

ZERO-TEMPERATURE LIMIT OF THE KAWASAKI DYNAMICS FOR THE ISING LATTICE GAS IN A LARGE TWO-DIMENSIONAL TORUS

BY B. GOIS AND C. LANDIM

IMPA, and IMPA and Université de Rouen

We consider the Kawasaki dynamics at inverse temperature β for the Ising lattice gas on a two-dimensional square of length $2L + 1$ with periodic boundary conditions. We assume that initially the particles form a square of length n , which may increase, as well as L , with β . We show that in a proper time scale the particles form almost always a square and that the center of mass of the square evolves as a Brownian motion when the temperature vanishes.

1. Introduction and main results. We introduced in [1, 4] a general method to describe the asymptotic evolution of one-parameter families of continuous-time Markov chains among the ground states or from high energy sets to lower energy sets. This method has been successfully applied in two situations. For zero-range dynamics on a finite set which exhibit condensation [3, 19], and for random walks evolving among random traps [17, 18]. In the first model, the chain admits a finite number of ground sets and one can prove that in an appropriate time scale the process evolves as a finite state Markov chain, each state corresponding to a ground set of the original chain. In the second model, there is a countable number of ground states and one can prove that in a certain time scale the process evolves as a continuous-time Markov chain in a countable state space, each point representing one of the ground states. In this paper, which follows [2, 5], we investigate a third case, where the limit dynamics is a continuous process.

We consider the Ising lattice gas on a torus subjected to a Kawasaki dynamics at inverse temperature β . Let $\mathbb{T}_L = \{-L, \dots, L\}^2$, $L \geq 1$, be a square with periodic boundary conditions. Denote by \mathbb{T}_L^* the set of bonds of \mathbb{T}_L . This is the set of unordered pairs $\{x, y\}$ of \mathbb{T}_L such that $\|x - y\| = 1$, where $\|\cdot\|$ stands for the Euclidean distance. The configurations are denoted by $\eta = \{\eta(x) : x \in \mathbb{T}_L\}$, where $\eta(x) = 1$ if site x is occupied and $\eta(x) = 0$ if site x is vacant. The Hamiltonian \mathbb{H} , defined on the state space $\Omega_L = \{0, 1\}^{\mathbb{T}_L}$, is given by

$$-\mathbb{H}(\eta) = \sum_{\{x,y\} \in \mathbb{T}_L^*} \eta(x)\eta(y).$$

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The Gibbs measure at inverse temperature β associated to the Hamiltonian \mathbb{H} , denoted by μ_β , is given by

$$\mu_\beta(\eta) = \frac{1}{Z_\beta} e^{-\beta\mathbb{H}(\eta)},$$

where Z_β is the normalizing partition function.

We consider the continuous-time Markov chain $\{\eta_t^\beta : t \geq 0\}$ on Ω_L whose generator L_β acts on functions $f : \Omega_L \rightarrow \mathbb{R}$ as

$$(L_\beta f)(\eta) = \sum_{\{x,y\} \in \mathbb{T}_L^*} c_{x,y}(\eta) [f(\sigma^{x,y}\eta) - f(\eta)],$$

where $\sigma^{x,y}\eta$ is the configuration obtained from η by exchanging the occupation variables $\eta(x)$ and $\eta(y)$:

$$(\sigma^{x,y}\eta)(z) = \begin{cases} \eta(z), & \text{if } z \neq x, y, \\ \eta(y), & \text{if } z = x, \\ \eta(x), & \text{if } z = y. \end{cases}$$

The rates $c_{x,y}$ are given by

$$c_{x,y}(\eta) = \exp\{-\beta[\mathbb{H}(\sigma^{x,y}\eta) - \mathbb{H}(\eta)]_+\},$$

and $[a]_+, a \in \mathbb{R}$, stands for the positive part of a : $[a]_+ = \max\{a, 0\}$. We sometimes represent η_t^β by $\eta^\beta(t)$ and we frequently omit the index β of η_t^β .

A simple computation shows that the Markov process $\{\eta_t : t \geq 0\}$ is reversible with respect to the Gibbs measures $\mu_\beta, \beta > 0$, and ergodic on each irreducible component formed by the configurations with a fixed total number of particles. Let $\Omega_{L,K} = \{\eta \in \Omega_L : \sum_{x \in \mathbb{T}_L} \eta(x) = K\}, 0 \leq K \leq |\mathbb{T}_L|$, and denote by $\mu_K = \mu_{\beta,K}$ the Gibbs measure μ_β conditioned on $\Omega_{L,K}$:

$$\mu_K(\eta) = \frac{1}{Z_{\beta,K}} e^{-\beta\mathbb{H}(\eta)}, \quad \eta \in \Omega_{L,K},$$

where $Z_K = Z_{\beta,K}$ is the normalizing constant $Z_K = \sum_{\eta \in \Omega_{L,K}} \exp\{-\beta\mathbb{H}(\eta)\}$.

Let $D(\mathbb{R}_+, \Omega_{L,K})$ be the space of right-continuous with left limits trajectories $e : \mathbb{R}_+ \rightarrow \Omega_{L,K}$ endowed with the Skorohod topology. For each configuration $\eta \in \Omega_{L,K}$, denote by \mathbf{P}_η^β the probability measure on $D(\mathbb{R}_+, \Omega_{L,K})$ induced by the Markov process $\{\eta_t : t \geq 0\}$ starting from η . Expectation with respect to \mathbf{P}_η^β is represented by \mathbf{E}_η^β . Sometimes we omit the index β in the notation. The reader should be warned that we consider in this article two types of asymptotics involving the probability measure \mathbf{P}_η^β . In Section 3, for example, we examine the limit of certain events in the case where the set \mathbb{T}_L and the number of particles K are fixed while $\beta \uparrow \infty$. In contrast, in all theorem stated in this section L and K depend on β .

Assume from now on that $K = n^2$ for some $n \geq 1$. Denote by Q the square $\{0, \dots, n - 1\} \times \{0, \dots, n - 1\}$. For $\mathbf{x} \in \mathbb{T}_L$, let $Q_{\mathbf{x}} = \mathbf{x} + Q$ and let $\eta^{\mathbf{x}}$ be the configuration in which all sites of the square $Q_{\mathbf{x}}$ are occupied. Denote by Γ the set of square configurations

$$\Gamma = \{\eta^{\mathbf{x}} : \mathbf{x} \in \mathbb{T}_L\}.$$

A simple computation [5] shows that if $L > 2n$ the ground states of the energy \mathbb{H} in $\Omega_{L,K}$ are the square configurations

$$\mathbb{H}_{\min} := \min_{\eta \in \Omega_{L,K}} \mathbb{H}(\eta) = \mathbb{H}(\eta^{\mathbf{x}}) = -2n(n - 1),$$

and $\mathbb{H}(\eta) > -2n(n - 1)$ for all $\eta \in \Omega_{L,K} \setminus \Gamma$.

We examine in this article the asymptotic evolution of the Markov process $\{\eta_t : t \geq 0\}$ among the $|\mathbb{T}_L|$ ground states $\{\eta^{\mathbf{x}} : \mathbf{x} \in \mathbb{T}_L\}$ as the temperature vanishes, while the volume and the number of particles increase not too fast.

The trace process on Γ . Denote by $\xi(t)$ the trace of the process $\eta(t)$ on Γ and by $r_{\beta}(\mathbf{x}, \mathbf{y}) = R_{\xi}(\eta^{\mathbf{x}}, \eta^{\mathbf{y}})$ the jump rates of the trace process $\xi(t)$. By symmetry, it is clear that $r_{\beta}(\mathbf{x}, \mathbf{y}) = r_{\beta}(0, \mathbf{y} - \mathbf{x}) =: r_{\beta}(\mathbf{y} - \mathbf{x})$.

Let $\mathbf{X}(\eta^{\mathbf{x}}) = \mathbf{x}$, $\mathbf{x} \in \mathbb{T}_L$, so that $X(t) = \mathbf{X}(\xi(t))$ is a random walk on \mathbb{T}_L which jumps from \mathbf{x} to \mathbf{y} at rate $r_{\beta}(\mathbf{y} - \mathbf{x})$ and which starts from the origin. Denote by $X_1(t)$, $X_2(t)$ the components of $X(t)$ and by $N_t^{\mathbf{x}}$ the number of jumps of size \mathbf{x} performed by the random walk $X(s)$ in the time interval $[0, t]$. Since, by symmetry, $r_{\beta}(-\mathbf{x}) = r_{\beta}(\mathbf{x})$,

$$X(t) - X(0) = \sum_{\mathbf{x} \in \mathbb{T}_L} \mathbf{x} N_t^{\mathbf{x}} = \sum_{\mathbf{x} \in \mathbb{T}_L} \mathbf{x} \{N_t^{\mathbf{x}} - r_{\beta}(\mathbf{x})t\} = \sum_{\mathbf{x} \in \mathbb{T}_L} \mathbf{x} M_t^{\mathbf{x}},$$

where $M_t^{\mathbf{x}}$, $\mathbf{x} \in \mathbb{T}_L$, are orthogonal martingales with quadratic variation $\langle M^{\mathbf{x}} \rangle_t$ equal to $r_{\beta}(\mathbf{x})t$. Hence, if we represent by $\langle X_1 \rangle_t$, $\langle X_2 \rangle_t$ the predictable quadratic variations of the components of the random walk $X(t)$,

$$(1.1) \quad \langle X_1 \rangle_t + \langle X_2 \rangle_t = \sum_{\mathbf{x} \in \mathbb{T}_L} \|\mathbf{x}\|^2 r_{\beta}(\mathbf{x})t.$$

Let

$$(1.2) \quad \frac{1}{\theta_{\beta}} := \frac{1}{t} \mathbf{E}_{\eta^0} [\|X(t)\|^2] = \frac{1}{t} \mathbf{E}_{\eta^0} [\langle X_1 \rangle_t + \langle X_2 \rangle_t] = \sum_{\mathbf{x} \in \mathbb{T}_L} \|\mathbf{x}\|^2 r_{\beta}(\mathbf{x}).$$

Hence, θ_{β}^{-1} is the diffusion constant of the centre of the square, that is, the mean-square displacement per unit of time. We prove in Section 9 that under the assumption that

$$(1.3) \quad \lim_{\beta \rightarrow \infty} n^4 L^2 e^{-\beta} = 0,$$

there exist constants $0 < c_0 < C_0 < \infty$, independent of β , such that

$$(1.4) \quad c_0 \frac{n}{L^2} e^{2\beta} \leq \theta_\beta \leq C_0 n^2 e^{2\beta}.$$

We adopt this convention throughout the paper, $0 < c_0 < C_0 < \infty$ are constants independent of β whose value may change from line to line. Let

$$(1.5) \quad \mathbf{e}(\beta) = L e^{-\beta/2} + \sqrt{n^7 [n^4 e^{-\beta} + n L e^{-\beta/2}]},$$

which is the order of the error which appears in Proposition 5.2. Under the stronger assumption that

$$(1.6) \quad \limsup_{\beta \rightarrow \infty} \frac{L^2}{n^2} \mathbf{e}(\beta) < \infty,$$

we have that

$$(1.7) \quad c_0 \frac{1}{n^2} e^{2\beta} \leq \theta_\beta \leq C_0 n^2 e^{2\beta}.$$

We comment below in Remark 1.4 on the exact order of θ_β . Entropy effects, evidenced by the presence of many paths of the same cost connecting two ground states, turn the estimation of θ_β into hard problem.

THEOREM 1.1. *Assume that $\eta_0 = \eta^0$ and that $L = L(\beta)$, $n = n(\beta)$ depend on the temperature. Consider a sequence $\ell = \ell(\beta)$ such that $1 \ll \ell \leq L$, and assume that*

$$(1.8) \quad \lim_{\beta \rightarrow \infty} \ell L \theta_\beta e^{-2\beta} \mathbf{e}(\beta) = 0,$$

and that for all $\delta > 0$,

$$(1.9) \quad \lim_{\beta \rightarrow \infty} \ell(\ell + n) \theta_\beta e^{-2\beta} e^{-\delta \ell / n} = 0.$$

Let $Z^\beta(t) = X(t\ell^2\theta_\beta)/\ell$. Then, as $\beta \rightarrow \infty$, $Z^\beta(t)$ converges in the Skorohod topology to a Brownian motion.

If $\ell = L$, the limiting Brownian motion evolves on the two-dimensional torus $[-1, 1]^2$, while if $\ell \ll L$, the limiting Brownian motion evolves on \mathbb{R}^2 . We discuss in Remark 1.5 the assumptions (1.8) and (1.9).

Energy landscape. Denote by \mathbb{H}_j , $j \geq 0$, the set of configurations with energy equal to $\mathbb{H}_{\min} + j = -2n(n - 1) + j$:

$$\mathbb{H}_j = \{ \eta \in \Omega_{L,K} : \mathbb{H}(\eta) = \mathbb{H}_{\min} + j \},$$

and let

$$(1.10) \quad \Delta_j = \{ \eta \in \Omega_{L,K} : \mathbb{H}(\eta) > \mathbb{H}_{\min} + j \},$$

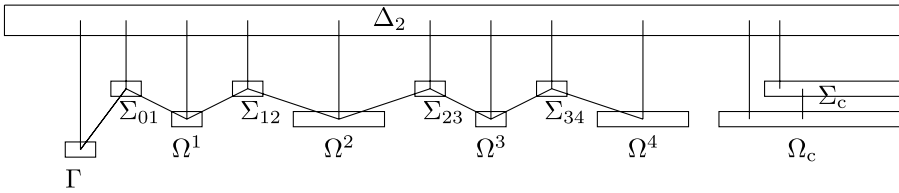


FIG. 1. The energy landscape of the Kawasaki dynamics at low temperature. Γ represents the set of ground states, Ω^j , $1 \leq j \leq 4$, disjoint subsets of \mathbb{H}_1 , $\Sigma_{i,i+1}$, $0 \leq i \leq 3$, disjoint subsets of \mathbb{H}_2 , and $\Omega_c = \mathbb{H}_1 \setminus [\cup_{1 \leq j \leq 4} \Omega^j]$, $\Sigma_c = \mathbb{H}_2 \setminus [\cup_{0 \leq i \leq 3} \Sigma_{i,i+1}]$. At low temperatures, during an excursion between two ground states, with probability very close to 1, the process does not visit the set Δ_2 and all the analysis is reduced to the lower-left portion of the picture.

so that $\mathbb{H}_0 = \Gamma$ and $\{\mathbb{H}_{0j}, \Delta_j\}$ forms a partition of the set $\Omega_{L,K}$, where $\mathbb{H}_{ij} = \cup_{i \leq k \leq j} \mathbb{H}_k$.

We have shown in [5] that the energy landscape of the Kawasaki dynamics at low temperature starting from a ground state configuration has the form illustrated by Figure 1. There are subsets Ω^j , $1 \leq j \leq 4$, of \mathbb{H}_1 and subsets $\Sigma_{j,j+1}$, $0 \leq j \leq 3$, of \mathbb{H}_2 , defined in Section 2, satisfying the following properties. Let $\Omega_c = \mathbb{H}_1 \setminus [\cup_j \Omega^j]$, $\Sigma_c = \mathbb{H}_2 \setminus [\cup_j \Sigma_{j,j+1}]$. Denote by Π , Π' generic sets appearing in Figure 1. The process η_t may jump from a configuration in Π to a configuration in Π or to a configuration in Π' if Π and Π' are joined by an edge in Figure 1. In particular, starting from Γ , the process η_t may only reach Ω_c by crossing the set Δ_2 .

Let \mathfrak{E}_\star be the set of configurations which can be reached from η^0 without crossing the set Δ_2 , $\mathfrak{E}_\star = \Gamma \cup_{1 \leq i \leq 4} \Omega^i \cup_{0 \leq j \leq 3} \Sigma_{j,j+1}$ in the notation of Figure 1. This set is described in the next section and in Section 7.

THEOREM 1.2. Assume that

$$(1.11) \quad \lim_{\beta \rightarrow \infty} \ell^2 \theta_\beta e^{-2\beta} (n^8 + L^2) e^{-\beta} = 0.$$

Then, for every $t > 0$,

$$(1.12) \quad \lim_{\beta \rightarrow \infty} \mathbf{P}_{\eta^0}[\eta(s) \notin \mathfrak{E}_\star \text{ for some } 0 \leq s \leq t \ell^2 \theta_\beta] = 0.$$

The set \mathfrak{E}_\star is small. In the next sections, we show that this set has less than $C_0 L^2 (n^8 + L^2)$ elements and that it is constituted of configurations whose particles form $n \times n$ squares, $(n + 1) \times (n - 1)$ and $(n + 2) \times (n - 2)$ rectangles.

Configurations whose particles form a $(n + k) \times (n - k)$ rectangle, $4 < k^2 < n + k$, and whose remaining k^2 particles form a $1 \times k^2$ rectangle attached to one side of the large rectangle belong to $\mathbb{H}_1 \cap \mathfrak{E}_\star^c$. It follows from this observation and from the definition of the set \mathfrak{E}_\star , presented in the next section, that even the cardinality of $\mathfrak{E}_\star \cap \mathbb{H}_1$ is small compared to the cardinality of \mathbb{H}_1 .

The previous two remarks show that in the time scale $\ell^2\theta_\beta$ the Kawasaki dynamics is confined to a tiny portion of the state space. It is this rather simple energy landscape which permits the analysis of the evolution of the center of mass. In dimension 3, for example, the energy landscape is much more complicated inso-much that, though we believe that a similar result holds, the proof requires a much more intricated argument.

Evolution of the center of mass. Due to the periodic boundary conditions, the center of mass of a configuration may not be well defined. However, the particles of a configuration $\eta \in \Xi_\star \setminus (\Sigma_{0,1} \cup \Sigma_{2,3})$ form a connected set and have, therefore, a well defined center of mass. On the other hand, the particles of a configuration η in $\Sigma_{0,1}$ form a square in which a particle at a corner of the square has been moved to a site whose neighbors are vacant or became vacant after the displacement of the particle at the corner. The detached particle may create an ambiguity in the definition of the center of mass of η . To avoid this problem we define the center of mass of a configuration $\eta \in \Sigma_{0,1}$ as the center of mass of the large connected component. Of course, if n is large and L is not too large compared to n^2 the detached particle does not affect the center of mass. This analysis extends to configurations in $\Sigma_{2,3}$. With the previous convention and since by Theorem 1.2 the process is confined to Ξ_\star in the time scale $\ell^2\theta_\beta$, we may examine the evolution of the center of mass without ambiguity.

Denote by $M(\eta)$ the center of mass of a configuration $\eta \in \Xi_\star$ and set, say, $M(\xi) = \mathbf{0}$ for configurations $\xi \notin \Xi_\star$. Denote by $[a]$ the integer part of $a > 0$.

THEOREM 1.3. *Assume that $\eta(0) = \eta^{-\mathbf{n}}$, where $\mathbf{n} = ([n/2], [n/2])$, and that the hypotheses of Theorems 1.1 and 1.2 are in force. Suppose also that $(n^7 + L)e^{-\beta} \rightarrow 0$. Let $M^\beta(t) = M(\eta(t\ell^2\theta_\beta))/\ell$. There exists $c_0 > 0$ such that if $n \geq c_0$ for all β large enough, then, as $\beta \uparrow \infty$, $M^\beta(t)$ converges in the Skorohod topology to a Brownian motion.*

As in Theorem 1.1, the limiting Brownian motion evolves on the torus $[-1, 1]^2$ if $\ell = L$ and on \mathbb{R}^2 if $\ell \ll L$.

Dynamics of the trace process $\xi(t)$. To examine the evolution of the trace of η_t on the set Γ of ground states, we consider the trace of η_t on the larger space $\Xi_1 = \Gamma \cup \Omega^1 \cup \Omega^3$. The set Ω^1 , defined in Section 2, is formed by configurations whose particles form a $n \times n$ square in which one particle at the corner of the square has been displaced and attached to one of the sides of the square. Figure 3 shows configurations in Ω^1 . The set Ω^3 contains all configurations whose particles form a $(n \pm 1) \times (n \mp 1)$ rectangle with an extra particle attached to one of the sides. Figure 5 shows configurations in Ω^3 .

Figure 2 illustrates the dynamics of the trace of η_t on Ξ_1 in the case $n = 6$, denoted by $\zeta_1(t)$. A 1×1 square has been placed at each vertex occupied by a particle and the gray square indicates the origin. The $n \times n$ squares represent the set of configurations formed by a ground state, say $\eta^{\mathbf{x}}$, as well as the $4(4n - 2)$

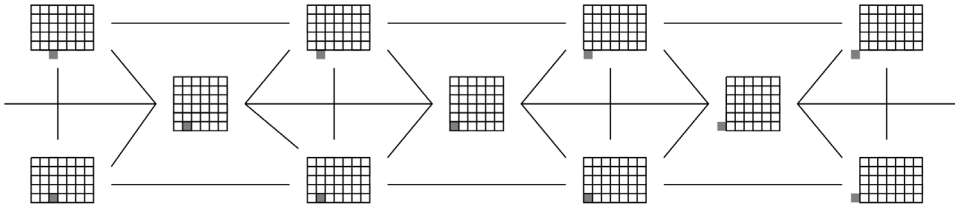


FIG. 2. The dynamics of the trace of η_t on $\Gamma \cup \Omega^1 \cup \Omega^3$.

configurations in Ω^1 obtained from η^x by displacing a particle at the corner of the square to one of the sides. There are 16 different types of such configurations, as we may choose a particle from 4 corners and move it to 4 sides. Each rectangle represents the set of $4n$ configurations whose particles form a $(n + 1) \times (n - 1)$ rectangle with an extra particle attached to one of the sides.

Figure 2 presents a piece of a periodic horizontal band of the graph which describes the dynamics of $\zeta_1(t)$. To form the complete graph, one has first to replicate this periodic horizontal band in the vertical direction to obtain a graph also periodic in the vertical direction. Denote by \mathbb{G} the graph obtained. To complete the construction, one has to superpose \mathbb{G} with the graph \mathbb{G} rotated by 90° . In this rotated graph, the $(n + 1) \times (n - 1)$ rectangles become $(n - 1) \times (n + 1)$ rectangles.

Recall that each square or rectangle in Figure 2 represents a set of configurations. Two sets Π_1, Π_2 are joined by a line if the process $\zeta_1(t)$ may jump from one configuration of Π_1 to a configuration in Π_2 . For example, $\zeta_1(t)$ jumps from a square to the rectangle on the right and above the square by sliding the small squares at the bottom of the square to its right side. In fact, all jumps occur from such displacements of particle along the sides.

Figure 2 also illustrates why the trace process $\xi(t)$ may have long jumps. For this to happen, it is enough that the process $\zeta_1(t)$ jumps from a square configuration to an upper rectangle and then move from rectangle to rectangle in the horizontal direction. The figure indicates that the probability that $\xi(t)$ performs a jump of length x should decay exponentially in x at a rate independent of β .

The diagrams in Figures 1, 2 have been used extensively in the literature. We refer to [6, 7, 9, 16] in the conservative case.

REMARK 1.4 (The asymptotic behavior of θ_β). We believe that the order of magnitude of θ_β is given by

$$\frac{1}{\mu_\zeta(\eta^0)} \text{cap}_\zeta(\eta^0, \Pi) \sim \frac{1}{\mu_\beta(\eta^0)} \text{cap}(\eta^0, \Pi).$$

In this formula, $\zeta(t)$ is the trace process introduced in Section 3, μ_ζ its stationary state, cap_ζ the associated capacity and Π the union of all sets joined by a line to η^0 in the graph partially drawn in Figure 2. If this insight is correct, by the

computation presented at the end of Section 6 the order of magnitude of θ_β lies between $ne^{2\beta}$ and $n^2e^{2\beta}$.

By definition (1.2) and by formula (9.1),

$$(1.13) \quad \theta_\beta^{-1} = \lambda_\zeta(\eta^0) \sum_{\mathbf{x} \in \mathbb{T}_L} \|\mathbf{x}\|^2 \mathbf{P}_{\eta^0}[H_\Gamma^+ = H_{\eta^{\mathbf{x}}}],$$

where $\lambda_\zeta(\eta^0)$ represents the holding rate at η^0 of the trace process $\zeta(t)$, and H_Π , H_Π^+ the hitting time of Π and the return time to Π , respectively. All these notions are defined in Section 3. It follows from the previous equation and from the definition of the capacity that

$$\theta_\beta^{-1} \geq \lambda_\zeta(\eta^0) \mathbf{P}_{\eta^0}[H_\Gamma^+ < H_{\eta^0}^+] = \frac{1}{\mu_\zeta(\eta^0)} \text{cap}_\zeta(\eta^0, \Gamma \setminus \eta^0).$$

By Lemma 9.1, this expression is bounded below by $c_0e^{-2\beta}n^{-2}$, which is the upper bound presented in (1.4). On the other hand, $\text{cap}_\zeta(\eta^0, \Gamma \setminus \eta^0)$ should be of the order of $\text{cap}_\zeta(\eta^0, \Pi)$ which explains the first half of our claim.

To obtain an upper bound for θ_β^{-1} , note that by (1.13),

$$\theta_\beta^{-1} \leq \lambda_\zeta(\eta^0) \mathbf{P}_{\eta^0}[H_\Pi < H_{\eta^0}^+] \sum_{\mathbf{x} \in \mathbb{T}_L} \|\mathbf{x}\|^2 \max_{\eta \in \Pi} \mathbf{P}_\eta[H_\Gamma = H_{\eta^{\mathbf{x}}}] .$$

We have argued right before this remark that $\mathbf{P}_\eta^s[H_\Gamma^+ = H_{\eta^{\mathbf{x}}}]$ decreases exponentially fast in $\|\mathbf{x}\|$, at a rate independent of β . In particular, the previous sum should be bounded by a finite constant independent of β . On the other hand, by the definition of the capacity

$$\lambda_\zeta(\eta^0) \mathbf{P}_{\eta^0}[H_\Pi < H_{\eta^0}^+] = \frac{1}{\mu_\zeta(\eta^0)} \text{cap}_\zeta(\eta^0, \Pi).$$

By the proof of Lemma 9.1, this expression is bounded above by $C_0n^{-1}e^{-2\beta}$.

The exact asymptotic behavior of θ_β requires sharp estimates of the jump rates of the process described in Figure 2. By symmetry, there are only few different rates. To estimate these rates, one has to examine the dynamics in which particles slide along the sides of the square. This process, denoted by $\zeta(t)$ in Section 3, is a reversible, symmetric Markov process on a seven-dimensional graph in which each vertex has degree less than or equal to eight. Such estimates are certainly not easy, but are conceivable. In possession of these estimates, one still has to compute the probability that the process described in Figure 2 starting from η^0 return to Γ at $\eta^{\mathbf{x}}$.

REMARK 1.5 (The hypotheses and the strategy). Condition (1.8) has to be understood as follows. $\ell^2\theta_\beta$ is the time scale. Each time the process visits a square (a configuration in Γ) it remains there an exponential time whose mean is of order $e^{2\beta}$. Therefore, $\ell^2\theta_\beta e^{-2\beta}$ represents the number of excursions between two

consecutive visits to Γ . In condition (1.8), one ℓ has been replaced by L in the argument due to our incapacity to estimate sharply the hitting probabilities (1.13).

The main idea of the article consists in taking a temperature low enough, with respect to n and L , to be allowed to discard all jumps of lower order in β . For example, a square configuration has 8 jumps whose rates are $e^{-2\beta}$ and $4(n-2)$ jumps whose rates are $e^{-3\beta}$. Hence, if $\ell^2\theta_\beta e^{-2\beta} n e^{-\beta}$ vanishes as $\beta \uparrow \infty$, a particle which is not at the corner of a square jumps in a time interval $[0, t\ell^2\theta_\beta]$, $t > 0$, with a probability converging to 0.

If we discard such jumps, from a square the process $\eta(t)$ can only reach a configuration in which a particle at the corner of the square has detached itself from the square. At this point, the detached particle performs a rate one symmetric random walk on \mathbb{T}_L until it reaches a side of this square. While this particle moves, other particles may also jump. There are a few rate $e^{-\beta}$ and rate $e^{-2\beta}$ jumps and $O(n)$ rate $e^{-3\beta}$ jumps. In order to apply the approach presented here, we need to guarantee that the detached particle returns to the square before any another particle moves. We have seen that a particle in the square jumps at rate at most $e^{-\beta}$ while the detached particle returns to the square in a time of order L^2 . Hence, the probability that a lower order jump occurs before the detached particle returns should be of order $L^2 e^{-\beta}$. We are only able to estimate this probability, in Proposition 4.2, by $Le^{-\beta/2}$. Thus, as in the previous paragraph, to guarantee that no lower order jump occurs in the time scale in which the center of mass evolves we need to impose that $\ell^2\theta_\beta e^{-2\beta} Le^{-\beta/2}$ vanishes. This explains the factor $Le^{-\beta/2}$ appearing in the error term $\mathbf{e}(\beta)$ introduced in (1.5).

