L, that is, for example,
$$\mathrm{ex}_{V_L}(x, \cdot; \breve{\Pi}^g_{L,r}) = \mathrm{ex}_{V_L}(x, \cdot; \hat{\Pi}^g_{L,r}).$$
- The old transition kernels are finer in the sense that the (new) Green's functions $\breve{G}$, $\breve{G}^g$, $\breve{g}$ are pointwise bounded from above by $\hat{G}$, $\hat{G}^g$ and $\hat{g}$, respectively. In particular, we obtain with the same constants as in Lemma 5.2,

LEMMA 5.5. (i) *For all $q \in \mathcal{P}^{\mathrm{s}}_\kappa$,*
$$\breve{g}^{(q)}_{L,r} \preceq C_1 \Gamma_{L,r}.$$

(ii) *Assume* **Cond**(δ, L_0, L_1), *and let* $L_1 \le L \le L_1(\log L_1)^2$. *For* $\delta > 0$ *small*,

$$\breve{G}^g_{L,r} \preceq C\Gamma_{L,r}.$$

For the new goodified Green's function, we have

COROLLARY 5.1. *Assume* **Cond**(δ, L_0, L_1), *and let* $L_1 \le L \le L_1(\log L_1)^2$. *There exists a constant* $C > 0$ *such that*:

(i) *On* OneBad_L, *if* $\mathcal{B}_L \cap \mathrm{Sh}_L(r_L) = \varnothing$ *or for general* $\mathcal{B}_L$ *in the case* $r = r_L$,

$$\sup_{x \in V_L} \breve{G}^g_{L,r}(x, \mathcal{B}_L) \le C.$$

On OneBad_L, *if* $\mathcal{B}_L \not\subset V_{L/4}$, *then*, *with* $t = \mathrm{d}(\mathcal{B}_L, \partial V_L)$,

$$\sup_{x \in V_{L/5}} \breve{G}^g_{L,r}(x, \mathcal{B}_L) \le C\left(\frac{s_L \wedge (t \vee r_L)}{L}\right)^{d-2}.$$

(ii) *On* $(\mathrm{BdBad}_{L,r})^c$, $\sup_{x \in V_{2L/3}} \breve{G}^g_{L,r}(x, \mathcal{B}^\partial_{L,r}) \le C(\log r)^{-1/2}$.

(iii) *For* $\omega \in \mathrm{OneBad}_L$ *bad on level at most* 3 *with* $\mathcal{B}_L \cap \mathrm{Sh}_L(r_L) = \varnothing$, *or for* ω *bad on level* 4 *with* $\mathcal{B}_L \subset V_{L/2}$, *putting* $\Delta = 1_{V_L}(\breve{\Pi}_{L,r} - \breve{\Pi}^g_{L,r})$,

$$\sup_{x \in V_L} \sum_{k=0}^{\infty} \left\| \left(\breve{G}^g_{L,r} 1_{\mathcal{B}_L} \Delta\right)^k (x, \cdot)\right\|_1 \le C.$$

PROOF. (i) The set $\mathcal{B}_L$ is contained in a neighborhood $D \in \mathcal{D}_L$. As $\breve{G}^g \preceq C\Gamma$, we have

$$\breve{G}^g(x, \mathcal{B}_L) \le C\Gamma^{(2)}(x, D). \tag{5.12}$$

From this, the first statement of (i) follows. Now let x be inside $V_{L/5}$, and $\mathcal{B}_L \not\subset V_{L/4}$. If the midpoint z of D can be chosen to lie inside $V_L \setminus \mathrm{Sh}(r_L)$, $a(\cdot)/h_L(z)$ and $h_L(z)/a(\cdot)$ are bounded on D. Then, the second statement of (i) is again a consequence of (5.12). If $z \in \mathrm{Sh}(r_L)$, we have

$$\breve{G}^g(x, \mathcal{B}_L) \le C\Gamma^{(1)}(x, D) \le C \sum_{j=0}^{2r_L} \sum_{y \in \mathcal{L}_j \cap D} \frac{L}{a(y)L^d}$$

$$\le CL^{-d+1} \sum_{j=0}^{2r_L} \frac{r_L^{d-1}}{j \vee r} \le C(\log L)\left(\frac{r_L}{L}\right)^{d-1}.$$

(ii) Recall the notation of Section 3.4. In order to bound $\breve{G}^g(x, \mathcal{B}^\partial_{L,r})$ uniformly in $x \in V_{2L/3}$, we look at the different bad sets $D_{j,r} \in \mathcal{Q}_{j,r}$ of layer Λ_j, $0 \le j \le J_1$. Estimating $\breve{G}^g$ by $\Gamma^{(1)}$, we have

$$\breve{G}^g(x, D_{j,r}) \le C\left(r2^j\right)^{d-1} L^{-d+1}.$$

On $(\mathrm{BdBad}_{L,r})^c$, the number of bad sets in layer Λ_j is bounded by

$$C(\log r + j)^{-3/2}\big(L/\big(r2^j\big)\big)^{d-1}.$$

Therefore,

$$\breve{G}^g\big(x, \mathcal{B}^{\partial}_{L,r} \cap \Lambda_j\big) \le C(\log r + j)^{-3/2}.$$

Summing over $0 \le j \le J_1$, this shows

$$\breve{G}^g\big(x, \mathcal{B}^{\partial}_{L,r}\big) \le C(\log r)^{-1/2}.$$

(iii) Assume $\omega \in \mathrm{Good}_L$ or ω is bad on level $i = 1, 2, 3$. Then $1_{\mathcal{B}_L}\Delta = 1_{\mathcal{B}_L}(\hat{\Pi} - \hat{\pi})$. Furthermore, if $\mathcal{B}_L \cap \mathrm{Sh}_L(r_L) = \varnothing$, we have $\|\breve{G}^g 1_{\mathcal{B}_L}\Delta(x,\cdot)\|_1 \le C\delta$. By choosing δ small enough, the claim follows. If ω is bad on level 4 and $\mathcal{B}_L \subset V_{L/2}$, we do not gain a factor δ from $\|1_{\mathcal{B}_L}\Delta(y,\cdot)\|_1$. However, thanks to our modified transition kernels, using (5.11), $\|1_{\mathcal{B}_L}\Delta\breve{G}^g 1_{\mathcal{B}_L}(y,\cdot)\|_1 \le 1/5$ (recall that $\breve{G}^g \le \hat{G}^g$ pointwise), so that (ii) follows in this case, too. $\square$

REMARK 5.3. All $\delta_0 > 0$ and L_0 appearing in the next sections are understood to be chosen in such a way that if we take $\delta \in (0, \delta_0]$ and $L \ge L_0$, then the conclusions of Lemmata 5.2, 5.5 and Corollary 5.1 are valid.

6. Exit distributions from the ball. In this part, we prove the main estimates on exit measures that are required to propagate condition $\mathbf{Cond}(\delta, L_0, L_1)$. First, in Section 6.1 we collect some preliminary results involving the kernel p_L. Then, in Section 6.2, we estimate the total variation norm of the globally smoothed difference $D^*_{L,p_L,\psi,q}$, while in Section 6.3, we prove the estimates for the nonsmoothed quantity D^*_{L,p_L}.

We work with both the original kernels $\hat{\Pi}$, $\hat{\Pi}^g$, $\hat{\pi}$ as well as with the modified kernels $\breve{\Pi}$, $\breve{\Pi}^g$, $\breve{\pi}$ from Section 5.3. For the goodified exit measure from V_L, we write

$$\Pi^g{}_L = \mathrm{ex}_{V_L}\big(x, \cdot; \hat{\Pi}^g_{L,r}\big) = \mathrm{ex}_{V_L}\big(x, \cdot; \breve{\Pi}^g_{L,r}\big).$$

Throughout this part, for reasons of readability, we put $p = p_{s_L/20}$.

6.1. *Preliminaries.* We start with a generalization of Lemma 5.1 which forms one of the key steps in transferring condition **Cond** from one level to the next.

LEMMA 6.1. *Assume* $\mathbf{Cond}(\delta, L_0, L_1)$, *and let* $L_1 \le L \le L_1(\log L_1)^2$ *Then for all* $x \in V_L \setminus \mathrm{Sh}_L(2r)$, *with* $H(x) = \max\{L_0, h_{L,r}(x)\}$,

$$\begin{aligned}&\big\|\big(\hat{\Pi}^g_{L,r} - \hat{\pi}^{(p_L)}_{L,r}\big)\hat{\pi}^{(p_L)}_{L,r}(x,\cdot)\big\|_1\\ &\qquad \le C\min\big\{\log\big(s_L/H(x)\big)\big(\log H(x)\big)^{-9}, \big(\log H(x)\big)^{-8}\big\}\end{aligned}$$

and

$$\left\|\left(\hat{\Pi}^g_{L,r}-\hat{\pi}^{(p_L)}_{L,r}\right)(x,\cdot)\right\|_1\le 3\delta.$$

PROOF. Let $\Delta = 1_{V_L}(\hat{\Pi}^g-\hat{\pi}^{(p_L)})$. With $B=V_L\setminus \mathrm{Sh}_L(2r)$,

$$1_B\Delta = 1_B\big(\hat{\Pi}^g-\hat{\pi}^{(p)}\big)+1_B\big(\hat{\pi}^{(p)}-\hat{\pi}^{(p_L)}\big).$$

Using Lemma 5.1, the first term is bounded in total variation by 2δ. For the second term, Lemma 3.2 in combination with Lemma 4.5 yields the bound $C(\log H(x))^{-9}$. Similarly,

$$\begin{aligned}1_B\Delta\hat{\pi}^{(p_L)} = 1_B\big[&\big(\hat{\Pi}^g-\hat{\pi}^{(p)}\big)\hat{\pi}^{(p)}+\hat{\pi}^{(p)}\big(\hat{\pi}^{(p)}-\hat{\pi}^{(p_L)}\big)\\ &+\big(\hat{\pi}^{(p)}-\hat{\pi}^{(p_L)}\big)\hat{\pi}^{(p_L)}+\hat{\Pi}^g\big(\hat{\pi}^{(p_L)}-\hat{\pi}^{(p)}\big)\big].\end{aligned}$$

Here, the last three terms on the right are bounded in total variation by $C(\log H(x))^{-9}$, and for the first one can use Lemma 5.1. □

The next lemma is useful for the globally smoothed exit distributions.

LEMMA 6.2. *Assume* **Cond**(δ, L_0, L_1), *and let* $L_1\le L\le L_1(\log L_1)^2$. *Put* $\Delta = 1_{V_L}(\hat{\Pi}^g_{L,r_L}-\hat{\pi}^{(p_L)}_{L,r_L})$. *Then, for some* $C>0$, *for all* $\psi\in\mathcal{M}_L$, $q\in\mathcal{P}^{\mathrm{s}}_\iota$, *with* $\phi_{L,p_L,\psi,q}=\pi^{(p_L)}_L\hat{\pi}^{(q)}_\psi$ *as in Section* 4,

$$\sup_{x\in V_L}\sup_{z\in\mathbb{Z}^d}\left|\Delta\phi_{L,p_L,\psi,q}(x,z)\right|\le C(\log L)^{-12}L^{-d}.$$

PROOF. Write $\hat{\pi}$ for $\hat{\pi}^{(p_L)}$, ϕ for $\phi_{L,p_L,\psi,q}$. Using $\Delta\phi=\Delta\hat{\pi}\phi$ and the fact that $\Delta\hat{\pi}(x,\cdot)$ sums up to zero,

$$\begin{aligned}\left|\Delta\phi(x,z)\right| &= \left|\sum_{y\in V_L\cup\partial V_L}\Delta\hat{\pi}(x,y)\big(\phi(y,z)-\phi(x,z)\big)\right|\\ &\le \left\|\Delta\hat{\pi}(x,\cdot)\right\|_1\sup_{y:\,|\Delta\hat{\pi}(x,y)|>0}\left|\phi(y,z)-\phi(x,z)\right|.\end{aligned}$$

For $x\in V_L\setminus\mathrm{Sh}_L(2r_L)$, we have by Lemma 6.1

$$\left\|\Delta\hat{\pi}(x,\cdot)\right\|_1\le C\log\big(s_L/h_L(x)\big)(\log L)^{-9}.$$

Moreover, notice that $|\Delta\hat{\pi}(x,y)|>0$ implies $|y-x|\le Ch_L(x)$. Bounding $|\phi(y,z)-\phi(x,z)|$ by Lemma 4.7(iii), the statement follows for those x. If $x\in\mathrm{Sh}_L(2r_L)$, we simply bound $\|\Delta\hat{\pi}(x,\cdot)\|_1$ by 2. Now we can restrict the supremum to those $y\in V_L$ with $|x-y|\le 3r_L$, so the claim again follows from Lemma 4.7(iii). □

We defined the kernel p_L in terms of averaged variances of $\mathbb{E}[\hat{\Pi}_{L,r}]$. Combined with Lemma 3.1, this shows that the covariances of $\hat{\pi}^{(p_L)}_{L,r}$ agree with those of $\mathbb{E}[\hat{\Pi}_{L,r}]$ up to an error of order $O(s_L^{-1})$. The same holds true with $\hat{\Pi}_{L,r}$ replaced by $\hat{\Pi}^g_{L,r}$.

LEMMA 6.3. *Assume* $\mathbf{Cond}(\delta, L_0, L_1)$. *There exists a constant* $C = C(d)$ *such that for* $L_1 \le L \le L_1(\log L_1)^2$, *we have*

$$\left|\sum_{y \in V_L} (\mathbb{E}[\hat{\Pi}^g_{L,r}] - \hat{\pi}^{(p_L)}_{L,r})(0, y)\frac{y_i^2}{|y|^2}\right| \le C(\log L)^3 L^{-1} \qquad \textit{for all } i = 1, \dots, d.$$

PROOF. Note that under $\mathbf{Cond}(\delta, L_0, L_1)$,

$$\mathbb{E}\big[\|(\hat{\Pi} - \hat{\Pi}^g)(0, \cdot)\|_1\big] \le 2\mathbb{P}(0 \in \mathcal{B}_L) \le C \exp\big(-(1/2)(\log L)^2\big).$$

Therefore,

$$2p_L(e_i) = \sum_y \mathbb{E}[\hat{\Pi}^g](0, y)\frac{y_i^2}{|y|^2} + O\big(\exp\big(-(1/2)(\log L)^2\big)\big).$$

On the other hand, as in (3.1) with p replaced by p_L,

$$2p_L(e_i) = \sum_y \hat{\pi}^{(p_L)}(0, y)\frac{y_i^2}{|y|^2} + O(s_L^{-1}),$$

and the statement follows. $\square$

6.2. *Globally smoothed exits*. Our objective here is to establish the estimates for the smoothed difference $D^*_{L,p_L,\psi,q}$. In this section, we only work with coarse graining schemes corresponding to the choice $r = r_L$. Lemma 6.4 compares the "goodified" smoothed exit distribution from the ball of radius L with that of symmetric random walk with transition kernel p_L. In particular, it provides an estimate for $D^*_{L,p_L,\psi,q}$ on Good_L.

LEMMA 6.4. *Assume* **A1**. *For every* (*small*) *constant* c_0, *there exist* $\delta_0 > 0$ *and* $L_0 \in \mathbb{N}$ *such that if* $\delta \in (0, \delta_0]$ *and* $L_1 \ge L_0$, *then* $\mathbf{Cond}(\delta, L_0, L_1)$ *implies that for* $L_1 \le L \le L_1(\log L_1)^2$, *for all* $\psi \in \mathcal{M}_L$ *and all* $q \in \mathcal{P}^{\mathrm{s}}_\iota$,

$$\mathbb{P}\Big(\sup_{x \in V_L} \|(\Pi^g{}_L - \pi^{(p_L)}_L)\hat{\pi}^{(q)}_\psi(x, \cdot)\|_1 \ge c_0(\log L)^{-9}\Big) \le \exp\big(-(\log L)^{7/3}\big).$$

PROOF. Clearly, the claim follows if we show that

$$\sup_{x \in V_L} \sup_{z \in \mathbb{Z}^d} \mathbb{P}\big(\big|(\Pi^g - \pi^{(p_L)})\hat{\pi}^{(q)}_\psi(x, z)\big| \ge c_0(\log L)^{-9} L^{-d}\big) \le \exp\big(-(\log L)^{5/2}\big). \tag{6.1}$$

We use the abbreviations $\phi = \pi_L^{(p_L)} \hat{\pi}_\psi^{(q)}$, $\Delta = 1_{V_L}(\hat{\Pi}^g - \hat{\pi}^{(p_L)})$, $\hat{g} = \hat{g}^{(p_L)}$ and $\hat{\pi} = \hat{\pi}^{(p_L)}$. By the perturbation expansion [cf. (2.9)],

$$(\Pi^g - \pi^{(p_L)})\hat{\pi}_\psi^{(q)} = \hat{G}^g \Delta\phi.$$

Set $S = \mathrm{Sh}_L(2L/(\log L)^2)$, and write

$$\hat{G}^g \Delta\phi = \hat{G}^g 1_S \Delta\phi + \hat{G}^g 1_{S^c} \Delta\phi. \tag{6.2}$$

Using $\hat{G}^g \preceq C\Gamma$, Lemma 5.4(iv) (with $r = r_L$) and Lemma 6.2 yield the estimate

$$\left|\hat{G}^g 1_S \Delta\phi(x,z)\right| \le \sup_{x\in V_L} \hat{G}^g(x,S) \sup_{y\in V_L} \left|\Delta\phi(y,z)\right| \le (\log L)^{-19/2} L^{-d}$$

for L large. It remains to bound $|\hat{G}^g 1_{S^c} \Delta\phi(x,z)|$. With $B = V_L \setminus \mathrm{Sh}_L(2r_L)$,

$$\hat{G}^g = \hat{g} 1_B \Delta \hat{G}^g + \hat{g} 1_{B^c} \Delta \hat{G}^g + \hat{g}.$$

By replacing successively $\hat{G}^g$ in the first summand on the right-hand side,

$$\begin{aligned} \hat{G}^g 1_{S^c} \Delta\phi &= \left(\sum_{k=0}^\infty (\hat{g} 1_B \Delta)^k \hat{g} + \sum_{k=0}^\infty (\hat{g} 1_B \Delta)^k \hat{g} 1_{B^c} \Delta \hat{G}^g\right) 1_{S^c} \Delta\phi \\ &= F 1_{S^c} \Delta\phi + F 1_{B^c} \Delta \hat{G}^g 1_{S^c} \Delta\phi, \end{aligned} \tag{6.3}$$

where $F = \sum_{k=0}^\infty (\hat{g} 1_B \Delta)^k \hat{g}$. With $R = \sum_{k=1}^\infty (1_B \Delta)^k \hat{\pi}$, expansion (2.7) shows

$$F = \hat{g} \sum_{m=0}^\infty (R\hat{g})^m \sum_{k=0}^\infty (1_B \Delta)^k.$$

From the proof of Lemma 5.2(ii) we learn that $|F| \preceq C\Gamma$. By Lemma 5.4(iv), (v) and again Lemma 6.2, we see that for large L, uniformly in $x \in V_L$ and $z \in \mathbb{Z}^d$,

$$\begin{aligned} &\left|F 1_{B^c} \Delta \hat{G}^g 1_{S^c} \Delta\phi(x,z)\right| \\ &\quad \le C\Gamma\big(x, \mathrm{Sh}_L(2r_L)\big) \sup_{v\in \mathrm{Sh}_L(3r_L)} \Gamma\big(v, S^c \cap V_L\big) \sup_{w\in V_L} \left|\Delta\phi(w,z)\right| \\ &\quad \le (\log L)^{-11} L^{-d}. \end{aligned}$$

Thus, the second summand of (6.3) is harmless. However, with the first summand one has to be more careful. With $\xi = \hat{g} \sum_{k=0}^\infty (1_B \Delta)^k 1_{S^c} \Delta\phi$, we have

$$F 1_{S^c} \Delta\phi = \xi + \hat{g} \sum_{m=0}^\infty (R\hat{g})^m R\xi = \xi + F 1_B \Delta \hat{\pi} \xi.$$

Clearly, $|F 1_B \Delta \hat{\pi}(x,y)| \le C(\log L)^{-3}$, so it remains to estimate $\xi(y,z)$, uniformly in y and z. Set $N = N(L) = \lceil \log\log L \rceil$. For small δ, the summands of

ξ with $k \geq N$ are readily bounded by

$$\sup_{y \in V_L} \sup_{z \in \mathbb{Z}^d} \sum_{k=N}^{\infty} \left| \hat{g}(1_B \Delta)^k 1_{S^c} \Delta\phi(y, z) \right| \leq C(\log L)^6 \sum_{k=N}^{\infty} \delta^k (\log L)^{-12} L^{-d}$$
$$\leq (\log L)^{-10} L^{-d}.$$

Now we look at the summands with $k < N$. Since the coarse grained walk cannot bridge a gap of length $L/(\log L)^2$ in less than N steps, we can drop the kernel 1_B. Defining $S' = \mathrm{Sh}_L(3L/(\log L)^2)$, we thus have

$$\hat{g}(1_B \Delta)^k 1_{S^c} \Delta\phi = \hat{g} 1_{S'} \Delta^k 1_{S^c} \Delta\phi + \hat{g} 1_{S'^c} \Delta^k 1_{S^c} \Delta\phi.$$

The first summand is bounded in the same way as $\hat{G}^g 1_S \Delta\phi$ from (6.2), and we can drop the kernel 1_{S^c} in the second summand. Therefore, (6.1) follows if we show

$$\sup_{x \in V_L} \sup_{z \in \mathbb{Z}^d} \mathbb{P}\left(\left| \sum_{k=1}^{N} \hat{g} 1_{S'^c} \Delta^k \phi(x, z) \right| \geq \frac{1}{2} c_0 (\log L)^{-9} L^{-d} \right) \leq \exp\bigl(-(\log L)^{5/2}\bigr).$$

