The Annals of Probability
2014, Vol. 42, No. 5, 2113–2160
DOI: 10.1214/14-AOP936

SOBOLEV REGULARITY FOR A CLASS OF SECOND ORDER ELLIPTIC PDE'S IN INFINITE DIMENSION

BY GIUSEPPE DA PRATO AND ALESSANDRA LUNARDI

Scuola Normale Superiore and Università di Parma

We consider an elliptic Kolmogorov equation $\lambda u - Ku = f$ in a separable Hilbert space H. The Kolmogorov operator K is associated to an infinite dimensional convex gradient system: $dX = (AX - DU(X))\,dt + dW(t)$, where A is a self-adjoint operator in H, and U is a convex lower semicontinuous function. Under mild assumptions we prove that for $\lambda > 0$ and $f \in L^2(H, \nu)$ the weak solution u belongs to the Sobolev space $W^{2,2}(H, \nu)$, where ν is the log-concave probability measure of the system. Moreover maximal estimates on the gradient of u are proved. The maximal regularity results are used in the study of perturbed nongradient systems, for which we prove that there exists an invariant measure. The general results are applied to Kolmogorov equations associated to reaction–diffusion and Cahn–Hilliard stochastic PDEs.

1. Introduction. Let H be an infinite dimensional separable Hilbert space (norm $\|\cdot\|$, inner product $\langle\cdot,\cdot\rangle$). We are concerned with the differential equation

$$\lambda u - \tfrac{1}{2}\operatorname{Tr}[D^2 u] - \langle Ax - DU(x), Du\rangle = f, \tag{1.1}$$

where $A : D(A) \subset H \to H$ is a linear self-adjoint negative operator, and such that A^{-1} is of trace class, $U : H \to \mathbb{R} \cup \{+\infty\}$ is convex, proper, lowerly bounded, and lower semicontinuous. The data are $\lambda > 0$ and $f : H \to \mathbb{R}$, the unknown is $u : H \to \mathbb{R}$. Du and D^2u represent first and second derivatives of u, and $\operatorname{Tr}[D^2u]$ is the trace of D^2u.

Equation (1.1) is the elliptic Kolmogorov equation corresponding to the differential stochastic equation

$$dX = \big(AX - DU(X)\big)\,dt + dW(t), \tag{1.2}$$

$$X(0) = x, \tag{1.3}$$

where $W(t), t \geq 0$, is an H-valued cylindrical Wiener process. Equation (1.2) is a typical example of gradient system. Under suitable assumptions, it has a unique invariant measure $\nu(dx) = Z^{-1}e^{-2U(x)}\mu(dx)$, where $Z = \int_H e^{-2U(y)}\mu(dy)$ and μ is the Gaussian measure in H with zero mean and covariance $Q = -\frac{1}{2}A^{-1}$. This

Received August 2012; revised November 2013.
MSC2010 subject classifications. 35R15, 37L40, 35B65.
Key words and phrases. Kolmogorov operators in infinite dimensions, maximal Sobolev regularity, invariant measures.

is the reason to assume A^{-1} of trace class. Z is just a normalization constant in order to have a probability measure. Moreover system (1.2) is reversible; that is, if the law of $X(0)$ coincides with ν, the reversed process $Y(t) = X(T-t)$, $t \in [0, T]$ fulfills again (1.2); see, for example, [17]. In statistical mechanics ν is called a Gibbs measure.

The above assumptions do not guarantee well-posedness of problem (1.2)–(1.3); however, under suitable additional assumptions, a solution in a weak sense may be constructed, using the general strategy presented in [22] and applied in [12]. But in this paper we shall concentrate on the solutions of the Kolmogorov equation (1.1) only. The precise relation between the weak solution to (1.1) and the solution to (1.2)–(1.3) is established in the case of Lipschitz continuous DU, and in the example of Section 5. In such cases we prove that the expected formula

$$u = \int_0^{+\infty} e^{-\lambda t} \mathbb{E}\big(f\big(X(t, \cdot)\big)\big)\, dt$$

holds for every $f \in C_b(H)$.

Throughout the paper we assume that U belongs to a suitable Sobolev space. Then, the measure ν symmetrizes the operator

$$\mathcal{K}u := \tfrac{1}{2}\operatorname{Tr}\big[D^2 u\big] + \langle Ax - DU(x), Du\rangle,$$

since for good functions u, v (e.g., smooth cylindrical functions) we have

$$\int_H \mathcal{K}u v \, d\nu = -\frac{1}{2}\int_H \langle Du, Dv\rangle \, d\nu.$$

Accordingly, we say that $u \in W^{1,2}(H, \nu)$ is a *weak solution* of equation (1.1) if

$$\lambda \int_H u\varphi \, d\nu + \frac{1}{2}\int_H \langle Du, D\varphi\rangle \, d\nu = \int_H f\varphi \, d\mu \qquad \forall \varphi \in W^{1,2}(H, \nu). \tag{1.4}$$

For every $\lambda > 0$, the weak solutions to (1.1) when f runs in $L^2(H, \nu)$ are precisely the elements of the domain of the self-adjoint realization K of $\mathcal{K}$ associated to the quadratic form $(u, v) \mapsto \frac{1}{2}\int_H \langle Du, D\varphi\rangle \, d\nu$. See Section 3.1 for the definition of K.

Existence and uniqueness of a weak solution to (1.1) have been extensively studied, even in more general situations. We quote [1] for the Dirichlet form approach and [12] where it was proved that the restriction of $\mathcal{K}$ to exponential functions is essentially m-dissipative in $L^2(H, \nu)$. However, in all these papers only $W^{1,2}$ regularity of solutions was considered.

Our main concern is the investigation of the second derivative of the weak solution and of other maximal regularity results. In Section 3 we shall prove that the weak solution u of equation (1.1) has the following properties:

$$\text{(i) } u \in W^{2,2}(H, \nu), \qquad \text{(ii) } \int_H \big\|(-A)^{1/2} Du\big\|^2 \, d\nu < \infty,$$

and under further assumptions,

$$\text{(iii)} \int_H \langle D^2 U\, Du, Du\rangle\, d\nu < \infty.$$

Regularity of the second derivative of u and sharp estimates for Du are challenging problems for the theory of elliptic equations, even in finite dimensions. (i) Is a "natural" maximal regularity result for elliptic equations, both in finite and in infinite dimensions, while (ii) is typical of the infinite dimensional setting; see, for example, [15, 23] for the Ornstein–Uhlenbeck operator, when $U \equiv 0$. (iii) Is meaningful in the case that $D^2 U$ is unbounded; otherwise it is contained in (i). It was known only in finite dimensions [19].

Properties (i)–(iii) allow us to study some perturbations of $\mathcal{K}$ of the type $\mathcal{K}_1 = \mathcal{K} + \mathcal{B}$, where

$$\mathcal{B}u(x) = \langle B(x), Du(x)\rangle,$$

and $B : H \to H$ is possibly unbounded. This is the subject of Section 4. Taking advantage of (i)–(iii), we can solve

$$\lambda u - Ku - \langle B, Du\rangle = f, \tag{1.5}$$

under reasonable assumptions on B, when λ is sufficiently large. The perturbed operator inherits some of the properties of K. For instance, it generates an analytic semigroup that preserves positivity. In some cases we can solve (1.5) for every $\lambda > 0$, in a different L^2 setting. More precisely, adapting arguments from [14] that involve positivity preserving and compactness, we are able to prove the existence of $\rho \in L^2(H, \nu)$ such that a suitable realization of $\widetilde{K}_1$ of $\mathcal{K}_1$ is m-dissipative in $L^2(H, \zeta)$ where $\zeta(dx) = \rho(x)\nu(dx)$. Then, equation (1.5) can be solved for any $\lambda > 0$ and any $f \in L^2(H, \zeta)$, and we prove that ζ is an invariant measure for the semigroup generated by $\widetilde{K}_1$ in $L^2(H, \zeta)$.

It is worth to note that $\mathcal{K}_1$ is the Kolmogorov operator corresponding to system

$$dX = \big(AX - DU(X) + B(X)\big)\, dt + dW(t), \qquad X(0) = x, \tag{1.6}$$

which is not a gradient system in general. It may be useful in the study of nonequilibrium problems arising in statistical mechanics; see, for example, [18]. Another possible application of the regularity of the second derivative of the solution u of (1.5) could be to the pathwise uniqueness of (1.6) (see the recent paper [11]), through the Veretennikov transform. This will be the object of future investigations.

In Sections 5 and 6 we show that the general theory may be applied to Kolmogorov equations of reaction–diffusion and Cahn–Hilliard stochastic PDEs.

2. Notation and preliminaries. In this section we fix notation and collect several preliminary results needed in the sequel. Though essentially known, they are scattered in different papers, so we will give details for the reader's convenience. Readers familiar with Sobolev spaces in infinite dimensions may jump to Section 3.

Let H be a separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, endowed with a Gaussian measure $\mu := \mathcal{N}_{0,Q}$ on the Borel sets of H, where $Q \in \mathcal{L}(H)$ is a self-adjoint positive operator with finite trace. We choose once and for all an orthonormal basis $\{e_k : k \in \mathbb{N}\}$ of H such that $Qe_k = \lambda_k e_k$ for $k \in \mathbb{N}$ and set $x_k = \langle x, e_k\rangle$ for each $x \in H$. We denote by P_n the orthogonal projection on the linear span of $e_1, \dots, e_n$. For each $k \in \mathbb{N} \cup \{+\infty\}$ we denote by $\mathcal{FC}_b^k(H)$ the set of the cylindrical functions $\varphi(x) = \phi(x_1, \dots, x_n)$ for some $n \in \mathbb{N}$, with $\phi \in C_b^k(\mathbb{R}^n)$.

2.1. *Sobolev spaces with respect to μ.* For $p > 1$ we set as usual $p' := p/(p-1)$. If a function $\varphi : H \mapsto \mathbb{R}$ is Fréchet differentiable at $x \in H$, we denote by $D\varphi(x)$ its gradient at x. Moreover, we denote by $D_k\varphi(x) = \langle D\varphi(x), e_k\rangle$ its derivative in the direction of e_k, for every $k \in \mathbb{N}$.

For $0 \le \theta \le 1$ and $p > 1$ the Sobolev spaces $W_\theta^{1,p}(H,\mu)$ are the completions of $\mathcal{FC}_b^1(H)$ in the Sobolev norms

$$\|\varphi\|^p_{W_\theta^{1,p}(H,\mu)} := \int_H \left(|\varphi|^p + \|Q^\theta D\varphi\|^p\right) d\mu = \int_H |\varphi|^p + \left(\sum_{k=1}^\infty (\lambda_k^\theta D_k\varphi)^2\right)^{p/2} d\mu.$$

For $\theta = 1/2$ they coincide with the usual Sobolev spaces of the Malliavin Calculus; see, for example, [3], Chapter 5; for $\theta = 0$ and $p = 2$ they are the spaces considered in [15]. Such completions are identified with subspaces of $L^p(H,\mu)$ since the integration by parts formula

$$\int_H D_k\varphi\psi\, d\mu = -\int_H D_k\psi\varphi\, d\mu + \frac{1}{\lambda_k}\int_H x_k\varphi\psi\, d\mu, \qquad \varphi, \psi \in \mathcal{FC}_b^1(H), \tag{2.1}$$

allows us to easily show that the operators $Q^\theta D : \mathcal{FC}_b^1(H) \mapsto L^p(H,\mu;H)$ are closable in $L^p(H,\mu)$, and the domains of their closures coincide with $W_\theta^{1,p}(H,\mu)$.

Moreover, since $x \mapsto x_k \in L^s(H,\mu)$ for every $s \ge 1$, (2.1) is extended by density to all $\varphi \in W_\theta^{1,q}(H,\mu)$, $\psi \in W_\theta^{1,p}(H,\mu)$ such that $1/p + 1/q < 1$. In fact, extending [15], Lemma 9.2.7, to the case $p \ge 2$ it is possible to see that it holds for $1/p + 1/q = 1$ too.

The spaces $W_\theta^{1,p}(H,\mu;H)$ are defined in a similar way, replacing $\mathcal{FC}_b^1(H)$ by linear combinations of functions of the type φe_k, with $\varphi \in \mathcal{FC}_b^1(H)$.

2.2. *Sobolev spaces with respect to ν.* Concerning U we shall assume the following:

HYPOTHESIS 2.1. *$U : H \to \mathbb{R} \cup \{+\infty\}$ is convex, lower semicontinuous and bounded from below. Moreover $U \in W^{1,2}_{1/2}(H, \mu)$.*

We denote by ν the log-concave measure $\nu(dx) = Z^{-1}e^{-2U(x)}\mu(dx)$. Since e^{-2U} is bounded, $\nu(H) = 1$.

LEMMA 2.2. *For every $p \geq 1$, $\mathcal{FC}^\infty_b(H)$ is dense in $L^p(H, \nu)$.*

PROOF. Since H is separable, then $C_b(H)$ is dense in $L^p(H, \nu)$. Any $f \in C_b(H)$ may be approached in $L^p(H, \nu)$ by the sequence $f_n(x) := f(P_n x)$, by the dominated convergence theorem. In its turn, the cylindrical functions f_n are approached by their (finite dimensional) convolutions with smooth mollifiers that belong to $\mathcal{FC}^\infty_b(H)$. $\square$

We may apply the integration by parts formula (2.1) with ψ replaced by ψe^{-2U}, that belongs to $W^{1,2}_{1/2}(H, \mu)$ for $\psi \in \mathcal{FC}^1_b(H)$. We get, for φ, $\psi \in \mathcal{FC}^1_b(H)$ and $h \in \mathbb{N}$,

$$\int_H D_h\varphi\psi\, d\nu + \int_H D_h\psi\varphi\, d\nu = 2\int_H D_h U\varphi\psi\, d\nu + \frac{1}{\lambda_h}\int_H x_h\varphi\psi\, d\nu. \tag{2.2}$$

Once again, the Sobolev spaces associated to the measure ν are introduced in a standard way with the help of the integration by parts formula (2.2). We recall that $\mathcal{L}_2(H)$ is the space of the Hilbert–Schmidt operators that are the bounded linear operators $L : H \mapsto H$ such that $\|L\|^2_{\mathcal{L}_2(H)} := \sum_{h,k=1}^\infty \langle Le_h, e_k\rangle^2 < \infty$.

LEMMA 2.3. *For all $q \geq 2$ the operators*

$$\begin{aligned} &D : \mathcal{FC}^1_b(H) \mapsto L^q(H, \nu; H), \\ &Q^{\pm 1/2}D : \mathcal{FC}^1_b(H) \mapsto L^q(H, \nu; H), \end{aligned} \tag{2.3}$$

$$\bigl(D, D^2\bigr) : \mathcal{FC}^2_b(H) \mapsto L^q(H, \nu; H) \times L^q\bigl(H, \nu; \mathcal{L}_2(H)\bigr) \tag{2.4}$$

are closable in $L^q(H, \nu)$.

PROOF. Let $(\varphi_n) \subset \mathcal{FC}^1_b(H)$ converge to 0 in $L^q(H, \nu)$ and be such that $Q^\theta D\varphi_n \to W$ in $L^q(H, \nu; H)$, with $\theta = 0$ or $\theta = 1/2$ or $\theta = -1/2$. Then for every $h \in \mathbb{N}$ the sequence $(\langle Q^\theta D\varphi_n, e_h\rangle) = (\lambda_h^\theta D_h\varphi_n)$ converges to $\langle W, e_h\rangle$ in $L^q(H, \nu)$. By formula (2.2) for each $\psi \in \mathcal{FC}^1_b(H)$ we have

$$\int_H D_h\varphi_n\psi\, d\nu + \int_H D_h\psi\varphi_n\, d\nu = 2\int_H D_h U\varphi_n\psi\, d\nu + \frac{1}{\lambda_k}\int_H x_h\varphi_n\psi\, d\nu, \tag{2.5}$$

and letting $n \to \infty$, we get

$$\lim_{n\to\infty} \int_H D_h\varphi_n \psi \, d\nu = \lim_{n\to\infty} \int_H \lambda_h^{-\theta} \langle W, e_h\rangle \psi \, d\nu = 0.$$

Since $\mathcal{F}\mathcal{C}_b^1(H)$ is dense in $L^{q'}(H, \nu)$, then $\langle W, e_h\rangle = 0$ ν-a.e. for every $h \in \mathbb{N}$, hence $W = 0$ ν-a.e., and the first statement is proved.

The proof of the second statement is similar. If $(\varphi_n) \subset \mathcal{F}\mathcal{C}_b^2(H)$ converge to 0 in $L^q(H, \nu)$ and $D\varphi_n \to W$ in $L^q(H, \nu; H)$, $D^2\varphi_n \to \mathcal{Q}$ in $L^q(H, \nu; \mathcal{L}_2(H))$, by the first part of the proof we have $W = 0$, so that for every $k \in \mathbb{N}$, $D_k\varphi_n \to 0$ in $L^q(H, \nu)$. On the other hand, for each $h, k \in \mathbb{N}$, $\langle D^2\varphi_n e_h, e_k\rangle = D_{hk}\varphi_n$ goes to $\langle \mathcal{Q}e_h, e_k\rangle$ in $L^q(H, \nu)$. Formula (2.2) applied to $D_k\varphi_n$ instead of φ reads as

$$\int_H D_{hk}\varphi_n \psi \, d\nu + \int_H D_h\psi D_k\varphi_n \, d\nu = 2\int_H D_hU D_k\varphi_n \psi \, d\nu + \frac{1}{\lambda_k} \int_H x_k D_k\varphi_n \psi \, d\nu,$$

for all $\psi \in \mathcal{F}\mathcal{C}_b^1(H)$. Letting $n \to \infty$ we get

$$\lim_{n\to\infty} \int_H D_{hk}\varphi_n \psi \, d\nu = \lim_{n\to\infty} \int_H \langle \mathcal{Q}e_h, e_k\rangle \psi \, d\nu = 0.$$

Then, $\langle \mathcal{Q}e_h, e_k\rangle = 0$ a.e. for each h and k, so that $\mathcal{Q} = 0$, ν-a.e. □

REMARK 2.4. We remark that the restriction $q \geq 2$ comes from the integral $\int_H D_hU\varphi_n\psi \, d\nu$ in (2.5), where $D_hU \in L^2(H, \nu)$ as a consequence of Hypothesis 2.1. If $\|DU\| \in L^p(H, \mu)$ for some $p > 2$ the proof of Lemma 2.3 works for any $q \geq p'$.

DEFINITION 2.5. For $q \geq 2$ we still denote by D, $Q^{1/2}D$, $Q^{-1/2}D$, and by (D, D^2) the closures in $L^q(H, \nu)$ of the operators defined in (2.3), (2.4).

We denote by $W^{1,q}(H, \nu)$ and by $W^{1,q}_{1/2}(H, \nu)$, $W^{1,q}_{-1/2}(H, \nu)$, the domains of D, $Q^{1/2}D$, $Q^{-1/2}D$ in $L^q(H, \nu)$, respectively, and by $W^{2,q}(H, \nu)$ the domain of (D, D^2) in $L^q(H, \nu)$.

Then, $W^{1,q}(H, \nu)$, $W^{1,q}_{\pm 1/2}(H, \nu)$ and $W^{2,q}(H, \nu)$ are Banach spaces with the norms

$$\|u\|^q_{W^{1,q}(H,\nu)} = \int_H |u|^q \, d\nu + \int_H \|Du\|^q d\nu,$$

$$\|u\|^q_{W^{1,q}_{\pm 1/2}(H,\nu)} = \int_H |u|^q \, d\nu + \int_H \|Q^{\pm 1/2}Du\|^q \, d\nu,$$

$$\|u\|^q_{W^{2,q}(H,\nu)} = \|u\|^q_{W^{1,q}(H,\nu)} + \int_H \|D^2u\|^q_{\mathcal{L}_2(H)} \, d\nu.$$

Denoting by $D_k u := \lambda_k^{-\theta}\langle Q^\theta Du, e_k\rangle$, with $\theta \in \{0, 1/2, -1/2\}$, $D_{hk}u := \langle D^2 u e_h, e_k\rangle$, the above Sobolev norms may be written in a more explicit way as

$$\|u\|^q_{W^{1,q}(H,\nu)} = \int_H |u|^q\,d\nu + \int_H \Big(\sum_{k\in\mathbb{N}} (D_k u)^2\Big)^{q/2}\,d\nu,$$

$$\|u\|^q_{W^{1,q}_{\pm 1/2}(H,\nu)} = \int_H |u|^q\,d\nu + \int_H \Big(\sum_{k\in\mathbb{N}} \lambda_k^{\pm 1}(D_k u)^2\Big)^{q/2}\,d\nu,$$

$$\begin{aligned}\|u\|^q_{W^{2,q}(H,\nu)} &= \|u\|^q_{W^{1,q}(H,\nu)} + \int_H \Big(\sum_{h,k\in\mathbb{N}} (D_{hk} u)^2\Big)^{q/2}\,d\nu \\ &= \|u\|^q_{W^{1,q}(H,\nu)} + \int_H \mathrm{Tr}\big([D^2 u]^2\big)\,d\nu.\end{aligned}$$

For $q = 2$, such spaces are Hilbert spaces with the respective scalar products

$$\langle u, v\rangle_{W^{1,2}(H,\nu)} = \int_H uv\,d\nu + \int_H \sum_{k\in\mathbb{N}} D_k u D_k v\,d\nu,$$

$$\langle u, v\rangle_{W^{1,2}_{\pm 1/2}(H,\nu)} = \int_H uv\,d\nu + \int_H \sum_{k\in\mathbb{N}} \lambda_k^{\pm 1} D_k u D_k v\,d\nu,$$

$$\langle u, v\rangle_{W^{2,2}(H,\nu)} = \langle u, v\rangle_{W^{1,2}(H,\nu)} + \int_H \sum_{h,k\in\mathbb{N}} D_{hk} u D_{hk} v\,d\nu.$$

REMARK 2.6. Let us make some remarks about the above definitions.

