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ON THE RANGE OF A RANDOM WALK IN A TORUS AND RANDOM INTERLACEMENTS

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Let a simple random walk run inside a torus of dimension three or higher for a number of steps which is a constant proportion of the volume. We examine geometric properties of the range, the random subgraph induced by the set of vertices visited by the walk. Distance and mixing bounds for the typical range are proven that are a k-iterated log factor from those on the full torus for arbitrary k. The proof uses hierarchical renormalization and techniques that can possibly be applied to other random processes in the Euclidean lattice. We use the same technique to bound the heat kernel of a random walk on random interlacements.

1. Introduction. Consider a discrete torus of side length N in dimension $d \geq 3$. Let a simple random walk run in the torus until it fills a constant proportion of the torus and examine the *range*, the random subgraph induced by the set of vertices visited by the walk. How well does this range capture the geometry of the torus? Viewing the range as a random perturbation of the torus, we can draw hope that at least some geometric properties of the torus are retained, by considering results on a more elementary random perturbation, Bernoulli percolation.

It is now known that various properties of the Euclidean lattice "survive" Bernoulli percolation with density $p > p_c(\mathbb{Z}^d)$. In [1], Antal and Pisztora proved that there is a finite $C(p,d)$ such that the graph distance between any two vertices in the infinite cluster is more than C times their l_2 distance, with probability exponentially low in this distance. Isoperimetric bounds for the largest connected cluster in a fixed box of side n were given by Benjamini and Mossel for p sufficiently close to 1 in [2], and by Mathieu and Remy for $p > p_c$ in [12]. A consequence is that the mixing time for a random walk on this cluster has the same order bound, $\theta(n^2)$, as on the full box. In [14], Pete extends this result to more general graphs.

Returning to our process, in Figure 1 simulation pictures are shown that give heuristical support to the view that although the range for $d \geq 3$ has long range dependence, it bears some similarities to i.i.d. site percolation. Indeed, one can see that the middle picture, a 2d slice of the range of a walk that filled 30% of a 3d torus, is "in between," dependence-wise, the i.i.d. picture on the right and

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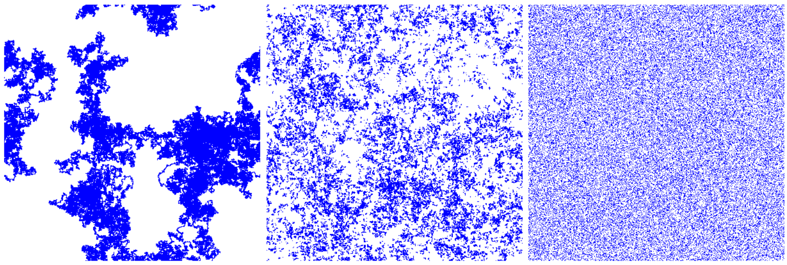

FIG. 1. *From left to right, the range in* 2 *dimensions, a slice in* 3 *dimensions and Bernoulli percolation, all of density* 0.3.

the highly dependent picture on the left where the effect of two-dimensional recurrence is evident. Thus, one might expect analogous geometric behavior of the range for $d \geq 3$ and i.i.d. percolation. This partially turns out to be the case.

In [3], the complement of the range, called the *vacant set*, is investigated by Benjamini and Sznitman. For positive u, it is shown uN^d is indeed the proper timescale to generate percolative behavior of the vacant set. Starting at the uniform distribution, it is easily shown that for some $c(u,d) > 0$, the probability a given vertex in the torus is visited by the walk is between c and $1 - c$, independently of N. A more difficult result is that for small u, the vacant set typically contains a connected component that is larger than some constant proportion of the torus. Indeed, simulations support the existence of a phase transition in u of the vacant set geometry, where below some critical $u_c > 0$, a unique giant component appears, and above it all clusters are microscopic.

The range, unlike the vacant set, does not display an obvious phase transition in u. It is connected for all positive u, and fills a $c'(u,d) > 0$ proportion of the torus with high probability. Despite the analogy to percolation being flawed in this respect, the range does display some percolative behavior due to the Markov property and uniform transience of a random walk in $d > 2$. Roughly, conditioning on the vertices by which the walk enters and exits a small box makes the path in between them independent from the walk outside this box. Using this idea and facts from percolation theory gathered in Section 4, we prove the range does capture the distance and isoperimetric bounds of the torus, though our methods require an iterated logarithmic correction to the bounds of the full torus. In Section 6, it is shown that for arbitrarily small $u > 0$, the range asymptotically dominates a recursive structure, defined in Section 2, which can roughly be described as a finite-level supercritical fractal percolation. From this structure, we extract distance bounds (Appendix B) and mixing bounds (Section 3) that are a $\log^{(k)}(N) = \log(\log(\cdots(\log(N)\cdots k\cdots)))$ factor from those on the torus.

Let us expand a bit on the heuristics presented in the previous paragraph. Since the holes in the range are larger than those in i.i.d. percolation (see the last comment in [3]), one can never hope to dominate it. Instead, we formulate a notion of density of a box of side n, which essentially means that it is crossed top to bottom (*traversed*) by the random walk an order of n^{d-2} times. A union bound then gives that w.h.p. all $\log^4 N$-sided "first-level" boxes in the torus possess this property. Next, given this condition, for each fixed first-level box, all internal "second-level" boxes of side $c\log^4(\log N)$ are dense w.h.p., and independently from other disjoint first-level boxes. The probability for the denseness of the second-level boxes is not high enough for a union bound on all of them, however, it is enough such that first-level boxes whose second-level boxes are all dense dominate p-percolation for arbitrarily high $p < 1$. This is the basis of the hierarchical renormalization used below to prove the same fact for "k-level" boxes with arbitrary k. A drawback of this method is that the density of boxes becomes diluted by a constant factor from level to level, preventing us from continuing this rescaling to reach boxes of a bounded size. This dilution is the main source of the $\log^{(k)}(N)$ correction. We believe this correction is an artifact of the method and that the true bounds should be the same as those on the torus.

A central technical concept introduced in the paper is the recursively defined k-goodness of a box, which is roughly that the $(k-1)$-good smaller scale boxes inside satisfy some typical supercritical Percolation properties. The main demand from 0-good boxes is that the range is connected in their interior. This provides a useful way to analyze the range but perhaps a better formulated notion will get sharper bounds. A second technique worth mentioning is the propagation of isoperimetric bounds through multiple scales in Lemma 3.3. This has been done for one level in [12], but it is not clear how to extend the method there to more than one level. Last, getting rid of dependence on time in the random walk when moving to smaller scale boxes is not trivial. To do this, we prove the domination of the k-good recursive structure mentioned above simultaneously for all $\{\mathcal{R}_N(t)\}_{t\geq uN^d}$, where $\mathcal{R}_N(t)$ is the range of the walk up to time t. This is facilitated by results on conditioned random walks from Section 5, in particular by Lemma 5.11. The lemma shows that given any fixed "boundary-connected-path" $f(t)$ in a dense box (see definition above Lemma 5.3), the random walk traversals will merge it w.h.p. into a single connected component, for all $t \geq 0$.

Using the results proved for the random walk on the torus, we prove a bound on the Heat kernal of random walk on Random Interlacements. In Appendix C, we write a short introduction on Random Interlacements where one can find the notation used in Section 7.

It should be mentioned that while all sections ahead require the terminology introduced in Section 2, all remaining sections apart from Section 6 may be read quite independently from one another. Section 6 also relies on random walk definitions from Section 5. For reading convenience, one can find an index of symbols in Appendix D.

2. Result and notation. Let $\mathcal{T}(N,d)$ be the discrete d-dimensional torus with side length N, for $d \geq 3$. Fixing d, $\mathcal{T}(V,E)$ is a graph with

$$V(N) = \{\mathbf{x} \in \mathbb{Z}^d : 0 \leq x_i < N, 1 \leq i \leq d\}$$

and

$$E(N) = \{\{\mathbf{x},\mathbf{y}\} \subset V(\mathcal{T}(N)) : \Pi_N(\mathbf{x}-\mathbf{y}) \in \{\pm\mathbf{e}_1,\dots,\pm\mathbf{e}_d\}\},$$

where $\Pi_N : \mathbb{Z}^d \to V(N)$ for $\mathbf{x} \in \mathbb{Z}^d$ is $\Pi_N(\mathbf{x}) = (x_1 \bmod N, \dots, x_d \bmod N)$ and $\{\mathbf{e}_i\}_{i=1}^d$ is the standard basis of $\mathbb{Z}^d$.

Note that if $S(\cdot)$ is a simple random walk (SRW) in $\mathbb{Z}^d$, $S_N(\cdot) = \Pi_N \circ S(\cdot)$ is a SRW in $\mathcal{T}(N)$. Let $\mathcal{R}(t_1,t_2) = \{S(s) : t_1 \leq s < t_2\}$ and call $\mathcal{R}(t) = \mathcal{R}(0,t)$ the *range* (until time t) of the walk. We consider $\mathcal{R}_N(t)$, the random connected subgraph of $\mathcal{T}$ induced by $\Pi_N \circ \mathcal{R}(t)$, where we include only edges traversed by the random walk. Throughout the paper, when no ambiguity is present, we identify a graph with its vertices.

Let $\mathbf{P}_\mathbf{x}[\cdot]$ be the law that makes $S(\cdot)$ an independent SRW starting at $\mathbf{x} \in \mathbb{Z}^d$. Below are the main three results of the paper.

THEOREM 2.1. *Set $u > 0$ and for a graph G, let $d_G(\cdot,\cdot)$ denote graph distance. Then for any k,*

$$\lim_{N\to\infty} \mathbf{P}_\mathbf{0}\left[\max_{t \geq uN^d}\left\{\frac{d_{\mathcal{R}_N(t)}(\mathbf{x},\mathbf{y})}{d_{\mathcal{T}(N)}(\mathbf{x},\mathbf{y})} : \mathbf{x},\mathbf{y} \in \mathcal{R}_N(t), d_{\mathcal{T}(N)}(\mathbf{x},\mathbf{y}) > (\log N)^{5d}\right\} > \log^{(k)} N\right] = 0,$$

where $\log^{(k)} N$ *is* $\log(\cdot)$ *iterated k-times of N.*

Since this paper was uploaded to the arXiv on 2010, the distance bounds where improved in [5] by Cernỳ and Popov. They managed to get a tight result without the log correction. Due to the improvement, the proof of Theorem 2.1 is postponed to Appendix B. Note that since distance bounds require finding one good path and isoperimetric bounds require a uniform bound on all subsets, the rest of the results in this paper do not follow the techniques of [5].

THEOREM 2.2. *Set $u > 0$ and let $\tau(G)$ be the (e.g., uniform) mixing time of a simple random walk on a graph G. Then for any k,*

$$\lim_{N\to\infty} \mathbf{P}_\mathbf{0}\left[\max_{t \geq uN^d} \frac{\tau(\mathcal{R}_N(t))}{N^2} > \log^{(k)} N\right] = 0.$$

The two theorems are a direct consequence of Theorem 6.1 and Theorems B.1, 3.1, respectively.

Using the same techniques for proving Theorem 2.1 and Theorem 2.2, we can show the next result for a random walk on the range of random interlacements (see Appendix C for notation).

THEOREM 2.3. *Let $u > 0$ and $k \in \mathbb{N}$. Then there exists a constant $C(u, k)$ such that for $\mathbb{P}_0^u$ almost every $\mathcal{I}^u$, and for all n large enough*

$$\mathbf{P}_0^u[0, n] \le \frac{C \cdot \log^{(k)}(n)}{n^{d/2}}.$$

This theorem quantifies the result of Ráth and Sapozhnikov in [16]. Ráth and Sapozhnikov proved the graph of random interlacements is transient a.s.

The main purpose of the remainder of the section is to define a k-good configuration, and to establish notation used throughout the paper.

2.1. *Graph notation.* Given a graph G, we identify a subset of vertices V with its induced subgraph in G. We denote $G \setminus V$, the complement of V relative to G, by V_G^c. Writing $d_G(\cdot, \cdot)$ for the graph distance in G, we let $d_G(\mathbf{v}, V) = \inf\{d_G(\mathbf{v}, \mathbf{x}) : \mathbf{x} \in V\}$. For the outer and inner boundary, we respectively write

$$\partial_G(V) = \{\mathbf{v} \in G : d_G(\mathbf{v}, V) = 1\},$$
$$\partial_G^{\text{in}}(V) = \partial_G(V_G^c) = \{\mathbf{v} \in G : d(\mathbf{v}, V_G^c) = 1\}.$$

We often omit G from the notation when the ambient graph is clear. We say V *is connected in* G if any two vertices in V have a path in G connecting them. $V_1, V_2 \subset G$ are connected in G if $V_1 \cup V_2$ is connected in G. Given $V \subset G$, we call a set that is connected in V and is maximal to inclusion a *component of* V.

As noted above, we identify graphs and their vertices. Thus, $\mathbb{Z}^d$ denotes the d-dimensional integers as well as the graph on these vertices in which two vertices are connected if they differ by a unit vector.

Last, if $V \subset \mathbb{Z}^d$, $\mathbf{z} \in \mathbb{Z}^d$ then $V \pm \mathbf{z} = \{\mathbf{x} \pm \mathbf{z} : \mathbf{x} \in V\}$.