For $j \in \mathbb{Z}$, consider the interval $I_j = (jNs_L, (j+1)Ns_L] \cap \mathbb{Z}$. We divide $S'^c \cap V_L$ into subsets $W_{\mathbf{j}} = (S'^c \cap V_L) \cap (I_{j_1} \times \cdots \times I_{j_d})$, where $\mathbf{j} = (j_1, \ldots, j_d) \in \mathbb{Z}^d$. Let J be the set of those $\mathbf{j}$ for which $W_{\mathbf{j}} \neq \varnothing$. Then we can find a constant K depending only on the dimension and a disjoint partition of J into sets $J_1, \ldots, J_K$, such that for any $1 \leq \ell \leq K$,

$$\mathbf{j}, \mathbf{j}' \in J_\ell, \qquad \mathbf{j} \neq \mathbf{j}' \quad \Longrightarrow \quad \mathrm{d}(W_{\mathbf{j}}, W_{\mathbf{j}'}) > Ns_L. \tag{6.4}$$

For $x \in V_L$, $z \in \mathbb{Z}^d$, we set

$$\xi_{\mathbf{j}} = \xi_{\mathbf{j}}(x, z) = \sum_{y \in W_{\mathbf{j}}} \sum_{k=1}^{N} \hat{g}(x, y) \Delta^k \phi(y, z),$$

and further $t = t(d, c_0, L) = (1/2) c_0 (\log L)^{-9} L^{-d}$. Assume that we can prove

$$\left| \sum_{\mathbf{j} \in J} \mathbb{E}[\xi_{\mathbf{j}}] \right| \leq \frac{t}{2}. \tag{6.5}$$

Then

$$\mathbb{P}\left(\left| \sum_{\mathbf{j} \in J} \xi_{\mathbf{j}} \right| \geq t \right) \leq \mathbb{P}\left(\left| \sum_{\mathbf{j} \in J} \xi_{\mathbf{j}} - \mathbb{E}[\xi_{\mathbf{j}}] \right| \geq \frac{t}{2} \right)$$
$$\leq K \max_{1 \leq \ell \leq K} \mathbb{P}\left(\left| \sum_{\mathbf{j} \in J_\ell} \xi_{\mathbf{j}} - \mathbb{E}[\xi_{\mathbf{j}}] \right| \geq \frac{t}{2K} \right).$$

Due to (6.4), the random variables $\xi_{\mathbf{j}} - \mathbb{E}[\xi_{\mathbf{j}}]$, $\mathbf{j} \in J_\ell$, are independent and centered. Hoeffding's inequality yields, with $\|\xi_{\mathbf{j}}\|_\infty = \sup_{\omega \in \Omega} |\xi_{\mathbf{j}}(\omega)|$,

$$\mathbb{P}\left(\left| \sum_{\mathbf{j} \in J_\ell} \xi_{\mathbf{j}} - \mathbb{E}[\xi_{\mathbf{j}}] \right| \geq \frac{t}{2K} \right) \leq 2 \exp\left(-c \frac{L^{-2d} (\log L)^{-18}}{\sum_{\mathbf{j} \in J_\ell} \|\xi_{\mathbf{j}}\|_\infty^2} \right) \tag{6.6}$$

for some constant $c > 0$. In order to control the sup-norm of the $\xi_{\mathbf{j}}$, we use the estimates

$$\hat{g}(x, W_{\mathbf{j}}) \leq C\Gamma^{(2)}(x, W_{\mathbf{j}}) \leq \frac{CN^d s_L^d}{s_L^2(s_L + \mathrm{d}(x, W_{\mathbf{j}}))^{d-2}} = CN^d\left(1 + \frac{\mathrm{d}(x, W_{\mathbf{j}})}{s_L}\right)^{2-d},$$

and, by Lemmata 6.1 and 6.2, for $y \in W_{\mathbf{j}}$, $|\Delta^k\phi(y, z)| \leq C\delta^{k-1}k(\log L)^{-12}L^{-d}$. Altogether we arrive at

$$\|\xi_{\mathbf{j}}\|_\infty \leq C\left(1 + \frac{\mathrm{d}(x, W_{\mathbf{j}})}{s_L}\right)^{2-d} N^d(\log L)^{-12}L^{-d},$$

uniformly in z. If we put the last display into (6.6), we get, using $d \geq 3$ in the last line,

$$\begin{aligned}\mathbb{P}\left(\left|\sum_{\mathbf{j}\in J_\ell}\xi_{\mathbf{j}} - \mathbb{E}[\xi_{\mathbf{j}}]\right| \geq \frac{t}{2K}\right) &\leq 2\exp\left(-c\frac{(\log L)^6}{N^4\sum_{r=1}^{C(\log L)^3/N} r^{-d+3}}\right)\\ &\leq 2\exp\left(-c\frac{(\log L)^3}{N^3}\right).\end{aligned}$$

It follows that for L large enough, uniformly in x and z,

$$\mathbb{P}\left(\left|\sum_{\mathbf{j}\in J}\xi_{\mathbf{j}}\right| \geq \frac{1}{2}c_0(\log L)^{-9}L^{-d}\right) \leq \exp(-(\log L)^{5/2}).$$

It remains to prove (6.5). We have

$$\left|\sum_{\mathbf{j}\in J}\mathbb{E}[\xi_{\mathbf{j}}]\right| \leq \sum_{y\in S'^c}\hat{g}(x, y)\left|\sum_{y'\in V_L}\mathbb{E}\left[\sum_{k=1}^N \Delta^k\hat{\pi}(y, y')\right]\phi(y', z)\right|.$$

The sum over the Green's function is estimated by $\hat{g}(x, S'^c) \leq C(\log L)^6$. In the innermost sum, we treat the summands with $k \geq 2$ and $k = 1$ in different ways. For each summand with $k \geq 2$, we use Proposition 4.1 applied to $\nu(\cdot) = \mathbb{E}[\Delta^k\hat{\pi}(y, y + \cdot)]$; see Remark 6.1. Recalling Lemma 6.1, we obtain by choosing δ small enough, uniformly in $y \in S'^c$,

$$\begin{aligned}\left|\sum_{y'\in V_L}\mathbb{E}[\Delta^k\hat{\pi}(y, y')]\phi(y', z)\right| &\leq C\delta^{k-1}(\log L)^{-9}(ks_L)^2L^{-(d+2)}\\ &\leq (c_0/8)(1/2)^{k-1}(\log L)^{-15}L^{-d}.\end{aligned}$$

For the summand $\nu(\cdot) = \mathbb{E}[\Delta\hat{\pi}(y, y + \cdot)]$ corresponding to $k = 1$, the proof of Proposition 4.1 shows

$$\begin{aligned}&\left|\sum_{y'\in V_L}\mathbb{E}[\Delta\hat{\pi}(y, y')]\phi(y', z)\right|\\ &\quad\leq \left|\frac{1}{2}\sum_{i=1}^d\frac{\partial^2}{\partial x_i^2}\phi^{\mathrm{B}}(y, z)\sum_{y'}\nu(y')(y_i')^2\right| + C(\log L)^{-18}L^{-d}.\end{aligned}$$

By translational invariance and Lemma 6.3 in the last step,

$$\left|\sum_{y'} \nu(y')(y_i')^2\right| = \left|\sum_{y} (\mathbb{E}[\hat{\Pi}^g] - \hat{\pi})\hat{\pi}(0,y)y_i^2\right| = \left|\sum_{y} (\mathbb{E}[\hat{\Pi}^g] - \hat{\pi})(0,y)y_i^2\right|$$
$$\leq CL/(\log L)^3.$$

Together with Lemma 4.7(ii), this shows (6.5) and completes the proof. □

REMARK 6.1. Note that for $y \in S'^c$, the signed measure ν fulfills the requirement of Proposition 4.1. Indeed, after $N = \lceil \log\log L\rceil$ steps away from y, the coarse grained walks are still in the interior part of V_L, where the coarse graining radius did not start to shrink. Therefore, the symmetry condition **A1** carries over to the signed measure $\mathbb{E}[\sum_{k=1}^N (1_{V_L}(\hat{\Pi} - \hat{\pi}))^k \hat{\pi}(y, y+\cdot)]$. Replacing $\hat{\Pi}$ by $\hat{\Pi}^g$ does not destroy the symmetries of this measure, so that Proposition 4.1 can be applied to ν.

Next, we estimate $D^*_{L,p_L,\psi,q}$ on environments with bad points. Again, we make the choice $r = r_L$.

LEMMA 6.5. *In the setting of Lemma* 6.4, *with a possibly smaller value of* δ_0 *and a larger* L_0, *we have for* $i = 1, 2, 3, 4$,

$$\mathbb{P}\Big(\sup_{x\in V_{L/5}} \big\|(\Pi_L - \pi_L^{(p_L)})\hat{\pi}_\psi^{(q)}(x,\cdot)\big\|_1 > (\log L)^{-9+9(i-1)/4}; \mathrm{OneBad}_L^{(i)}\Big)$$
$$\leq \exp\big(-(\log L)^{7/3}\big).$$

PROOF. For the ease of readability, we write $\hat{\pi}_\psi$ for $\hat{\pi}_\psi^{(q)}$ and let $\phi = \pi^{(p_L)}\hat{\pi}_\psi$. By the triangle inequality,

$$\begin{aligned}\tag{6.7} &\big\|(\Pi - \pi^{(p_L)})\hat{\pi}_\psi(x,\cdot)\big\|_1 \\ &\quad\leq \big\|(\Pi - \Pi^g)\hat{\pi}_\psi(x,\cdot)\big\|_1 + \big\|(\Pi^g - \pi^{(p_L)})\hat{\pi}_\psi(x,\cdot)\big\|_1.\end{aligned}$$

The second summand on the right can be bounded by Lemma 6.4. For the first term we have by the perturbation expansion [see (2.9)], with $\Delta = 1_{V_L}(\breve{\Pi} - \breve{\Pi}^g)$,

$$(\Pi - \Pi^g)\hat{\pi}_\psi = \breve{G}^g 1_{\mathcal{B}_L}\Delta\Pi\hat{\pi}_\psi.$$

Note that since we are on OneBad_L, the set $\mathcal{B}_L$ is contained in a small region. First assume that $\mathcal{B}_L \subset \mathrm{Sh}_L(L/(\log L)^{10})$. Then $\sup_{x\in V_{L/5}} \breve{G}^g(x, \mathcal{B}_L) \leq C(\log L)^{-10}$ by Corollary 5.1, which bounds the first summand of (6.7). Next assume ω bad on level 4 and $\mathcal{B}_L \not\subset V_{L/2}$. Then $\sup_{x\in V_{L/5}} \breve{G}^g(x, \mathcal{B}_L) \leq C(\log L)^{-3}$ by the same corollary, which is enough for this case.

It remains to consider the cases ω bad on level at most 3 with $\mathcal{B}_L \not\subset \mathrm{Sh}_L(L/(\log L)^{10})$, or ω bad on level 4 with $\mathcal{B}_L \subset V_{L/2}$. We expand

$$\begin{aligned}(\Pi - \Pi^g)\hat{\pi}_\psi &= (\breve{G}^g 1_{\mathcal{B}_L} \Delta\Pi)\hat{\pi}_\psi \\ &= \sum_{k=1}^{\infty} (\breve{G}^g 1_{\mathcal{B}_L} \Delta)^k \Pi^g \hat{\pi}_\psi \\ &= \sum_{k=1}^{\infty} (\breve{G}^g 1_{\mathcal{B}_L} \Delta)^k \phi + \sum_{k=1}^{\infty} (\breve{G}^g 1_{\mathcal{B}_L} \Delta)^k (\Pi^g - \pi^{(p_L)})\hat{\pi}_\psi \\ &= F_1 + F_2.\end{aligned}$$

By Corollary 5.1,

$$\begin{aligned}\|F_1(x,\cdot)\|_1 &\le \sum_{k=0}^{\infty} \|(\breve{G}^g 1_{\mathcal{B}_L} \Delta)^k (x,\cdot)\|_1 \sup_{v \in V_L} \breve{G}^g(v, \mathcal{B}_L) \sup_{w \in \mathcal{B}_L} \|\Delta\phi(w,\cdot)\|_1 \\ &\le C \sup_{w \in \mathcal{B}_L} \|\Delta\phi(w,\cdot)\|_1.\end{aligned}$$

Proceeding as in Lemma 6.2, for $w \in \mathcal{B}_L$,

$$\|\Delta\phi(w,\cdot)\|_1 \le \|\Delta\hat{\pi}^{(p_L)}(w,\cdot)\|_1 \sup_{w' : |\Delta\hat{\pi}^{(p_L)}(w,w')|>0} \|\phi(w',\cdot) - \phi(w,\cdot)\|_1. \tag{6.8}$$

Lemma 4.7(iii) bounds the second factor on the right by $C(\log L)^{-3}$. If ω is bad on level 4, we simply bound the first factor by 2. If ω is bad on level at most 3, we have on $\mathcal{B}_L$ the equality $\Delta = \hat{\Pi} - \hat{\pi}^{(p)}$. With $p' = p_{h_L(w)}$, the triangle inequality gives, for every $i = 1, 2, 3$,

$$\begin{aligned}\|\Delta\hat{\pi}^{(p_L)}(w,\cdot)\|_1 &\le \|(\hat{\Pi} - \hat{\pi}^{(p')})\hat{\pi}^{(p')}(w,\cdot)\|_1 + \|(\hat{\pi}^{(p')} - \hat{\pi}^{(p)})(w,\cdot)\|_1 \\ &\quad + \sup_{y \in V_L \setminus \mathrm{Sh}_L(r_L)} \|(\hat{\pi}^{(p_L)} - \hat{\pi}^{(p')})(y,\cdot)\|_1.\end{aligned}$$

By definition, the first summand is bounded by $C(\log L)^{-9+9i/4}$. For the second and third summand, we use Lemma 4.5 and the fact that by Lemma 3.2, $\|p' - p_L\|_1 \le C(\log\log L)(\log L)^{-9}$, and similarly for $\|p - p'\|_1$. For all $i = 1, 2, 3, 4$, we arrive at $\|F_1(x,\cdot)\|_1 \le C(\log L)^{-12+9i/4}$. For F_2, we obtain once more with Corollary 5.1,

$$\|F_2(x,\cdot)\|_1 \le C \sup_{y \in V_L} \|(\Pi^g - \pi^{(p_L)})\hat{\pi}_\psi(y,\cdot)\|_1.$$

This term is again estimated by Lemma 6.4 (with c_0 small enough), which completes the proof. $\square$

6.3. *Nonsmoothed and locally smoothed exits.* Here, we aim at bounding the total variation distance of the exit measures without additional smoothing (Lemma 6.6), as well as in the case where the exit measures are convoluted with a kernel of constant smoothing radius s (Lemma 6.7).

Throughout this part, we work with transition kernels coming from coarse graining schemes with constant parameter r. We always assume L large enough such that $r < r_L$. The right choice of r depends on the deviations δ and η we are shooting for and will become clear from the proofs. In either case, we choose $r \geq r_0$, where r_0 is the constant from Section 3.4. The value of r will then also influence the choice of the perturbation ε_0 in Lemma 6.6 and the smoothing radius ℓ in Lemma 6.7, respectively.

We recall the partition of bad points into the sets $\mathcal{B}_L$, $\mathcal{B}_{L,r}$, $\mathcal{B}^{\partial}_{L,r}$, $\mathcal{B}^{\star}_{L,r}$ and the classification of environments into Good_L, OneBad_L and $\mathrm{BdBad}_{L,r}$ from Section 3.

The bounds for $\mathrm{ManyBad}_L$ (Lemma 3.3) and for $\mathrm{BdBad}_{L,r}$ (Lemma 3.4) ensure that we may restrict ourselves to environments

$$\omega \in \mathrm{OneBad}_L \cap (\mathrm{BdBad}_{L,r})^c.$$

For such environments, we introduce two disjoint sets $Q^1_{L,r}(\omega)$, $Q^2_{L,r}(\omega) \subset V_L$ as follows:

- If $\mathcal{B}_L(\omega) \subset V_{L/2}$, set $Q^1_{L,r}(\omega) = \mathcal{B}_L(\omega)$ and $Q^2_{L,r}(\omega) = \mathcal{B}^{\partial}_{L,r}(\omega)$.
- If $\mathcal{B}_L(\omega) \not\subset V_{L/2}$, set $Q^1_{L,r}(\omega) = \varnothing$ and $Q^2_{L,r}(\omega) = \mathcal{B}^{\star}_{L,r}(\omega)$.

Of course, on Good_L, we have $Q^1_{L,r}(\omega) = \varnothing$ and $Q^2_{L,r}(\omega) = \mathcal{B}^{\partial}_{L,r}(\omega)$.

Recall that we write p for $p_{s_L/20}$.

LEMMA 6.6. *There exists $\delta_0 > 0$ such that if $\delta \in (0, \delta_0]$, there exist $\varepsilon_0 = \varepsilon_0(\delta) > 0$ and $L_0 = L_0(\delta) > 0$ with the following property: if $\varepsilon \leq \varepsilon_0$ and $L_1 \geq L_0$, then* $\mathbf{A0}(\varepsilon)$, $\mathbf{Cond}(\delta, L_0, L_1)$ *imply that for $L_1 \leq L \leq L_1(\log L_1)^2$,*

$$\mathbb{P}\Big(\sup_{x \in V_{L/5}} \big\|(\Pi_L - \pi_L^{(p_L)})(x, \cdot)\big\|_1 > \delta\Big) \leq \exp\Big(-\frac{9}{5}(\log L)^2\Big).$$

PROOF. We choose $\delta_0 > 0$ according to Remark 5.3 and take $\delta \in (0, \delta_0]$. The right choice of ε_0 and L_0 will be clear from the course of the proof. From Lemmata 3.3 and 3.4 we learn that if we take $L_1 \geq L_0$ for L_0 large and $L_1 \leq L \leq L_1(\log L_1)^2$, then under $\mathbf{Cond}(\delta, L_0, L_1)$,

$$\mathbb{P}(\mathrm{ManyBad}_L \cup \mathrm{BdBad}_{L,r}) \leq \exp\big(-\tfrac{9}{5}(\log L)^2\big).$$

Therefore, the claim follows if we show that on $\mathrm{OneBad}_L \cap (\mathrm{BdBad}_{L,r})^c$, we have for all sufficiently small ε and all large L, $x \in V_{L/5}$,

$$\big\|(\Pi - \pi^{(p_L)})(x, \cdot)\big\|_1 \leq \delta.$$

Let $\omega \in \mathrm{OneBad}_L \cap (\mathrm{BdBad}_{L,r})^c$. We use the partition of $\mathcal{B}^{\star}_{L,r}$ into the sets Q^1, Q^2 described above. With $\Delta = 1_{V_L}(\breve{\Pi} - \breve{\Pi}^g)$, we have inside V_L,

$$\Pi = \breve{G}^g 1_{Q^1} \Delta \Pi + \breve{G}^g 1_{Q^2} \Delta \Pi + \Pi^g.$$

By replacing successively Π in the first summand on the right-hand side, we arrive at

$$\Pi = \sum_{k=0}^{\infty} \left(\breve{G}^g 1_{Q^1} \Delta\right)^k \Pi^g + \sum_{k=0}^{\infty} \left(\breve{G}^g 1_{Q^1} \Delta\right)^k \breve{G}^g 1_{Q^2} \Delta \Pi.$$

Since with $\Delta' = 1_{V_L}(\breve{\Pi}^g - \breve{\pi}^{(p_L)})$, $\Pi^g = \pi^{(p_L)} + \breve{G}^g \Delta' \pi^{(p_L)}$, we obtain

$$\begin{aligned}
\Pi - \pi^{(p_L)} &= \sum_{k=1}^{\infty} \left(\breve{G}^g 1_{Q^1} \Delta\right)^k \pi^{(p_L)} + \sum_{k=0}^{\infty} \left(\breve{G}^g 1_{Q^1} \Delta\right)^k \breve{G}^g 1_{Q^2} \Delta \Pi \\
&\quad + \sum_{k=0}^{\infty} \left(\breve{G}^g 1_{Q^1} \Delta\right)^k \breve{G}^g \Delta' \pi^{(p_L)} \qquad (6.9) \\
&= F_1 + F_2 + F_3.
\end{aligned}$$

We will now prove that each of the three parts F_1, F_2, F_3 is bounded by $\delta/3$. If $Q^1 \neq \varnothing$, then $Q^1 = \mathcal{B}_L \subset V_{L/2}$ and $Q^2 = \mathcal{B}^{\partial}_{L,r}$. Using Corollary 5.1 in the second and Lemma 4.1(ii) in the third inequality,

$$\begin{aligned}
\|F_1(x,\cdot)\|_1 &\leq \sum_{k=0}^{\infty} \left\|\left(\breve{G}^g 1_{\mathcal{B}_L} \Delta\right)^k (x,\cdot)\right\|_1 \sup_{y \in V_L} \breve{G}^g(y, \mathcal{B}_L) \sup_{z \in \mathcal{B}_L} \left\|\Delta' \pi^{(p_L)}(z,\cdot)\right\|_1 \\
&\leq C \sup_{z \in V_{L/2}} \left\|\Delta' \pi^{(p_L)}(z,\cdot)\right\|_1 \leq C(\log L)^{-3} \qquad (6.10) \\
&\leq C(\log L_0)^{-3} \leq \delta/3
\end{aligned}$$

for $L_0 = L_0(\delta)$ large enough, $L \geq L_0$. Regarding F_2, we have in the case $Q^1 \neq \varnothing$ by Corollary 5.1(ii),

$$\|F_2(x,\cdot)\|_1 \leq C \sup_{y \in V_{2L/3}} \breve{G}^g\left(y, \mathcal{B}^{\partial}_{L,r}\right) \leq C(\log r)^{-1/2}.$$

On the other hand, if $Q^1 = \varnothing$, then $\mathcal{B}_L$ is outside $V_{L/3}$, so that by Corollary 5.1(i), (ii)

$$\|F_2(x,\cdot)\|_1 \leq 2\breve{G}^g\left(x, \mathcal{B}^{\partial}_{L,r} \cup \mathcal{B}_L\right) \leq C\left((\log L)^{-3} + (\log r)^{-1/2}\right).$$