(1) It follows immediately from the definition that for every $u \in W^{1,p}(H,\nu)$ and $\varphi \in C_b^1(\mathbb{R})$, the superposition $\varphi\circ u$ belongs to $W^{1,p}(H,\nu)$, and $D(\varphi\circ u) = (\varphi'\circ u)Du$. This fact will be used frequently in the sequel.

(2) Formula (2.2) holds for each $\varphi \in \mathcal{FC}_b^1(H)$, $\psi \in W^{1,q}(H,\nu)$ with $q \geq 2$. Indeed, it is sufficient to approach ψ by a sequence of cylindrical functions in $\mathcal{FC}_b^1(H)$, and to use (2.2) for the approximating functions, recalling that $D_h U$, $x_h \in L^2(H,\nu)$.

(3) Similarly, (2.2) holds for $\varphi \in W^{1,p}(H,\nu)$, $\psi \in W^{1,q}(H,\nu)$ such that $1/p + 1/q \leq 1/2$.

2.2.1. *Positive and negative parts of elements of* $W^{1,2}(H,\nu)$. The following technical lemma will be used later to study positivity of solutions of (1.1).

LEMMA 2.7. *Let* $u \in W^{1,2}(H,\nu)$. *Then* $|u|$ (*and consequently*, $u^+ = \sup\{u, 0\}$, $u^- = \sup\{-u, 0\}$) *belongs to* $W^{1,2}(H,\nu)$, *and* $D|u| = \operatorname{sign} u\, Du$. *Moreover* $Du = 0$ *a.e. in the set* $u^{-1}(0)$, *and* $Du^+ = Du\mathbb{1}_{\{u\geq 0\}} = Du\mathbb{1}_{\{u>0\}}$, $Du^- = -Du\mathbb{1}_{\{u\leq 0\}} = -Du\mathbb{1}_{\{u<0\}}$.

PROOF. Set $f_n(\xi) = \sqrt{\xi^2 + 1/n}$, $\xi \in \mathbb{R}$. If (u_n) is a sequence of functions in $\mathcal{FC}_b^1(H)$ that approach u in $W^{1,2}(H, \nu)$ and pointwise a.e., the functions $f_n \circ u_n$ belong to $\mathcal{FC}_b^1(H)$ and approach $|u|$ in $W^{1,2}(H, \nu)$. Indeed, they converge to $|u|$ in $L^2(H, \nu)$ by the dominated convergence theorem, and $D(f_n \circ u_n) = f_n' \circ u_n Du_n$ converge to $\operatorname{sign} u Du$ in $L^2(H, \nu; H)$. The first statement follows.

Let us prove that Du vanishes a.e. in the kernel of u. It is sufficient to prove that for every $u \in W^{1,2}(H, \nu)$ and $i \in \mathbb{N}$ we have

$$\int_{\{u=0\}} D_i u \varphi \, d\nu = 0, \qquad \varphi \in \mathcal{FC}_b^1(H). \tag{2.6}$$

Indeed, since $\mathcal{FC}_b^1(H)$ is dense in $L^2(H, \nu)$, (2.6) implies that $D_i u \mathbb{1}_{\{u=0\}}$ is orthogonal to all elements of $L^2(H, \nu)$, hence it vanishes a.e.

Let $\theta : \mathbb{R} \mapsto \mathbb{R}$ be a smooth function with support contained in $[-1, 1]$, with values in $[0, 1]$ and such that $\theta(0) = 1$. For $\varepsilon > 0$ set $\theta_\varepsilon(\xi) = \theta(\xi/\varepsilon)$. The functions $\theta_\varepsilon \circ u$ have values in $[0, 1]$ and converge pointwise to $\mathbb{1}_{\{u=0\}}$. Moreover, they belong to $W^{1,2}(H, \nu)$ and we have $D_i(\theta_\varepsilon \circ u) = (\theta_\varepsilon' \circ u) D_i u = (\theta' \circ u/\varepsilon) D_i u/\varepsilon$. Integrating we obtain

$$\begin{aligned}\int_H D_i u \varphi(\theta_\varepsilon \circ u)\, d\nu = &- \int_H u D_i \varphi(\theta_\varepsilon \circ u)\, d\nu \\ &- \int_H u\varphi D_i(\theta_\varepsilon \circ u)\, d\nu + 2\int_H u\varphi(\theta_\varepsilon \circ u) D_i U\, d\nu \\ &+ \frac{1}{\lambda_i}\int_H x_i u \varphi(\theta_\varepsilon \circ u)\, d\nu.\end{aligned}$$

As $\varepsilon \to 0$ we obtain by the dominated convergence theorem

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_H D_i u\varphi(\theta_\varepsilon \circ u)\, d\nu &= \int_{\{u=0\}} D_i u \varphi\, d\nu, \\ \lim_{\varepsilon\to 0}\int_H u D_i\varphi(\theta_\varepsilon \circ u)\, d\nu &= \int_{\{u=0\}} u D_i \varphi\, d\nu = 0, \\ \lim_{\varepsilon\to 0}\int_H u\varphi(\theta_\varepsilon \circ u) D_i U\, d\nu &= \int_{\{u=0\}} u\varphi D_i U\, d\nu = 0, \\ \lim_{\varepsilon\to 0}\frac{1}{\lambda_i}\int_H x_i u\varphi(\theta_\varepsilon \circ u)\, d\nu &= \frac{1}{\lambda_i}\int_{\{u=0\}} x_i u\varphi\, d\nu = 0.\end{aligned}$$

The integral $\int_H u\varphi D_i(\theta_\varepsilon \circ u)\, d\nu$ vanishes too as $\varepsilon \to 0$, by the dominated convergence theorem. Indeed the support of $u\varphi D_i(\theta_\varepsilon \circ u)$ is contained in $u^{-1}([-\varepsilon, \varepsilon])$ so that its modulus is bounded by $\|\theta'\|_\infty \|\varphi\|_\infty$. Moreover it converges to 0 pointwise as $\varepsilon \to 0$. So, letting $\varepsilon \to 0$ we obtain (2.6).

Once we know that Du vanishes a.e. in the kernel of u, the formulas for Du^+ and Du^- follow from the equalities $u^+ = (|u| + u)/2$, $u^- = (|u| - u)/2$. $\square$

2.2.2. *Functional inequalities and embeddings*. Under some additional assumptions important functional inequalities hold in the space $W^{1,2}(H,\nu)$.

HYPOTHESIS 2.8. *$U \in W_0^{1,2}(H,\mu)$ and $\|DU\| \in L^p(H,\mu)$ for some $p > 2$.*

We recall that since A is invertible, and $-A^{-1}$ is nonnegative and compact, then

$$-\omega := \sup\{\langle Ax, x\rangle : x \in D(A)\} < 0.$$

PROPOSITION 2.9. *Let Hypotheses 2.1 and 2.8 hold. Then the following Poincaré and Logarithmic Sobolev inequalities hold*:

$$\int_H \left(\varphi - \int_H \varphi \, d\nu\right)^2 d\nu \le \frac{1}{2\omega}\int_H \|D\varphi\|^2 \, d\nu, \qquad \varphi \in W^{1,2}(H,\nu), \tag{2.7}$$

$$\begin{aligned}\int_H \varphi^2 \log(\varphi^2)\, d\nu \le \frac{1}{\omega}\int_H \|D\varphi\|^2 \, d\nu + \int_H \varphi^2 \, d\nu \log\left(\int_H \varphi^2 \, d\nu\right),\\ \varphi \in W^{1,2}(H,\nu).\end{aligned} \tag{2.8}$$

For the proof we refer to [15], Section 12.3.1.

Another useful property is the compact embedding of $W^{1,2}(H,\nu)$ in $L^2(H,\nu)$; see [10].

PROPOSITION 2.10. *Under Hypotheses 2.1 and 2.8, $W^{1,2}(H,\nu)$ is compactly embedded in $L^2(H,\nu)$.*

PROOF. Let (f_n) be a bounded sequence in $W^{1,2}(H,\nu)$. We look for a subsequence that converges in $L^2(H,\nu)$. By the Log-Sobolev inequality (2.8) the sequence is uniformly integrable, and hence it is sufficient to find a subsequence that converges almost everywhere.

The sequence $(f_n e^{-U})$ is bounded in $W_0^{1,q}(H,\mu)$, with $q = 2p/(2+p) \in (1,2)$. Indeed, it is bounded in $L^2(H,\mu)$, and hence it is bounded in $L^q(H,\mu)$, moreover $D(f_n e^{-U}) = Df_n e^{-U} - f_n DU e^{-U}$. Once again, $\|Df_n e^{-U}\|$ is bounded in $L^2(H,\mu)$, while the second addendum $f_n DU e^{-U}$ satisfies

$$\begin{aligned}\int_H \|f_n DU e^{-U}\|^q \, d\mu &\le \left(\int_H f_n^2 e^{-2U}\, d\mu\right)^{q/2}\left(\int_H \|DU\|^{2q/(2-q)}\, d\mu\right)^{(2-q)/q}\\ &= \|f_n\|^q_{L^2(H,\nu)}\left(\int_H \|DU\|^p \, d\mu\right)^{(2-q)/q}\end{aligned}$$

so that it is bounded in $L^q(H,\mu)$.

Since the embedding $W_0^{1,q}(H,\mu) \subset L^q(H,\mu)$ is compact [5], there exists a subsequence that converges in $L^q(H,\mu)$ and a further subsequence that converges pointwise μ-a.e. and also ν-a.e., since ν is absolutely continuous with respect to μ. $\square$

2.3. *Moreau–Yosida approximations*. An important tool in our analysis are the Moreau–Yosida approximations of U defined for $\alpha > 0$ by

$$U_\alpha(x) = \inf\left\{U(y) + \frac{|x-y|^2}{2\alpha}, y \in H\right\}, \qquad x \in H. \tag{2.9}$$

We recall that $U_\alpha(x) \le U(x)$ and $U_\alpha(x)$ converges monotonically to $U(x)$ for each x as $\alpha \to 0$. Moreover, each U_α is differentiable at any point, DU_α is Lipschitz continuous and $\|DU_\alpha\|$ converges monotonically to $\|D_0U\|$, at any x such that the subdifferential of $U(x)$ is not empty. Here, $D_0U(x)$ is the element with minimal norm in the subdifferential of $U(x)$. At such points we have

$$\|DU_\alpha(x) - D_0U(x)\|^2 \le \|D_0U(x)\|^2 - \|DU_\alpha(x)\|^2; \tag{2.10}$$

see, for example, [4], Chapter 2. If in addition $U \in C^2$, then $D_0U = DU$, and we have convergence of the second order derivatives, as the next lemma shows.

LEMMA 2.11. *Let $U : H \mapsto \mathbb{R}$ be convex and C^2. Then $\lim_{\alpha\to 0} D^2U_\alpha(x) = D^2U(x)$ in $\mathcal{L}(H)$ for all $x \in H$.*

PROOF. For each $x \in H$ set $y_\alpha(x) = (I + \alpha DU)^{-1}(x)$, so that

$$y_\alpha(x) + \alpha DU\big(y_\alpha(x)\big) = x, \tag{2.11}$$

and by [4], Chapter 2,

$$DU_\alpha(x) = DU(y_\alpha). \tag{2.12}$$

Since U is convex, then $\langle DU(x) - DU(y_\alpha(x)), \alpha DU(y_\alpha(x))\rangle = \langle DU(x) - DU(y_\alpha(x)), x - y_\alpha(x)\rangle \ge 0$. Taking the scalar product with $DU(y_\alpha(x))$ yields $\|DU(y_\alpha(x))\| \le \|DU(x)\|/(1-\alpha)$, and letting $\alpha \to 0$ in (2.11) we get

$$\lim_{\alpha\to 0} y_\alpha(x) = x \qquad \forall x \in H.$$

Now it is clear that y_α is of class C^1, and differentiating (2.11) yields

$$y'_\alpha(x) + \alpha D^2U\big(y_\alpha(x)\big)y'_\alpha(x) = I. \tag{2.13}$$

Since U is convex,

$$\|y'_\alpha(x)\|_{\mathcal{L}(H)} \le 1,$$

so that, letting $\alpha \to 0$ in (2.13) and recalling that D^2U is continuous, we obtain

$$\lim_{\alpha\to 0} y'_\alpha(x) = I.$$

On the other hand, differentiating identity (2.12) gives $D^2U_\alpha(x) = D^2U(y_\alpha(x)) \cdot y'_\alpha(x)$ which yields the statement. □

3. Elliptic problems. This section is devoted to the main result of the paper. In Section 3.1 we prove existence and uniqueness of a weak solution u of equation (1.1). Section 3.2 is devoted to the particular case that DU is Lipschitz continuous. This is an intermediate step in order to prove in Section 3.3 that under Hypothesis 2.1 we have

$$u \in W^{2,2}(H, \nu) \cap W^{1,2}_{-1/2}(H, \nu).$$

In Section 3.4 we show that if in addition U is twice continuously differentiable, then

$$\int_H \langle D^2U(x)Du(x), Du(x)\rangle \nu(dx) < \infty.$$

3.1. *Weak solutions*. We consider a Kolmogorov operator defined on $\mathcal{FC}_b^2(H)$ by

$$\mathcal{K}\varphi = \tfrac{1}{2}\operatorname{Tr}[D^2\varphi] + \tfrac{1}{2}\langle x, Q^{-1}D\varphi\rangle - \langle DU(x), D\varphi\rangle. \tag{3.1}$$

Using the partial derivatives D_k and D_{kk}, $\mathcal{K}$ may be rewritten as

$$\mathcal{K}\varphi(x) = \frac{1}{2}\sum_{k=1}^{\infty} D_{kk}\varphi(x) - \frac{1}{2}\sum_{k=1}^{\infty} \lambda_k^{-1} x_k D_k\varphi(x) - \sum_{k=1}^{\infty} D_kU(x)D_k\varphi(x).$$

The measure ν enjoys the following important symmetrizing property:

PROPOSITION 3.1. *For all $\varphi \in \mathcal{FC}_b^2(H)$, $\psi \in \mathcal{FC}_b^1(H)$ we have*

$$\int_H \mathcal{K}\varphi\psi\, d\nu = -\frac{1}{2}\int_H \langle D\varphi, D\psi\rangle\, d\nu. \tag{3.2}$$

PROOF. Recalling (2.2) we get

$$\begin{aligned}\frac{1}{2}\int_H \sum_{k=1}^{\infty} D_{kk}\varphi(x)\psi(x)\, d\nu &= -\frac{1}{2}\int_H \sum_{k=1}^{\infty} D_k\varphi(x)D_k\psi(x)\, d\nu \\ &\quad + \int_H \sum_{k=1}^{\infty}\Big(D_kU(x)D_k\varphi(x) + \frac{1}{2\lambda_k}x_kD_k\varphi(x)\Big)\, d\nu,\end{aligned}$$

and the conclusion follows (note that all series are finite sums in our case). □

Let $f \in L^2(H, \nu)$, $\lambda > 0$. Taking into account formula (3.2), we say that $u \in W^{1,2}(H, \nu)$ is a weak solution of equation (1.1) if we have

$$\lambda \int_H u\varphi\, d\nu + \frac{1}{2}\int_H \langle Du, D\varphi\rangle\, d\nu = \int_H f\varphi\, d\nu \qquad \forall \varphi \in W^{1,2}(H, \nu). \tag{3.3}$$

Since $\mathcal{FC}_b^1(H)$ is dense in $W^{1,2}(H, \nu)$, it is enough that the above equality is satisfied for every $\varphi \in \mathcal{FC}_b^1(H)$.

The function $\mathcal{A}:(W^{1,2}(H,\nu))^2 \mapsto \mathbb{R}$, $\mathcal{A}(u,\varphi) = \lambda \int_H u\varphi\, d\nu + \frac{1}{2}\int_H \langle Du, D\varphi\rangle\, d\nu$ is bilinear, continuous and coercive, while the function $F : W^{1,2}(H,\nu) \mapsto \mathbb{R}$, $F(\varphi) = \int_H f\varphi\, d\nu$, is linear and continuous. By the Lax–Milgram theorem there exists a unique $u \in W^{1,2}(H,\nu)$ such that $\mathcal{A}(u,\varphi) = F(\varphi)$ for each $\varphi \in W^{1,2}(H,\nu)$; namely equation (1.1) has a unique weak solution $u \in W^{1,2}(H,\nu)$.

We denote by $K : D(K) \subset L^2(H,\nu) \mapsto L^2(H,\nu)$ the operator associated to the quadratic form $\mathcal{A}$ in $W^{1,2}(H,\nu)$. So, the domain $D(K)$ consists of all $u \in W^{1,2}(H,\nu)$ such that there exists $v \in L^2(H,\nu)$ satisfying

$$\frac{1}{2}\int_H \langle Du, D\varphi\rangle\, d\nu = -\langle v, \varphi\rangle_{L^2(H,\nu)}$$

for all $\varphi \in W^{1,2}(H,\nu)$, or equivalently for all $\varphi \in \mathcal{F}\mathcal{C}^1_b(H)$. In this case, $v = Ku$. The weak solution u to (1.1) belongs to $D(K)$, and it is just $(\lambda I - K)^{-1} f$.

REMARK 3.2. We have $\mathcal{F}\mathcal{C}^2_b(H) \subset D(K)$. In fact, for $u \in \mathcal{F}\mathcal{C}^2_b(H)$, integrating by parts we obtain

$$\frac{1}{2}\int_H \langle Du, D\varphi\rangle\, d\nu = -\int_H (\mathcal{K}u(x))\varphi(x)\nu(dx), \tag{3.4}$$

for all $\varphi \in \mathcal{F}\mathcal{C}^1_b(H)$. Here $\mathcal{K}u \in L^2(H,\nu)$ since it consists of the sum of a finite number of addenda, each of them in $L^2(H,\nu)$. Hence, $u \in D(K)$ and $Ku = \mathcal{K}u$.

To study the domain of K it is convenient to introduce a family of approximating problems, with U replaced by its Moreau–Yosida approximations U_α defined in (2.9). Since DU_α is Lipschitz continuous, in the next section we consider the case of functions U with Lipschitz gradient.

3.2. *The case of Lipschitz continuous DU.* Here we assume that $U : H \mapsto \mathbb{R}$ is a differentiable convex function bounded from below and with Lipschitz continuous gradient. Since DU is Lipschitz, it has at most linear growth, and U has at most quadratic growth. Therefore, it satisfies Hypothesis 2.1.

The aim of this section is to show that for every $f \in L^2(H,\nu)$ the weak solution to (1.1) belongs to $W^{2,2}(H,\nu) \cap W^{1,2}_{-1/2}(H,\nu)$ and the estimate

$$\begin{aligned} &\lambda \int_H |Du|^2\, d\nu + \frac{1}{2}\int_H \mathrm{Tr}[(D^2u)^2]\, d\nu + \int_H \|Q^{-1/2}Du\|^2\, d\nu \\ &\quad + \int_H \langle D^2U\, Du, Du\rangle\, d\nu \le 4\int_H f^2\, d\nu \end{aligned} \tag{3.5}$$

holds.

Note that $U \notin W^{2,2}(H,\mu)$ in general. The term $\langle D^2U\, Du, Du\rangle$ in the last integral is meant as follows: since H is separable, and μ is nondegenerate, by [21], Theorem 6, $DU : H \mapsto H$ is Gateaux differentiable ν almost everywhere. The

Gateaux second order derivatives $D_{hk}U$ are bounded by a constant independent of h, k, since DU is Lipschitz continuous so that the Lipschitz constant of each D_kU is bounded by a constant independent of k. Since $u \in W^{1,2}_{-1/2}(H,\nu)$ the double series $\sum_{h,k} D_{hk}U\, D_h u\, D_k u$ is well defined and belongs to $L^1(H,\nu)$. Indeed,

$$\left|\sum_{h,k=1}^{\infty} D_{hk}U\, D_h u\, D_k u\right| \le C\left(\sum_{k=1}^{\infty} |D_k u|\right)^2 = C\left(\sum_{k=1}^{\infty} \lambda_k^{-1/2}|D_k u|\lambda_k^{1/2}\right)^2$$
$$\le C\left\|Q^{-1/2}Du\right\|^2 \operatorname{Tr} Q.$$

Moreover, we shall show that the weak solution is also a strong solution in the Friedrichs sense.