2.2. *Box notation.* For $\mathbf{x} \in \mathbb{Z}^d$, $n > 0$, let

$$B(\mathbf{x}, n) = \{\mathbf{y} \in \mathbb{Z}^d : \forall i, 1 \le i \le d, -n/2 \le \mathbf{x}(i) - \mathbf{y}(i) < n/2\}.$$

We write $B(n)$ if $\mathbf{x}$ is the origin, and when length and center are unambiguous we often just write B. Occasionally, we use lowercase b for a smaller instance of a box. We denote the side length of a box by $\|B\|$, that is,

$$\|B\| = |B|^{1/d}.$$

Let $\text{sp}\{B(\mathbf{x}, n)\} = \{B(\mathbf{x} + \sum_i \mathbf{e}_i k_i n, n) : (k_1, \dots, k_d) \in \mathbb{Z}^d\}$ where $\mathbf{e}_1, \dots, \mathbf{e}_d$ are the unit vectors in $\mathbb{Z}^d$, that is, all the nonintersecting translations of B in $\mathbb{Z}^d$. We attach a graph structure to $\mathrm{sp}\{B(\mathbf{x},n)\}$ by defining the neighbors of a box $B(\mathbf{x},n)$ as $B(\mathbf{x}\pm\mathbf{e}_i n, n)$, $1\le i\le d$. Henceforth, any graph operators on a subset of some $\mathrm{sp}\{B\}$ refer to this graph structure. Observe that $\mathrm{sp}\{B(\mathbf{x},n)\}$ is isomorphic as a graph to $\mathbb{Z}^d$. We fix an isomorphism $\Delta:\mathrm{sp}\{B\}\to\mathbb{Z}^d$, $\Delta(B(\mathbf{x}+\sum_i \mathbf{e}_i k_i n, n)) = \mathbf{x}+\sum_i \mathbf{e}_i k_i$. Using Δ, we extend the definitions of a box to boxes as well. Thus, for a box $b=b(n)$ and an integer $m>0$, $B_\Delta(b,m)$ is a set of m^d boxes. We use a big union symbol to denote internal union, that is, $\bigcup \mathbf{A}=\{\mathbf{x}\in A : A\in\mathbf{A}\}$. So in the preceding example, we have $\bigcup B_\Delta(b,m)=B(mn)$.

To ease the reading, we often refer to boxes that are neighbors under the above relationship as Δ-*neighbors*, a connected set of boxes as Δ-*connected*, and a component under Δ-neighbor relationship a Δ-*component*.

DEFINITION 2.4. Given a box $B(\mathbf{x},n)$, and $\alpha>0$, we write B^α for $B(\mathbf{x},\alpha n)$. Let

$$s(n)=\lceil\log n\rceil^4.$$

We write $s^{(i)}(n)$ to denote $s(\cdot)$ iterated i times.

DEFINITION 2.5. Let

$$\sigma\bigl(B(\mathbf{x},n)\bigr)=\mathrm{sp}\bigl\{b\bigl(\mathbf{x},s(n)\bigr)\bigr\}\cap\bigl\{b\bigl(\mathbf{y},s(n)\bigr):\mathbf{y}\in B\bigl(\mathbf{x},5n+3\lceil\log n\rceil^6\bigr)\bigr\}$$

be the *subboxes* of $B(\mathbf{x},n)$. Note that $B^5\subset\bigcup\sigma(B)$. $\sigma(B)$ is a collection of subboxes of side length $s(n)$ covering B^5; see Figure 2 for visualization.

We write 2^A for the power set of a set A, that is, the collection of subsets of A. We refer to finite subsets of $\mathbb{Z}^d$ as configurations.

2.3. *Percolating configurations.* Let c_a, c_b be fixed positive constants dependent only on dimension (c_a, c_b are determined in Lemma 4.8 and Corollary 4.6, resp.). $\omega\in 2^{B(n)}$ is a *percolating* configuration, denoted by $\omega\in\mathcal{P}(n)$, if there exists a subset which we call a *good cluster* $\mathcal{C}=\mathcal{C}(\omega)\subset\omega$, connected in ω (not necessarily maximal) for which the following properties hold:

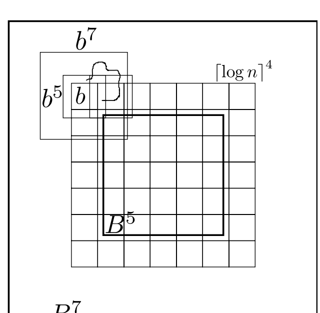

FIG. 2. 0-*good configuration*.

1. $|\mathcal{C}| > (1 - 10^{-d})|B(n)|$.
2. The largest component in $B(n) \setminus \mathcal{C}$ is of size less than $(\log n)^2$.
3. For any $\mathbf{v}, \mathbf{w} \in \mathcal{C} \cap B(n - c_a \log n)$ we have $d_{\mathcal{C}}(\mathbf{v}, \mathbf{w}) < c_a(d_B(\mathbf{v}, \mathbf{w}) \vee \log n)$.

Moreover, a configuration $\omega \in \mathcal{P}(n)$ admits an isoperimetry property:

4. Let $T \subset B(n)$ satisfy $n^{1/5d} < |T| \leq n^d/2$, and assume both T and $B(n) \setminus T$ are connected in $B(n)$. Then $|\partial_B T \cap \omega|, |\partial_B^c T \cap \omega| > c_b |T|^{(d-1)/d}$.

The following claim is easy to check.

CLAIM 2.1. *$\mathcal{P}(n)$ is a* monotone *set, that is, if $\omega \in \mathcal{P}(n)$ and $\omega \subset \omega^+ \subset B(n)$ then $\omega^+ \in \mathcal{P}(n)$.*

2.4. *k-good configurations.* Let c_h be a fixed positive constant dependent only on dimension (c_h is determined in Theorem 5.12 below). For $n \in \mathbb{N}$, $\rho > 0$, and setting $B = B(n)$, a configuration $\omega \subset B^7$ belongs to $\mathcal{G}_0^\rho(n)$ if and only if the following properties hold:

1. For each $b \in \sigma(B)$, $|\omega \cap b| > (\rho c_h \wedge \frac{1}{2})|b|$.
2. For each $b \in \sigma(B)$, $\omega \cap b^5$ is connected in $\omega \cap b^7$.

REMARK 2.6. If $\omega \in \mathcal{G}_0^\rho(n)$, then for all $n > (\rho c_h)^{-1/d}$: (i) ω intersects all $b \in \sigma(B)$ (property 1), and (ii) for any two Δ-neighbors $b_1, b_2 \in \sigma(B)$, since $b_2 \subset b_1^5$, $\omega \cap b_1$ and $\omega \cap b_2$ are connected in $\omega \cap b_1^7$ (property 2). In particular, $\omega \cap B^5$ is connected in $\omega \cap B^7$. See Figure 2 for a graphical explanation.

Let Λ be a fixed positive constant dependent only on dimension (Λ is determined in Theorem 5.8). For $k > 0$, $\mathcal{G}_k^\rho(n)$ is defined recursively. Given $\omega \subset \mathbb{Z}^d$, $i \in \mathbb{N}$ and a box $b(\mathbf{x}, m)$, we say b is *(ω, i, ρ)-good* if $(\omega \cap b^7) - \mathbf{x} \in \mathcal{G}_i^\rho(m)$. Let

$$\mathcal{S} = \{b \in \sigma(B) : b \text{ is } (\omega, k-1, \rho\Lambda)\text{-good}\},$$

and let $\sigma_B = \|\Delta(\sigma(B))\| = |\sigma(B)|^{1/d}$. Then $\omega \in \mathcal{G}_k^\rho(n)$ if $\omega \in \mathcal{G}_0^\rho(n)$ and $\Delta(\mathcal{S}) \in \mathcal{P}(\sigma_B)$. See Figure 3 for a graphical explanation.

2.5. *k-good torus.* Let $\mathcal{T} = \mathcal{T}(N)$ and fix $\omega \subset \mathcal{T}$. Let $k \geq 0$, $\rho > 0$. We define *(ω, k, ρ)-goodness of a torus*. Let $n = \lceil N/10 \rceil$. We call

$$\mathbf{T} = \operatorname{sp}\{B(n)\} \cap \{B(\mathbf{y}, n) : \mathbf{y} \in B(N)\}$$

the *top-level* boxes for $\mathcal{T}$. Then $\mathcal{T}$ is a (ω, k, ρ)-good torus if all boxes in $\mathbf{T}$ are $(\Pi_N^{-1}\omega, k, \rho)$-good.

Remark 2.6 therefore implies the following.

REMARK 2.7. If $\mathcal{T}(N)$ is a (ω, k, ρ)-good torus, then ω is connected for all $N > C(\rho)$.

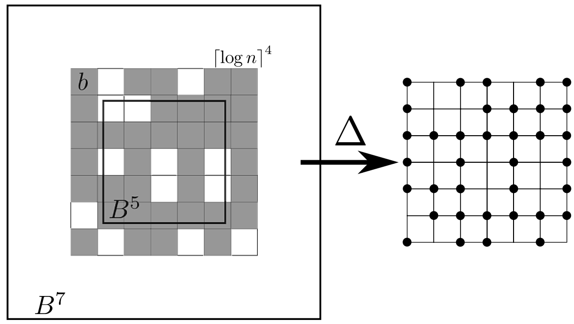

FIG. 3. *k-good configuration. All the grey subboxes are $k-1$-good, that is,* $\omega \cap b \in \mathcal{G}_{k-1}^{\rho\Lambda}(\lceil \log n \rceil^4)$. *The configuration on the right is in* $\mathcal{P}(\sigma_B)$.

2.6. *Constants.* All constants are dependent on dimension by default and independent of any other parameter not appearing in their definition. Constants like c, C may change their value from use to use. Numbered constants (e.g., c_1, C_2) retain their value in a proof but no more than that, and constants tagged by a letter (c_a, c_Λ) represent the same value throughout the paper.

3. Mixing bound. Given a finite connected graph G, let $X(t)$ be a lazy random walk on G. That is, denoting the walk's transition matrix by $p(\cdot,\cdot)$, for any $\mathbf{v} \in G$ of degree m, $p(\mathbf{v},\mathbf{v}) = 1/2$ and $p(\mathbf{v},\mathbf{w}) = 1/2m$ for any neighbor $\mathbf{w} \in \partial\{\mathbf{v}\}$. We write $\tau(G)$ for the mixing time of $X(t)$ on G, that is,

$$\tau(G) = \min\left\{n : \left|\frac{p^n(x,y) - \pi(y)}{\pi(y)}\right| \le \frac{1}{4},\ \forall x, y \in V(G)\right\},$$

where π is the stationary measure of the random walk on G. See [13] a thorough introduction on mixing times.

THEOREM 3.1. *Let $\omega_0 \subset \mathcal{T}(N)$, $\rho > 0$, $k \ge 1$. There is a $C(k,\rho)$ such that if $\mathcal{T}(N)$ is a (ω_0, k, ρ)-good torus then*

$$\tau(\omega_0) < CN^2 \log^{(k-1)} N,$$

where $\log^{(m)} N$ is $\log(\cdot)$ iterated m times of N.

We begin by stating and proving propositions required for Corollary 3.4, then using the corollary we prove Theorem 3.1.

Recall the definition of $\mathcal{G}_l^\rho(n)$ from Section 2.4. Let $c_\rho = (\rho c_h \wedge \frac{1}{2})/3$. We assume n is large enough such that $\mathcal{G}_l^\rho(n)$ is nonempty, and that for any $\omega \in \mathcal{G}_l^\rho(n)$, $\omega \cap B^5(n)$ is connected in ω and satisfies $|\omega \cap B^5(n)| > 3c_\rho n^d$ (see property 1 of $\mathcal{G}_0^\rho$ in Section 2.4 and Remark 2.6). In particular, there exists a set $S \subset \omega$, $|S \cap B^5(n)| \wedge |(\omega \setminus S) \cap B^5(n)| \ge c_\rho n^d$.

Since $\omega \cap B^5(n)$ is connected in ω, we have the following.

PROPOSITION 3.2. *For any $l \geq 0$ and all large n, and $S \subset \omega \in \mathcal{G}_l^\rho(n)$*

$$|\partial_\omega S| \geq 1.$$

Next, we bound $|\partial S|$ more accurately. The next theorem is one of the main results and techniques introduced in this paper. The theorem proves an almost tight isoperimetric inequality (up to an iterated log). The main idea of the proof is induction on the number of iterations (which provide the iterated log) and analyzing the geometry of renormalized subsets, that is, use the geometrical properties of the percolation configuration of good subboxes.

THEOREM 3.3. *Let $l \geq 0$, $\rho > 0$, $\omega \in \mathcal{G}_l^\rho$ and $S \subset \omega$ such that $|S \cap B^5(n)| \wedge |(\omega \setminus S) \cap B^5(n)| = r \geq n^{1/3}$. There exists a constant $c_1(l, \rho) > 0$, such that*

$$|\partial_\omega S| > c_1(l, \rho) r^{(d-1)/d} \big(s^{(l)}(n)\big)^{1-d}. \tag{1}$$

PROOF. The proof is by induction on l. For $l = 0$, since $s^{(0)}(n) = n$, $|B^5(n)|^{(d-1)/d} s^{(0)}(n)^{1-d}$ is less than some C_1 for any $r \leq |B^5(n)|$. Thus, the base case of $l = 0$ is given in Proposition 3.2 and the connectedness of ω with $c_1(0) = C_1^{-1}$. Now fix $l > 0$, $\rho > 0$ and assume (1) is true for $l - 1$ with constant $c_1(l-1, \rho\Lambda) > 0$, for all large n and $n^{1/3} \leq r \leq |B^5(n)|$.

Our default ambient graph for S is ω. Thus, for $S \subset \omega$, $S^c = \omega \setminus S$ and $\partial S = \partial_\omega S$. Note that as $|S| \geq r$, if $|\partial S| > |S|^{(d-1)/d}$ we are done. W.l.o.g. assume $|S^c \cap B^5| \geq |S \cap B^5|$ since $|\partial_\omega S^c| \sim |\partial_\omega S|$.