Altogether, for all $L \geq L_0$, by choosing $r = r(\delta)$ and $L_0 = L_0(\delta, r)$ large enough,

$$\|F_2(x,\cdot)\|_1 \leq C\left((\log L_0)^{-3} + (\log r)^{-1/2}\right) \leq \delta/3. \qquad (6.11)$$

It remains to handle F_3. Once again with Corollary 5.1(iii) for some $C_3 > 0$,

$$\|F_3(x,\cdot)\|_1 \leq C_3 \sup_{y \in V_{2L/3}} \left\|\hat{G}^g \Delta' \pi^{(p_L)}(y,\cdot)\right\|_1.$$

We write

$$\breve{G}^g\Delta'\pi^{(p_L)} = \breve{G}^g 1_{V_L\setminus Q^1}\Delta'\pi^{(p_L)} + \breve{G}^g 1_{Q^1}\Delta'\pi^{(p_L)}. \tag{6.12}$$

As in (6.10), we deduce that

$$\|\breve{G}^g 1_{Q^1}\Delta'\pi^{(p_L)}\|_1 \le C_3^{-1}\delta/21 \tag{6.13}$$

for L_0 large enough, $L \ge L_0$. Concerning the first term of (6.12), we note that on $V_L \setminus Q^1$, by definition,

$$\Delta'\pi^{(p_L)} = (\hat{\Pi}^g - \hat{\pi}^{(p_L)})\hat{\pi}^{(p_L)}\pi^{(p_L)}.$$

For $z \in V_L \setminus (\mathrm{Sh}_L(2r_L) \cup Q^1)$, we obtain with Lemma 6.1,

$$\|\Delta'\pi^{(p_L)}(z,\cdot)\|_1 \le C(\log\log L)(\log L)^{-9}.$$

Since $\hat{G}^g(y, V_L) \le C(\log L)^6$, it follows that

$$\sup_{y\in V_{2L/3}} \|\hat{G}^g 1_{V_L\setminus(Q^1\cup \mathrm{Sh}_L(2r_L))}\Delta'\pi^{(p_L)}(y,\cdot)\|_1 \le C(\log L)^{-2} \le C_3^{-1}\delta/21 \tag{6.14}$$

for L large. Recall the definition of the layers Λ_j from Section 3.4. For $z \in \Lambda_j$, $1 \le j \le J_1$, we have again by Lemma 6.1,

$$\|\Delta'\pi^{(p_L)}(z,\cdot)\|_1 \le C(\log r + j)^{-8}.$$

Using Lemma 5.4(iii), it follows that $\hat{G}^g(y, \Lambda_j) \le C$ for some constant C, independent of r and j. Thus

$$\sup_{y\in V_{2L/3}} \|\hat{G}^g 1_{\bigcup_{j=1}^{J_1}\Lambda_j}\Delta'\pi^{(p_L)}(y,\cdot)\|_1 \le C(\log r)^{-7} \le C_3^{-1}\delta/21, \tag{6.15}$$

if r is chosen large enough. For the first layer Λ_0, there is a constant C_0 with

$$\sup_{y\in V_{2L/3}} \|\hat{G}^g 1_{\Lambda_0}\Delta'\pi^{(p_L)}(y,\cdot)\|_1 \le C_0 \sup_{z\in\Lambda_0}\|\Delta'(z,\cdot)\|_1.$$

Now, with $p_o \equiv 1/(2d)$ denoting the transition kernel of simple random walk,

$$\begin{aligned}\|\Delta'(z,\cdot)\|_1 \le{}& \|(\hat{\Pi} - \hat{\pi}^{(p_o)})(z,\cdot)\|_1 + \|(\hat{\pi}^{(p_o)} - \hat{\pi}^{(p_L)})(z,\cdot)\|_1\\ &+ \|(\hat{\Pi}^g - \hat{\Pi})(z,\cdot)\|_1.\end{aligned} \tag{6.16}$$

Concerning the second summand of (6.16), recall that by Lemma 3.2, $\|p_o - p_L\|_1 \le (\log L_0)^{-7}$. Therefore, choosing L_0 large enough (r is now fixed), we can guarantee that

$$\|(\hat{\pi}^{(p_o)} - \hat{\pi}^{(p_L)})(z,\cdot)\|_1 \le C_0^{-1}C_3^{-1}\delta/21,$$

uniformly in $z \in \Lambda_0$. The third summand of (6.16) vanishes if $z \in \Lambda_0 \setminus \mathcal{B}^\star_{L,r}$. For $z \in \mathcal{B}^\star_{L,r}$,

$$\begin{aligned}\|(\hat{\Pi}^g - \hat{\Pi})(z,\cdot)\|_1 &= \|(\hat{\pi}^{(p)} - \hat{\Pi})(z,\cdot)\|_1\\ &\le \|(\hat{\Pi} - \hat{\pi}^{(p_o)})(z,\cdot)\|_1 + \|(\hat{\pi}^{(p_o)} - \hat{\pi}^{(p)})(z,\cdot)\|_1.\end{aligned}$$

The last term on the right is bounded in the same way as the second term of (6.16). For the first summand on the right of the inequality [and also for the first summand of (6.16)], we may simply choose $\varepsilon_0 = \varepsilon_0(\delta, r)$ small enough such that for $\varepsilon \le \varepsilon_0$,

$$\sup_{z\in\Lambda_0} \|(\hat{\Pi} - \hat{\pi}^{(p_o)})(z, \cdot)\|_1 \le C_0^{-1} C_3^{-1} \delta/21.$$

Altogether we have shown that $\|F_3(x, \cdot)\|_1 \le \delta/3$, and the claim is proved. □

REMARK 6.2. As the proof shows, we do not have to assume condition $\mathbf{Cond}(\delta, L_0, L_1)$ for the desired deviation δ. We could instead assume $\mathbf{Cond}(\delta', L_0, L_1)$ for some arbitrary $0 < \delta' \le \delta_0$. However, L_0 depends on the chosen δ. This observation will be useful in the next lemma.

LEMMA 6.7. *There exists $\delta_0 > 0$ with the following property: for each $\eta > 0$, there exist a smoothing radius $\ell_0 = \ell_0(\eta)$ and $L_0 = L_0(\eta)$ such that if $L_1 \ge L_0$, $\ell \ge \ell_0$ and $\mathbf{Cond}(\delta, L_0, L_1)$ holds for some $\delta \in (0, \delta_0]$, then for $L_1 \le L \le L_1(\log L_1)^2$ and $\psi \equiv \ell$, for all $q \in \mathcal{P}_\iota^{\mathrm{s}}$*

$$\mathbb{P}\Big(\sup_{x\in V_{L/5}} \|(\Pi_L - \pi_L^{(p_L)})\hat{\pi}_\psi^{(q)}(x, \cdot)\|_1 > \eta\Big) \le \exp\Big(-\frac{9}{5}(\log L)^2\Big).$$

PROOF. The proof is based on a modification of the computations in the foregoing lemma. Let δ_0 be as in Lemma 6.6. We fix an arbitrary $0 < \delta \le \delta_0$ and assume $\mathbf{Cond}(\delta, L_0, L_1)$ for some $L_1 \ge L_0$, where $L_0 = L_0(\eta)$ will be chosen later. In the following, "good" and "bad" is always to be understood with respect to δ. Again, for $L_1 \le L \le L_1(\log L_1)^2$,

$$\mathbb{P}(\mathrm{ManyBad}_L \cup \mathrm{BdBad}_{L,r}) \le \exp\big(-\tfrac{9}{5}(\log L)^2\big).$$

For $\omega \in \mathrm{OneBad}_L \cap (\mathrm{BdBad}_{L,r})^c$, we use the splitting (6.9) of $\Pi - \pi^{(p_L)}$ into the parts F_1, F_2, F_3. For the summands F_1 and F_2, we do not need the additional smoothing by $\hat{\pi}_\psi^{(q)}$, since by (6.10),

$$\|F_1(x, \cdot)\|_1 \le C(\log L)^{-3} \le \eta/3,$$

and by (6.11),

$$\|F_2(x, \cdot)\|_1 \le C\big((\log L)^{-3} + (\log r)^{-1/2}\big) \le \eta/3,$$

if $L \ge L_0$ and r, L_0 are chosen large enough, depending on d and η. We turn to F_3. By (6.13), (6.14) and (6.15) we have, with $\Delta' = 1_{V_L}(\breve{\Pi}^g - \breve{\pi}^{(p_L)})$ as in the proof of Lemma 6.6, writing again $\hat{\pi}_\psi$ for $\hat{\pi}_\psi^{(q)}$,

$$\begin{aligned} &\|F_3\hat{\pi}_s(x, \cdot)\|_1 \\ &\quad \le C\Big(\sup_{y\in V_{2L/3}} \|\breve{G}^g 1_{V_L\setminus\Lambda_0}\Delta'\pi^{(p_L)}(y, \cdot)\|_1 + \sup_{z\in\Lambda_0}\|\Delta'\pi^{(p_L)}\hat{\pi}_\psi(z, \cdot)\|_1\Big) \end{aligned} \tag{6.17}$$

$$\leq C\Big((\log L)^{-3} + (\log r)^{-8} + \sup_{z\in\Lambda_0} \big\|\Delta'\pi^{(p_L)}\hat{\pi}_\psi(z,\cdot)\big\|_1\Big)$$

$$\leq \eta/6 + C_1 \sup_{z\in\Lambda_0} \big\|\Delta'\pi^{(p_L)}\hat{\pi}_\psi(z,\cdot)\big\|_1,$$

if $L \geq L_0$ and r, L_0 are sufficiently large. Regarding the second summand of (6.17), set $m = 3r$ and define for $K \in \mathbb{N}$

$$\vartheta_K(z) = \min\{n \in \mathbb{N} : |X_n^z - z| > Km\} \in [0, \infty],$$

where X_n^z denotes symmetric random walk with law P_{z,p_L}. By the invariance principle for symmetric random walk, we can clearly choose K so large such that

$$\sup_{z\in V_L : \mathrm{d}_L(z)\leq m} \mathrm{P}_{z,q}\big(\vartheta_K(z) \leq \tau_L\big) \leq \frac{\eta}{24C_1}$$

uniformly in $L \geq L_0$ and $q \in \mathcal{P}_\iota^{\mathrm{s}}$, where C_1 is the constant from (6.17). If $z \in \Lambda_0$, $z' \in V_L \cup \partial V_L$ with $\Delta'(z,z') \neq 0$, we have $\mathrm{d}_L(z') \leq m$ and $|z - z'| \leq m$. Thus, using Lemma A.2(iii) of the Appendix with $\psi \equiv \ell$,

$$\begin{aligned}
&C_1 \sup_{z\in\Lambda_0} \big\|\Delta'\pi^{(p_L)}\hat{\pi}_\psi(z,\cdot)\big\|_1 \\
&\quad = C_1 \sup_{z\in\Lambda_0} \Bigg\| \sum_{z'\in V_L\cup\partial V_L} \Delta'(z,z') \sum_{\substack{w\in\partial V_L:\\ |z'-w|>Km}} \pi^{(p_L)}(z',w)\big(\hat{\pi}_\psi(w,\cdot) - \hat{\pi}_\psi(z,\cdot)\big) \\
&\qquad\qquad + \sum_{z'\in V_L\cup\partial V_L} \Delta'(z,z') \\
&\qquad\qquad\qquad \times \sum_{\substack{w\in\partial V_L:\\ |z'-w|\leq Km}} \pi^{(p_L)}(z',w)\big(\hat{\pi}_\psi(w,\cdot) - \hat{\pi}_\psi(z,\cdot)\big)\Bigg\|_1 \\
&\quad \leq \frac{\eta}{6} + C(K+1)m\frac{\log\ell}{\ell} \leq \eta/3,
\end{aligned}$$

if we choose $\ell = \ell(\eta, r)$ large enough. This proves the lemma. $\square$

7. Proofs of the main results.

PROOF OF PROPOSITION 2.1. We take δ small enough and choose $L_0 = L_0(\delta)$ large enough according to Remark 5.3 and the statements of Section 6. In the course of this proof, we might enlarge L_0 further. (ii) is a consequence of Lemma 6.7, so we have to prove (i). Let $L_1 \geq L_0$, and assume that condition $\mathbf{Cond}(\delta, L_0, L_1)$ holds. Then the first point of $\mathbf{Cond}(\delta, L_0, L_1(\log L_1)^2)$ is trivially fulfilled. Now let $L_1 < L \leq L_1(\log L_1)^2$, and consider first $L' = L$ and

$i = 1, 2, 3$. Take $\psi \in \mathcal{M}_L$, $q \in \mathcal{P}^{\mathrm{s}}_{\iota}$. For simplicity, write D^*_L for $D^*_{L,p_L,\psi,q}$. Then by Lemma 3.3,

$$\begin{aligned}
&b_i(L, p_L, \psi, q, \delta)\\
&\quad\le \mathbb{P}\bigl(D^*_L > (\log L)^{-9+9(i-1)/4}\bigr)\\
&\quad\le \mathbb{P}(\mathrm{ManyBad}_L) + \mathbb{P}\bigl(D^*_L > (\log L)^{-9+9(i-1)/4}; \mathrm{OneBad}_L\bigr)\\
&\quad\le \exp\bigl(-\tfrac{19}{10}(\log L)^2\bigr) + \mathbb{P}\bigl(D^*_L > (\log L)^{-9+9(i-1)/4}; \mathrm{OneBad}_L\bigr).
\end{aligned}$$

For the last summand, we have by Lemmata 6.4 and 6.5, under $\mathbf{Cond}(\delta, L_0, L_1)$,

$$\begin{aligned}
&\mathbb{P}\bigl(D^*_L > (\log L)^{-9+9(i-1)/4}; \mathrm{OneBad}_L\bigr)\\
&\quad\le \mathbb{P}\bigl(D^*_L > (\log L)^{-9}; \mathrm{Good}_L\bigr)\\
&\qquad + \sum_{j=1}^{4} \mathbb{P}\bigl(D^*_L > (\log L)^{-9+9(i-1)/4}; \mathrm{OneBad}^{(j)}_L\bigr)\\
&\quad\le \exp\bigl(-(\log L)^{7/3}\bigr) + \sum_{j=1}^{i} \mathbb{P}\bigl(D^*_L > (\log L)^{-9+9(i-1)/4}; \mathrm{OneBad}^{(j)}_L\bigr)\\
&\qquad + \sum_{j=i+1}^{4} \mathbb{P}\bigl(\mathrm{OneBad}^{(j)}_L\bigr)\\
&\quad\le 4\exp\bigl(-(\log L)^{7/3}\bigr) + CL^d s^d_L \exp\bigl(-((3+i+1)/4)\bigl(\log(r_L/20)\bigr)^2\bigr).
\end{aligned}$$

Therefore, for L large,

$$\mathbb{P}\bigl(D^*_L > (\log L)^{-9+9(i-1)/4}; \mathrm{OneBad}_L\bigr) \le \tfrac{1}{8}\exp\bigl(-((3+i)/4)(\log L)^2\bigr)$$

and

$$b_i(L, p_L, \psi, q, \delta) \le \tfrac{1}{4}\exp\bigl(-((3+i)/4)(\log L)^2\bigr).$$

For the case $i = 4$, notice that

$$b_4(L, p_L, \psi, q, \delta) \le \mathbb{P}\bigl(D^*_L > (\log L)^{-9/4}\bigr) + \mathbb{P}\bigl(D^*_{L,p_L} > \delta\bigr).$$

The first summand can be estimated as the corresponding terms in the case $i = 1, 2, 3$, while for the last term we use Lemma 6.6.

It remains to show that for all L with $L_1 < L \le L_1(\log L_1)^2$ and all $L' \in (L, 2L]$, for all $\psi \in \mathcal{M}_{L'}$ and all $q \in \mathcal{P}^{\mathrm{s}}_{\iota}$, all $i = 1, 2, 3, 4$,

$$b_i\bigl(L', p_L, \psi, q, \delta\bigr) \le \tfrac{1}{4}\exp\bigl(-((3+i)/4)\bigl(\log L'\bigr)^2\bigr).$$

This, however, is easily deduced from the estimates above, by a slight change of the coarse graining scheme. Indeed, defining for $L' \in (L, 2L]$ the coarse graining function $\tilde{h}_{L',r} : \overline{C}_{L'} \to \mathbb{R}_+$ as

$$\tilde{h}_{L',r}(x) = \frac{1}{20} \max\left\{ s_L h\left(\frac{\mathrm{d}_{L'}(x)}{s_{L'}}\right), r \right\},$$

it follows that $\tilde{h}_{L',r}(x) = h_{L,r}(0) = s_L/20$ for $x \in V_{L'}$ with $\mathrm{d}_{L'}(x) \geq 2s_{L'}$. With an analogous definition of good and bad points within $V_{L'}$, using the coarse graining function $\tilde{h}_{L',r}$ instead of $h_{L',r}$ and the transition kernels corresponding to $\tilde{h}_{L',r}$, clearly all the statements of Sections 4 and 5 remain true. Moreover, we can use the kernel p_L to obtain the results of Section 6 for the radius L', noticing that in the proofs at most the constants change. Arguing now exactly as above for the choice $L' = L$, we conclude that also the second point of condition $\mathbf{Cond}(\delta, L_0, L_1(\log L_1)^2)$ holds true, provided L_0 is large, $L_1 \geq L_0$. $\square$

PROOF OF PROPOSITION 1.1. According to Proposition 2.1(i), for δ, $\varepsilon > 0$ small and $L_0 = L_0(\delta)$ large, conditions **A1**, $\mathbf{A0}(\varepsilon)$ and $\mathbf{Cond}(\delta, L_0, L_0)$ imply $\mathbf{Cond}(\delta, L_0, L)$ for all $L \geq L_0$. By shrinking ε if necessary, we may further guarantee that $\mathbf{Cond}(\delta, L_0, L_0)$ holds true, as it was explained below the statement of the proposition. Therefore, we can assume that $\mathbf{Cond}(\delta, L_0, L)$ is satisfied for all $L \geq L_0$. By Lemma 3.2, the limit $\lim_{L\to\infty} p_L(e_i)$ exists for each $i = 1, \dots, d$. Now let $\ell = s_L/20$. Using the definition of p_L in the first, $\mathbf{Cond}(\delta, L_0, L)$ in the second and fourth and Lemma 3.1 in the third equality,

$$\begin{aligned}
2p_L(e_i) &= \sum_y \mathbb{E}[\hat{\Pi}_{L,r}(0, y)] \frac{|y|^2}{y_i^2} \\
&= \sum_y \hat{\pi}_{L,r}^{(p_\ell)}(0, y) \frac{|y|^2}{y_i^2} + O\big((\log L)^{-9}\big) \\
&= \sum_y \pi_\ell^{(p_\ell)}(0, y) \frac{|y|^2}{y_i^2} + O\big((\log L)^{-9}\big) \\
&= \sum_y \mathbb{E}[\Pi_\ell(0, y)] \frac{|y|^2}{y_i^2} + O\big((\log L)^{-9}\big).
\end{aligned}$$

From this, the first statement of the proposition follows. Moreover, $p_\infty \in \mathcal{P}_\iota^{\mathrm{s}}$, and recalling that we may first choose L_0 as large as we wish and then choose ε sufficiently small, we see that $\|p_\infty - p_o\|_1 \to 0$ as $\varepsilon \downarrow 0$. $\square$

PROOF OF THEOREM 1.1. As in the proof of Proposition 1.1, if the parameters are appropriately chosen, $\mathbf{Cond}(\delta/2, L_0, L)$ holds true for all $L \geq L_0$. This implies

$$\mathbb{P}\big(D_{L,p_L}^* > \delta/2\big) \leq \exp\big(-(\log L)^2\big). \tag{7.1}$$

With p_∞ defined in Proposition 1.1, we obtain by the triangle inequality

$$
\begin{aligned}
D^*_{L,p_\infty} &= \sup_{x\in V_{L/5}} \|(\Pi - \pi^{(p_\infty)})(x,\cdot)\|_1 \\
&\le D^*_{L,p_L} + \sup_{x\in V_{L/5}} \|(\pi^{(p_\infty)} - \pi^{(p_L)})(x,\cdot)\|_1.
\end{aligned}
\tag{7.2}
$$

The claim therefore follows if we show that the second summand is bounded by $\delta/2$ if $L \ge L_0$ and L_0 is large enough. To prove this bound, we move inside V_L according to the coarse grained transition kernels $\hat\pi^{(p_L)}_{L,r}$ and $\hat\pi^{(p_\infty)}_{L,r}$, respectively, where similarly to Section 6.3, r is a large but fixed number. First, we recall from the proof of Proposition 1.1 that $p_\infty = \lim_{L\to\infty} p_L$. We therefore deduce from Lemma 3.2,

$$
\|p_\infty - p_L\|_1 \le C\sum_{k=0}^{\infty}\bigl(\log(2^k L)\bigr)^{-9} \le C(\log L)^{-8}.
\tag{7.3}
$$

This implies by Lemma 4.5 that for $x \in V_L$ with $\mathrm{d}_L(x) > (1/10)r$,

$$
\|(\hat\pi^{(p_L)} - \hat\pi^{(p_\infty)})(x,\cdot)\|_1 \le C\max\{h_{L,r}(x)^{-1/4}, (\log L)^{-8}\}.
\tag{7.4}
$$

Now by the perturbation expansion, with $\Delta = 1_{V_L}(\hat\pi^{(p_L)} - \hat\pi^{(p_\infty)})$,

$$
\begin{aligned}
\pi^{(p_\infty)} - \pi^{(p_L)} &= \hat g^{(p_L)}\Delta\pi^{(p_\infty)} \\
&= \hat g^{(p_L)} 1_{V_L\setminus \mathrm{Sh}_L(2r_L)}\Delta\pi^{(p_\infty)} + \hat g^{(p_L)} 1_{\mathrm{Sh}_L(2r_L)}\Delta\pi^{(p_\infty)}.
\end{aligned}
$$

Since $\hat g^{(p_L)}(x, V_L) \preceq C\Gamma(x, V_L) \le C(\log L)^6$, we obtain with (7.4)

$$
\sup_{x\in V_L} \|\hat g^{(p_L)} 1_{V_L\setminus \mathrm{Sh}_L(2r_L)}\Delta\pi^{(p_\infty)}(x,\cdot)\|_1 \le C(\log L)^{-2} \le \delta/4,
$$

if L is large enough. Moreover, as below (6.15), for $x \in V_{L/5}$, using again (7.4),

$$
\begin{aligned}
&\|\hat g^{(p_L)} 1_{\mathrm{Sh}_L(2r_L)}\Delta\pi^{(p_\infty)}(x,\cdot)\|_1 \\
&\quad\le \|\hat g^{(p_L)} 1_{\bigcup_{j=1}^{J_1}\Lambda_j}\Delta(x,\cdot)\|_1 + \|\hat g^{(p_L)} 1_{\Lambda_0}\Delta(x,\cdot)\|_1 \\
&\quad\le Cr^{-1/4} + C_0 \sup_{z\in\Lambda_0}\|\Delta(z,\cdot)\|_1.
\end{aligned}
$$

We choose r so large such that $Cr^{-1/4} \le \delta/8$. Now that r is fixed, the difference over the first layer $\mathrm{Sh}_L(2r)$ is also bounded by $C_0^{-1}\delta/4$ if the difference between p_L and p_∞ is small enough, that is, if L is large enough. This proves that the second summand of (7.2) is bounded by $\delta/2$, for $L \ge L_0$ and L_0 large. Applying (7.1), we conclude that

$$
\mathbb{P}(D^*_{L,p_\infty} > \delta) \le \mathbb{P}(D^*_{L,p_L} > \delta/2) \le \exp(-(\log L)^2). \qquad \square
$$

Since Theorem 1.2 is proved in a similar way, using the second part of Proposition 2.1, we may safely omit the details and turn now to the proof of the local estimates.