Proceeding in this way, we face many cases presented in Sections 2, 3 and 4. The error term $\mathbf{e}(\beta)$ summarizes all the constraints, it represents the bound obtained in Proposition 5.2 for the probability that a lower order jump occurs before the process returns to a square configuration when it starts from a square configuration. This estimate is not sharp and the conditions on the growth of L and n can certainly be improved.

Condition (1.9) is spurious and comes from the fact that we are not able to estimate correctly the hitting probabilities appearing in (1.13). It can certainly be suppressed, but its removal depends on the estimation of hitting probabilities of a seven dimensional random walk.

Finally, note that in Theorems 1.1 and 1.3, n has to be large, but does not need to increase with β .

REMARK 1.6. The approach presented in this article can be applied to the case in which the total number of particles is $n(n+k)$, for some integer k independent of β , and the particles of the initial configuration form a $n \times (n+k)$ rectangle. The energy landscape becomes more complex for k large and computations harder. For example, for $k=2$, a $(n+1) \times (n+1)$ square without a particle at a corner is a ground state.

Since its origins, the Ising model has been known to properly represent the condensation phenomena in two-dimensional systems formed by the adsorption of gases on the surfaces of crystals [24, 25]. There is a huge literature on this subject. In this perspective, the present work can be seen as an investigation of the evolution of the condensate.

The metastable behavior of the Ising lattice gas evolving under the nonconservative Glauber dynamics at very low temperature has been examined 20 years ago and is well understood [23]. In contrast, there are not many results on the asymptotic behavior of the Ising lattice gas under the conservative Kawasaki dynamics at very low temperature.

This problem has been examined in two situations. Den Hollander et al. [9] and Gaudillière et al. [16] described the critical droplet, the nucleation time and the typical trajectory followed by the process during the transition from a metastable set to the stable set in a two-dimensional Ising lattice gas evolving under the Kawasaki dynamics at very low temperature and very low density in a finite square in which particles are created and destroyed at the boundary. This result has been extended to the anisotropic case by Nardi et al. [22] and to three dimensions by den Hollander et al. [8]. Bovier et al. [6] presented the detailed geometry of the set of critical droplets and provided sharp estimates for the expectation of the nucleation time for this model in dimension two and three.

More recently, Gaudillière et al. [15] proved that the dynamics of particles evolving according to the Kawasaki dynamics at very low temperature and very low density in a two-dimensional torus, whose length increases as the temperature decreases, can be approximated by the evolution of independent particles. Bovier et al. [7] obtain sharp estimates for the expectation of the nucleation time for this model where the length of the square increases as the temperature vanishes.

Finally, Funaki [12, 13] examined, in the zero-temperature limit, the motion of a rigid body formed by a system of interacting Brownian particles with a pairwise potential in \mathbb{R}^d , and the nucleation in the one-dimensional case.

The article is divided as follows. In the next section, we introduce a subset Ξ of $\Omega_{L,K}$. In Sections 3 and 4, we obtain sharp estimates for the jump rates of $\zeta(t)$, the trace of $\eta(t)$ on Ξ , and we introduce a process $\widehat{\zeta}(t)$ whose rate jumps are close to the ones of $\zeta(t)$. In Section 5, we couple $\zeta(t)$ and $\widehat{\zeta}(t)$ and we obtain an estimate on the time these processes remain coupled. In Section 6, we prove Theorem 1.1, in Section 7, Theorem 1.2, and in Section 8, Theorem 1.3. We conclude the article in Section 9 with estimates on the time-scale θ_β .

2. The subset of configurations Ξ . We introduce in this section a subset Ξ of \mathbb{H}_{01} , and we estimate, in the next section, the jump rates of the trace of $\eta(t)$ on Ξ .

2.1. *The configurations $\eta_{\mathbf{x}}^{i,j}$.* For a subset B of \mathbb{T}_L , denote by $\partial_- B, \partial_+ B$ the inner, outer boundary of B , respectively. These are the set of sites which are at distance one from B^c, B , respectively,

$$\begin{aligned} \partial_- B &= \{\mathbf{x} \in B : \exists \mathbf{y} \in B^c, |\mathbf{y} - \mathbf{x}| = 1\}, \\ \partial_+ B &= \{\mathbf{x} \in \mathbb{T}_L \setminus B : \exists \mathbf{y} \in B, |\mathbf{y} - \mathbf{x}| = 1\}, \end{aligned}$$

where $|\cdot|$ stands for the sum norm, $\mathbf{x} = (x_1, x_2), |\mathbf{x}| = |x_1| + |x_2|$.

Let $Q^i = Q \setminus \{\mathbf{w}_i\}, 0 \leq i \leq 3$, where

$$\begin{aligned} \mathbf{w}_0 = \mathbf{w} &= (0, 0), & \mathbf{w}_1 &= (n - 1, 0), \\ \mathbf{w}_2 &= (n - 1, n - 1), & \mathbf{w}_3 &= (0, n - 1) \end{aligned}$$

are the corners of the square Q . For $\mathbf{x} \in \mathbb{T}_L, 0 \leq i \leq 3$, let $\mathbf{x}_i = \mathbf{x} + \mathbf{w}_i, Q_{\mathbf{x}}^i = \mathbf{x} + Q^i$. We refer to the sets $Q_{\mathbf{x}}^i$ as quasi-squares, a square with a corner missing.

Let $e_1 = (1, 0), e_2 = (0, 1)$ be the canonical basis of \mathbb{R}^2 , and denote by $\partial_j Q_{\mathbf{x}}^i, 0 \leq j \leq 3$, the j th outer boundary of $Q_{\mathbf{x}}^i$:

$$\begin{aligned} \partial_j Q_{\mathbf{x}}^i &= \{\mathbf{z} \in \partial_+ Q_{\mathbf{x}}^i : \exists \mathbf{y} \in Q_{\mathbf{x}}^i; \mathbf{y} - \mathbf{z} = (1 - j)e_2\}, & j &= 0, 2, \\ \partial_j Q_{\mathbf{x}}^i &= \{\mathbf{z} \in \partial_+ Q_{\mathbf{x}}^i : \exists \mathbf{y} \in Q_{\mathbf{x}}^i; \mathbf{y} - \mathbf{z} = (j - 2)e_1\}, & j &= 1, 3. \end{aligned}$$

Let $Q_{\mathbf{x}}^{i,j} = \partial_j Q_{\mathbf{x}}^i \setminus Q_{\mathbf{x}}, 0 \leq i, j \leq 3, \mathbf{x} \in \mathbb{T}_L$, let $\mathcal{E}_{\mathbf{x}}^{i,j}$ be the set of configurations in which all sites of the set $Q_{\mathbf{x}}^i$ are occupied with an extra particle at some location of $Q_{\mathbf{x}}^{i,j}$ and let $\Omega^1 = \Omega_{L,K}^1$ be the set of all such configurations

$$\mathcal{E}_{\mathbf{x}}^{i,j} = \{\sigma^{\mathbf{x}_i, \mathbf{z}} \eta^{\mathbf{x}} : \mathbf{z} \in Q_{\mathbf{x}}^{i,j}\}, \quad \Omega^1 = \bigcup_{\mathbf{x} \in \mathbb{T}_L} \bigcup_{i,j=0}^3 \mathcal{E}_{\mathbf{x}}^{i,j}.$$

We proved in [5] that for any configuration $\xi \in \mathcal{E}_{\mathbf{x}}^{i,j}$, the triple $(\mathcal{E}_{\mathbf{x}}^{i,j}, \mathcal{E}_{\mathbf{x}}^{i,j} \cup \Delta_1, \xi)$ is a valley of depth e^β in the sense of [1], Definition 2.1. To select a representative configuration of the well $\mathcal{E}_{\mathbf{x}}^{i,j}$, denote by $\hat{\mathbf{w}}_j, 0 \leq j \leq 3$, the point in the inner boundary of Q whose distance to \mathbf{w}_j and to \mathbf{w}_{j+1} is equal to $(n - 1)/2$ if n is odd, and whose distance to $\mathbf{w}_j, \mathbf{w}_{j+1}$ is equal to $(n/2) - 1, n/2$, respectively, if n is even. Let \mathbf{w}_j^* the site on the outer boundary of Q at distance one from $\hat{\mathbf{w}}_j$, and let $\eta_{\mathbf{x}}^{i,j}, \mathbf{x} \in \mathbb{T}_L$, be the configuration $\sigma^{\mathbf{x} + \mathbf{w}_i, \mathbf{x} + \mathbf{w}_j^*} \eta^{\mathbf{x}}$. Denote by $\hat{\Omega}^1$ the space of such configurations

$$\hat{\Omega}_{\mathbf{x}}^1 = \{\eta_{\mathbf{x}}^{i,j} : 0 \leq i, j \leq 3\}, \quad \hat{\Omega}^1 = \bigcup_{\mathbf{x} \in \mathbb{T}_L} \hat{\Omega}_{\mathbf{x}}^1.$$

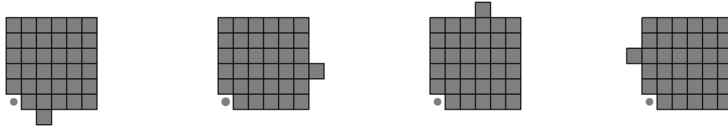


FIG. 3. The configurations $\eta_{\mathbf{x}}^{0,0}, \eta_{\mathbf{x}}^{0,1}, \eta_{\mathbf{x}}^{0,2}$ and $\eta_{\mathbf{x}}^{0,3}$ for $n = 6$. The gray dot indicates the site \mathbf{x} . We placed a square $[-1/2, 1/2]^2$ around each particle.

2.2. The configurations $\eta_{\mathbf{x}}^{\mathfrak{a},(\mathbf{k},\ell)}$. Let $R^{\mathfrak{l}}, R^{\mathfrak{s}}$ be the rectangles $R^{\mathfrak{l}} = \{1, \dots, n - 1\} \times \{1, \dots, n - 2\}$, $R^{\mathfrak{s}} = \{1, \dots, n - 2\} \times \{1, \dots, n - 1\}$, where \mathfrak{l} stands for lying and \mathfrak{s} for standing. Let $n_0^{\mathfrak{s}} = n_2^{\mathfrak{s}} = n - 2$, $n_1^{\mathfrak{s}} = n_3^{\mathfrak{s}} = n - 1$ be the length of the sides of the standing rectangle $R^{\mathfrak{s}}$. Similarly, denote by $n_i^{\mathfrak{l}}, 0 \leq i \leq 3$, the length of the sides of the lying rectangle $R^{\mathfrak{l}}$: $n_i^{\mathfrak{l}} = n_{i+1}^{\mathfrak{s}}$, where the sum over the index i is performed modulo 4.

Denote by $\mathbb{I}_{\mathfrak{a}}$, $\mathfrak{a} \in \{\mathfrak{s}, \mathfrak{l}\}$, the set of pairs $(\mathbf{k}, \ell) = (k_0, \ell_0; k_1, \ell_1; k_2, \ell_2; k_3, \ell_3)$ such that:

- $0 \leq k_i \leq \ell_i \leq n_i^{\mathfrak{a}}$,
- if $k_j = 0$, then $\ell_{j-1} = n_{j-1}^{\mathfrak{a}}$.

For $(\mathbf{k}, \ell) \in \mathbb{I}_{\mathfrak{a}}$, $\mathfrak{a} \in \{\mathfrak{s}, \mathfrak{l}\}$, let $R^{\mathfrak{l}}(\mathbf{k}, \ell), R^{\mathfrak{s}}(\mathbf{k}, \ell)$ be the sets

$$\begin{aligned}
 R^{\mathfrak{l}}(\mathbf{k}, \ell) &= R^{\mathfrak{l}} \cup \{(a, 0) : k_0 \leq a \leq \ell_0\} \cup \{(n, b) : k_1 \leq b \leq \ell_1\} \\
 &\quad \cup \{(n - a, n - 1) : k_2 \leq a \leq \ell_2\} \cup \{(0, n - 1 - b) : k_3 \leq b \leq \ell_3\}, \\
 R^{\mathfrak{s}}(\mathbf{k}, \ell) &= R^{\mathfrak{s}} \cup \{(a, 0) : k_0 \leq a \leq \ell_0\} \cup \{(n - 1, b) : k_1 \leq b \leq \ell_1\} \\
 &\quad \cup \{(n - 1 - a, n) : k_2 \leq a \leq \ell_2\} \cup \{(0, n - b) : k_3 \leq b \leq \ell_3\}.
 \end{aligned}$$

Note that a hole between particles on a side of a rectangle is not allowed in the sets $R^{\mathfrak{a}}(\mathbf{k}, \ell), R^{\mathfrak{s}}(\mathbf{k}, \ell)$.

Denote by $I_{\mathfrak{a}}$, $\mathfrak{a} \in \{\mathfrak{s}, \mathfrak{l}\}$, the set of pairs $(\mathbf{k}, \ell) \in \mathbb{I}_{\mathfrak{a}}$ such that $|R^{\mathfrak{a}}(\mathbf{k}, \ell)| = n^2$. For $(\mathbf{k}, \ell) \in I_{\mathfrak{a}}$, denote by $M_i(\mathbf{k}, \ell)$ the number of particles attached to the side i of the rectangle $R^{\mathfrak{a}}(\mathbf{k}, \ell)$:

$$M_i(\mathbf{k}, \ell) = \begin{cases} \ell_i - k_i + 1, & \text{if } k_{i+1} \geq 1, \\ \ell_i - k_i + 2, & \text{if } k_{i+1} = 0. \end{cases}$$

Clearly, for $(\mathbf{k}, \ell) \in I_{\mathfrak{a}}$, $\sum_{0 \leq i \leq 3} M_i(\mathbf{k}, \ell) = 3n - 2 + A$, where A is the number of occupied corners, which are counted twice since they are attached to two sides.

Denote by $I_{\mathfrak{a}}^* \subset I_{\mathfrak{a}}$, the set of pairs $(\mathbf{k}, \ell) \in I_{\mathfrak{a}}$ whose rectangles $R^{\mathfrak{a}}(\mathbf{k}, \ell)$ have at least two particles on each side: $M_i(\mathbf{k}, \ell) \geq 2, 0 \leq i \leq 3$. Note that if (\mathbf{k}, ℓ) belongs to $I_{\mathfrak{a}}^*$, for all $\mathbf{x} \in R^{\mathfrak{a}}(\mathbf{k}, \ell)$, there exist $\mathbf{y}, \mathbf{z} \in R^{\mathfrak{a}}(\mathbf{k}, \ell), \mathbf{y} \neq \mathbf{z}$, with the property $|\mathbf{x} - \mathbf{y}| = |\mathbf{x} - \mathbf{z}| = 1$.

For $(\mathbf{k}, \ell) \in I_{\mathfrak{a}}, \mathfrak{a} \in \{\mathfrak{s}, \mathfrak{l}\}, \mathbf{x} \in \mathbb{T}_L$, let $R_{\mathbf{x}}^{\mathfrak{a}}(\mathbf{k}, \ell) = \mathbf{x} + R^{\mathfrak{a}}(\mathbf{k}, \ell)$, and let $\eta_{\mathbf{x}}^{\mathfrak{a},(\mathbf{k},\ell)}$ represent the configurations defined by

$$\eta_{\mathbf{x}}^{\mathfrak{a},(\mathbf{k},\ell)}(\mathbf{y}) = 1 \quad \text{if and only if } \mathbf{y} \in R_{\mathbf{x}}^{\mathfrak{a}}(\mathbf{k}, \ell).$$

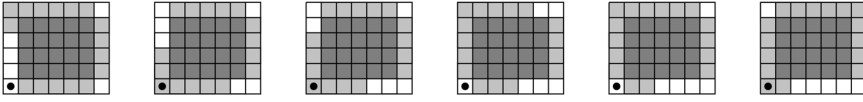


FIG. 4. Some configurations $\eta_{\mathbf{x}}^{l,(\mathbf{k},\ell)}$ for $n = 6$. The first one corresponds to the vector $(\mathbf{k}, \ell) = ((1, 5); (1, 4); (1, 6); (0, 1))$ and the last one to the vector $(\mathbf{k}, \ell) = ((0, 1); (1, 5); (0, 5); (1, 5))$. The inner gray rectangle represents the set $\mathbf{x} + R^1$ and the black dot the site \mathbf{x} .

Figure 4 presents examples of configurations $\eta_{\mathbf{x}}^{l,(\mathbf{k},\ell)}$. The configurations $\eta_{\mathbf{x}}^{a,(\mathbf{k},\ell)}$, $(\mathbf{k}, \ell) \in I_a$, belong to \mathbb{H}_1 . Moreover, the configurations $\eta_{\mathbf{x}}^{a,(\mathbf{k},\ell)}$, $(\mathbf{k}, \ell) \in I_a \setminus I_a^*$, belong to Ω^1 or form a $(n - 1) \times (n + 1)$ rectangle of particles with one extra particle attached to a side of length $n + 1$. Let $\Omega^2 = \Omega_{L,K}^2$, be the set of configurations associated to the pairs (\mathbf{k}, ℓ) in I_a^* :

$$\Omega_{\mathbf{x}}^2 = \{\eta_{\mathbf{x}}^{a,(\mathbf{k},\ell)} : a \in \{\mathfrak{s}, \mathfrak{l}\}, (\mathbf{k}, \ell) \in I_a^*\}, \quad \Omega^2 = \bigcup_{\mathbf{x} \in \mathbb{T}_L} \Omega_{\mathbf{x}}^2.$$

2.3. The configurations $\zeta_{\mathbf{x}}^{a,j}$. Let $T^{\mathfrak{l}}, T^{\mathfrak{s}}$ be the rectangles $T^{\mathfrak{l}} = \{0, \dots, n\} \times \{0, \dots, n - 2\}$, $T^{\mathfrak{s}} = \{0, \dots, n - 2\} \times \{0, \dots, n\}$, where \mathfrak{l} stands for lying and \mathfrak{s} for standing. Denote by $T_{\mathbf{x}}^a$, $a \in \{\mathfrak{s}, \mathfrak{l}\}$, $\mathbf{x} \in \mathbb{T}_L$, the rectangle T^a translated by \mathbf{x} : $T_{\mathbf{x}}^a = \mathbf{x} + T^a$, and by $\eta_{\mathbf{x}}^{\mathbf{x}}$ the configuration in which all sites of $T_{\mathbf{x}}^a$ are occupied. Note that $\eta_{\mathbf{x}}^{\mathbf{x}}$ belongs to $\Omega_{L,K-1}$ and not to $\Omega_{L,K}$.

For $a \in \{\mathfrak{s}, \mathfrak{l}\}$, $\mathbf{x} \in \mathbb{T}_L$, $\mathbf{z} \in \partial_+ T_{\mathbf{x}}^a$, denote by $\eta_{\mathbf{x}}^{\mathbf{x},\mathbf{z}}$ the configuration in which all sites of the rectangle $T_{\mathbf{x}}^a$ and the site \mathbf{z} are occupied: $\eta_{\mathbf{x}}^{\mathbf{x},\mathbf{z}} = \eta_{\mathbf{x}}^{\mathbf{x}} + \mathfrak{d}_{\mathbf{z}}$, where \mathfrak{d}_y , $y \in \mathbb{T}_L$, represents the configuration with a unique particle at y and summation of configurations is performed componentwise.

Denote by $\partial_j T_{\mathbf{x}}^a$, $0 \leq j \leq 3$, the j th outer boundary of $T_{\mathbf{x}}^a$:

$$\begin{aligned} \partial_j T_{\mathbf{x}}^a &= \{\mathbf{z} \in \partial_+ T_{\mathbf{x}}^a : \exists \mathbf{y} \in T_{\mathbf{x}}^a; \mathbf{y} - \mathbf{z} = (1 - j)\mathbf{e}_2\}, & j = 0, 2, \\ \partial_j T_{\mathbf{x}}^a &= \{\mathbf{z} \in \partial_+ T_{\mathbf{x}}^a : \exists \mathbf{y} \in T_{\mathbf{x}}^a; \mathbf{y} - \mathbf{z} = (j - 2)\mathbf{e}_1\}, & j = 1, 3. \end{aligned}$$

Let $\mathcal{E}_{\mathbf{x}}^{a,j}$ be the set of configurations in which all sites of the set $T_{\mathbf{x}}^a$ are occupied with an extra particle at some location of $\partial_j T_{\mathbf{x}}^a$ and let $\Omega^3 = \Omega_{L,K}^3$ be the set of all such configurations:

$$\mathcal{E}_{\mathbf{x}}^{a,j} = \{\eta_{\mathbf{x}}^{\mathbf{x},\mathbf{z}} : \mathbf{z} \in \partial_j T_{\mathbf{x}}^a\}, \quad \Omega^3 = \bigcup_{\mathbf{x} \in \mathbb{T}_L} \bigcup_{a=\mathfrak{s},\mathfrak{l}} \bigcup_{j=0}^3 \mathcal{E}_{\mathbf{x}}^{a,j}.$$

We proved in [5] that for any configuration $\xi \in \mathcal{E}_{\mathbf{x}}^{a,j}$, the triple $(\mathcal{E}_{\mathbf{x}}^{a,j}, \mathcal{E}_{\mathbf{x}}^{a,j} \cup \Delta_1, \xi)$ is a valley of depth e^β . We select a representative of the well $\mathcal{E}_{\mathbf{x}}^{a,j}$.

Fix $a = \mathfrak{s}, \mathfrak{l}$, and denote by $\mathbf{w}_{\mathfrak{l},j}^*$, $0 \leq j \leq 3$, the mid-points of the outer boundary of T^a . Hence, if n is even $\mathbf{w}_{\mathfrak{l},0}^* = (n/2, -1)$, $\mathbf{w}_{\mathfrak{l},1}^* = (n + 1, (n - 2)/2)$, $\mathbf{w}_{\mathfrak{l},2}^* = (n/2, n - 1)$, $\mathbf{w}_{\mathfrak{l},3}^* = (-1, (n - 2)/2)$, and if n is odd $\mathbf{w}_{\mathfrak{l},0}^* = ((n - 1)/2, -1)$, $\mathbf{w}_{\mathfrak{l},1}^* = (n + 1, (n - 3)/2)$, $\mathbf{w}_{\mathfrak{l},2}^* = ((n + 1)/2, n - 1)$, $\mathbf{w}_{\mathfrak{l},3}^* = (-1, (n - 1)/2)$, with

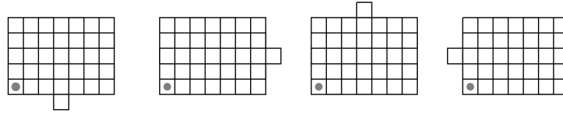


FIG. 5. The configurations $\zeta_{\mathbf{x}}^{l,i}$, $0 \leq i \leq 3$, for $n = 6$. The gray dot represents the site \mathbf{x} .

a similar definition when \mathfrak{s} replaces l . Let $\zeta_{\mathbf{x}}^{a,j}$, $\mathbf{x} \in \mathbb{T}_L$, $a \in \{\mathfrak{s}, l\}$, $0 \leq j \leq 3$, be the configuration $\eta_{\mathbf{a}}^{\mathbf{x}} + \partial_{\mathbf{x}+\mathbf{w}_{\mathbf{a}}^*}$ and denote by $\widehat{\Omega}^3$ the space of such configurations:

$$\widehat{\Omega}_{\mathbf{x}}^3 = \{\zeta_{\mathbf{x}}^{a,j} : a \in \{\mathfrak{s}, l\}, 0 \leq j \leq 3\}, \quad \widehat{\Omega}^3 = \bigcup_{\mathbf{x} \in \mathbb{T}_L} \widehat{\Omega}_{\mathbf{x}}^3.$$

2.4. The configurations $\zeta_{\mathbf{x}}^{a,(\mathbf{k},\ell)}$. Let $R^{2,l}$, $R^{2,\mathfrak{s}}$ be the rectangles $R^{2,l} = \{1, \dots, n\} \times \{1, \dots, n-3\}$, $R^{2,\mathfrak{s}} = \{1, \dots, n-3\} \times \{1, \dots, n\}$. Let $n_0^{2,\mathfrak{s}} = n_2^{2,\mathfrak{s}} = n-3$, $n_1^{2,\mathfrak{s}} = n_3^{2,\mathfrak{s}} = n$ be the length of the sides of the standing rectangle $R^{2,\mathfrak{s}}$. Similarly, denote by $n_i^{2,l}$, $0 \leq i \leq 3$, the length of the sides of the lying rectangle $R^{2,l}$: $n_i^{2,l} = n_{i+1}^{2,\mathfrak{s}}$, where the sum over the index i is performed modulo 4.

Denote by $\mathbb{I}_{2,a}$, $a \in \{\mathfrak{s}, l\}$, the set of pairs (\mathbf{k}, ℓ) such that:

- $0 \leq k_i \leq \ell_i \leq n_i^{2,a}$,
- if $k_j = 0$, then $\ell_{j-1} = n_{j-1}^{2,a}$.

For $(\mathbf{k}, \ell) \in \mathbb{I}_{2,a}$, $a \in \{\mathfrak{s}, l\}$, let $R^{2,l}(\mathbf{k}, \ell)$, $R^{2,\mathfrak{s}}(\mathbf{k}, \ell)$ be the sets

$$\begin{aligned} R^{2,l}(\mathbf{k}, \ell) &= R^{2,l} \cup \{(a, 0) : k_0 \leq a \leq \ell_0\} \cup \{(n+1, b) : k_1 \leq b \leq \ell_1\} \\ &\quad \cup \{(n+1-a, n-2) : k_2 \leq a \leq \ell_2\} \cup \{(0, n-2-b) : k_3 \leq b \leq \ell_3\}, \\ R^{2,\mathfrak{s}}(\mathbf{k}, \ell) &= R^{2,\mathfrak{s}} \cup \{(a, 0) : k_0 \leq a \leq \ell_0\} \cup \{(n-2, b) : k_1 \leq b \leq \ell_1\} \\ &\quad \cup \{(n-2-a, n+1) : k_2 \leq a \leq \ell_2\} \cup \{(0, n+1-b) : k_3 \leq b \leq \ell_3\}. \end{aligned}$$

Denote by $I_{2,a}$, $a \in \{\mathfrak{s}, l\}$, the set of pairs $(\mathbf{k}, \ell) \in \mathbb{I}_{2,a}$ such that $|R^{2,a}(\mathbf{k}, \ell)| = n^2$. For $(\mathbf{k}, \ell) \in I_{2,a}$, denote by $M_i^{2,a}(\mathbf{k}, \ell)$ the number of particles attached to the side i of the rectangle $R^{2,a}(\mathbf{k}, \ell)$:

$$M_i^{2,a}(\mathbf{k}, \ell) = \begin{cases} \ell_i - k_i + 1, & \text{if } k_{i+1} \geq 1, \\ \ell_i - k_i + 2, & \text{if } k_{i+1} = 0. \end{cases}$$

Clearly, for $(\mathbf{k}, \ell) \in I_{2,a}$, $\sum_{0 \leq i \leq 3} M_i^{2,a}(\mathbf{k}, \ell) = 3n + A$, where A is the number of occupied corners, which are counted twice since they are attached to two sides.

Denote by $I_{2,a}^* \subset I_{2,a}$, the set of pairs $(\mathbf{k}, \ell) \in I_{2,a}$ whose rectangles $R^{2,a}(\mathbf{k}, \ell)$ have at least two particles on each side: $M_i^{2,a}(\mathbf{k}, \ell) \geq 2$, $0 \leq i \leq 3$. Note that if

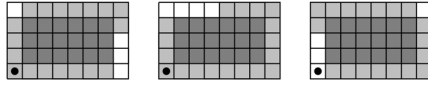


FIG. 6. Examples of configurations in Ω_x^4 for $n = 6$. In general, $3n$ particles (or $n - 2$ holes) have to be placed around the rectangle, respecting the constraints introduced above. The black dot represents the site x .

(\mathbf{k}, ℓ) belongs to $I_{2,\alpha}^*$, for all $x \in R^{2,\alpha}(\mathbf{k}, \ell)$, there exist $y, z \in R^{2,\alpha}(\mathbf{k}, \ell)$, $y \neq z$, with the property $|x - y| = |x - z| = 1$.