Let $B = B(n)$ and let $m = s(n)$. For $0 < \alpha < 1$, let

$$\mathbf{F} = \mathbf{F}(\omega, S, \alpha) = \big\{b \in \sigma(B) : |b \cap S| \geq \alpha |b \cap \omega|\big\},$$

be the α-filled subboxes. By the pigeon hole principle, there are $\alpha(\rho) < 1$, $c_2(\rho) > 0$, such that

$$|\mathbf{F}| < (1 - c_2)\big|\sigma(B)\big|. \tag{2}$$

Let $\mathbf{T} = \mathbf{T}(\omega, S) = \{b \in \sigma(B) : b \cap S \neq \varnothing\}$, then $|\mathbf{T}| \geq |S| m^{-d}$. The proof is separated into cases depending on the size of F. We begin with the case that $|\mathbf{F}|$ is small.

If $|\mathbf{F}| \leq \frac{1}{2}|S| m^{-d}$ then by the trivial lower bound on $\mathbf{T}$, $|\mathbf{T} \setminus \mathbf{F}| \geq \frac{1}{2}|S| m^{-d}$. For any box $b \in \mathbf{T} \setminus \mathbf{F}$, we have $\mathbf{x}, \mathbf{y} \in b$ such that $\mathbf{x} \in S, \mathbf{y} \in S^c$. Since $\mathbf{x}, \mathbf{y}$ are connected in $\omega \cap b^7$ (property 2 of $\mathcal{G}_0^\rho$), $\partial S \cap b^7 \neq \varnothing$. For any box $b \in \sigma(B)$, there are at most $50d$ boxes $b' \in \sigma(B)$ such that $b^7 \cap b'^7 \neq \phi$. Since $|S| \geq n^{1/3}$ and m^d is $o(n^{1/4d})$ we have for all large n,

$$|\partial S| \geq \frac{1}{50d} |\mathbf{T} \setminus \mathbf{F}| \geq \frac{1}{100d} |S| m^{-d} > |S|^{1-3/(4d)} > |S|^{(d-1)/d},$$

and are done with this case.

Our default ambient graph for sets of subboxes is $\sigma(B)$ with the box (Δ) neighbor relationship (see Section 2.2). Thus, for $\mathbf{A} \subset \sigma(B)$, $\mathbf{A}^c = \sigma(B) \setminus \mathbf{A}$, $\partial \mathbf{A} = \partial_{\sigma(B)} \mathbf{A}$, $\partial^{\text{in}} \mathbf{A} = \partial^{\text{in}}_{\sigma(B)} \mathbf{A}$. We introduce edge boundary notation

$$\partial^e(\mathbf{Q}) = \{\{b, b'\} : b \sim b', b \in \mathbf{Q}, b' \in \mathbf{Q}^c\}.$$

In the case that remains, $|\mathbf{F}| > \frac{1}{2}|S|m^{-d}$. Note that any box $b \in \partial \mathbf{F}$ satisfies $|b^5 \cap S| \wedge |b^5 \cap S^c| > c'(\rho)m^d$. Hence, if we knew that $\mathbf{F}$ was a single Δ-connected component with a connected complement, we could lower bound $|\partial \mathbf{F}|$ and use the fact that $\partial \mathbf{F}$ is a typical set (Percolation property 4) to get that a constant proportion of $\partial \mathbf{F}$ are $(\omega, l-1, \rho\Lambda)$-good boxes. Together with our induction hypothesis, this would complete the proof.

$$\begin{aligned} |\partial S| &\geq \left|\partial S \cap \partial \mathbf{F} \cap_{b \in \mathcal{G}^{\lambda\rho}_{l-1}(m)} \{b\}\right| \geq c|\partial \mathbf{F}| m^{d-1} \left(s^{l-1}(m)\right)^{1-d} \\ &\geq c|\mathbf{F}|^{(d-1)/d} m^{d-1} \left(s^l(n)\right)^{1-d} \geq \frac{c}{2}|S|^{(d-1)/d} m^{1-d} m^{d-1} \left(s^{l-1}(m)\right)^{1-d} \\ &= \frac{c}{2}|S|^{(d-1)/d} \left(s^{l-1}(m)\right)^{1-d}. \end{aligned} \tag{3}$$

$\mathbf{F}$ is not in general so nice. However, being of size greater than $\frac{1}{2}|S|m^{-d}$ implies there is a $c_3(\rho) > 0$ and a set $\mathbb{K} = \mathbb{K}(\mathbf{F}) \subset 2^{\sigma(B)}$ with the following properties for all large n, allowing us to make a similar isoperimetric statement:

$$\sum_{\mathbf{f} \in \mathbb{K}} (|\mathbf{f}| \wedge |\mathbf{f}^c|) \geq c_3 |S| m^{-d}, \tag{4}$$

$$\forall \mathbf{f} \in \mathbb{K}, \qquad \partial \mathbf{f} \subset \mathbf{F}^c, \qquad \partial^{\text{in}} \mathbf{f} \subset \mathbf{F}, \tag{5}$$

$$\forall \mathbf{f}_1, \mathbf{f}_2 \in \mathbb{K}, \qquad \mathbf{f}_1 \neq \mathbf{f}_2 \quad \Longrightarrow \quad \partial^e \mathbf{f}_1 \cap \partial^e \mathbf{f}_2 = \varnothing, \tag{6}$$

$$\forall \mathbf{f} \in \mathbb{K}, \qquad \mathbf{f}, \mathbf{f}^c \text{ are } \Delta\text{-connected}, \tag{7}$$

$$n^{1/5d} < |\mathbf{f}| \wedge |\mathbf{f}^c| \leq |\sigma(B)|/2. \tag{8}$$

First, we show how the proof follows from the existence of $\mathbb{K}$. Let $\mathbf{G} = \mathbf{G}(\omega, l, \rho)$ be the set of $(\omega, l-1, \rho\Lambda)$-good subboxes in $\sigma(B)$. By (7), (8) and Percolation property 4 (see Section 2.3), for all large enough n, for any $\mathbf{f} \in \mathbb{K}$, $|\partial \mathbf{f} \cap \mathbf{G}| > c_b(|\mathbf{f}| \wedge |\mathbf{f}^c|)^{(d-1)/d}$. Let $\mathbb{K}^\partial = \{\partial \mathbf{f} : \mathbf{f} \in \mathbb{K}\}$. By (6), for any $b \in \sigma(B)$, $|\{\mathbf{f} \in \mathbb{K}^\partial : b \in \mathbf{f}\}| \leq 2d$. Thus,

$$\left|\bigcup \mathbb{K}^\partial \cap \mathbf{G}\right| \geq \frac{1}{2d} \sum_{\mathbf{f} \in \mathbb{K}} c_b \left(|\mathbf{f}| \wedge |\mathbf{f}^c|\right)^{(d-1)/d}.$$

By subadditivity of x^β where $\beta < 1$ and (4) this gives

$$\left|\bigcup \mathbb{K}^\partial \cap \mathbf{G}\right| \geq c\left[\sum_{\mathbf{f} \in \mathbb{K}} (|\mathbf{f}| \wedge |\mathbf{f}^c|)\right]^{(d-1)/d} \geq c' \frac{|S|^{(d-1)/d}}{m^{d-1}}. \tag{9}$$

Let $\mathbf{A} \subset \bigcup \mathbb{K}^\partial \cap \mathbf{G}$, be a subset of size $|A| > c|\bigcup \mathbb{K}^\partial \cap \mathbf{G}|$, satisfying that for any distinct $b_1, b_2 \in \mathbf{A}$, $b_1^7 \cap b_2^7 = \varnothing$, for example, $A = (\bigcup \mathbb{K}^\partial \cap \mathbf{G}) \cap \Delta^{-1}(20 \cdot \mathbb{Z}^d)$. By (5), for any $b \in \mathbf{A}$, $b \in \mathbf{F}^c$ but has a Δ-neighbor $b' \in \mathbf{F}$, implying $|S \cap b^5| \wedge |S^c \cap b^5| \geq c(\hat{\alpha}, \rho)m^d = \hat{c}m^d$. Since $\mathbf{A} \subset \mathbf{G}$, using our induction assumption and that $|S| > r$,

$$
\begin{aligned}
|\partial S| &\geq \left|\partial S \cap \bigcup \mathbf{F}\right| \overset{(5)}{\geq} |\partial S \cap A| \overset{(9)}{\geq} c' \frac{|S|^{(d-1)/d}}{m^{d-1}} m^{d-1} \big(s^{l-1}(m)\big)^{1-d} \\
&= c'|S|^{(d-1)/d} \big(s^l(n)\big)^{1-d}
\end{aligned} \tag{10}
$$

and we are done.

We return to proving the existence of $\mathbb{K}$.

Recall, a Δ-component of a set $\mathbf{Q} \subset \sigma(B)$ is a maximal connected component in $\mathbf{Q}$ according to the box neighbor relationship (see Section 2.2). Let $\mathbb{F}$ be the set of Δ-components of $\mathbf{F}$. Since $\mathbf{F} \neq \sigma(B)$, for any $\mathbf{f} \in \mathbb{F}$, there exists $b \in \mathbf{f}$ with a Δ-neighbor $b' \in \mathbf{f}^c$, such that $b' \subset b^5$. As before, by property 2 of $\mathcal{G}_0^\rho$ (see Section 2.4), $b^7 \cap \partial S \neq \varnothing$. Letting $\mathbf{F}^\partial = \{b \in \mathbf{F} : b^7 \cap \partial S \neq \varnothing\}$, we then have $|\mathbf{F}^\partial| \geq |\mathbb{F}|$. Since we can extract a subset $\mathbf{A} \subset \mathbf{F}^\partial$ where $|\mathbf{A}| > c|\mathbf{F}^\partial|$, and for any distinct $b_1, b_2 \in \mathbf{A}$, $b_1^7 \cap b_2^7 = \varnothing$, we only need deal with the case $|\mathbb{F}| < |S|^{1-1/(2d)}$. Let $\mathbb{H}$ be the set of Δ-components of $\mathbf{F}^c$. In the same way, we may assume $|\mathbb{H}| < |S|^{1-1/(2d)}$. By (2), $|\mathbf{F}^c| > c_2|\sigma(B)| > 2c_3|S|m^{-d}$. We also assumed $|\mathbf{F}| > \frac{1}{2}|S|m^{-d}$, so w.l.o.g. $c_3 < 1/4$ and

$$
\left|\mathbf{F}^c\right|, |\mathbf{F}| > 2c_3|S|m^{-d}. \tag{11}
$$

Let $\widehat{\mathbb{F}} = \{\mathbf{f} \in \mathbb{F} : |\mathbf{f}| \geq c_3|S|^{1/(2d)}m^{-d}\}$ and let $\widehat{\mathbb{H}} = \{\mathbf{h} \in \mathbb{H} : |\mathbf{h}| \geq c_3|S|^{1/(2d)}m^{-d}\}$. We assumed $|\mathbb{F}|, |\mathbb{H}| < |S|^{1-1/(2d)}$, and thus $\bigcup(\mathbb{F} \setminus \widehat{\mathbb{F}}), \bigcup(\mathbb{H} \setminus \widehat{\mathbb{H}}) < c_3|S|m^{-d}$. So, from (11), we get

$$
\left|\bigcup \widehat{\mathbb{F}}\right|, \left|\bigcup \widehat{\mathbb{H}}\right| > c_3|S|m^{-d}. \tag{12}
$$

Let

$$
\mathbb{K} = \left\{\mathbf{f} \subset \sigma(B) : \mathbf{f} \text{ is a } \Delta\text{-component of } \mathbf{h}^c, \mathbf{h} \in \mathbb{H}, |\mathbf{f}| \wedge \left|\mathbf{f}^c\right| > c_3|S|^{1/(2d)}m^{-d}\right\}.
$$

Let $U : \mathbb{K} \to \mathbb{H}$ where for $\mathbf{f} \in \mathbb{K}$, $U(\mathbf{f})$ is the unique element in $\mathbb{H}$ for which $\mathbf{f}$ is a Δ-component of $U(\mathbf{f})^c$. For each $\mathbf{f} \in \mathbb{K}$, $\partial \mathbf{f} \subset U(\mathbf{f}) \subset \mathbf{F}^c$ and because $U(\mathbf{f})$ is a component of $\mathbf{F}^c$, $\partial^{\text{in}}\mathbf{f} \subset \mathbf{F}$, giving us (5). Let $\mathbf{h} \in \widehat{\mathbb{H}}$. For any $\mathbf{f} \in \widehat{\mathbb{F}}$, $\mathbf{f} \subset \mathbf{h}^c$ and thus $\mathbf{f}$ is contained in some Δ-component of $\mathbf{h}^c$ which we denote $\hat{\mathbf{f}}$. Since $\mathbf{h} \subset \hat{\mathbf{f}}^c$ and $\mathbf{f} \subset \hat{\mathbf{f}}$ we get $\hat{\mathbf{f}} \in \mathbb{K}$ and in particular, $\hat{\mathbf{f}} \in U^{-1}(\mathbf{h})$. Thus, for any $\mathbf{h} \in \widehat{\mathbb{H}}$, $\bigcup \widehat{\mathbb{F}} \subset \bigcup U^{-1}(\mathbf{h})$. In Figure 4, we give an example of some $\mathbf{F}$ and the resulting $\mathbb{K}$.