PROOF OF THEOREM 1.3. We choose $\delta > 0$ small and $L_0(\delta)$ large enough according to Proposition 2.1 such that $\mathbf{A0}(\varepsilon)$ and $\mathbf{A1}$ imply $\mathbf{Cond}(\delta, L_0, L)$ for all $L \geq L_0$. We let $r = r_L$. Recall the definition of Good_L from Section 3. With $A_L = \mathrm{Good}_L$, we note that similar to Lemma 3.3, if $L \geq L_0$,

$$\mathbb{P}(A_L^c) \leq \exp(-(1/2)(\log L)^2).$$

For the rest of the proof, we take $\omega \in A_L$. On such environments, $\hat{G}$ equals $\hat{G}^g$ by our choice of $r = r_L$. We fix $z \in \partial V_L$ and simply write W_t for $W_t(z) = V_t(z) \cap \partial V_L$.

Let us now prove part (i). We recall that $t \geq r$, and observe that W_t can be covered by $K\lfloor (t/r)^{d-1} \rfloor$ many neighborhoods $V_{3r}(y_i)$, $y_i \in \mathrm{Sh}_L(r)$, as defined in Section 5.2, where K depends on the dimension only. Applying Lemma 5.2(ii), we deduce that

$$\begin{aligned}\Pi_L(x, W_t) = \hat{G}^g(x, W_t) &\leq \sum_{i=1}^{K\lfloor (t/r)^{d-1} \rfloor} \hat{G}^g(x, V_{3r}(y_i)) \\ &\leq C \sum_{i=1}^{K\lfloor (t/r)^{d-1} \rfloor} \Gamma(x, V_{3r}(y_i)) \leq C\left(\frac{t}{L}\right)^{d-1},\end{aligned}$$

where for the last inequality we have used that $\Gamma(x, V_{3r}(y_i)) \leq C(r/L)^{d-1}$ for some constant $C = C(d, \eta)$ (recall that we assume $x \in V_{\eta L}$ for some $0 < \eta < 1$). From Lemma 4.1(i) we know that if $x \in V_{\eta L}$, then there is a constant $c = c(d, \eta)$ such that

$$\pi^{(p_o)}(x, z) \geq c L^{-(d-1)}.$$

Together with the preceding inequality, this shows (i).

(ii) We recall that for (ii), we assume $t \geq L/(\log L)^6$. If not explicitly mentioned otherwise, the underlying one-step transition kernel is in the following given by p_L defined in (2.8), which we omit from notation. Set $\ell = (\log L)^{13/2} r$, and consider the smoothing kernel $\hat{\pi}_\psi$ with $\psi \equiv \ell \in \mathcal{M}_\ell$. Let

$$U_\ell(W_t) = \{y \in \mathbb{Z}^d : \mathrm{d}(y, W_t) \leq 2\ell\}.$$

We claim that

$$\Pi(x, W_t) - \pi(x, W_{t+6\ell}) \leq (\Pi - \pi)\hat{\pi}_\psi(x, U_\ell(W_t)), \tag{7.5}$$

$$\pi(x, W_{t-6\ell}) - \Pi(x, W_t) \leq (\pi - \Pi)\hat{\pi}_\psi(x, U_\ell(W_{t-6\ell})). \tag{7.6}$$

Concerning the first inequality,

$$\Pi\hat{\pi}_\psi(x, U_\ell(W_t)) \geq \sum_{y \in W_t} \Pi(x, y)\hat{\pi}_\psi(y, U_\ell(W_t)) = \Pi(x, W_t),$$

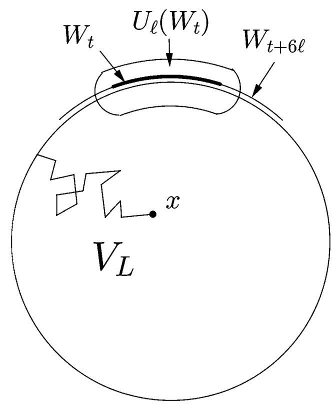

FIG. 5. *On the proof of Theorem* 1.3(ii). *If the walk exits V_L through $\partial V_L \setminus W_{t+6\ell}$, it cannot enter $U_\ell(W_t)$ in one step with $\hat{\pi}_\psi$, $\psi \equiv \ell$.*

since $\hat{\pi}_\psi(y, U_\ell(W_t)) = 1$ for $y \in W_t$. Also,

$$\pi\hat{\pi}_\psi\big(x, U_\ell(W_t)\big) = \sum_{y \in W_{t+6\ell}} \pi(x, y)\hat{\pi}_\psi\big(y, U_\ell(W_t)\big) \leq \pi(x, W_{t+6\ell}),$$

since $\hat{\pi}_\psi(y, U_\ell(W_t)) = 0$ for $y \in \partial V_L \setminus W_{t+6\ell}$; see Figure 5. This proves (7.5), while (7.6) is entirely similar. In the remainder of this proof, we often write $|F|(x, y)$ for $|F(x, y)|$. If we show

$$\big|(\pi - \Pi)\hat{\pi}_\psi\big|\big(x, U_\ell(W_t)\big) \leq O\big((\log L)^{-5/2}\big)\pi(x, W_t), \tag{7.7}$$

then by (7.5),

$$\begin{aligned}
\Pi(x, W_t) &\leq \pi(x, W_{t+6\ell}) + O\big((\log L)^{-5/2}\big)\pi(x, W_t) \\
&= \pi(x, W_t) + \pi(x, W_{t+6\ell} \setminus W_t) + O\big((\log L)^{-5/2}\big)\pi(x, W_t) \\
&= \pi(x, W_t)\big(1 + O\big(\max\{\ell/t, (\log L)^{-5/2}\}\big)\big) \\
&= \pi(x, W_t)\big(1 + O\big((\log L)^{-5/2}\big)\big),
\end{aligned}$$

where in the next-to-last line, we used that $\pi(x, W_{t+6\ell}) \leq C(\ell/t)\pi(x, W_t)$ by Lemma 4.1(i). On the other hand, by (7.6) and still assuming (7.7),

$$\begin{aligned}
\Pi(x, W_t) &\geq \pi(x, W_{t-6\ell}) - O\big((\log L)^{-5/2}\big)\pi(x, W_t) \\
&= \pi(x, W_t)\big(1 - O\big((\log L)^{-5/2}\big)\big),
\end{aligned}$$

so that indeed

$$\Pi(x, W_t) = \pi(x, W_t)\big(1 + O\big((\log L)^{-5/2}\big)\big),$$

provided we prove (7.7). In that direction, set $B = V_L \setminus \mathrm{Sh}_L(5r)$ and write, with $\Delta = 1_{V_L}(\hat{\Pi}^g - \hat{\pi}^{(p_L)})$,

$$(\pi - \Pi)\hat{\pi}_\psi = \hat{G}^g \Delta\pi\hat{\pi}_\psi = \hat{G}^g 1_B \Delta\pi\hat{\pi}_\psi + \hat{G}^g 1_{\mathrm{Sh}_L(5r)}\Delta\pi\hat{\pi}_\psi. \tag{7.8}$$

Looking at the first summand we have

$$\left|\hat{G}^g 1_B \Delta\pi \hat{\pi}_\psi\right|\left(x, U_\ell(W_t)\right) \le \left(\hat{G}^g 1_B |\Delta\hat{\pi}|\pi\right)(x, W_{t+6\ell}).$$

By Lemma 6.1, we have

$$\left\|1_B \Delta\pi(x, \cdot)\right\|_1 \le C(\log\log L)(\log L)^{-9}.$$

Furthermore, a slight modification of the proof of Proposition 5.4(ii) shows

$$\hat{G}^g 1_B |\Delta\hat{\pi}|\Gamma(x, z) \le C(\log L)^{-5/2}\Gamma(x, z).$$

Together with $\pi \preceq C\Gamma$ and $\pi(x, z) \ge c(d, \eta)L^{-(d-1)}$ this yields the bound

$$\hat{G}^g 1_B |\Delta\hat{\pi}|\pi(x, W_{t+6\ell}) \le C(\log L)^{-5/2}\Gamma(x, W_t) \le C(\log L)^{-5/2}\pi(x, W_t).$$

To obtain (7.7), it remains to handle the second summand of (7.8); that is, we have to bound

$$\left|\hat{G}^g 1_{\mathrm{Sh}_L(5r)} \Delta\pi \hat{\pi}_\psi\right|\left(x, U_\ell(W_t)\right).$$

We abbreviate $S = \mathrm{Sh}_L(5r)$ and split into

$$\begin{aligned}
&\hat{G}^g 1_S \Delta\pi \hat{\pi}_\psi\left(x, U_\ell(W_t)\right) \\
&\quad = \sum_{y \in S} \hat{G}^g(x, y) \sum_{z \in \partial V_L} \Delta\pi(y, z)\left(\hat{\pi}_\psi\left(z, U_\ell(W_t)\right) - \hat{\pi}_\psi\left(y, U_\ell(W_t)\right)\right) \\
&\quad = \sum_{y \in S} \hat{G}^g(x, y) \sum_{z \in W_{t+6\ell}} \Delta\pi(y, z)\left(\hat{\pi}_\psi\left(z, U_\ell(W_t)\right) - \hat{\pi}_\psi\left(y, U_\ell(W_t)\right)\right) \\
&\qquad - \sum_{y \in S} \hat{G}^g(x, y) \sum_{z \in \partial V_L \setminus W_{t+6\ell}} \Delta\pi(y, z)\hat{\pi}_\psi\left(y, U_\ell(W_t)\right).
\end{aligned}$$

First note that since $\hat{\pi}_\psi(y, z') = 0$ if $|y - z'| > 2\ell$,

$$\begin{aligned}
&\left|\sum_{y \in S} \hat{G}^g(x, y) \sum_{z \in \partial V_L \setminus W_{t+6\ell}} \Delta\pi(y, z)\hat{\pi}_\psi\left(y, U_\ell(W_t)\right)\right| \\
&\qquad \le \hat{G}^g 1_{U_{2\ell}(W_t) \cap S}|\Delta\pi|(x, \partial V_L \setminus W_{t+6\ell}).
\end{aligned}$$

For $y \in U_{2\ell}(W_t) \cap S$, we apply Lemma 4.2(iii) together with Lemma 4.6 and obtain

$$\begin{aligned}
|\Delta\pi|(y, \partial V_L \setminus W_{t+6\ell}) &\le \sup_{y' : \mathrm{d}(y', U_{2\ell}(W_t) \cap S) \le r} \pi\left(y', \partial V_L \setminus W_{t+6\ell}\right) \\
&\le C\frac{r}{\ell} \le C(\log L)^{-13/2}.
\end{aligned}$$

Since $\hat{G}^g \preceq \Gamma$ and $\pi(x, z) \ge cL^{-d-1}$, $\hat{G}^g(x, U_{2\ell}(W_t) \cap S) \le C\pi(x, W_t)$, and thus

$$\hat{G}^g 1_{U_{2\ell}(W_t) \cap S}|\Delta\pi|(x, \partial V_L \setminus W_{t+6\ell}) \le C(\log L)^{-13/2}\pi(x, W_t).$$

It remains to bound

$$\left|\sum_{y\in S}\hat{G}^g(x,y)\sum_{z\in W_{t+6\ell}}\Delta\pi(y,z)\big(\hat{\pi}_\psi\big(z,U_\ell(W_t)\big)-\hat{\pi}_\psi\big(y,U_\ell(W_t)\big)\big)\right|.$$

Set $D^1(y)=\{z\in W_{t+6\ell}:|z-y|\le\ell(\log\ell)^{-4}\}$. If $D^1(y)\neq\varnothing$, then $\mathrm{d}(y,W_t)\le 7\ell$. Using Lemma A.2 for the difference of the smoothing steps and the standard estimate on $\hat{G}^g$,

$$\begin{aligned}&\sum_{y\in S}\hat{G}^g(x,y)\sum_{z\in D^1(y)}|\Delta\pi(y,z)|\big|\hat{\pi}_\psi\big(z,U_\ell(W_t)\big)-\hat{\pi}_\psi\big(y,U_\ell(W_t)\big)\big|\\&\quad\le C\frac{t^{d-1}}{L^{d-1}}(\log\ell)^{-3}\le C(\log L)^{-5/2}\pi(x,W_t).\end{aligned}$$

The region $W_{t+6\ell}\setminus D^1(y)$ we split into

$$\begin{aligned}&B_0(y)=\{z\in W_{t+6\ell}:|z-y|\in\big(\ell(\log\ell)^{-4},t\big]\},\\&B_i(y)=\{z\in W_{t+6\ell}:|z-y|\in\big(it,(i+1)t\big]\},\qquad i=1,2,\dots,\lfloor 2L/t\rfloor.\end{aligned}$$

Moreover, let

$$S_i=\{y\in S:B_i(y)\neq\varnothing\},\qquad i=0,1,\dots,\lfloor 2L/t\rfloor.$$

Then

$$\sum_{y\in S}\hat{G}^g(x,y)\sum_{z\in W_{t+6\ell}\setminus D^1(y)}|\Delta\pi(y,z)|\le C\sum_{i=0}^{\lfloor 2L/t\rfloor}\hat{G}^g(x,S_i)\sup_{y\in S_i}|\Delta\pi|\big(y,B_i(y)\big).$$

If $i\ge 1$ and $y\in S_i$, then by Lemma 4.2(iii),

$$|\Delta\pi|\big(y,B_i(y)\big)\le\sup_{y':|y'-y|\le r}\pi\big(y',B_i(y)\big)\le C\frac{rt^{d-1}}{(it)^d}\le C\frac{r}{i^dt},$$

while in the case $i=0$, using the same lemma and additionally Lemma 4.6,

$$\begin{aligned}\sup_{y':|y'-y|\le r}\pi\big(y',B_i(y)\big)&\le Cr\sum_{z\in\partial V_L}\frac{1}{((1/2)\ell(\log\ell)^{-4}+|y-z|)^d}\\&\le C\frac{r(\log\ell)^4}{\ell}\le C(\log L)^{-5/2}.\end{aligned}$$

For the Green's function, we use the estimates

$$\hat{G}^g(x,S_0)\le C\frac{t^{d-1}}{L^{d-1}},\qquad\hat{G}^g\Big(x,\bigcup_{i\ge(1/10)L/t}S_i\Big)\le C,$$

while for $i=1,2,\dots,\lfloor(1/10)L/t\rfloor$, it holds that $|S_i|\le Cr(it)^{d-2}t$, whence

$$\hat{G}^g(x,S_i)\le C\frac{i^{d-2}t^{d-1}}{L^{d-1}}.$$

Altogether, we obtain

$$\begin{aligned}
&\sum_{i=0}^{\lfloor 2L/t\rfloor} \hat{G}^g(x, S_i) \sup_{y\in S_i} |\Delta\pi|\big(y, B_i(y)\big) \\
&\quad \le C\left((\log L)^{-5/2}\frac{t^{d-1}}{L^{d-1}} + \left(\frac{r}{t}\frac{t^{d-1}}{L^{d-1}}\sum_{i=1}^{\lfloor (1/10)L/t\rfloor}\frac{1}{i^2}\right) + \frac{t^{d-1}}{L^{d-1}}\frac{r}{L}\right) \\
&\quad \le C(\log L)^{-5/2}\frac{t^{d-1}}{L^{d-1}}.
\end{aligned}$$

This finishes the proof of part (ii). $\square$

Let us finally show how to obtain transience of the RWRE.

PROOF OF COROLLARY 1.1. As in the proof of Theorem 1.3, we assume that **Cond**(δ, L_0, L) holds true for all $L \ge L_0$. Fix numbers $\rho \ge 3$, $\alpha \in (0, (4\rho)^{-1})$ to be specified below. With these parameters and $n \ge 1$, we put $p = p_{\alpha\rho^n}$ and choose $\psi = (m_x)_{x\in\mathbb{Z}^d}$ constant in x, namely $m_x = \alpha\rho^n$. Define

$$A_n = \bigcap_{|x| \le \rho^{n^{3/2}}} \bigcap_{t\in[\alpha\rho^n, 2\alpha\rho^n]} \big\{\big\|\big(\Pi_{V_t(x)} - \pi^{(p)}_{V_t(x)}\big)\hat{\pi}^{(p)}_\psi(x, \cdot)\big\|_1 \le (\log t)^{-9}\big\}.$$

Under **Cond**(δ, L_0, L), we then have

$$\mathbb{P}(A_n^c) \le C\alpha^d \rho^{(d+1)n^{3/2}} \exp\big(-\big(\log(\alpha\rho^n)\big)^2\big).$$

Therefore, for any choice of α, ρ it holds that

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n^c) < \infty,$$

whence by Borel–Cantelli,

$$\mathbb{P}\Big(\liminf_{n\to\infty} A_n\Big) = 1. \tag{7.9}$$

We set $q_{n,\alpha,\rho} = \hat{\pi}^{(p)}_\psi$ and denote the coarse-grained RWRE transition kernel by

$$Q_{n,\alpha,\rho}(x, \cdot) = \frac{1}{\alpha\rho^n}\int_{\mathbb{R}_+} \varphi\left(\frac{t}{\alpha\rho^n}\right)\Pi_{V_t(x)}(x, \cdot)\,\mathrm{d}t.$$

If $|x| \le \rho^{n^{3/2}}$, we have on A_n

$$\big\|(Q_{n,\alpha,\rho} - q_{n,\alpha,\rho})q_{n,\alpha,\rho}(x, \cdot)\big\|_1 \le \big(\log(\alpha\rho^n)\big)^{-9} \le C(\alpha, \rho)n^{-9}.$$

Now assume $|x| \le \rho^n + 1$. On A_n, for N fixed, it follows that for $1 \le M \le N$,

$$\big\|\big((Q_{n,\alpha,\rho})^M - (q_{n,\alpha,\rho})^M\big)q_{n,\alpha,\rho}(x, \cdot)\big\|_1 \le C(\alpha, \rho)Mn^{-9}. \tag{7.10}$$

For fixed ω, let $(\xi_k)_{k\geq 0}$ be the Markov chain running with transition kernel $Q_{n,\alpha,\rho} = Q_{n,\alpha,\rho}(\omega)$. Clearly, $(\xi_k)_{k\geq 0}$ can be obtained by observing the basic RWRE $(X_k)_{k\geq 0}$ at randomized stopping times. Then

$$\begin{aligned}
&\mathrm{P}_{x,\omega}(\xi_{N-1} \in V_{\rho^{n+1}+2\alpha\rho^n}) \\
&\quad \leq (Q_{n,\alpha,\rho})^{N-1} q_{n,\alpha,\rho}(x, V_{\rho^{n+1}+4\alpha\rho^n}) \\
&\quad \leq \big\| \big((Q_{n,\alpha,\rho})^{N-1} - (q_{n,\alpha,\rho})^{N-1}\big) q_{n,\alpha,\rho}(x, \cdot)\big\|_1 + (q_{n,\alpha,\rho})^N(x, V_{2\rho^{n+1}}).
\end{aligned}$$

Using Proposition 5.1, we can find $N = N(\alpha, \rho) \in \mathbb{N}$, depending not on n, such that for any x with $|x| \leq \rho^n + 1$, it holds that $(q_{n,\alpha,\rho})^N(x, V_{2\rho^{n+1}}) \leq 1/10$. With (7.10), we conclude that for such x, $n \geq n_0(\alpha, \rho, N)$ large enough and $\omega \in A_n$,

$$\mathrm{P}_{x,\omega}(\xi_{N-1} \in V_{\rho^{n+1}+2\alpha\rho^n}) \leq C(\alpha, \rho) N n^{-9} + 1/10 \leq 1/5. \tag{7.11}$$

On the other hand, if x is outside $V_{\rho^{n-1}+2\alpha\rho^n}$,

$$\begin{aligned}
&\mathrm{P}_{x,\omega}(\xi_M \in V_{\rho^{n-1}+2\alpha\rho^n} \text{ for some } 0 \leq M \leq N-1) \\
&\quad \leq \sum_{M=1}^{N-1} (Q_{n,\alpha,\rho})^M q_{n,\alpha,\rho}(x, V_{\rho^{n-1}+4\alpha\rho^n}) \\
&\quad \leq \sum_{M=1}^{N-1} \big\| \big((Q_{n,\alpha,\rho})^M - (q_{n,\alpha,\rho})^M\big) q_{n,\alpha,\rho}(x, \cdot)\big\|_1 + \sum_{k=2}^{N} (q_{n,\alpha,\rho})^k(x, V_{2\rho^{n-1}}).
\end{aligned}$$