DEFINITION 3.3. A function $u \in L^2(H,\nu)$ is called *strong solution* (in the Friedrichs sense) to (1.1) if there is a sequence (u_n) of $\mathcal{FC}^2_b(H)$ functions that converge to u in $L^2(H,\nu)$ and such that $\lambda u_n - \mathcal{K}u_n \to f$ in $L^2(H,\nu)$.

In fact, we begin with the strong solution. The procedure is the following: we show that the operator $\mathcal{K}: \mathcal{FC}^3_b(H) \mapsto L^2(H,\nu)$ is dissipative, so that it is closable. Then we show that $(\lambda - \mathcal{K})(\mathcal{FC}^3_b(H))$ is dense in $L^2(H,\nu)$ for every $\lambda > 0$. This implies that the closure $\overline{\mathcal{K}}$ of $\mathcal{K}$ generates a contraction semigroup in $L^2(H,\nu)$, and $\mathcal{FC}^3_b(H)$ is a core, that is, it is dense in $D(\overline{\mathcal{K}})$ endowed with the graph norm. In particular, for every $f \in L^2(H,\nu)$ and $\lambda > 0$, equation (1.1) has a unique solution $u \in D(\overline{\mathcal{K}})$, which is a strong solution by definition. Then we show that $D(\overline{\mathcal{K}}) \subset W^{2,2}(H,\nu)$ and that (3.5) holds. Eventually, we prove that the strong solution coincides with the weak solution.

3.2.1. $\mathcal{K}: \mathcal{FC}^3_b(H) \mapsto L^2(H,\nu)$ *is dissipative*. This is just a simple consequence of the integration formula (3.4), taking $u = \varphi \in \mathcal{FC}^3_b(H)$.

3.2.2. $(\lambda I - \mathcal{K})(\mathcal{FC}^3_b(H))$ *is dense in* $L^2(H,\nu)$. We shall approach every element $f \in \mathcal{FC}^\infty_b(H)$ by functions g of the type $g = \lambda v - \mathcal{K}v$, first with $v \in \mathcal{FC}^2_b(H)$ and then with $v \in \mathcal{FC}^3_b(H)$. This will be done using existence and regularity results for differential equations in finite dimensions. Since $\mathcal{FC}^\infty_b(H)$ is dense in $L^2(H,\nu)$, our aim will be achieved.

We recall that P_n is the orthogonal projection on the linear span of $e_1, \dots, e_n$. We identify $P_n(H)$ with $\mathbb{R}^n$, by the obvious isomorphism $\mathbb{R}^n \mapsto P_n(H)$, $\xi \mapsto \sum_{k=1}^n \xi_k e_k$. The induced Gaussian measure in $\mathbb{R}^n$ is just $\mathcal{N}_{0,Q_n}$ where $Q_n = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$.

For any function $v: H \mapsto \mathbb{R}$ we identify $v \circ P_n$ with the function $v_n: \mathbb{R}^n \mapsto \mathbb{R}$, $v_n(\xi) := v(\sum_{k=1}^n \xi_k e_k)$. In particular, we identify $U \circ P_n: H \mapsto \mathbb{R}$ with the function $U_n: \mathbb{R}^n \mapsto \mathbb{R}$, $U_n(\xi) := U(\sum_{k=1}^n \xi_k e_k)$. U_n is convex, and DU_n is Lipschitz continuous, and hence U_n belongs to $W^{2,\infty}(\mathbb{R}^n, d\xi) \subset W^{2,\infty}(\mathbb{R}^n, \mathcal{N}_{0,Q_n})$.

For $\lambda > 0$ let us consider the problem

$$\lambda v_n - \mathcal{L} v_n + \langle DU_n, Dv_n \rangle = f_n, \tag{3.6}$$

where the Ornstein–Uhlenbeck operator $\mathcal{L}$ in $\mathbb{R}^n$ is defined by

$$\mathcal{L}\varphi(\xi) = \frac{1}{2}\sum_{k=1}^{n}\big(D_{kk}\varphi(\xi) - \lambda_k^{-1}\xi_k D_k\varphi(\xi)\big), \qquad \xi \in \mathbb{R}^n.$$

Since DU_n is Lipschitz continuous, (3.6) has a unique solution $v_n \in \bigcup_{\alpha\in(0,1)} C_b^{2+\alpha}(\mathbb{R}^n)$. A reference is [20], Theorem 1. In fact [20], Theorem 1, deals with large λ's, but a standard application of the maximum principle (e.g., [20], Lemma 2.4) and of the Schauder estimates of [20], Theorem 1, show that (3.6) is uniquely solvable in $C_b^{2+\theta}(\mathbb{R}^n)$ for each $\lambda > 0$. Moreover, an estimate for the first order derivatives of v_n,

$$\big\| |Dv_n| \big\|_\infty \le \frac{1}{\lambda}\big\| |Df_n| \big\|_\infty, \tag{3.7}$$

follows from the well-known probabilistic representation formula for v_n,

$$v_n(\xi) = \int_0^\infty e^{-\lambda t}\mathbb{E}\big(f\big(X_n(t,\xi)\big)\big)\,dt, \qquad \xi \in \mathbb{R}^n, \tag{3.8}$$

$X_n(t,\xi)$ being the solution to the stochastic ode in $\mathbb{R}^n$

$$\begin{cases} dX_n(t,\xi) = -\frac{1}{2}Q_n^{-1}X_n(t,\xi)\,dt - DU_n\big(X_n(t,\xi)\big)\,dt + dW_n(t), \\ X_n(0,\xi) = \xi, \end{cases}$$

where $W_n(t) = P_n W(t)$ is a standard Brownian motion in $\mathbb{R}^n$. Indeed, (3.7) follows taking into account that

$$\begin{aligned} d\big(X_n(t,x) - X_n(t,y)\big) = -\tfrac{1}{2}\big(Q_n^{-1}\big(X_n(t,x) - X_n(t,y)\big)\,dt \\ - \big(DU_n\big(X_n(t,x)\big) - DU_n\big(X_n(t,y)\big)\big)\big)\,dt \end{aligned}$$

so that $X_n(\cdot,x) - X_n(\cdot,y)$ is almost surely differentiable, and taking the scalar product by $X_n(t,x) - X_n(t,y)$ we get $\frac{d}{dt}\|X_n(t,x) - X_n(t,y)\|^2 \le 0$, by the monotonicity of DU_n. This implies $\|X_n(t,x) - X_n(t,y)\| \le \|x - y\|$ and consequently $|v_n^\varepsilon(x) - v_n^\varepsilon(y)| \le \|f_n\|_{\mathrm{Lip}}\|x-y\|/\lambda$.

Going back to infinite dimensions, we set

$$V_n(x) := v_n(x_1,\dots,x_n), \qquad x \in H. \tag{3.9}$$

Then $V_n \in \mathcal{F}C_b^2(H)$, and

$$\lambda V_n - \mathcal{K}V_n = f \circ P_n + \big\langle DU - D(U\circ P_n), DV_n\big\rangle, \tag{3.10}$$

where $f \circ P_n = f$ for n large enough, since f is cylindrical. The right-hand side converges to f as $n \to \infty$ since estimate (3.7) implies

$$\big|\big\langle DU(x) - D(U\circ P_n)(x), DV_n^\varepsilon(x)\big\rangle\big| \le \frac{1}{\lambda}\sup_{y\in H}\big\|Df(y)\big\|\,\big\|DU(x) - D(U\circ P_n)(x)\big\|,$$

which goes to 0 pointwise, since DU is continuous, and in $L^2(H,\nu)$ by the dominated convergence theorem, since

$$\|D(U\circ P_n)(x)\| \le [DU]_{\mathrm{Lip}}\|P_n x\| + \|DU(0)\| \le [DU]_{\mathrm{Lip}}\|x\| + \|DU(0)\|,$$

for each $n\in\mathbb{N}$. Therefore, $\lambda V_n - \mathcal{K}V_n$ converges to f in $L^2(H,\nu)$, which implies that $(\lambda I - \mathcal{K})(\mathcal{F}\mathcal{C}_b^2(H))$ is dense in $L^2(H,\nu)$.

This will be used later, in the proof of Proposition 3.8; however, it is not enough for our aims. This is because the formula, (3.20), which is the starting point of all our optimal estimates, is obtained differentiating $\lambda u - \mathcal{K}u$ for a cylindrical u, and we need that u has third order derivatives. So, we shall approximate using $\mathcal{F}\mathcal{C}_b^3$ functions instead of only $\mathcal{F}\mathcal{C}_b^2$ functions.

To be able to use regularity theorems for elliptic equations in $\mathbb{R}^n$ that yield C^3 solutions, we need regular coefficients, so we approach U_n in a standard way by convolution with smooth mollifiers. Precisely, we fix once and for all a function $\theta\in C_c^\infty(\mathbb{R}^n)$ with support contained in the ball $B(0,1)$ of center 0 and radius 1, such that $\int_{\mathbb{R}^n}\theta(\xi)\,d\xi = 1$, and for $\varepsilon>0$ we set

$$U_n^\varepsilon(\xi) = \int_{\mathbb{R}^n} U_n(\xi-\varepsilon y)\theta(y)\,dy, \qquad \xi\in\mathbb{R}^n.$$

Then U_n^ε is smooth and convex, and DU_n^ε is Lipschitz continuous. Moreover,

$$\begin{aligned}
|DU_n(\xi) - D\mathcal{U}_n^\varepsilon(\xi)| &= \left|\int_{\mathbb{R}^n}\big(DU_n(\xi) - DU_n(\xi-\varepsilon y)\big)\theta(y)\,dy\right| \\
&\le \varepsilon[DU_n]_{\mathrm{Lip}}\int_{\mathbb{R}^n}|y|\theta(y)\,dy \qquad (3.11)\\
&\le \varepsilon[DU_n]_{\mathrm{Lip}} \le \varepsilon[DU]_{\mathrm{Lip}}, \qquad \xi\in\mathbb{R}^n.
\end{aligned}$$

For $\lambda>0$ and $\varepsilon>0$ let us consider the problem

$$\lambda v_n^\varepsilon - \mathcal{L}v_n^\varepsilon + \langle DU_n^\varepsilon, Dv_n^\varepsilon\rangle = f_n. \qquad (3.12)$$

As before, since DU_n^ε are Lipschitz continuous, (3.12) has a unique solution $v_n^\varepsilon\in\bigcup_{\alpha\in(0,1)}C_b^{2+\alpha}(\mathbb{R}^n)$, again by [20], Theorem 1. The functions v_n^ε are represented by

$$v_n^\varepsilon(x) = \int_0^\infty e^{-\lambda t}\mathbb{E}\big(f_n\big(X^\varepsilon(t,x)\big)\big)\,dt, \qquad (3.13)$$

where $X^\varepsilon(t,x)$ is the solution to the stochastic ode

$$\begin{cases} dX^\varepsilon(t,x) = -\frac{1}{2}Q_n^{-1}X^\varepsilon(t,x)\,dt - DU_n^\varepsilon\big(X^\varepsilon(t,x)\big)\,dt + dW_n(t),\\ X^\varepsilon(0,x) = x,\end{cases}$$

and $W_n(t)$ is a standard Brownian motion in $\mathbb{R}^n$. The representation formula (3.13) yields the sup norm estimates

$$\|v_n^\varepsilon\|_\infty \le \frac{1}{\lambda}\|f_n\|_\infty, \tag{3.14}$$

$$\||Dv_n^\varepsilon|\|_\infty \le \frac{1}{\lambda}\||Df_n|\|_\infty. \tag{3.15}$$

Equation (3.14) is immediate, while (3.15) follows arguing as in the proof of (3.7), since DU_n^ε is monotonic as well.

We want to show that $v_n^\varepsilon \in C_b^3(\mathbb{R}^n)$. Since DU_n^ε is smooth, then v_n^ε belongs to $C^\infty(\mathbb{R}^n)$ by local elliptic regularity, and we need only to prove that its third order derivatives are bounded. To this end we differentiate both sides of (3.12) with respect to x_i, getting

$$\lambda D_i v_n^\varepsilon - \mathcal{L} D_i v_n^\varepsilon + \frac{1}{\lambda_i} D_i v_n^\varepsilon + \langle DU_n^\varepsilon, D(D_i v_n^\varepsilon)\rangle = D_i f_n - \langle D(D_i U_n^\varepsilon), Dv_n^\varepsilon\rangle.$$

The right-hand side is Hölder continuous and bounded. Applying once again the Schauder theorem [20], Theorem 1, we obtain $D_i v_n^\varepsilon \in C_b^{2+\alpha}(\mathbb{R}^n)$ for each $\alpha \in (0,1)$. In particular, $v_n^\varepsilon \in C_b^3(\mathbb{R}^n)$.

Let us go back to infinite dimensions and set

$$V_n^\varepsilon(x) := v_n^\varepsilon(x_1,\dots,x_n), \qquad \mathcal{U}_n^\varepsilon(x) = U_n^\varepsilon(x_1,\dots,x_n), \qquad x \in H. \tag{3.16}$$

Then $V_n^\varepsilon \in \mathcal{FC}_b^3(H)$ and

$$\lambda V_n^\varepsilon - \mathcal{K} V_n^\varepsilon = f \circ P_n + \langle DU - D\mathcal{U}_n^\varepsilon, DV_n\rangle. \tag{3.17}$$

Concerning the right-hand side, taking into account (3.15) and (3.11), we get

$$\begin{aligned}
&|\langle DU(x) - D\mathcal{U}_n^\varepsilon(x), DV_n^\varepsilon(x)\rangle| \\
&\quad \le \frac{1}{\lambda}\sup_{y\in H}\|Df(y)\|\big(\|DU(x) - D(U\circ P_n)(x)\| \\
&\qquad\qquad + \|D(U\circ P_n)(x) - D\mathcal{U}_n^\varepsilon(x)\|\big) \\
&\quad \le \frac{1}{\lambda}\sup_{y\in H}\|Df(y)\|\big(\|DU(x) - D(U\circ P_n)(x)\| + \varepsilon[DU]_{\mathrm{Lip}(X)}\big)
\end{aligned}$$

so that

$$\begin{aligned}
&\|\langle DU - D\mathcal{U}_n^\varepsilon, DV_n^\varepsilon\rangle\|_{L^2(H,\nu)}^2 \\
&\quad \le \left(\frac{1}{\lambda}\sup_{y\in H}\|Df(y)\|\right)^2 2\left(\int_H \|DU - D(U\circ P_n)\|^2\,d\nu + \big(\varepsilon[DU]_{\mathrm{Lip}(X)}\big)^2\right),
\end{aligned}$$

where the first integral $\int_H \|DU - D(U\circ P_n)\|^2\,d\nu$ vanishes as $n \to \infty$, as we already remarked. Therefore, $\|\langle DU - D\mathcal{U}_n^\varepsilon, DV_n^\varepsilon\rangle\|_{L^2(H,\nu)}$ is as small as we wish provided we take n large and ε small, and the same holds for $\lambda V_n^\varepsilon - \mathcal{K} V_n^\varepsilon - f$.

Summarizing, we have proved the following proposition.

PROPOSITION 3.4. *The closure $\overline{\mathcal{K}}$ of the operator $\mathcal{K}: \mathcal{FC}_b^3(H) \mapsto L^2(H,\nu)$ is m-dissipative, so that it generates a strongly continuous contraction semigroup in $L^2(H,\nu)$. In particular, for every $\lambda > 0$ and $f \in L^2(H,\nu)$ problem* (1.1) *has a unique strong solution u, that is: there is a sequence $(u_n) \subset \mathcal{FC}_b^3(H)$ such that $u_n \to u$ and $\lambda u_n - \mathcal{K}u_n \to f$ in $L^2(H,\nu)$.*

3.2.3. *$W^{2,2}(H,\nu)$ regularity of the strong solution and other estimates.* To prove our estimates it is sufficient to consider functions $u \in \mathcal{FC}_b^3(H)$, which is dense in the domain of $\overline{\mathcal{K}}$. So, we fix $u \in \mathcal{FC}_b^3(H)$, $\lambda > 0$, and we set

$$\lambda u - \mathcal{K}u = f.$$

Estimates on u and on Du in terms of f are elementary. They are obtained multiplying both sides by u and taking into account (3.2).

LEMMA 3.5. *We have*

$$\int_H \left(\lambda u^2 + \frac{1}{2}\|Du\|^2\right) d\nu = \int_H uf\, d\nu,$$

and therefore

$$\int_H u^2\, d\nu \le \frac{1}{\lambda^2}\int_H f^2\, d\nu \tag{3.18}$$

and

$$\int_H \|Du\|^2\, d\nu \le \frac{2}{\lambda}\int_H f^2\, d\nu. \tag{3.19}$$

Estimates on the second order derivatives are less obvious. They are a consequence of the following proposition.

PROPOSITION 3.6. *For each $u \in \mathcal{FC}_b^3(H)$ we have*

$$\begin{aligned} &\lambda \int_H \|Du\|^2\, d\nu + \frac{1}{2}\int_H \mathrm{Tr}\big[(D^2u)^2\big]\, d\nu + \frac{1}{2}\int_H \big\|Q^{-1/2}Du\big\|^2\, d\nu \\ &\quad + \int_H \langle D^2U\, Du, Du\rangle\, d\nu = \int_H \langle Du, Df\rangle\, d\nu = 2\int_H (\lambda u - f) f\, d\nu. \end{aligned} \tag{3.20}$$

PROOF. As in Section 3.2.2, we differentiate the equality $\lambda u - \mathcal{K}u = f$ with respect to x_i, then we multiply by $D_i u$ and sum up. We obtain

$$\lambda\|Du\|^2 - \sum_{i=1}^{\infty}(\mathcal{K}D_i u)D_i u + \sum_{i=1}^{\infty}\frac{(D_i u)^2}{2\lambda_i} + \sum_{i,j=1}^{\infty} D_{ij}U\, D_i u\, D_j u = \langle Df, Du\rangle,$$

where the series are in fact finite sums. Integrating on H and taking (3.1) into account, (3.20) follows. □

As a corollary of Lemma 3.5 and Proposition 3.6 we obtain estimates on the strong solution to (1.1).

PROPOSITION 3.7. *Let* $\lambda > 0$, $f \in L^2(H,\nu)$, *and let* u *be the strong solution to* (1.1). *Then* $u \in W^{2,2}(H,\nu) \cap W^{1,2}_{-1/2}(H,\nu)$, *and*

$$\begin{aligned}\lambda \int_H \|Du\|^2\,d\nu + \frac{1}{2}\int_H \mathrm{Tr}\big[(D^2u)^2\big]\,d\nu + \frac{1}{2}\int_H \|Q^{-1/2}Du\|^2\,d\nu \\ + \int_H \langle D^2U\,Du, Du\rangle\,d\nu \le 4\int_H f^2\,d\nu.\end{aligned} \tag{3.21}$$

In addition, if $f \in \mathcal{F}\mathcal{C}_b^\infty(H)$, *then* u *is* ν-*essentially bounded, and we have*

$$\operatorname*{ess\,sup}_{x\in H}|u(x)| \le \frac{1}{\lambda}\sup_{x\in H}|f(x)|. \tag{3.22}$$

PROOF. Let $u_j \in \mathcal{F}\mathcal{C}_b^3(H)$ approach u in $D(\overline{\mathcal{K}})$. By estimate (3.19), $Du_j \to Du$ in $L^2(H,\nu;H)$. By Proposition 3.6, equality (3.20) holds, with u_j replacing u, and $f_j := \lambda u_j - \mathcal{K}u_j$ replacing f. Then

$$\begin{aligned}\lambda \int_H \|Du_j\|^2\,d\nu + \frac{1}{2}\int_H \mathrm{Tr}\big[(D^2u_j)^2\big]\,d\nu + \frac{1}{2}\int_H \|Q^{-1/2}Du_j\|^2\,d\nu \\ + \int_H \langle D^2U\,Du_j, Du_j\rangle\,d\nu \le 2\int_H (\lambda u_j - f_j)f_j\,d\nu \le 4\|f_j\|^2_{L^2(H,\nu)},\end{aligned}$$

while by (3.18) we have $\lambda\|u_j\|_{L^2(H,\nu)} \le \|f_j\|_{L^2(H,\nu)}$. Since $f_j \to f$ in $L^2(H,\nu)$ as $j \to \infty$, (u_j) is a Cauchy sequence in $W^{2,2}(H,\nu)$ and in $W^{1,2}_{-1/2}(H,\nu)$. So, u belongs to such spaces, and letting $j \to \infty$ estimate (3.21) follows.

To prove the last statement, for $f \in \mathcal{F}\mathcal{C}_b^\infty(H)$ we approach u by the functions used in the proof of Proposition 3.4. Then (3.22) follows from (3.14), taking into account that for a suitable sequence (j_k), (u_{j_k}) converges to u, ν-a.e. □

3.2.4. *Weak = strong*. For $\lambda > 0$ and $f \in L^2(H,\nu)$, let u be the strong solution to (1.1) given by Proposition 3.4. Let $u_n \in \mathcal{F}\mathcal{C}_b^3(H)$ be such that $u_n \to u$ and $f_n := \lambda u_n - \mathcal{K}u_n \to f$ in $L^2(H,\nu)$. As we remarked in the proof of Proposition 3.7, $u_n \to u$ in $W^{1,2}(H,\nu)$.