We regroup terms in the sum and use the fact that for any $\mathbf{h} \in \mathbb{H}, \mathbf{f} \in U^{-1}(\mathbf{h})$, we have $\mathbf{h} \subset \mathbf{f}^c$ to get:

$$
\sum_{\mathbf{f} \in \mathbb{K}} \big(|\mathbf{f}| \wedge \left|\mathbf{f}^c\right|\big) \geq \sum_{\mathbf{h} \in \widehat{\mathbb{H}}} \sum_{\mathbf{f} \in U^{-1}(\mathbf{h})} \big(|\mathbf{f}| \wedge \left|\mathbf{f}^c\right|\big) \geq \sum_{\mathbf{h} \in \widehat{\mathbb{H}}} \sum_{\mathbf{f} \in U^{-1}(\mathbf{h})} \big(|\mathbf{f}| \wedge |\mathbf{h}|\big).
$$

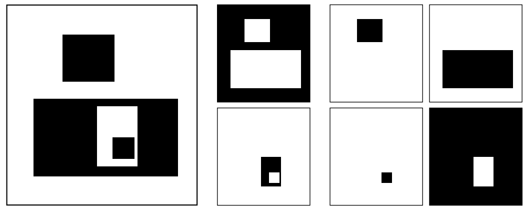

FIG. 4. *Example of* $\mathbf{F}$ *and resulting* $\mathbb{K} = \{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3, \mathbf{f}_4\}$. $\left[\mathbf{F}\ \begin{smallmatrix}\mathbf{h}_1 & \mathbf{f}_1 & \mathbf{f}_2\\ \mathbf{h}_2 & \mathbf{f}_3 & \mathbf{f}_4\end{smallmatrix}\right]$ *where the sets are in black and* $\mathbf{h}_1 = U(\mathbf{f}_1) = U(\mathbf{f}_2)$, $\mathbf{h}_2 = U(\mathbf{f}_3) = U(\mathbf{f}_4)$.

If there exists $\mathbf{h}^* \in \widehat{\mathbb{H}}$ such that for any $f \in U^{-1}(\mathbf{h}^*)$, $|\mathbf{h}^*| \ge |\mathbf{f}|$, we have

$$\sum_{\mathbf{f}\in\mathbb{K}}(|\mathbf{f}| \wedge |\mathbf{f}^c|) \ge \sum_{\mathbf{f}\in U^{-1}(\mathbf{h}^*)} |\mathbf{f}| = \Big|\bigcup U^{-1}(\mathbf{h}^*)\Big| \ge \Big|\bigcup \widehat{\mathbb{F}}\Big|.$$

If none such exists, then

$$\sum_{\mathbf{f}\in\mathbb{K}}(|\mathbf{f}| \wedge |\mathbf{f}^c|) \ge \sum_{\mathbf{h}\in\widehat{\mathbb{H}}} |\mathbf{h}| = \Big|\bigcup \widehat{\mathbb{H}}\Big|.$$

Thus, from (12), we get (4). Next, for $\mathbf{f}_1 \in \mathbb{K}$, any edge $\{b, \hat{b}\} \in \partial^e \mathbf{f}_1$ satisfies w.l.o.g. $\hat{b} \in U(\mathbf{f}_1)$ and $b \in \mathbf{f}_1$. Thus, if $\mathbf{f}_2 \in \mathbb{K}$ shares the edge $\{b, \hat{b}\}$ with $\mathbf{f}_1$, then $U(\mathbf{f}_1) = U(\mathbf{f}_2)$ and since $b \in \mathbf{f}_1 \cap \mathbf{f}_2$ and both are Δ-components of $U(\mathbf{f}_1)^c$, we have $\mathbf{f}_1 = \mathbf{f}_2$, giving us (6). To get (7), let $\mathbf{h} \in \mathbb{H}$, and let $\mathbf{h}^c = \mathbf{f}_1 \cup \cdots \cup \mathbf{f}_n$ where $\mathbf{f}_i$ are the Δ-components of $\mathbf{h}^c$. Then $\forall i$, $\partial \mathbf{f}_i \subset \mathbf{h}$, and since $\mathbf{h}$ is connected, $\mathbf{f}_i, \mathbf{f}_j$ are connected in $\mathbf{f}_i \cup \mathbf{f}_j \cup \mathbf{h}$ for any i, j. This implies $\mathbf{f}_i^c = \mathbf{h} \cup \mathbf{f}_1 \cup \cdots \cup \mathbf{f}_n \setminus \mathbf{f}_i$ is Δ-connected for any i. Last, since $|\mathbf{f}| \wedge |\mathbf{f}^c| > c_3|S|^{1/(2d)} m^{-d}$ and m^d is $o(n^{1/20d})$, we get (8). $\square$

In the below corollary, we transfer the isoperimetric bounds on φ from the setting of a box to a torus. The main idea of the proof is to show that given any large set S in a (ω, k, ρ)-good torus, there are two neighboring top-level boxes which have a large intersection with S and $\omega \setminus S$.

COROLLARY 3.4. *Let* $\omega \subset \mathcal{T}(N)$. *If* $\mathcal{T}(N)$ *is a* (ω, k, ρ)*-good torus then for all large enough* N, *and* $r \ge N$

$$\begin{aligned}\hat{\phi}(r) &= \inf\left\{\frac{|\partial_\omega S|}{|S|} : S \subset \omega, N^{1/3} \le |S| \le r \wedge \left(1 - \frac{1}{4d}\right)|\omega|\right\}\\ &> c(k, \rho)\frac{r^{-1/d}}{(s^{(k)}(N))^{d-1}}.\end{aligned}$$

PROOF. Let $\omega^+ = \Pi_N^{-1}(\omega) \cap B^3(N)$. Recall from Section 2.5 that all top-level boxes for $\mathcal{T}(N)$ are (ω^+, k, ρ)-good, so by property 1 of $\mathcal{G}_0^\rho$, for any top-level box

B, there is a $c_1(\rho) > 0$ such that

$$|B \cap \omega^+| > c_1 N^d. \tag{13}$$

Fix $r \geq N$. By construction, $\frac{1}{2d}|\omega^+| = |\omega| \geq |B \cap \omega^+|$ for any top-level box B. We assume that N is large enough so that $c_1 N^{d-1} > 4d$, and $|B \cap \omega^+| > 4dN$. In particular, this implies that the infimum is not on an empty set. Let S satisfy the conditions to be a candidate for the infimum in $\hat{\phi}(r)$ and extend it to $S^+ = \Pi_N^{-1}(S) \cap B^3(N)$. Let $\hat{r} = |S| \wedge |\omega \setminus S|$. Again by (13), for each top-level box B, $|B \cap S^+| \vee |B \cap (\omega^+ \setminus S^+)| \geq \frac{1}{2} c_1 N^d > c_2 \hat{r}$. On the other hand, since there are 10^d top-level boxes whose union covers $B(N)$, by the pigeonhole principle, there must be some box B for which $|B \cap S^+| \geq 10^{-d}|S|$ and likewise a box B' for which $|B' \cap (\omega^+ \setminus S^+)| \geq 10^{-d}|(\omega \setminus S)|$. Let $c_3 = c_2 \wedge 10^{-d}$. Since the top-level boxes are Δ-connected, there are two Δ-neighboring top-level boxes B_1, B_2 such that $|B_1 \cap S^+|, |B_2 \cap (\omega^+ \setminus S^+)| \geq c_3 \hat{r}$. This implies $|B_1^5 \cap S^+| \wedge |B_1^5 \cap (\omega^+ \setminus S^+)| \geq c_3 \hat{r}$. By construction, $|\partial_{B_1^7 \cap \omega^+} S^+| \leq |\partial_\omega S|$. Since B_1 is (ω^+, k, ρ)-good, we can use Theorem 3.3 to lower bound $|\partial_{B_1^7 \cap \omega^+} S^+|$ by $c\hat{r}^{(d-1)/d}(s^{(k)}(N))^{1-d}$ for all large N. Note that as $|\omega| > 4dN$, implying $|\omega \setminus S| \geq N$, we have $\hat{r} \geq N$. Since $|\omega \setminus S| \geq \frac{1}{4d}|\omega| > \frac{1}{4d}|S|$, we can bound $|S|$, the denominator in the infimum, from above by $4d\hat{r}$, giving us $\frac{|\partial_\omega S|}{|S|} \geq c\hat{r}^{-1/d}(s^{(k)}(N))^{1-d}$. Since $\hat{r} \leq r$ we are done. $\square$

We now proceed to prove the main theorem of this section.

PROOF OF THEOREM 3.1. The following proof makes assumptions which are valid for all but a finite number of N, and those are resolved by the large constant above. Note that ω_0 is viewed as a subgraph of $\mathcal{T}(N)$ as far as connectivity is concerned. We present an upper bound to the mixing time τ of $X(t)$ using average conductance, a method developed in [11] and refined in subsequent papers.

We follow notation of [13]. Let $\pi(\cdot)$ be the stationary distribution of $X(t)$ and for $\mathbf{x}, \mathbf{y} \in \omega_0$ let $Q(\mathbf{x}, \mathbf{y}) = \pi(\mathbf{x})p(\mathbf{x}, \mathbf{y})$. For $S, A \subset \omega_0$ let $Q(S, A) = \sum_{\mathbf{s} \in S, \mathbf{a} \in A} Q(\mathbf{s}, \mathbf{a})$. Let $\Phi_S = \frac{Q(S, S^c)}{\pi(S)}$ and let $\Phi(u) = \inf\{\Phi_S : 0 < \pi(S) \leq u \wedge \frac{1}{2}\}$. Let $\pi_* = \min_{\mathbf{x} \in \omega_0} \pi(\mathbf{x})$.

By [13],

$$\tau = \tau\left(\omega_0, \frac{1}{4}\right) \leq 1 + \int_{4\pi_*}^{16} \frac{4\,du}{u\Phi^2(u)}. \tag{14}$$

Recall the notation from Section 2.1. In this proof, our ambient graph is ω_0 and thus $S^c = \omega_0 \setminus S$ and $\partial S = \partial_{\omega_0} S$. To simplify notation in the proof, we restate (14) in terms of internal volume and boundary size.

For $S \subset \omega_0$, if $\pi(S) \leq u$, then we have by definition $u \geq \sum_{\mathbf{v} \in S} \deg(\mathbf{v}) \times [\sum_{\mathbf{v} \in \omega_0} \deg(\mathbf{v})]^{-1}$. Using the bound on degree and connectedness of ω_0, we get

$|S| \le 2ud|\omega_0|$. In the same way, $2d\frac{|S^c|}{|\omega_0|} > \pi(S^c) \ge 1-u$ which gives $|S| \le (1-\frac{1}{2d}(1-u))|\omega_0|$, and thus for $u \le \frac{1}{2}$,

$$|S| \le 2ud|\omega_0| \wedge \left(1-\frac{1}{4d}\right)|\omega_0|. \tag{15}$$

Let $\phi_S = \frac{|\partial S|}{|S|}$. Since ω_0 is a bounded degree graph and $\mathbf{x} \sim \mathbf{y} \iff \frac{1}{4d} \le p(\mathbf{x},\mathbf{y}) \le \frac{1}{2}$, for some $C(d)$ and all $S \subset \omega_0$ we have $\phi_S < C\Phi_S$. Let $\phi(r) = \inf\{\phi_S : 0 < |S| \le r \wedge (1-\frac{1}{4d})|\omega_0|\}$. Then by (15) the infimum in $\phi(2ud|\omega_0|)$ is on a larger set than the infimum in $\Phi(u)$ giving us $\phi(2ud|\omega_0|) < C\Phi(u)$. Thus, by the change of variables $r = 2ud|\omega_0|$ in (14), we get

$$\tau < C\int_1^{32dN^d} \frac{dr}{r\phi^2(r)}. \tag{16}$$

We continue by showing that for our purposes, a rough estimate of ϕ_S for sufficiently small sets S is enough. Let

$$\hat{\phi}(r) = \inf\left\{\phi_S : N^{1/3} \le |S| \le r \wedge \left(1-\frac{1}{4d}\right)|\omega_0|\right\},$$

where the infimum of an empty set is ∞. Since ω_0 is connected (see Remark 2.7), $\phi(r) \ge 1/r$ for any $1 \le r < |\omega_0|$. For large N, by property 1 of $\mathcal{G}_0^\rho$ (see Section 2.4), $N < (1-\frac{1}{4d})|\omega_0|$. Thus,

$$\begin{aligned}\phi(r) &= \inf\{\phi_S : |S| \le r \wedge N^{1/3}\} \wedge \hat{\phi}(r)\\ &\ge [r^{-1} \vee N^{-1/3}] \wedge \hat{\phi}(r).\end{aligned}$$

By Corollary 3.4 below, $\hat{\phi}(r) > c(k,\rho)(s^{(k)}(N))^{1-d}r^{-1/d}$. Integrating (16) with the above lower bound for $\phi(r)$, we thus get

$$\begin{aligned}\tau &< C\int_1^{32N^d} \frac{dr}{r(N^{-1/3})^2} + C\int_{10^d N^{1/3}}^{32N^d} \frac{dr}{r\hat{\phi}^2(r)}\\ &< o(N^2) + C(s^{(k)}(N))^{2d-2}N^2 = o((\log^{(k-1)} N))N^2\end{aligned}$$

as required. □

4. High density percolation percolates. This section presents results used in the renormalization arguments of Section 6. See Section 2.3 for the properties of percolating configuration. Note that many of the lemmas in this section deal with i.i.d. Bernoulli percolation.

LEMMA 4.1. *For $n \in \mathbb{N}$, let $\{Y(\mathbf{z})\}_{\mathbf{z}\in B(n)}$ be i.i.d. $\{0,1\}$ r.v.'s, and write $\mathcal{S}(n) = \{\mathbf{z} \in B(n) : Y(\mathbf{z}) = 1\}$ for the random support of Y. Then there are dimensional dependent constants, $C > 0$ and $p_b < 1$, such that if* $\Pr[Y(\mathbf{0}) = 1] = p_b$,

$$\Pr[\mathcal{S}(n) \in \mathcal{P}(n)] \ge 1 - \frac{C(\log n)^{d-1}}{n^d}.$$

PROOF. Lemmas 4.3, 4.7, 4.8 and Corollary 4.6 prove Percolation properties 1–4, respectively. □

The next lemma assures a percolation configuration given a finite range dependance requirement.