For $(\mathbf{k}, \ell) \in I_{2,\alpha}$, $\alpha \in \{s, l\}$, $x \in \mathbb{T}_L$, let $R_x^{2,\alpha}(\mathbf{k}, \ell) = x + R^{2,\alpha}(\mathbf{k}, \ell)$, and let $\zeta_x^{\alpha,(\mathbf{k},\ell)}$ represent the configurations defined by

$$\zeta_x^{\alpha,(\mathbf{k},\ell)}(y) = 1 \quad \text{if and only if } y \in R_x^{2,\alpha}(\mathbf{k}, \ell).$$

Figure 6 presents examples of configurations $\zeta_x^{\alpha,(\mathbf{k},\ell)}$. The configurations $\zeta_x^{\alpha,(\mathbf{k},\ell)}$, $(\mathbf{k}, \ell) \in I_{2,\alpha}$, have at least four particles attached to the longer side, and the configurations $\zeta_x^{\alpha,(\mathbf{k},\ell)}$, $(\mathbf{k}, \ell) \in I_{2,\alpha} \setminus I_{2,\alpha}^*$, belong to Ω^3 , forming a $(n - 1) \times (n + 1)$ rectangle of particles with one extra particle attached to a side of length $n - 1$. Let $\Omega^4 = \Omega_{L,K}^4$, be the set of configurations associated to the pairs (\mathbf{k}, ℓ) in $I_{2,\alpha}^*$:

$$\Omega_x^4 = \{\zeta_x^{\alpha,(\mathbf{k},\ell)} : \alpha \in \{s, l\}, (\mathbf{k}, \ell) \in I_{2,\alpha}^*\}, \quad \Omega^4 = \bigcup_{x \in \mathbb{T}_L} \Omega_x^4.$$

Let $\Xi = \Xi(L, n)$ be the set of configurations

$$\Xi = \Gamma \cup \widehat{\Omega}^1 \cup \Omega^2 \cup \widehat{\Omega}^3 \cup \Omega^4.$$

The set Ξ has less than $C_0 L^2 n^7$ configurations, the main contributions coming from Ω^2 and Ω^4 :

$$(2.1) \quad |\Xi| \leq C_0 L^2 n^7.$$

3. The trace of $\eta(t)$ on Ξ . We define in this section the trace of $\eta(t)$ on Ξ , denoted by $\zeta(t)$, and we introduce a process $\widehat{\zeta}(t)$ which approximates well the trace process. The parameters n and L are fixed and $\beta \uparrow \infty$.

Denote by $\zeta(t)$ the trace of the process $\eta(t)$ on Ξ . By Propositions 6.1 and 6.3 in [1], $\zeta(t)$ is a irreducible, reversible process on Ξ , whose stationary measure, denoted by μ_ζ , is the measure μ_K conditioned to Ξ : $\mu_\zeta(\xi) = \mu_K(\xi) / \mu_K(\Xi) = \mu_\beta(\xi) / \mu_\beta(\Xi)$, $\xi \in \Xi$.

For a subset Π of Ξ , denote by H_Π (resp., H_Π^+) the hitting time of Π (resp., the return time to Π):

$$(3.1) \quad \begin{aligned} H_\Pi &:= \inf\{t > 0 : \zeta(t) \in \Pi\}, \\ H_\Pi^+ &:= \inf\{t > 0 : \zeta(t) \in \Pi, \zeta(s) \neq \zeta(0) \text{ for some } 0 < s < t\}. \end{aligned}$$

Denote by $R_\zeta(\eta, \xi)$ the jump rates of the trace process $\zeta(t)$, and by L_ζ its generator:

$$(L_\zeta f)(\eta) = \sum_{\xi \in \Xi} R_\zeta(\eta, \xi)[f(\xi) - f(\eta)].$$

Let $\mathbf{P}_\xi^\zeta, \xi \in \Xi$, be the probability measure on the path space $D(\mathbb{R}_+, \Xi)$ induced by the Markov process $\zeta(t)$ starting from ξ . Expectation with respect to \mathbf{P}_ξ^ζ is represented by \mathbf{E}_ξ^ζ . For two disjoint subsets A, B of Ξ , denote by $\text{cap}_\zeta(A, B)$ the capacity between A and B for the $\zeta(t)$ process

$$\text{cap}_\zeta(A, B) = \sum_{\eta \in A} \mu_\zeta(\eta) \lambda_\zeta(\eta) \mathbf{P}_\eta^\zeta[H_B < H_A^+],$$

where $\lambda_\zeta(\eta)$ stands for the holding rates of the process $\zeta(t)$, $\lambda_\zeta(\eta) = \sum_{\xi \in \Xi} R_\zeta(\eta, \xi)$.

We will introduce below several Markov processes obtained from the trace process $\zeta(t)$, either by reflecting it on a subset of Ξ or by modifying its jump rates. All processes will be denoted by some Greek letter with, sometimes, an accent. If the process is represented by χ , the jump rates, the invariant measure, the holding rates, and the capacities are denoted by $R_\chi, \mu_\chi, \lambda_\chi$ and cap_χ , respectively.

The rates $\mathbf{R}(\eta, \xi)$. The trace process $\zeta(t)$ may jump from any configuration of Ξ to any other configuration. Most of these jumps, however, have very small rates when β is large. The main results of the next section assert that in the zero-temperature limit the process $\zeta(t)$ is very well approximated by a simple Markov process denoted by $\hat{\zeta}(t)$. We define in this subsection the jump rates $\mathbf{R}(\cdot, \cdot)$ of $\hat{\zeta}(t)$ and derive in the next subsection some properties of \mathbf{R} .

Let $\{\mathbf{x}_t : t \geq 0\}$ be the nearest-neighbor, symmetric random walk on \mathbb{T}_L which jumps from a site \mathbf{x} to $\mathbf{x} \pm e_i$ at rate 1. Denote by $\mathbb{P}_\mathbf{y}^\mathbf{x}, \mathbf{y} \in \mathbb{T}_L$, the probability measure on $D(\mathbb{R}_+, \mathbb{T}_L)$ induced by \mathbf{x}_t starting from \mathbf{y} . Denote by $\mathfrak{p}(\mathbf{x}, \mathbf{y}, G), \mathbf{x} \in \mathbb{T}_L, \mathbf{y} \in G, G \subset \mathbb{T}_L$, the probability that the random walk starting from \mathbf{x} reaches G at \mathbf{y} :

$$(3.2) \quad \mathfrak{p}(\mathbf{y}, \mathbf{z}, G) := \mathbb{P}_\mathbf{y}^\mathbf{x}[\mathbf{x}_{H_G} = \mathbf{z}].$$

By extension, for a subset \mathbf{A} of \mathbb{T}_L , let $\mathfrak{p}(\mathbf{x}, \mathbf{A}, G) = \sum_{\mathbf{y} \in \mathbf{A}} \mathfrak{p}(\mathbf{x}, \mathbf{y}, G)$. Moreover, when $G = \partial_+ Q$, we omit the set G in the notation

$$\mathfrak{p}(\mathbf{x}, \mathbf{A}) := \mathfrak{p}(\mathbf{x}, \mathbf{A}, \partial_+ Q).$$

Let $\mathbf{y}_t = (\mathbf{y}_t^1, \mathbf{y}_t^2)$ be the continuous-time Markov chain on $D_n = \{(j, k) : 0 \leq j < k \leq n - 1\} \cup \{(0, 0)\}$ which jumps from a site \mathbf{y} to any of its nearest neighbor sites $\mathbf{z}, |\mathbf{y} - \mathbf{z}| = 1$, at rate 1. Let $D_n^o = \{(0, 0)\} \cup \{(j, n - 1) : 0 \leq j < n - 1\}$ and let

$$(3.3) \quad \mathfrak{q}_n = \mathbb{P}_{(0,1)}^\mathbf{y}[H_{D_n^o} < H_{(0,0)}],$$

where $\mathbb{P}_{(0,1)}^y$ stands for the distribution of \mathbf{y}_t starting from $(0, 1)$.

Let $\mathbf{z}_t = (\mathbf{z}_t^1, \mathbf{z}_t^2)$ be the continuous-time Markov chain on $E_n = \{0, \dots, n - 1\}^2 \cup \{\partial\}$ which jumps from a site \mathbf{z} to any of its nearest neighbor sites \mathbf{z}' , $|\mathbf{z}' - \mathbf{z}| = 1$, at rate 1, and which jumps from $(1, 1)$ (resp., from ∂) to ∂ [resp., to $(1, 1)$] at rate 1. Let $E_n^+ = \{(j, n - 1) : 0 \leq j \leq n - 1\}$, $E_n^- = \{(j, 0) : 1 \leq j \leq n - 1\}$, $\partial E_n = E_n^+ \cup E_n^- \cup \{(0, 0)\}$ and for $0 \leq k \leq n - 1$, let

$$(3.4) \quad \begin{aligned} \tau_n^+ &= \mathbb{P}_{(0,1)}^z[H_{\partial E_n} = H_{E_n^+}], & \tau_n^- &= \mathbb{P}_{(0,1)}^z[H_{\partial E_n} = H_{E_n^-}], \\ \tau_n^0(k) &= \mathbb{P}_{(k,1)}^z[H_{\partial E_n} = H_{(0,0)}], & \tau_n &= \tau_n^+ + \tau_n^-, \end{aligned}$$

where $\mathbb{P}_{(k,1)}^z$ stands for the distribution of \mathbf{z}_t starting from $(k, 1)$.

We are finally in a position to define $\mathbf{R}(\eta^x, \xi)$. Let

$$(3.5) \quad \mathbf{R}(\eta^0, \eta_0^{0,0}) = \frac{1 + \mathfrak{A}_{0,3} + \tau_n^- + q_n}{4 + q_n + \tau_n - \mathfrak{A}}, \quad \mathbf{R}(\eta^0, \eta_0^{0,1}) = \frac{\mathfrak{A}_{1,2} + \tau_n^+}{4 + q_n + \tau_n - \mathfrak{A}},$$

where

$$\begin{aligned} \mathfrak{A} &= p(\mathbf{w}_0 - 2e_2, \{\mathbf{w}_0 - e_2, \mathbf{w}_0 - e_1\}) + p(\mathbf{w}_0 - e_1 - e_2, \{\mathbf{w}_0 - e_2, \mathbf{w}_0 - e_1\}), \\ \mathfrak{A}_{0,3} &= p(\mathbf{w}_0 - 2e_2, Q_0^{0,0} \cup Q_0^{0,3}) + p(\mathbf{w}_0 - e_1 - e_2, Q_0^{0,0} \cup Q_0^{0,3}), \\ \mathfrak{A}_{1,2} &= p(\mathbf{w}_0 - 2e_2, Q_0^{0,1} \cup Q_0^{0,2}) + p(\mathbf{w}_0 - e_1 - e_2, Q_0^{0,1} \cup Q_0^{0,2}). \end{aligned}$$

For $\mathbf{x} \in \mathbb{T}_L$, let

$$(3.6) \quad \mathcal{V}(\eta^x) = \{\eta_x^{i,j} : 0 \leq i, j \leq 3\}, \quad \mathcal{G}_x = \{\eta^x\} \cup \mathcal{V}(\eta^x).$$

We extend $\mathbf{R}(\eta^0, \cdot)$ to $\mathcal{V}(\eta^0)$ by symmetry, and we set

$$\mathbf{R}(\eta^x, \eta_x^{i,j}) = \mathbf{R}(\eta^0, \eta_0^{i,j}), \quad \mathbf{R}(\eta^x, \xi) = 0, \quad 0 \leq i, j \leq 3, \xi \notin \mathcal{V}(\eta^x).$$

The rates $\mathbf{R}(\eta, \xi)$ on $\Xi \setminus \Gamma$. Denote by $\mathcal{N}(\mathcal{E}_x^{i,j})$, \mathcal{N} for neighborhood, $0 \leq i, j \leq 3$, $\mathbf{x} \in \mathbb{T}_L$, the configurations in \mathbb{H}_2 which can be reached from a configuration in $\mathcal{E}_x^{i,j}$ by a rate $e^{-\beta}$ jump. The set $\mathcal{N}(\mathcal{E}_w^{2,2})$, for instance, has the following $3n$ elements. There are $n + 1$ configurations obtained when the top particle detaches itself from the others: $\sigma^{\mathbf{w}_2, \mathbf{z}} \eta^w$, where $\mathbf{z} = (-1, n)$, $(a, n + 1)$, $0 \leq a \leq n - 2$, $(n - 1, n)$. There are $n - 1$ configurations obtained when the particle at $\mathbf{w}_2 - e_2$ moves upward: $\sigma^{\mathbf{w}_2 - e_2, \mathbf{z}} \eta^w$, $\mathbf{z} = (a, n)$, $0 \leq a \leq n - 2$. There are $n - 2$ configurations obtained when the particle at $\mathbf{w}_2 - e_1$ moves to the right: $\sigma^{\mathbf{w}_2 - e_1, \mathbf{z}} \eta^w$, $\mathbf{z} = (a, n)$, $0 \leq a \leq n - 3$. To complete the description of the set $\mathcal{N}(\mathcal{E}_w^{2,2})$, we have to add the configurations $\sigma^{\mathbf{w}_3, \mathbf{w}_3 + e_2} \sigma^{\mathbf{w}_2, \mathbf{w}_3 + e_1 + e_2} \eta^w$ and $\sigma^{\mathbf{w}_2 - e_1, \mathbf{w}_2 - e_1 + e_2} \sigma^{\mathbf{w}_2, \mathbf{w}_2 - 2e_1 + e_2} \eta^w$.

Recall that L and $K = n^2$ are fixed in this section. We proved in [5] that for each $\zeta \in \mathcal{N}(\mathcal{E}_x^{i,j})$, there exists a probability measure $\mathbb{M}(\zeta, \cdot)$ concentrated on $\Gamma \cup \Omega^1 \cup \Omega^2$ such that

$$\lim_{\beta \rightarrow \infty} \mathbf{P}_\zeta[H_{\mathbb{H}_{01}} = H_\xi] = \mathbb{M}(\zeta, \xi).$$

Define the rates $\mathbf{R}(\eta_{\mathbf{x}}^{i,j}, \cdot)$ by

$$\mathbf{R}(\eta_{\mathbf{x}}^{i,j}, \xi) := \begin{cases} \sum_{\zeta \in \mathcal{N}(\mathcal{E}_{\mathbf{x}}^{i,j})} \mathbb{M}(\zeta, \mathcal{E}_{\mathbf{y}}^{k,l}), & \text{if } \xi \in \widehat{\Omega}^1, \xi = \eta_{\mathbf{y}}^{k,l}, \\ \sum_{\zeta \in \mathcal{N}(\mathcal{E}_{\mathbf{x}}^{i,j})} \mathbb{M}(\zeta, \xi), & \text{if } \xi \in \Gamma \cup \Omega^2, \\ 0, & \text{if } \xi \in \widehat{\Omega}^3 \cup \Omega^4. \end{cases}$$

Note that the rates $\mathbf{R}(\eta_{\mathbf{x}}^{i,j}, \cdot)$ depend on n and L , but not on β . In the next sections, however, n and L will be taken as functions of β and $\mathbf{R}(\eta_{\mathbf{x}}^{i,j}, \cdot)$ will depend on β through n, L . This remark holds for all rates $\mathbf{R}(\eta, \xi)$ defined below in this section.

For each $\eta \in \Omega^2 \cup \Omega^4$, denote by $\mathcal{N}(\eta)$ the set of configurations in \mathbb{H}_2 which can be reached from η by a rate $e^{-\beta}$ jump. Each set $\mathcal{N}(\eta)$ has at most 8 configurations. Fix $\eta \in \Omega^2$ (resp., $\eta \in \Omega^4$). We proved in [5] that for each $\zeta \in \mathcal{N}(\eta)$, there exists a probability measure, denoted by $\mathbb{M}(\zeta, \cdot)$, concentrated on $\Omega^1 \cup \Omega^2 \cup \Omega^3$ (resp., $\Omega^3 \cup \Omega^4$) such that

$$\lim_{\beta \rightarrow \infty} \mathbf{P}_{\zeta}[H_{\mathbb{H}_{01}} = H_{\xi}] = \mathbb{M}(\zeta, \xi).$$

For $\eta \in \Omega^2$, let

$$\mathbf{R}(\eta, \xi) = \begin{cases} \sum_{\zeta \in \mathcal{N}(\eta)} \mathbb{M}(\zeta, \xi), & \text{if } \xi \in \Omega^2, \\ \sum_{\zeta \in \mathcal{N}(\eta)} \mathbb{M}(\zeta, \mathcal{E}_{\mathbf{y}}^{k,l}), & \text{if } \xi \in \widehat{\Omega}^1, \xi = \eta_{\mathbf{y}}^{k,l}, \\ \sum_{\zeta \in \mathcal{N}(\eta)} \mathbb{M}(\zeta, \mathcal{E}_{\mathbf{y}}^{\alpha,k}), & \text{if } \xi \in \widehat{\Omega}^3, \xi = \zeta_{\mathbf{y}}^{\alpha,k}, \\ 0, & \text{if } \xi \in \Gamma \cup \Omega^4, \end{cases}$$

and for $\eta \in \Omega^4$, let

$$\mathbf{R}(\eta, \xi) = \begin{cases} \sum_{\zeta \in \mathcal{N}(\eta)} \mathbb{M}(\zeta, \xi), & \text{if } \xi \in \Omega^4, \\ \sum_{\zeta \in \mathcal{N}(\eta)} \mathbb{M}(\zeta, \mathcal{E}_{\mathbf{y}}^{\alpha,k}), & \text{if } \xi \in \widehat{\Omega}^3, \xi = \zeta_{\mathbf{y}}^{\alpha,k}, \\ 0, & \text{if } \xi \in \Gamma \cup \widehat{\Omega}^1 \cup \Omega^2. \end{cases}$$

For each $\mathbf{x} \in \mathbb{T}_L$, $\alpha \in \{l, s\}$, $0 \leq j \leq 3$, denote by $\mathcal{N}(\mathcal{E}_{\mathbf{x}}^{\alpha,j})$ the set of configurations in \mathbb{H}_2 which can be reached from a configuration in $\mathcal{E}_{\mathbf{x}}^{\alpha,j}$ by a rate $e^{-\beta}$ jump. We proved in [5] that for each $\zeta \in \mathcal{N}(\mathcal{E}_{\mathbf{x}}^{\alpha,j})$, there exists a probability measure $\mathbb{M}(\zeta, \cdot)$ concentrated on $\Omega^2 \cup \Omega^3 \cup \Omega^4$ such that

$$\lim_{\beta \rightarrow \infty} \mathbf{P}_{\zeta}[H_{\mathbb{H}_{01}} = H_{\xi}] = \mathbb{M}(\zeta, \xi).$$

Define the rates $\mathbf{R}(\zeta_{\mathbf{x}}^{a,j}, \xi)$ by

$$\mathbf{R}(\zeta_{\mathbf{x}}^{a,j}, \xi) = \begin{cases} \sum_{\zeta \in \mathcal{N}(\mathcal{E}_{\mathbf{x}}^{a,j})} \mathbb{M}(\zeta, \xi), & \text{if } \xi \in \Omega^2 \cup \Omega^4, \\ \sum_{\zeta \in \mathcal{N}(\mathcal{E}_{\mathbf{y}}^{a,k})} \mathbb{M}(\zeta, \mathcal{E}_{\mathbf{y}}^{a,k}), & \text{if } \xi \in \widehat{\Omega}^3, \xi = \zeta_{\mathbf{y}}^{a,k}, \\ 0, & \text{if } \xi \in \Gamma \cup \widehat{\Omega}^1. \end{cases}$$

In [5], we expressed the probability measures \mathbb{M} introduced above in terms of hitting probabilities of simple finite-state Markov processes. We recall below some of these explicit expressions.

The process $\widehat{\zeta}(t)$. Denote by $\widehat{\zeta}(t)$ the continuous-time Markov chain on Ξ whose jump rates, denoted by $R_{\widehat{\zeta}}(\eta, \xi)$, are given by

$$R_{\widehat{\zeta}}(\eta, \xi) = \begin{cases} e^{-\beta} \mathbf{R}(\eta, \xi), & \eta \in \Xi \setminus \Gamma, \xi \in \Xi, \\ e^{-2\beta} \mathbf{R}(\eta, \xi), & \eta \in \Gamma, \xi \in \Xi. \end{cases}$$

In contrast to the jump rates of the trace process, the jump rates $\mathbf{R}(\eta, \xi)$ vanish for a great number of pair of configurations. In the next section, we show that the trace process $\zeta(t)$ is well approximated by the Markov process $\widehat{\zeta}(t)$. In contrast to $\zeta(t)$, the process $\widehat{\zeta}(t)$ may not be reversible with respect to μ_{ζ} since the detailed balance conditions between configurations in Γ and configurations in $\Xi \setminus \Gamma$ may fail. The reflection of $\widehat{\zeta}(t)$ on subsets of $\Xi \cap \mathbb{H}_1$ is however reversible with respect to the uniform measure in view of (3.7) below.

Denote by $\mathbf{P}_{\xi}^{\widehat{\zeta}}$, $\xi \in \Xi$, the probability measure on the path space $D(\mathbb{R}_+, \Xi)$ induced by the Markov process $\widehat{\zeta}(t)$ starting from ξ . Expectation with respect to $\mathbf{P}_{\xi}^{\widehat{\zeta}}$ is represented by $\mathbf{E}_{\xi}^{\widehat{\zeta}}$.

Properties of the rates $\mathbf{R}(\eta, \xi)$. We claim that for each fixed n and L ,

$$(3.7) \quad \mathbf{R}(\eta, \xi) = \mathbf{R}(\xi, \eta), \quad \eta, \xi \in \Xi \setminus \Gamma.$$

These identities can be checked directly from the definition of the rates $\mathbf{R}(\xi, \eta)$ or can be deduced from the results of the previous section. Indeed, since the trace process $\zeta(t)$ is reversible, $\mu_K(\eta)R_{\zeta}(\eta, \xi) = \mu_K(\xi)R_{\zeta}(\xi, \eta)$. For $\eta, \xi \in \Xi \setminus \Gamma$, $\mu_K(\eta) = \mu_K(\xi)$ so that $R_{\zeta}(\eta, \xi) = R_{\zeta}(\xi, \eta)$. The results of the previous section are in force if we fix n and L and let $\beta \uparrow \infty$. Therefore, by Proposition 4.6, $e^{-\beta} \mathbf{R}(\eta, \xi) = e^{-\beta} \mathbf{R}(\xi, \eta) + o(e^{-\beta})$, from which the claim follows.

Let $\xi_1^* = \sigma^{\mathbf{w}_0, \mathbf{w}_0 - e_2} \eta^0$, $\xi_2^* = \sigma^{\mathbf{w}_0, \mathbf{w}_0 - e_1} \eta^0$ and for $j = 1, 2$, let

$$(3.8) \quad \mathbb{M}_j(\xi) = \lim_{\beta \rightarrow \infty} \mathbf{P}_{\xi_j^*}^{\widehat{\zeta}}[H_{\mathbb{H}_{01}} = H_{\xi}].$$

We have computed \mathbb{M}_j in [5], with the difference that the sets $\mathcal{E}_x^{i,j}$ in [5] have been replaced here by the configurations $\eta_x^{i,j}$ so that $\mathbb{M}_j(\mathcal{E}_x^{i,j})$ in [5] corresponds to $\mathbb{M}_j(\eta_x^{i,j})$ here. We have that

$$\begin{aligned} \mathbb{M}_1(\eta_0^{0,0}) &= \frac{1}{2} \left\{ \frac{1 + \tau_n^- + \mathfrak{A}_{0,3}}{4 + q_n + \tau_n - \mathfrak{A}} + \frac{1 + \tau_n^- + \mathfrak{A}_0 - \mathfrak{A}_3}{4 + q_n + \tau_n + \mathfrak{A}(e_1) - \mathfrak{A}(e_2)} \right\}, \\ \mathbb{M}_2(\eta_0^{0,0}) &= \frac{1}{2} \left\{ \frac{1 + \tau_n^- + \mathfrak{A}_{0,3}}{4 + q_n + \tau_n - \mathfrak{A}} - \frac{1 + \tau_n^- + \mathfrak{A}_0 - \mathfrak{A}_3}{4 + q_n + \tau_n + \mathfrak{A}(e_1) - \mathfrak{A}(e_2)} \right\}, \end{aligned}$$

where

$$\begin{aligned} \mathfrak{A}_j &= p(\mathbf{w}_0 - 2e_2, Q_0^{0,j}) + p(\mathbf{w}_0 - e_1 - e_2, Q_0^{0,j}), \\ \mathfrak{A}(e_i) &= p(\mathbf{w}_0 - 2e_2, \mathbf{w}_0 - e_i) + p(\mathbf{w}_0 - e_1 - e_2, \mathbf{w}_0 - e_i). \end{aligned}$$

LEMMA 3.1. For all $0 \leq i, j \leq 3, \mathbf{x} \in \mathbb{T}_L$,

$$(3.9) \quad \begin{aligned} \mathbf{R}(\eta_x^{i,j}, \eta_x^{i,j \pm 1}) &\geq (1/4)^3, \quad \mathbf{R}(\eta_x^{i,i-1}, \eta^x) = \mathbf{R}(\eta_x^{i,i}, \eta^x) \geq 1/4, \\ \mathbf{R}(\eta_x^{i,i}, \eta_x^{i-1,i}) &= \mathbf{R}(\eta_x^{i-1,i}, \eta_x^{i,i}) \geq 1. \end{aligned}$$

Moreover,

$$(3.10) \quad \mathbf{R}(\eta_x^{i,i}, \Omega^2) = \frac{1}{n}, \quad \mathbf{R}(\eta_x^{i,i+1}, \Omega^2) = \frac{1}{n} + \frac{1}{n-1}.$$

PROOF. To fix ideas consider the case $i = j = 0, \mathbf{x} = \mathbf{w}$. Recall the explicit form of \mathbb{M} given in [5]. As we have seen in Section 2, the set $\mathcal{N}(\mathcal{E}_w^{0,0})$ has $3n$ elements which can be divided into five classes. Consider first the set $\mathcal{A}_1 = \{\sigma^{\mathbf{w}+e_2, \mathbf{z}} \eta^w : \mathbf{z} \in Q^{0,0}\}$ which has $n - 1$ configurations. By [5], Lemma 4.2,

$$(3.11) \quad \sum_{\zeta \in \mathcal{A}_1} \mathbb{M}(\zeta, \xi) = \mathbf{1}\{\xi = \eta_w^{3,0}\} + \sum_{k=1}^{n-1} \tau_n^0(k) \mathbb{M}_1(\xi),$$

where $\tau_n^0(k)$ has been defined in (3.4) and $\mathbb{M}_1(\xi)$ in (3.8).

The second class consists of the set of configurations $\mathcal{A}_2 = \{\sigma^{\mathbf{w}+e_1, \mathbf{z}} \eta^w : \mathbf{z} = (k, -1), 2 \leq k \leq n - 1\}$ which has $n - 2$ elements. By [5], Lemma 4.2,

$$\mathbb{M}(\zeta, \eta_w^{0,0}) = 1, \quad \zeta \in \mathcal{A}_2.$$

In particular, this subset of $\mathcal{N}(\mathcal{E}_w^{0,0})$ does not contribute to the rate $\mathbf{R}(\eta_x^{0,0}, \cdot)$.

Two special configurations form each class:

$$(3.12) \quad \begin{aligned} &\mathbb{M}(\sigma^{\mathbf{w}, \mathbf{w}_1 - e_1 - e_2} \sigma^{\mathbf{w}_1, \mathbf{w}_1 - e_2} \eta^0, \xi) \\ &= \frac{1}{n} \mathbf{1}\{\xi = \sigma^{\mathbf{w}, \mathbf{w}_1 - e_1 - e_2} \sigma^{\mathbf{w}_2, \mathbf{w}_1 - e_2} \eta^0\} + \left(1 - \frac{1}{n}\right) \mathbf{1}\{\xi = \eta_w^{0,0}\} \end{aligned}$$

and

$$\mathbb{M}(\sigma^{\mathbf{w}, \mathbf{w}+e_1-e_2} \sigma^{\mathbf{w}+e_1, \mathbf{w}+2e_1-e_2} \eta^{\mathbf{0}}, \eta_{\mathbf{w}}^{0,0}) = 1.$$

This latter configuration also does not contribute to the rate $\mathbf{R}(\eta_{\mathbf{0}}^{0,0}, \cdot)$.

The last set is $\mathcal{A}_5 = \{\sigma^{\mathbf{w}, \mathbf{z}} \eta^{\mathbf{0}} : \mathbf{z} \in \Gamma_{0,0}\}$, where $\Gamma_{0,0}$ stands for the set of sites $\mathbf{z} \in \mathbb{T}_L$ which do not belong to \mathcal{Q} and are at distance 1 from $\mathcal{Q}^{0,0}$. This set has $n + 1$ elements and

$$\begin{aligned} \sum_{\zeta \in \mathcal{A}_5} \mathbb{M}(\zeta, \xi) &= \sum_{\mathbf{z} \in \Gamma_{0,0}} \sum_{j=0}^3 \mathbf{p}(\mathbf{z}, \mathcal{Q}^{0,j}) \mathbf{1}\{\xi = \eta_{\mathbf{0}}^{0,j}\} \\ (3.13) \quad &+ \sum_{\mathbf{z} \in \Gamma_{0,0}} \{\mathbf{p}(\mathbf{z}, \mathbf{w} - e_2) \mathbb{M}_1(\xi) + \mathbf{p}(\mathbf{z}, \mathbf{w} - e_1) \mathbb{M}_2(\xi)\}, \end{aligned}$$

where $\mathbb{M}_1(\xi)$ and $\mathbb{M}_2(\xi)$ have been defined in (3.8).