If $\rho^n - 1 \leq |x|$, then $(q_{n,\alpha,\rho})^k(x, V_{2\rho^{n-1}}) = 0$ as long as $k \leq (1 - 3/\rho)/(2\alpha)$. By first choosing ρ large enough, then α small enough and estimating the higher summands again with Proposition 5.1, we deduce that for such x and all large n,

$$\sum_{k=1}^{\infty} (q_{n,\alpha,\rho})^k(x, V_{2\rho^{n-1}}) \leq 1/10.$$

Together with (7.10), we have for large n, $\omega \in A_n$ and $\rho^n - 1 \leq |x| \leq \rho^n + 1$,

$$\begin{aligned}
&\mathrm{P}_{x,\omega}(\xi_M \in V_{\rho^{n-1}+2\alpha\rho^n} \text{ for some } 0 \leq M \leq N-1) \\
&\quad \leq C(\alpha, \rho) N^2 n^{-9} + 1/10 \leq 1/5.
\end{aligned} \tag{7.12}$$

Let B be the event that the walk $(\xi_k)_{k\geq 0}$ leaves $V_{\rho^{n+1}+2\alpha\rho^n}$ before reaching $V_{\rho^{n-1}+2\alpha\rho^n}$. From (7.11) and (7.12) we deduce that $\mathrm{P}_{x,\omega}(B) \geq 3/5$, provided n is large enough, $\omega \in A_n$ and $\rho^n - 1 \leq |x| \leq \rho^n + 1$. But on B, the underlying basic RWRE $(X_k)_{k\geq 0}$ clearly leaves $V_{\rho^{n+1}}$ before reaching $V_{\rho^{n-1}}$; see Figure 6. Hence if $\omega \in \liminf A_n$, there exists $m_0 = m_0(\omega) \in \mathbb{N}$ such that

$$\mathrm{P}_{x,\omega}(\tau_{V_{\rho^{n+1}}} < T_{V_{\rho^{n-1}}}) \geq 3/5$$

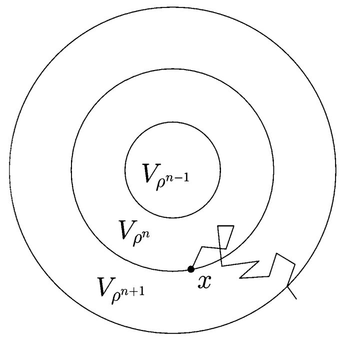

FIG. 6. *On a set of environments with mass* 1, *the RWRE started at any x with $|x| \geq \rho^n - 1$ leaves the ball $V_{\rho^{n+1}}$ before hitting $V_{\rho^{n-1}}$ with probability at least* 3/5. *This implies transience of the RWRE.*

for all $n \geq m_0$, x with $|x| \geq \rho^n - 1$ (of course, we may now drop the constraint $|x| \leq \rho^n + 1$). From this property, transience easily follows. Indeed, for m, M, $k \in \mathbb{N}$ satisfying $M > m \geq m_0$ and $0 \leq k \leq M + 1 - m$, set

$$h_M(k) = \sup_{x : |x| \geq \rho^{m+k} - 1} \mathrm{P}_{x,\omega}(T_{V_{\rho^m}} < \tau_{V_{\rho^M}}).$$

Then h_M solves the difference inequality

$$h_M(k) \leq \tfrac{2}{5} h_M(k-1) + \tfrac{3}{5} h_M(k+1)$$

with boundary conditions $h_M(0) = 1$, $h_M(M + 1 - m) = 0$. By either applying a discrete maximum principle or by a direct computation, we see that $h_M \leq \overline{h}_M$, where $\overline{h}_M$ is the solution of the difference equality

$$\overline{h}_M(k) = \tfrac{2}{5} \overline{h}_M(k-1) + \tfrac{3}{5} \overline{h}_M(k+1) \tag{7.13}$$

with boundary conditions $\overline{h}_M(0) = 1$, $\overline{h}_M(M + 1 - m) = 0$. Solving (7.13), we get

$$\overline{h}_M(k) = \frac{1}{1 - (3/2)^{M+1-m}} + \frac{1}{1 - (2/3)^{M+1-m}} \left(\frac{2}{3}\right)^k.$$

Letting $M \to \infty$, we deduce that for $|x| \geq \rho^{m+k}$,

$$\mathrm{P}_{x,\omega}(T_{V_{\rho^m}} < \infty) \leq \lim_{M \to \infty} \overline{h}_M(k) = \left(\frac{2}{3}\right)^k. \tag{7.14}$$

Together with (7.9), this proves that for almost all $\omega \in \Omega$, the random walk is transient under $\mathrm{P}_{\cdot,\omega}$. $\square$

APPENDIX

A.1. Some difference estimates. In this section we collect some difference estimates of (non)smoothed exit distributions which are mainly needed to prove Lemma 4.7(i) and (iii). The first technical lemma connects the exit distributions of a symmetric random walk with one-step distribution $p \in \mathcal{P}_\kappa^{\mathrm{s}}$ to those of Brownian motion with covariance matrix Λ_p. As always, $\kappa > 0$ can be chosen arbitrarily small.

LEMMA A.1. *Let $p \in \mathcal{P}_\kappa^{\mathrm{s}}$, and let $\beta, \eta > 0$ with $3\eta < \beta < 1$. For large L, there exists a constant $C > 0$ such that for $A \subset \mathbb{R}^d$, $A^\beta = \{y \in \mathbb{R}^d : \mathrm{d}(y, A) \le L^\beta\}$ and $x \in V_L$ with $\mathrm{d}_L(x) > L^\beta$, the following holds*:

(i) $\pi_L^{(p)}(x, A) \le \pi_L^{\mathrm{B}(p)}(x, A^\beta)(1 + CL^{-(\beta-3\eta)}) + L^{-(d+1)}$.
(ii) $\pi_L^{\mathrm{B}(p)}(x, A) \le \pi_L^{(p)}(x, A^\beta)(1 + CL^{-(\beta-3\eta)}) + L^{-(d+1)}$.

PROOF. (i) Set $L' = L + L^\eta$, $L'' = L + 2L^\eta$, and denote by $A_{\prime}^\beta$ the image of $\{y \in \partial C_L : \mathrm{d}(y, A) \le L^\beta/2\}$ on $\partial C_{L'}$ under the map $y \mapsto (L'/L)y$. We write π_L^{B} for $\pi_L^{\mathrm{B}(p)}$. Then, denoting by $\mathrm{P}_x^{\mathrm{B}}$ the law of a d-dimensional Brownian motion W_t with covariance Λ_p, conditioned on $W_0 = x$, and by $\tilde{\tau}_L = \inf\{t \ge 0 : W_t \notin C_L\}$ the first exit time from C_L,

$$\pi_{L'}^{\mathrm{B}}(x, A_{\prime}^\beta) \le \pi_L^{\mathrm{B}}(x, A^\beta) + \mathrm{P}_x^{\mathrm{B}}(W_{\tilde{\tau}_{L'}} \in A_{\prime}^\beta, W_{\tilde{\tau}_L} \notin A^\beta).$$

Let us first assume that we have already proved

$$\mathrm{P}_x^{\mathrm{B}}(W_{\tilde{\tau}_{L'}} \in A_{\prime}^\beta, W_{\tilde{\tau}_L} \notin A^\beta) \le CL^{\eta-\beta}\pi_{L'}^{\mathrm{B}}(x, A_{\prime}^\beta), \tag{A.1}$$

so that

$$\pi_L^{\mathrm{B}}(x, A^\beta) \ge \pi_{L'}^{\mathrm{B}}(x, A_{\prime}^\beta)(1 - CL^{-(\beta-\eta)}). \tag{A.2}$$

As a consequence of [24], Theorem 2, for each $k \in \mathbb{N}$ there exists a positive constant $C_1 = C_1(k)$ such that for each integer $n \ge 1$, one can construct on the same probability space a Brownian motion W_t with covariance matrix $d^{-1}\Lambda_p$ as well as a symmetric random walk X_n with one-step probability p, both starting in x and satisfying (with $\mathbb{Q}$ denoting the probability measure on that space)

$$\mathbb{Q}\Big(\max_{0 \le m \le n} |X_m - W_m| > C_1 \log n\Big) \le C_1 n^{-k}. \tag{A.3}$$

Choose $k > (2/5)(d+1)$, and let $C_1(k)$ be the corresponding constant. The following arguments hold for sufficiently large L. By standard results on the oscillation of Brownian paths,

$$\mathbb{Q}\Big(\sup_{0 \le t \le L^{5/2}} |W_{\lfloor t \rfloor} - W_t| > (5/2)C_1 \log L\Big) \le (1/3)L^{-(d+1)}. \tag{A.4}$$

With

$$B_1 = \Big\{\sup_{0 \le t \le L^{5/2}} |X_{\lfloor t \rfloor} - W_t| \le 5C_1 \log L\Big\},$$

we deduce from (A.3) and (A.4) that

$$\mathbb{Q}(B_1^c) \leq (2/3)L^{-(d+1)}.$$

Let $B_2 = \{\tilde{\tau}_{L'} \vee \tau_{L''} \leq L^{5/2}\}$. We claim that

$$\mathbb{Q}(B_2^c) \leq (1/3)L^{-(d+1)}. \tag{A.5}$$

By the central limit theorem, one finds a constant $c > 0$ with

$$\mathbb{Q}(\tau_{L''} \leq (L'')^2) \geq c \qquad \text{for } L \text{ large}.$$

By the Markov property, we obtain $\mathbb{Q}(\tau_{L''} > L^{5/2}) \leq (1-c)^{L^{1/3}}$. The probability $\mathbb{Q}(\tilde{\tau}_{L'} > L^{5/2})$ decays in the same way, and (A.5) follows. Since π_L^{B} is unchanged if the Brownian motion is replaced by a Brownian motion with covariance $d^{-1}\Lambda_p$, we have (P_x denotes the law of X_n)

$$\begin{aligned} &\pi_{L'}^{\mathrm{B}}(x, A_{\prime}^{\beta}) \\ &\quad \geq \mathbb{Q}(X_{\tau_L} \in A, W_{\tilde{\tau}_{L'}} \in A_{\prime}^{\beta}) \\ &\quad \geq \mathrm{P}_x(X_{\tau_L} \in A) - \mathbb{Q}(X_{\tau_L} \in A, W_{\tilde{\tau}_{L'}} \notin A_{\prime}^{\beta}, B_1 \cap B_2) - L^{-(d+1)}. \end{aligned} \tag{A.6}$$

Let $U = \{z \in \mathbb{Z}^d : \mathrm{d}(z, (\partial C_{L'} \setminus A_{\prime}^{\beta})) \leq 5C_1 \log L\}$. Then

$$\mathbb{Q}(X_{\tau_L} \in A, W_{\tilde{\tau}_{L'}} \notin A_{\prime}^{\beta}, B_1 \cap B_2) \leq \mathrm{P}_x(X_{\tau_L} \in A, T_U < \tau_{L''}).$$

By the strong Markov property,

$$\mathrm{P}_x(X_{\tau_L} \in A, T_U < \tau_{L''}) \leq \mathrm{P}_x(X_{\tau_L} \in A) \sup_{y \in A} \mathrm{P}_y(T_U < \tau_{L''}).$$

Furthermore, there exists a constant $c > 0$ such that for $y \in A$ and $z \in U$, we have $|y - z| \geq cL^{\beta}$ and $\mathrm{d}_{L''}(z) \leq \mathrm{d}_{L''}(y) \leq 2L^{\eta}$. Therefore, an application of first Lemma 4.2(ii) and then Lemma 4.6 yields

$$\mathrm{P}_y(T_U < \tau_{L''}) \leq CL^{2\eta} \sum_{z \in U} \frac{1}{|y - z|^d} \leq CL^{2\eta}(\log L)L^{-\beta} \leq CL^{-(\beta - 3\eta)},$$

uniformly in $y \in A$. Going back to (A.6), we arrive at

$$\pi_{L'}^{\mathrm{B}}(x, A_{\prime}^{\beta}) \geq \pi_L(x, A)(1 - CL^{-(\beta - 3\eta)}) - L^{-(d+1)}.$$

Together with (A.2), this shows (i), but we still have to prove (A.1). First, by Lemma 4.3(i) (all integrals are surface integrals)

$$\begin{aligned} &\mathrm{P}_x^{\mathrm{B}}(W_{\tilde{\tau}_{L'}} \in A_{\prime}^{\beta}, W_{\tilde{\tau}_L} \notin A^{\beta}) \\ &\quad \leq \int_{\partial C_L \setminus A^{\beta}} \pi_L^{\mathrm{B}}(x, dy) \pi_{L'}^{\mathrm{B}}(y, A^{\beta}) \\ &\quad \leq C \mathrm{d}_L(x) L^{\eta} \int_{\partial C_L \setminus A^{\beta}} \frac{1}{|x - y|^d} \int_{A_{\prime}^{\beta}} \frac{1}{|y - z|^d} \, \mathrm{d}z \, \mathrm{d}y. \end{aligned}$$

Fix $z \in A_{\prime}^{\beta} \subset \partial C_{L'}$ and put

$$D_1 = \{y \in \partial C_L \setminus A^{\beta} : |y - z| > |x - z|/2\},$$
$$D_2 = \partial C_L \setminus (A^{\beta} \cup D_1).$$

Then, using $\mathrm{d}_L(x) \geq L^{\beta}$ and Lemma 4.6 in the last step,

$$\int_{D_1} \frac{1}{|x-y|^d} \frac{1}{|y-z|^d} \, \mathrm{d}y \leq \frac{C}{|x-z|^d} \int_{D_1} \frac{1}{|x-y|^d} \, \mathrm{d}y \leq \frac{CL^{-\beta}}{|x-z|^d}.$$

For the integral over D_2 we obtain the same bound, by using that $|x - y| \geq (1/2)|x - z|$ for $y \in D_2$, the fact that $|y - z| \geq cL^{\beta}$ if $y \in \partial C_L \setminus A^{\beta}$, $z \in A_{\prime}^{\beta}$ and Lemma 4.6. Altogether,

$$\mathrm{P}_x^{\mathrm{B}}(W_{\tilde{\tau}_{L'}} \in A_{\prime}^{\beta}, W_{\tilde{\tau}_L} \notin A^{\beta}) \leq \frac{C\mathrm{d}_L(x)L^{\eta-\beta}}{|x-z|^d}.$$

Integrating over $z \in A_{\prime}^{\beta}$, we obtain with the lower bound of Lemma 4.3(i),

$$\mathrm{P}_x^{\mathrm{B}}(W_{\tilde{\tau}_{L'}} \in A_{\prime}^{\beta}, W_{\tilde{\tau}_L} \notin A^{\beta}) \leq CL^{\eta-\beta}\pi_{L'}^{\mathrm{B}}(x, A_{\prime}^{\beta}),$$

as claimed.

(ii) One can follow the same steps, interchanging the role of Brownian motion and the random walk. □

Again, let $p \in \mathcal{P}_{\kappa}^{\mathrm{s}}$. We write $\hat{\pi}_{\psi}^{\mathrm{B}(p)}(x,z)$ for the density of $\hat{\pi}_{\psi}^{\mathrm{B}(p)}(x,dz)$ with respect to d-dimensional Lebesgue measure, that is, for $\psi = (m_x) \in \mathcal{M}_L$,

$$\hat{\pi}_{\psi}^{\mathrm{B}(p)}(x,z) = \frac{1}{m_x}\varphi\left(\frac{|z-x|}{m_x}\right)\pi_{C_{|z-x|}}^{\mathrm{B}(p)}(0, z-x). \tag{A.7}$$

For ease of notation, we write in the following $\hat{\pi}_{\psi}^{\mathrm{B}}$ for $\hat{\pi}_{\psi}^{\mathrm{B}(p)}$ and $\hat{\pi}_{\psi}$ for $\hat{\pi}_{\psi}^{(p)}$.

LEMMA A.2. *Let $p \in \mathcal{P}_{\kappa}^{\mathrm{s}}$. There exists a constant $C > 0$ such that for large L, $\psi = (m_y) \in \mathcal{M}_L$, $x, x' \in \mathcal{U}_L \cap \mathbb{Z}^d$ and any $z, z' \in \mathbb{Z}^d$*:

(i) $\hat{\pi}_{\psi}(x,z) \leq CL^{-d}$;

(ii) $\hat{\pi}_{\psi}^{\mathrm{B}}(x,z) \leq CL^{-d}$;

(iii) $|\hat{\pi}_{\psi}(x,z) - \hat{\pi}_{\psi}(x',z)| \leq C|x-x'|L^{-(d+1)}\log L$;

(iv) $|\hat{\pi}_{\psi}(x,z) - \hat{\pi}_{\psi}(x,z')| \leq C|z-z'|L^{-(d+1)}\log L$;

(v) $|\hat{\pi}_{\psi}^{\mathrm{B}}(x,z) - \hat{\pi}_{\psi}^{\mathrm{B}}(x',z)| \leq C|x-x'|L^{-(d+1)}\log L$;

(vi) $|\hat{\pi}_{\psi}^{\mathrm{B}}(x,z) - \hat{\pi}_{\psi}^{\mathrm{B}}(x,z')| \leq C|z-z'|L^{-(d+1)}\log L$;

(vii) $|\hat{\pi}_{\psi}(x,z) - \hat{\pi}_{\psi}^{\mathrm{B}}(x,z)| \leq L^{-(d+1/4)}$.

COROLLARY A.1. *In the situation of the preceding lemma*:

(i)

$$\begin{aligned}&\left|\hat{\pi}_\psi(x,z)-\hat{\pi}_\psi(x',z)\right|\\&\quad\le C\min\left\{\left|x-x'\right|L^{-(d+1)}\log L,\left|x-x'\right|L^{-(d+1)}+L^{-(d+1/4)}\right\};\end{aligned}$$

(ii)

$$\begin{aligned}&\left|\hat{\pi}_\psi(x,z)-\hat{\pi}_\psi(x,z')\right|\\&\quad\le C\min\left\{\left|z-z'\right|L^{-(d+1)}\log L,\left|z-z'\right|L^{-(d+1)}+L^{-(d+1/4)}\right\}.\end{aligned}$$

PROOF. Combine (iii)–(vii). □

PROOF OF LEMMA A.2. (i) and (ii) follow from the definitions of $\hat{\pi}_\psi$ and $\hat{\pi}_\psi^{\mathrm{B}}$ together with part (i) of Lemma 4.1 and Lemma 4.3, respectively.

(iii) and (iv) We can restrict ourselves to the case $|x-x'|=1$, as otherwise we take a shortest path connecting x with x' inside $\mathcal{U}_L\cap\mathbb{Z}^d$, and apply the result for distance one $O(|x-x'|)$ times. We have

$$\begin{aligned}&\hat{\pi}_\psi(x,z)-\hat{\pi}_\psi(x',z)\\&\quad=\left(1-\frac{m_x}{m_{x'}}\right)\hat{\pi}_\psi(x,z)+\frac{1}{m_{x'}}\int_{\mathbb{R}_+}\left(\varphi\left(\frac{t}{m_x}\right)-\varphi\left(\frac{t}{m_{x'}}\right)\right)\pi_{V_t(x)}(x,z)\,\mathrm{d}t\\&\qquad+\frac{1}{m_{x'}}\int_{\mathbb{R}_+}\varphi\left(\frac{t}{m_{x'}}\right)\left(\pi_{V_t(x)}(x,z)-\pi_{V_t(x')}\left(x',z\right)\right)\mathrm{d}t\\&\quad=I_1+I_2+I_3.\end{aligned}$$

Using the fact that $\psi\in\mathcal{M}_L$ and part (i) for $\hat{\pi}_\psi(x,z)$, it follows that $|I_1|\le CL^{-(d+1)}$. Using additionally the smoothness of φ and, by Lemma 4.1(i), $|\pi_{V_t(x)}(x,z)|\le CL^{-(d-1)}$, we also have $|I_2|\le CL^{-(d+1)}$. It remains to handle I_3. By translation invariance of the random walk, $\pi_{V_t(x)}(x,z)=\pi_{V_t}(0,z-x)$. In particular, both (iii) and (iv) will follow if we prove that

$$\left|\int_{\mathbb{R}_+}\varphi\left(\frac{t}{m_x}\right)\left(\pi_{V_t}(0,z-x)-\pi_{V_t}\left(0,z-x'\right)\right)\mathrm{d}t\right|\le CL^{-d}\log L\tag{A.8}$$

for x,x' with $|x-x'|=1$. By definition of $\mathcal{M}_L$, $m_x\in(L/10,5L)$. We may therefore assume that $L/10<|y-z|<10L$ for $y=x,x'$. Due to the smoothness of φ and the fact that the integral is over an interval of length at most 2, (A.8) will follow if we show

$$\left|\int_{L/10}^{10L}\left(\pi_{V_t}(0,z-x)-\pi_{V_t}\left(0,z-x'\right)\right)\mathrm{d}t\right|\le CL^{-d}\log L.$$

We set $J=\{t>0:z-x\in\partial V_t\}$ and $J'=\{t>0:z-x'\in\partial V_{t'}\}$, where

$$t'=t'(t)=\left|t\frac{(z-x)}{|z-x|}-\left(x'-x\right)\right|.$$

J is an interval of length at most 1, and J' has the same length up to order $O(L^{-1})$. Furthermore, $|(J \setminus J') \cup (J' \setminus J)|$ is of order $O(L^{-1})$, and $|\frac{\mathrm{d}}{\mathrm{d}t} t'| = 1 + O(L^{-1})$. Using that both $\pi_{V_t}(0, z - x)$ and $\pi_{V_t}(0, z - x')$ are of order $O(L^{-(d-1)})$, it therefore suffices to prove

$$\left| \int_{J \cap J'} \bigl(\pi_{V_t(x)}(x, z) - \pi_{V_{t'}(x')}(x', z)\bigr) \,\mathrm{d}t \right| \le C L^{-d} \log L. \tag{A.9}$$

Write V for $V_t(x)$ and V' for $V_{t'}(x')$. By a first exit decomposition,

$$\pi_V(x, z) \le \pi_{V'}(x, z) + \sum_{y \in V \setminus V'} \mathrm{P}_{x,p}(T_y < \tau_V) \pi_V(y, z).$$

By Lemma 4.1(ii), we can replace $\pi_{V'}(x, z)$ by $\pi_{V'}(x', z) + O(L^{-d})$. For $y \in V \setminus V'$ we have by Lemma 4.2(ii) $\pi_V(y, z) = O(|y - z|^{-d})$ and $\mathrm{P}_{x,p}(T_y < \tau_V) = O(L^{-(d-1)})$, uniformly in $t \in J \cap J'$. Using $|x - x'| = 1$, we have with $r = |z - x|$

$$\bigcup_{t \in J \cap J'} (V \setminus V') \subset V_r(x) \setminus V_{r-2}(x') \subset x + \mathrm{Sh}_r(3),$$

and for any $y \in x + \mathrm{Sh}_r(3)$, it follows by a geometric consideration that

$$\int_{J \cap J'} 1_{\{y \in V \setminus V'\}} \,\mathrm{d}t \le C \frac{|y - z|}{L}.$$

Altogether, applying Lemma 4.6 in the last step,

$$\begin{aligned}
&\int_{J \cap J'} \pi_V(x, z) \,\mathrm{d}t \\
&\qquad \le \int_{J \cap J'} \pi_{V'}(x', z) \,\mathrm{d}t + O(L^{-d}) + C L^{-(d-1)} \sum_{y \in x + \mathrm{Sh}_r(3)} \frac{1}{|y - z|^d} \frac{|y - z|}{L} \\
&\qquad \le \int_{J \cap J'} \pi_{V'}(x', z) \,\mathrm{d}t + C L^{-d} \log L.
\end{aligned}$$

The reverse inequality, proved in the same way, then implies (A.9).