COROLLARY 4.2. *For $n \in \mathbb{N}$, let $\{Y(\mathbf{z})\}_{\mathbf{z}\in B(n)}$ be $\{0, 1\}$ r.v.'s, not necessarily i.i.d., and write $\mathcal{S}(n) = \{\mathbf{z} \in B(n) : Y(\mathbf{z}) = 1\}$ for the random support of Y. Assume the r.v.'s have the property that for any $\mathbf{x} \in B(n)$ and any $A \subset B(n) \setminus b(\mathbf{x}, 20)$,*

$$\Pr\bigl[Y(\mathbf{x}) = 1 | \mathcal{S} \cap B(n) \setminus b(\mathbf{x}, 20) = A\bigr] > p_d,$$

where $p_d < 1$ is a fixed constant dependent only on p_b (from Lemma 4.1) and dimension.

Then for all $p < 1$, there is a $C(p) < \infty$ such that for all $n > C$,

$$\Pr\bigl[\mathcal{S}(n) \in \mathcal{P}(n)\bigr] > p.$$

PROOF. The domination of product measures result of Liggett, Schonmann and Stacey [10], implies there is a $p_d < 1$ for which $\mathcal{S}(n)$ stochastically dominates an i.i.d. product field with density p_b on $B(n)$. Lemma 4.1 tells us that the probability such an i.i.d. field belongs to $\mathcal{P}(n)$ approaches one as n tends to infinity. Since Percolation properties are monotone (Claim 2.1), we are done. □

Write $\mathbf{P}_p[\cdot]$ for the law that makes $\{Y(\mathbf{z})\}_{\mathbf{z}\in\mathbb{Z}^d}$ i.i.d. $\{0, 1\}$ r.v.'s where $Y(\mathbf{z}) = 1$ w.p. p. Let $B = B(n)$ and write $\mathcal{S} = Y^{-1}(1) \cap B$ for the random set of open sites in B. Denote by $\mathcal{C}$ the largest connected component in $\mathcal{S}$.

We write a consequence of Theorem 1.1 of [6]. One can find the proof in the appendix of [15].

LEMMA 4.3. *There is a $p_0(d) < 1$ such that for every $p > p_0$, there exists a $c > 0$ such that*

$$\mathbf{P}_p\bigl[|\mathcal{C}| < \bigl(1 - 10^{-d}\bigr)\bigl|B(n)\bigr|\bigr] \le ce^{-cn}.$$

DEFINITION 4.4. Let B^* be the graph of $B(n)$ where we add edges between any two vertices in B of l_∞ distance one. We call a set A in B *$*$-connected*, if it is connected in B^*.

LEMMA 4.5. *There is a $\beta_1, \beta_2, c_d(d) > 0, \beta_3, p_1(d) < 1$ and $C(d) < \infty$ such that for any $p > p_1$*

$$\mathbf{P}_p\bigl[\exists A, *\text{-connected}, |A| > C \log n, |A \cap \mathcal{S}| < c_d|A|\bigr] \le \beta_1 e^{-\beta_2 n^{\beta_3}}. \tag{17}$$

PROOF. Fix a vertex $\mathbf{v} \in B$ and let A be $*$-connected such that $\mathbf{v} \in A$ and $|A| = k$. The number of such components is bounded by $(3^d - 1)^{2k} < e^{\hat{c}k}$. To see this, fix a spanning tree for each such set and explore the tree starting at $\mathbf{v}$ using a depth first search. Each edge is crossed at most twice and at each step the number of directions is bounded by the degree. Using Cramér's theorem for i.i.d. (large deviations), for large enough $p_1(d) < 1$ and small enough $p_1(d) > c_d(d) > 0$, $\mathbf{P}_p[|A \cap \mathcal{S}| < c_d|A|] < \exp(-2\hat{c}|A|)$. To bound the probability of the event in (17), we union bound over $*$-connected components larger than $n^{1/3}$ that contain a fixed vertex in B to get

$$n^d \sum_{k \geq n^{1/3}} e^{\hat{c}k} e^{-2\hat{c}k},$$

which is smaller than $\beta_1 e^{-\beta_2 n^{\beta_3}}$ for appropriate constants. $\square$

COROLLARY 4.6. *There is a $c_b > 0$, $C_b < \infty$ such that for all $p > p_1(d)$, with probability greater than $1 - \beta_1 e^{-\beta_2 n^{\beta_3}}$, any connected set $A \subset B$ such that $B \setminus A$ is also connected and $C_b \log^{d/(d-1)} n < |A| \leq n^d/2$.*

$$|\partial_B A \cap \mathcal{S}|, |\partial_B^{\text{in}} A \cap \mathcal{S}| > c_b |A|^{(d-1)/d}.$$

PROOF. By Lemma 2.1(ii) in [6], $\partial_B A$, $\partial_B^c A$ are $*$-connected. By well-known isoperimetric inequalities for the grid; see, for example, Proposition 2.2 in [6], there is a $c_I > 0$ such that for $|A| \leq n^d/2$, $|\partial_B A|, |\partial_B^c A| > c_I |A|^{(d-1)/d}$. For appropriate C_b, $c_I|A|^{(d-1)/d} > C_a \log n$, and thus Lemma 4.5 gives the result with $c_b = c_I c_d$. $\square$

LEMMA 4.7. *Let $\mathcal{K}$ denote the largest connected component in $B \setminus \mathcal{C}$. There are $c > 0$, $\gamma < 1$ and $p_2(d) < 1$ such that for all $p > p_2$,*

$$\mathbf{P}_p\big[|\mathcal{K}| > \log^2 n\big] \leq e^{-cn^\gamma}.$$

PROOF. Choose a component $\mathcal{K}$ of $B \setminus \mathcal{C}$. Since $\mathcal{C}$ is connected and $\mathcal{K}$ is maximal, $B \setminus \mathcal{K}$ is also connected. This easy fact is proved in Theorem 3.3. From Lemma 4.3, we have for $p > p_0$, $k = |\mathcal{K}| < |B|/2$. It is not true in general that $Y(\mathcal{K}) = 0$ but since $\partial_B^{\text{in}} \mathcal{K}$ separates $\mathcal{K}$ from $\mathcal{C}$, $Y(\partial_B^{\text{in}} \mathcal{K}) = 0$. Thus, from Corollary 4.6, for $p_2 > p_1$, w.h.p., $|\mathcal{K}| < C_b \log^{d/(d-1)} n$. $\square$

LEMMA 4.8. *There is a $c_a > 0$ such that for $p > p_1 > p_c$*

$$\mathbf{P}_p\big[\exists \mathbf{v}, \mathbf{w} \in \mathcal{C} \cap B(n - c_a \log n), d_{\mathcal{C}}(\mathbf{v}, \mathbf{w}) > c_a\big(d_B(\mathbf{v}, \mathbf{w}) \vee \log n\big)\big] \leq \frac{C(\log n)^{d-1}}{n^d}.$$

PROOF. Recall $Y(\mathbf{z})$ are defined for all $\mathbf{z} \in \mathbb{Z}^d$. Let $\mathcal{C}_\infty$ be the infinite component of $Y^{-1}(1)$. We start by showing that w.h.p., $\mathcal{C}$, the largest cluster in $Y^{-1}(1) \cap B$ is contained in $\mathcal{C}_\infty$. By Lemma 4.3, the diameter of $\mathcal{C}$ is of order n w.h.p. If in this case $\mathcal{C} \nsubseteq \mathcal{C}_\infty$, then $\mathcal{C}$ is a finite cluster in $Y^{-1}(1)$ of diameter n. In the supercritical phase ($p > p_c$), the probability for such a cluster at a fixed vertex decays exponentially in n (see, e.g., 8.4 in [7]). Thus we may union bound over the vertices of B to get that w.h.p.

$$\mathcal{C} \subset \mathcal{C}_\infty. \tag{18}$$

We assume henceforth that this is the case.

Next, by Theorem 1.1 of [1], we have that for some $0 < k, K_0, K < \infty$, dependent on dimension and p_1,

$$\mathbf{P}_p\bigl[d_{\mathcal{C}_\infty}(\mathbf{x},\mathbf{y}) > K_0 m | \mathbf{x},\mathbf{y} \in \mathcal{C}_\infty, d(\mathbf{x},\mathbf{y}) = m\bigr] < K \exp(-km).$$

We use this to show that for appropriate $K_1 < \infty$, the probability of the following event decays to 0. Let

$$\mathcal{A} = \bigl\{\exists \mathbf{x},\mathbf{y} \in B \cap \mathcal{C}_\infty, K_1 \log n < d(\mathbf{x},\mathbf{y}) < K_0^{-1} d_{\mathcal{C}_\infty}(\mathbf{x},\mathbf{y})\bigr\}.$$

Using a union bound,

$$\mathbf{P}_p[\mathcal{A}] < n^d \sum_{m=K_1 \log n}^{\infty} C m^{d-1} \exp(-km) < C n^d (\log n)^{d-1} n^{-2d}.$$

Let $B^- = B(n - 4dK_0K_1 \log n)$. We now show that $\mathcal{A}$ not occurring implies the event $\mathcal{B}$.

$$\mathcal{B} = \left\{\forall \mathbf{x},\mathbf{y} \in \mathcal{C} \cap B^- \text{ s.t. } 1 < \frac{d_{\mathcal{C}}(\mathbf{x},\mathbf{y})}{K_1 \log n} < 4d, d_{\mathcal{C}}(\mathbf{x},\mathbf{y}) < 4dK_0K_1 \log n\right\}.$$

From $\mathcal{A}$ not occurring and (18), we get that for any $\mathbf{x},\mathbf{y}$ satisfying the condition in $\mathcal{B}$, $d_{\mathcal{C}_\infty}(\mathbf{x},\mathbf{y}) < 4dK_0K_1 \log n$. Since $\mathbf{x},\mathbf{y} \in B^-$, a path connecting $\mathbf{x}$ to $\mathbf{y}$ in $\mathcal{C}_\infty$ realizing this distance is too short to reach $\partial^{\text{in}} B$, and thus by (18) is contained in $\mathcal{C}$.

Next, for any $\mathbf{x},\mathbf{y} \in B^-$, there is a sequence of boxes $b_1,\dots,b_m$ where $\mathbf{x} \in b_1, \mathbf{y} \in b_m$ and the following conditions hold. For all i for which it is defined, $\|b_i\| = \lceil K_1 \log n\rceil$, the diameter of $b_i \cup b_{i+1}$ is less than $4dK_1 \log n$, $d(b_i, b_{i+1}) > K_1 \log n$ and for some $K_2 < \infty$, $m < K_2 d(x,y)/\log n + 2$. The left term in the bound for m can be achieved for example by placing boxes with order $\log n$ spacing in lines parallel to the coordinate axes. The constant 2 appears for the case where $d(\mathbf{x},\mathbf{y}) < K_1 \log n$ and we use an intermediary box.

Lemma 4.7 tells us that for all large n, w.h.p. every box b with $\|b\| \geq \log n$ intersects $\mathcal{C}$. Assuming that this and the high probability $\mathcal{B}$ event occur, we have that for $\mathbf{x},\mathbf{y}$ as in $\mathcal{B}^-$, $d_{\mathcal{C}}(\mathbf{x},\mathbf{y}) < 4dK_0K_1(K_2 d(x,y) + 2\log n)$, and we are done. $\square$

5. Goodness of random walk range.

5.1. *Random walk definitions and notation.* Given a box B, consider the two faces of ∂B^7 for which the first coordinate is constant. We call the one for which this coordinate is larger the *top face* and call the other one the *bottom face.* Let $\mathrm{Top}^+(B)$, $\mathrm{Bot}(B)$ be the projection of B^3 on the top and bottom faces, respectively. Let $\mathrm{Top}(B)$ be the neighbors of $\mathrm{Top}^+(B)$ inside B^7. Thus, $\mathrm{Top}(B) \subset \partial^{\mathrm{in}} B^7$ is a translation along the first coordinate of $\mathrm{Bot}(B) \subset \partial B^7$.

Let $\mathbf{P_x}[\cdot]$ be the law that makes $S(\cdot)$ an independent SRW starting at $\mathbf{x} \in \mathbb{Z}^d$. For a set $A \subset \mathbb{Z}^d$, let $\tau_A = \inf\{t \geq 0 : S(t) \in A\}$ be the first hitting time of A, and for a single vertex $\mathbf{v}$, we write $\tau_{\mathbf{v}} = \tau_{\{\mathbf{v}\}}$. For $\mathbf{a} \in \mathrm{Top}(B)$, $\mathbf{z} \in \mathrm{Bot}(B)$, we call the ordered pair $\eta = (\mathbf{a}, \mathbf{z})$ a B*-traversal*. We write $\mathbf{P}^\eta[\cdot] = \mathbf{P_a}[\cdot | \tau_{\partial B^7} = \tau_{\mathbf{z}}]$.

Let $H = (\eta_1, \eta_2, \dots, \eta_k)$ be an ordered sequence of B-traversals. We call H a B*-itinerary* and write $\mathbb{P}_H = \mathbf{P}^{\eta_1} \times \cdots \times \mathbf{P}^{\eta_k}$ for the product probability space. For each $\eta \in H$, we denote the associated independent conditioned random walk by $S_\eta(\cdot)$, write $\mathcal{R}_\eta(t_1, t_2) = \{S_\eta(s) : t_1 \leq s \leq t_2\}$ and simply $\mathcal{R}_\eta$ for $\mathcal{R}_\eta(0, \tau_{\partial B^7})$. We say H is ρ*-dense* if $|H| \geq \rho \|B\|^{d-2}$.

For a $B(\mathbf{x}, n)$-itinerary H, we abbreviate notation inside $\mathbb{P}_H[\cdot]$ by writing $\mathcal{G}_k^\rho$ instead of $\mathcal{G}_k^\rho(n) + \mathbf{x}$.

For a SRW $S(\cdot)$, we write $S(t_1, t_2)$ for the sequence $(S(t_1), \dots, S(t_2))$.

For $Q \subset H$ a set (subsequence) of B-traversals, let $\mathcal{R}_Q = \bigcup_{\eta \in Q} \mathcal{R}_\eta$. When in use under the law $\mathbb{P}_H$, we write $\mathcal{R}$ for $\mathcal{R}_H$.