Fix $\varphi \in \mathcal{F}\mathcal{C}_b^1(H)$. Multiplying both sides of $\lambda u_n - \mathcal{K}u_n = f_n$ by φ, integrating over H and recalling (3.2), we obtain

$$\lambda\int_H u_n\varphi\,d\nu + \frac{1}{2}\int_H \langle Du_n, D\varphi\rangle\,d\nu = \int_H f_n\varphi\,d\nu.$$

Letting $n \to \infty$ yields that u is the weak solution to (1.1). So, weak and strong solutions to (1.1) do coincide.

As a consequence of coincidence of strong and weak solutions we obtain a probabilistic representation formula for the weak solution to (1.1). Let $W(t)$ be any H-valued cylindrical Wiener process defined in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A construction of such a process may be found, for example, in [13], Section 4.3. For each $x \in H$ consider the stochastic differential equation

$$dX = \big(AX - DU(X)\big)\,dt + dW(t), \qquad X(0) = x. \tag{3.23}$$

We recall that a mild solution to (3.23) is a $\mathcal{F}_t$ adapted, H-continuous process that satisfies

$$X(t) = e^{tA}x - \int_0^t e^{(t-s)A} DU\big(X(s)\big)\,ds + \int_0^t e^{(t-s)A}\,dW(s), \qquad t \ge 0,$$

where $\mathcal{F}_t$ is the natural filtration of $W(t)$. Existence and uniqueness of a mild solution to (3.23) follow, for example, from [14], Theorem 5.5.8; see also Remark 5.5.7 of [14].

PROPOSITION 3.8. *For $\lambda > 0$ and $f \in C_b(H)$, let u be the weak solution to* (1.1). *Then*

$$u = \int_0^{+\infty} e^{-\lambda t}\,\mathbb{E} f\big(X(t,\cdot)\big)\,dt. \tag{3.24}$$

PROOF. As a first step, let $f \in \mathcal{F}C_b^\infty(H)$, let V_n be the functions defined in (3.9) and set $f_n := \lambda V_n - KV_n$. In Section 3.2.2 we have shown that $\lim_{n\to\infty} f_n = f$ in $L^2(H,\nu)$. Therefore, $u = R(\lambda, K)f = \lim_{n\to\infty} R(\lambda,K) f_n = \lim_{n\to\infty} V_n$. On the other hand, we have $V_n(x) = v_n(x_1, \dots, x_n)$, where the functions v_n solve (3.6). This implies that V_n satisfies

$$V_n(x) = \int_0^{+\infty} e^{-\lambda t}\,\mathbb{E} f\big(X_n(t,x)\big)\,dt, \qquad x \in H, \tag{3.25}$$

where X_n is the mild solution to

$$dX_n = \big(AX_n - D(U \circ P_n)(X_n)\big)\,dt + dW(t), \qquad X_n(0) = P_n x, \tag{3.26}$$

and for every $t > 0$, $x \in X$ we have $\lim_{n\to\infty} X_n(t,x) = X(t,x)$, a.s. Letting $n \to \infty$ in (3.25), the left-hand side goes to u in $L^2(H,\nu)$. The right-hand side converges to $\int_0^{+\infty} e^{-\lambda t}\mathbb{E} f(X(t,x))\,dt$ pointwise and in $L^2(H,\nu)$ by the dominated convergence theorem. Indeed, for each $x \in H$ and $t > 0$ we have $\lim_{n\to\infty} f(X_n(t,x)) = f(X(t,x))$ a.s., and $|f(X_n(t,x))| \le \|f\|_\infty$. Therefore, the statement holds if $f \in \mathcal{F}C_b^\infty(H)$.

If $f \in C_b(H)$, it is possible to approach it, pointwise and in $L^2(H,\nu)$, by a sequence (f_n) of functions belonging to $\mathcal{F}C_b^\infty(H)$. For instance, one can take approximations by convolution of $f \circ P_n$. Then, $u_n := R(\lambda,K) f_n$ satisfy (3.24) with f replaced by f_n and converge to $u = R(\lambda,K)f$ in $L^2(H,\nu)$. The right-hand sides converge to $\int_0^{+\infty} e^{-\lambda t}\mathbb{E} f(X(t,\cdot))\,dt$ in $L^2(H,\nu)$, again by the dominated convergence theorem, and the statement follows. □

3.3. *The general case.* Here we apply the results of Section 3.2 to prove our main result.

THEOREM 3.9. *Under Hypothesis 2.1, for every $\lambda > 0$ and $f \in L^2(H,\nu)$, the weak solution u to (1.1) belongs to $W^{2,2}(H,\nu) \cap W^{1,2}_{-1/2}(H,\nu)$, and it satisfies*

$$\int_H u^2\,d\nu \le \frac{1}{\lambda^2}\int_H f^2\,d\nu, \tag{3.27}$$
$$\int_H \|Du\|^2\,d\nu \le \frac{2}{\lambda}\int_H f^2\,d\nu,$$

$$\frac{1}{2}\int_H \mathrm{Tr}\big[(D^2u)^2\big]\,d\nu + \int_H \|Q^{-1/2}Du\|^2\,d\nu \le 4\int_H f^2\,d\nu. \tag{3.28}$$

PROOF. Let U_α be the Moreau–Yosida approximations of U, defined in (2.9). Since DU_α is Lipschitz continuous, we may use the results of Sections 3.2.3 and 3.2.4 for problem

$$\lambda u_\alpha - \mathcal{L}u_\alpha + \langle DU_\alpha, Du_\alpha\rangle = f. \tag{3.29}$$

Let $Z_\alpha = \int_H e^{-2U_\alpha(x)}\mu(dx)$ and $\nu_\alpha := e^{-2U_\alpha}\mu/Z_\alpha$. Fix any $f \in \mathcal{F}\mathcal{C}_b^\infty(H)$, $\lambda > 0$, and let u_α be the strong solution to (3.29) in the space $L^2(H,\nu_\alpha)$. By Lemma 3.5,

$$\int_H u_\alpha^2 e^{-2U_\alpha}\,d\mu \le \frac{1}{\lambda^2}\int_H f^2 e^{-2U_\alpha}\,d\mu, \tag{3.30}$$
$$\int_H \|Du_\alpha\|^2 e^{-2U_\alpha}\,d\mu \le \frac{2}{\lambda}\int_H f^2 e^{-2U_\alpha}\,d\mu,$$

and by Proposition 3.7,

$$\begin{aligned}\frac{1}{2}\int_H \mathrm{Tr}\big[(D^2u_\alpha)^2\big]e^{-2U_\alpha}\,d\mu + \frac{1}{2}\int_H \big\|Q^{-1/2}Du_\alpha\big\|^2 e^{-2U_\alpha}\,d\mu\\ + \int_H \langle D^2U_\alpha Du_\alpha, Du_\alpha\rangle e^{-2U_\alpha}\,d\mu \le 4\int_H f^2 e^{-2U_\alpha}\,d\mu.\end{aligned} \tag{3.31}$$

The right-hand sides of (3.30) and (3.31) are bounded by a constant independent of α, since $U_\alpha \ge \inf U$ so that

$$\int_H f^2 e^{-2U_\alpha}\,d\mu \le \|f\|_\infty^2 e^{-2\inf U}. \tag{3.32}$$

Since $U_\alpha \le U$, then $e^{-2U} \le e^{-2U_\alpha}$, and it follows that $u_\alpha \in W^{2,2}(H,\nu)$ and their $W^{2,2}(H,\nu)$ norms are bounded by a constant independent of α. A sequence (u_{α_n}), with $\lim_{n\to\infty}\alpha_n = 0$, converges weakly in $W^{2,2}(H,\nu)$ and in $W^{1,2}_{-1/2}(H,\nu)$ to a limit function denoted by u. Letting $n \to \infty$ yields that u satisfies (3.27) and (3.28). Our aim is to show that u coincides with the weak solution to (1.1). For every n we have

$$\lambda \int_H u_{\alpha_n}\varphi e^{-2U_{\alpha_n}}\,d\mu + \frac{1}{2}\int_H \langle Du_{\alpha_n}, D\varphi\rangle e^{-2U_{\alpha_n}}\,d\mu = \int_H f\varphi e^{-2U_{\alpha_n}}\,d\mu, \qquad \varphi \in \mathcal{FC}_b^1(H).$$

Letting $n \to \infty$, the right-hand side converges to $\int_H f\varphi e^{-2U}\,d\mu$. Let us split the left-hand side as

$$\begin{aligned}
&\int_H \Big(\lambda u_{\alpha_n}\varphi + \frac{1}{2}\langle Du_{\alpha_n}, D\varphi\rangle\Big)e^{-2U_{\alpha_n}}\,d\mu \\
&\quad = \int_H \Big(\lambda u_{\alpha_n}\varphi + \frac{1}{2}\langle Du_{\alpha_n}, D\varphi\rangle\Big)e^{-2U}\,d\mu \\
&\qquad + \int_H \Big(\lambda u_{\alpha_n}\varphi + \frac{1}{2}\langle Du_{\alpha_n}, D\varphi\rangle\Big)(1 - e^{-2U+2U_{\alpha_n}})e^{-2U_{\alpha_n}}\,d\mu.
\end{aligned}$$

The first integral converges to $\int_H (\lambda u\varphi + \frac{1}{2}\langle Du, D\varphi\rangle)e^{-2U}\,d\mu$. We claim that the second integral too vanishes as $n \to \infty$. Indeed, by the Hölder inequality with respect to the measure $e^{-2U_{\alpha_n}}\,d\mu$, its modulus is bounded by

$$\begin{aligned}
&\bigg(\int_H \Big(\lambda u_{\alpha_n}\varphi + \frac{1}{2}\langle Du_{\alpha_n}, D\varphi\rangle\Big)^2 e^{-2U_{\alpha_n}}\,d\mu\bigg)^{1/2} \\
&\quad \times \bigg(\int_H (1 - e^{-2U+2U_{\alpha_n}})^2 e^{-2U_{\alpha_n}}\,d\mu\bigg)^{1/2} \\
&\quad \le \|\varphi\|_{C_b^1(H)}\Big(\|\lambda u_{\alpha_n}\|_{L^2(H, e^{-2U_{\alpha_n}}\mu)} + \frac{1}{2}\big\|\|Du_{\alpha_n}\|\big\|_{L^2(H, e^{-2U_{\alpha_n}}\mu)}\Big) \\
&\qquad \times \bigg(\int_H (1 - e^{-2U+2U_{\alpha_n}})^2 e^{-2U_{\alpha_n}}\,d\mu\bigg)^{1/2}.
\end{aligned}$$

Recalling (3.32), (3.30) implies now that

$$\|\lambda u_{\alpha_n}\|_{L^2(H, e^{-2U_{\alpha_n}}\mu)} + \tfrac{1}{2}\big\|\|Du_{\alpha_n}\|\big\|_{L^2(H, e^{-2U_{\alpha_n}}\mu)}$$

is bounded by a constant independent of n. Moreover $\int_H (1 - e^{-2U+2U_{\alpha_n}})^2 \times e^{-2U_{\alpha_n}}\,d\mu$ vanishes as $n \to \infty$ by the dominated convergence theorem, and the claim is proved.

Therefore, u satisfies (3.3) for every $\varphi \in \mathcal{FC}_b^1(H)$, and hence it is the weak solution to (1.1).

If $f \in L^2(H, \nu)$, there is a sequence of $\mathcal{FC}_b^\infty(H)$ functions that converge to f in $L^2(H, \nu)$. The sequence $(R(\lambda, K)f_k)$ of the weak solutions to (1.1) with f replaced by f_k converge to the weak solution $u = R(\lambda, K)f$ of (1.1), and it is a Cauchy sequence in $W^{2,2}(H, \nu)$ and in $W^{1,2}_{-1/2}(H, \nu)$ by estimate (3.28). Then $u \in W^{2,2}(H, \nu) \cap W^{1,2}_{-1/2}(H, \nu)$, and it satisfies (3.28) too. □

3.4. *Another maximal estimate*. Under further assumptions we may recover the full estimate on Du that holds in the case that DU is Lipschitz continuous. In fact, we shall show below that

$$\int_H \langle D^2U\,Du, Du\rangle d\nu \le 4\int_H f^2\,d\nu, \tag{3.33}$$

in the case where $U \in C^2(H)$, while in Section 4.2 it will be proved in a specific example with $U \notin C^2(H)$. Here and in the following, we denote by $C^2(H)$ the space of the twice Fréchet differentiable functions from H to $\mathbb{R}$, with continuous second order derivative.

We need a preliminary result.

LEMMA 3.10. *Under Hypothesis 2.1, for each $f \in C_b(H)$ there is $\alpha_n \to 0$ such that $u_{\alpha_n} \to u$ in $W^{1,2}(H,\nu)$ as $n \to \infty$.*

PROOF. We already know that there exists a sequence (u_{α_n}) weakly convergent to u in $W^{1,2}(H,\nu)$. So, it is enough to show that

$$\limsup_{n\to\infty} |u_{\alpha_n}|_{W^{1,2}(H,\nu)} \le |u|_{W^{1,2}(H,\nu)} \tag{3.34}$$

for some equivalent norm $|\cdot|_{W^{1,2}(H,\nu)}$ in $W^{1,2}(H,\nu)$.

By Lemma 3.5 we have

$$\int_H \left(\lambda|u_{\alpha_n}|^2 + \frac{1}{2}\|Du_{\alpha_n}\|^2\right)e^{-2U_{\alpha_n}}\,d\mu = \int_H f u_{\alpha_n} e^{-2U_{\alpha_n}}\,d\mu.$$

We claim that the right-hand side converges to $Z\int_H f u\,d\nu$ as $n \to \infty$. In fact we have

$$\int_H f u_{\alpha_n} e^{-2U_{\alpha_n}}\,d\mu = \int_H f u_{\alpha_n} e^{-2U}\,d\mu + \int_H f u_{\alpha_n}(1 - e^{2U_{\alpha_n}-2U})e^{-2U_{\alpha_n}}\,d\mu,$$

where the first addendum tends to $Z\int_H f u\,d\nu$, and the second one is estimated by

$$\left|\int_H f u_{\alpha_n}(1 - e^{2U_{\alpha_n}-2U})e^{-2U_{\alpha_n}}\,d\mu\right|$$
$$\le \|f\|_\infty \|u_{\alpha_n}\|_{L^2(H,e^{-2U_{\alpha_n}}\mu)} \int_H (1 - e^{2U_{\alpha_n}-2U})^2 e^{-2U_{\alpha_n}}\,d\mu,$$

which vanishes as $n \to \infty$ because $\|u_{\alpha_n}\|_{L^2(H,e^{-2U_{\alpha_n}}\mu)}$ is bounded and

$$\lim_{n\to\infty} \int_H (1 - e^{2U_{\alpha_n}-2U})^2 e^{-2U_{\alpha_n}}\,d\mu = 0$$

by the dominated convergence theorem.

Therefore we have

$$\limsup_{n\to\infty}\int_H\Big(\lambda u_{\alpha_n}^2+\frac{1}{2}\|Du_{\alpha_n}\|^2\Big)e^{-2U}\,d\mu$$
$$\le\limsup_{n\to\infty}\int_H\Big(\lambda|u_{\alpha_n}|^2+\frac{1}{2}\|Du_{\alpha_n}\|^2\Big)e^{-2U_{\alpha_n}}\,d\mu=Z\int_H fu\,d\nu.$$

Moreover

$$\int_H fu\,d\nu=\int_H\Big(\lambda u^2+\frac{1}{2}\|Du\|^2\Big)\,d\nu,$$

so that

$$\limsup_{n\to\infty}\int_H\Big(\lambda|u_{\alpha_n}|^2+\frac{1}{2}\|Du_{\alpha_n}\|^2\Big)\,d\nu\le\int_H\Big(\lambda u^2+\frac{1}{2}\|Du\|^2\Big)\,d\nu,$$

and (3.34) follows. □

Now we can prove estimate (3.33).

THEOREM 3.11. *Let U be a C^2 function satisfying Hypothesis 2.1. Then (3.33) is fulfilled for all $f\in L^2(H,\nu)$.*

PROOF. Since $C_b(H)$ is dense in $L^2(H,\nu)$ it is sufficient to prove (3.33) when $f\in C_b(H)$. In this case, let $\alpha_n\to 0$ be such that $u_{\alpha_n}\to u$ in $W^{1,2}(H,\nu)$ (Lemma 3.10). Then $Du_{\alpha_n}\to Du$ in $L^2(H,\nu;H)$ and so [possibly replacing (α_n) by a subsequence] $Du_{\alpha_n}(x)\to Du(x)$ for almost all x. Using Lemma 2.11, for these x we have

$$\lim_{n\to\infty}\langle D^2U_{\alpha_n}(x)Du_{\alpha_n}(x),Du_{\alpha_n}(x)\rangle e^{-2U_{\alpha_n}(x)}=\langle D^2U(x)Du(x),Du(x)\rangle e^{-2U(x)},$$

and by Fatou's lemma,

$$\int_H\langle D^2U(x)Du(x),D(x)\rangle\,d\nu$$
$$=\int_H\langle D^2U(x)Du(x),D(x)\rangle e^{-2U(x)}\,d\mu$$
$$\le\liminf_{n\to\infty}\int_H\langle D^2U_{\alpha_n}(x)Du_{\alpha_n}(x),Du_{\alpha_n}(x)\rangle e^{-2U_{\alpha_n}(x)}\,d\mu$$
$$\le 4\liminf_{n\to\infty}\int_H f^2e^{-2U_{\alpha_n}}\,d\mu=4\int_H f^2\,d\nu.$$
□

4. Perturbations. The regularity results and estimates of Section 3 open the way to new results for nonsymmetric Kolmogorov operators, by perturbation. Here we consider the operator K_1 in the space $L^2(H,\nu)$ defined by

$$D(K_1)=D(K),\qquad K_1v:=Kv+\langle B(x),Dv(x)\rangle\tag{4.1}$$

with a (possibly) nongradient field $B : H \mapsto H$.

We shall give two perturbation results, the first one in the general case (Section 4.1) and the second one in the case where the weak solution to (1.1) satisfies (3.33) (Section 4.2). In both cases we shall use the next proposition and a part of its proof.

PROPOSITION 4.1. *Let A be a self-adjoint dissipative operator in $L^2(H, \nu)$, and let $\mathcal{B} : D(A) \mapsto L^2(H, \nu)$ be a linear operator such that*

$$\|\mathcal{B}v\|^2_{L^2(H,\nu)} \le a\|Av\|^2_{L^2(H,\nu)} + b\|v\|^2_{L^2(H,\nu)}, \qquad v \in D(A), \tag{4.2}$$

for some $a < 1/(\sqrt{2}+1)^2$ and $b > 0$. Then the operator

$$A_1 : D(A) \mapsto L^2(H, \nu), \qquad A_1 v = Av + \mathcal{B}v$$

generates an analytic semigroup in $L^2(H, \nu)$.

PROOF. Let us denote by $\mathcal{X} = L^2(H, \nu; \mathbb{C})$ the complexification of $L^2(H, \nu)$ and by $\mathcal{A}$ the complexification of A, $\mathcal{A}(u + iv) = Au + iAv$. Then the spectrum of $\mathcal{A}$ is contained in $(-\infty, 0]$, and we have $\|\lambda R(\lambda, \mathcal{A})\|_{\mathcal{L}(\mathcal{X})} \le 1/\cos(\theta/2)$ for $\lambda \in \mathbb{C} \setminus (-\infty, 0]$, with $\theta = \arg\lambda$. Hence, for $\operatorname{Re}\lambda > 0$ we have $\|\lambda R(\lambda, \mathcal{A})\|_{\mathcal{L}(\mathcal{X})} \le \sqrt{2}$.

A standard general perturbation result for analytic semigroups in Banach spaces states that if the generator $\mathcal{A}$ of an analytic semigroup in a complex Banach space $\mathcal{X}$ satisfies $\|\lambda R(\lambda, \mathcal{A})\|_{\mathcal{L}(\mathcal{X})} \le M$ for $\operatorname{Re}\lambda > \omega$, then for any linear perturbation $\mathcal{B} : D(\mathcal{A}) \mapsto \mathcal{X}$ that satisfies

$$\|\mathcal{B}v\|_{\mathcal{X}} \le c_1\|Av\|_{\mathcal{X}} + c_2\|v\|_{\mathcal{X}}, \qquad v \in D(\mathcal{A}),$$

with $c_1 < 1/(M+1)$ and $c_2 \in \mathbb{R}$, the sum $\mathcal{A}+\mathcal{B} : D(\mathcal{A}) \mapsto \mathcal{X}$ generates an analytic semigroup in $\mathcal{X}$. We write down a proof, which will be used later.