(v) and (vi) are the analogous statements of (iii) and (iv) for Brownian motion with covariance matrix Λ_p and can be proved in the same way.

(vii) Fix $\alpha = 2/3$, $\beta = 1/3$, and let $0 < \eta < 1/40$. Set $A = C_{L^\alpha}(z)$ and $A^{\mathbb{Z}} = A \cap \mathbb{Z}^d$. By part (iv), we have

$$\hat{\pi}_\psi(x, z) \le \frac{1}{|A^{\mathbb{Z}}|} \hat{\pi}_\psi(x, A^{\mathbb{Z}}) + C L^{-(d+1-\alpha)} \log L. \tag{A.10}$$

Moreover,

$$\hat{\pi}_\psi(x, A^{\mathbb{Z}}) = \frac{1}{m_x} \int_{L/10}^{10L} \varphi\left(\frac{t}{m_x}\right) \pi_{V_t(x)}(x, A^{\mathbb{Z}}) \,\mathrm{d}t. \tag{A.11}$$

By Lemma A.1(i), it follows that for $t \in (L/10, 10L)$,

$$\pi_{V_t(x)}\big(x, A^{\mathbb{Z}}\big) \le \pi^{\mathrm{B}}_{V_t(x)}\big(x, A^{\beta}\big)\big(1 + CL^{-(\beta-3\eta)}\big) + CL^{-(d+1)},$$

where $A^{\beta} = C_{L^{\alpha}+L^{\beta}}(z)$ and the constant C is uniform in t. If we plug the last line into (A.11), and use parts (ii) and (vi), we arrive at

$$\begin{aligned}\hat{\pi}_{\psi}\big(x, A^{\mathbb{Z}}\big) &\le \hat{\pi}^{\mathrm{B}}_{\psi}\big(x, A^{\beta}\big)\big(1 + CL^{-(\beta-3\eta)}\big) + CL^{-(d+1)}\\ &\le \hat{\pi}^{\mathrm{B}}_{\psi}(x, A)\big(1 + CL^{-(\beta-3\eta)}\big) + CL^{-d}L^{(d-1)\alpha+\beta}\\ &\le |A| \cdot \hat{\pi}^{\mathrm{B}}_{\psi}(x, z) + CL^{d\alpha}L^{-(d+\beta-3\eta)}.\end{aligned}$$

Notice that in our notation, $|A|$ is the volume of A, while $|A^{\mathbb{Z}}|$ is the cardinality of $A^{\mathbb{Z}}$. From Gauss's circle problem we have learned that $|A| = |A^{\mathbb{Z}}| + O(L^{(d-1)\alpha})$. Going back to (A.10), this implies

$$\hat{\pi}_{\psi}(x, z) \le \hat{\pi}^{\mathrm{B}}_{\psi}(x, z) + L^{-(d+1/4)},$$

as claimed. To prove the reverse inequality, we can follow the same steps, replacing the random walk estimates by those of Brownian motion and vice versa. □

A.2. Proof of Lemma 4.7. For simplicity, let us write ϕ for $\phi_{L,p,\psi,q}$ and ϕ^{B} for $\phi^{\mathrm{B}}_{L,p,\psi,q}$.

PROOF OF LEMMA 4.7. (i) Set $\alpha = 2/3$, $\beta = 1/3$ and $\eta = \mathrm{d}(x, \partial V_L)$. Choose $y_1 \in \partial V_L$ such that $|x - y_1| = \eta$. First assume $\eta \le L^{\beta}$. The following estimates are valid for L large. First,

$$\begin{aligned}\phi(x, z) &= \sum_{\substack{y\in\partial V_L:\\ |y-y_1|\le L^{\alpha}}} \pi^{(p)}_L(x, y)\hat{\pi}^{(q)}_{\psi}(y, z) + \sum_{\substack{y\in\partial V_L:\\ |y-y_1|> L^{\alpha}}} \pi^{(p)}_L(x, y)\hat{\pi}^{(q)}_{\psi}(y, z)\\ &= I_1 + I_2.\end{aligned}$$

For I_2, notice that $|y - y_1| > L^{\alpha}$ implies $|y - x| > L^{\alpha}/2$. Using Lemmata A.2(i) and 4.2(iii) in the first and Lemma 4.6 in the second inequality, we have

$$I_2 \le C\eta L^{-d} \sum_{\substack{y\in\partial V_L:\\ |y-y_1|> L^{\alpha}}} \frac{1}{|x - y|^d} \le C\eta L^{-(d+\alpha)} \le L^{-(d+1/4)}. \tag{A.12}$$

For I_1, we first use Lemma A.2(iii) to deduce

$$\hat{\pi}^{(q)}_{\psi}(y, z) \le \hat{\pi}^{(q)}_{\psi}(y_1, z) + CL^{-(d+1-\alpha)}\log L.$$

Therefore by part (vii),

$$I_1 \le \hat{\pi}^{(q)}_{\psi}(y_1, z) + L^{-(d+1/4)} \le \hat{\pi}^{\mathrm{B}(q)}_{\psi}(y_1, z) + 2L^{-(d+1/4)}.$$

A similar argument as in (A.12), using Lemma 4.3(i), yields

$$\int_{y\in\partial C_L:|y-y_1|>L^\alpha} \pi_L^{\mathrm{B}(p)}(x,dy) \le L^{-1/4}.$$

Using Lemma A.2(ii) in the first and (v) in the second inequality, we obtain

$$\begin{aligned}\hat{\pi}_\psi^{\mathrm{B}(q)}(y_1,z) &\le \hat{\pi}_\psi^{\mathrm{B}(q)}(y_1,z)\int_{y\in\partial C_L:|y-y_1|\le L^\alpha} \pi_L^{\mathrm{B}(p)}(x,dy) + CL^{-(d+1/4)}\\ &\le \int_{y\in\partial C_L:|y-y_1|\le L^\alpha} \pi_L^{\mathrm{B}(p)}(x,dy)\hat{\pi}_\psi^{\mathrm{B}(q)}(y,z) + CL^{-(d+1/4)}\\ &\le \phi^{\mathrm{B}}(x,z) + CL^{-(d+1/4)}.\end{aligned}$$

Together with (A.12), we have shown that $\phi(x,z) \le \phi^{\mathrm{B}}(x,z) + CL^{-(d+1/4)}$ when $\eta \le L^\beta$.

Now we look at the case $\eta > L^\beta$. We take a cube U_1 of radius L^α, centered at y_1, and set $W_1 = U_1 \cap \partial V_L$. Then we can find a partition of $\partial V_L \setminus W_1$ into disjoint sets $W_i = U_i \cap \partial V_L$, $i = 2,\dots,k_L$, where U_i is a cube such that for some $c_1, c_2 > 0$ depending only on d,

$$c_1 L^{\alpha(d-1)} \le |W_i| \le c_2 L^{\alpha(d-1)}.$$

For $i \ge 2$, we fix an arbitrary $y_i \in W_i$. Let $W_i^\beta = \{y \in \mathbb{R}^d : \mathrm{d}(y, W_i) \le L^\beta\}$. Applying first Lemma A.2(iii) and then Lemma A.1(i) gives

$$\begin{aligned}\phi(x,z) &\le \sum_{i=1}^{k_L} \pi_L^{(p)}(x,W_i)\hat{\pi}_\psi^{(q)}(y_i,z) + L^{-(d+1/4)}\\ &\le \sum_{i=1}^{k_L} \pi_L^{\mathrm{B}(p)}(x,W_i^\beta)\hat{\pi}_\psi^{(q)}(y_i,z)(1+L^{-1/4}) + L^{-(d+1/4)}.\end{aligned} \tag{A.13}$$

As the W_i^β overlap, we refine them as follows: set $\tilde{W}_1 = W_1^\beta \cap \partial C_L$, and split $\partial C_L \setminus \tilde{W}_1$ into a collection of disjoint measurable sets $\tilde{W}_i \subset W_i^\beta \cap \partial C_L$, $i = 2,\dots,k_L$, such that $\bigcup_{i=1}^{k_L} \tilde{W}_i = \partial C_L$ and $|(W_i^\beta \cap \partial C_L) \setminus \tilde{W}_i| \le C_1 L^{\alpha(d-2)+\beta}$ for some $C_1 = C_1(d)$. By construction we can find constants $c_3, c_4 > 0$ such that $|\tilde{W}_i| \ge c_3 L^{\alpha(d-1)}$ and, for $i = 2,\dots,k_L$,

$$\inf_{y\in W_i^\beta} |x-y| \ge c_4 \sup_{y\in\tilde{W}_i} |x-y|,$$

which implies by Lemma 4.3(i) that

$$\sup_{y\in W_i^\beta} \pi_L^{\mathrm{B}(p)}(x,y) \le C \inf_{y\in\tilde{W}_i} \pi_L^{\mathrm{B}(p)}(x,y).$$

For $i = 1,\dots,k_L$ we then have

$$\pi_L^{\mathrm{B}(p)}(x,W_i^\beta) \le \pi_L^{\mathrm{B}(p)}(x,\tilde{W}_i)(1+CL^{\beta-\alpha}) \le \pi_L^{\mathrm{B}(p)}(x,\tilde{W}_i)(1+L^{-1/4}).$$

Plugging the last line into (A.13),

$$\phi(x,z) \le \sum_{i=1}^{k_L} \pi_L^{\mathrm{B}(p)}(x, \tilde{W}_i)\hat{\pi}_\psi^{(q)}(y_i, z)\big(1 + L^{-1/4}\big) + L^{-(d+1/4)}.$$

A reapplication of Lemma A.2(iii), (vii) and then (ii) yields

$$\begin{aligned}\phi(x,z) &\le \sum_{i=1}^{k_L} \int_{\tilde{W}_i} \pi_L^{\mathrm{B}(p)}(x, dy)\hat{\pi}_\psi^{(q)}(y, z) + L^{-(d+1/4)} \\ &\le \sum_{i=1}^{k_L} \int_{\tilde{W}_i} \pi_L^{\mathrm{B}(p)}(x, dy)\hat{\pi}_\psi^{\mathrm{B}(q)}(y, z)\big(1 + L^{-1/4}\big) + L^{-(d+1/4)} \\ &= \phi^{\mathrm{B}}(x,z) + CL^{-(d+1/4)}.\end{aligned}$$

The reverse inequalities in both the cases $\eta \le L^\beta$ and $\eta > L^\beta$ are obtained similarly.

(ii) Let $\psi = (m_y) \in \mathcal{M}_L$ and $z \in \mathbb{Z}^d$. For $y \in \mathcal{U}_L$ we set

$$g(y,z) = \frac{1}{m_y}\varphi\left(\frac{|z-y|}{m_y}\right)\pi_{C_{|z-y|}}^{\mathrm{B}(q)}(0, z-y). \tag{A.14}$$

Then

$$\phi^{\mathrm{B}}(x,z) = \int_{\partial C_L} \pi_L^{\mathrm{B}(p)}(x, dy)g(y,z).$$

Choose a cutoff function $\chi \in C^\infty(\mathbb{R}^d)$ with compact support in $\{x \in \mathbb{R}^d : 1/2 < |x| < 2\}$ such that $\chi \equiv 1$ on $\{x \in \mathbb{R}^d : 2/3 \le |x| \le 3/2\}$. Setting $m_v = 1$ for $v \notin \mathcal{U}_L$, we define

$$\tilde{g}(y,z) = g(Ly, z)\chi(y), \qquad y, z \in \mathbb{R}^d.$$

By Brownian scaling,

$$\tilde{g}(y,z) = \frac{1}{m_{Ly}}\varphi\left(\frac{|z-Ly|}{m_{Ly}}\right)\frac{1}{|z-Ly|^{d-1}}\pi_{C_1}^{\mathrm{B}(q)}\left(0, \frac{z-y}{|z-y|}\right)\chi(y).$$

Notice that $\tilde{g}(\cdot, z) \in C^4(\mathbb{R}^d)$, with $\tilde{g}(y,z) = 0$ if $|y| \notin (1/2, 2)$ or $|z - Ly| \notin (L/10, 10L)$. Let $\mathcal{L} = p(e_1)\partial^2/\partial x_1^2 + \cdots + p(e_d)\partial^2/\partial x_d^2$. Then $u(\overline{x}, z) = \phi^{\mathrm{B}}(x,z)$, $x = L\overline{x}$, solves

$$\begin{cases} \mathcal{L}u(\cdot, z) = 0, & \text{in } C_1, \\ u(\cdot, z) = \tilde{g}(\cdot, z), & \text{on } \partial C_1. \end{cases}$$

By Corollary 6.5.4 of Krylov [14], $u(\cdot, z)$ is smooth on $\overline{C}_1$. Write

$$|u(\cdot, z)|_k = \sum_{i=0}^{k} \|D^i u(\cdot, z)\|_{C_1}.$$

By Theorem 6.3.2 in the same book, there exists $C > 0$ independent of z such that

$$|u(\cdot, z)|_3 \leq C|\tilde{g}(\cdot, z)|_4.$$

A direct calculation shows that $\sup_{z\in\mathbb{R}^d} |\tilde{g}(\cdot, z)|_4 \leq CL^{-d}$. Now the claim follows from

$$\|D^i\phi^{\mathrm{B}}(\cdot, z)\|_{C_L} = L^{-i}\|D^i u(\cdot, z)\|_{C_1}.$$

(iii) Let $x, x' \in V_L \cup \partial V_L$. Choose $\tilde{x} \in V_L$ next to x so that $|\tilde{x} - x| = 1$ if $x \in \partial V_L$ and $\tilde{x} = x$ otherwise. Similarly, choose $\tilde{x}' \in V_L$ next to x'. By the triangle inequality,

$$\begin{aligned} &|\phi(x, z) - \phi(x', z)| \\ &\quad \leq |\phi(x, z) - \phi(\tilde{x}, z)| + |\phi(\tilde{x}, z) - \phi(\tilde{x}', z)| + |\phi(\tilde{x}', z) - \phi(x', z)|. \end{aligned} \tag{A.15}$$

By parts (i) and (ii) combined with the mean value theorem, we get for the middle term

$$\begin{aligned} &|\phi(\tilde{x}, z) - \phi(\tilde{x}', z)| \\ &\quad \leq |\phi(\tilde{x}, z) - \phi^{\mathrm{B}}(\tilde{x}, z)| + |\phi^{\mathrm{B}}(\tilde{x}, z) - \phi^{\mathrm{B}}(\tilde{x}', z)| + |\phi^{\mathrm{B}}(\tilde{x}', z) - \phi(\tilde{x}', z)| \\ &\quad \leq C(L^{-(d+1/4)} + |x - x'|L^{-(d+1)}). \end{aligned}$$

If $x \in \partial V_L$, then $\phi(x, z) = \hat{\pi}_\psi^{(q)}(x, z)$, so that the first term of (A.15) can be written as

$$|\phi(x, z) - \phi(\tilde{x}, z)| = \left|\sum_{y\in\partial V_L} \pi_L^{(p)}(\tilde{x}, y)(\hat{\pi}_\psi^{(q)}(y, z) - \hat{\pi}_\psi^{(q)}(x, z))\right|.$$

Set $A = \{y \in \partial V_L : |x - y| > L^{1/4}\}$. Then by Lemmata 4.2(iii) and 4.6,

$$\pi_L^{(p)}(\tilde{x}, A) \leq C \sum_{y\in A} \frac{1}{|x - y|^d} \leq CL^{-1/4}.$$

For all $y \in \partial V_L$, we have by Lemma A.2(i) $|\hat{\pi}_\psi^{(q)}(y, z) - \hat{\pi}_\psi^{(q)}(x, z)| \leq CL^{-d}$. If $y \in \partial V_L \setminus A$, then part (iii) gives $|\hat{\pi}_\psi^{(q)}(y, z) - \hat{\pi}_\psi^{(q)}(x, z)| \leq CL^{-(d+3/4)} \log L$. Altogether,

$$|\phi(x, z) - \phi(\tilde{x}, z)| \leq CL^{-(d+1/4)}.$$

The third term of (A.15) is treated in exactly the same way. $\square$

A.3. Proofs of Lemmata 4.1 and 4.2. We let

$$V_L^p(y) = \{x \in \mathbb{Z}^d : |\Lambda_p^{-1/2}(x-y)| \le L\}$$

and $V_L^p = V_L^p(0)$. Note that

$$V^p_{(1+2d\kappa)^{-1/2}L} \subset V_L \subset V^p_{(1-2d\kappa)^{-1/2}L}. \tag{A.16}$$

We will make use of the following:

LEMMA A.3. (i) *Let* $0 < \ell < L$ *and* $x \in V_L^p$ *with* $\ell < |\Lambda_p^{-1/2}x| < L$. *Then*

$$\mathrm{P}_{x,p}(\tau_{V_L^p} < T_{V_\ell^p}) = \frac{\ell^{-d+2} - |\Lambda_p^{-1/2}x|^{-d+2} + O(\ell^{-d+1})}{\ell^{-d+2} - L^{-d+2}}.$$

(ii) *There exists* $C > 0$ *such that for all* $\theta \in \mathbb{R}^d$ *with* $|\theta| = 1$ *and* $\ell \ge 0$,

$$\mathrm{P}_{0,p}(X_n \cdot \theta \ge -\ell \text{ for all } 0 \le n \le \tau_L) \le C(\ell+1)L^{-1}.$$

PROOF. (i) For the case of simple random walk, that is, $p = p_o$, this is Proposition 1.5.10 of [16]. For the case of general $p \in \mathcal{P}_\kappa^{\mathrm{s}}$, one can use Proposition 6.3.1 of [17] for the Green's function, and then the proof is exactly the same.

(ii) This is a version of the gambler's ruin estimate; see, for example, [17], Exercise 7.5. □

We turn to the proof of Lemma 4.1. We constantly write P_x for $\mathrm{P}_{x,p}$ and π_L for $\pi_L^{(p)}$, but stress that V_L^p is not the same set as V_L.

PROOF OF LEMMA 4.1. (i) $\pi_L(\cdot, z)$ is p-harmonic inside V_L; that is, for $x \in V_L$,

$$\pi_L(x,z) = \sum_{e \in \mathbb{Z}^d : |e|=1} p(e)\pi_L(x+e,z).$$

Applying a discrete Harnack inequality, as, for example, provided by Theorem 6.3.9 in the book of Lawler and Limic [17], we obtain $C^{-1}\pi_L(0,z) \le \pi_L(\cdot,z) \le C\pi_L(0,z)$ on $V_{\eta L}$ for some $C = C(d,\eta)$, and it remains to show that $\pi_L(0,z)$ has the right order of magnitude. Note that we cannot directly apply Lemma 6.3.7 in this book, since we look at the exit distribution from V_L, not from V_L^p. However, by a last-exit decomposition as in Lemma 6.3.7, with $g_{V_L}(x,y) = \sum_{k=0}^\infty (1_{V_L}p)^k(x,y)$ and $\tilde{\tau}_A = \inf\{n \ge 1 : X_n \notin A\}$,

$$\pi_L(0,z) = \sum_{y \in V_{L/2}} g_{V_L}(0,y)\mathrm{P}_z(X_{\tilde{\tau}_{V_L \setminus V_{L/2}}} = y).$$

Using (A.16), we have for $y \in V_{L/2}$, with $L_1 = (1+2d\kappa)^{-1/2}L$ and $L_2 = (1-2d\kappa)^{-1/2}L$,

$$g_{V_{L_1}^p}(0,y) \le g_{V_L}(0,y) \le g_{V_{L_2}^p}(0,y).$$

For $y \in V_{L/2} \cap \partial(V_L \setminus V_{L/2})$ both outer Green's functions are of order L^{-d+2}, by Proposition 6.3.5 of [17]. Now by translation invariance and Lemma A.3(ii), with $\theta = -z/|z|$,

$$\mathrm{P}_z(X_{\tilde{\tau}_{V_L \setminus V_{L/2}}} \in V_{L/2}) \le \mathrm{P}_0(X_n \cdot \theta \ge 0 \text{ for all } 0 \le n \le \tau_{L/2}) \le CL^{-1},$$

which proves that $\pi_L(0,z) \le CL^{-d+1}$. For the lower bound, if κ is small enough, we find an ellipsoid $V_{L_3}^p(y)$ with $L_3 \ge (9/10)L$, centered at some $y \in V_L$ and lying completely inside V_L such that $z \in \partial V_{L_3}^p(y) \cap \partial V_L$. Also, $V_{L_3/5}^p(y) \subset V_{L/2}$ if κ is small. Therefore,

$$\mathrm{P}_z(X_{\tilde{\tau}_{V_L \setminus V_{L/2}}} \in V_{L/2}) \ge \mathrm{P}_z\bigl(X_{\tilde{\tau}_{V_{L_3}^p(y) \setminus V_{L_3/5}^p(y)}} \in V_{L_3/5}^p(y)\bigr).$$

With positive probability, the random walk starting at z enters $V_{L_3}^p(y)$ in the next step and then visits a point w with $\mathrm{d}_L(w) \ge 1$, staying inside $V_{L_3}^p(y)$. Thus

$$\mathrm{P}_z\bigl(X_{\tilde{\tau}_{V_{L_3}^p(y) \setminus V_{L_3/5}^p(y)}} \in V_{L_3/5}^p(y)\bigr) \ge c\mathrm{P}_w(T_{V_{L_3/5}^p(y)} < \tau_{V_{L_3}^p(y)}) \ge cL^{-1},$$

where the last inequality follows from bounding the expression obtained in Lemma A.3(i). With the estimate on g_{V_L}, this proves that $\pi_L(0,z)$ is bounded from below by $C^{-1}L^{-d+1}$, and (i) follows.