Now that we have explicit formulas for the rates $\mathbf{R}(\eta_{\mathbf{0}}^{0,0}, \cdot)$, we may prove the bounds claim in the lemma. By (3.11) $\mathbf{R}(\eta_{\mathbf{0}}^{0,0}, \eta_{\mathbf{0}}^{3,0}) \geq 1$, and by (3.13),

$$\mathbf{R}(\eta_{\mathbf{0}}^{0,0}, \eta_{\mathbf{0}}^{0,1}) \geq \mathbf{p}(\mathbf{w}_1 + e_1 - e_2, \mathbf{w}_1 + e_1) = \frac{1}{4},$$

where \mathbf{p} represents the jump probabilities of a nearest-neighbor, symmetric, discrete-time random walk on \mathbb{T}_L . By similar reasons, $\mathbf{R}(\eta_{\mathbf{0}}^{0,0}, \eta_{\mathbf{0}}^0) \geq 1/4$ and $\mathbf{R}(\eta_{\mathbf{0}}^{0,0}, \eta_{\mathbf{0}}^{0,3}) \geq (1/4)^3$. By symmetry, $\mathbf{R}(\eta_{\mathbf{0}}^{0,3}, \eta_{\mathbf{0}}^0) = \mathbf{R}(\eta_{\mathbf{0}}^{0,0}, \eta_{\mathbf{0}}^0)$. This proves (3.9) in view of (3.7).

To prove (3.10), note that $\mathbb{M}(\zeta, \Omega^2)$ vanishes for all configurations but for $\zeta = \sigma^{\mathbf{w}, \mathbf{w}_1-e_1-e_2} \sigma^{\mathbf{w}_1, \mathbf{w}_1-e_2} \eta^{\mathbf{0}}$ in which case it is equal to $1/n$, as observed in (3.12). A similar assertion holds for $\mathbb{M}(\zeta, \Omega^2)$, $\zeta \in \mathcal{N}(\mathcal{E}_{\mathbf{w}}^{0,1})$, but in this case there are two configurations which contribute to $\mathbf{R}(\eta_{\mathbf{0}}^{0,1}, \Omega^2)$. We leave the details to the reader. □

LEMMA 3.2. *For $n \geq 9$, we have that*

$$\max_{\eta \in \mathcal{V}(\eta^{\mathbf{x}})} \mathbf{P}_{\eta}^{\widehat{\zeta}}[H_{\mathcal{G}_{\widehat{\zeta}}} < H_{\eta^{\mathbf{x}}}] \leq \frac{23}{n}.$$

PROOF. Fix $\mathbf{x} = \mathbf{0}$ and let $\chi(t)$ be the Markov process $\chi(t) = \widehat{\zeta}(t \wedge H_{\mathcal{B}})$, where $\mathcal{B} = \mathcal{V}(\eta^{\mathbf{0}})^c$. We identify all configurations in $\mathcal{G}_{\mathbf{0}}^c$, turning $\chi(t)$ a Markov process on $\mathcal{G}_{\mathfrak{d}} = \mathcal{G}_{\mathbf{0}} \cup \{\mathfrak{d}\}$, where \mathfrak{d} replaces $\mathcal{G}_{\mathbf{0}}^c$.

There are two types of configurations in $\mathcal{V}(\eta^{\mathbf{0}})$, configurations in which the attached particle is on the same side of the empty corner and configurations in which this is not the case. We denote by $\mathcal{C}_1 = \{\eta_{\mathbf{0}}^{i,i}, \eta_{\mathbf{0}}^{i+1,i} : 0 \leq i \leq 3\}$ the first set, and by $\mathcal{C}_2 = \{\eta_{\mathbf{0}}^{i+2,i}, \eta_{\mathbf{0}}^{i+3,i} : 0 \leq i \leq 3\}$ the second one. Define $F : \mathcal{G}_{\mathfrak{d}} \rightarrow \{0, 1, 2, 3\}$ by

$$F(\eta^{\mathbf{0}}) = 0, \quad F(\mathfrak{d}) = 3, \quad F(\eta) = i, \quad \eta \in \mathcal{C}_i, i = 1, 2.$$

By symmetry $\widehat{\chi}(t) = F(\chi(t))$ is a four state Markov process and by construction

$$\mathbf{P}_\eta^{\widehat{\zeta}}[H_{G_0^c} < H_{\eta^0}] = \mathbf{P}_i^{\widehat{\chi}}[H_3 < H_0]$$

if $\eta \in \mathcal{C}_i$, where $\mathbf{P}_i^{\widehat{\chi}}$ represents the distribution of the process $\widehat{\chi}(t)$ starting from i .

Denote by $R_{\widehat{\chi}}$ the jump rates of the process $\widehat{\chi}(t)$. Since $\mathbf{R}(\eta, \xi) = \mathbf{R}(\xi, \eta)$ for $\eta \in \mathcal{C}_1, \xi \in \mathcal{C}_2$, and since $|\mathcal{C}_1| = |\mathcal{C}_2|$, $R_{\widehat{\chi}}(1, 2) = R_{\widehat{\chi}}(2, 1)$. By Lemma 3.1, $R_{\widehat{\chi}}(1, 0) = \mathbf{R}(\eta_0^{0,0}, \eta^0) \geq 1/4$, $R_{\widehat{\chi}}(1, 2) \geq \mathbf{R}(\eta_0^{0,0}, \eta_0^{3,0}) \geq 1$, and $R_{\widehat{\chi}}(i, 3) \leq n^{-1} + (n - 1)^{-1}$ for $i = 1, 2$.

Consider the Markov process Y_t on $\{0, 1, 2, 3\}$ with jump rates r given by $a := r(1, 2) = r(2, 1) = R_{\widehat{\chi}}(1, 2) \geq 1, r(1, 0) = 1/4$ and $r(1, 3) = r(2, 3) = 2/(n - 1)$. A coupling shows that for $i = 1, 2$,

$$\mathbf{P}_i^{\widehat{\chi}}[H_3 < H_0] \leq \mathbf{P}_i^Y[H_3 < H_0] \leq \frac{4\varepsilon[2a + (1/4) + \varepsilon]}{a + 4\varepsilon[2a + (1/4) + \varepsilon]},$$

where $\varepsilon = 2/(n - 1)$ and \mathbf{P}_i^Y represents the distribution of the process $\widehat{\chi}(t)$ starting from i . At last inequality is an identity if $i = 2$. The right-hand side of the previous displayed equation is uniformly bounded in $a \geq 1$ by $(9 + 4\varepsilon)\varepsilon \leq 10\varepsilon$ provided $\varepsilon \leq 1/4$. This proves the lemma. \square

The next result states that the chain $\widehat{\zeta}(t)$ jumps with a probability bounded below by a positive constant independent of β from a configuration $\zeta_{\mathbf{x}}^{1,3}$ to the configuration $\zeta_{\mathbf{x}}^{1,2}$.

LEMMA 3.3. *Denote by $\mathcal{V}(\zeta_{\mathbf{x}}^{1,3})$ the set of configurations ξ in Ξ such that $\mathbf{R}(\zeta_{\mathbf{x}}^{1,3}, \xi) > 0, \mathcal{V}(\zeta_{\mathbf{x}}^{1,3}) = \{\xi \in \Xi : \mathbf{R}(\zeta_{\mathbf{x}}^{1,3}, \xi) > 0\}$. There exists $c_0 > 0$ such that*

$$\mathbf{P}_{\zeta_{\mathbf{x}}^{1,3}}^{\widehat{\zeta}}[H_{\zeta_{\mathbf{x}}^{1,2}} = H_{\mathcal{V}(\zeta_{\mathbf{x}}^{1,3})}] \geq c_0.$$

PROOF. By definition of the chain $\widehat{\zeta}(t)$,

$$\mathbf{P}_{\zeta_{\mathbf{x}}^{1,3}}^{\widehat{\zeta}}[H_{\zeta_{\mathbf{x}}^{1,2}} = H_{\mathcal{V}(\zeta_{\mathbf{x}}^{1,3})}] = \frac{\mathbf{R}(\zeta_{\mathbf{x}}^{1,3}, \zeta_{\mathbf{x}}^{1,2})}{\sum_{\xi \in \Xi} \mathbf{R}(\zeta_{\mathbf{x}}^{1,3}, \xi)}.$$

By definition of the rates $\mathbf{R}(\eta, \xi), \mathbf{R}(\zeta_{\mathbf{x}}^{1,3}, \zeta_{\mathbf{x}}^{1,2}) = \mathbf{R}(\zeta_{\mathbf{x}}^{1,3}, \zeta_{\mathbf{x}}^{1,0}) \geq c_0$, while $\mathbf{R}(\zeta_{\mathbf{x}}^{1,3}, \Omega^2) = 0$ and $\mathbf{R}(\zeta_{\mathbf{x}}^{1,3}, \Omega^4) \leq C_0/n$. By [20], Proposition 2.4.5, $\mathbf{R}(\zeta_{\mathbf{x}}^{1,3}, \zeta_{\mathbf{x}}^{1,1})$, whose value involves the sum of n terms, is bounded above by C_0 . This proves the lemma. \square

The bounds (3.14) and (3.15) below are needed to estimate the capacity between the configurations η^0 and η^{e_i} for the trace process $\zeta(t)$.

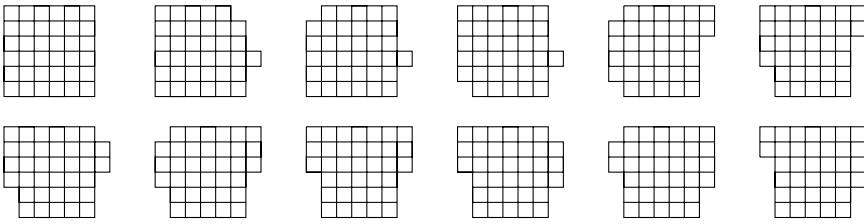


FIG. 7. The first 12 configurations of the path from η^0 to η^{e_1} when $n = 6$. Observe the subpath starting from the sixth configuration and ending at the ninth, as well as the one starting from the ninth configuration and ending at the twelfth.

Consider the path $\xi_0 = \eta^0, \xi_1, \dots, \xi_M = \eta^{e_1}, M = 3n - 2$, from η^0 to η^{e_1} in Ξ obtained by sliding particles around three sides of the square Q : $\xi_0 = \eta^0, \xi_1 = \eta_0^{2,1}, \xi_2 = \eta_0^{3,1}, \xi_3 = \eta_0^{0,1}, \xi_4 = \eta_0^{l,((1,n-1),(n-2,n-1),(0,n-1),(1,n-2))}, \xi_5 = \eta_0^{l,((1,n-1),(n-2,n-1),(0,n),(0,n-3))}, \dots, \xi_{M-4} = \eta_0^{l,((1,n-1),(1,n-2),(1,n-1),(0,1))}, \xi_{M-3} = \eta_0^{1,3}, \xi_{M-2} = \eta_{e_1}^{2,3}, \xi_{M-1} = \eta_{e_1}^{3,3}, \xi_M = \eta^{e_1}$. Figure 7 presents the first configurations of this sequence in the case $n = 6$.

We claim that

$$(3.14) \quad \begin{aligned} \mathbf{R}(\xi_0, \xi_1) &\geq \frac{1}{6}, & \mathbf{R}(\xi_1, \xi_2) &\geq 1, \\ \mathbf{R}(\xi_2, \xi_3) &\geq 1, & \mathbf{R}(\xi_3, \xi_4) &\geq \frac{1}{n}. \end{aligned}$$

The first inequality follows from the definition (3.5) of $\mathbf{R}(\eta^0, \cdot)$ and from the fact that $q_n \leq 1, \tau_n \leq 1$. The other three estimates have been proven in Lemma 3.1. Similar estimates hold for the last jumps of the sequence (ξ_i) . On the other hand, in view of the definition of the jump rates \mathbf{R} , it is clear that

$$(3.15) \quad \mathbf{R}(\xi_i, \xi_{i+1}) \geq \frac{1}{n}$$

for the middle terms. The expression n^{-1} comes from the probability of a symmetric nearest-neighbor random walk on $\{0, 1, \dots, n\}$ starting from 1 to reach n before 0.

4. The jump rates of $\zeta(t)$ and $\hat{\zeta}(t)$. The main results of this section, Propositions 4.2 and 4.6, provide estimates on the difference between the jump rates $R_\zeta(\eta, \xi)$ and $\mathbf{R}(\eta, \xi)$. The dependence of the errors on the parameters n and L are explicit to allow them depend on β .

The first lemma presents an explicit formula for the jump rates R_ζ of the trace process. Denote by $\text{cap}_K(A, B)$ the capacity between two disjoint subsets A and

B of $\Omega_{L,K}$ for the $\eta(t)$ process

$$\text{cap}_K(A, B) = \sum_{\eta \in A} \mu_K(\eta) \lambda(\eta) \mathbf{P}_\eta^\beta [H_B < H_A^+],$$

where $\lambda(\eta)$ stands for the holding rates of the process $\eta(t)$ and $R(\eta, \xi) = R_\beta(\eta, \xi)$ for the jump rates,

$$R(\eta, \xi) = \begin{cases} c_{x,y}(\eta), & \text{if } \xi = \sigma^{x,y}\eta \text{ for some } x, y \in \mathbb{T}_L, |y - x| = 1, \\ 0, & \text{otherwise,} \end{cases}$$

$\lambda(\eta) = \sum_{\xi \in \Omega_{L,K}} R(\eta, \xi)$. Denote by D_K the Dirichlet form associated to the process $\eta(t)$ on the ergodic component $\Omega_{L,K}$: For any function $f : \Omega_{L,K} \rightarrow \mathbb{R}$,

$$D_K(f) = \frac{1}{2} \sum_{\eta, \xi \in \Omega_{L,K}} \mu_K(\eta) R(\eta, \xi) [f(\xi) - f(\eta)]^2.$$

LEMMA 4.1. For $\eta, \xi \in \Xi, \xi \neq \eta$,

$$(4.1) \quad R_\zeta(\eta, \xi) = \frac{\text{cap}_K(\eta, \Delta_1)}{\mu_K(\eta)} \sum_{\zeta \in \Delta_1} \mathbf{P}_\eta [H_{\Delta_1} = H_\zeta] \mathbf{P}_\zeta [H_\Xi = H_\xi].$$

PROOF. The proof of this lemma relies on the fact that the process $\eta(t)$ has to cross the set Δ_1 when going from a configuration $\eta \in \Xi$ to a configuration $\xi \in \Xi, \xi \neq \eta, \mathbf{P}_\eta [H_{\Delta_1} < H_\xi] = 1$ if $\xi \neq \eta, \eta, \xi \in \Xi$. In contrast, there are configurations $\eta \in \Xi$ which can be hit from η without crossing the set Δ_1 , for example, the configuration $\eta_0^{0,0}$. With a positive probability which does not vanish in the zero-temperature limit, the attached particle may jump to the right and then return to its original position while all the other particles remain still. In this event, $H_\eta^+ < H_{\Delta_1}$.

Fix $\eta, \xi \in \Xi, \xi \neq \eta$. By [1], Proposition 6.1,

$$(4.2) \quad R_\zeta(\eta, \xi) = \lambda(\eta) \mathbf{P}_\eta [H_\Xi^+ = H_\xi].$$

The process $\eta(t)$ has to cross Δ_1 when going from a configuration η in Ξ to a configuration $\xi \neq \eta$ in Ξ . Therefore, on the event $\{H_\Xi^+ = H_\xi\}$ we have that $H_{\Delta_1} < H_\Xi^+$, and by the strong Markov property

$$\mathbf{P}_\eta [H_\Xi^+ = H_\xi] = \sum_{\zeta \in \Delta_1} \mathbf{P}_\zeta [H_\Xi = H_\xi] \mathbf{P}_\eta [H_\zeta = H_{\Delta_1} < H_\Xi^+].$$

We have seen that $H_{\Delta_1} < H_{\Xi \setminus \{\eta\}}$ almost surely with respect to \mathbf{P}_η . Hence, the event $\{H_{\Delta_1} < H_\Xi^+\}$ can be rewritten as $\{H_{\Delta_1} < H_\eta^+\}$ and the previous sum becomes

$$\sum_{\zeta \in \Delta_1} \mathbf{P}_\zeta [H_\Xi = H_\xi] \mathbf{P}_\eta [H_\zeta = H_{\Delta_1 \cup \{\eta\}}^+].$$

On the other hand, since

$$\mathbf{P}_\eta [H_\zeta = H_{\Delta_1}] = \mathbf{P}_\eta [H_\eta^+ < H_{\Delta_1}, H_\zeta = H_{\Delta_1}] + \mathbf{P}_\eta [H_\eta^+ > H_{\Delta_1}, H_\zeta = H_{\Delta_1}],$$

observing that the event on the second term of the right-hand side is $H_\zeta = H_{\Delta_1 \cup \{\eta\}}^+$, and applying the strong Markov property to the first term, we obtain that

$$\mathbf{P}_\eta[H_\zeta = H_{\Delta_1 \cup \{\eta\}}^+] = \mathbf{P}_\eta[H_\zeta = H_{\Delta_1}] \mathbf{P}_\eta[H_{\Delta_1} < H_\eta^+].$$

Therefore, in view of (4.2) and the previous identities,

$$R_\zeta(\eta, \xi) = \lambda(\eta) \mathbf{P}_\eta[H_{\Delta_1} < H_\eta^+] \sum_{\zeta \in \Delta_1} \mathbf{P}_\zeta[H_\Xi = H_\xi] \mathbf{P}_\eta[H_\zeta = H_{\Delta_1}].$$

To conclude the proof of the lemma, it remains to recall the definition of the capacity. \square

The jump rates of $\zeta(t)$ on Γ . We examine in this subsection the jump rates of order $e^{-2\beta}$ of the trace process $\zeta(t)$. The main result reads as follows.

PROPOSITION 4.2. *There exists a finite constant C_0 independent of β such that*

$$\max_{\xi \in \mathcal{G}_x} |R_\zeta(\eta^x, \xi) - e^{-2\beta} \mathbf{R}(\eta^x, \xi)| \leq e^{-2\beta} \kappa_1,$$

where κ_1 is a remainder absolutely bounded by $C_0 L e^{-\beta/2}$. Moreover, for all $x \in \mathbb{T}_L$,

$$R_\zeta(\eta^x, \Xi \setminus \mathcal{G}_x) \leq e^{-2\beta} \kappa_1.$$

It follows from this proposition that

$$(4.3) \quad \frac{1}{R_\zeta(\eta^x, \xi)} \leq \frac{e^{2\beta}}{\mathbf{R}(\eta^x, \xi)} \left\{ 1 + \frac{\kappa_1}{\mathbf{R}(\eta^x, \xi) - \kappa_1} \right\}.$$

The proof of Proposition 4.2 relies on the next two lemmas. Recall the definition of $\mathbb{M}_j(\xi)$ given in (3.8).

LEMMA 4.3. *Let $\xi_1^* = \sigma^{\mathbf{w}_0, \mathbf{w}_0 - e_2} \eta^0$, $\xi_2^* = \sigma^{\mathbf{w}_0, \mathbf{w}_0 - e_1} \eta^0$ and let*

$$P_j(\xi) = \mathbf{P}_{\xi_j^*}[H_\Xi = H_\xi], \quad j = 1, 2.$$

Then, there exists a finite constant C_0 such that for all $\xi \in \Xi$

$$|P_j(\xi) - \mathbb{M}_j(\xi)| \leq C_0 L e^{-\beta/2}.$$

This bound, as well as the next ones, hold for all β, L and $K = n^2$. While $\mathbb{M}_j(\xi)$ depends only on L and n , $P_j(\xi)$ also depends on β . This estimate will be used to replace a two step limit in which we first send $\beta \uparrow \infty$, and then $n, L \uparrow \infty$, by a diagonal procedure in which all parameters diverge simultaneously.

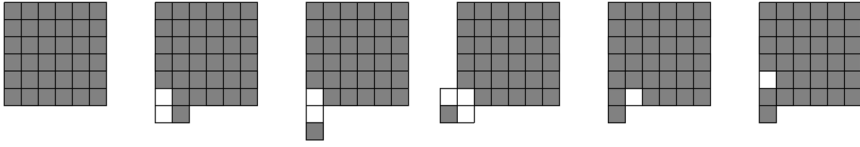


FIG. 8. The six configurations which can be reached from ξ_1^* by a rate 1 jump. The first two belong to \mathbb{H}_{01} .

PROOF OF LEMMA 4.3. We prove this lemma for $\xi = \eta_0^{0,0}$ and leave the other cases to the reader. Set $P_j = P_j(\eta_0^{0,0})$, $j = 1, 2$, and recall the definition of q_n, τ_n^\pm introduced in (3.3), (3.4). We claim that

$$(4.4) \quad (4 + q_n + \tau_n - 2)(P_1 + P_2) = 1 + 2\mathfrak{A}_{0,3} + \tau_n^- + L\varepsilon(\beta/2),$$

where $\varepsilon(\beta)$ represents here and below an error term absolutely bounded by $C_0e^{-\beta}$.

To prove (4.4), observe that the configuration ξ_1^* may jump at rate 1 to 6 configurations, at rate $e^{-\beta}$ to 1 configuration, at rate $e^{-2\beta}$ to 7 configurations, and at rate $e^{-3\beta}$ to $O(n)$ configurations. Figure 8 shows the six configurations which can be attained from ξ_1^* with a jump of rate 1. Among the configurations which can be reached at rate 1, two belong to $\Gamma \cup \mathbb{H}_1$, one of them being η^0 . Hence, if we denote by $\mathcal{N}_2(\xi_1^*)$ the set of the remaining four configurations which can be reached from ξ_1^* by a rate 1 jump, and by $\eta_1^* = \sigma^{w_0, w_0 - e_2 + e_1} \eta^0$ the configuration in $\Gamma \cup \mathbb{H}_1$ which is not η^0 , decomposing $\mathbf{P}_{\xi_1^*}[H_\Xi = H_\xi]$ according to the first jump we obtain that

$$(4.5) \quad 6\mathbf{P}_{\xi_1^*}[H_\Xi = H_\xi] = \mathbf{P}_{\eta_1^*}[H_\Xi = H_\xi] + \sum_{\eta' \in \mathcal{N}_2(\xi_1^*)} \mathbf{P}_{\eta'}[H_\Xi = H_\xi] + \varepsilon(\beta).$$

The particle at position $-e_2 + e_1$ of the configuration $\eta_1^* = \sigma^{w_0, w_0 - e_2 + e_1} \eta^0$, is attached to the side $Q_0^{0,0}$ of the quasi-square Q_0^0 . This particle moves along the side of the square at rate 1, while all the other jumps occur at rate at most $e^{-\beta}$. Let H_{hit} be the first time the attached particle reaches the center of the side, and let H_{mv} be the first time a jump of rate $e^{-\beta}$ or less occurs. The random time H_{mv} is bounded below by a mean C_0e^β exponential random time ϵ_β independent of the lateral displacement of the attached particle. Therefore,

$$\mathbf{P}_{\eta_1^*}[H_{\text{mv}} \leq H_{\text{hit}}] \leq \mathbf{P}_{\eta_1^*}[\epsilon_\beta \leq H_{\text{hit}}] = \int_0^\infty \mathbf{P}_{\eta_1^*}[H_{\text{hit}} \geq t] \alpha e^{-\alpha t} dt,$$

where $\alpha^{-1} = C_0e^\beta$. It is not difficult to bound this integral by $2\{\alpha \mathbf{E}_{\eta_1^*}[H_{\text{hit}}]\}^{1/2}$. We shall use repeatedly this estimate. Since $\mathbf{E}_{\eta_1^*}[H_{\text{hit}}] \leq C_0n^2$, the last integral is bounded by $C_0ne^{-\beta/2}$.

On the event $\{H_{\text{hit}} \leq H_{\text{mv}}\}$, $H_\Xi = H_\xi$, where $\xi = \eta_0^{0,0}$ is the configuration obtained from η_1^* by moving the particle attached to the side of the square to the middle position. Therefore,

$$\mathbf{P}_{\eta_1^*}[H_\Xi = H_\xi] = 1 + n\varepsilon(\beta/2),$$

and (4.5) becomes

$$(4.6) \quad 6\mathbf{P}_{\xi_1^*}[H_{\Xi} = H_{\xi}] = 1 + \sum_{\eta' \in \mathcal{N}_2(\xi_1^*)} \mathbf{P}_{\eta'}[H_{\Xi} = H_{\xi}] + n\varepsilon(\beta/2).$$

We examine the four configurations of $\mathcal{N}_2(\xi_1^*)$ separately. In two configurations, $\sigma^{\mathbf{w}_0, \mathbf{w}_0 - 2e_2} \eta^0$ and $\sigma^{\mathbf{w}_0, \mathbf{w}_0 - e_2 - e_1} \eta^0$, a particle is detached from the quasi-square Q_0^0 . The detached particle performs a rate 1 symmetric random walk on \mathbb{T}_L until it reaches the boundary of the square Q_0 . Among the remaining particles, two jumps have rate $e^{-\beta}$ and all the other ones have rate at most $e^{-2\beta}$. Denote by H_{hit} the time the detached particle hits the outer boundary of the square Q_0 and by H_{mv} the first time a particle on the quasi-square moves.

As above, H_{mv} can be bounded below by a mean $C_0 e^{\beta}$ exponential random variable ϵ_{β} independent of the displacement of the detached particle. Therefore, for $\eta' = \sigma^{\mathbf{w}_0, \mathbf{w}_0 - 2e_2} \eta^0, \sigma^{\mathbf{w}_0, \mathbf{w}_0 - e_2 - e_1} \eta^0$,

$$\mathbf{P}_{\eta'}[H_{\text{mv}} \leq H_{\text{hit}}] \leq \mathbf{P}_{\eta'}[\epsilon_{\beta} \leq H_{\text{hit}}] = \int_0^{\infty} \mathbf{P}_{\eta'}[H_{\text{hit}} \geq t] \alpha e^{-\alpha t} dt,$$

where $\alpha^{-1} = C_0 e^{\beta}$. Since, by [21], Proposition 10.13, $\mathbf{E}_{\eta'}[H_{\text{hit}}] \leq C_0 L^2$, this last integral is bounded by $C_0 L e^{-\beta/2}$. Hence,

$$(4.7) \quad \mathbf{P}_{\eta'}[H_{\Xi} = H_{\xi}] = \mathbf{P}_{\eta'}[H_{\Xi} = H_{\xi}, H_{\text{hit}} \leq H_{\text{mv}}] + L\varepsilon(\beta/2).$$

On the event $H_{\text{hit}} \leq H_{\text{mv}}$ the configuration $\eta(H_{\text{hit}})$ is either ξ_1^*, ξ_2^* or it belongs to \mathbb{H}_1 . Therefore, by the strong Markov property, the previous expression is equal to

$$\begin{aligned} & \sum_{i=1}^2 \mathbf{P}_{\eta'}[\eta(H_{\text{hit}}) = \xi_i^*] \mathbf{P}_{\xi_i^*}[H_{\Xi} = H_{\xi}] \\ & + \sum_{j=0}^3 \mathbf{E}_{\eta'}[\mathbf{1}\{\eta(H_{\text{hit}}) \in \mathcal{E}_0^{0,j}\} \mathbf{P}_{\eta(H_{\text{hit}})}[H_{\Xi} = H_{\xi}]] + L\varepsilon(\beta/2). \end{aligned}$$

Once the detached particle hits the side of the quasi-square, as we have seen above, it attains the middle position of the side before anything else happens with a probability close to 1. Since $\xi = \eta_0^{0,0}$, the previous sum over j is equal to

$$\mathbf{P}_{\eta'}[\eta(H_{\text{hit}}) \in \mathcal{E}_0^{0,0}] + n\varepsilon(\beta/2).$$

By definition (3.2) of \mathbf{p} , the contribution of the terms $\eta' = \sigma^{\mathbf{w}_0, \mathbf{w}_0 - 2e_2} \eta^0$ and $\eta' = \sigma^{\mathbf{w}_0, \mathbf{w}_0 - e_2 - e_1} \eta^0$ to the sum appearing on the right-hand side of (4.6) is

$$(4.8) \quad \begin{aligned} & \{\mathbf{p}(\mathbf{w}_0 - 2e_2, \mathbf{w}_0 - e_2) + \mathbf{p}(\mathbf{w}_0 - e_1 - e_2, \mathbf{w}_0 - e_2)\} \mathbf{P}_{\xi_1^*}[H_{\Xi} = H_{\xi}] \\ & + \{\mathbf{p}(\mathbf{w}_0 - 2e_2, \mathbf{w}_0 - e_1) + \mathbf{p}(\mathbf{w}_0 - e_1 - e_2, \mathbf{w}_0 - e_1)\} \mathbf{P}_{\xi_2^*}[H_{\Xi} = H_{\xi}] \\ & + \mathbf{p}(\mathbf{w}_0 - 2e_2, Q_0^{0,0}) + \mathbf{p}(\mathbf{w}_0 - e_1 - e_2, Q_0^{0,0}) + L\varepsilon(\beta/2). \end{aligned}$$

In this equation, we have replaced $n\varepsilon(\beta/2)$ by $L\varepsilon(\beta/2)$ because $n \leq 2L$.