For $\operatorname{Re}\lambda > \omega$ the resolvent equation $\lambda u - (\mathcal{A} + \mathcal{B})u = f$ is equivalent (setting $\lambda u - \mathcal{A}u = v$) to the fixed point problem $v = Tv$, with $T : \mathcal{X} \mapsto \mathcal{X}$, $Tv = \mathcal{B}R(\lambda, \mathcal{A})v + f$. We have

$$\begin{aligned}\|Tv\| &\le c_1\|\mathcal{A}R(\lambda, A)v\| + c_2\|R(\lambda, \mathcal{A})v\|\\ &\le c_1(M+1)\|v\| + \frac{c_2 M}{|\lambda|}\|v\|, \qquad v \in \mathcal{X}.\end{aligned}$$

Fix $\omega_0 > \omega$ such that $C := c_1(M+1) + c_2M/\omega_0 < 1$. Then for every λ in the halfplane $\operatorname{Re}\lambda \ge \omega_0$ T is a contraction with constant C, the equation $v = Tv$ has a unique solution $v \in \mathcal{X}$ and $\|v\| \le \|f\|/(1-C)$, and the resolvent equation $\lambda u - A_1 u = f$ has a unique solution $u = R(\lambda, A)v$ with $\|u\| \le M\|f\|/|\lambda|(1-C)$, and the statement follows.

In our case we can take $\omega = 0$ and $M = \sqrt{2}$. Assumption (4.2) implies that $\|\mathcal{B}v\|_{\mathcal{X}} \le \sqrt{a}\|Av\|_{\mathcal{X}} + \sqrt{b}\|v\|_{\mathcal{X}}$, for every $v \in D(\mathcal{A})$, so we require $a < 1/(\sqrt{2}+1)^2$. Once we know that $\mathcal{A} + \mathcal{B}$ generates an analytic semigroup $T(t)$ in $L^2(H, \nu; \mathbb{C})$, it is sufficient to remark that the restriction of $T(t)$ to $L^2(H, \nu)$ preserves $L^2(H, \nu)$, and it is an analytic semigroup in $L^2(H, \nu)$. □

4.1. *First perturbation.*

PROPOSITION 4.2. *Let U satisfy Hypothesis 2.1. Let $B : H \mapsto H$ be μ-measurable (hence, ν-measurable) and such that there exist $c_1 \in (0, 1/2(\sqrt{2}+1))$, $c_2 > 0$ such that for a.e. $x \in H$ we have*

$$\big|\langle B(x), y\rangle\big| \le c_1 \|Q^{-1/2} y\| + c_2 \|y\|, \qquad y \in Q^{1/2}(H). \tag{4.3}$$

Then the operator K_1 defined in (4.1) generates an analytic semigroup in $L^2(H, \nu)$. In particular, there exist $\lambda_0 \ge 0$, $C > 0$ such that for every $\lambda > \lambda_0$ and for every $f \in L^2(H, \nu)$ the equation $\lambda v - K_1 v = f$ has a unique solution $v \in D(K)$, and

$$\|v\|_{D(K)} \le C \|f\|_{L^2(H,\nu)}.$$

PROOF. In view of Proposition 4.1, it is sufficient to show that the operator $\mathcal{B}$ defined in $D(K)$ by

$$\mathcal{B}u(x) = \big\langle B(x), Du(x)\big\rangle, \qquad x \in H,$$

satisfies estimate

$$\|\mathcal{B}v\|^2_{L^2(H,\nu)} \le a\|Kv\|^2_{L^2(H,\nu)} + b\|v\|^2_{L^2(H,\nu)}, \qquad v \in D(K), \tag{4.4}$$

for some $a < (\sqrt{2}+1)^{-2}$. We note that for every $u \in D(K)$ we have

$$\int_H \|Du\|^2 \, d\nu \le 4\lambda \int_H u^2 \, d\nu + \frac{4}{\lambda} \int_H (Ku)^2 \, d\nu \qquad \forall \lambda > 0, \tag{4.5}$$

$$\int_H \big\|Q^{-1/2} Du\big\|^2 \, d\nu \le 4 \int_H (Ku)^2 \, d\nu. \tag{4.6}$$

Estimate (4.5) follows from (3.27), taking $f = \lambda u - Ku$. Estimate (4.6) follows from (3.28) taking again $f = \lambda u - Ku$, and letting $\lambda \to 0$. Using (4.5) and (4.6), for each $\varepsilon \in (0,1)$ and $\lambda > 0$ we get

$$\begin{aligned}
\int_H \langle B, Du\rangle^2 \, d\nu &\le \int_H \big(c_1 \|Q^{-1/2} Du\| + c_2 \|Du\|\big)^2 \, d\nu \\
&\le c_1^2 (1+\varepsilon) \int_H \big\|Q^{-1/2} Du\big\|^2 \, d\nu + c_2^2 \Big(1 + \frac{1}{\varepsilon}\Big) \int_H \|Du\|^2 \, d\nu \\
&\le 4c_1^2 (1+\varepsilon) \int_H (Ku)^2 \, d\nu \\
&\quad + c_2^2 \Big(1 + \frac{1}{\varepsilon}\Big) \Big(4\lambda \int_H u^2 \, d\nu + \frac{4}{\lambda} \int_H (Ku)^2 \, d\nu\Big).
\end{aligned}$$

Since $4c_1^2 < 1/(\sqrt{2}+1)^2$, there is $\varepsilon > 0$ such that $4c_1^2(1+\varepsilon) < 1/(\sqrt{2}+1)^2$. Fixed such ε, choose λ big enough, such that $a := 4c_1^2(1+\varepsilon) + 4c_2^2(1+1/\varepsilon)/\lambda < 1/(\sqrt{2}+1)^2$. With these choices estimate (4.4) is satisfied with $a < 1/(\sqrt{2}+1)^2$, and the statement follows from Proposition 4.1. $\square$

REMARK 4.3. The assumptions of Proposition 4.2 are satisfied if $x \mapsto Q^\alpha B(x) \in L^\infty(H, \nu; H)$ for some $\alpha < 1/2$. Indeed, in this case for $y \in Q^{1/2}(H)$ and a.e. $x \in H$, we have

$$|\langle B(x), y\rangle| = |\langle Q^\alpha B(x), Q^{-\alpha} y\rangle| \le \|Q^\alpha B(\cdot)\|_\infty (\varepsilon \|Q^{-1/2}y\| + c(\varepsilon)\|y\|), \qquad x \in H, \varepsilon > 0,$$

and choosing ε small enough, (4.3) is satisfied with $c_1 < 1/2(\sqrt{2}-1)$.

In the case that $x \mapsto Q^{1/2}B(x) \in L^\infty(H, \nu; H)$ we need some restriction in order that the assumptions of Proposition 4.2 be satisfied. For instance, they are satisfied if $B = B_1 + B_2$, with $B_1 \in L^\infty(H, \nu; H)$ and $Q^{1/2}B_2 \in L^\infty(H, \nu; H)$, $\|Q^{1/2}B_2\|_\infty \le c_1 < 1/2(\sqrt{2}+1)$.

4.2. *Second perturbation.* In the case that $U \in C^2(H)$ we have also estimate (3.33), which is useful when

$$\langle D^2U(x)y, y\rangle \ge C(x)\|y\|^2, \qquad x, y \in H, \tag{4.7}$$

and the function $C(x)$ is unbounded from above [if C is bounded from above, (3.33) does not add much information to (3.27)].

PROPOSITION 4.4. *Let $U \in C^2(H)$ satisfy Hypothesis 2.1. Assume moreover that (4.7) holds for some unbounded $C(x)$ and that for every $\lambda > 0$ and $f \in L^2(H, \nu)$ the weak solution u to (1.1) satisfies (3.33). Moreover, let $B : H \mapsto H$ be μ-measurable and such that there exist $c_1, c_2, c_3 > 0$ with $c_1^2 + c_2^2 < 1/8(\sqrt{2}+1)^2$, and for a.e. $x \in H$, we have*

$$|\langle B(x), y\rangle| \le c_1\|Q^{-1/2}y\| + c_2\sqrt{C(x)}\|y\| + c_3\|y\|, \qquad y \in Q^{1/2}(H). \tag{4.8}$$

Then the operator K_1 defined in (4.1) generates an analytic semigroup in $L^2(H, \nu)$. In particular, there exist $\lambda_0 \ge 0$, $C > 0$ such that for every $\lambda > \lambda_0$ and for every $f \in L^2(H, \nu)$ the equation $\lambda v - K_1 v = f$ has a unique solution $v \in D(K)$, and

$$\|v\|_{D(K)} \le C\|f\|_{L^2(H,\nu)}.$$

PROOF. We argue as in the proof of Proposition 4.2. Here, besides estimates (4.5) and (4.6), we also use

$$\int_H \langle D^2U\,Du, Du\rangle\, d\nu \le 4\int_H (Ku)^2\, d\nu, \qquad u \in D(K), \tag{4.9}$$

which follows from (3.33) taking $f = \lambda u - Ku$ and letting $\lambda \to 0$. By (4.8) for each $u \in D(K)$ we have

$$\int_H \langle B, Du\rangle^2\, d\nu \le \int_H \bigl(c_1\|Q^{-1/2}Du\| + c_2\sqrt{C(x)}\|Du\| + c_3\|Du\|\bigr)^2\, d\nu.$$

Using the inequalities $(a+b+c)^2 \le a^2(2+\varepsilon)+b^2(2+\varepsilon)+c^2(1+2/\varepsilon)$ for each $\varepsilon \in (0,1)$, and

$$\int_H C(x)\|Du\|^2\,d\nu \le \int_H \langle D^2U\,Du, Du\rangle\,d\nu \le 4\int_H (Ku)^2\,d\nu$$

that follows from (4.7) and (4.9), we obtain, recalling (4.5) and (4.6),

$$\begin{aligned}
&\int_H \langle B, Du\rangle^2\,d\nu \\
&\quad \le c_1^2(2+\varepsilon)\int_H \|Q^{-1/2}Du\|^2\,d\nu + c_2^2(2+\varepsilon)\int_H C(x)\|Du\|^2\,d\nu \\
&\qquad + c_3^2\Big(1+\frac{2}{\varepsilon}\Big)\int_H \|Du\|^2\,d\nu \\
&\quad \le 4\big(c_1^2+c_2^2\big)(2+\varepsilon)\int_H (Ku)^2\,d\nu \\
&\qquad + c_3^2\Big(1+\frac{2}{\varepsilon}\Big)\Big(4\lambda\int_H u^2\,d\nu + \frac{4}{\lambda}\int_H (Ku)^2\,d\nu\Big).
\end{aligned}$$

As in the proof of Proposition 4.2, we may choose ε small and then λ large, in such a way that for every $u \in D(K)$, we have $\int_H \langle B, Du\rangle^2\,d\nu \le a\int_H (Ku)^2\,d\nu + b\int_H u^2\,d\nu$ with $a < 1/(\sqrt{2}+1)^2$, and the statement follows from Proposition 4.1. □

REMARK 4.5. Assumption (4.8) is satisfied if $B = B_1 + B_2$, where $x \mapsto Q^\alpha B_1(x) \in L^\infty(H,\nu;H)$ for some $\alpha \in [1/2)$ and there are $b < 1/2(2+\sqrt{2})$, $c > 0$ such that $\|B_2(x)\| \le bC(x)+c$ for almost every $x \in H$.

Theorem 3.11 allows to use Proposition 4.4 when $U \in C^2(H)$. In some specific examples the result of Proposition 4.4 holds when U is not C^2, but belongs to a suitable Sobolev space. See Section 5.2.

We emphasize that the domain of the perturbed operator K_1 coincides with $D(K)$. Therefore, under the assumptions of Proposition 4.2 for every $u \in D(K_1)$ we have

$$u \in W^{2,2}(H,\nu), \qquad \int_H \|A^{-1/2}Du\|^2\,d\nu < \infty,$$

and if the assumptions of Proposition 4.4 hold, then for every $u \in D(K_1)$ we have also

$$\int_H \langle D^2U\,Du, Du\rangle\,d\nu < \infty.$$

An important feature of the semigroup generated by K_1 is positivity preserving. If $B \equiv 0$, that is $K_1 = K$, Lemma 2.7 implies that K satisfies the Beurling–Deny conditions that yield positivity preserving (e.g., [6], Sections 1.3, 1.4).

PROPOSITION 4.6. *Let the assumptions of Proposition 4.2 or of Proposition 4.4 hold, and let λ_0 be given by Proposition 4.2 or 4.4. Then for every $\lambda > \lambda_0$ and $f \in L^2(H,\nu)$ such that $f(x) \geq 0$ a.e., $R(\lambda, K_1)f(x) \geq 0$ a.e.*

PROOF. Let us introduce the approximations

$$B_n(x) := nR(n,A)B(x)\mathbb{1}_{\{x\in H:\,\|B(x)\|\leq n\}}, \qquad n \in \mathbb{N}, x \in H,$$

that are μ-measurable and bounded in H.

If the assumptions of Proposition 4.2 hold, then each B_n satisfies (4.2) with the same constants a, b of B. Indeed, since $\|nR(n,A)\|_{\mathcal{L}(H)} \leq 1$, then for every $x \in H$ and $y \in Q^{1/2}(H)$ we have

$$\begin{aligned}|\langle B_n(x), y\rangle| &= |\langle B(x), nR(n,A)y\rangle|\mathbb{1}_{\{x\in H:\,\|B(x)\|\leq n\}}\\ &\leq a\|Q^{-1/2}nR(n,A)y\| + b\|nR(n,A)y\|\\ &= a\|nR(n,A)Q^{-1/2}y\| + b\|nR(n,A)y\| \leq a\|Q^{-1/2}y\| + b\|y\|.\end{aligned}$$

Similarly, if the assumptions of Proposition 4.4 hold, then B_n satisfies (4.8) with the same constants c_1, c_2, c_3 as B. Moreover B_n converges to B ν-a.e., since

$$B_n(x) - B(x) = nR(n,A)B(x) - B(x) \qquad \text{if } \|B(x)\| \leq n.$$

For each $f \in L^2(H,\nu)$ we may approach $R(\lambda, K_1)f$ by the solutions $u_n \in D(K)$ of problems

$$\lambda u_n - Ku_n - \langle B_n(x), Du_n\rangle = f \tag{4.10}$$

that still exist for $\lambda > \lambda_0$ since the functions B_n satisfy the assumptions of Proposition 4.1 (or, of Proposition 4.4) with the same constants as B. By the proof of Propositions 4.2 and 4.4, u_n is obtained as $R(\lambda, K)(I - T_n)^{-1}$ where

$$T_n v = \langle B_n(\cdot), DR(\lambda, K)v\rangle, \qquad v \in L^2(H,\nu),$$

and $(I - T_n)^{-1}$ exists because T is a contraction. We may use the principle of contractions depending on a parameter since

$$\|T_n v - Tv\|^2_{L^2(H,\nu)} \leq \int_H |\langle B - B_n, DR(\lambda, K)v\rangle|^2\, d\nu$$

that vanishes as $n \to \infty$ by the dominated convergence theorem. Indeed, for ν-almost every x we have $\lim_{n\to\infty} B_n(x) = B(x)$ and

$$|\langle B_n(x), DR(\lambda,K)v(x)\rangle| \leq a\|Q^{-1/2}DR(\lambda,K)v(x)\| + b\|DR(\lambda,K)v(x)\|,$$

if the assumptions of Proposition 4.2 hold, and

$$\begin{aligned}|\langle B_n(x), DR(\lambda,K)v(x)\rangle| \leq c_1&\|Q^{-1/2}DR(\lambda,K)v(x)\|\\ &+ c_2\sqrt{C(x)}\|DR(\lambda,K)v(x)\|\\ &+ c_3\|DR(\lambda,K)v(x)\|,\end{aligned}$$

if the assumptions of Proposition 4.4 hold. In both cases, the right-hand sides belong to $L^2(H,\nu)$.

It follows that for $\lambda > \lambda_0$ we have $\lim_{n\to\infty} u_n = R(\lambda, K_1)f$, in $L^2(H,\nu)$. To finish the proof we show that if $f \geq 0$ ν-a.e., then $u_n \geq 0$ ν-a.e. This will yield the statement.

Let us multiply both sides of (4.10) by u_n^-, that belongs to $W^{1,2}(H,\nu)$ by Lemma 2.7, and integrate over H. We get

$$\lambda \int_H u_n u_n^- \, d\nu + \frac{1}{2}\int_H \langle Du_n, Du_n^-\rangle \, d\nu - \int_H \langle B_n, Du_n\rangle u_n^- \, d\nu = \int_H f u_n^- \, d\nu,$$

and recalling that $u_n u_n^- = -(u_n^-)^2$, $\langle Du_n, Du_n^-\rangle = -\|Du_n^-\|^2$ by Lemma 2.7, we obtain

$$-\lambda \int_H (u_n^-)^2 \, d\nu - \frac{1}{2}\int_H \|Du_n^-\|^2 \, d\nu - \int_H \langle B_n, Du_n\rangle u_n^- \, d\nu \geq 0.$$

Now we estimate

$$\begin{aligned}
\left|\int_H \langle B_n, Du_n\rangle u_n^- \, d\nu\right| &= \left|\int_{\{u_n\leq 0\}} \langle B_n, Du_n\rangle u_n^- \, d\nu\right| \\
&= \left|\int_H \langle B_n, Du_n^-\rangle u_n^- \, d\nu\right| \\
&\leq \|B_n\|_\infty \left(\int_H \|Du_n^-\|^2 \, d\nu\right)^{1/2} \left(\int_H (u_n^-)^2 \, d\nu\right)^{1/2} \\
&\leq \frac{1}{2}\int_H \|Du_n^-\|^2 \, d\nu + 2\|B_n\|_\infty \int_H (u_n^-)^2 \, d\nu.
\end{aligned}$$

If $\lambda > C_n := 2\|B_n\|_\infty$, we get

$$-(\lambda - C_n)\|u_n^-\|^2_{L^2(H,\nu)} \geq 0$$

which implies $u_n^- \equiv 0$, namely $u_n \geq 0$ a.e. So, the resolvent of $K_n := K + \langle B_n, D\cdot\rangle$ preserves positivity for λ large, possibly depending on n. Since K_n generates a C_0 semigroup, its resolvent preserves positivity for every λ bigger than the type of the semigroup, in particular for every $\lambda > \lambda_0$. Then, $R(\lambda, K_1)$ preserves positivity for $\lambda > \lambda_0$. $\square$

Now we discuss the existence of an invariant measure $\zeta(dx) = \rho(x)\nu(dx)$ for the semigroup generated by K_1 in $L^2(H,\nu)$. An important step is the following proposition.

PROPOSITION 4.7. *Let the assumptions of Proposition 4.2 or of Proposition 4.4 hold. Let in addition Hypothesis 2.8 hold. Then the kernel of K_1^* [the adjoint of K_1 in $L^2(H,\nu)$] contains a nonnegative function $\rho \not\equiv 0$.*

PROOF. The function $\mathbb{1}$ identically equal to 1 belongs to the domain of K_1, and $K_1\mathbb{1} = 0$. Then for any $\lambda > \lambda_0$, $\mathbb{1}$ is an eigenvector of $R(\lambda, K_1)$ with eigenvalue $1/\lambda$. Since $D(K_1) = D(K)$ is compactly embedded in $L^2(H, \nu)$ by Proposition 2.10, then $R(\lambda, K_1)$ is a compact operator, and $1/\lambda$ is an eigenvalue of $R(\lambda, K_1)^* = R(\lambda, K_1^*)$ too. Hence, 0 is an eigenvalue of K_1^*, so that the kernel of K_1^* contains nonzero elements. Note that since $R(\lambda, K_1)$ preserves positivity for large λ, then $R(\lambda, K_1^*)$ too preserves positivity for large λ, hence the semigroup $e^{tK_1^*}$ generated by K_1^* preserves positivity for every $t > 0$.

Let us check that the kernel of K_1^* is a lattice, that is, if $\varphi \in \operatorname{Ker} K_1^*$, then $|\varphi| \in \operatorname{Ker} K_1^*$. Assume that $\varphi \in \operatorname{Ker} K_1^*$. Then $\varphi = e^{tK_1^*}\varphi$ for every $t > 0$, and since $e^{tK_1^*}$ preserves positivity, then

$$\left|\varphi(x)\right| = \left|e^{tK_1^*}\varphi(x)\right| \le \left(e^{tK_1^*}|\varphi|\right)(x), \qquad \nu\text{-a.e. } x \in H.$$

We claim that for every $t > 0$,

$$\left|\varphi(x)\right| = e^{tK_1^*}\left(|\varphi|\right)(x), \qquad \nu\text{-a.e. } x \in H. \tag{4.11}$$

Assume by contradiction that there are $t > 0$ and a Borel subset $I \subset H$ such that $\nu(I) > 0$ and $|\varphi(x)| < e^{tK_1^*}(|\varphi|)(x)$ for $x \in I$. Then we have

$$\int_H \left|\varphi(x)\right| \nu(dx) < \int_H \left(e^{tK_1^*}|\varphi|\right)(x)\nu(dx).$$

On the other hand, since $\mathbb{1} \in \operatorname{Ker} K_1$, then $e^{tK_1^*}\mathbb{1} = \mathbb{1}$. Hence

$$\int_H e^{tK_1^*}|\varphi|\,d\nu = \langle e^{tK_1^*}|\varphi|, \mathbb{1}\rangle_{L^2(H,\nu)} = \langle |\varphi|, e^{tK_1}\mathbb{1}\rangle_{L^2(H,\nu)} = \int_H |\varphi|\,d\nu,$$

which is a contradiction. Then (4.11) holds and it yields $|\varphi| \in \operatorname{Ker} K_1^*$. □

A realization of $\mathcal{K}_1$ in $L^2(H, \rho\nu)$ is m-dissipative, as the next proposition shows.