(ii) By the triangle inequality,

$$\bigl|\pi_L(x,z) - \pi_L(x',z)\bigr| \le C|x - x'| \max_{u,v \in V_{\eta L} : |u-v| \le 1} \bigl|\pi_L(u,z) - \pi_L(v,z)\bigr|.$$

For $u \in V_{\eta L}$, the function $\pi_L(u + \cdot, z)$ is p-harmonic inside $V_{(1-\eta)L}$. The claim now follows from (6.19) of Theorem 6.3.8 in [17], together with (i). $\square$

Before we start with the proof of Lemma 4.2, we prove a further auxiliary lemma, which already includes the upper bound of part (iii).

LEMMA A.4. *Let $x \in V_L$, $y \in \partial V_L$, and set $t = |x - y|$.*

(i)

$$\mathrm{P}_x(X_{\tau_L} = y) \le C\mathrm{d}_L(x)^{-d+1}.$$

(ii)

$$\mathrm{P}_x(X_{\tau_L} = y) \le C\frac{\max\{1, \mathrm{d}_L(x)\}}{t} \max_{x' \in \partial V_{t/3}(y) \cap V_L} \mathrm{P}_{x'}(X_{\tau_L} = y).$$

(iii)

$$\mathrm{P}_x(X_{\tau_L} = y) \le C\frac{\max\{1, \mathrm{d}_L(x)\}}{|x - y|^d}.$$

PROOF. (i) We can assume that $s = \mathrm{d}_L(x) \geq 6$. If $s' = \lfloor s/3 \rfloor$, then $\partial V_{s'}(x) \subset V_{L-s'}$. Using Lemma A.3(ii), we compute for any $y' \in V_L$ with $|y - y'| = 1$, $\theta = -y'/|y'|$,

$$\mathrm{P}_{y'}(T_{\partial V_{s'}(x)} < \tau_L) \leq \mathrm{P}_0(X_n \cdot \theta \geq -1 \text{ for all } 0 \leq n \leq \tau_{V_{s'}}) \leq Cs^{-1}.$$

By Lemma 4.2(i) it follows that uniformly in $z \in \partial V_{s'}(x)$,

$$\mathrm{P}_z(T_x < \tau_L) \leq \mathrm{P}_z(T_x < \infty) \leq C\bigl(s'\bigr)^{-d+2} \leq Cs^{-d+2}.$$

Thus, by the strong Markov property at $T_{\partial V_{s'}(x)}$,

$$\mathrm{P}_{y'}(T_x < \tau_L) \leq Cs^{-d+1}.$$

Since by time reversibility of symmetric random walk

$$\begin{aligned} \mathrm{P}_x(X_{\tau_L} = y) &= \sum_{\substack{y' \in V_L, \\ |y'-y|=1}} \mathrm{P}_x\bigl(X_{\tau_L} = y, X_{\tau_L - 1} = y'\bigr) \\ &= \frac{1}{2d} \sum_{\substack{y' \in V_L, \\ |y'-y|=1}} \mathrm{P}_{y'}(T_x < \tau_L), \end{aligned}$$

the claim is proved.

(ii) We may assume that $t = |x - y| > 100d$ and $\mathrm{d}_L(x) < t/100$. Choose a point $x' \in \mathbb{Z}^d$ such that $V_{t/10}(x') \cap V_L = \varnothing$ and $|x - x'| \leq \mathrm{d}_L(x) + t/10 + \sqrt{d}$. Then $|x - x'| \leq t/5$. Furthermore, since $|x' - y| \geq 4t/5$,

$$\bigl(V_{t/4}\bigl(x'\bigr) \cup \partial V_{t/4}\bigl(x'\bigr)\bigr) \cap V_{t/3}(y) = \varnothing.$$

We apply twice the strong Markov property and obtain

$$\mathrm{P}_x(X_{\tau_L} = y) \leq \mathrm{P}_x(\tau_{V_{t/4}(x')} < \tau_L) \max_{z \in \partial V_{t/3}(y) \cap V_L} \mathrm{P}_z(X_{\tau_L} = y).$$

Arguing much as in (i), Lemma A.3(ii) shows

$$\mathrm{P}_x(\tau_{V_{t/4}(x')} < \tau_L) \leq C \frac{\max\{1, \mathrm{d}_L(x)\}}{t},$$

which completes the proof of part (ii).

(iii) By (ii) it suffices to prove that for some constant K and for all $\ell \geq 1$,

$$\max_{z \in \partial V_{\ell/3}(y) \cap V_L} \mathrm{P}_z(X_{\tau_L} = y) \leq K\ell^{-d+1}. \tag{A.17}$$

Let c_1 and c_2 be the constants from (i) and (ii), respectively. Define $\eta = 3^{-d}c_2^{-1}$ and $K = \max\{3^{d(d-1)}c_2^{d-1}, c_1\eta^{-d+1}\}$. For $\ell \leq 3^d c_2$, there is nothing to prove since $K\ell^{-d+1} \geq 1$. Thus let $\ell > 3^d c_2$, and choose ℓ_0 with $\ell_0 < \ell \leq 2\ell_0$. Assume that (A.17) is proved for all $\ell' \le \ell_0$. We show that (A.17) also holds for ℓ. For z with $\mathrm{d}_L(z) \ge \eta\ell$, it follows from (i) that

$$\mathrm{P}_z(X_{\tau_L} = y) \le c_1 \eta^{-d+1} \ell^{-d+1} \le K\ell^{-d+1}.$$

If $1 \le \mathrm{d}_L(z) < \eta\ell$, then by (ii) and the fact that $\ell/3 \le \ell_0$

$$\begin{aligned}\mathrm{P}_z(X_{\tau_L} = y) &\le c_2 \frac{\max\{1, \mathrm{d}_L(z)\}}{|z-y|} \max_{z' \in \partial V_{l/9}(y) \cap V_L} \mathrm{P}_z(X_{\tau_L} = y) \\ &\le c_2 3\eta K(\ell/3)^{-d+1} \le K\ell^{-d+1}.\end{aligned}$$

If $\mathrm{d}_L(z) < 1$, then again by (i)

$$\mathrm{P}_z(X_{\tau_L} = y) \le c_2 3\ell^{-1} K(\ell/3)^{-d+1} \le K\ell^{-d+1}.$$

This proves the claim. $\square$

PROOF OF LEMMA 4.2. (i) follows from Proposition 6.4.2 of [17].

(ii) We consider different cases. If $|x-y| \le \mathrm{d}_L(y)/2$, then $\mathrm{d}_L(x) \ge \mathrm{d}_L(y)/2$, and thus by Lemma 4.2(i),

$$\mathrm{P}_x(T_{V_a(y)} < \tau_L) \le \mathrm{P}_x(T_{V_a(y)} < \infty) \le C\left(\frac{a}{|x-y|}\right)^{d-2} \le C\frac{a^{d-2}\mathrm{d}_L(y)\mathrm{d}_L(x)}{|x-y|^d}.$$

For the rest of the proof we assume that $|x-y| > \mathrm{d}_L(y)/2$. Set $a' = \mathrm{d}_L(y)/5$. First we argue that for the case $1 \le a \le a'$, we only have to prove the bound for a'. Indeed, if $\mathrm{d}_L(y)/6 \le a < a'$, we get an upper bound by replacing a by a'. For $1 \le a < \mathrm{d}_L(y)/6$, the strong Markov property together with Lemma 4.2(i) yields

$$\begin{aligned}\mathrm{P}_x(T_{V_a(y)} < \tau_L) &\le \max_{z \in \partial(\mathbb{Z}^d \setminus V_{a'}(y))} \mathrm{P}_z(T_{V_a(y)} < \tau_L)\mathrm{P}_x(T_{V_{a'}(y)} < \tau_L) \\ &\le C\left(\frac{a}{a'-1}\right)^{d-2} \frac{(a')^{d-2}\mathrm{d}_L(y)\max\{1, \mathrm{d}_L(x)\}}{|x-y|^d} \\ &\le C\frac{a^{d-2}\mathrm{d}_L(y)\max\{1, \mathrm{d}_L(x)\}}{|x-y|^d}.\end{aligned}$$

Now we prove the claim for $a = \mathrm{d}_L(y)/5$. We take a point $y' \in \partial V_L$ closest to y. If $|x-z| \ge |x-y|/2$ for all $z \in V_a(y')$, then by Lemma A.4(iii),

$$\max_{z \in V_a(y')} \mathrm{P}_x(X_{\tau_L} = z) \le C 2^d \frac{\max\{1, \mathrm{d}_L(x)\}}{|x-y|^d}.$$

As a subset of $\mathbb{Z}^d$, $V_a(y') \cap \partial V_L$ contains on the order of $\mathrm{d}_L(y)^{d-1}$ points. Therefore, by Lemma 4.1(i), we deduce that there exists some $\delta > 0$ such that

$$\min_{x' \in V_a(y)} \mathrm{P}_{x'}\big(X_{\tau_L} \in V_a(y')\big) \ge \delta.$$

We conclude that

$$\begin{aligned}
&\frac{a^{d-1}\max\{1,\mathrm{d}_L(x)\}}{|x-y|^d}\\
&\quad\geq c\mathrm{P}_x\bigl(X_{\tau_L}\in V_a\bigl(y'\bigr)\bigr)\geq c\mathrm{P}_x\bigl(X_{\tau_L}\in V_a\bigl(y'\bigr),T_{V_a(y)}<\tau_L\bigr)\\
&\quad= c\sum_{x'\in V_a(y)}\mathrm{P}_x\bigl(X_{T_{V_a(y)}}=x',T_{V_a(y)}<\tau_L\bigr)\mathrm{P}_{x'}\bigl(X_{\tau_L}\in V_a\bigl(y'\bigr)\bigr)\\
&\quad\geq c\delta\cdot\mathrm{P}_x(T_{V_a(y)}<\tau_L).
\end{aligned}\tag{A.18}$$

On the other hand, if $|x-z|<|x-y|/2$ for some $z\in V_a(y')$, then

$$|x-y|\leq|x-z|+\bigl|z-y'\bigr|+\bigl|y'-y\bigr|\leq 2\mathrm{d}_L(y)+|x-y|/2$$

and thus

$$\mathrm{d}_L(y)/2<|x-y|\leq 4\mathrm{d}_L(y).\tag{A.19}$$

If $\mathrm{d}_L(x)\geq 4\mathrm{d}_L(y)/5$, we use again Lemma 4.2(i). For $\mathrm{d}_L(x)<4\mathrm{d}_L(y)/5$, we get by Lemma A.3(ii),

$$\mathrm{P}_x(T_{V_a(y)}<\tau_L)\leq\mathrm{P}_x(T_{V_{L-4\mathrm{d}_L(y)/5}}<\tau_L)\leq C\frac{\max\{1,\mathrm{d}_L(x)\}}{\mathrm{d}_L(y)}.$$

Together with (A.19), this proves the claim for $a=\mathrm{d}_L(y)/5$. It remains to handle the case $\max\{1,\mathrm{d}_L(y)/5\}\leq a$. If $z\in V_{6a}(y)$, we have

$$|x-y|\leq|x-z|+6a$$

and thus, using $|x-y|>7a$,

$$|x-y|\leq 7|x-z|.$$

Therefore Lemma A.4(iii) yields

$$\max_{z\in V_{6a}(y)}\mathrm{P}_x(X_{\tau_L}=z)\leq C\frac{\max\{1,\mathrm{d}_L(x)\}}{|x-z|^d}\leq 7^dC\frac{\max\{1,\mathrm{d}_L(x)\}}{|x-y|^d}.$$

Again by Lemma 4.1(i), we find some $\delta>0$ such that

$$\min_{x'\in V_a(y)}\mathrm{P}_{x'}\bigl(X_{\tau_L}\in V_{6a}(y)\bigr)\geq\delta.$$

A similar argument as in (A.18), with $V_a(y')$ replaced by $V_{6a}(y)$, finishes the proof of (ii).

(iii) It only remains to prove the lower bound. Let $t=|x-z|$. First assume $t\geq L/2$. By replacing V_L and $V_{2L/3}$ by appropriate ellipsoids as in the proof of the lower bound of Lemma 4.1(i), we deduce with part (i) of Lemma A.3 that

$$\mathrm{P}_x(T_{V_{2L/3}}<\tau_L)\geq c\frac{\mathrm{d}_L(x)}{t}.$$

The claim then follows from the strong Markov property at time $T_{V_{2L/3}}$ and Lemma 4.1(i). Now assume $t < L/2$. We can restrict ourselves to the case $10\sqrt{d} \le t < L/2$. We find $t' \in [t, t + \sqrt{d}]$ and $x' \in V_L$ such that $V_{t'}(x') \subset V_L$ and $z \in \partial V_{t'}(x')$. If $\mathrm{d}_L(x) > t/2$, a simple geometric consideration and Lemma 4.1(i) show that there is a strictly positive probability to exit the ball $V_{t/2}(x)$ within $V_{2t/3}(x')$. Since by the same lemma,

$$\inf_{y \in V_{2t/3}(x')} \mathrm{P}_y(\tau_L = z) \ge ct^{-(d-1)}, \tag{A.20}$$

we obtain the claim in this case again by applying the strong Markov property. Finally, assume $\mathrm{d}_L(x) \le t/2$. Once more by Lemma A.3(i),

$$\mathrm{P}_x(T_{V_{L-t/3}} < \tau_L) \ge c\frac{\mathrm{d}_L(x)}{t}$$

and

$$\begin{aligned} \mathrm{P}_x(\tau_L = z) &\ge \mathrm{P}_x(\tau_L = z, T_{L-t/3} < \tau_L, T_{V_{2t/3}(x')} < \tau_L) \\ &\ge c\frac{\mathrm{d}_L(x)}{t}\mathrm{P}_x(\tau_L = z | T_{L-t/3} < \tau_L, T_{V_{2t/3}(x')} < \tau_L) \\ &\quad \times \mathrm{P}_x(T_{V_{2t/3}(x')} < \tau_L | T_{L-t/3} < \tau_L). \end{aligned} \tag{A.21}$$

By (A.20), we deduce

$$\begin{aligned} \mathrm{P}_x(\tau_L = z | T_{L-t/3} < \tau_L, T_{V_{2t/3}(x')} < \tau_L) &= \mathrm{P}_x(\tau_L = z | T_{V_{2t/3}(x')} < \tau_L) \\ &\ge ct^{-(d-1)}. \end{aligned}$$

Finally, a geometric argument and Lemma 4.1(i) show that the probability in (A.21) is bounded from below by some $\delta > 0$. □

A.4. Proofs of Lemmata 4.3 and 4.4.

PROOF OF LEMMA 4.3. By Brownian scaling, we may restrict ourselves to the case $L = 1$. Let

$$E_p = \{y \in \mathbb{R}^d : |\Lambda_p^{1/2} y| < 1\}, \tag{A.22}$$

and fix $x \in C_1$, $z \in \partial C_1$. Set $z' = \Lambda_p^{-1/2} z$, $x' = \Lambda_p^{-1/2} x$. Again by Brownian scaling and the fact that $\det \Lambda_p^{-1/2} = 1 + O(\kappa)$, the (continuous versions of the) densities satisfy

$$\pi_{C_1}^{\mathrm{B}(p)}(x, z) = \pi_{E_p}^{\mathrm{B}(p_o)}(x', z')(1 + O(\kappa)).$$

Now (i) follows from Theorem 1 of Krantz [13] (with $\Omega = E_p$), and for (ii), one can use the derivative estimates in Section 2 of the same paper. □

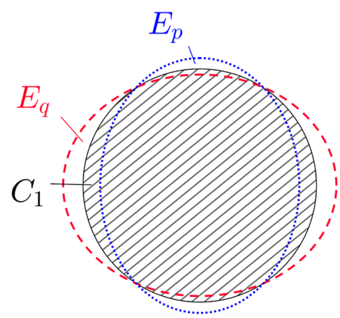

FIG. 7. *The ball C_1 (shaded), and the ellipsoids E_p and E_q.*

PROOF OF LEMMA 4.4. We can assume $L = 1$. Let $\eta = \|q - p\|_1$. Define E_p as in (A.22), and similarly E_q, see Figure 7. Let $x \in C_{2/3}$, $z \in \partial C_1$, and put $x' = \Lambda_p^{-1/2} x$, $x'' = \Lambda_q^{-1/2} x$ and $z' = \Lambda_p^{-1/2} z$, $z'' = \Lambda_q^{-1/2} z$. If κ is small, both x' and x'' lie in $C_{3/4} \subset C_{4/5} \subset E_p \cap E_q$. For the rest of the proof, we write $\pi^{\mathrm{B}}_{E_p}$ instead of $\pi^{\mathrm{B}(p_o)}_{E_p}$, and similarly $\pi^{\mathrm{B}}_{E_q}$ for $\pi^{\mathrm{B}(p_o)}_{E_q}$. If γ is a parametrization of the unit sphere ∂C_1, then $\Lambda_p^{-1/2} \circ \gamma$ and $\Lambda_q^{-1/2} \circ \gamma$ are parametrizations of E_p and E_q, respectively. Since the coefficients of the covariance matrices satisfy $(\Lambda_p)_{i,i} = (\Lambda_q)_{i,i} + O(\eta)$, we obtain by Brownian scaling,

$$\left|\pi^{\mathrm{B}(p)}_{C_1}(x,z) - \pi^{\mathrm{B}(q)}_{C_1}(x,z)\right| \le C\left|\pi^{\mathrm{B}}_{E_p}(x',z') - \pi^{\mathrm{B}}_{E_q}(x'',z'')\right| + O(\eta). \tag{A.23}$$

Clearly

$$\left|z' - z''\right| = \left|\left(\Lambda_p^{-1/2} - \Lambda_q^{-1/2}\right)z\right| \le C\eta, \tag{A.24}$$

and also $|x' - x''| \le C\eta$. By the derivative estimate of Lemma 4.3(ii),

$$\left|\pi^{\mathrm{B}}_{E_q}(x',z'') - \pi^{\mathrm{B}}_{E_q}(x'',z'')\right| \le C\eta.$$

With (A.23), the claim will therefore follow if we show that

$$\left|\pi^{\mathrm{B}}_{E_p}(x,z') - \pi^{\mathrm{B}}_{E_q}(x,z'')\right| \le C\eta$$

uniformly in $x \in C_{3/4}$, $z' \in \partial E_p$, $z'' \in \partial E_q$ with $|z' - z''| \le C\eta$. In this direction, recall that the Green's function on $\mathbb{R}^d$ of standard d-dimensional Brownian motion is given by

$$\Phi(x,y) = \frac{c_d}{|x-y|^{d-2}},$$

where $c_d = \boldsymbol{\Gamma}(d/2 - 1)/(2\pi^{d/2})$; cf. [17], page 241. The Green's function of standard Brownian motion killed outside E_p is given by (see, e.g., Evans [10])

$$\Phi(x,y) - \Phi^{(p)}_x(y), \qquad x, y \in E_p, x \neq y,$$

where the corrector function $\Phi_x^{(p)}$ solves the Dirichlet problem (x is fixed)

$$\begin{cases} \Delta\Phi_x^{(p)} = 0, & \text{in } E_p, \\ \Phi_x^{(p)} = \Phi(x,\cdot), & \text{on } \partial E_p. \end{cases} \tag{A.25}$$

Furthermore, the density $\pi^{\mathrm{B}}_{E_p}(x,z')$ with respect to surface measure on ∂E_p is given by the normal derivative of the Green's function in the direction of the inward unit normal vector $\nu_p = \nu_p(z')$ on ∂E_p, that is,

$$\begin{aligned} \pi^{\mathrm{B}}_{E_p}(x,z') &= \partial_{\nu_p}\big(\Phi(x,z') - \Phi_x^{(p)}(z')\big) \\ &= \nabla_{z'}\big(\Phi(x,z') - \Phi_x^{(p)}(z')\big)\cdot\nu_p(z'), \qquad z' \in \partial E_p. \end{aligned}$$

With ν_q denoting the inward unit normal on ∂E_q, we therefore have to show that

$$\big|\partial_{\nu_p}\big(\Phi(x,z') - \Phi_x^{(p)}(z')\big) - \partial_{\nu_q}\big(\Phi(x,z'') - \Phi_x^{(q)}(z'')\big)\big| \le C\eta \tag{A.26}$$

uniformly in $x \in C_{3/4}$, $z' \in \partial E_p$, $z'' \in \partial E_q$ with $|z'-z''| \le C\eta$. First, note that

$$\begin{aligned} &\big|\partial_{\nu_p}\Phi(x,z') - \partial_{\nu_q}\Phi(x,z'')\big| \\ &\qquad \le \big|\nabla_{z'}\Phi(x,z') - \nabla_{z''}\Phi(x,z'')\big| + \big|\nabla_{z'}\Phi(x,z')\cdot\big(\nu_p(z') - \nu_q(z'')\big)\big|. \end{aligned}$$

Using (A.24) and

$$\partial_{z'_i}\Phi(x,z') = c_d(2-d)\frac{(x-z')_i}{|x-z'|^d},$$

we easily obtain

$$\big|\nabla_{z'}\Phi(x,z') - \nabla_{z''}\Phi(x,z'')\big| \le C\eta.$$

Moreover, with $\Lambda = \Lambda_q^{-1/2}\Lambda_p^{1/2}$, we have

$$\nu_q(z'') = \frac{\Lambda^{-1}\nu_p(z')}{|\Lambda^{-1}\nu_p(z')|}.$$

The coefficients of the diagonal matrix Λ are of order $1 + O(\eta)$, which shows

$$\big|\nabla_{z'}\Phi(x,z')\cdot\big(\nu_p(z') - \nu_q(z'')\big)\big| \le C\big|\nu_p(z') - \nu_q(z'')\big| \le C\eta. \tag{A.27}$$

In view of (A.26), it remains to prove that

$$\big|\partial_{\nu_p}\Phi_x^{(p)}(z') - \partial_{\nu_q}\Phi_x^{(q)}(z'')\big| \le C\eta.$$

First recall that $\Phi_x^{(p)}$ solves the Dirichlet problem (A.25). The boundary function $\Phi(x,\cdot)$ is smooth in a tubular neighborhood of ∂E_p, with bounded derivatives up to arbitrary order. By multiplying with an appropriate smooth cutoff function equal to 1 near the boundary, we obtain a smooth function in $\mathbb{R}^d$. By Corollary 6.5.4 of

Krylov [14], we see that $\Phi_x^{(p)}$ is a smooth function in $\overline{E}_p$, and similarly $\Phi_x^{(q)}$ is smooth in $\overline{E}_q$.