It remains to analyze the two configurations of $\mathcal{N}'_2(\eta)$, $\eta' = \sigma^{\mathbf{w}_0+e_2, \mathbf{w}_0-e_2}\eta^{\mathbf{0}}$ and $\eta' = \sigma^{\mathbf{w}_0+e_1, \mathbf{w}_0-e_2}\eta^{\mathbf{0}}$. In the first one, if we denote by \mathbf{z}_t^1 the horizontal position of the particle attached to the side Q^0 of the square Q and by \mathbf{z}_t^2 the vertical position of the hole on the side Q^3 , it is not difficult to check that $(\mathbf{z}_t^1, \mathbf{z}_t^2)$ evolves as the Markov chain described just before (3.4) with initial condition $(\mathbf{z}_0^1, \mathbf{z}_0^2) = (0, 1)$.

Denote by H_{hit} the time the hole hits 0 or $n - 1$ and by H_{mv} the first time a jump of rate $e^{-\beta}$ or less occurs. The arguments presented above in this proof show that the probability $\mathbf{P}_{\eta'}[H_{\text{mv}} \leq H_{\text{hit}}]$ is bounded by $n\varepsilon(\beta/2)$. Hence, (4.7) holds with this new meaning for H_{mv} , H_{hit} , and with n in place of L . On the event $H_{\text{hit}} \leq H_{\text{mv}}$ three situations can occur. If the process $(\mathbf{z}_t^1, \mathbf{z}_t^2)$ reached $(0, 0)$ (resp., E_n^+, E_n^-), the process $\eta(t)$ returned to the configuration ξ_1^* (resp., hit a configuration in $\mathcal{E}_0^{3,0}, \mathcal{E}_0^{0,0}$). At this point, we may repeat the previous arguments to replace the set $\mathcal{E}_0^{3,0}, \mathcal{E}_0^{0,0}$ by the configurations $\eta_0^{3,0}, \eta_0^{0,0}$, respectively. Hence, by definition of τ_n^\pm ,

$$\mathbf{P}_{\eta'}[H_\Xi = H_\xi] = \tau_n^- + (1 - \tau_n)\mathbf{P}_{\xi_1^*}[H_\Xi = H_\xi] + n\varepsilon(\beta/2)$$

for $\eta' = \sigma^{\mathbf{w}_0+e_2, \mathbf{w}_0-e_2}\eta^{\mathbf{0}}, \xi = \eta_0^{0,0}$.

Assume now that $\eta' = \sigma^{\mathbf{w}_0+e_1, \mathbf{w}_0-e_2}\eta^{\mathbf{0}}$. In this case, if we denote by \mathbf{y}^1 the horizontal position of the particle attached to the side of the quasi-square and by \mathbf{y}^2 the horizontal position of the hole, the pair $(\mathbf{y}_t^1, \mathbf{y}_t^2)$ evolve according to the Markov process introduced just before (3.3) with initial condition $(\mathbf{y}_0^1, \mathbf{y}_0^2) = (0, 1)$. Denote by H_{hit} the time the hole hits 0 or $n - 1$ and by H_{mv} the first time a jump of rate $e^{-\beta}$ or less occurs. The arguments presented above in this proof show that the probability $\mathbf{P}_{\eta'}[H_{\text{mv}} \leq H_{\text{hit}}]$ is bounded by $n\varepsilon(\beta/2)$. Hence, (4.7) holds with this new meaning for H_{mv} , H_{hit} , and with n in place of L . On the event $H_{\text{hit}} \leq H_{\text{mv}}$, if $\mathbf{y}_{H_{\text{hit}}}^2 = 0$, at time H_{hit} the process $\eta(t)$ has returned to the configuration ξ_1^* , while if $\mathbf{y}_{H_{\text{hit}}}^2 = n - 1$, at time H_{hit} the process $\eta(t)$ has reached a configuration in $\mathcal{E}_0^{1,0}$. At this point, we may repeat the previous arguments to replace the set $\mathcal{E}_0^{1,0}$ by the configurations $\eta_0^{1,0}$. Since the random walk reaches $n - 1$ before 0 with probability q_n , by the strong Markov property

$$\mathbf{P}_{\eta'}[H_\Xi = H_\xi] = (1 - q_n)\mathbf{P}_{\xi_1^*}[H_\Xi = H_\xi] + n\varepsilon(\beta/2)$$

for $\eta' = \sigma^{\mathbf{w}_0+e_1, \mathbf{w}_0-e_2}\eta^{\mathbf{0}}, \xi = \eta_0^{0,0}$.

Therefore, the contribution of the last two configurations of $\mathcal{N}'_2(\xi_1^*)$ to the sum on the right-hand side of (4.6) is

$$(4.9) \quad \tau_n^- + (2 - \tau_n - q_n)\mathbf{P}_{\xi_1^*}[H_\Xi = H_\xi] + n\varepsilon(\beta/2).$$

Equations (4.6), (4.8) and (4.9) yield a linear equation for $P_1 = \mathbf{P}_{\xi_1^*}[H_\Xi = H_\xi]$ in terms of P_1 and $P_2 = \mathbf{P}_{\xi_2^*}[H_\Xi = H_\xi]$. Analogous arguments provide a similar equation for P_2 in terms of P_1 and P_2 . Adding these two equations, we obtain (4.4), while subtracting them gives an expression for the difference $P_1 - P_2$. The assertion of the lemma follows from these equations for $P_1 + P_2$ and $P_1 - P_2$. \square

LEMMA 4.4. *We have that*

$$8(4 + q_n + \tau_n - \mathfrak{A})\mathbf{P}_{\eta^0}[H_{\Xi}^+ = H_{\eta_0^{0,0}}] = 1 + \mathfrak{A}_{0,3} + \tau_n^- + q_n + \kappa_1,$$

$$8(4 + q_n + \tau_n - \mathfrak{A})\mathbf{P}_{\eta^0}[H_{\Xi}^+ = H_{\eta_0^{0,1}}] = \mathfrak{A}_{1,2} + \tau_n^+ + \kappa_1,$$

$$(4 + q_n + \tau_n - \mathfrak{A})\mathbf{P}_{\eta^0}[H_{\Xi}^+ = H_{\eta^0}] = 1 + \kappa_1,$$

where κ_1 is a remainder absolutely bounded by $C_0Le^{-\beta/2}$.

Note that using the symmetry of the model we can deduce from this result the values of $\mathbf{P}_{\eta^0}[H_{\Xi}^+ = H_{\eta_0^{i,j}}]$ for all $0 \leq i, j \leq 3$. For example,

$$\mathbf{P}_{\eta^0}[H_{\Xi}^+ = H_{\eta_0^{0,3}}] = \mathbf{P}_{\eta^0}[H_{\Xi}^+ = H_{\eta_0^{0,0}}],$$

$$\mathbf{P}_{\eta^0}[H_{\Xi}^+ = H_{\eta_0^{0,2}}] = \mathbf{P}_{\eta^0}[H_{\Xi}^+ = H_{\eta_0^{0,1}}].$$

Moreover, summing over all configurations in \mathcal{G}_0 , we get that

$$\mathbf{P}_{\eta^0}[H_{\Xi}^+ \neq H_{\mathcal{G}_0}] = \kappa_1.$$

PROOF OF LEMMA 4.4. We prove the lemma for $\xi = \eta_0^{0,0}$. Decomposing the event $\{H_{\Xi}^+ = H_{\xi}\}$ according to the first jump of the process $\eta(t)$ we obtain that

$$\mathbf{P}_{\eta^0}[H_{\Xi}^+ = H_{\xi}] = \frac{1}{8} \sum_{\eta \in \mathcal{N}(\eta^0)} \mathbf{P}_{\eta}[H_{\Xi} = H_{\xi}] + n\varepsilon(\beta),$$

where $\mathcal{N}(\eta^0)$ is the set of 8 configurations which can be obtained from η^0 by a jump of rate $e^{-2\beta}$. Lemma 4.3 provides a formula $\mathbf{P}_{\eta}[H_{\Xi} = H_{\xi}]$ for each η in $\mathcal{N}(\eta^0)$, which concludes the proof of the lemma. \square

COROLLARY 4.5. *There exists a finite constant C_0 such that*

$$p_{\zeta}(\eta^0, \eta_0^{0,0}) = \frac{1}{8} \frac{1 + \mathfrak{A}_{0,3} + \tau_n^- + q_n}{1 + \mathfrak{A}_{0,3} + \mathfrak{A}_{1,2} + \tau_n + q_n} + \kappa_1,$$

$$p_{\zeta}(\eta^0, \eta_0^{0,1}) = \frac{1}{8} \frac{\mathfrak{A}_{1,2} + \tau_n^+}{1 + \mathfrak{A}_{0,3} + \mathfrak{A}_{1,2} + \tau_n + q_n} + \kappa_1,$$

where κ_1 is a remainder absolutely bounded by $C_0Le^{-\beta/2}$.

Here again, using the symmetry of the model, we can deduce from this result the values of $p_{\zeta}(\eta^{\mathbf{x}}, \eta_{\mathbf{x}}^{i,j})$ for all $0 \leq i, j \leq 3, \mathbf{x} \in \mathbb{T}_L$. Moreover, it follows from this result that

$$(4.10) \quad \sum_{\xi \notin \mathcal{V}(\eta^{\mathbf{x}})} p_{\zeta}(\eta^{\mathbf{x}}, \xi) \leq \kappa_1.$$

PROOF OF COROLLARY 4.5. By the displayed equation after (6.9) in [1],

$$p_\zeta(\eta^0, \xi) = \mathbf{P}_{\eta^0}[H_{\Xi \setminus \{\eta^0\}} = H_\xi].$$

Intersecting the previous event with the partition $\{H_\Xi^+ = H_{\{\eta^0\}}^+\}, \{H_\Xi^+ < H_{\{\eta^0\}}^+\}$, we obtain by the strong Markov property that

$$\mathbf{P}_{\eta^0}[H_{\Xi \setminus \{\eta^0\}} = H_\xi] = \frac{\mathbf{P}_{\eta^0}[H_\Xi^+ = H_\xi]}{\mathbf{P}_{\eta^0}[H_\Xi^+ \neq H_{\{\eta^0\}}^+]}$$

It remains to recall the statement of Lemma 4.4 to conclude the proof of the corollary. \square

PROOF OF PROPOSITION 4.2. Since $R_\zeta(\eta^0, \xi) = \lambda(\eta^0)\mathbf{P}_{\eta^0}[H_\Xi^+ = H_\xi]$ and since $\lambda(\eta^0) = 8e^{-2\beta}(1 + ne^{-\beta})$, the assertion follows from Lemma 4.4. \square

The jump rates of $\zeta(t)$ on $\Xi \setminus \Gamma$. We examine in this subsection the jump rates of order $e^{-\beta}$ of the trace process $\zeta(t)$. Let

$$\kappa_2 = n^4 e^{-\beta} + nLe^{-(1/2)\beta}.$$

PROPOSITION 4.6. For $\eta \in \Xi \setminus \Gamma$, let $\mathcal{V}(\eta) = \{\xi \in \Xi : \mathbf{R}(\eta, \xi) > 0\}$. There exists a finite constant C_0 such that for all $\eta \in \Xi \setminus \Gamma$,

$$\max_{\xi \in \mathcal{V}(\eta)} |R_\zeta(\eta, \xi) - e^{-\beta} \mathbf{R}(\eta, \xi)| \leq C_0 e^{-\beta} \kappa_2,$$

$$\max_{\eta \in \Xi \setminus \Gamma} R_\zeta(\eta, \mathcal{V}(\eta)^c) \leq C_0 e^{-\beta} \kappa_2.$$

It follows from this result and some simple algebra that there exists a finite constant C_0 such that for all $\eta \in \Xi \setminus \Gamma, \xi \in \Xi$,

$$(4.11) \quad \frac{1}{R_\zeta(\eta, \xi)} \leq \frac{e^\beta}{\mathbf{R}(\eta, \xi)} \left\{ 1 + \frac{C_0 \kappa_2}{\mathbf{R}(\eta, \xi) - C_0 \kappa_2} \right\}.$$

The proof of Proposition 4.6 is divided in several lemmas.

LEMMA 4.7. There exists a finite constant C_0 such that for all $\mathbf{x} \in \mathbb{T}_L, 0 \leq i, j \leq 3$,

$$\begin{aligned} \max_{\xi \in \mathcal{V}(\eta_{\mathbf{x}}^{i,j})} \left| R_\zeta(\eta_{\mathbf{x}}^{i,j}, \xi) - e^{-\beta} \sum_{\zeta \in \mathcal{N}(\mathcal{E}_{\mathbf{x}}^{i,j})} \mathbb{M}(\zeta, \xi) \right| &\leq C_0 e^{-\beta} \kappa_2, \\ R_\zeta(\eta_{\mathbf{x}}^{i,j}, \mathcal{V}(\eta_{\mathbf{x}}^{i,j})^c) &\leq C_0 e^{-\beta} \kappa_2. \end{aligned}$$

PROOF. Let $\eta = \eta_{\mathbf{x}}^{i,j}$ and fix a configuration $\xi \in \Xi$, $\xi \neq \eta$. Recall the formula for the jump rate $R_{\zeta}(\eta, \xi)$ provided by Lemma 4.1. We first claim that

$$(4.12) \quad |\mu_K(\eta)^{-1} \text{cap}_K(\eta, \Delta_1) - |\mathcal{N}(\mathcal{E}_{\mathbf{x}}^{i,j})|e^{-\beta}| \leq C_0 n^3 e^{-2\beta}$$

for some finite constant C_0 . Indeed, let W be the equilibrium potential: $W(\zeta) = \mathbf{P}_{\zeta}[H_{\eta} < H_{\Delta_1}]$, $\zeta \in \mathcal{E}_{\mathbf{x}}^{i,j}$. Note that ζ in the previous formula is a subscript and not a superscript. The probability refers to the process $\eta(t)$ and not to the trace process $\zeta(t)$. By the Dirichlet principle [10], $\text{cap}_K(\eta, \Delta_1) = D_K(W)$. To fix ideas, assume that $i = j = 2$, $\mathbf{x} = \mathbf{w}$. Denote by ζ_j , $1 \leq j \leq n - 1$, the configuration of $\mathcal{E}_{\mathbf{w}}^{2,2}$ which has a particle at site $\mathbf{u}_j = (j - 1, n)$. By the strong Markov property,

$$\{1 + 4e^{-\beta} + \delta_1\}W(\zeta_1) = W(\zeta_2),$$

where $\delta_1 \in (0, C_0 e^{-2\beta})$ represents the rate at which the process jumps from ζ_1 to configurations in Δ_2 . We may write similar identities for $W(\zeta_j)$, $1 \leq j \leq n - 1$. Since $0 \leq W \leq 1$ and since $W(\eta) = 1$, we deduce from these identities that $|W(\zeta_{j+1}) - W(\zeta_j)| \leq 5ne^{-\beta}$ and that $|W(\zeta_j) - 1| \leq 5n^2 e^{-\beta}$. To conclude the proof of (4.12), it remains to recall the explicit expression for the Dirichlet form $D_K(W)$.

We turn to the second term in (4.1). We claim that

$$(4.13) \quad \max_{\zeta \in \Delta_1} \left| \mathbf{P}_{\eta}[H_{\zeta} = H_{\Delta_1}] - \frac{1}{|\mathcal{N}(\mathcal{E}_{\mathbf{x}}^{i,j})|} \mathbf{1}\{\zeta \in \mathcal{N}(\mathcal{E}_{\mathbf{x}}^{i,j})\} \right| \leq C_0 n^2 e^{-\beta}$$

for some finite constant C_0 . To fix ideas, assume again that $i = j = 2$, $\mathbf{x} = \mathbf{w}$, and observe that $|\mathcal{N}(\mathcal{E}_{\mathbf{w}}^{2,2})| = 3n$. It is clear that $\mathbf{P}_{\eta}[H_{\zeta} = H_{\Delta_1}]$ is bounded by $C_0 e^{-\beta}$ if ζ does not belong to $\mathcal{N}(\mathcal{E}_{\mathbf{w}}^{2,2})$. Fix $\zeta \in \mathcal{N}(\mathcal{E}_{\mathbf{w}}^{2,2})$ and let $V(\zeta_j) = \mathbf{P}_{\zeta_j}[H_{\zeta} = H_{\Delta_1}]$, $1 \leq j \leq n - 1$. By the strong Markov property,

$$(4.14) \quad \{1 + 4e^{-\beta} + \delta_1\}V(\zeta_1) = V(\zeta_2) + e^{-\beta} \mathbf{1}(\zeta, \zeta_1),$$

where $0 < \delta_1 < C_0 e^{-2\beta}$ represents the rate at which the process jumps from ζ_1 to configurations in Δ_2 , and $\mathbf{1}(\zeta, \zeta_1)$ is equal to 1 if ζ can be obtained from ζ_1 by moving one particle, and is equal to 0 otherwise. Similar identities can be obtained for ζ_j , $2 \leq j \leq n - 1$. For some configurations, the factor 4 is replaced by 3. Summing all these identities and dividing by $e^{-\beta}$, we obtain that

$$3 \sum_{j=1}^{n-1} V(\zeta_j) + V(\zeta_1) + V(\zeta_2) + V(\zeta_{n-2}) + e^{\beta} \sum_{j=1}^{n-1} \delta_j V(\zeta_j) = 1.$$

It also follows from the identities (4.14) and from the bound $0 \leq V \leq 1$ that $|V(\zeta_2) - V(\zeta_1)| \leq 6e^{-\beta}$, $|V(\zeta_{j+2}) - V(\zeta_{j+1})| \leq |V(\zeta_{j+1}) - V(\zeta_j)| + 6e^{-\beta}$ for $1 \leq j \leq n - 3$ and for β large. Iterating these inequalities, we obtain that $|V(\zeta_k) - V(\zeta_j)| \leq 6n^2 e^{-\beta}$, $1 \leq j \leq k \leq n - 1$. Hence, summing and subtracting $V(\eta)$ in the last displayed formula we get that

$$\left| V(\eta) - \frac{1}{3n} \right| \leq C_0 n^2 e^{-\beta},$$

which proves (4.13) in the case $i = j = 2$.

It remains to evaluate the third term in the sum (4.1). We claim that

$$(4.15) \quad \max_{\zeta \in \mathcal{N}(\mathcal{E}_x^{i,j})} \max_{\xi \in \Xi} |\mathbf{P}_\zeta[H_\Xi = H_\xi] - \mathbb{M}(\zeta, \xi)| \leq C_0 L e^{-\beta/2}.$$

We prove (4.15) in the case $i = j = 2$, $\mathbf{x} = \mathbf{w}$, $\zeta = \sigma^{\mathbf{w}_2, \mathbf{z}} \eta^{\mathbf{w}}$ for some site \mathbf{z} at distance 1 from $Q_{\mathbf{w}}^{2,2}$. The other cases, which are simpler, are left to the reader. For $\zeta = \sigma^{\mathbf{w}_2, \mathbf{z}} \eta^{\mathbf{w}}$, the probability measure $\mathbb{M}(\zeta, \cdot)$ gives positive weight only to the configurations $\eta^{\mathbf{0}}$, $\eta_{\mathbf{w}}^{2,j}$, $0 \leq j \leq 3$, $\eta_{\mathbf{w}}^{1,2}$, $\eta_{\mathbf{w}}^{3,2}$, $\eta_{\mathbf{w}}^{1,1}$ and $\eta_{\mathbf{w}}^{3,1}$. We examine the case $\xi = \eta_{\mathbf{w}}^{2,2}$ and leave the others to the reader.

Until the particle at site \mathbf{z} hits the outer boundary of the square Q , it evolves as a rate 1, symmetric, nearest-neighbor random walk, while all the other particles move at rate at most $e^{-\beta}$. Denote by H_{hit} the time the particle initially at site \mathbf{z} hits $\partial_+ Q$ and by H_{mv} the time of the first jump of a particle sitting in the quasi-square $Q_{\mathbf{0}}^{2,2}$.

The random time H_{mv} is bounded below by a mean $C_0 e^\beta$ random time ϵ_β independent of the motion of the detached particle. Hence, as in the proof of Lemma 4.4,

$$(4.16) \quad \mathbf{P}_\zeta[H_{\text{mv}} < H_{\text{hit}}] \leq \mathbf{P}_\zeta[H_{\text{hit}} > \epsilon_\beta] = \int_0^\infty \mathbf{P}_\zeta[H_{\text{hit}} > t] \alpha e^{-\alpha t} dt,$$

where $\alpha^{-1} = C_0 e^\beta$. Since, by [21], Proposition 10.13, $\mathbf{E}_\zeta[H_{\text{hit}}] \leq C_0 L^2$, this last integral is bounded by $C_0 L e^{-\beta/2}$. Therefore,

$$(4.17) \quad \mathbf{P}_\zeta[H_\Xi = H_\xi] = \mathbf{P}_\zeta[H_\Xi = H_\xi, H_{\text{hit}} < H_{\text{mv}}] + L\epsilon(\beta/2).$$

On the set $H_{\text{hit}} < H_{\text{mv}}$, $\eta(H_{\text{hit}})$ belongs to the sets $\mathcal{E}_{\mathbf{0}}^{2,j}$, $0 \leq j \leq 3$, or is equal to the configurations $\eta_1^* = \sigma^{\mathbf{w}_2, \mathbf{w}_2 + e_2} \eta^{\mathbf{w}}$, $\eta_2^* = \sigma^{\mathbf{w}_2, \mathbf{w}_2 + e_1} \eta^{\mathbf{w}}$. Hence, by the strong Markov property, the previous expression is equal to

$$\begin{aligned} & \sum_{j=0}^3 \mathbf{E}_\zeta[\mathbf{1}\{H_{\text{hit}} < H_{\text{mv}}\} \mathbf{1}\{\eta(H_{\text{hit}}) \in \mathcal{E}_{\mathbf{0}}^{2,j}\} \mathbf{P}_{\eta(H_{\text{hit}})}[H_\Xi = H_\xi]] \\ & + \sum_{i=1}^2 \mathbf{P}_\zeta[H_{\text{hit}} < H_{\text{mv}}, \eta(H_{\text{hit}}) = \eta_i^*] \mathbf{P}_{\eta_i^*}[H_\Xi = H_\xi] + L\epsilon(\beta/2). \end{aligned}$$

Proceeding as in the proof of Lemma 4.4, we may replace the sets $\mathcal{E}_{\mathbf{0}}^{2,j}$ by the configurations $\eta_{\mathbf{0}}^{2,j}$ with an error bounded by $C_0 n e^{-\beta/2} \leq C_0 L e^{-\beta/2}$. Since $\xi = \eta_{\mathbf{0}}^{2,2}$, only the term $j = 2$ gives a positive contribution. By symmetry, $\mathbf{P}_{\eta_i^*}[H_\Xi = H_\xi] = P_i(\xi_{\mathbf{w}}^{0,0})$, $i = 1, 2$, where P_i has been introduced in Lemma 4.3. Hence, by the definition (3.2) of \mathbf{p} , the previous sum is equal to

$$\mathbf{p}(\mathbf{z}, Q_{\mathbf{0}}^{2,2}) + \mathbf{p}(\mathbf{z}, \mathbf{w}_2 + e_2) P_1(\xi_{\mathbf{w}}^{0,0}) + \mathbf{p}(\mathbf{z}, \mathbf{w}_2 + e_1) P_2(\xi_{\mathbf{w}}^{0,0}) + L\epsilon(\beta/2).$$

At this point, (4.15) follows from Lemma 4.3.

The assertion of the lemma is a consequence of Lemma 4.1 and of estimates (4.12), (4.13) and (4.15). We first use estimate (4.12) and the fact that $\mathbf{P}_\eta[H_{\Delta_1} = H_\zeta]$ is a probability measure in ζ to replace $\mu_K(\eta)^{-1} \text{cap}_K(\eta, \Delta_1)$ by $|\mathcal{N}(\mathcal{E}_x^{i,j})|e^{-\beta}$ with an error bounded by $C_0n^3e^{-2\beta}$. Then we consider separately the sum over $\zeta \in \mathcal{N}(\mathcal{E}_x^{i,j})$ and the sum over $\zeta \in \Delta_1 \setminus \mathcal{N}(\mathcal{E}_x^{i,j})$. For the first one, we use (4.13) to replace $\mathbf{P}_\eta[H_{\Delta_1} = H_\zeta]$ by $|\mathcal{N}(\mathcal{E}_x^{i,j})|^{-1} \mathbf{1}\{\zeta \in \mathcal{N}(\mathcal{E}_x^{i,j})\}$ with an error bounded by $C_0n^4e^{-2\beta}$ because $|\mathcal{N}(\mathcal{E}_x^{i,j})|$ is less than or equal to C_0n . To estimate the sum over $\zeta \in \Delta_1 \setminus \mathcal{N}(\mathcal{E}_x^{i,j})$, we proceed as follows. Since $|\mathcal{N}(\mathcal{E}_x^{i,j})| \leq C_0n$, we have that

$$\begin{aligned} & |\mathcal{N}(\mathcal{E}_x^{i,j})|e^{-\beta} \sum_{\zeta \in \Delta_1 \setminus \mathcal{N}(\mathcal{E}_x^{i,j})} \mathbf{P}_\eta[H_{\Delta_1} = H_\zeta] \mathbf{P}_\zeta[H_\Xi = H_\xi] \\ & \leq C_0ne^{-\beta} \sum_{\zeta \in \Delta_1 \setminus \mathcal{N}(\mathcal{E}_x^{i,j})} \mathbf{P}_\eta[H_{\Delta_1} = H_\zeta]. \end{aligned}$$

This last sum can be written as

$$1 - \sum_{\zeta \in \mathcal{N}(\mathcal{E}_x^{i,j})} \mathbf{P}_\eta[H_{\Delta_1} = H_\zeta] = \sum_{\zeta \in \mathcal{N}(\mathcal{E}_x^{i,j})} \left\{ \frac{1}{|\mathcal{N}(\mathcal{E}_x^{i,j})|} - \mathbf{P}_\eta[H_{\Delta_1} = H_\zeta] \right\}.$$

By (4.13), this expression is bounded by $C_0n^3e^{-\beta}$. Finally, we use (4.15) to replace $\mathbf{P}_\zeta[H_\Xi = H_\xi]$ by $\mathbb{M}(\zeta, \xi)$ with an error bounded by $C_0nLe^{-(3/2)\beta}$ because $|\mathcal{N}(\mathcal{E}_x^{i,j})|$ is less than or equal to C_0n . This concludes the proof of the first assertion of the lemma. The proof of the second assertion is analogous. \square

LEMMA 4.8. *There exists a finite constant C_0 such that for all $\mathbf{x} \in \mathbb{T}_L$, $\mathfrak{a} \in \{\mathfrak{s}, \mathfrak{l}\}$, $(\mathbf{k}, \ell) \in I_\mathfrak{a}^*$*

$$\max_{\xi \in \Xi} \left| R_\zeta(\eta_{\mathbf{x}}^{\mathfrak{a}, (\mathbf{k}, \ell)}, \xi) - e^{-\beta} \sum_{\zeta \in \mathcal{N}(\eta_{\mathbf{x}}^{\mathfrak{a}, (\mathbf{k}, \ell)})} \mathbb{M}(\zeta, \xi) \right| \leq C_0n^{1/2}e^{-(3/2)\beta}.$$

PROOF. Fix $\mathbf{x} \in \mathbb{T}_L$, $\mathfrak{a} \in \{\mathfrak{s}, \mathfrak{l}\}$, $(\mathbf{k}, \ell) \in I_\mathfrak{a}^*$ and let $\eta = \eta_{\mathbf{x}}^{\mathfrak{a}, (\mathbf{k}, \ell)}$. In view of (4.1), we need to compute three expressions.