PROPOSITION 4.8. *Under the assumptions of Proposition 4.7, let ρ be a nonnegative function belonging to* $\operatorname{Ker} K_1^* \setminus \{0\}$. *Then the operator*

$$\mathcal{D} := \left\{u \in D(K_1) \cap L^2(H, \rho\nu) : K_1 u \in L^2(H, \rho\nu)\right\} \mapsto L^2(H, \rho\nu), \qquad u \mapsto K_1 u$$

is dissipative in $L^2(H, \rho\nu)$ and the range of $\lambda I - K_1 : \mathcal{D} \mapsto L^2(H, \rho\nu)$ is dense in $L^2(H, \rho\nu)$ for $\lambda > 0$. Then its closure $\widetilde{K}_1$ generates a contraction semigroup $\widetilde{T}_1(t)$ in $L^2(H, \rho\nu)$, and the measure $\rho\nu$ is invariant for $\widetilde{T}_1(t)$.

PROOF. As a first step we prove dissipativity, through estimates on $R(\lambda, K_1)$.

We remark that Lemma 2.2 holds for the measure $\rho\nu$ as well, with the same proof. In particular, $C_b(H)$ is dense in $L^1(H, \rho\nu)$.

Let $\lambda > \lambda_0$ and let $f \in C_b(H)$. Set $u = R(\lambda, K_1) f$. We recall that, since $\rho \in D(K_1^*)$ and $K_1^*\rho = 0$, then for every $u \in D(K_1)$ we have $\int_H K_1 u \rho\, d\nu =$

$\int_H uK_1^*\rho\, d\nu = 0$. So, multiplying both sides of $\lambda u - K_1 u = f$ by ρ and integrating we obtain

$$\int_H \lambda u\rho\, d\nu = \int_H f\rho\, d\nu.$$

If f has nonnegative values ν-a.e., by Proposition 4.6 u has nonnegative values ν-a.e., and the above equality implies

$$\|u\|_{L^1(H,\rho\nu)} \le \frac{1}{\lambda}\|f\|_{L^1(H,\rho\nu)}. \tag{4.12}$$

In general, we split f as $f = f^+ - f^-$. Since $u = R(\lambda, K_1)f^+ - R(\lambda, K_1)f^- = u^+ - u^-$, (4.12) follows for every $f \in C_b(H)$. Since $C_b(H)$ is dense in $L^1(H, \rho\nu)$, the resolvent $R(\lambda, K_1)$ may be extended to a bounded operator [still denoted by $R(\lambda, K_1)$] to $L^1(H, \rho\nu)$, and

$$\|R(\lambda, K_1)f\|_{L^1(H,\rho\nu)} \le \frac{1}{\lambda}\|f\|_{L^1(H,\rho\nu)}, \qquad f \in L^1(H, \rho\nu). \tag{4.13}$$

Let now $f \in L^\infty(H, \rho\nu)$. f is in fact an equivalence class of functions, that contains a Borel bounded element. Indeed, for each element $\varphi \in f$, setting $\tilde f(x) = \varphi(x)$ if $|\varphi(x)| \le \|f\|_{L^\infty(H,\rho\nu)}$, $\tilde f(x) = 0$ if $|\varphi(x)| > \|f\|_{L^\infty(H,\rho\nu)}$, the function $\tilde f$ is Borel and bounded, and $\|f\|_{L^\infty(H,\rho\nu)} = \sup_{x\in H}|\tilde f(x)|$.

Let us go back to the resolvent equation, $\lambda u - K_1 u = \tilde f$. Since $\tilde f$ is Borel and bounded, it can be seen as an element of $L^\infty(H, \nu)$, identifying it with its equivalence class.[1] Moreover, $\|\tilde f\|_{L^\infty(H,\nu)} = \sup_{x\in H}|\tilde f(x)| = \|\tilde f\|_{L^\infty(H,\rho\nu)}$.

Since $\sup|\tilde f| - \tilde f(x) \ge 0$ for every x, still by Proposition 4.6 we have $R(\lambda, K_1)(\sup|\tilde f| - \tilde f) = \sup|\tilde f|/\lambda - u \ge 0$, ν-a.e. Similarly, since $\tilde f(x) + \sup|\tilde f| \ge 0$ for every x, then $u + \sup|\tilde f|/\lambda \ge 0$, ν-a.e. So, we get an L^∞ estimate, $\|u\|_{L^\infty(H,\nu)} \le \sup|\tilde f|/\lambda$. Hence

$$\begin{aligned}\|R(\lambda, K_1)f\|_{L^\infty(H,\rho\nu)} \le \|R(\lambda, K_1)\tilde f\|_{L^\infty(H,\nu)} \le \frac{1}{\lambda}\|f\|_{L^\infty(H,\rho\nu)},\\ f \in L^\infty(H, \rho\nu).\end{aligned} \tag{4.14}$$

By interpolation, $R(\lambda, K_1)$ may be extended to $L^2(H, \rho\nu)$ [and, in fact, to all spaces $L^p(H, \rho\nu)$], in such a way that the norm of the extension does not exceed $1/\lambda$. In particular,

$$\begin{aligned}\|R(\lambda, K_1)f\|_{L^2(H,\rho\nu)} \le \frac{1}{\lambda}\|f\|_{L^2(H,\rho\nu)},\\ f \in L^2(H, \rho\nu) \cap L^2(H, \nu).\end{aligned} \tag{4.15}$$

[1] Note that ρ may vanish on some set with positive measure, so that f does not belong necessarily to $L^\infty(H, \nu)$, and even it does, its $L^\infty(H, \nu)$ norm may be bigger than its $L^\infty(H, \rho\nu)$ norm.

Let now $u \in \mathcal{D}$. For $\lambda > \lambda_0$ estimate (4.15) gives

$$\lambda \|u\|_{L^2(H,\rho\nu)} \le \|\lambda u - K_1 u\|_{L^2(H,\rho\nu)}$$

and squaring the norms of both sides, we obtain

$$\langle u, K_1 u\rangle_{L^2(H,\rho\nu)} \le \frac{1}{2\lambda} \|K_1 u\|^2_{L^2(H,\rho\nu)}.$$

Letting $\lambda \to \infty$ yields $\langle u, K_1 u\rangle_{L^2(H,\rho\nu)} \le 0$, namely the restriction of K_1 to $\mathcal{D}$ is dissipative in $L^2(H,\rho\nu)$.

We remark that $\mathcal{D}$ is dense in $L^2(H,\rho\nu)$ since it contains $\mathcal{FC}_b^\infty(H)$ which is dense by the extension of Lemma 2.2 to $L^2(H,\rho\nu)$. Moreover $(\lambda I - K_1)(\mathcal{D})$ is dense for $\lambda > \omega_0$, since it contains $\mathcal{FC}_b^\infty(H)$. Indeed, if $f \in \mathcal{FC}_b^\infty(H)$, then $u = R(\lambda, K_1)f$ belongs to $\mathcal{D}$ and $\lambda u - K_1 u = f$.

Let us denote by $\widetilde{K}_1 : D(\widetilde{K}_1) \mapsto L^2(H,\rho\nu)$ the closure of $K_1 : \mathcal{D} \mapsto L^2(H,\rho\nu)$. By the Lumer–Phillips theorem, $\widetilde{K}_1$ generates a strongly continuous contraction semigroup in $L^2(H,\rho\nu)$, and $\mathcal{D}$ is a core for $\widetilde{K}_1$. So, for every $\varphi \in D(\widetilde{K}_1)$ there is a sequence of functions $\varphi_n \in \mathcal{D}$ such that $\varphi_n \to \varphi$ and $K_1\varphi_n \to \widetilde{K}_1\varphi$ in $L^2(H,\rho\nu)$. For every n we have

$$\int_H K_1\varphi_n \rho \, d\nu = \int_H \varphi_n K_1^* \rho \, d\nu = 0$$

and letting $n \to \infty$ we obtain $\int_H \widetilde{K}_1 \varphi\rho \, d\nu = 0$. This proves the last statement. $\square$

5. Kolmogorov equations of stochastic reaction–diffusion equations. Let $H = L^2((0,1), d\xi)$, and let A be the realization of the second order derivative with Dirichlet boundary condition, that is, $D(A) = W^{2,2}((0,\pi), d\xi) \cap W_0^{1,2}((0,\pi), d\xi)$, $Ax = x''$.

We consider the Gaussian measure μ in H with mean 0 and covariance $Q := -\frac{1}{2}A^{-1}$. A canonical orthonormal basis of H consists of the functions $e_k(\xi) := \sqrt{2}\sin(k\pi\xi)$, $k \in \mathbb{N}$, that are eigenfunctions of Q with eigenvalues $\lambda_k := 1/(2k^2\pi^2)$.

Let $\Phi : \mathbb{R} \mapsto \mathbb{R}$ be any convex lowerly bounded function, with (at most) polynomial growth at infinity, say

$$|\Phi(t)| \le C(1 + |t|^{p_1}), \qquad t \in \mathbb{R}, \tag{5.1}$$

for some $C > 0$, $p_1 \ge 2$. We set

$$U(x) = \begin{cases} \displaystyle\int_0^1 \Phi(x(\xi))\, d\xi, & x \in L^{p_1}(0,1), \\ +\infty, & x \notin L^{p_1}(0,1). \end{cases} \tag{5.2}$$

Section 5.1 is devoted to check that U satisfies Hypotheses 2.1 and 2.8, so that we can apply Theorem 3.9 to obtain regularity results for the solution u to (1.1). Then in Section 5.2 we show that under an additional assumption u fulfills (3.33) too.

5.1. *Checking Hypotheses* 2.1 *and* 2.8. We first note that U is finite μ-a.e., thanks to the next lemma. Its statement should be well known; however, we write down a simple proof for the reader's convenience.

LEMMA 5.1. *For every $p \ge 2$ we have*

$$\int_H \int_0^1 |x(\xi)|^p \, d\xi \, d\mu < \infty, \tag{5.3}$$

and hence $\mu(L^p(0,1)) = 1$. Moreover, $x \mapsto \|x\|_{L^p(0,1)} \in L^q(H,\mu)$ for every $q \ge 1$.

PROOF. Let P_n be the orthogonal projection on the subspace spanned by $e_1, \dots, e_n$. For every $\xi \in (0,1)$ and $m < n \in \mathbb{N}$, the function $x \mapsto P_n x(\xi) - P_m x(\xi)$ is a Gaussian random variable $N_{0, \sum_{k=m+1}^n \lambda_k e_k(\xi)^2}$. Then, for $p \ge 1$,

$$\begin{aligned}
\int_H |P_n x(\xi) - P_m x(\xi)|^p \, d\mu &= \int_{\mathbb{R}} |\eta|^p N_{0, \sum_{k=m+1}^n \lambda_k e_k(\xi)^2}(d\eta) \\
&= c_p \left(\sum_{k=m+1}^n \lambda_k e_k(\xi)^2 \right)^{p/2} \\
&\le \tilde{c}_p \left(\sum_{k=m+1}^n \lambda_k \right)^{p/2},
\end{aligned}$$

with $\tilde{c}_p = 2^{p/2} c_p$, so that

$$\begin{aligned}
\int_H \int_0^1 |P_n x(\xi) - P_m x(\xi)|^p \, d\xi \, d\mu &= \int_0^1 \int_H |P_n x(\xi) - P_m x(\xi)|^p \, d\mu \, d\xi \\
&\le \tilde{c}_p \left(\sum_{k=m+1}^n \lambda_k \right)^{p/2}.
\end{aligned}$$

This implies that the sequence $(x,\xi) \mapsto P_n x(\xi)$ converges in $L^p(H \times (0,1), \mu \times d\xi)$ to a limit function u that belongs to $L^p(H \times (0,1), \mu \times d\xi)$ for every p. Let us show that $u(x,\xi) = x(\xi)$ taking $p = 2$: indeed, $\int_0^1 |P_n x(\xi) - x(\xi)|^2 \, d\xi$ vanishes for every $x \in H$ as $n \to \infty$, and it is bounded by $\|x\|^2$ which belongs to $L^1(H,\mu)$, so that by the dominated convergence theorem, $\int_H \int_0^1 |P_n x(\xi) - x(\xi)|^2 \, d\xi \, d\mu$ vanishes as $n \to \infty$. Then $u(x,\xi) = x(\xi)$, and (5.3) follows. It implies that $\mu(L^p(H,\mu)) = 1$ for every $p \ge 2$ and that $x \mapsto \|x\|_{L^p(0,1)} \in L^p(H,\mu)$. For $q > p$ and $x \in L^q(0,1)$, the Hölder inequality yields $\|x\|_{L^p(0,1)} \le \|x\|_{L^q(0,1)}$ so that $x \mapsto \|x\|_{L^p(0,1)} \in L^q(H,\mu)$. $\square$

The function U defined by (5.2) is convex and bounded from below because Φ is. Using the Fatou lemma, it is easily seen to be lowerly semicontinuous. By assumption (5.1) and Lemma 5.1, $U \in L^p(H,\mu)$ for every $p \ge 1$, and the measures

μ and $\nu = e^{-2U}\mu/\int_H e^{-2U}\,d\mu$ are equivalent. For U belong to some Sobolev space it is sufficient that also Φ' has at most polynomial growth, as the next proposition shows.

PROPOSITION 5.2. *Let $\Phi:\mathbb{R}\mapsto\mathbb{R}$ be any C^1 convex lowerly bounded function such that*

$$|\Phi'(t)| \le C(1+|t|^{p_2}), \qquad t\in\mathbb{R}, \tag{5.4}$$

for some $C>0$, $p_2\ge 1$. Then the function U defined in (5.2) belongs to $W_0^{1,p}(H,\mu)$ for every $p\ge 1$, and $DU(x)=\Phi'\circ x$ for a.e. $x\in H$ [namely, for each $x\in L^{2p_2}(0,1)$].

PROOF. By (5.4), Φ satisfies (5.1) with $p_1=p_2+1$, so that $U\in L^p(H,\mu)$ for every p by Lemma 5.1. To prove that $U\in W_0^{1,p}(H,\mu)$ we shall approach U by its Moreau–Yosida approximations U_α defined in (2.9). Each U_α is continuously differentiable and DU_α is Lipschitz continuous, hence $U_\alpha\in W_0^{1,p}(H,\mu)$ for every p. This can be easily proved arguing as in the case $p=2$ of [8], Proposition 10.11.

Since $U_\alpha(x)$ converges monotonically to $U(x)$ at each x such that $U(x)<\infty$, by Lemma 5.1 U_α converges to U, μ-a.e. Since

$$\begin{aligned}\inf U \le U_\alpha(x) &\le U(x)\\ &\le C\left(1+\int_0^1 |x(\xi)|^{p_1}\,d\xi\right)\\ &\le C\left(1+\left(\int_0^1 |x(\xi)|^{p_1 p}\,d\xi\right)^{1/p}\right),\end{aligned}$$

by Lemma 5.1 and the dominated convergence theorem, $U_\alpha\to U$ in $L^p(H,\mu)$.

Let $x\in L^{2p_2}(0,1)$. Then the subdifferential $\partial U(x)$ is not empty. Indeed, since Φ is convex, for each $y\in H$ we have

$$\begin{aligned}U(y)-U(x) &= \int_0^\pi [\Phi(x(\xi))-\Phi(y(\xi))]\,d\xi\\ &\ge \int_0^\pi \Phi'(x(\xi))(x(\xi)-y(\xi))\,d\xi,\end{aligned}\tag{5.5}$$

which implies that the function $\Phi'\circ x\in H$ belongs to $\partial U(x)$. In fact, $\Phi'\circ x\in H$ is the unique element of $\partial U(x)$; see, for example, [2], Proposition 2.5. By Lemma 5.1, $x\mapsto \|\Phi'\circ x\|\in L^p(H,\mu)$, and again by the dominated convergence theorem $\int_H \|DU_\alpha(x)-\Phi'\circ x\|^p\,d\mu\to 0$ as $\alpha\to 0$, which shows that $U\in W_0^{1,p}(H,\mu)$ and $DU(x)=\Phi'\circ x$, μ-a.e. $\square$

If the assumptions of Proposition 5.2 hold, then U satisfies Hypotheses 2.1 and 2.8, and consequently the results of Theorem 3.9 and of Propositions 4.7 and 4.8 hold.

5.2. *Further estimates of Du.* We are going to show that for every $\lambda > 0$ and $f \in L^2(H, \nu)$, the solution of (1.1) satisfies estimate (3.33) as well, under reasonable additional assumptions on Φ. We use the following preliminary result.

PROPOSITION 5.3. *Let $g \in C^2(\mathbb{R})$ be such that*

$$|g''(t)| \le C(1 + |t|^m), \qquad t \in \mathbb{R} \tag{5.6}$$

for some $C > 0$, $m \ge 1$. Then the function $F(x) := g \circ x$ belongs to $W^{1,q}_{1/2}(H, \mu; H)$ for all $q > 1$. If in addition $g_\alpha : \mathbb{R} \mapsto \mathbb{R}$ are C^2 functions fulfilling (5.6) with constant C independent of $\alpha > 0$ and g_α, g'_α pointwise converge to g, g', respectively, as $\alpha \to 0^+$, then $F_\alpha(x) := g_\alpha \circ x$ converges to F in $W^{1,q}_{1/2}(H, \mu; H)$ as $\alpha \to 0^+$ for all $q > 1$.

PROOF. As first step we show that for each $x \in L^{2m}(0, 1)$ (hence, μ-a.e.), F is differentiable in any direction $h \in Q^{1/2}(H) = H^1_0(0, 1)$ and that $\frac{\partial F(x)}{\partial h} = g' \circ x \cdot h$. We have in fact for all $h \in H^1_0(0, 1)$, $\xi \in (0, 1)$ and all $0 < |t| \le 1$,

$$\begin{aligned}
&\left| \frac{g(x + th)(\xi) - g(x(\xi))}{t} - g'(x(\xi))h(\xi) \right| \\
&\quad = \left| \int_0^1 [g'(x(\xi) + t\sigma h(\xi)) - g'(x(\xi))] h(\xi)\, d\sigma \right| \\
&\quad = \left| \int_0^1 \int_0^1 g''(x(\xi) + t\sigma\eta h(\xi)) t\sigma h(\xi)^2\, d\eta\, d\sigma \right| \\
&\quad \le t\|h\|^2_\infty C(1 + 2^{m-1}|(|x(\xi)|^m + \|h\|^m_\infty)).
\end{aligned}$$

Now, taking the square and integrating over $(0, 1)$, yields

$$\left\| \frac{F(x + th) - F(x)}{t} - g' \circ x \cdot h \right\|_H \le tC(h)(1 + \|x\|^m_{L^{2m}}).$$

This implies that for each $x \in L^{2m}(0, 1)$, F is differentiable at x in any direction $h \in H^1_0(0, 1)$ and that

$$\frac{\partial F(x)}{\partial h} = g' \circ x \cdot h.$$

Let us notice that F, $\partial F/\partial h$ belong to $L^q(H, \mu; H)$ for every $q \ge 1$. Indeed, (5.6) implies that $|g(t)| \le M(1 + |t|^{m+2})$, $|g'(t)| \le M(1 + |t|^{m+1})$ for every $t \in \mathbb{R}$ and for some $M > 0$, so that $|F(x(\xi))| \le M(1 + |x(\xi)|^{m+2})$, $|\partial F(x)/\partial h(\xi)| \le M(1 + |x(\xi)|^{m+1})\|h\|_\infty$ and then

$$\|F(x)\|^2_H \le \int_0^1 M^2(1 + |x(\xi)|^{m+2})^2\, d\xi,$$

$$\left\| \frac{\partial F(x)}{\partial h}(x) \right\|^2_H \le \|h\|^2_\infty \int_0^1 M^2(1 + |x(\xi)|^{m+1})^2\, d\xi,$$

and the right-hand sides belong to $L^q(H,\mu)$ for every q. It follows from [3], Section 5.2, that F belongs to $G^{q,1}(H,\mu;H)$ [i.e., F belongs to $L^q(H,\mu;H)$, it is weakly differentiable in all directions of the Cameron–Martin space $H_0^1(0,1)$ and any weak derivative $\frac{\partial F(x)}{\partial h}$ with $h\in H_0^1(0,1)$ can be expressed as $\Psi(x)h$, where $\Psi\in L^q(H,\mu;\mathcal{L}(H_0^1(0,1),H))$ is such that $\partial F(x)/\partial h=\Psi(x)(h)$]. To show that $F\in W^{1,q}_{1/2}(H,\mu;H)$ we have still to check that ([3], Proposition 5.4.6, Corollary 5.4.7)

$$\int_H\biggl(\sum_{h,k\in\mathbb{N}}\lambda_h\lambda_k\langle\partial F(x)/\partial e_h,e_k\rangle^2\biggr)^{q/2}d\mu<\infty.$$

This is because a canonical orthonormal basis of $H_0^1(0,1)$ is just the set $\{\sqrt{\lambda_k}e_k:k\in\mathbb{N}\}$. Recalling that $\|e_k\|_\infty=\sqrt{2}$ for every k, we get

$$\begin{aligned}\bigl|\langle\partial F(x)/\partial e_h,e_k\rangle\bigr|&=\biggl|\int_0^1 g'\bigl(x(\xi)\bigr)e_h(\xi)e_k(\xi)\,d\xi\biggr|\\&\le 2M\int_0^1\bigl(1+\bigl|x(\xi)\bigr|^{m+1}\bigr)\,d\xi\\&=2M\bigl(1+\|x\|^{m+1}_{L^{m+1}}\bigr)\end{aligned}$$

for each $h,k\in\mathbb{N}$, which implies

$$\int_H\biggl(\sum_{h,k\in\mathbb{N}}\lambda_h\lambda_k\langle\partial F(x)/\partial e_h,e_k\rangle^2\biggr)^{q/2}d\mu\le 2M\int_H(\operatorname{Tr}Q)^q\|x\|^{q(m+1)}_{L^{m+1}}\,d\mu<\infty,$$

so that $F\in W^{1,q}_{1/2}(H,\mu;H)$.