5.2. *Independence of a random walk traversing a box.* Let $\mathcal{I}_N(\cdot) = \Pi_N^{-1} \circ \Pi_N(\cdot)$ and for $b = b(\lceil N/10 \rceil)$ let $\mathcal{I}_N^*(\cdot) = b^7 \cap \mathcal{I}_N(\cdot)$. Since $\|b^7\| < N$, $\mathcal{I}_N(b^7)$ is an infinite disconnected union of translated copies of b^7. Thus, we have that for any $\mathbf{x} \in \mathcal{I}_N(b^7)$, $\mathcal{I}_N^*$ is a graph isomorphism between b^7 and $\beta_{\mathbf{x}}$, the component of $\mathbf{x}$ in $\mathcal{I}_N(b^7)$.

Given $S(\cdot)$, a simple random walk in $\mathbb{Z}^d$, we define the following random set of triplets.

$$\begin{aligned} \mathfrak{T}_N = \big\{(\gamma, \gamma^+, \beta) : {}& 0 < \gamma < \gamma^+, \beta \text{ a box}, \beta^7 \text{ is a component of } \mathcal{I}_N(b^7), \\ & S(\gamma - 1) \in \partial\beta^7, \\ & \qquad S(\gamma) \in \mathrm{Top}(\beta), \mathcal{R}(\gamma, \gamma^+ - 1) \subset \beta^7, S(\gamma^+) \in \mathrm{Bot}(\beta)\big\}. \end{aligned}$$

For any two distinct copies of b^7 in $\mathcal{I}_N(b^7) - \beta, \hat{\beta}$ we have $\partial\beta^7 \cap \partial\hat{\beta}^7 = \varnothing$. Thus, for any two distinct triplets $(\gamma, \gamma^+, \beta)$, $(\hat{\gamma}, \hat{\gamma}^+, \hat{\beta})$ either $\gamma > \hat{\gamma}^+$ or $\hat{\gamma} > \gamma^+$. Ordering the triplets by increasing first coordinate, we write $(\gamma_i, \gamma_i^+, \beta_i)$ for the ith triplet by this order.

Since $\mathfrak{T}_N$ may be defined in terms of the finite state Markov process $S_N(\cdot)$, $\mathbf{P_x}[|\mathfrak{T}_N| = \infty] = 1$. Thus, for $\rho > 0$, $\gamma^+_{\lceil \rho n^{d-2} \rceil}$ is well defined.

DEFINITION 5.1. Let $\tau_\rho(b) = \gamma^+_{\lceil \rho n^{d-2} \rceil}$.

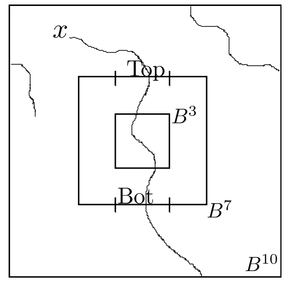

FIG. 5. *Traversal and Top, Bot definition.*

The next lemma claims the following: Run a SRW up to time uN^d from a point $x \in B^{10}$. There exists a constant $\rho(u)$ such that with high probability there are at least ρN^{d-2} traversals from Top to Bot. See Figure 5 for graphical representation.

LEMMA 5.2. *For any $u > 0$, there is a $\rho(u) > 0$ such that*

$$\mathbf{P}_{\mathbf{x}}\bigl[\tau_\rho(b) < uN^d\bigr] \xrightarrow{N} 1,$$

uniformly for any $\mathbf{x} \in \mathbb{Z}^d$.

PROOF. Let $n = \lceil N/10 \rceil$ and let $b = b(n)$. By the central limit theorem, there is a $c_1 > 0$ such that $\mathbf{P}_{\mathbf{x}}[\tau_{\mathcal{I}_N(b)} < N^2/2] > c_1$ uniformly in $\mathbf{x}$. For $\mathbf{y} = (y_1, \dots, y_d) \in \mathbb{Z}^d$, define $\mathcal{B}(\mathbf{y})$ to be the event that in $N^2/2$ steps the first coordinate of a d-dimensional random walk hits $y_1 + 4n$ and then hits $y_1 - 8n$, while the maximal change in the other coordinates is less than n. By the invariance principle, there is a $c_2 > 0$ such that for all large N, $\mathbf{P}_{\mathbf{y}}[\mathcal{B}(\mathbf{y})] > c_2$.

Let $\tau_i^b = \inf\{t \geq iN^2 : S(t) \in \mathcal{I}_N(b)\}$, let $\mathcal{A}_i = \{\tau_i^b < (i + \frac{1}{2})N^2\}$ and let χ_i be the indicators of $\mathcal{A}_i$ occurring for $S(t)$ and $\mathcal{B}(S(\tau_i^b))$ occurring for $S(\tau_i^b + t)$. Note that χ_i implies there is a $(\gamma, \gamma^+, \beta) \in \mathfrak{T}_N$, $iN^2 \leq \gamma < \gamma^+ < (i+1)N^2$.

By the Markov property, χ_i dominates i.i.d. Bernoulli r.v.'s that are 1 w.p. $c_3 > 0$ for all large N. Thus, by the law of large numbers, $\sum_{i=1}^{\lfloor uN^{d-2} \rfloor} \chi_i > c_3 u N^{d-2}/2$, w.h.p. This event implies that $\tau_\rho(b) \leq uN^d$ for $\rho < c_3 u/2$, which completes the proof. $\square$

Given a box b, and a set $\omega \subset b$, we call ω *b-boundary-connected* if any $\mathbf{x} \in \omega \cap b^7$ is connected in ω to $\partial^{\text{in}} b^7$. We call $F : \mathbb{Z}^{\geq 0} \to 2^{b^7}$ a *b-boundary-connected-path* if $F(t) \subset F(t+1)$ and $F(t)$ is b-boundary-connected for all $t \geq 0$.

We write $\mathbf{S}$ for the set of all finite paths in $\mathbb{Z}^d$. That is,

$$\mathbf{S} = \bigl\{s = (\mathbf{v}_0, \dots, \mathbf{v}_n) : \mathbf{P}_{\mathbf{v}_0}\bigl[S(0, n) = s\bigr] > 0\bigr\}.$$

For $s = (\mathbf{v}_0, \dots, \mathbf{v}_n) \in \mathbf{S}$, we let $s_i = \mathbf{v}_i$ and write $\|s\|$ for n, the number of edges traversed by the path s. The next lemma attains stochastic domination between the range of the random walk and a ρ-dense b-itinerary.

LEMMA 5.3. *For $N > 0$, fix a box $b = b(\lceil N/10 \rceil)$, $\rho > 0$ and $\mathbf{x} \in \mathbb{Z}^d$. Then for any $\mathcal{A} \subset 2^{b^7}$ there is a ρ-dense b-itinerary $H = H(\mathcal{A})$ and a b-boundary-connected-path $F(t) = F(\mathbf{x}, t)$ such that*

$$\mathbf{P_x}\big[\{\mathcal{I}_N^* \circ \mathcal{R}(t) : t \geq \tau_\rho(b)\} \subset \mathcal{A}\big] \geq \mathbb{P}_H\big[\{\mathcal{R} \cup F(t) : t \geq 0\} \subset \mathcal{A}\big].$$

PROOF. Let $n = \lceil N/10 \rceil$ and let $M = \lceil \rho n^{d-2} \rceil$. For $1 \leq i \leq M+1$ fix $s_i \in \mathbf{S}$. Let

$$\tau_\Box = \inf\{t \geq \tau_\rho : \mathcal{T}(N) = \mathcal{R}_N(\tau_\rho, t)\}$$

that is, the first time after τ_ρ, the random walk (starting at time τ_ρ) covers the torus. Since $\mathcal{R}_N$ takes values in the finite state space $\mathcal{T}(N)$, $\mathbf{P_x}[\tau_\Box < \infty] = 1$. With the convention that $\gamma_0^+ = 0$, we partition the probability space of $S(\cdot)$ to events

$$\mathcal{B} = \mathcal{B}(s_1, \ldots, s_{M+1}) = \left\{\bigcap_{i=1}^{M} S(\gamma_{i-1}^+, \gamma_i) = s_i\right\} \cap S(\gamma_M^+, \tau_\Box) = s_{M+1}$$

satisfying $\mathbf{P_x}[\mathcal{B}(s_1, \ldots, s_{M+1})] > 0$. For $i = 1, \ldots, M$ let $\alpha(i) = s_i(\|s_i\|)$, $\zeta(i) = s_{i+1}(0)$, that is, the end point of the path s_i and the starting point of the path s_{i+1}. By the Markov property (see Proposition A.1), $\{S(\gamma_i, \gamma_i^+)\}_{i=1}^M$ under $\mathbf{P_x}[\cdot|\mathcal{B}]$ are independent random vectors with the distribution of $S(0, \tau_{\partial\beta_i^7})$ under $\mathbf{P}_{\alpha(i)}[\cdot|\tau_{\partial\beta_i^7} = \tau_{\zeta(i)}]$. Let $\mathbf{a}(i) = \mathcal{I}_N^*(\alpha(i))$, $\mathbf{z}(i) = \mathcal{I}_N^*(\zeta(i))$ and let H be a b-itinerary, $H = (\eta_1, \ldots, \eta_M)$ where $\eta_i = (\mathbf{a}(i), \mathbf{z}(i))$. Since $\mathcal{I}_N^*$ is an isomorphism between β_i^+ and b^7, $\mathcal{I}_N^* \circ S(0, \tau_{\partial\beta_i^7})$ under $\mathbf{P}_{\alpha(i)}[\cdot|\tau_{\partial\beta_i^7} = \tau_{\zeta(i)}]$ is distributed the same as $S(0, \tau_{\partial b^7})$ under $\mathbf{P}^{\eta_i}[\cdot]$. Thus, $\bigcup_{i=1}^M \{\mathcal{I}_N^* \circ \mathcal{R}(\gamma_i, \gamma_i^+)\}$ under $\mathbf{P_x}[\cdot|\mathcal{B}]$ is distributed like $\mathcal{R}_H$ under $\mathbb{P}_H[\cdot]$. Let

$$\hat{F}(t) = \bigcup_{i=1}^{M} \mathcal{R}(\gamma_{i-1}^+, \gamma_i) \cup \mathcal{R}(\gamma_M^+, (\gamma_M^+ + t) \wedge \tau_\Box).$$

Since $\tau_\rho = \gamma_M^+$, we have $\mathcal{R}(\tau_\rho + t) = \hat{F}(t) \cup \bigcup_{i=1}^M \{\mathcal{R}(\gamma_i, \gamma_i^+)\}$ for all $t \geq 0$. Given $\mathcal{B}$, $\hat{F}(t)$ is uniquely determined. Let $F(t) = \mathcal{I}_N^* \circ \hat{F}(t)$. Since $\mathcal{I}_N^*$ is either a local isomorphism to b^7 or else gives the empty set, $F(t)$ is a b-boundary-connected-path. Thus, for any $\mathcal{A} \subset 2^{b^7}$

$$\mathbf{P_x}\big[\{\mathcal{I}_N^* \circ \mathcal{R}(t) : t \geq \tau_\rho(b)\} \subset \mathcal{A}|\mathcal{B}\big] = \mathbb{P}_{H(\mathcal{B})}\big[\{\mathcal{R} \cup F_{(\mathcal{B})}(t) : t \geq 0\} \subset \mathcal{A}\big],$$

which proves the lemma (see Proposition A.2). $\Box$

For a box B and a B-itinerary H, we proceed to define the event $\mathcal{D}_\rho^\sigma = \mathcal{D}_\rho^\sigma(H, B)$. Roughly, $\mathcal{D}_\rho^\sigma$ is the event that all subboxes are crossed a correct order of times by B-traversals. First, given a box B and $b \in \sigma(B)$, let us define for a random walk $S(\cdot)$ the event $\mathcal{J}_B[b]$

$$\mathcal{J}_B[b] = \{\exists t, t^+ : 0 < t < t^+ < \tau_{\partial B^7} : S(t-1) \in \partial b^7, S(t) \in \mathrm{Top}(b),$$
$$\mathcal{R}(t, t^+ - 1) \subset b^7, S(t^+) \in \mathrm{Bot}(b)\}.$$

Given H a B-itinerary, $\eta \in H$, and a subbox $b \in \sigma(B)$, we write $\mathcal{J}_\eta[b]$ for the event $\mathcal{J}_B[b]$ occurring on the random walk $S_\eta(\cdot)$.

Next, we would like to assign each box $b \in \sigma(B)$ a subset $H[b] \subset H$ with the property that if two distinct subboxes intersect, they have disjoint $H[\cdot]$ sets. Let us do this by first fixing a function $(\cdot)_{50} : \mathbb{Z}^d \to \{0, 1, \dots, 50^d - 1\}$ with the property that any distinct $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^d$ with $(\mathbf{x})_{50} = (\mathbf{y})_{50}$ are a distance of at least 50 in the l_∞ norm. This can be induced by any bijection from $(\mathbb{Z}/50\mathbb{Z})^d$ to $\{0, 1, \dots, 50^d - 1\}$.

Recall that Δ is the isomorphism mapping $\sigma(B)$ into $\mathbb{Z}^d$, and that $H = (\eta_1, \dots, \eta_k)$ is an ordered sequence. We write

$$H[b] = \{\eta_i \in H : i \equiv (\Delta b)_{50} (\operatorname{mod} 50^d)\}.$$

Next, for each $b \in \sigma(B)$ define the random set of B-traversals $\psi_H[b] = \{\eta \in H[b] : \mathcal{J}_\eta[b]\}$. Since $2\|b^7\| < 50^d \|b\|$, we get the following desired property.