We first claim that

$$|\mu_K(\eta)^{-1} \text{cap}_K(\eta, \Delta_1) - |\mathcal{N}(\eta)|e^{-\beta}| \leq C_0e^{-2\beta}$$

for some finite constant C_0 . Indeed, let W be the equilibrium potential: $W(\xi) = \mathbf{P}_\xi[H_\eta < H_{\Delta_1}]$. By the Dirichlet principle, $\text{cap}_K(\eta, \Delta_1) = D_K(W)$. From η , the process jumps at rate $e^{-\beta}$ (resp., $e^{-2\beta}$, $e^{-3\beta}$) to $|\mathcal{N}(\eta)|$ (resp., at most C_0 , $4n$) configurations in Δ_1 . Hence, since $ne^{-\beta} \ll 1$, $\mu_K(\eta)^{-1} D_K(W) = |\mathcal{N}(\eta)|e^{-\beta} \pm C_0e^{-2\beta}$, proving the claim.

By similar reasons,

$$\max_{\zeta \in \Delta_1} \left| \mathbf{P}_\eta[H_\zeta = H_{\Delta_1}] - \frac{1}{|\mathcal{N}(\eta)|} \mathbf{1}\{\zeta \in \mathcal{N}(\eta)\} \right| \leq C_0 e^{-\beta}$$

for some finite constant C_0 . Finally, we claim that

$$\max_{\zeta \in \mathcal{N}(\eta)} \max_{\xi \in \Xi} |\mathbf{P}_\zeta[H_\Xi = H_\xi] - \mathbb{M}(\zeta, \xi)| \leq C_0 \sqrt{ne^{-\beta}}.$$

There are two different cases. Assume that ζ is obtained from η by moving a particle on a side which has $3 \leq m \leq n$ particles. In this case, the configuration ζ has a hole in one of the sides of the square which moves according to a rate 1 nearest-neighbor, symmetric random walk until it reaches the boundary of the set of m particles. Let H_{hit} be the time the hole attains the boundary and let H_{mv} be the first time a jump of rate $e^{-\beta}$ or less occurs. H_{mv} is bounded below by a mean $C_0 e^\beta$ exponential random variable ϵ_β , independent of the displacement of the hole. Since $\mathbf{E}_\zeta[H_{\text{hit}}] = (m - 1)/2$, by an argument repeatedly used in the previous proofs, $\mathbf{P}_\zeta[H_{\text{mv}} < H_{\text{hit}}] \leq C_0 (ne^{-\beta})^{1/2}$. Hence,

$$\mathbf{P}_\zeta[H_\Xi = H_\xi] = \mathbf{P}_\zeta[H_\Xi = H_\xi, H_{\text{hit}} < H_{\text{mv}}] + \sqrt{n}\epsilon(\beta/2).$$

On the set $\{H_{\text{hit}} < H_{\text{mv}}\}$, $H_{\text{hit}} = H_\Xi$ and either $\eta(H_{\text{hit}})$ is the original configuration or it is the one in which a row or a column of particles in a side of the rectangle has been translated by one unit. The first case has probability $(m - 1)/m$ and the second one $1/m$.

This argument can be extended to the case where ζ is obtained from η by moving a particle on a side which has 2 particles.

To conclude the proof, it remains to put together the previous three estimates, as in the previous lemma. \square

The proof of the next lemma is similar to that of Lemma 4.7 and the proof of the following one is identical to that of Lemma 4.8.

LEMMA 4.9. *There exists a finite constant C_0 such that for all $\mathbf{x} \in \mathbb{T}_L$, $\mathbf{a} \in \{\mathfrak{s}, \mathfrak{l}\}$, $0 \leq j \leq 3$,*

$$\max_{\xi \in \Xi} \left| R_\zeta(\zeta_{\mathbf{x}}^{\mathbf{a},j}, \xi) - e^{-\beta} \sum_{\zeta \in \mathcal{N}(\xi_{\mathbf{x}}^{\mathbf{a},j})} \mathbb{M}(\zeta, \xi) \right| \leq C_0 e^{-2\beta}.$$

LEMMA 4.10. *There exists a finite constant C_0 such that for all $\mathbf{x} \in \mathbb{T}_L$, $\mathbf{a} \in \{\mathfrak{s}, \mathfrak{l}\}$, $(\mathbf{k}, \ell) \in I_{2,\mathbf{a}}^*$,*

$$\max_{\xi \in \Xi} \left| R_\zeta(\zeta_{\mathbf{x}}^{\mathbf{a},(\mathbf{k},\ell)}, \xi) - e^{-\beta} \sum_{\zeta \in \mathcal{N}(\zeta_{\mathbf{x}}^{\mathbf{a},(\mathbf{k},\ell)})} \mathbb{M}(\zeta, \xi) \right| \leq C_0 ne^{-(3/2)\beta}.$$

5. Coupling $\zeta(t)$ and $\widehat{\zeta}(t)$. In the previous section, we estimate the difference between the jump rates of the trace process $\zeta(t)$ and of the asymptotic process $\widehat{\zeta}(t)$. We use these estimates here to compare the hitting times of these processes. The first result of this section shows that we may couple the processes $\zeta(t)$ and $\widehat{\zeta}(t)$ in such a way that they stay together for a long amount of time. Consider two continuous-time Markov processes X^1, X^2 taking values on the same countable state space E . Denote by R_1, R_2 the respective jump rates.

LEMMA 5.1. *Assume that*

$$\sup_{\eta \in E} \sum_{\xi \in E} |R_1(\eta, \xi) - R_2(\eta, \xi)| \leq \alpha$$

for some $\alpha < \infty$. Then, for every $\eta \in E$, there exists a coupling $\bar{\mathbf{P}}$ of the two processes such that $X^1(0) = X^2(0) = \eta$ $\bar{\mathbf{P}}$ -a.s., and such that for all $t > 0$

$$\bar{\mathbf{P}}[T_{cp} \leq t] \leq P[\epsilon_\alpha \leq t],$$

where T_{cp} is the first time the processes separate, $T_{cp} = \inf\{t > 0 : X_t^1 \neq X_t^2\}$, and where ϵ_α is a mean α^{-1} exponential random variable.

The proof of this result is elementary and left to the reader. By this lemma and by Propositions 4.6 and 4.2, for each configuration $\eta \in \Xi$, there exists a coupling of the processes $\zeta(t)$ and $\widehat{\zeta}(t)$, denoted by $\mathbf{P}_\eta^{\zeta, \widehat{\zeta}}$, such that $\mathbf{P}_\eta^{\zeta, \widehat{\zeta}}[\zeta(0) = \widehat{\zeta}(0) = \eta] = 1$ and such that for all $t > 0$

$$(5.1) \quad \mathbf{P}_\eta^{\zeta, \widehat{\zeta}}[T_{cp} \leq t] \leq P[\epsilon_\beta \leq t],$$

where ϵ_β is a mean $e^\beta \kappa_2^{-1}$ exponential random variable.

PROPOSITION 5.2. *Recall from (1.5) the definition of the error $\mathbf{e}(\beta)$. For a subset A of Λ_L , let $\Gamma_A = \{\eta^{\mathbf{x}} \in \Omega_{L,K} : \mathbf{x} \in A\}$. Assume that $\lim_{\beta} \kappa_1 = 0$. Then*

$$\max_{A \subset \Lambda_L} |\mathbf{P}_{\eta^0}^\zeta[H_\Gamma^+ = H_{\Gamma_A}^+] - \mathbf{P}_{\eta^0}^{\widehat{\zeta}}[H_\Gamma^+ = H_{\Gamma_A}^+]| \leq \mathbf{e}(\beta).$$

PROOF. Fix a subset A of Λ_L and let τ_1 be the time of the first jump so that

$$\mathbf{P}_{\eta^0}^\zeta[H_\Gamma^+ = H_{\Gamma_A}^+] = \sum_{\eta \in \Xi} \mathbf{P}_{\eta^0}^\zeta[\zeta(\tau_1) = \eta] \mathbf{P}_\eta^\zeta[H_\Gamma = H_{\Gamma_A}].$$

By Proposition 4.2, this expression is equal to

$$\sum_{\eta \in \mathcal{V}(\eta^0)} \mathbf{P}_{\eta^0}^{\widehat{\zeta}}[\widehat{\zeta}(\tau_1) = \eta] \mathbf{P}_\eta^\zeta[H_\Gamma = H_{\Gamma_A}] + R(\beta),$$

where $R(\beta)$ is a remainder absolutely bounded by $e^{-2\beta}\kappa_1[\lambda_\zeta(\eta^0)^{-1} + \lambda_{\widehat{\zeta}}(\eta^0)^{-1}]$, and $\lambda_\zeta(\eta^0)$, $\lambda_{\widehat{\zeta}}(\eta^0)$ are the holding rates at η^0 of the processes $\zeta(t)$, $\widehat{\zeta}(t)$, respectively. By (3.5) and by Proposition 4.2, $\lambda_{\widehat{\zeta}}(\eta^0) \geq (1/4)e^{-2\beta}$ and $\lambda_\zeta(\eta^0) \geq e^{-2\beta}\{(1/4) - \kappa_1\}$. This proves that $|R(\beta)| \leq \kappa_1$.

We estimate $\mathbf{P}_\eta^\zeta[H_\Gamma = H_{\Gamma_A}]$ for a fixed $\eta \in \mathcal{V}(\eta^0)$. Recall the definition of the coupling $\mathbf{P}_\eta^{\zeta, \widehat{\zeta}}$ introduced after Lemma 5.1. Under this coupling, on the set $\{H_\Gamma \leq T_{\text{cp}}\}$ the processes $\zeta(t)$ and $\widehat{\zeta}(t)$ reach the set Γ at the same time and at the same configuration. Hence,

$$\mathbf{P}_\eta^\zeta[H_\Gamma = H_{\Gamma_A}] = \mathbf{P}_\eta^{\zeta, \widehat{\zeta}}[H_\Gamma^\zeta = H_{\Gamma_A}^\zeta] \leq \mathbf{P}_\eta^{\widehat{\zeta}}[H_\Gamma = H_{\Gamma_A}] + \mathbf{P}_\eta^{\zeta, \widehat{\zeta}}[H_\Gamma^\zeta \geq T_{\text{cp}}].$$

In this formula, H^ζ represents the hitting time for the process $\zeta(t)$. A similar inequality holds interchanging the roles of ζ and $\widehat{\zeta}$. Therefore,

$$|\mathbf{P}_\eta^\zeta[H_\Gamma = H_{\Gamma_A}] - \mathbf{P}_\eta^{\widehat{\zeta}}[H_\Gamma = H_{\Gamma_A}]| \leq \mathbf{P}_\eta^{\zeta, \widehat{\zeta}}[H_\Gamma^\zeta \geq T_{\text{cp}}] + \mathbf{P}_\eta^{\zeta, \widehat{\zeta}}[H_\Gamma^{\widehat{\zeta}} \geq T_{\text{cp}}].$$

We estimate the first probability on the right-hand side, the arguments needed for the second one being similar.

Consider a sequence T_β to be chosen later. By Lemma 5.1 and by (5.1),

$$\mathbf{P}_\eta^{\zeta, \widehat{\zeta}}[H_\Gamma^\zeta \geq T_{\text{cp}}] \leq \mathbf{P}_\eta^\zeta[H_\Gamma^\zeta \geq T_\beta] + P[\epsilon_\beta \leq T_\beta],$$

where ϵ_β is a mean $e^\beta \kappa_2^{-1}$ exponential random variable. Since $1 - e^{-x} \leq x$, $x > 0$, the second term is bounded by $e^{-\beta} T_\beta \kappa_2$. On the other hand, by Chebyshev's inequality and by Lemma 5.3 below,

$$\mathbf{P}_\eta^\zeta[H_\Gamma^\zeta \geq T_\beta] \leq \frac{1}{T_\beta} \mathbf{E}_\eta^\zeta[H_\Gamma] \leq \frac{C_0 |\Xi| e^\beta}{L^2 T_\beta}.$$

Choosing $T_\beta = e^\beta (|\Xi|/\kappa_2)^{1/2}$ we obtain that

$$|\mathbf{P}_\eta^\zeta[H_\Gamma = H_{\Gamma_A}] - \mathbf{P}_\eta^{\widehat{\zeta}}[H_\Gamma = H_{\Gamma_A}]| \leq \sqrt{|\Xi| \kappa_2 / L^2}.$$

Estimating $\mathbf{P}_\eta^{\zeta, \widehat{\zeta}}[H_\Gamma^\zeta \geq T_{\text{cp}}]$ in a similar way, we complete the proof of the proposition. \square

The trace process on equivalent classes. Denote by $\{\tau_x : x \in \mathbb{Z}^2\}$ the group of translations in \mathbb{Z}^2 : For any configuration $\eta \in \Omega_{L,K}$, $\tau_x \eta$ is the configuration defined by $(\tau_x \eta)(y) = \eta(x + y)$, where the summation is performed modulo L . Two configurations η, ξ , are said to be equivalent, $\eta \sim \xi$, if $\eta = \tau_x \xi$ for some $x \in \mathbb{Z}^2$. Denote by $\widetilde{\Xi}$ the equivalence classes of Ξ , and by η, ξ the equivalent classes, that is, the elements of $\widetilde{\Xi}$.

Let $\widetilde{\Psi} : \Xi \rightarrow \widetilde{\Xi}$ be the function which associates to a configuration its equivalence class. Since the dynamics is translation invariant, for all $\xi \in \widetilde{\Xi}$

$$R_\zeta(\eta, \widetilde{\Psi}^{-1}(\xi)) = R_\zeta(\eta', \widetilde{\Psi}^{-1}(\xi)) \quad \text{if } \widetilde{\Psi}(\eta) = \widetilde{\Psi}(\eta').$$

In particular,

$$\zeta(t) = \tilde{\Psi}(\zeta(t))$$

is a Markov chain which jumps from a class η to a class ξ at rate $\mathbf{R}_\zeta(\eta, \xi) := R_\zeta(\eta, \xi) = \sum_{\xi \in \tilde{\Xi}} R_\zeta(\eta, \xi)$ if η belongs to the equivalent class η . Moreover, the probability measure μ_ζ on $\tilde{\Xi}$ defined by $\mu_\zeta(\eta) = \mu_\zeta(\Psi^{-1}(\eta))$ is reversible for the Markov chain $\zeta(t)$.

An auxiliary dynamics. Denote by η^0 the equivalence class of the configurations in Γ . We introduce in this subsection a process $\tilde{\zeta}(t)$ which behaves as $\zeta(t)$ until $\zeta(t)$ hits the class η^0 . Let $\tilde{\zeta}(t)$ be the Markov process on $\tilde{\Xi}$ whose jump rates, denoted by $\mathbf{R}_{\tilde{\zeta}}(\eta, \xi)$, are defined as follows:

$$\mathbf{R}_{\tilde{\zeta}}(\eta, \xi) = \mathbf{R}_\zeta(\eta, \xi), \quad \eta \in \tilde{\Xi} \setminus \{\eta^0\}, \xi \in \tilde{\Xi}.$$

It is clear that we may couple $\zeta(t)$ and $\tilde{\zeta}(t)$ in such a way that starting from a configuration $\eta \in \tilde{\Xi} \setminus \{\eta^0\}$, both processes evolve together until they reach simultaneously η^0 . Let $\mu_{\tilde{\zeta}}$ be the uniform measure on $\tilde{\Xi}$. Since $\mu_\zeta(\eta) = \mu_\zeta(\xi)$ for $\eta, \xi \in \tilde{\Xi} \setminus \{\eta^0\}$, and since the process $\zeta(t)$ is reversible, the measure $\mu_{\tilde{\zeta}}$ satisfies the detailed balance conditions on $\tilde{\Xi} \setminus \{\eta^0\}$ for the rates $\mathbf{R}_{\tilde{\zeta}}$. We define $\mathbf{R}_{\tilde{\zeta}}(\eta^0, \xi)$ $\xi \in \tilde{\Xi} \setminus \{\eta^0\}$ to fulfill the detailed balance conditions with respect to the uniform measure $\mu_{\tilde{\zeta}}$:

$$\mathbf{R}_{\tilde{\zeta}}(\eta^0, \xi) = \mathbf{R}_{\tilde{\zeta}}(\xi, \eta^0), \quad \xi \in \tilde{\Xi} \setminus \{\eta^0\}.$$

LEMMA 5.3. *There exists a finite constant C_0 such that*

$$\max_{\eta \in \mathcal{V}(\eta^0)} \mathbf{E}_\eta^\zeta[H_\Gamma] \leq C_0(|\Xi|/L^2)e^\beta.$$

PROOF. Fix a configuration η in $\mathcal{V}(\eta^0)$ and denote by η the equivalence class of η . Recall the definition of the processes $\zeta(t)$, $\tilde{\zeta}(t)$ introduced above. Let \mathbf{P}_η^ζ , $\mathbf{P}_\eta^{\tilde{\zeta}}$ be the distributions of the processes $\zeta(t)$, $\tilde{\zeta}(t)$, respectively, starting from η . Expectations with respect to \mathbf{P}_η^ζ , $\mathbf{P}_\eta^{\tilde{\zeta}}$ are represented by \mathbf{E}_η^ζ , $\mathbf{E}_\eta^{\tilde{\zeta}}$, respectively.

Denote by H_{η^0} the time the processes $\zeta(t)$, $\tilde{\zeta}(t)$ hit the equivalence class η^0 . Since $H_{\eta^0} = H_\Gamma$ and since the processes $\zeta(t)$, $\tilde{\zeta}(t)$ evolve together until the hitting time of η^0 ,

$$\mathbf{E}_\eta^\zeta[H_\Gamma] = \mathbf{E}_\eta^\zeta[H_{\eta^0}] = \mathbf{E}_\eta^{\tilde{\zeta}}[H_{\eta^0}].$$

Since the chain $\tilde{\zeta}(t)$ is reversible, by [1], Proposition 6.10,

$$\mathbf{E}_\eta^{\tilde{\zeta}}[H_{\eta^0}] \leq \frac{1}{\text{cap}_{\tilde{\zeta}}(\eta, \eta^0)},$$

where $\text{cap}_{\tilde{\zeta}}$ stands for the capacity associated to the process $\tilde{\zeta}$. To fix ideas, assume that $\eta = \eta_0^{0,2}$ and consider the unitary flow Φ from η to η^0 given by $\Phi(\eta_0^{0,2}, \eta_0^{0,1}) = \Phi(\eta_0^{0,1}, \eta_0^{0,0}) = \Phi(\eta_0^{0,0}, \eta^0) = 1$, where $\eta_0^{i,j}$ represents the equivalence class of $\eta_0^{i,j}$. By Thomson's principle [14], since $\mathbf{R}_{\tilde{\zeta}}(\eta, \xi) = \mathbf{R}_{\zeta}(\eta, \xi) \geq R_{\zeta}(\eta, \xi)$, $\eta \in \Xi \setminus \Gamma$, and since $\mu_{\tilde{\zeta}}$ is the uniform measure on $\tilde{\Xi}$,

$$\begin{aligned} \frac{1}{\text{cap}_{\tilde{\zeta}}(\eta, \eta^0)} &\leq \frac{1}{\mu_{\tilde{\zeta}}(\eta)} \left\{ \frac{1}{\mathbf{R}_{\zeta}(\eta_0^{0,2}, \eta_0^{0,1})} + \frac{1}{\mathbf{R}_{\zeta}(\eta_0^{0,1}, \eta_0^{0,0})} + \frac{1}{\mathbf{R}_{\zeta}(\eta_0^{0,0}, \eta^0)} \right\} \\ &\leq |\tilde{\Xi}| \left\{ \frac{1}{R_{\zeta}(\eta_0^{0,2}, \eta_0^{0,1})} + \frac{1}{R_{\zeta}(\eta_0^{0,1}, \eta_0^{0,0})} + \frac{1}{R_{\zeta}(\eta_0^{0,0}, \eta^0)} \right\}. \end{aligned}$$

By (4.11) and (3.9), the previous sum is bounded by $C_0 |\tilde{\Xi}| e^{\beta}$ for some finite constant C_0 , which proves the lemma since $|\tilde{\Xi}| = |\Xi|/L^2$. \square

6. Proof of Theorem 1.1. We prove in this section Theorem 1.1. Instead of working in the torus \mathbb{T}_L we will consider the process $X(t)$ as a random walk on \mathbb{Z}^2 . Let $\bar{X}(t)$ the random walk on \mathbb{Z}^2 which jumps from $\mathbf{x} \in \mathbb{Z}^2$ to $\mathbf{y} \in \mathbb{Z}^2$ at rate $r_{\beta}(\mathbf{y} - \mathbf{x})$ and let $\bar{Z}^{\beta}(t) = \bar{X}(t\theta_{\beta}\ell^2)/\ell$. To prove Theorem 1.1, it is enough to show that $\bar{Z}^{\beta}(t)$ converges in the Skorohod topology to a two-dimensional Brownian motion as $\beta \rightarrow \infty$.

In view of the representation of the process $\bar{X}(t)$ in terms of the orthogonal martingales $M_t^{\mathbf{x}}$, $\bar{Z}^{\beta}(t) = (\bar{Z}_1^{\beta}(t), \bar{Z}_2^{\beta}(t))$ is a two-dimensional martingale. Since $r_{\beta}(x_1, x_2) = r_{\beta}(\pm x_1, \pm x_2)$, the predictable quadratic covariation of $\bar{Z}_i^{\beta}(t), \bar{Z}_j^{\beta}(t)$, $1 \leq i, j \leq 2$, denoted by $\langle \bar{Z}_i^{\beta}, \bar{Z}_j^{\beta} \rangle_t$, is given by

$$\langle \bar{Z}_i^{\beta}, \bar{Z}_j^{\beta} \rangle_t = t\theta_{\beta} \sum_{\mathbf{x}=(x_1, x_2) \in \Lambda_L} x_i x_j r_{\beta}(\mathbf{x}) = t\theta_{\beta} \delta_{i,j} \sum_{\mathbf{x}=(x_1, x_2) \in \Lambda_L} x_i^2 r_{\beta}(\mathbf{x}),$$

where $\Lambda_L = \{-L, \dots, L\}^2$. Hence, by the definition of θ_{β} ,

$$(6.1) \quad \langle \bar{Z}_i^{\beta}, \bar{Z}_j^{\beta} \rangle_t = (1/2)t\delta_{i,j}.$$

Let $\Delta \bar{Z}^{\beta}(t) = \bar{Z}^{\beta}(t) - \bar{Z}^{\beta}(t-)$, $t \geq 0$. Since $\bar{Z}^{\beta}(0) = 0$, by the martingale central limit theorem, [11], Theorem VII.1.4, in view of (6.1), to prove that $\bar{Z}^{\beta}(t)$ converges to a two-dimensional Brownian motion with diffusion matrix equal to $(1/2)t\mathbb{I}$, we have to show that

$$(6.2) \quad \lim_{\beta \rightarrow \infty} \mathbf{E} \left[\sup_{s \leq t} \|\Delta \bar{Z}^{\beta}(s)\| \right] = 0$$

for all $t > 0$.

In the remaining of this section, we prove (6.2). Fix $\delta > 0$. By definition of $\bar{Z}^{\beta}(s)$, $\sup_{s \leq t} \|\Delta \bar{Z}^{\beta}(s)\| = \sup_{s \leq t} \ell^2 \theta_{\beta} \|\Delta \bar{X}(s)\|/\ell$. Since the Poisson processes

$N_t^{\mathbf{x}}$ have no common jumps, for all $s \geq 0$

$$\|\Delta \bar{X}(s)\| = \left\| \sum_{\mathbf{x} \in \Lambda_L} \mathbf{x} [N_s^{\mathbf{x}} - N_{s-}^{\mathbf{x}}] \right\| = \sum_{\mathbf{x} \in \Lambda_L} \|\mathbf{x}\| [N_s^{\mathbf{x}} - N_{s-}^{\mathbf{x}}]$$

and

$$\sum_{\mathbf{x} \in \Lambda_{\delta\ell}} \|\mathbf{x}\| [N_s^{\mathbf{x}} - N_{s-}^{\mathbf{x}}] \leq 2\delta\ell \sum_{\mathbf{x} \in \Lambda_{\delta\ell}} [N_s^{\mathbf{x}} - N_{s-}^{\mathbf{x}}] \leq 2\delta\ell,$$

where $\Lambda_k = \{-k, \dots, k\}^2$. The expectation appearing in (6.2) is thus bounded by

$$2\delta + \sum_{\mathbf{x} \notin \Lambda_{\delta\ell}} \frac{\|\mathbf{x}\|}{\ell} \mathbf{E} \left[\sup_{s \leq t\ell^2\theta_\beta} [N_s^{\mathbf{x}} - N_{s-}^{\mathbf{x}}] \right] = 2\delta + \sum_{\mathbf{x} \notin \Lambda_{\delta\ell}} \frac{\|\mathbf{x}\|}{\ell} \mathbf{P}[N_{t\ell^2\theta_\beta}^{\mathbf{x}} \geq 1].$$

Since $\mathbf{P}[N_{t\ell^2\theta_\beta}^{\mathbf{x}} \geq 1] \leq \mathbf{E}[N_{t\ell^2\theta_\beta}^{\mathbf{x}}] = t\ell^2\theta_\beta r_\beta(\mathbf{x})$, to prove (6.2) it is enough to show that for all $\delta > 0$

$$(6.3) \quad \lim_{\beta \rightarrow \infty} \ell\theta_\beta \sum_{\|\mathbf{x}\| > \delta\ell} \|\mathbf{x}\| r_\beta(\mathbf{x}) = 0.$$

Replacing the Euclidean norm $\|\cdot\|$ by the sum norm $|\cdot|$ and recalling formula (9.1) for $r_\beta(\mathbf{x})$, we bound the sum appearing in (6.3) by

$$(6.4) \quad \ell\theta_\beta \sum_{|\mathbf{x}| > \delta\ell} |\mathbf{x}| R_\xi(\eta^0, \eta^{\mathbf{x}}) = \ell\theta_\beta \lambda_\zeta(\eta^0) \sum_{|\mathbf{x}| > \delta\ell} |\mathbf{x}| \mathbf{P}_{\eta^0}^\zeta [H_\Gamma^+ = H_{\eta^{\mathbf{x}}}] .$$

By equation (6.9) in [1], $\lambda_\zeta(\eta^0) \leq \lambda(\eta^0) = 8e^{-2\beta}(1 + O(n)e^{-\beta})$. The sum appearing in (6.3) is thus bounded by

$$C_0 \ell\theta_\beta e^{-2\beta} \sum_{|\mathbf{x}| > \delta\ell} |\mathbf{x}| \mathbf{P}_{\eta^0}^\zeta [H_\Gamma^+ = H_{\eta^{\mathbf{x}}}] = C_0 \ell\theta_\beta e^{-2\beta} \sum_{k=[\delta\ell]+1}^{2L} k \mathbf{P}_{\eta^0}^\zeta [H_\Gamma^+ = H_{\diamond_k}],$$

where $\diamond_k = \{\eta^{\mathbf{x}} \in \Omega_{L,K} : |\mathbf{x}| = k\}$ and $[a]$ stands for the integer part of $a > 0$. Let $\blacklozenge_k = \{\eta^{\mathbf{x}} \in \Omega_{L,K} : |\mathbf{x}| \geq k\}$. With this notation we may rewrite the previous sum as

$$\begin{aligned} & \sum_{k=[\delta\ell]+1}^{2L} k \{ \mathbf{P}_{\eta^0}^\zeta [H_\Gamma^+ = H_{\blacklozenge_k}] - \mathbf{P}_{\eta^0}^\zeta [H_\Gamma^+ = H_{\blacklozenge_{k+1}}] \} \\ & \leq C_0 \ell \mathbf{P}_{\eta^0}^\zeta [H_\Gamma^+ = H_{\blacklozenge_{[\delta\ell]}}] + \sum_{k=[\delta\ell]+1}^{2L} \mathbf{P}_{\eta^0}^\zeta [H_\Gamma^+ = H_{\blacklozenge_k}], \end{aligned}$$

where we performed a summation by parts in the last step. By Proposition 5.2 and Lemma 6.1, the previous expression is bounded by

$$L\kappa_3 + C_0 \gamma^{([\delta\ell]-4)/3} \left\{ \ell + \frac{\gamma^{1/3}}{1 - \gamma^{1/3}} \right\},$$

where $\kappa_3 = \kappa_1 + \sqrt{\kappa_2|\Xi|/L^2}$ and γ is given by (6.6) below. By (6.12), $\gamma \leq e^{-c_0/2n}$. The previous expression is thus bounded by $L\kappa_3 + (\ell + n)e^{-c_0\delta\ell/n}$.