Now we can show that $F_\alpha\to F$ as $\alpha\to 0$. In fact, since (5.6) is fulfilled with constant independent of α, there is $M_1>0$ independent of α such that

$$\bigl|g_\alpha(t)\bigr|\le M_1\bigl(1+|t|^{m+2}\bigr),\qquad \bigl|g_\alpha'(t)\bigr|\le M_1\bigl(1+|t|^{m+1}\bigr),\qquad t\in\mathbb{R}.$$

Concerning the convergence of $g_\alpha\circ x$ to $g\circ x$ in $L^q(H,\mu;H)$ we have

$$\begin{aligned}\int_H\|g_\alpha\circ x-g\circ x\|_H^q\,d\mu&=\int_H\biggl(\int_0^1\bigl|g_\alpha\bigl(x(\xi)\bigr)-g\bigl(x(\xi)\bigr)\bigr|^2\,d\xi\biggr)^{q/2}d\mu\\&\le\int_H\int_0^1\bigl|g_\alpha\bigl(x(\xi)\bigr)-g\bigl(x(\xi)\bigr)\bigr|^q\,d\xi\,d\mu,\end{aligned}$$

and the last integral goes to 0 as $\alpha\to 0$ by the dominated convergence theorem. Therefore $F_\alpha(x)=g_\alpha\circ x$ converges to F in $L^q(H,\mu;H)$. Concerning the convergence in $W^{1,q}_{1/2}(H,\mu;H)$ we have

$$\begin{aligned}
&\int_H\Bigl(\sum_{h,k\in\mathbb{N}}\lambda_h\lambda_k\bigl\langle\partial(g_\alpha\circ x)/\partial e_h-\partial(g\circ x)/\partial e_h,e_k\bigr\rangle^2\Bigr)^{q/2}\,d\mu\\
&\quad=\int_H\Bigl(\sum_{h,k\in\mathbb{N}}\lambda_h\lambda_k\Bigl(\int_0^1\bigl(g'_\alpha(x(\xi))-g'(x(\xi))\bigr)e_h(\xi)e_k(\xi)\,d\xi\Bigr)^2\Bigr)^{q/2}\,d\mu\\
&\quad\le C_q\int_H\Bigl(\sum_{h,k\in\mathbb{N}}\lambda_h\lambda_k\int_0^1\bigl|g'_\alpha(x(\xi))-g'(x(\xi))\bigr|^2\,d\xi\Bigr)^{q/2}\,d\mu\\
&\quad\le C_q(\operatorname{Tr}Q)^q\int_H\int_0^1\bigl|g'_\alpha(x(\xi))-g'(x(\xi))\bigr|^q\,d\xi\,d\mu,
\end{aligned}$$

and the last integral vanishes as $\alpha\to 0$ again by the dominated convergence theorem. □

We shall use Proposition 5.3 to prove that the Moreau–Yosida approximations U_α converge to U in $W^{2,q}_{1/2}(H,\mu)$ for every q [for the moment, we only know convergence in $W^{1,q}(H,\mu)$].

PROPOSITION 5.4. *Let $\Phi:\mathbb{R}\mapsto\mathbb{R}$ be any C^3 convex lowerly bounded function such that*

$$|\Phi'''(t)|\le C(1+|t|^m),\qquad t\in\mathbb{R},\tag{5.7}$$

for some $C,m>0$. Then $U\in W^{2,q}_{1/2}(H,\mu)$ for all $q>1$, and we have

$$\lim_{\alpha\to0}U_\alpha=U\qquad\text{in } W^{2,q}_{1/2}(H,\mu)\ \forall q>1.$$

PROOF. Let us apply Proposition 5.3 to $F(x)=DU(x)=g\circ x$ with $g=\Phi'$. Since g'' has polynomial growth, $F\in W^{2,q}_{1/2}(H,\mu;H)$ for all q, so that $U\in W^{2,q}_{1/2}(H,\mu)$ for all q. Moreover $DU_\alpha(x)=D_0U(y_\alpha)$, where y_α is the solution of

$$y_\alpha+\alpha D_0U(y_\alpha)=x,$$

that is

$$y_\alpha+\alpha\Phi'(y_\alpha)=x.$$

Therefore

$$y_\alpha(\xi)=\bigl(I+\alpha\Phi'\bigr)^{-1}\bigl(x(\xi)\bigr),\qquad 0<\xi<1,$$

and so

$$DU_\alpha(x)=\Phi'\circ\bigl(I+\alpha\Phi'\bigr)^{-1}\circ x.$$

Setting $g_\alpha(t) = \Phi' \circ (I + \alpha\Phi')^{-1}(t)$, we see that g_α converges pointwise to $g = \Phi'$, and

$$g'_\alpha = \frac{\Phi'' \circ (I + \alpha\Phi')^{-1}}{(1 + \alpha\Phi'' \circ (I + \alpha\Phi')^{-1})}$$

converges pointwise to $g' = \Phi''$.

Moreover we notice that there exists $M > 0$, independent of $\alpha \in (0, 1)$ such that $|(I + \alpha\Phi')^{-1}(t)| \le M + |t|$ for all $t \in \mathbb{R}$. (5.7) implies that Φ' and Φ'' have polynomial growth as well; in particular $|\Phi'(t)| \le c_1(1 + |t|^{m+2})$, so that $|g_\alpha(t)| \le c_1(1 + (M + |t|)^{m+2})$. A similar estimate with $m + 1$ instead of $m + 2$ holds also for $|g'_\alpha(t)|$. By the second part of Proposition 5.3, DU_α converges to DU in $W^{1,q}_{1/2}(H, \mu; H)$ as $\alpha \to 0$, thereby U_α converges to U in $W^{2,q}_{1/2}(H, \mu)$. $\square$

As a final step, we can show that the solution to (1.1) satisfies (3.33) under the assumptions of Proposition 5.4.

PROPOSITION 5.5. *Let U be defined by* (5.2) *with $\Phi : \mathbb{R} \mapsto \mathbb{R}$ convex, bounded from below, of class C^3 and satisfying* (5.7). *Then for every $\lambda > 0$ and $f \in L^2(H, \nu)$ the weak solution u of* (1.1) *satisfies* (3.33).

PROOF. It is sufficient to prove the statement for $f \in C_b(H)$, which is dense in $L^2(H, \nu)$. By Lemma 3.10 there is a sequence $(\alpha_n) \to 0$ such that $u_{\alpha_n} \to u$ in $W^{1,2}(H, \nu)$. Then $Du_{\alpha_n} \to Du$ in $L^2(H, \nu; H)$ so that (up to a subsequence) $Du_{\alpha_n}(x) \to Du(x)$ for almost all x. By Proposition 5.4, U_{α_n} converges to U in $W^{2,2}_{1/2}(H, \mu)$, thereby for all fixed $h, k \in \mathbb{N}$ we have $D_{hk}U_{\alpha_n} \to D_{hk}U$ in $L^2(H, \mu)$. Let us fix $N \in \mathbb{N}$. Possibly choosing a further subsequence, we have $D_{hk}U_{\alpha_n} \to D_{hk}U$ pointwise a.e. for all $h, k \le N$. Therefore for μ-a.e. $x \in H$ we have

$$\begin{aligned}\lim_{n\to\infty} \sum_{h,k=1}^{N} D_{hk}U_{\alpha_n}(x) D_h u_{\alpha_n}(x) D_k u_{\alpha_n}(x) e^{-2U_{\alpha_n}(x)} \\ = \sum_{h,k=1}^{N} D_{hk}U(x) D_h u(x) D_k(x) e^{-2U(x)}\end{aligned}$$

and by Fatou's lemma,

$$\begin{aligned}&\int_H \sum_{h,k=1}^{N} D_{hk}U(x) D_h u(x) D_k u(x)\, d\nu \\ &\quad = \int_H \sum_{h,k=1}^{N} D_{hk}U(x) D_h u(x) D_k u(x) e^{-2U(x)}\, d\mu\end{aligned}$$

$$\leq \liminf_{n\to\infty} \int_H \sum_{h,k=1}^N D_{hk} U_{\alpha_n}(x) D_h u_{\alpha_n}(x) D_k u_{\alpha_n}(x) e^{-2U_{\alpha_n}(x)}\, d\mu$$

$$\leq 4 \liminf_{n\to\infty} \int_H f^2 e^{-2U_{\alpha_n}}\, d\mu = 4 \int_H f^2\, d\nu.$$

Now by Theorem 3.9 we know that $x \mapsto \|Du(x)\|_{H_0^1(0,1)} = \|Q^{-1/2} Du(x)\|_H/\sqrt{2} \in L^2(H,\mu)$, therefore for almost any $x \in H$, $Du(x) \in H_0^1(0,1)$, whereas by Proposition 5.4 it follows that $x \mapsto \sum_{h,k=1}^\infty \lambda_h \lambda_k (D_{hk} U(x))^2$ belongs to $L^1(H,\mu)$, that is $x \mapsto \|D^2 U(x)\|_{\mathcal{L}_2(H_0^1(0,1))} \in L^2(H,\mu)$. Therefore for almost $x \in H$, $D^2 U(x) \in \mathcal{L}_2(H_0^1(0,1))$. It follows that for almost any $x \in H$ the sequence $\sum_{h,k=1}^N D_{hk} U(x) D_k u(x) D_k u(x)$ converges to $\sum_{h,k=1}^\infty D_{hk} U(x) D_k u(x) D_k u(x)$. Using once again Fatou's lemma we can conclude that

$$\int_H \sum_{h,k=1}^\infty D_{hk} U(x) D_h u(x) D_k u(x)\, d\nu$$

$$= \int_H \lim_{N\to\infty} \sum_{h,k=1}^N D_{hk} U(x) D_h u(x) D_k u(x)\, d\nu$$

$$\leq \liminf_{N\to\infty} \int_H \sum_{h,k=1}^N D_{hk} U(x) D_h u(x) D_k u(x)\, d\nu \leq 4 \int_H f^2\, d\nu. \qquad \square$$

Then we can apply all the results of Sections 3 and 4. In particular, we have the following theorem.

THEOREM 5.6. *Let $\Phi : \mathbb{R} \mapsto \mathbb{R}$ be any convex C^1 lowerly bounded function satisfying* (5.4), *and let U be defined by* (5.2). *Then for every $\lambda > 0$ and $f \in L^2(H,\nu)$ the weak solution u to* (1.1) *belongs to $W^{2,2}(H,\nu) \cap W^{1,2}_{-1/2}(H,\nu)$, and it satisfies* (3.27), (3.28). *If in addition Φ is C^3 and satisfies* (5.7), *then u satisfies* (3.33) *as well.*

With our choice of U, the stochastic differential equation (1.2) in H reads as

$$dX = \big(AX - \Phi'(X)\big)\, dt + dW(t), \qquad X(0) = x, \tag{5.8}$$

and hence it is a reaction–diffusion SPDE, whose Kolmogorov operator is just $\mathcal{K}$. As in Section 3.2.4, $W(t)$ is any H-valued cylindrical Wiener process defined in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The connection between (5.8) and (1.1) is stated in the next proposition. The definition of mild solution to (5.8) is the same as in the case of Lipschitz continuous DU.

PROPOSITION 5.7. *Let $\Phi:\mathbb{R}\mapsto\mathbb{R}$ be a convex lowerly bounded function satisfying* (5.4) *for some $p_2\geq 1$. Then for every $x\in L^{2p_2}(0,1)$ (hence, for μ-a.e. $x\in H$) problem* (5.8) *has a unique mild solution X. For every $f\in C_b(H)$ we have*

$$u(x)=\int_0^\infty e^{-\lambda t}\mathbb{E}\big(f\big(X(t,x)\big)\big)\,dt, \tag{5.9}$$

μ-a.e. $x\in H$, where u is the weak solution to (1.1).

PROOF. Existence of a unique mild solution to (5.8) follows from [14], Theorem 5.5.8, that deals with Cauchy problems such as $dX=(AX+F(X))\,dt+dW(t)$, $X(0)=x$. In our case, $F(x)=-DU(x)=-\Phi'(x)$ satisfies the assumptions of [14], Theorem 5.5.8, with $K=L^{2p_2}(0,1)$. In particular, Hypothesis 5.5 is satisfied, since in [7], Proposition 4.3, it is proved that $(t,\xi)\mapsto\int_0^t e^{(t-s)A}\,dW(s)(\xi)$ is a.s. continuous.

The mild solution is obtained as the limit of mild solutions to approximating problems,

$$dX_\alpha=\big(AX_\alpha-DU_\alpha(X)\big)\,dt+dW(t),\qquad X(0)=x,$$

as $\alpha\to 0$, where DU_α are the Yosida approximations of DU, and for each $T>0$ we have $\lim_{\alpha\to 0}\sup_{0\leq t\leq T}\|X_\alpha(t)-X(t)\|=0$, $\mathbb{P}$-a.e. By Proposition 3.8, for every $\lambda>0$,

$$R(\lambda,K_\alpha)f=\int_0^\infty e^{-\lambda t}\mathbb{E}\big(f\big(X_\alpha(t,\cdot)\big)\big)\,dt. \tag{5.10}$$

We recall that $R(\lambda,K_\alpha)f=u_\alpha$ is the weak solution to (3.29), and that a sequence u_{α_n} with $\alpha_n\to 0$ converges to u in $L^2(H,\mu)$ as $n\to\infty$, by Lemma 3.10. Moreover, $\int_0^\infty e^{-\lambda t}\mathbb{E}(f(X_{\alpha_n}(t,\cdot)))\,dt$ goes to $\int_0^\infty e^{-\lambda t}\mathbb{E}(f(X(t,x))\,dt$ pointwise μ-a.e. and also in $L^2(H,\mu)$, by the dominated convergence theorem. Taking $\alpha=\alpha_n$ in (5.10) and letting $n\to\infty$ formula (5.9) follows. □

Concerning perturbed equations,

$$dX=\big(AX-\Phi'(X)+B(X)\big)\,dt+dW(t), \tag{5.11}$$

we do not know about existence of invariant measures except in the case of bounded perturbations of Ornstein–Uhlenbeck equations. See [14], Chapter 8. If B is a bounded Borel function, Proposition 4.8 yields that the corresponding Kolmogorov semigroup e^{tK_1} has an invariant measure ν. The verification of formula (5.9) where now $X(t,x)$ is the mild solution to (5.11) and $u=R(\lambda,K_1)$ is not obvious. In fact, even existence of a mild solution is not obvious. It could be done through the Girsanov transform, but the argument is quite delicate and we hope to be able to treat the subject in a future paper.

6. Kolmogorov equations of stochastic Cahn–Hilliard-type problems. In Section 5 we have seen that the superposition $x\mapsto\Phi'\circ x$ may be seen as the gradient of a suitable function U in the space $L^2(0,1)$. This is no longer true for operators of the type $x \mapsto \frac{d}{d\xi}(\Phi' \circ x)$ or $x \mapsto \frac{d^2}{d\xi^2}(\Phi' \circ x)$. However they may be still interpreted as gradients, with suitable choices of the space H.

Here we set $V := \{x \in H^1(0,1) : \int_0^1 x(\xi)\, d\xi = 0\}$, with scalar product $\langle x, y\rangle_V = \int_0^1 x'(\xi)y'(\xi)\, d\xi$, and we choose H to be the dual space of V, endowed with the dual norm. We consider the spaces $\widetilde{L}^p(0,1) := \{x \in L^p(0,1) : \int_0^1 x(\xi)\, d\xi = 0\}$ as subspaces of H, identifying any $x \in L^p(0,1)$ with zero mean value with the element $y \mapsto \int_0^1 x(\xi)y(\xi)\, d\xi$ of H.

The standard extension B of the negative second order derivative on V with values in H is defined by

$$Bx(y) = \int_0^1 x'(\xi)y'(\xi)\, d\xi, \qquad y \in V.$$

If $x \in V \cap H^2(0,1)$ and $x'(0) = x'(1) = 0$, then $Bx(y) = -\int_0^1 x''(\xi)y(\xi)\, d\xi$ so that, with the above identification, B is an extension of (minus) the second order derivative with Neumann boundary condition. The operator B is an isometry between V and H, since $\|Bx\|_H = \sup_{y\neq 0}\langle x, y\rangle_V / \|y\|_V = \|x\|_V$. Moreover, if $z \in \widetilde{L}^2(0,1)$ and $x \in V$, then $\langle z, Bx\rangle_H = \langle z, x\rangle_{L^2(0,1)}$.

Let $e_k(\xi) := \sqrt{2}\cos(k\pi\xi)$. Then $\{e_k : k \in \mathbb{N}\}$ is an orthonormal basis of $\widetilde{L}^2(0,1)$, $Be_k = k^2\pi^2 e_k$, and setting $f_k = k\pi e_k$, the set $\{f_k : k \in \mathbb{N}\}$ is an orthonormal basis of H. We recall that P_n is the orthogonal projection on the subspace spanned by the first n elements of the basis,

$$P_n x = \sum_{k=1}^{n} \langle x, f_k\rangle_H f_k.$$

REMARK 6.1. Note that the restriction of P_n to $\widetilde{L}^2(0,1)$ is the orthogonal projection in $\widetilde{L}^2(0,1)$ on the subspace spanned by $e_1, \dots, e_n$. Indeed, for every $x \in \widetilde{L}^2(0,1)$ and $k \in \mathbb{N}$ we have

$$\langle x, f_k\rangle_H f_k = \langle x, B^{-1} f_k\rangle_{L^2} f_k = \left\langle x, \frac{e_k}{k\pi}\right\rangle_{L^2} k\pi e_k = \langle x, e_k\rangle_{L^2} e_k.$$

Here we set $A = -B^2$ and, as usual, we denote by μ the Gaussian measure on H with zero mean and covariance $Q = -A^{-1}/2$. Note that the eigenvalues of Q are now $\lambda_k := 1/2\pi^4 k^4$, and $B = \sqrt{2}Q^{1/2}$.

We consider a function $\Phi : \mathbb{R} \mapsto \mathbb{R}$ satisfying the following assumptions.

HYPOTHESIS 6.2. *$\Phi : \mathbb{R} \mapsto \mathbb{R}$ is a C^1 convex lowerly bounded function, satisfying* (5.4) *and*

$$\lim_{r\to\pm\infty} \frac{\Phi(r)}{|r|} = +\infty. \tag{6.1}$$

Setting $p_1 = p_2 + 1$, we define U as in Section 5.1, by

$$U(x) = \begin{cases} \displaystyle\int_0^1 \Phi\big(x(\xi)\big)\,d\xi, & x \in \widetilde{L}^{p_1}(0,1), \\ +\infty, & x \notin \widetilde{L}^{p_1}(0,1). \end{cases} \tag{6.2}$$

U is obviously convex and bounded from below, moreover by [2], Proposition 2.8, it is lower semicontinuous. To be more precise, in [2] the space H is the dual space of $H_0^1(0,1)$, but the argument goes as well in our case. The subdifferential of U is not empty at each $x \in \widetilde{L}^1(0,1)$ such that $\Phi' \circ x \in V$ and it consists of the unique element $D_0U(x) = B(\Phi' \circ x)$.

We shall see that $U \in W^{1,2}_{1/2}(H,\mu)$, while $U \notin W^{1,2}_0(H,\mu)$. For the proof, instead of approaching U by its Moreau–Yosida approximations, we shall approach it by the sequence $U \circ P_n$; namely we set

$$U_n(x) = \int_0^1 \Phi\big(P_n x(\xi)\big)\,d\xi, \qquad x \in H.$$

By (5.4), Φ satisfies (5.1), and we have $U(x) \le C(1 + \|x\|^{p_1}_{L^{p_1}(0,1)})$, $U_n(x) \le C(1 + \|P_n x\|^{p_1}_{L^{p_1}(0,1)})$. So, the starting point of our analysis is the study of the functions $x \mapsto \|x\|_{L^p(0,1)}$, $x \mapsto \|P_n x\|_{L^p(0,1)}$ for $p \ge 2$.