CLAIM 5.4. *For any distinct $b_0, b_1 \in \sigma(B)$ satisfying $b_0^7 \cap b_1^7 \neq \varnothing$ we have $\psi_H[b_0] \cap \psi_H[b_1] = \varnothing$.*

DEFINITION 5.5. Let $\mathcal{D}_\rho^\sigma$ be the event that for each $b \in \sigma(B)$, $|\psi_H[b]| \geq \rho \|b\|^{d-2}$.

The next lemma identifies, given some set, an itinerary which minimizes the probability to be contained in the set. The sets in mind are non good sets.

LEMMA 5.6. *Fix a box B, a B-itinerary H, a B-boundary-connected-path $F(t)$, and a subbox $b \in \sigma(B)$. Let $\mathcal{H} = \mathcal{H}(\mathcal{E}) = \mathcal{D}_\rho^\sigma, \mathcal{R}_H \setminus b^7 \in \mathcal{E}$ where $\mathcal{E}$ is a fixed subset of 2^{B^7}. Assume $\mathbb{P}_H[\mathcal{H}] > 0$. Then for any $\mathcal{A} \subset 2^{b^7}$, there is a ρ-dense b-itinerary $h = h(\mathcal{A})$ and a b-boundary-connected-path $f(t) = f(\mathcal{A}, F)(t) \subset b^7$ satisfying*

$$\mathbb{P}_H[\{(\mathcal{R} \cup F(t)) \cap b^7 : t \geq 0\} \subset \mathcal{A} | \mathcal{H}] \geq \mathbb{P}_h[\{\mathcal{R} \cup f(t) : t \geq 0\} \subset \mathcal{A}].$$

PROOF. If for $\eta \in H$ the event $\mathcal{J}_\eta[b]$ occurs, then we know there exists at least one time pair (t, t^+), $0 < t < t^+ < \tau_{\partial B^7}$ satisfying the requirements of $\mathcal{J}_\eta[b]$—roughly that b^7 is crossed top to bottom by S_η. Since these time pairs must be disjoint, we can consider the first, which we shall denote by (t_η, t_η^+).

Fix $Q \subset H[b]$, $s_\eta^1, s_\eta^2 \in \mathbf{S}$ for each $\eta \in Q$ and $s_\eta^0 \in \mathbf{S}$ for each $\eta \in H \setminus Q$ and define the event

$$\mathcal{B} = \mathcal{B}(Q, s_\eta^i) = \{\psi_H[b] = Q\} \cap \bigcap_{\eta \in Q} \{S_\eta(0, t_\eta) = s_\eta^1, S_\eta(t_\eta^+, \tau_{\partial B^7}) = s_\eta^2\}$$
$$\cap \bigcap_{\eta \in H \setminus Q} \{S_\eta(0, \tau_{\partial B^7}) = s_\eta^0\}.$$

We partition $\{S_\eta(\cdot):\eta \in H\}$ to such $\mathcal{B}(Q, s^i_\eta)$ events satisfying $\mathbb{P}_H[\mathcal{B}(Q, s^i_\eta)] > 0$. Any two distinct $\mathcal{B}(Q, s^i_\eta), \mathcal{B}(\hat{Q}, \hat{s}^i_\eta)$ have an empty intersection because either $Q \neq \hat{Q}$ or if $Q = \hat{Q}$ then $s^i_\eta \neq \hat{s}^i_\eta$ for some $\eta \in Q$. Observe that $\mathcal{R}_H \setminus b^7$ is determined by $\mathcal{B}$, and that by our construction (Claim 5.4), so is $\psi_H[\cdot]$. Since $\mathcal{H}$ is $(\mathcal{R}_H \setminus b^7, \psi_H[\cdot])$ measurable, and the $\mathcal{B}$ events are a partition of the entire probability space, those for which $\mathbb{P}_H[\mathcal{B}, \mathcal{H}] > 0$ form a partition of $\mathcal{H}$. $\mathcal{H} \subset \mathcal{D}^\sigma_\rho$ so any positive probability $\mathcal{B}(Q, s^i_\eta) \subset \mathcal{H}$ has $|Q| \geq \rho \|b\|^{d-2}$. For each $\eta \in Q$ let $\mathbf{a}(\eta) = s^1_\eta(\|s^1_\eta\|), \mathbf{z}(\eta) = s^2_\eta(0)$ and let h be a b-itinerary, $h = (\mathbf{a}(\eta), \mathbf{z}(\eta))_{\eta \in Q}$ with order inherited from H. Since S_η are independent and $\mathcal{B}$ is a product of events on $\{S_\eta\}_{\eta \in H}$ ($\{\psi_H[b] = Q\}$ can be factored to each $\{S_\eta\}_{\eta \in H}$), we have by the Markov property (see Proposition A.1), that $\{S_\eta(t_\eta, t^+_\eta)\}_{\eta \in Q}$ under $\mathbb{P}_H[\cdot|\mathcal{B}]$ are independent random vectors with the distribution of $S(0, \tau_{\partial b^7})$ under $\mathbf{P}_{\mathbf{a}(\eta)}[\cdot|\tau_{\partial b^7} = \tau_{\mathbf{z}(\eta)}]$. Thus $\bigcup_{\eta \in Q} \mathcal{R}_\eta(t_\eta, t^+_\eta)$ under $\mathbb{P}_H[\cdot|\mathcal{B}]$ is distributed like $\mathcal{R}_h$ under $\mathbb{P}_h[\cdot]$. Let $\hat{f} = \bigcup_{\eta \in H \setminus Q} s^0_\eta \cup \bigcup_{\eta \in Q} s^1_\eta \cup \bigcup_{\eta \in Q} s^2_\eta$ and let $f(t) = (\hat{f} \cup F(t)) \cap b^7$. Since all elements in the union are B-boundary-connected, $f(t)$ is a b^7-boundary-connected-path. As $(\mathcal{R}_H \cup F(t)) \cap b^7 = f(t) \cup \bigcup_{\eta \in Q} \{\mathcal{R}_\eta(t_\eta, t^+_\eta)\}$, we have for any $\mathcal{A} \subset \mathrm{P}(b^7)$

$$\mathbb{P}_H\big[\{(\mathcal{R} \cup F(t)) \cap b^7 : t \geq 0\} \subset \mathcal{A}|\mathcal{B}\big] = \mathbb{P}_h\big[\{\mathcal{R} \cup f(t) : t \geq 0\} \subset \mathcal{A}\big].$$

Since $|h| = |Q| \geq \rho \|b\|^{d-2}$, h is ρ-dense, and as $\mathcal{B}$ is an arbitrary partition element of $\mathcal{H}$, this proves the lemma by Proposition A.2. $\square$

5.3. *Properties of the range of a random walk.* We will require the following large deviation estimate for sums of independent indicators, a weak version of Lemma 4.3 from [4].

LEMMA 5.7. *Let Q be a finite sum of independent indicator* ($\{0, 1\}$*-valued*) *random variables with mean $\mu > 0$. There is a $0 < c_f < 1$ such that*

$$\Pr[Q < \mu/2], \Pr[Q > 2\mu] < \exp(-c_f \mu).$$

Recall $\mathcal{D}^\sigma_\rho$ from Definition 5.5.

THEOREM 5.8. *There is a $\Lambda(d) > 0$ such that for any $q > 0, \varrho > 0$ there is a $C(q, \varrho) < \infty$ such that if $n > C$ and H is a ϱ-dense $B(n)$-itinerary,*

$$\mathbb{P}_H\big[\mathcal{D}^\sigma_{\Lambda\varrho}\big] > 1 - q.$$

PROOF. Fix $b \in \sigma(B)$ and let $m = \|b\| = s(n)$. Lemma A.3 tells us that for any B-traversal $\eta \in H$,

$$\mathbf{P}^\eta\big[\mathcal{J}_\eta[b]\big] > c_\Lambda \left(\frac{m}{n}\right)^{d-2}.$$

Let $Q = \sum_{\eta \in H[b]} \mathbb{1}_{\mathcal{J}_\eta[b]}$. For all large enough n, $|H[b]| \geq 60^{-d} \varrho n^{d-2}$, so by linearity,

$$\mathbb{E}_H\big[|\psi_H[b]|\big] = \mathbb{E}_H[Q] > \varrho 60^{-d} c_\Lambda m^{d-2}.$$

Since the random walks S_η are mutually independent, by Lemma 5.7 there is a c_f such that

$$\mathbb{P}_H\big[Q \leq \mathbb{E}[Q]/2\big] \leq \exp\big(-c_f \mathbb{E}[Q]\big) \leq e^{-cm^{d-2}}.$$

Let $\Lambda = 60^{-d} c_\Lambda / 2$. For $m \geq [\frac{2d}{c_\Lambda c_f 60^{-d} \varrho} \log n]^{1/(d-2)}$ we have

$$\mathbb{P}_H\big[Q \leq \varrho \Lambda m^{d-2}\big] < n^{-4(d-2)}. \tag{19}$$

If $b = b(\mathbf{x}, m) \in \sigma(B)$ then $\mathbf{x} \in B^6$ and $m = s(n) = \lceil \log^4 n \rceil$ which is $\omega(\log^{1/(d-2)} n)$ but $o(n)$. Thus, for all large n, a union bound on $\sigma(B)$ gives the result. $\square$

REMARK 5.9. One can obtain any polynomial decay in (19) by taking $m = \log^k n$, for large enough k.

The below lemma shows that w.h.p., the union of those ranges of a dense itinerary which intersect an interior set of low density, has size of greater order than the size of the set itself.

LEMMA 5.10. *Let $b = b(n)$, and let $M \subset b^6$ where $|M| \geq \beta n^\alpha$ with $\alpha, \beta > 0$. Let h be a ϱ-dense b-itinerary, $\varrho > 0$. Then for $\gamma = 1 + 2(\alpha^{-1} - d^{-1})$, $c_g(d) > 0$ and all large n,*

$$\mathbb{P}_h\bigg[\big|\mathcal{R}\big(\{\eta \in h : \mathcal{R}(\eta) \cap M \neq \varnothing\}\big) \cap b\big| < c_g \varrho \beta n^{\gamma\alpha} \wedge \frac{n^d}{2}\bigg] \leq \exp\big(-c_g \varrho \beta n^{\alpha(1-2/d)}\big).$$

PROOF. Fix $\eta = (\mathbf{a}, \mathbf{z}) \in h$ and $Q \subset b$. Let $\mathcal{B}(\eta, Q) = \{|Q| < n^d/2\} \cup \{|\mathcal{R}(\eta) \cap Q| > c_0 n^2\}$, $c_0 > 0$ determined below. Let $\tau_M(\eta)$ be the first hitting time of M by S_η, and let $\tau_{\mathcal{B}}(\eta) = \inf\{t \geq 0 : S_\eta(0, t) \subset \mathcal{B}\}$ be the first time the occurrence of $\mathcal{B}$ is implied by $S_\eta(t)$. By Proposition A.5 for some $\mathbf{x} \in M$

$$\mathbf{P}_\mathbf{a}[\tau_M, \tau_{\mathcal{B}} < \tau_{\partial B^7} | \tau_{\partial B^7} = \tau_\mathbf{z}] \geq \mathbf{P}_\mathbf{a}[\tau_M < \tau_{\partial B^7} | \tau_{\partial B^7} = \tau_\mathbf{z}] \mathbf{P}_\mathbf{x}[\tau_{\mathcal{B}} < \tau_{\partial B^7} | \tau_{\partial B^7} = \tau_\mathbf{z}].$$

By Lemma A.4 and Corollary A.11,

$$\mathbf{P}_\mathbf{a}[\tau_M < \tau_{\partial B^7} | \tau_{\partial B^7} = \tau_\mathbf{z}] > c|M|^{1-2/d} n^{2-d} \geq c_1 \beta n^{\alpha(1-2/d)+(2-d)}.$$

Since $M \subset b^6$ and assuming $|Q| \geq n^d/2$ for the nontrivial case, again by Lemma A.4 and Corollary A.11, we get for some $c_0, c_2 > 0$,

$$\mathbf{P}_\mathbf{x}\big[\mathcal{R}(0, \tau_{\partial B^7}) \cap Q > c_0 n^2 | \tau_{\partial B^7} = \tau_\mathbf{z}\big] > c_2.$$

Recall, $h = (\eta_1, \eta_2, \ldots)$. Let $\mathcal{R}_k^M = \mathcal{R}(\{\eta_i \in h : 1 \le i \le k, \mathcal{R}(\eta_i) \cap M \neq \varnothing\})$, let $\chi(\eta, Q)$ be the indicator variable for the event $\{\tau_M(\eta), \tau_{\mathcal{B}}(\eta) < \tau_{\partial B^7}(\eta)\}$ and let $\mathcal{S}_k = \sum_{i=1}^k \chi(\eta_i, b \setminus \mathcal{R}_{i-1}^M)$. Then $|\mathcal{R}_k^M|$ stochastically dominates $c_0 n^2 \mathcal{S}_k \wedge \frac{n^d}{2}$. By above bounds, and independence of traversals, the sequence $\chi(\eta_i, b \setminus \mathcal{R}_{i-1}^M)$ dominates i.i.d. Bernoulli r.v.'s that are 1 w.p. $c_1 c_2 \beta n^{\alpha(1-2/d)+(2-d)}$. Thus, by concentration of i.i.d. Bernoulli r.v.'s, for example, as stated in Lemma 5.7,

$$\mathbb{P}_h\left[|\mathcal{R}_{|h|}^M| < \frac{c_0}{2} n^2 \mathbb{E}_h[\mathcal{S}_{|h|}] \wedge \frac{n^d}{2}\right] < \exp\bigl(-c_f \mathbb{E}_h[\mathcal{S}_{|h|}]\bigr).$$