Up to this point, we have shown that the sum appearing in (6.3) is bounded by

$$\ell\theta_\beta e^{-2\beta} \{L\kappa_3 + (\ell + n)e^{-c_0\delta\ell/n}\}.$$

In view of the assumptions of the theorem and of (6.2), this proves (6.3) and concludes the argument. \square

We conclude this section with two results used in the proof of Theorem 1.1. Recall the definition of \mathcal{G}_x given in (3.6). Let $\mathfrak{E}_x, x \in \mathbb{T}_L$, be the event that the process $\widehat{\zeta}(t)$ leaves the set \mathcal{G}_x without visiting the configuration η^x ,

$$\mathfrak{E}_x = \{H_{\mathcal{G}_x}^{\widehat{\zeta}} < H_{\eta^x}^{\widehat{\zeta}}\},$$

where $H_C^{\widehat{\zeta}}, C \subset \Xi$, stands for the hitting time of C by $\widehat{\zeta}$:

$$H_C^{\widehat{\zeta}} = \inf\{t > 0 : \widehat{\zeta}(t) \in C\}.$$

Let

$$(6.5) \quad \varrho := \max_{\eta \in \mathcal{V}(\eta^0)} \mathbf{P}_\eta^{\widehat{\zeta}}[\mathfrak{E}_0].$$

For $x \in \mathbb{T}_L$, let $\mathcal{L}_x = \{\zeta_x^{l,j}, 0 \leq j \leq 3\}, \mathcal{S}_x = \{\zeta_x^{s,j}, 0 \leq j \leq 3\}$, and

$$\mathcal{G} = \bigcup_{x \in \mathbb{T}_L} \mathcal{G}_x, \quad \mathcal{L} = \bigcup_{x \in \mathbb{T}_L} \mathcal{L}_x, \quad \mathcal{S} = \bigcup_{x \in \mathbb{T}_L} \mathcal{S}_x.$$

Define

$$(6.6) \quad \gamma := \varrho + (1 - \varrho) \max_{\xi \in \mathcal{L}_0} \mathbf{P}_\xi^{\widehat{\zeta}}[H_{\mathcal{L} \setminus \mathcal{L}_0} < H_{\mathcal{G}}].$$

LEMMA 6.1. *For every $k \geq 5$,*

$$\mathbf{P}_{\eta^0}^{\widehat{\zeta}}[H_\Gamma^+ = H_{\blacklozenge_k}] \leq \gamma^{(k-4)/3}.$$

PROOF. As the chain $\widehat{\zeta}(t)$ jumps from η^0 only to $\mathcal{V}(\eta^0)$, it is enough to show that

$$\max_{\eta \in \mathcal{V}(\eta^0)} \mathbf{P}_\eta^{\widehat{\zeta}}[H_\Gamma = H_{\blacklozenge_k}] \leq \gamma^{(k-4)/3}.$$

Fix $\eta \in \mathcal{V}(\eta^0)$. To prove the lemma, we deplete Ξ of its inner configurations, and we keep only the extremal ones. Let

$$\Xi_e = \bigcup_{x \in \mathbb{T}_L} \{\mathcal{G}_x \cup \mathcal{L}_x \cup \mathcal{S}_x\}.$$

Denote by $\widehat{\chi}(t)$ the trace of $\widehat{\zeta}(t)$ on Ξ_e . An inspection shows that the process $\widehat{\chi}(t)$ may only jump from a configuration in \mathcal{G}_x to a configuration in $\mathcal{G}_x \cup \mathcal{G}_{x\pm e_1} \cup \mathcal{L}_x \cup \mathcal{L}_{x+e_2} \cup \mathcal{L}_{x-e_1} \cup \mathcal{L}_{x-e_1+e_2} \cup \mathcal{S}_x \cup \mathcal{S}_{x+e_1} \cup \mathcal{S}_{x-e_2} \cup \mathcal{S}_{x+e_1-e_2}$. Similarly, the process $\widehat{\chi}(t)$ may only jump from a configuration in \mathcal{L}_x to a configuration in $\mathcal{L}_x \cup \mathcal{L}_{x\pm e_1} \cup \mathcal{L}_{x\pm e_2} \cup \mathcal{G}_x \cup \mathcal{G}_{x+e_1} \cup \mathcal{G}_{x-e_2} \cup \mathcal{G}_{x+e_1-e_2}$. Finally, the process $\widehat{\chi}(t)$ may only jump from a configuration in \mathcal{S}_x to a configuration in $\mathcal{S}_x \cup \mathcal{S}_{x\pm e_1} \cup \mathcal{S}_{x\pm e_2} \cup \mathcal{G}_x \cup \mathcal{G}_{x-e_1} \cup \mathcal{G}_{x+e_2} \cup \mathcal{G}_{x-e_1+e_2}$. In all cases, the distance from the index x of the starting set to the index y of the ending set is at most 2. The neighborhood of a configuration in \mathcal{L}_x is illustrated in Figure 9.

Recall that $\widehat{\zeta}(0) = \eta^0$. Let $\Pi_j = \bigcup_{x: |x| \geq j} \{\mathcal{G}_x \cup \mathcal{L}_x \cup \mathcal{S}_x\}$, $j \geq 1$, and denote by τ_j the hitting time of the set Π_j by the process $\widehat{\zeta}(t)$:

$$\tau_j = \inf\{t > 0 : \widehat{\zeta}(t) \in \Pi_j\}, \quad j \geq 0.$$

It follows from the conclusions of the previous paragraph that $\widehat{\chi}(t) \in \Xi_e \setminus \Pi_{j+2}$ if $\widehat{\chi}(t-) \in \Xi_e \setminus \Pi_j$. In particular, $\tau_j < \tau_{j+2}$ for $j \geq 1$.

Let $Y: \Xi_e \rightarrow \mathbb{T}_L$ be given by

$$Y(\eta) = \sum_{x \in \mathbb{T}_L} x \mathbf{1}\{\eta \in \mathcal{G}_x \cup \mathcal{L}_x \cup \mathcal{S}_x\},$$

and let $Y_j = Y(\widehat{\zeta}(\tau_j))$. In the formulas below, $\mathcal{A} = \mathcal{L}$ if $a = l$ and $\mathcal{A} = \mathcal{S}$ if $a = s$. Consider the events

$$\mathfrak{E}_j^1 = \{\widehat{\zeta}(\tau_j) \in \mathcal{G}\} \cap \mathfrak{E}_{Y_j},$$

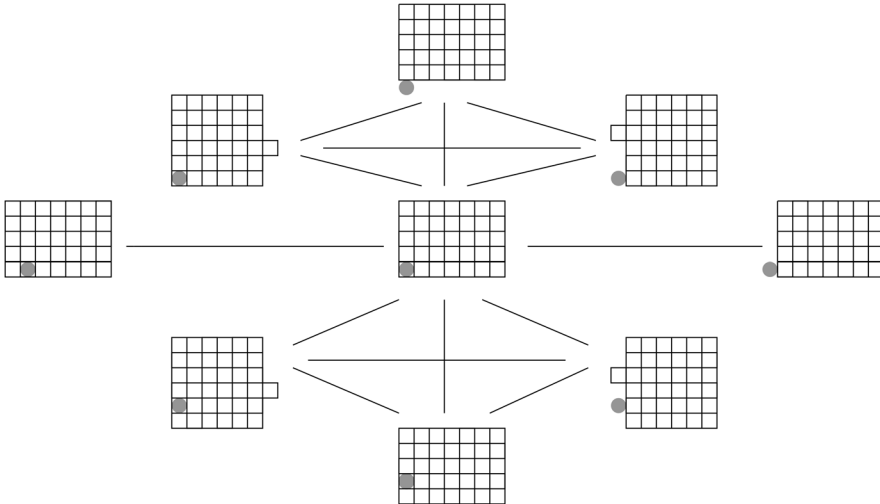


FIG. 9. The neighborhood of a configuration ζ_0^l . We omitted the extra square of the configurations ζ_x^l . The gray dot represents the site 0 .

$$\begin{aligned} \mathfrak{E}_j^{2,a} &= \{\widehat{\zeta}(\tau_j) \in \mathcal{A}\} \cap \{H_{\mathcal{A} \setminus \mathcal{A}_{Y_j}} \circ \theta_{\tau_j} < H_G \circ \theta_{\tau_j}\}, \\ \mathfrak{E}_j^{3,a} &= \{\widehat{\zeta}(\tau_j) \in \mathcal{A}\} \cap \{H_{\mathcal{A} \setminus \mathcal{A}_{Y_j}} \circ \theta_{\tau_j} > H_G \circ \theta_{\tau_j}\} \cap \mathfrak{E}_{Y(\widehat{\zeta}(H_G \circ \theta_{\tau_j}))}, \\ \mathfrak{E}_j &= \mathfrak{E}_j^1 \cup \bigcup_{a=l,s} \{\mathfrak{E}_j^{2,a} \cup \mathfrak{E}_j^{3,a}\}. \end{aligned}$$

Note that \mathfrak{E}_j is measurable with respect to the σ -algebra induced by the stopping time τ_{j+3} of the Markov process $\widehat{\zeta}(t)$, which we denote by $\mathcal{F}_{\tau_{j+3}}$. Moreover, $\{H_\Gamma^+ = H_{\blacklozenge_k}\} \subset \bigcap_{1 \leq j \leq k-4} \mathfrak{E}_j$. Hence,

$$\mathbf{P}_\eta^{\widehat{\zeta}}[H_\Gamma = H_{\blacklozenge_k}] \leq \mathbf{P}_\eta^{\widehat{\zeta}} \left[\bigcap_{j=1}^{k-4} \mathfrak{E}_j \right].$$

Fix a positive integer $m \geq 1$. Since \mathfrak{E}_{3j} belongs to $\mathcal{F}_{\tau_{3m}}$ for $j < m$, taking conditional expectation with respect to $\mathcal{F}_{\tau_{3m}}$, by the strong Markov property we obtain that

$$\mathbf{P}_\eta^{\widehat{\zeta}} \left[\bigcap_{j=1}^m \mathfrak{E}_{3j} \right] = \mathbf{E}_\eta^{\widehat{\zeta}} \left[\mathbf{1} \left\{ \bigcap_{j=1}^{m-1} \mathfrak{E}_{3j} \right\} \gamma_0(\widehat{\zeta}(\tau_{3m})) \right],$$

where

$$\begin{aligned} (6.7) \quad \gamma_0(\xi) &= \mathbf{1}\{\xi \in \mathcal{G}\} \mathbf{P}_\xi^{\widehat{\zeta}}[\mathfrak{E}_{Y(\xi)}] + \sum_{A=\mathcal{L},\mathcal{S}} \mathbf{1}\{\xi \in \mathcal{A}\} \mathbf{P}_\xi^{\widehat{\zeta}}[H_{\mathcal{A} \setminus \mathcal{A}_{Y(\xi)}} < H_G] \\ &+ \sum_{A=\mathcal{L},\mathcal{S}} \mathbf{1}\{\xi \in \mathcal{A}\} \mathbf{P}_\xi^{\widehat{\zeta}}[H_G < H_{\mathcal{A} \setminus \mathcal{A}_{Y(\xi)}}] \mathfrak{E}_{Y(\widehat{\zeta}(H_G))}. \end{aligned}$$

By symmetry, the first probability is bounded by ϱ , defined in (6.5), while by the strong Markov property, the third one is bounded by

$$\begin{aligned} \varrho \sum_{A=\mathcal{L},\mathcal{S}} \mathbf{1}\{\xi \in \mathcal{A}\} \mathbf{P}_\xi^{\widehat{\zeta}}[H_G < H_{\mathcal{A} \setminus \mathcal{A}_{Y(\xi)}}] \\ = \varrho \sum_{A=\mathcal{L},\mathcal{S}} \mathbf{1}\{\xi \in \mathcal{A}\} \{1 - \mathbf{P}_\xi^{\widehat{\zeta}}[H_{\mathcal{A} \setminus \mathcal{A}_{Y(\xi)}} < H_G]\}. \end{aligned}$$

Summing this expression with the second one on the right-hand side of (6.7), we obtain by symmetry that

$$\max_{\xi \in \Xi} \gamma_0(\xi) \leq \varrho + (1 - \varrho) \max_{\xi \in \mathcal{L}_0} \mathbf{P}_\xi^{\widehat{\zeta}}[H_{\mathcal{L} \setminus \mathcal{L}_0} < H_G] = \gamma.$$

Iterating this argument $(k - 4)/3$ times, we get that

$$\mathbf{P}_\eta^{\widehat{\zeta}}[H_\Gamma = H_{\blacklozenge_k}] \leq \gamma^{(k-4)/3},$$

which proves the lemma. \square

LEMMA 6.2. *There exists a positive constant $c_0 > 0$ such that*

$$\max_{\eta \in \mathcal{L}} \mathbf{P}_{\eta}^{\widehat{\zeta}}[H_{\mathcal{L} \setminus \mathcal{L}_{Y(n)}} < H_G] \leq 1 - \frac{c_0}{n}.$$

PROOF. By definition of the set \mathcal{L}_0 ,

$$\max_{\eta \in \mathcal{L}_0} \mathbf{P}_{\eta}^{\widehat{\zeta}}[H_{\mathcal{L} \setminus \mathcal{L}_0} < H_G] = \max_{0 \leq j \leq 3} \mathbf{P}_{\xi_0^{l,j}}^{\widehat{\zeta}}[H_{\mathcal{L} \setminus \mathcal{L}_0} < H_G].$$

Consider the case $j = 2$ which, by symmetry, is equivalent to the case $j = 0$. The cases $j = 1, 3$ are examined after. Let $\eta = \xi_0^{1,2}$, $\mathcal{L}_0^c = \mathcal{L} \setminus \mathcal{L}_0$. Intersect the event $\{H_{\mathcal{L}_0^c} < H_G\}$ with the partition $\{H_{\eta}^+ < H_{\mathcal{L}_0^c \cup G}\}$, $\{H_{\mathcal{L}_0^c \cup G} < H_{\eta}^+\}$, and apply the strong Markov property to obtain that

$$\mathbf{P}_{\eta}^{\widehat{\zeta}}[H_{\mathcal{L}_0^c} < H_G] = \frac{\mathbf{P}_{\eta}^{\widehat{\zeta}}[H_{\mathcal{L}_0^c} < H_{G \cup \{\eta\}}^+]}{\mathbf{P}_{\eta}^{\widehat{\zeta}}[H_{\mathcal{L}_0^c \cup G} < H_{\eta}^+]}$$

Since $\mathbf{P}_{\eta}^{\widehat{\zeta}}[H_{\mathcal{L}_0^c \cup G} < H_{\eta}^+] = \mathbf{P}_{\eta}^{\widehat{\zeta}}[H_{\mathcal{L}_0^c} < H_{G \cup \{\eta\}}^+] + \mathbf{P}_{\eta}^{\widehat{\zeta}}[H_G < H_{\mathcal{L}_0^c \cup G}^+]$, we obtain that

$$(6.8) \quad \mathbf{P}_{\eta}^{\widehat{\zeta}}[H_{\mathcal{L}_0^c} < H_G] = \frac{\alpha}{1 + \alpha} \quad \text{where } \alpha = \frac{\mathbf{P}_{\eta}^{\widehat{\zeta}}[H_{\mathcal{L}_0^c} < H_{G \cup \{\eta\}}^+]}{\mathbf{P}_{\eta}^{\widehat{\zeta}}[H_G < H_{\mathcal{L}_0^c \cup G}^+]}$$

It remains to bound α .

For this purpose, we introduce a new process which corresponds to reflect the process $\widehat{\zeta}(t)$ at some configurations. Let \mathcal{N} be the set of neighbors in Ξ_e of \mathcal{L}_0 . As illustrated in Figure 9,

$$\mathcal{N} = \mathcal{L}_{\pm e_i} \cup \mathcal{G}_0 \cup \mathcal{G}_{e_1} \cup \mathcal{G}_{-e_2} \cup \mathcal{G}_{e_1 - e_2}.$$

Let \mathcal{W} be the set of configuration which can be reached by $\widehat{\zeta}(t)$ before hitting \mathcal{N} :

$$\mathcal{W} = \left\{ \xi \in \Xi : \max_{\zeta \in \mathcal{L}_0} \mathbf{P}_{\zeta}^{\widehat{\zeta}}[H_{\xi} \leq H_{\mathcal{N}}] > 0 \right\}.$$

Note that only the four configuration $\eta_0^{i,1}$, $0 \leq i \leq 3$, of \mathcal{G}_0 belong to \mathcal{N} . Denote by $\check{\zeta}(t)$ the process $\widehat{\zeta}(t)$ reflected at \mathcal{W} . The jump rates of $\check{\zeta}(t)$, denoted by $R_{\check{\zeta}}(\xi, \zeta)$, are given by

$$R_{\check{\zeta}}(\xi, \zeta) = \begin{cases} e^{-\beta} \mathbf{R}(\xi, \zeta), & \text{if } \xi, \zeta \in \mathcal{W}, \\ 0, & \text{otherwise.} \end{cases}$$

For $\xi, \zeta \in \mathcal{W}$, $R_{\check{\zeta}}(\xi, \zeta) = e^{-\beta} \mathbf{R}(\xi, \zeta) = e^{-\beta} \mathbf{R}(\zeta, \xi) = e^{-\beta} R_{\check{\zeta}}(\zeta, \xi)$. Therefore, the process $\check{\zeta}(t)$ is reversible with respect to the uniform measure on \mathcal{W} , denoted

by μ_ζ . Moreover, since the rates coincide on \mathcal{W} , we may couple $\widehat{\zeta}(t)$ and $\check{\zeta}(t)$ in such a way that they evolve together until they reach \mathcal{N} . In particular,

$$\alpha = \frac{\mathbf{P}_\eta^{\widehat{\zeta}}[H_{\mathcal{L}_0^c} < H_{\mathcal{G} \cup \{\eta\}}^+]}{\mathbf{P}_\eta^{\widehat{\zeta}}[H_{\mathcal{G}} < H_{\mathcal{L}_0^c \cup \{\eta\}}^+]} = \frac{\mathbf{P}_\eta^{\check{\zeta}}[H_{\mathcal{L}_0^c} < H_{\mathcal{G} \cup \{\eta\}}^+]}{\mathbf{P}_\eta^{\check{\zeta}}[H_{\mathcal{G}} < H_{\mathcal{L}_0^c \cup \{\eta\}}^+]},$$

where $\mathbf{P}_\eta^{\check{\zeta}}$ stands for the distribution of the process $\check{\zeta}(t)$ starting from η .

Recall that $\eta = \zeta_0^{l,2}$. By definition of the capacity,

$$(6.9) \quad \mu_\zeta(\eta)\lambda_\zeta(\eta)\mathbf{P}_\eta^{\check{\zeta}}[H_{\mathcal{L}_0^c} < H_{\mathcal{G} \cup \{\eta\}}^+] \leq \text{cap}_\zeta(\mathcal{L}_0^c, \mathcal{G} \cup \{\eta\}).$$

Let $\mathcal{G}_+ = \mathcal{G}_x \cup \mathcal{G}_{x+e_1}$ and $\mathcal{G}_- = \mathcal{G}_{x-e_2} \cup \mathcal{G}_{x+e_1-e_2}$. For the denominator, we claim that

$$(6.10) \quad 2\mu_\zeta(\eta)\lambda_\zeta(\eta)\mathbf{P}_\eta^{\check{\zeta}}[H_{\mathcal{G}} < H_{\mathcal{L}_0^c \cup \{\eta\}}^+] \geq \text{cap}_\zeta(\mathcal{G}_+, \{\zeta_{e_2}^{l,0}, \eta\}).$$

Indeed, from η the process may only jump to a configuration in \mathcal{L}_0 or to a configuration in Ω^2 . If it jumps to a configuration in \mathcal{L}_0 , since the process is reflected at \mathcal{W} , to reach \mathcal{G}_+ the process $\check{\zeta}(t)$ necessarily passes by η . Therefore,

$$\mathbf{P}_\eta^{\check{\zeta}}[H_{\mathcal{G}} < H_{\mathcal{L}_0^c \cup \{\eta\}}^+] \geq \mathbf{P}_\eta^{\check{\zeta}}[H_{\mathcal{G}_+} < H_{\{\zeta_{e_2}^{l,0}, \eta\}}^+].$$

By symmetry, $\mathbf{P}_\eta^{\check{\zeta}}[H_{\mathcal{G}_+} < H_{\{\zeta_{e_2}^{l,0}, \eta\}}^+] = \mathbf{P}_{\zeta_{e_2}^{l,0}}^{\check{\zeta}}[H_{\mathcal{G}_+} < H_{\{\zeta_{e_2}^{l,0}, \eta\}}^+]$. Since μ_ζ is the uniform measure and since $\lambda_\zeta(\eta) \geq e^{-\beta} \mathbf{R}(\eta, \Omega^2) = e^{-\beta} \mathbf{R}(\zeta_{e_2}^{l,0}, \Omega^2) = \lambda_\zeta(\zeta_{e_2}^{l,0})$, we obtain (6.10).

By (6.8), (6.9), (6.10) and Lemma 6.3 below,

$$(6.11) \quad \mathbf{P}_{\zeta_0^{l,2}}^{\widehat{\zeta}}[H_{\mathcal{L} \setminus \mathcal{L}_0} < H_{\mathcal{G}}] \leq 1 - \frac{c_0}{n}.$$

It remains to consider the case $j = 3$ which is equal to the case $j = 1$.

Let a be the probability that the chain $\widehat{\zeta}(t)$ jumps from $\zeta_0^{l,3}$ to $\zeta_0^{l,2}$. With the notation introduced in Lemma 3.3,

$$a = \mathbf{P}_{\zeta_0^{l,3}}^{\widehat{\zeta}}[H_{\zeta_0^{l,2}} = H_{\mathcal{V}(\zeta_0^{l,3})}].$$

By the strong Markov property and by symmetry,

$$\mathbf{P}_{\zeta_0^{l,3}}^{\widehat{\zeta}}[H_{\mathcal{L} \setminus \mathcal{L}_0} < H_{\mathcal{G}}] \leq (1 - 2a) + 2a\mathbf{P}_{\zeta_0^{l,2}}^{\widehat{\zeta}}[H_{\mathcal{L} \setminus \mathcal{L}_0} < H_{\mathcal{G}}].$$

By (6.11) and by Lemma 3.3, the previous expression is less than or equal to $1 - 2ac_0/n = 1 - c'_0/n$. This completes the proof of the lemma. \square

We are now in a position to obtain a bound for γ introduced in (6.6). By Lemma 3.2, $\rho \leq 1/2$ for $n \geq 46$. Hence, by Lemma 6.2,

$$(6.12) \quad \gamma \leq 1 - \frac{c_0}{2n} \leq e^{-c_0/2n}$$

for some positive constant c_0 .

The proof of Lemma 9.1 below provides bounds for the capacity associated to the process $\check{\zeta}(t)$. Let $U = \{\zeta_0^{l,j} : 0 \leq j \leq 3\} \cup \mathcal{G}_0 \cup \mathcal{G}_{-e_1} \cup \mathcal{G}_{e_2} \cup \mathcal{G}_{e_2-e_1} \cup \{\zeta_{e_2}^{l,0}, \zeta_{-e_1}^{l,1}, \zeta_{-e_2}^{l,2}, \zeta_{e_1}^{l,3}\}$.

LEMMA 6.3. *There exist constants $0 < c_0 < C_0 < \infty$, independent of β , such that for any disjoint subsets A, B of U ,*

$$\frac{c_0 e^{-\beta}}{n^2 |\mathcal{W}|} \leq \text{cap}_\zeta(A, B) \leq \frac{C_0 e^{-\beta}}{n |\mathcal{W}|}.$$

In this lemma, by a subset A of U , we understand the union of the sets forming U . This means that either all configurations of \mathcal{G}_x belongs to A or none.

7. Proof of Theorem 1.2. Recall that we denote by Ξ_\star be the set of configurations which can be reached from η^0 without crossing Δ_2 , the set of configurations with energy greater than $\mathbb{H}_{\min} + 2$, defined in (1.10). We first claim that

$$(7.1) \quad \lim_{\beta \rightarrow \infty} \mathbf{E}_{\eta^0} \left[\int_0^t \mathbf{1}\{\eta(s\ell^2\theta_\beta) \notin \Xi_\star\} ds \right] = 0.$$

Recall the definition of the set $\mathcal{N}(\eta^0)$ introduced in the proof of Lemma 4.4 and define $\mathcal{N}(\eta^x)$, $x \in \mathbb{T}_L$, as the set of configurations in $\mathcal{N}(\eta^0)$ translated by x . Denote by $N_x(t)$ the number of jumps from the configuration η^x to the set of configurations Δ_2 in the time interval $[0, t]$. This corresponds to the number of jumps from η^x to $\Omega_{L,K} \setminus \mathcal{N}(\eta^x)$ in the time interval $[0, t]$. It is clear that $N_x(t)$, $x \in \mathbb{T}_L$, are independent Poisson processes of intensity $4(n-2)e^{-3\beta} \mathbf{1}\{\eta = \eta^x\}$. In particular, if $N_0(t) = \sum_{x \in \mathbb{T}_L} N_x(t)$,

$$\mathbf{P}_{\eta^0}[N_0(t) \geq 1] \leq \mathbf{E}_{\eta^0}[N_0(t)] = 4(n-2)e^{-3\beta} \mathbf{E}_{\eta^0} \left[\int_0^t \mathbf{1}\{\eta(s) \in \Gamma\} ds \right].$$

By symmetry,

$$(7.2) \quad \begin{aligned} & \mathbf{E}_{\eta^0} \left[\int_0^{\ell^2\theta_\beta} \mathbf{1}\{\eta(s) \in \Gamma\} ds \right] \\ &= \frac{1}{|\mathbb{T}_L|} \sum_{y \in \mathbb{T}_L} \mathbf{E}_{\eta^y} \left[\int_0^{\ell^2\theta_\beta} \mathbf{1}\{\eta(s) \in \Gamma\} ds \right] \\ &= \frac{1}{|\mathbb{T}_L| \mu_K(\eta^w)} \sum_{y \in \mathbb{T}_L} \mu_K(\eta^y) \mathbf{E}_{\eta^y} \left[\int_0^{\ell^2\theta_\beta} \mathbf{1}\{\eta(s) \in \Gamma\} ds \right]. \end{aligned}$$

Clearly, the sum is bounded above by

$$\mathbf{E}_{\mu_K} \left[\int_0^{\ell^2 \theta_\beta} \mathbf{1}\{\eta(s) \in \Gamma\} ds \right] = \ell^2 \theta_\beta \mu_K(\Gamma),$$

while the numerator is equal to $\mu_K(\Gamma)$ so that

$$\mathbf{P}_{\eta^0}[N_0(\ell^2 \theta_\beta) \geq 1] \leq C_0 n \ell^2 \theta_\beta e^{-3\beta}.$$

By assumption (1.11), this expression vanishes as $\beta \uparrow \infty$. Hence, with a probability converging to 1, the process $\eta(t)$ does not hit Δ_2 in the time interval $[0, \ell^2 \theta_\beta]$ from a configuration in Γ . Moreover, on the event $N_0(\ell^2 \theta_\beta) = 0$, in the time interval $[0, \ell^2 \theta_\beta]$ the process $\eta(t)$ may only jump from a configuration in Γ to a configuration in $\bigcup_{\mathbf{x} \in \mathbb{T}_L} \mathcal{N}(\eta^{\mathbf{x}})$.

Denote by $\mathcal{P}_1(\mathbf{x})$, \mathcal{P} for plateau, the set of configurations in \mathbb{H}_2 which can be reached from configurations in $\mathcal{N}(\eta^{\mathbf{x}})$ by rate one jumps. These are the configurations in \mathbb{H}_2 which appeared in the proof of Lemma 4.4 and which have been denoted by $\Sigma_{0,1}$ in Figure 1.

For each configuration $\eta \in \mathcal{P}_1(\mathbf{x})$, denote by $N_1(\eta, t)$ the number of jumps from η to a configuration in Δ_2 in the time interval $[0, t]$. It is clear that $N_1(\eta, t)$, $\eta \in \mathcal{P}_1(\mathbf{x})$, are orthogonal Poisson processes of intensity bounded by $C_0 e^{-\beta} \mathbf{1}\{\eta(t) = \eta\}$. Hence, $N_1(t) = \sum_{\eta \in \mathcal{P}_1} N_1(\eta, t)$, $\mathcal{P}_1 = \bigcup_{\mathbf{x} \in \mathbb{T}_L} \mathcal{P}_1(\mathbf{x})$, is a Poisson process of intensity bounded by $C_0 e^{-\beta} \mathbf{1}\{\eta(t) \in \mathcal{P}_1\}$ and

$$\mathbf{P}_{\eta^0}[N_1(t) \geq 1] \leq \mathbf{E}_{\eta^0}[N_1(t)] \leq C_0 e^{-\beta} \mathbf{E}_{\eta^0} \left[\int_0^t \mathbf{1}\{\eta(s) \in \mathcal{P}_1\} ds \right].$$

Repeating the arguments presented in (7.2), we show that

$$\mathbf{P}_{\eta^0}[N_1(\ell^2 \theta_\beta) \geq 1] \leq \frac{C_0 \ell^2 \theta_\beta e^{-\beta} \mu_K(\mathcal{P}_1)}{\mu_K(\Gamma)} \leq C_0 (\ell/L)^2 \theta_\beta e^{-3\beta} |\mathcal{P}_1|.$$

Since $|\mathcal{P}_1|$ is bounded by $C_0 L^2 (n^2 + L^2)$, since $n \leq L$, the previous expression is less than or equal to $C_0 \ell^2 L^2 \theta_\beta e^{-3\beta}$, which vanishes as $\beta \uparrow \infty$ in view of (1.11). Therefore, with a probability converging to 1, the process $\eta(t)$ does not hit Δ_2 in the time interval $[0, \ell^2 \theta_\beta]$ from a configuration in \mathcal{P}_1 . Moreover, on the event $N_1(\ell^2 \theta_\beta) = 0$, in the time interval $[0, \ell^2 \theta_\beta]$ the process $\eta(t)$ may only leave the set \mathcal{P}_1 to a configuration in Γ or to a configuration in $\mathcal{E}_x^{i,j}$, $\mathbf{x} \in \mathbb{T}_L$, $0 \leq i, j \leq 3$.