PROPOSITION 6.3. *For each $p \ge 1$ there is $C_p > 0$ such that*

$$\int_H \int_0^1 \big|P_n x(\xi)\big|^p\,d\xi\,d\mu \le C_p \left(\sum_{k=1}^n \frac{1}{k^2\pi^2}\right)^{p/2}, \qquad n \in \mathbb{N}, \tag{6.3}$$

$$\int_H \int_0^1 \big|P_n x(\xi) - P_m x(\xi)\big|^p\,d\xi\,d\mu \le C_p \left(\sum_{k=m+1}^n \frac{1}{k^2\pi^2}\right)^{p/2}, \qquad m < n \in \mathbb{N}. \tag{6.4}$$

PROOF. First of all note that for every $x \in H$, $P_n x$ is a smooth function. Moreover for every $\xi \in (0,1)$ and $m < n \in \mathbb{N}$, the function $x \mapsto P_n x(\xi) - P_m x(\xi)$ is a Gaussian random variable $N_{0,\sum_{k=m+1}^n (1/(\pi^4 k^4)) f_k(\xi)^2}$. Then, for $p \ge 1$,

$$\begin{aligned}
&\int_H \big|P_n x(\xi) - P_m x(\xi)\big|^p\,d\mu \\
&\quad = \int_{\mathbb{R}} |\eta|^p N_{0,\sum_{k=m+1}^n (1/(\pi^4 k^4)) f_k(\xi)^2}(d\eta) \\
&\quad = c_p \left(\sum_{k=m+1}^n \frac{1}{k^2\pi^2} e_k(\xi)^2\right)^{p/2} \le 2^{p/2} c_p \left(\sum_{k=m+1}^n \frac{1}{k^2\pi^2}\right)^{p/2},
\end{aligned}$$

so that

$$\int_H \int_0^1 \left|P_n x(\xi) - P_m x(\xi)\right|^p d\xi\, d\mu$$
$$= \int_0^1 \int_H \left|P_n x(\xi) - P_m x(\xi)\right|^p d\mu\, d\xi \le 2^{p/2} c_p \left(\sum_{k=m+1}^{n} \frac{1}{k^2\pi^2}\right)^{p/2};$$

that is, (6.4) holds. The proof of (6.3) is the same. □

Proposition 6.3 has several consequences.

COROLLARY 6.4. $\mu(\widetilde{L}^p(0,1)) = 1$, *and the sequence of functions* $(x,\xi) \mapsto P_n x(\xi)$ *converges to* $(x,\xi) \mapsto x(\xi)$ *in* $L^p(H \times (0,1), \mu \times d\xi)$, *for every* $p \ge 1$.

PROOF. It is sufficient to prove that the statement holds for $p = 2$. Indeed, estimate (6.4) implies that the sequence $(x,\xi) \mapsto P_n x(\xi)$ converges in $L^p(H \times (0,1), \mu \times d\xi)$ for every p to a limit function, that we identify with the function $(x,\xi) \mapsto x(\xi)$ taking $p = 2$. Once we know that $\int_H \int_0^1 |x(\xi)|^p\, d\xi\, d\mu < \infty$, then $\mu(\widetilde{L}^p(0,1))$ is obviously 1.

So, fix $p = 2$. Since

$$\int_0^1 \left|P_n x(\xi)\right|^2 d\xi$$
$$= \int_0^1 \sum_{h,k=1}^{n} \langle x, f_k\rangle_H \langle x, f_h\rangle_H f_k(\xi) f_h(\xi)\, d\xi$$
$$= \int_0^1 \sum_{k=1}^{n} \langle x, f_k\rangle_H^2 f_k(\xi)^2\, d\xi,$$

then for every $x \in H$ the sequence $\int_0^1 |P_n x(\xi)|^2\, d\xi$ is increasing, it converges to $\|x\|_{L^2}^2$ if $x \in \widetilde{L}^2(0,1)$, and to $+\infty$ if $x \notin \widetilde{L}^2(0,1)$ by Remark 6.1. By monotone convergence and (6.3) with $p = 2$ the limit function belongs to $L^1(H,\mu)$, and this implies $\mu(\widetilde{L}^2(0,1)) = 1$. Consequently, the function $(x,\xi) \mapsto x(\xi)$ is defined a.e. in $H \times (0,1)$. Moreover,

$$\int_{\widetilde{L}^2(0,1)} \int_0^1 \left|P_n x(\xi) - x(\xi)\right|^2 d\xi\, d\mu$$
$$= \int_{\widetilde{L}^2(0,1)} \lim_{m\to\infty} \int_0^1 \left|P_n x(\xi) - P_m x(\xi)\right|^2 d\xi\, d\mu$$
$$\le \liminf_{m\to\infty} \int_{\widetilde{L}^2(0,1)} \int_0^1 \left|P_n x(\xi) - P_m x(\xi)\right|^2 d\xi\, d\mu.$$

For each $\varepsilon > 0$ there is $n_\varepsilon \in \mathbb{N}$ such that for $n, m \geq n_\varepsilon$ we have $\int_{\widetilde{L}^2(0,1)} \int_0^1 |P_n x(\xi) - P_m x(\xi)|^2 \, d\xi \, d\mu \leq \varepsilon$. Then for $n \geq n_\varepsilon$ we get $\int_{\widetilde{L}^2(0,1)} \int_0^1 |P_n x(\xi) - x(\xi)|^2 \, d\xi \, d\mu \leq \varepsilon$, and the statement follows. $\square$

PROPOSITION 6.5. *Under Hypothesis 6.2, $U \in W^{1,p}_{1/2}(H,\mu)$ and $\lim_{n\to\infty} U_n = U$ in $L^p(H,\mu)$, for every $p \geq 1$. Moreover, $D_k U(x) = \int_0^1 \Phi'(x(\xi)) f_k(\xi) \, d\xi$ for a.e. $x \in H$.*

PROOF. As a first step, we remark that the sequence of functions $x \mapsto \|P_n x\|^p_{L^p(0,1)}$ is bounded in $L^s(H,\mu)$ for every $s \geq 1$. Indeed, using the Hölder inequality we get

$$\int_0^1 |P_n x(\xi)|^p \, d\xi \leq \left(\int_0^1 |P_n x(\xi)|^{ps} \, d\xi \right)^{1/s}, \qquad s \geq 1,$$

and the right-hand side belongs to $L^s(H,\mu)$ with norm independent of n, by estimate (6.3).

We already remarked that $|U_n(x)| \leq \int_0^1 C(1 + |P_n x(\xi)|)^{p_1} \, d\xi$ with $p_1 = p_2 + 1$, so that U_n is bounded in $L^p(H,\mu)$ by a constant independent of n, for every $p \geq 1$. Let us prove that $U_n \to U$ in $L^p(H,\mu)$. Using (5.4) and the Hölder inequality we get

$$\begin{aligned}
|U_n(x) - U(x)|^p &\leq \left(\int_0^1 |\Phi(P_n x(\xi)) - \Phi(x(\xi))| \, d\xi \right)^p \\
&\leq C^p \left(\int_0^1 (1 + |x(\xi)| + |P_n x(\xi)|)^{p_2} |P_n x(\xi) - x(\xi)| \, d\xi \right)^p \\
&\leq C^p \left(\int_0^1 (1 + |x(\xi)| + |P_n x(\xi)|)^{2p_2 p} \, d\xi \right)^{1/2} \\
&\quad \times \left(\int_0^1 |P_n x(\xi) - x(\xi)|^{2p} \, d\xi \right)^{1/2}.
\end{aligned}$$

Since $x \mapsto \|1 + |x| + |P_n x|\|_{L^{2p_2 p}(0,1)}$ is bounded in $L^{2p_2 p}(H,\mu)$ by a constant independent of n, and $\|P_n x - x\|_{L^{2p}(0,1)}$ vanishes in $L^{2p}(H,\mu)$ as $n \to \infty$, by the Hölder inequality the right-hand side vanishes in $L^1(H,\mu)$ as $n \to \infty$. Hence, U in $L^p(H,\mu)$ and $U_n \to U$ in $L^p(H,\mu)$ as $n \to \infty$.

To prove that $U \in W^{1,p}_{1/2}(H,\mu)$ it is enough to show that the sequence U_n is bounded in $W^{1,p}_{1/2}(H,\mu)$ (e.g., [3], Lemma 5.4.4). We already know that it is bounded in $L^p(H,\mu)$. Moreover each U_n is continuously differentiable, since it is the composition of $x \mapsto P_n x$ which is smooth from H to $C([0,1])$, and $y \mapsto \int_0^1 \Phi(y(\xi)) \, d\xi$ which is continuously differentiable from $C([0,1])$ to $\mathbb{R}$, and

$$D_k U_n(x) = \int_0^1 \Phi'(P_n x(\xi)) f_k(\xi) \, d\xi, \qquad k \leq n, \tag{6.5}$$

while $D_k U_n(x) = 0$ for $k > n$. Using again assumption (5.4) and the Hölder inequality, we get

$$\begin{aligned}|D_k U_n(x)| &= \left| \int_0^1 \Phi'(P_n x(\xi)) f_k(\xi)\, d\xi \right| \le C \int_0^1 (1 + |P_n x(\xi)|)^{p_2} |f_k(\xi)|\, d\xi \\ &\le \frac{C}{\lambda_k^{1/4}} \| 1 + |P_n x| \|_{L^{2p_2}(0,1)}^{p_2},\end{aligned}$$

for $k \le n$. Then

$$\| Q^{1/2} D U_n(x) \|^2 = \sum_{k=1}^{n} \lambda_k |D_k U_n(x)|^2 \le C^2 \sum_{k=1}^{\infty} \lambda_k^{1/2} \| 1 + |P_n x| \|_{L^{2p_2}(0,1)}^{2p_2}.$$

By the first part of the proof we know that $x \mapsto \| P_n x \|_{L^{2p_2}(0,1)}^{2p_2}$ belongs to $L^1(H, \mu)$ with norm bounded by a constant independent of n. Since $\sum_{k=1}^{\infty} \lambda_k^{1/2} < \infty$, then U_n is bounded in $W^{1,p}(H, \mu)$ so that $U \in W^{1,p}(H, \mu)$.

Now we show that for every $k \in \mathbb{N}$, a subsequence of $D_k U_n$ converges to $\int_0^1 \Phi'(x(\xi)) f_k(\xi)\, d\xi$ in $L^2(H, \mu)$. Then the equality $D_k U(x) = \int_0^1 \Phi'(x(\xi)) \times f_k(\xi)\, d\xi$ μ-a.e. follows using the integration by parts formula (2.1).

We have

$$\begin{aligned}&\int_H \left| D_k U_n(x) - \int_0^1 \Phi'(x(\xi)) f_k(\xi)\, d\xi \right|^2 d\mu \\ &\qquad \le \int_H \int_0^1 |\Phi'(P_n x(\xi)) - \Phi'(x(\xi))|^2 f_k(\xi)^2\, d\xi\, d\mu.\end{aligned}$$

By Corollary 6.4, the sequence of functions $(x, \xi) \mapsto P_n x(\xi)$ converges to $x(\xi)$ in $L^2(H, \mu)$. Consequently, a subsequence converges μ-almost everywhere, and since Φ' is continuous, along such subsequence $(x, \xi) \mapsto (\Phi'(P_n x(\xi)) - \Phi'(x(\xi))) f_k(\xi)$ vanishes. Moreover, by assumption (5.4),

$$|\Phi'(P_n x(\xi)) - \Phi'(x(\xi))|^2 f_k(\xi)^2 \le C^2 (2 + |P_n x(\xi)|^{p_2} + |x(\xi)|^{p_2}) \| f_k \|_\infty^2$$

which belongs to $L^1(H \times (0,1), \mu \times d\xi)$ with norm bounded by a constant independent of n. The statement follows by the dominated convergence theorem. □

Then, U satisfies Hypothesis 2.1. So, the results of Theorem 3.9 and of Propositions 4.2, 4.6 hold.

We recall that the operator $Q^{1/2} D$ in the space $L^2(H, \nu; H)$ is the closure of the operator $\varphi \mapsto Q^{1/2} D\varphi$ defined in a set of smooth functions; see Definition 2.5. However, we can identify $Q^{1/2} D U(x)$: indeed, recalling that $B = Q^{-1/2}/\sqrt{2}$, we obtain

$$
\begin{aligned}
D_k U(x) &= \langle \Phi' \circ x, f_k \rangle_{L^2(0,1)} = \left\langle \Phi' \circ x - \int_0^1 \Phi'\big(x(\xi)\big)\, d\xi, B f_k \right\rangle_H \\
&= \frac{\lambda_k^{-1/2}}{\sqrt{2}} \left\langle \Phi' \circ x - \int_0^1 \Phi'\big(x(\xi)\big)\, d\xi, f_k \right\rangle_H
\end{aligned}
$$

for every $x \in \widetilde{L}^{2p_2}(0,1)$, so that

$$
\begin{aligned}
Q^{1/2} DU(x) &= \frac{1}{\sqrt{2}} \sum_{k=1}^{\infty} \left\langle \Phi' \circ x - \int_0^1 \Phi'\big(x(\xi)\big)\, d\xi, f_k \right\rangle_H f_k \\
&= \frac{\Phi' \circ x - \int_0^1 \Phi'(x(\xi))\, d\xi}{\sqrt{2}}.
\end{aligned}
$$

On the other hand, we already mentioned that if $\Phi' \circ x \in V$ [i.e., $\Phi' \circ x \in D(B)$], then $D_0 U(x) = B(\Phi' \circ x)$, so that, since $Q^{1/2} = B^{-1}/\sqrt{2}$, $Q^{1/2} D_0 U(x) = Q^{1/2} DU(x)$. For such x we have

$$
\begin{aligned}
\big\langle B\big(\Phi' \circ x\big), Du(x)\big\rangle &= \big\langle \Phi' \circ x, BDu(x)\big\rangle = \big\langle Q^{1/2} DU(x), Q^{-1/2} Du(x)\big\rangle \\
&= \big\langle DU(x), Du(x)\big\rangle.
\end{aligned}
$$

Then the stochastic differential equation (1.2) in H reads as

$$
dX(t) = \left(-\frac{\partial^4}{\partial \xi^4} X - \frac{\partial^2}{\partial \xi^2} \Phi'(X)\right) dt + dW(t), \qquad X(0) = x, \tag{6.6}
$$

and it is a stochastic Cahn–Hilliard equation, whose Kolmogorov operator is $\mathcal{K}$. It was studied in [16] and in several following papers, in particular in [9], where existence and uniqueness of weak solutions were proved for polynomial nonlinearities Φ. Here $W(t)$ is, as usual, any H-valued cylindrical Wiener process defined in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

We think that it is possible to relate the weak solution to (6.6) constructed in [9] to the solution of the Kolmogorov equation by formula (3.24), at least in the model case $\Phi(\xi) = \xi^{2m}$ with $m \in \mathbb{N}$. Indeed, for every $x \in H$ the weak solution given by [9], Theorem 2.1, is obtained through cylindrical approximations $X_n(t)$, solutions to

$$
dX_n = \big(A_n X_n + P_n B \Phi'(P_n X)\big)\, dt + P_n\, dW(t), \qquad X_n(0) = P_n x, \tag{6.7}
$$

with $A_n = A_{|P_n(H)} \in \mathcal{L}(P_n(H))$; identifying $P_n(H)$ with $\mathbb{R}^n$ the Kolmogorov operator $\mathcal{K}_n$ associated to (6.7) is

$$
\mathcal{K}_n \varphi = \frac{1}{2} \Delta \varphi - \sum_{k=1}^{n} \left(\frac{x_k}{2\lambda_k} + \int_0^1 \Phi'\left(\sum_{h=1}^{n} x_h f_h(\xi) \right) f_k(\xi)\, d\xi \right) D_k \varphi.
$$

Taking into account such explicit expressions, one should be able to follow the procedure of Proposition 3.8 (that deals with the case of Lipschitz continuous DU). However, many details should be fixed, and giving a complete proof goes beyond the aims of this paper.

Acknowledgement. We thank one of the referees for careful reading of the manuscript, and for several remarks that helped to improve the paper.

REFERENCES

[1] ALBEVERIO, S. and RÖCKNER, M. (1991). Stochastic differential equations in infinite dimensions: Solutions via Dirichlet forms. *Probab. Theory Related Fields* **89** 347–386. MR1113223

[2] BARBU, V. (2010). *Nonlinear Differential Equations of Monotone Types in Banach Spaces*. Springer, New York. MR2582280

[3] BOGACHEV, V. I. (1998). *Gaussian Measures*. Amer. Math. Soc., Providence, RI. MR1642391

[4] BRÉZIS, H. (1973). *Opérateurs Maximaux Monotones et Semi-groupes de Contractions dans les Espaces de Hilbert*. North-Holland, Amsterdam. MR0348562

[5] CHOJNOWSKA-MICHALIK, A. and GOŁDYS, B. (1995). Existence, uniqueness and invariant measures for stochastic semilinear equations on Hilbert spaces. *Probab. Theory Related Fields* **102** 331–356. MR1339737

[6] DAVIES, E. B. (1989). *Heat Kernels and Spectral Theory*. Cambridge Univ. Press, Cambridge. MR0990239

[7] DA PRATO, G. (2004). *Kolmogorov Equations for Stochastic PDEs*. Birkhäuser, Basel. MR2111320

[8] DA PRATO, G. (2006). *An Introduction to Infinite-Dimensional Analysis*. Springer, Berlin. MR2244975

[9] DA PRATO, G. and DEBUSSCHE, A. (1996). Stochastic Cahn–Hilliard equation. *Nonlinear Anal.* **26** 241–263. MR1359472

[10] DA PRATO, G., DEBUSSCHE, A. and GOLDYS, B. (2002). Some properties of invariant measures of nonsymmetric dissipative stochastic systems. *Probab. Theory Related Fields* **123** 355–380. MR1918538

[11] DA PRATO, G., FLANDOLI, F., PRIOLA, E. and RÖCKNER, M. (2013). Strong uniqueness for stochastic evolution equations in Hilbert spaces perturbed by a bounded measurable drift. *Ann. Probab.* **41** 3306–3344. MR3127884

[12] DA PRATO, G. and RÖCKNER, M. (2002). Singular dissipative stochastic equations in Hilbert spaces. *Probab. Theory Related Fields* **124** 261–303. MR1936019

[13] DA PRATO, G. and ZABCZYK, J. (1992). *Stochastic Equations in Infinite Dimensions*. *Encyclopedia of Mathematics and Its Applications* **44**. Cambridge Univ. Press, Cambridge. MR1207136

[14] DA PRATO, G. and ZABCZYK, J. (1996). *Ergodicity for Infinite-Dimensional Systems*. *London Mathematical Society Lecture Note Series* **229**. Cambridge Univ. Press, Cambridge. MR1417491

[15] DA PRATO, G. and ZABCZYK, J. (2002). *Second Order Partial Differential Equations in Hilbert Spaces*. *London Mathematical Society Lecture Notes* **293**. Cambridge Univ. Press, Cambridge. MR1985790

[16] ELEZOVIĆ, N. and MIKELIĆ, A. (1991). On the stochastic Cahn–Hilliard equation. *Nonlinear Anal.* **16** 1169–1200. MR1111627

[17] HAUSSMANN, U. G. and PARDOUX, É. (1986). Time reversal of diffusions. *Ann. Probab.* **14** 1188–1205. MR0866342

[18] JONA-LASINIO, G. and SÉNÉOR, R. (1991). On a class of stochastic reaction–diffusion equations in two space dimensions. *J. Phys. A* **24** 4123–4128. MR1126653

[19] LUNARDI, A., METAFUNE, G. and PALLARA, D. (2005). Dirichlet boundary conditions for elliptic operators with unbounded drift. *Proc. Amer. Math. Soc.* **133** 2625–2635. MR2146208

[20] LUNARDI, A. and VESPRI, V. (1998). Optimal L^∞ and Schauder estimates for elliptic and parabolic operators with unbounded coefficients. In *Reaction Diffusion Systems* (*Trieste*, 1995) (G. Caristi and E. Mitidieri, eds.). *Lecture Notes in Pure and Applied Mathematics* **194** 217–239. Dekker, New York. MR1472521

[21] PHELPS, R. R. (1978). Gaussian null sets and differentiability of Lipschitz map on Banach spaces. *Pacific J. Math.* **77** 523–531. MR0510938

[22] RÖCKNER, M. (1999). L^p-analysis of finite and infinite-dimensional diffusion operators. In *Stochastic PDE's and Kolmogorov Equations in Infinite Dimensions* (*Cetraro*, 1998) (G. Da Prato, ed.). *Lecture Notes in Math.* **1715** 65–116. Springer, Berlin. MR1731795

[23] SHIGEKAWA, I. (1992). Sobolev spaces over the Wiener space based on an Ornstein–Uhlenbeck operator. *J. Math. Kyoto Univ.* **32** 731–748. MR1194112

SCUOLA NORMALE SUPERIORE
PIAZZA DEI CAVALIERI, 7
56126 PISA
ITALY
E-MAIL: g.daprato@sns.it

DIPARTIMENTO DI MATEMATICA
UNIVERSITÀ DI PARMA
PARCO AREA DELLE SCIENZE, 53/A
43124 PARMA
ITALY
E-MAIL: alessandra.lunardi@unipr.it