Now, with Λ as before, we have equality of the sets $\Lambda\overline{E}_p = \overline{E}_q$. Fix $x \in C_{3/4} \subset E_p \cap E_q$, and let

$$u(y) = \Phi_x^{(q)}(\Lambda y) - \Phi_x^{(p)}(y), \qquad y \in \overline{E}_p.$$

With $f(\cdot) = \Delta_y \Phi_x^{(q)}(\Lambda \cdot)$ and $g(\cdot) = \Phi(x, \Lambda \cdot) - \Phi(x, \cdot)$, u solves

$$\begin{cases} \Delta u = f, & \text{in } E_p, \\ u = g, & \text{on } \partial E_p. \end{cases}$$

Recalling that the coefficients of Λ are of order $1 + O(\eta)$, we use harmonicity of $\Phi_x^{(q)}$ and boundedness of the derivatives to obtain

$$\|D^0 f\|_{E_p} + \|D^1 f\|_{E_p} \leq C\eta.$$

The function g is smooth in a tubular neighborhood U of ∂E_p, and a similar (explicit) calculation as above gives

$$\sum_{i=0}^{3} \|D^i g\|_U \leq C\eta.$$

We extend g to the interior of E_p such that $\sum_{i=0}^3 \|D^i g\|_{\overline{E}_p} \leq C\eta$. Then, applying a Schauder estimate as given by Theorem 6.3.2 of Krylov [14], we deduce that the derivatives of u up to order 2 are uniformly bounded by $C\eta$. But, similarly to above (A.27),

$$\begin{aligned} \left|\partial_{\nu_p} \Phi_x^{(p)}(z') - \partial_{\nu_q} \Phi_x^{(q)}(z'')\right| &\leq \left|\nabla_{z'} \Phi_x^{(p)}(z') - \nabla_{z''} \Phi_x^{(q)}(z'')\right| + C\eta \\ &\leq \left|\nabla_{z'} u(z')\right| + C\eta \\ &\leq C\eta, \end{aligned}$$

where in the next-to-last step we used $z'' = \Lambda z'$ and

$$\left|\nabla_{z''} \Phi_x^{(q)}(z'') - \nabla_{z'} \Phi_x^{(q)}(z'')\right| \leq C\eta. \qquad \square$$

A.5. Proofs of Propositions 5.1 and 5.2. By a small abuse of notation, we will in this part write $\hat{\pi}_m$ for $\hat{\pi}_{\psi_m}$. Since $\hat{\pi}_m(x, y) = \hat{\pi}_m(0, y - x)$, it suffices to look at $\hat{\pi}_m(x) = \hat{\pi}_m(0, x)$ and $\hat{g}_{m,\mathbb{Z}^d}(x) = \hat{g}_{m,\mathbb{Z}^d}(0, x)$. Recall the definitions of $\lambda_{m,i}$ and Λ_m from Section 5.1.

PROOF OF PROPOSITION 5.1. For bounded m, that is, $m \leq m_0$ for some m_0, the result is a special case of Theorem 2.1.1 in [17]. Also, for $n \leq n_0$ and all m, the statement follows from Lemma A.2(i). We therefore have to prove the proposition only for large n and m. To this end, let

$$B_m = [-\sqrt{\lambda_{m,1}}\pi, \sqrt{\lambda_{m,1}}\pi] \times \cdots \times [-\sqrt{\lambda_{m,d}}\pi, \sqrt{\lambda_{m,d}}\pi],$$

and for $\theta \in B_m$ set

$$\phi_m(\theta) = \sum_{y\in\mathbb{Z}^d} \mathrm{e}^{i\theta\cdot\Lambda_m^{-1/2}y}\hat{\pi}_m(y).$$

The Fourier inversion formula gives

$$\hat{\pi}_m^n(x) = \frac{1}{(2\pi)^d \det \Lambda_m^{1/2}} \int_{B_m} \mathrm{e}^{-ix\cdot\Lambda_m^{-1/2}\theta}\big[\phi_m(\theta)\big]^n \,\mathrm{d}\theta.$$

We decompose the integral into

$$(2\pi)^d \det \Lambda_m^{1/2} n^{d/2}\hat{\pi}_m^n(x) = I_0(n,m,x) + \cdots + I_3(n,m,x),$$

where, with $\beta = \sqrt{n}\theta$,

$$\begin{aligned}
I_0(n,m,x) &= \int_{\mathbb{R}^d} \mathrm{e}^{-ix\cdot\Lambda_m^{-1/2}\beta/\sqrt{n}}\mathrm{e}^{-|\beta|^2/2}\,\mathrm{d}\beta,\\
I_1(n,m,x) &= \int_{|\beta|\le n^{1/4}} \mathrm{e}^{-ix\cdot\Lambda_m^{-1/2}\beta/\sqrt{n}}\big(\big[\phi_m(\beta/\sqrt{n})\big]^n - \mathrm{e}^{-|\beta|^2/2}\big)\,\mathrm{d}\beta,\\
I_2(n,m,x) &= -\int_{|\beta|>n^{1/4}} \mathrm{e}^{-ix\cdot\Lambda_m^{-1/2}\beta/\sqrt{n}}\mathrm{e}^{-|\beta|^2/2}\,\mathrm{d}\beta,\\
I_3(n,m,x) &= n^{d/2}\int_{n^{-1/4}<|\theta|,\theta\in B_m} \mathrm{e}^{-ix\cdot\Lambda_m^{-1/2}\theta}\big[\phi_m(\theta)\big]^n\,\mathrm{d}\theta.
\end{aligned}$$

By completing the square in the exponential, we get

$$I_0(n,m,x) = (2\pi)^{d/2}\exp\biggl(-\frac{\mathcal{J}_m^2(x)}{2n}\biggr).$$

For I_1 and $|\beta| \le n^{1/4}$, we expand ϕ_m in a series around the origin

$$\begin{aligned}
\phi_m(\beta/\sqrt{n}) &= 1 - |\beta|^2/2n + |\beta|^4 O(n^{-2}),\\
\log\phi_m(\beta/\sqrt{n}) &= -|\beta|^2/2n + |\beta|^4 O(n^{-2}).
\end{aligned} \tag{A.28}$$

Therefore,

$$\big[\phi_m(\beta/\sqrt{n})\big]^n = \mathrm{e}^{-|\beta|^2/2}\big(1 + |\beta|^4 O(n^{-1})\big),$$

so that

$$\big|I_1(n,m,x)\big| \le O(n^{-1})\int_{|\beta|\le n^{1/4}} \mathrm{e}^{-|\beta|^2/2}|\beta|^4\,\mathrm{d}\beta = O(n^{-1}).$$

Similarly, I_2 is bounded by

$$\big|I_2(n,m,x)\big| \le C\int_{n^{1/4}}^{\infty} r^{d-1}\mathrm{e}^{-r^2/2}\,\mathrm{d}r = O(n^{-1}).$$

Concerning I_3, we follow closely the proof of [8], Proposition B1, and split the integral further into

$$\begin{aligned} n^{-d/2} I_3(n,m,x) &= \int_{n^{-1/4}<|\theta|\le a} + \int_{a<|\theta|\le A} + \int_{A<|\theta|\le m^\alpha} + \int_{m^\alpha<|\theta|,\theta\in B_m} \\ &= (I_{3,0}+I_{3,1}+I_{3,2}+I_{3,3})(n,m,x), \end{aligned}$$

where $0<a<A$ and $\alpha\in(0,1)$ are constants that will be chosen in a moment, independently of n and m. By (A.28), we can find $a>0$ such that for $|\beta|\le a\sqrt{n}$, $\log\phi_m(\theta)\le -|\theta|^2/3$ (recall that $\beta=\sqrt{n}\theta$). Then

$$\left|I_{3,0}(n,m,x)\right| \le C\int_{n^{-1/4}}^{\infty} r^{d-1}\mathrm{e}^{-nr^2/3}\,\mathrm{d}r = O\big(n^{-(d+2)/2}\big).$$

As a consequence of Lemma 4.1(i) and of our coarse graining, it follows that for any $0<a<A$, one has for some $0<\rho=\rho(a,A)<1$, uniformly in m (and $p\in\mathcal{P}_\kappa^{\mathrm{s}}$),

$$\sup_{a\le|\theta|\le A}\left|\phi_m(\theta)\right|\le\rho.$$

Using this fact,

$$\left|I_{3,1}(n,m,x)\right|\le CA^d\rho^n = O\big(n^{-(d+2)/2}\big).$$

To deal with the last two integrals is more delicate since we have to take into account the m-dependency. First,

$$\left|I_{3,2}(n,m,x)\right|\le\int_{A<|\theta|\le m^\alpha}\left|\phi_m(\theta)\right|^n\,\mathrm{d}\theta.$$

We bound the integrand pointwise. Since $\hat{\pi}_m(\cdot)$ is invariant under the maps $e_i\mapsto -e_i$, $e_j\mapsto e_j$ for $j\ne i$, it suffices to look at θ with all components positive. Assume $\theta_1=\max\{\theta_1,\dots,\theta_d\}$. Set $M=\lfloor 2\pi\sqrt{\lambda_{m,1}}/\theta_1\rfloor$ and $K=\lfloor 5m/M\rfloor$. Notice that $\hat{\pi}_m(x)>0$ implies $|x|<2m$. By taking A large enough, we can assume that on the domain of integration, $M\le m$. First,

$$\begin{aligned} \phi_m(\theta) = \sum_{(x_2,\dots,x_d)} &\Bigg(\exp\Bigg(\frac{i}{\sqrt{\lambda_{m,1}}}\sum_{s=2}^d x_s\theta_s\Bigg) \\ &\times\sum_{j=1}^K\sum_{x_1=-2m+(j-1)M}^{-2m+jM-1}\exp\Bigg(\frac{ix_1\theta_1}{\sqrt{\lambda_{m,1}}}\Bigg)\hat{\pi}_m(x)\Bigg). \end{aligned}$$

Inside the x_1-summation, we write for each j separately

$$\hat{\pi}_m(x)=\hat{\pi}_m(x)-\hat{\pi}_m\big(x^{(j)}\big)+\hat{\pi}_m\big(x^{(j)}\big),$$

where $x^{(j)}=(-2m+(j-1)M,x_2,\dots,x_d)$. By Corollary A.1,

$$\left|\hat{\pi}_m(x)-\hat{\pi}_m\big(x^{(j)}\big)\right|\le C\left|\frac{x_1+2m-(j-1)M}{m}\right|^{1/2} m^{-d}.$$

Thus

$$\left|\sum_{x_1=-2m+(j-1)M}^{-2m+jM-1} \exp\left(\frac{ix_1\theta_1}{\sqrt{\lambda_{m,1}}}\right)\left(\hat{\pi}_m(x)-\hat{\pi}_m\left(x^{(j)}\right)\right)\right| \le C\theta_1^{-3/2}m^{-d+1}$$

and

$$\left|\sum_{j=1}^{K}\sum_{x_1=-2m+(j-1)M}^{-2m+jM-1} \exp\left(\frac{ix_1\theta_1}{\sqrt{\lambda_m}}\right)\left(\hat{\pi}_m(x)-\hat{\pi}_m\left(x^{(j)}\right)\right)\right| \le C\theta_1^{-1/2}m^{-d+1}.$$

On our domain of integration, $0<(\theta_1/\sqrt{\lambda_{m,1}})\le Cm^{\alpha-1}<2\pi$ for large m. Therefore,

$$\begin{aligned}\left|\sum_{j=1}^{K}\hat{\pi}_m\left(x^{(j)}\right)\sum_{x_1=-2m+(j-1)M}^{-2m+jM-1} \exp\left(\frac{ix_1\theta_1}{\sqrt{\lambda_{m,1}}}\right)\right| &\le CKm^{-d}\left|\frac{1-\exp(i\theta_1 M/\sqrt{\lambda_{m,1}})}{1-\exp(i\theta_1/\sqrt{\lambda_{m,1}})}\right| \\ &\le C|\theta|m^{-d},\end{aligned}$$

and altogether for sufficiently large A, m and n,

$$\int_{A<|\theta|\le m^\alpha}\left|\phi_m(\theta)\right|^n \mathrm{d}\theta \le C_1^n\int_{A<|\theta|\le m^\alpha}\left(\frac{1}{\sqrt{|\theta|}}+\frac{|\theta|}{m}\right)^n \mathrm{d}\theta = O\left(n^{-(d+2)/2}\right).$$

For $I_{3,3}$ we again assume all components of θ positive and as before $\theta_1=\max\{\theta_1,\dots,\theta_d\}$. Since

$$\hat{\pi}_m(x)=\sum_{y=-2m}^{x_1}\left(\hat{\pi}_m(y,x_2,\dots,x_d)-\hat{\pi}_m(y-1,x_2,\dots,x_d)\right),$$

we have

$$\begin{aligned}\left|\phi_m(\theta)\right| &\le Cm^{d-1}\left|\sum_{x_1=-2m}^{2m}\exp\left(\frac{ix_1\theta_1}{\sqrt{\lambda_{m,1}}}\right)\right. \\ &\qquad\left.\times\sum_{y=-2m}^{x_1}\left(\hat{\pi}_m(y,x_2,\dots,x_d)-\hat{\pi}_m(y-1,x_2,\dots,x_d)\right)\right| \\ &\le Cm^{d-1}\sum_{y=-2m}^{2m}\left|\hat{\pi}_m(y,x_2,\dots,x_d)-\hat{\pi}_m(y-1,x_2,\dots,x_d)\right| \\ &\qquad\times\left|\sum_{x_1=y}^{2m}\exp\left(\frac{ix_1\theta_1}{\sqrt{\lambda_{m,1}}}\right)\right|.\end{aligned}$$

The sum over the exponentials is estimated by $Cm/|\theta|$, so that again with Corollary A.1,

$$\left|\phi_m(\theta)\right|\le C_2 m^{1/2}|\theta|^{-1}.$$

Hence, for α close to 1 and large n, m,

$$\int_{m^\alpha<|\theta|,\theta\in B_m} |\phi_m^n(\theta)|\,\mathrm{d}\theta \le C_2^n m^{n/2+\alpha(d-n)} = O(n^{-(d+2)/2}). \qquad \square$$

For Proposition 5.2, we need additionally a large deviation estimate.

LEMMA A.5 (Large deviation estimate). *Let $p \in \mathcal{P}_\kappa^{\mathrm{s}}$. There exist constants $c_1, c_2 > 0$ such that for $|x| \ge 3m$,*

$$\hat{\pi}_m^n(x) \le c_1 m^{-d} \exp\left(-\frac{|x|^2}{c_2 n m^2}\right).$$

PROOF. Write P for $\mathrm{P}_{0,\hat{\pi}_m}$ and E for the expectation with respect to P, and denote by X_n^j the jth component of the random walk X_n under P. For $r > 0$, since p is symmetric,

$$\begin{aligned}\sum_{y:|y|\ge r} \hat{\pi}_m^n(y) &\le \sum_{j=1}^d \mathrm{P}(|X_n^j| \ge d^{-1/2} r)\\ &\le 2d \max_{j=1,\dots,d} \mathrm{P}(X_n^j \ge d^{-1/2} r).\end{aligned}$$

We claim that

$$\mathrm{P}(X_n^j \ge d^{-1/2} r) \le \exp\left(-\frac{r^2}{8dnm^2}\right).$$

By the martingale maximal inequality for all $t, \lambda > 0$,

$$\mathrm{P}(X_n^j \ge \lambda) \le \mathrm{e}^{-t\lambda} \mathrm{E}[\exp(tX_n^j)] = \mathrm{e}^{-t\lambda}(\mathrm{E}[\exp(tX_1^j)])^n.$$

Since $X_1^j \in (-2m, 2m)$ and $x \mapsto \mathrm{e}^{tx}$ is convex, it follows that

$$\exp(tX_1^j) \le \frac{1}{2}\frac{(2m - X_1^j)}{2m}\mathrm{e}^{-2tm} + \frac{1}{2}\frac{(2m + X_1^j)}{2m}\mathrm{e}^{2tm}.$$

Therefore, using again symmetry of X_1^j,

$$\mathrm{E}[\exp(tX_n^j)] \le \left(\tfrac{1}{2}\mathrm{e}^{-2tm} + \tfrac{1}{2}\mathrm{e}^{2tm}\right)^n = \cosh^n(2tm) \le \mathrm{e}^{2nt^2m^2}$$

and

$$\mathrm{P}(X_n^j \ge d^{-1/2} r) \le \mathrm{e}^{-td^{-1/2}r}\mathrm{e}^{2nt^2m^2}.$$

Putting $t = r/(4\sqrt{d}nm^2)$ we get

$$\mathrm{P}(X_n^j \ge d^{-1/2} r) \le \exp\left(-\frac{r^2}{8dnm^2}\right).$$

From this it follows that

$$\hat{\pi}_m^n(x) = \sum_{y\,:\,|y|\geq|x|-2m} \hat{\pi}_m^{n-1}(y)\hat{\pi}_m(x-y) \leq \frac{c_1}{m^d}\exp\biggl(-\frac{(|x|-2m)^2}{8d(n-1)m^2}\biggr)$$

$$\leq \frac{c_1}{m^d}\exp\biggl(-\frac{|x|^2}{c_2 n m^2}\biggr). \qquad \square$$

PROOF OF PROPOSITION 5.2. (i) This follows from Proposition 5.1.

(ii) We set

$$N = N(x,m) = \frac{|x|^2}{m^2}\biggl(\log\frac{|x|^2}{m^2}\biggr)^{-2}.$$

We split $\hat{g}_{m,\mathbb{Z}^d}(x)$ into

$$\hat{g}_{m,\mathbb{Z}^d}(x) = \sum_{n=1}^{\infty}\hat{\pi}_m^n(x) = \sum_{n=1}^{\lfloor N\rfloor}\hat{\pi}_m^n(x) + \sum_{n=\lfloor N\rfloor+1}^{\infty}\hat{\pi}_m^n(x).$$

For the first sum on the right, we use the large deviation estimate from Lemma A.5,

$$\sum_{n=1}^{\lfloor N\rfloor}\hat{\pi}_m^n(x) \leq c_1 m^{-d}\sum_{n=1}^{\lfloor N\rfloor}\exp\biggl(-\frac{|x|^2}{c_2 n m^2}\biggr) = O\bigl(|x|^{-d}\bigr).$$

In the second sum, we replace the transition probabilities by the expressions obtained in Proposition 5.1. The error terms are estimated by

$$\sum_{n=\lfloor N\rfloor+1}^{\infty} O\bigl(m^{-d}n^{-(d+2)/2}\bigr) = O\biggl(|x|^{-d}\biggl(\log\frac{|x|^2}{m^2}\biggr)^d\biggr).$$

Putting $t_n = 2n\mathcal{J}_m^{-2}(x)$, we obtain for the main part

$$\begin{aligned}
&\sum_{n=\lfloor N\rfloor+1}^{\infty}\frac{1}{(2\pi n)^{d/2}\det\Lambda_m^{1/2}}\exp\biggl(-\frac{\mathcal{J}_m^2(x)}{2n}\biggr)\\
&\quad = \frac{\mathcal{J}_m^{-d+2}(x)}{2\pi^{d/2}\det\Lambda_m^{1/2}}\sum_{n=\lfloor N\rfloor+1}^{\infty} t_n^{-d/2}\exp(-1/t_n)(t_n - t_{n-1})\\
&\quad = \frac{\mathcal{J}_m^{-d+2}(x)}{2\pi^{d/2}\det\Lambda_m^{1/2}}\int_0^{\infty} t^{-d/2}\exp(-1/t)\,\mathrm{d}t + O\bigl(|x|^{-d}\bigr).
\end{aligned}$$

This proves the statement for $|x|\geq 3m$ with

$$c(d) = \frac{1}{2\pi^{d/2}}\int_0^{\infty} t^{-d/2}\exp(-1/t)\,\mathrm{d}t. \qquad \square$$

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