At this point, we repeat the reasoning developed above for the configurations in Γ to the configurations in $\mathcal{E}_x^{i,j}$. Proceeding in this manner, we complete the proof of (7.1). The main contribution, which explains the need of assumption (1.11), comes from the subsets of configurations which are crossed in a transition from a configuration in Ω^2 (resp., Ω^4) to another configuration in Ω^2 (resp., Ω^4). There are $C_0 L^2 n^8$ such configurations. The details are left to the reader.

We conclude this section with a similar estimate needed in the proof of Theorem 1.3. Under the assumptions of Theorem 1.2 and the hypothesis that $n^7 e^{-\beta} + L^2 e^{-2\beta} \rightarrow 0$, we claim that

$$(7.3) \quad \lim_{\beta \rightarrow \infty} \mathbf{E}_{\eta^0} \left[\int_0^t \mathbf{1}\{\eta(s\ell^2\theta_\beta) \notin \Gamma\} ds \right] = 0.$$

In view of Theorem 1.2, we may replace Γ^c by $\Xi_\star \cap \mathbb{H}_{12}$ in the previous formula. Repeating the arguments which led to (7.2), we obtain that the previous expectation is bounded by

$$\frac{t\mu_K(\Xi_\star \cap \mathbb{H}_{12})}{\mu_K(\Gamma)} = \frac{te^{-2\beta}|\Xi_\star \cap \mathbb{H}_2|}{L^2} + \frac{te^{-\beta}|\Xi_\star \cap \mathbb{H}_1|}{L^2}.$$

The main components of $\Xi_\star \cap \mathbb{H}_1$ are the subsets Ω^2 and Ω^4 whose cardinality are bounded by $C_0 L^2 n^7$. To estimate the first term in the previous formula, one has to go through the proofs of Section 4 and recollect all configurations of $\Xi_\star \cap \mathbb{H}_2$. An inspection shows that $|\Xi_\star \cap \mathbb{H}_2| \leq C_0 L^2 (L^2 + n^8)$. This completes the proof of (7.3).

8. Proof of Theorem 1.3. Recall that we denote by $\zeta(t)$ the trace of $\eta(t)$ on Ξ and that the center of mass, $\mathbf{M}(\eta)$ of a configuration η in Ξ is well defined.

We first claim that the process $\hat{\mathbf{M}}_\beta(t) = \mathbf{M}(\zeta(t\ell^2\theta_\beta))/\ell$ converges to a Brownian motion in the Skorohod topology. The proof of this assertion is divided in two steps. We first prove the tightness of the sequence and then we characterize the limit points.

Assume that $\zeta(0) = \eta^0$. Let $S_1 = 0$ and let T_1 be the time of the first jump of $\zeta(t)$. Define inductively $S_{j+1} = H_\Gamma \circ \vartheta_{T_j} + T_j$, $T_{j+1} = T_1 \circ \vartheta_{S_j} + S_j$, $j \geq 1$. Thus, $[S_j, T_j)$ represent the successive sojourn times at Γ . Fix $t > 0$ and let $n = \min\{k \geq 1 : T_k \geq t\theta_\beta \ell^2\}$.

Recall that $\xi(t)$ is the trace of $\zeta(t)$ on Γ , and fix $\delta > 0$, $\varepsilon > 0$. By the observation of the previous paragraph, if

$$\begin{aligned} \sup_{\substack{0 \leq s \leq t\theta_\beta \ell^2 \\ 0 \leq r \leq \delta\theta_\beta \ell^2}} \|\mathbf{M}(\xi(s+r)) - \mathbf{M}(\xi(r))\| &\leq \varepsilon \ell \quad \text{and} \\ \sup_{T_j \leq s \leq S_{j+1}} \|\mathbf{M}(\zeta(s)) - \mathbf{M}(\zeta(T_j))\| &\leq \varepsilon \ell \quad \text{for all } j \leq n, \end{aligned}$$

then, the inequality written in the first line of the above formula holds with ξ , ε replaces by ζ , 3ε , respectively. In particular, since we have shown in the proof of Theorem 1.1 that the chain $\tilde{\mathbf{M}}_\beta(t) = \mathbf{M}(\xi(t\ell^2\theta_\beta))/\ell$ is tight with respect to the uniform modulus of continuity, to prove the tightness of $\hat{\mathbf{M}}_\beta(t)$ it remains to show that

$$\lim_{\delta \rightarrow 0} \limsup_{\beta \rightarrow \infty} \mathbf{P}_{\eta^0}^\zeta \left[\bigcup_{j=1}^n \sup_{T_j \leq s \leq S_{j+1}} \|\mathbf{M}(\zeta(s)) - \mathbf{M}(\zeta(T_j))\| \geq \varepsilon \ell \right] = 0.$$

Let $\epsilon_j = T_j - S_j$, $j \geq 1$, so that $\{\lambda_\zeta(\eta^0)\epsilon_j : j \geq 1\}$ is a sequence of mean 1 i.i.d. exponential random variables. Clearly, $\sum_{1 \leq i \leq j} \epsilon_i \leq t\theta_\beta \ell^2$ if $n > j$. Since by equation (6.9) in [1], $\lambda_\zeta(\eta^0) \leq \lambda(\eta^0) = 8e^{-2\beta}(1 + O(n)e^{-\beta}) \leq 9e^{-2\beta}$,

$$\begin{aligned} \mathbf{P}_{\eta^0}^\zeta[n > j] &\leq \mathbf{P}_{\eta^0}^\zeta\left[\sum_{i=1}^j \lambda_\zeta(\eta^0)\epsilon_i \leq 9e^{-2\beta}t\theta_\beta \ell^2\right] \\ &\leq \frac{9e^{-2\beta}t\theta_\beta \ell^2}{j}. \end{aligned}$$

Therefore, to prove tightness of the chain $\hat{M}_\beta(t)$ with respect to the uniform modulus of continuity it remains to show that

$$\lim_{\delta \rightarrow 0} \limsup_{\beta \rightarrow \infty} e^{-2\beta}\theta_\beta \ell^2 \mathbf{P}_{\eta^0}^\zeta\left[\sup_{0 \leq s \leq H_\Gamma} \|\mathbf{M}(\zeta(s))\| \geq \varepsilon \ell\right] = 0.$$

This follows from the proof of Theorem 1.1.

Let $\tilde{M}_\beta(t) = \mathbf{M}(\xi(t\ell^2\theta_\beta))/\ell$, which converges to a Brownian motion by Theorem 1.1. We claim that for every $t > 0$, $\varepsilon > 0$,

$$(8.1) \quad \lim_{\beta \rightarrow \infty} \mathbf{P}_{\eta^0}^\zeta\left[\sup_{0 \leq s \leq t} \|\hat{M}_\beta(s) - \tilde{M}_\beta(s)\| > \varepsilon\right] = 0.$$

By definition of $\hat{M}_\beta(t)$, $\tilde{M}_\beta(t)$, it is enough to show that

$$(8.2) \quad \lim_{\beta \rightarrow \infty} \mathbf{P}_{\eta^0}^\zeta\left[\sup_{0 \leq s \leq t\ell^2\theta_\beta} \|\mathbf{M}(\zeta(s)) - \mathbf{M}(\xi(s))\| > \varepsilon \ell\right] = 0.$$

Recall that $\xi(s)$ is the trace of $\zeta(s)$ on Γ . Hence, if we define the additive functional $T(s)$ by

$$T(s) = \int_0^s \mathbf{1}\{\zeta(r) \in \Gamma\} dr, \quad s \geq 0,$$

and if $S(s)$ is the generalized inverse of T , $S(s) = \sup\{r \geq 0 | T(r) \leq s\}$, $\xi(s) = \zeta(S(s))$. We may therefore replace $\mathbf{M}(\zeta(s)) - \mathbf{M}(\xi(s))$ by $\mathbf{M}(\zeta(s)) - \mathbf{M}(\zeta(S(s)))$ in the previous formula.

We claim that for every $t > 0$, $\delta > 0$,

$$(8.3) \quad \lim_{\beta \rightarrow \infty} \mathbf{P}_{\eta^0}^\zeta[S(t\theta_\beta \ell^2) - t\ell^2\theta_\beta \geq \delta \ell^2\theta_\beta] = 0.$$

By definition of S , for every $\delta' > 0$ $t' > 0$, $S(t') < t' + \delta'$ if $T(t' + \delta') > t'$. Hence, taking $\delta' = \delta \ell^2\theta_\beta$ and $t' = t\ell^2\theta_\beta$, since $t' - T(t') = \int_{[0, t']} \mathbf{1}\{\zeta(s) \notin \Gamma\} ds$, we conclude that the previous probability is less than or equal to

$$\mathbf{P}_{\eta^0}^\zeta\left[\int_0^{(t+\delta)\theta_\beta \ell^2} \mathbf{1}\{\zeta(s) \notin \Gamma\} ds \geq \delta \ell^2\theta_\beta\right].$$

By Chebyshev’s inequality and by translation invariance and by the arguments used in (7.2), this probability is bounded by

$$\begin{aligned} & \frac{1}{\delta \ell^2 \theta_\beta} \mathbf{E}_{\eta^0}^\zeta \left[\int_0^{(t+\delta)\theta_\beta \ell^2} \mathbf{1}\{\zeta(s) \notin \Gamma\} ds \right] \\ & \leq \frac{1}{\delta \ell^2 \theta_\beta \mu_\zeta(\Gamma)} \mathbf{E}_{\mu_\zeta}^\zeta \left[\int_0^{(t+\delta)\theta_\beta \ell^2} \mathbf{1}\{\zeta(s) \notin \Gamma\} ds \right] = \frac{(t+\delta)\mu_\zeta(\Gamma^c)}{\delta \mu_\zeta(\Gamma)}. \end{aligned}$$

Since $\mu_\zeta(\Gamma^c)/\mu_\zeta(\Gamma)$ is bounded by $C_0 n^7 e^{-\beta}$, the previous expression vanishes as $\beta \rightarrow \infty$, proving (8.3).

Recall that we replaced $\mathbb{M}(\xi(s))$ by $\mathbb{M}(\zeta(S(s)))$ in (8.2). Since $S(r) - r$ is a nonnegative increasing function, in view of (8.3), to prove (8.1) it is enough to show that for every $\varepsilon > 0$,

$$\lim_{\delta \rightarrow 0} \limsup_{\beta \rightarrow \infty} \mathbf{P}_{\eta^0}^\zeta \left[\sup_{\substack{0 \leq s \leq t \ell^2 \theta_\beta \\ 0 \leq r \leq \delta \ell^2 \theta_\beta}} \|\mathbb{M}(\zeta(s)) - \mathbb{M}(\zeta(s+r))\| > \varepsilon \ell \right] = 0.$$

This follows from the tightness of the process $\mathbb{M}(\zeta(s))$.

To complete the proof of the theorem, we need to replace $\mathbb{M}(\zeta(s))$ by $\mathbb{M}(\eta(s))$. This is simpler and follows the same strategy of the first part of the proof. Denote by \mathbb{A} the event $\{\eta(s) \notin \Xi_\star \text{ for some } 0 \leq s \leq t \ell^2 \theta_\beta\}$. By Theorem 1.2, this event has a vanishing asymptotic probability. In view of our convention for the center of mass of configurations in $\Sigma_{0,1} \cup \Sigma_{2,3}$, on \mathbb{A} the center of mass $\mathbb{M}(\eta(s))$ does not change appreciably during excursions in \mathbb{H}_2 : on \mathbb{A} for all $0 \leq s \leq t \ell^2 \theta_\beta$,

$$\sup_{S_s \leq s \leq T_s} \max\{\|\mathbb{M}(\eta(s)) - \mathbb{M}(\eta(S_s))\|, \|\mathbb{M}(\eta(s)) - \mathbb{M}(\eta(T_s))\|\} \leq \frac{C_0}{n\ell},$$

where S_s (resp., T_s) is the last (resp., first) time before (resp., after) s in which $\eta(s)$ belongs to \mathbb{H}_{01} . Therefore,

$$\sup_{\substack{0 \leq s \leq t \ell^2 \theta_\beta \\ 0 \leq r \leq \delta \ell^2 \theta_\beta}} \|\mathbb{M}(\eta(s)) - \mathbb{M}(\eta(s+r))\| \leq \frac{C_0}{n\ell} + \sup_{\substack{0 \leq s \leq t \ell^2 \theta_\beta \\ 0 \leq r \leq \delta \ell^2 \theta_\beta}} \|\mathbb{M}(\zeta(s)) - \mathbb{M}(\zeta(s+r))\|.$$

This proves that the sequence of Markovs chains $M^\beta(t)$ is tight.

To characterize the limit points, we compare $M^\beta(t)$ to $\hat{M}_\beta(t)$ and we use the fact that $\zeta(t)$ is the trace of $\eta(t)$ on Ξ . We need to prove (8.1) with the obvious modifications. The main point in the proof of (8.1) is assertion (8.3) whose proof reduces to the estimate

$$\lim_{\beta \rightarrow \infty} \mathbf{E}_{\eta^0} \left[\int_0^{t+\delta} \mathbf{1}\{\eta(s\theta_\beta \ell^2) \notin \Xi\} ds \right] = 0.$$

This estimate has been derived in (7.3), which completes the proof of the theorem.

9. The time-scale θ_β . We prove in this section the bounds (1.4) on θ_β . The proof relies on the next lemma.

LEMMA 9.1. *Assume that (1.3) holds. Then there exist constants $0 < c_0 < C_0 < \infty$, independent of β , such that*

$$\frac{c_0 e^{-2\beta} \mu_\beta(\eta^0)}{n^2 \mu_\beta(\Xi)} \leq \text{cap}_\zeta(\eta^0, \Gamma \setminus \{\eta^0\}) \leq \frac{C_0 e^{-2\beta} \mu_\beta(\eta^0)}{n \mu_\beta(\Xi)}.$$

PROOF. On the one hand, by the Dirichlet principle

$$\text{cap}_\zeta(\eta^0, \Gamma \setminus \{\eta^0\}) \leq \langle (-L_\zeta f), f \rangle_{\mu_\zeta}$$

for any function $f : \Xi \rightarrow [0, 1]$ which vanishes on $\Gamma \setminus \{\eta^0\}$ and which is equal to 1 at η^0 . Taking $f = \mathbf{1}\{\mathcal{G}_0\}$, we obtain that

$$\text{cap}_\zeta(\eta^0, \Gamma \setminus \{\eta^0\}) \leq \sum_{\eta \in \mathcal{G}_0} \mu_\zeta(\eta) R_\zeta(\eta, \mathcal{G}_0^c).$$

By Propositions 4.2 and 4.6, Lemma 3.1 and since $\mu_\zeta(\eta) = \mu_K(\eta)/\mu_K(\Xi) = \mu_\beta(\eta)/\mu_\beta(\Xi)$, the previous sum is bounded by

$$\begin{aligned} & \frac{e^{-\beta} \mu_\beta(\eta^0)}{\mu_\beta(\Xi)} \sum_{\eta \in \mathcal{V}(\eta^0)} \{e^{-\beta} \mathbf{R}(\eta, \mathcal{G}_0^c) + C_0 \kappa_2 e^{-\beta}\} + \frac{\mu_\beta(\eta^0)}{\mu_\beta(\Xi)} e^{-2\beta} \kappa_1 \\ & \leq \frac{C_0 e^{-2\beta} \mu_\beta(\eta^0)}{n \mu_\beta(\Xi)} \{1 + n \kappa_2 + n \kappa_1\}, \end{aligned}$$

for some finite constant C_0 whose value change from line to line. By (1.3), $n(\kappa_1 + \kappa_2)$ vanishes as $\beta \uparrow \infty$. This proves the upper bound.

On the other hand, by Thomson’s principle [14], Proposition 3.2.2,

$$\frac{1}{\text{cap}_\zeta(\eta^0, \Gamma \setminus \{\eta^0\})} \leq \frac{1}{2} \sum_{\eta, \xi \in \Xi} \frac{1}{\mu_\zeta(\eta) R_\zeta(\eta, \xi)} \Phi(\eta, \xi)^2,$$

for any unitary flow Φ from η^0 to $\Gamma \setminus \{\eta^0\}$.

To construct such a flow, recall from (3.14) the path $\xi_0 = \eta^0, \xi_1, \dots, \xi_M = \eta^{e_1}$, $M = 3n - 2$, from η^0 to η^{e_1} obtained by sliding particles around the square Q . Let Ψ be the unitary flow from η^0 to η^{e_1} defined by $\Psi(\xi_i, \xi_{i+1}) = 1, 0 \leq i < M$. By (4.3) and (4.11),

$$\begin{aligned} & \frac{1}{2} \sum_{\eta, \xi \in \Xi} \frac{1}{\mu_\zeta(\eta) R_\zeta(\eta, \xi)} \Psi(\eta, \xi)^2 \\ & \leq \frac{\mu_\beta(\Xi)}{\mu_\beta(\eta^0)} \frac{e^{2\beta}}{\mathbf{R}(\xi_0, \xi_1)} \left\{ 1 + \frac{\kappa_1}{\mathbf{R}(\xi_0, \xi_1) - \kappa_1} \right\} \\ & \quad + \frac{\mu_\beta(\Xi) e^\beta}{\mu_\beta(\eta^0)} \sum_{i=1}^{M-1} \frac{e^\beta}{\mathbf{R}(\xi_i, \xi_{i+1})} \left\{ 1 + \frac{C_0 \kappa_2}{\mathbf{R}(\xi_i, \xi_{i+1}) - C_0 \kappa_2} \right\}. \end{aligned}$$

By (3.14) and (3.15), $\mathbf{R}(\xi_i, \xi_{i+1}) \geq n^{-1}$. The previous expression is thus less than or equal to

$$\frac{C_0 \mu_\beta(\Xi) n^2 e^{2\beta}}{\mu_\beta(\eta^0)} \left\{ 1 + \frac{n\kappa_2}{1 - n\kappa_2} + \frac{n\kappa_1}{1 - n\kappa_1} \right\}.$$

By (1.3), $n\kappa_1$ and $n\kappa_2$ vanishes as $\beta \uparrow \infty$. This proves the lower bound. \square

We are now in a position to prove the bounds (1.4). Recall that we denote by $R_\xi(\eta^{\mathbf{x}}, \eta^{\mathbf{y}})$, $\mathbf{x} \neq \mathbf{y} \in \mathbb{T}_L$, the jump rates of the trace process $\xi(t)$. Since $\zeta(t)$, $\xi(t)$ are the traces of the process $\eta(t)$ on Ξ , Γ , respectively, and since $\Xi \supset \Gamma$, $\xi(t)$ is also the trace of $\zeta(t)$ on Γ . In particular, by [1], Proposition 6.1,

$$(9.1) \quad R_\xi(\eta^0, \eta^{\mathbf{x}}) = \lambda_\zeta(\eta^0) \mathbf{P}_{\eta^0}^\zeta [H_\Gamma^+ = H_{\eta^{\mathbf{x}}}] .$$

Hence, by the definition (1.2) of θ_β ,

$$(9.2) \quad \theta_\beta^{-1} = \sum_{\mathbf{x} \in \mathbb{T}_L} \|\mathbf{x}\|^2 r_\beta(\mathbf{x}) = \lambda_\zeta(\eta^0) \sum_{\mathbf{x} \in \mathbb{T}_L} \|\mathbf{x}\|^2 \mathbf{P}_{\eta^0}^\zeta [H_\Gamma^+ = H_{\eta^{\mathbf{x}}}] ,$$

so that

$$\lambda_\zeta(\eta^0) \mathbf{P}_{\eta^0}^\zeta [H_{\Gamma \setminus \{\eta^0\}} < H_{\eta^0}^+] \leq \theta_\beta^{-1} \leq 2L^2 \lambda_\zeta(\eta^0) \mathbf{P}_{\eta^0}^\zeta [H_{\Gamma \setminus \{\eta^0\}} < H_{\eta^0}^+] .$$

Since

$$\text{cap}_\zeta(\eta^0, \Gamma \setminus \{\eta^0\}) = \mu_\zeta(\eta^0) \lambda_\zeta(\eta^0) \mathbf{P}_{\eta^0}^\zeta [H_{\Gamma \setminus \{\eta^0\}} < H_{\eta^0}^+] ,$$

we conclude that

$$\frac{1}{\mu_\zeta(\eta^0)} \text{cap}_\zeta(\eta^0, \Gamma \setminus \{\eta^0\}) \leq \theta_\beta^{-1} \leq \frac{2L^2}{\mu_\zeta(\eta^0)} \text{cap}_\zeta(\eta^0, \Gamma \setminus \{\eta^0\}) .$$

Assertion (1.4) follows now from Lemma 9.1.

We may improve the upper bound above using the arguments presented in the proof of Theorem 1.1. Assume that (1.6) is in force and recall the definition of the sets $\diamond_k, \blacklozenge_k$ introduced right after (6.4). By (9.2) and since $\lambda_\zeta(\eta^0) \leq C_0 e^{-2\beta}$,

$$\theta_\beta^{-1} \leq C_0 e^{-2\beta} \sum_{k=1}^{2L} k^2 \mathbf{P}_{\eta^0}^\zeta [H_\Gamma^+ = H_{\diamond_k}] \leq C_0 e^{-2\beta} \sum_{k=1}^{2L} k \mathbf{P}_{\eta^0}^\zeta [H_\Gamma^+ = H_{\blacklozenge_k}] ,$$

where we performed a summation by parts in the last step. By Proposition 5.2, the previous sum is bounded by

$$C_0 e^{-2\beta} L^2 \kappa_3 + C_0 e^{-2\beta} \sum_{k=1}^{2L} k \widehat{\mathbf{P}}_{\eta^0} [H_\Gamma^+ = H_{\blacklozenge_k}] ,$$

where $\kappa_3 = \kappa_1 + \sqrt{\kappa_2 |\Xi| / L^2}$. By Lemma 6.1 and (6.12), the second term is less than or equal to $C_0 e^{-2\beta} n^2$. By assumption (1.6), the first one is bounded above by $C_0 e^{-2\beta} n^2$. This proves (1.7).

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REFERENCES

- [1] BELTRÁN, J. and LANDIM, C. (2010). Tunneling and metastability of continuous time Markov chains. *J. Stat. Phys.* **140** 1065–1114. [MR2684500](#)
- [2] BELTRÁN, J. and LANDIM, C. (2011). Metastability of reversible finite state Markov processes. *Stochastic Process. Appl.* **121** 1633–1677. [MR2811018](#)
- [3] BELTRÁN, J. and LANDIM, C. (2012). Metastability of reversible condensed zero range processes on a finite set. *Probab. Theory Related Fields* **152** 781–807. [MR2892962](#)
- [4] BELTRÁN, J. and LANDIM, C. (2012). Tunneling and metastability of continuous time Markov chains II, the nonreversible case. *J. Stat. Phys.* **149** 598–618. [MR2998592](#)
- [5] BELTRÁN, J. and LANDIM, C. (2015). Tunneling of the Kawasaki dynamics at low temperatures in two dimensions. *Ann. Inst. Henri Poincaré Probab. Stat.* **51** 59–88. [MR3300964](#)
- [6] BOVIER, A., DEN HOLLANDER, F. and NARDI, F. R. (2006). Sharp asymptotics for Kawasaki dynamics on a finite box with open boundary. *Probab. Theory Related Fields* **135** 265–310. [MR2218873](#)
- [7] BOVIER, A., DEN HOLLANDER, F. and SPITONI, C. (2010). Homogeneous nucleation for Glauber and Kawasaki dynamics in large volumes at low temperatures. *Ann. Probab.* **38** 661–713. [MR2642889](#)
- [8] DEN HOLLANDER, F., NARDI, F. R., OLIVIERI, E. and SCOPPOLA, E. (2003). Droplet growth for three-dimensional Kawasaki dynamics. *Probab. Theory Related Fields* **125** 153–194. [MR1961341](#)
- [9] DEN HOLLANDER, F., OLIVIERI, E. and SCOPPOLA, E. (2000). Metastability and nucleation for conservative dynamics. *J. Math. Phys.* **41** 1424–1498. [MR1757966](#)
- [10] DOYLE, P. G. and SNELL, J. L. (1984). *Random Walks and Electric Networks*. *Carus Mathematical Monographs* **22**. Mathematical Association of America, Washington, DC. [MR0920811](#)
- [11] ETHIER, S. N. and KURTZ, T. G. (1986). *Markov Processes: Characterization and Convergence*. Wiley, New York. [MR0838085](#)
- [12] FUNAKI, T. (2004). Zero temperature limit for interacting Brownian particles. I. Motion of a single body. *Ann. Probab.* **32** 1201–1227. [MR2060296](#)
- [13] FUNAKI, T. (2004). Zero temperature limit for interacting Brownian particles. II. Coagulation in one dimension. *Ann. Probab.* **32** 1228–1246. [MR2060297](#)
- [14] GAUDILLIÈRE, A. (2009). Condenser physics applied to Markov chains: A brief introduction to potential theory. Available at <http://arxiv.org/abs/0901.3053>.
- [15] GAUDILLIÈRE, A., DEN HOLLANDER, F., NARDI, F. R., OLIVIERI, E. and SCOPPOLA, E. (2009). Ideal gas approximation for a two-dimensional rarefied gas under Kawasaki dynamics. *Stochastic Process. Appl.* **119** 737–774. [MR2499857](#)
- [16] GAUDILLIÈRE, A., OLIVIERI, E. and SCOPPOLA, E. (2005). Nucleation pattern at low temperature for local Kawasaki dynamics in two dimensions. *Markov Process. Related Fields* **11** 553–628. [MR2191966](#)
- [17] JARA, M., LANDIM, C. and TEIXEIRA, A. (2011). Quenched scaling limits of trap models. *Ann. Probab.* **39** 176–223. [MR2778800](#)
- [18] JARA, M., LANDIM, C. and TEIXEIRA, A. (2014). Universality of trap models in the ergodic time scale. *Ann. Probab.* **42** 2947–2557. [MR3265173](#)
- [19] LANDIM, C. (2014). Metastability for a non-reversible dynamics: The evolution of the condensate in totally asymmetric zero range processes. *Comm. Math. Phys.* **330** 1–32. [MR3215575](#)

- [20] LAWLER, G. F. (1991). *Intersections of Random Walks*. Birkhäuser, Boston, MA. [MR1117680](#)
- [21] LEVIN, D. A., PERES, Y. and WILMER, E. L. (2009). *Markov Chains and Mixing Times*. Amer. Math. Soc., Providence, RI. [MR2466937](#)
- [22] NARDI, F. R., OLIVIERI, E. and SCOPPOLA, E. (2005). Anisotropy effects in nucleation for conservative dynamics. *J. Stat. Phys.* **119** 539–595. [MR2149934](#)
- [23] OLIVIERI, E. and VARES, M. E. (2005). *Large Deviations and Metastability*. *Encyclopedia of Mathematics and Its Applications* **100**. Cambridge Univ. Press, Cambridge. [MR2123364](#)
- [24] ONSAGER, L. (1944). Crystal statistics. I. A two-dimensional model with an order-disorder transition. *Phys. Rev. (2)* **65** 117–149. [MR0010315](#)
- [25] PEIERLS, R. (1936). Statistical theory of adsorption with interaction between the adsorbed atoms. *Proc. Camb. Phil. Soc.* **32** 471.

IMPA
ESTRADA DONA CASTORINA 110
CEP 22460 RIO DE JANEIRO
BRASIL
E-MAIL: brunog@impa.br

IMPA
ESTRADA DONA CASTORINA 110
CEP 22460 RIO DE JANEIRO
BRASIL
AND
CNRS UMR 6085
UNIVERSITÉ DE ROUEN
AVENUE DE L'UNIVERSITÉ, BP.12
TECHNOPÔLE DU MADRILLET
F76801 SAINT-ÉTIENNE-DU-ROUVRAY
FRANCE
E-MAIL: landim@impa.br