Since

$$\mathbb{E}_h[\mathcal{S}_{|h|}] \ge \varrho n^{d-2} c_3 \beta n^{\alpha(1-2/d)+(2-d)},$$

we get

$$\mathbb{P}_h\left[|\mathcal{R}_{|h|}^M| < c_4 \varrho \beta n^{\alpha(1+2(\alpha^{-1}-d^{-1}))} \wedge \frac{n^d}{2}\right] < \exp\bigl(-c_f c_3 \varrho \beta n^{\alpha(1-2/d)}\bigr),$$

which proves the lemma. $\square$

LEMMA 5.11. *Let $b(n)$ be a box, let $F(t)$ be a b-boundary-connected-path, and let H be a ρ-dense b-itinerary, $\rho > 0$. There is a $c(\rho) > 0$ such that for all large n*

$$\mathbb{P}_H\bigl[\forall t \ge 0, \bigl(\mathcal{R} \cup F(t)\bigr) \cap b^5 \textit{ is connected in } \mathcal{R} \cup F(t)\bigr] > 1 - \exp\bigl(-cn^{1-2/d}\bigr).$$

PROOF. For any $h \subset H$, $F(t) \cup \mathcal{R}_h$ is also a b-boundary-connected-path and is independent from the traversals in $H \setminus h$. Thus, we may assume w.l.o.g. that $|H| = \lceil \rho n^{d-2} \rceil$. Let $H^{5.5} = \{\eta \in H : \mathcal{R}(\eta) \cap b^{5.5} \neq \varnothing\}$. We show that $\mathcal{R}(H^{5.5})$ is connected in $\mathcal{R} = \mathcal{R}_H$ w.h.p. Set $D = \lceil \log_{(1+d^{-1})} \frac{2d}{3} \rceil$. If $|H^{5.5}| \le 1$ we are done. Otherwise given distinct traversals $\zeta, \varphi \in H^{5.5}$, partition $H \setminus \{\zeta, \varphi\}$ into sets $H_1^\zeta, H_1^\varphi, \ldots, H_D^\zeta, H_D^\varphi$ and H_*, where each of the $2D + 1$ sets has size at least $|H|/3D > c_D(\rho, d) n^{d-2}$. Set $M_0^\zeta = \mathcal{R}(\zeta) \cap b^6$ and for $i = 1, \ldots, D$ recursively define

$$M_i^\zeta = \bigcup_{\eta \in H_i^\zeta : M_{i-1}^\zeta \cap \mathcal{R}(\eta) \neq \varnothing} \mathcal{R}(\eta) \cap b^6.$$

Define M_i^φ analogously. Thus, the event $\mathcal{M} = \{\exists \eta \in H_* : |\mathcal{R}(\eta) \cap M_D^\zeta|, |\mathcal{R}(\eta) \cap M_D^\varphi| > 0\}$ implies that $\mathcal{R}(\zeta)$ is connected to $\mathcal{R}(\varphi)$ in $\mathcal{R}$, an event we denote by $\zeta \leftrightarrow \varphi$. For $i = 0, \ldots, D$, let $\mathcal{M}_i^\zeta = \{|M_i^\zeta| \ge c_\rho^i n^{(1+d^{-1})^i}\}$ where $c_\rho = c_g \rho \wedge \frac{1}{2}$, with c_g from Lemma 5.10. Define $\mathcal{M}_i^\varphi$ analogously. By independence of $\mathcal{M}_D^\zeta$ and $\mathcal{M}_D^\varphi$,

$$\mathbb{P}_H[\zeta \leftrightarrow \varphi] \ge \mathbb{P}_H[\mathcal{M}] \ge \mathbb{P}_H[\mathcal{M} | \mathcal{M}_D^\zeta, \mathcal{M}_D^\varphi] \mathbb{P}_H[\mathcal{M}_D^\zeta] \mathbb{P}_H[\mathcal{M}_D^\varphi]. \tag{20}$$

By Proposition A.2, for some $F_D^\zeta(H_*), F_D^\varphi(H_*) \subset b^6$ with $|F_D^\zeta|, |F_D^\varphi| \geq c_\rho^D n^{2d/3}$,

$$\mathbb{P}_H[\mathcal{M}|\mathcal{M}_D^\zeta, \mathcal{M}_D^\varphi] \geq \mathbb{P}_H[\mathcal{M}|M_D^\zeta = F_D^\zeta, M_D^\varphi = F_D^\varphi] = 1 - q_D. \tag{21}$$

Given $\eta = (\mathbf{a}, \mathbf{z}) \in H_*$, let $\tau_\zeta(\eta), \tau_\varphi(\eta)$ be the first hitting times of F_D^ζ, F_D^φ by η, respectively. By Proposition A.5, for some $\mathbf{x} \in F_D^\zeta$,

$$\mathbf{P_a}[\tau_\zeta, \tau_\varphi < \tau_{\partial B^7}|\tau_{\partial B^7} = \tau_\mathbf{z}] \geq \mathbf{P_a}[\tau_\zeta < \tau_{\partial B^7}|\tau_{\partial B^7} = \tau_\mathbf{z}]\mathbf{P_x}[\tau_\varphi < \tau_{\partial B^7}|\tau_{\partial B^7} = \tau_\mathbf{z}].$$

Since $F_D^\zeta, F_D^\varphi \subset b^6$ and $|F_D^\zeta|, |F_D^\varphi| \geq c_\rho^D n^{2d/3}$, by Lemma A.4 and Corollary A.11 each term in the product above is at least

$$c_1\bigl(c_\rho^D n^{2d/3}\bigr)^{1-2/d} c n^{2-d} = c_2 n^{(2-d)/3}.$$

Let $\chi(\eta)$ be the indicator for the event $\{\tau_\zeta(\eta), \tau_\varphi(\eta) < \tau_{\partial B^7}(\eta)\}$, and write $\mathcal{S} = \sum_{\eta \in H_*} \chi(\eta)$. Then $q_D \leq \mathbb{P}_H[\mathcal{S} = 0]$. By concentration of independent indicators in Lemma 5.7, we get

$$q_D \leq \exp\bigl(-c_f c_D n^{d-2} c_2^2 n^{2(2-d)/3}\bigr) \leq \exp\bigl(-c_3(\rho) n^{(d-2)/3}\bigr). \tag{22}$$

We now lower bound $\mathbb{P}_H[\mathcal{M}_D^\zeta]$ and $\mathbb{P}_H[\mathcal{M}_D^\varphi]$ from (20). Since the bound is the same for both terms, we drop ζ, φ from the notation. Note that by connectedness of each traversal in $H^{5.5}$, $\mathcal{M}_0$ is of probability one, thus by chaining conditions

$$\mathbb{P}_H[\mathcal{M}_D] \geq \prod_{i=0}^{D-1} \mathbb{P}_H[\mathcal{M}_{i+1}|\mathcal{M}_i, \dots, \mathcal{M}_0]. \tag{23}$$

By Proposition A.2, for some $F_i(H_{i+1}) \subset b^6$ with $|F_i| \geq c_\rho^i n^{(1+d^{-1})^i}$ we have

$$\mathbb{P}_H[\mathcal{M}_{i+1}|\mathcal{M}_i, \dots, \mathcal{M}_0] \geq \mathbb{P}_H[\mathcal{M}_{i+1}|M_i = F_i].$$

By Lemma 5.10, for $0 \leq i \leq D-1$, if $F_i \subset b^6$ and $|F_i| \geq c_\rho^i n^{(1+d^{-1})^i}$, then for all large n and some $c_i(\rho) > 0$,

$$q_i(F_i) = 1 - \mathbb{P}_H[\mathcal{M}_{i+1}|M_i = F_i] \leq \exp\bigl(-c_i n^{1-2/d}\bigr). \tag{24}$$

Let $q_i = 1 - p_i$ for $i = 0, \dots, D$. Writing $\zeta \nleftrightarrow \varphi$ for the event that $\mathcal{R}(\zeta)$ is not connected to $\mathcal{R}(\varphi)$ in $\mathcal{R}$, and plugging (21), (23), (24) into (20) and using the bounds from (22), (24) we have for large n and $c_4(\rho)$

$$\mathbb{P}_H[\zeta \nleftrightarrow \varphi] \leq 1 - \prod_{i=0}^{D}(1-q_i)^2 \leq 2\sum_{i=0}^{D} q_i \leq 2D\exp\bigl(-c_4 n^{1-2/d}\bigr).$$

Since we assumed $|H| < 2\rho n^{d-2}$, we union bound the probability for $\{\zeta \nleftrightarrow \varphi\}$ over any two traversals in $H^{5.5}$, to get

$$\mathbb{P}_H\bigl[\mathcal{R}\bigl(H^{5.5}\bigr) \text{ is connected in } \mathcal{R}\bigr] \geq 1 - \exp\bigl(-c_5 n^{1-2/d}\bigr). \tag{25}$$

Let $\mathcal{F}$ be the event that for any $t \geq 0$, any $\mathbf{x} \in F(t) \cap b^5$ is connected to $\mathcal{R}(H^{5.5})$ in $\mathcal{R} \cup F(t)$. By (25), to prove the lemma it remains to show that $\mathcal{F}$ occurs w.h.p. Let $t_{\mathbf{x}} = \inf\{t \geq 0 : \mathbf{x} \in F(t)\}$ and denote by $M_{\mathbf{x}}$ the component of $\mathbf{x}$ in $F(t_{\mathbf{x}})$. If for fixed $t \geq 0$ $\mathbf{x} \in F(t)$, then $t_{\mathbf{x}} \leq t$ and $M_{\mathbf{x}} \subset F(t)$. Thus, $\mathcal{F}$ is implied by $\{\forall \mathbf{x} \in b^5 : M_{\mathbf{x}} \cap \mathcal{R}(H^{5.5}) \neq \varnothing\}$, which is in turn implied by $\{\forall \mathbf{x} \in b^5 : M_{\mathbf{x}} \cap b^{5.5} \cap \mathcal{R}_H \neq \varnothing\}$. Since $F(t)$ is b-boundary-connected for all t, $M_{\mathbf{x}} \cap b^{5.5}$ is of size at least $n/2$ for all $\mathbf{x} \in b^5$. By Lemma 5.10, the probability none of the traversals in H hit $M_{\mathbf{x}} \cap b^{5.5}$ decays exponentially in n. Thus, by union bound for some $c_6(\rho)$

$$\mathbb{P}_H[\exists \mathbf{x} \in b^5 : M_{\mathbf{x}} \cap b^{5.5} \cap \mathcal{R} = \varnothing] \leq |b^5| \exp(-cn^{1-2/d}) < \exp(-c'n^{1-2/d}),$$

and we are done. $\square$

THEOREM 5.12. *Fix $\rho > 0$. Let H be a $B(n)$-itinerary and let $F(t)$ be a B-boundary-connected-path. There is a $C(\rho)$, D such that for $n > C$*

$$\mathbb{P}_H[\{\mathcal{R} \cup F(t) : t \geq 0\} \subset \mathcal{G}_0^{\rho}(n) | \mathcal{D}^{\sigma}_{\Lambda\rho}] > 1 - \frac{D}{n^d}.$$

PROOF. See Section 2.4 for the properties each subbox must possess relative to $\mathcal{R} \cup F(t)$ for the above to hold. Using Lemma 5.6 and a union bound on $\sigma(B)$, it suffices to show that for any fixed $b \in \sigma(B)$, any $\Lambda\rho$-dense b-itinerary h and any b-boundary-connected-path $f(t)$,

$$\mathbb{P}_h\left[|\mathcal{R} \cap b| \geq \left(\rho c_h \wedge \frac{1}{2}\right)|b|\right] > 1 - n^{-2d}, \tag{26}$$

$$\mathbb{P}_h[\forall t \geq 0, (\mathcal{R} \cup f(t)) \cap b^5 \text{ is connected in } (\mathcal{R} \cup f(t)) \cap b^7] > 1 - n^{-2d}. \tag{27}$$

Let $m = \|b\|$. Using Lemma 5.10 with $M = b, \alpha = d, \beta = 1$ and $c_h = \Lambda c_g$, we get that the LHS of (26) is greater than $1 - \exp(-\rho c_h m^{d-2})$. By Lemma 5.11, the LHS of (27) is greater than $1 - \exp(-cm^{(d-2)/d})$.

Since $m = \lceil \log^4 n \rceil$ is $\omega(\log^{d/(d-2)} n)$, we are done. $\square$

6. Renormalization. Refer to Sections 5.1, 5.2 and Definition 5.5 for the definitions of τ_ρ, an itinerary, a boundary-connected-path and $\mathcal{D}^{\sigma}_{\rho}$, used in this section.

THEOREM 6.1. *For any $u > 0$, there is a $\rho(u) > 0$ such that for any $k > 0$,*

$$\mathbf{P_0}[\forall t \geq uN^d, \mathcal{T}(N) \text{ is a } (\mathcal{R}_N(t), k, \rho)\text{-good torus}] \xrightarrow{N} 1.$$

PROOF. Let $\mathcal{F}_N^T(b, k, \rho)$ be the event that a box b is $(\Pi_N^{-1} \circ \mathcal{R}_N(t), k, \rho)$-good for all $t \geq T$. Since the number of top-level boxes for $\mathcal{T}(N)$ is bounded, the theorem follows by definition of a good torus if we show that for some $\rho > 0$, $\mathbf{P_0}[\mathcal{F}_N^{uN^d}(b)] \xrightarrow{N} 1$ uniformly for an arbitrary top-level box b.

By translation invariance the above follows from showing that for $B = B(\mathbf{0}, \lceil N/10 \rceil)$

$$\mathbf{P_x}\big[\mathcal{F}_N^{uN^d}(B)\big] \xrightarrow{N} 1,$$

uniformly for $\mathbf{x} \in \mathbb{Z}^d$. For $\rho > 0$, $\tau_\rho = \tau_\rho(N)$ is a random function of $S_N(\cdot)$ defined in Section 5.2. Roughly, τ_ρ is the time it takes $S_N(\cdot)$ to make $\rho\|B\|^{d-2}$ top to bottom crossings of $\Pi_N(B^7)$. In Lemma 5.2, we show there is a $\rho(u) > 0$ for which $\mathbf{P_x}[\tau_\rho > uN^d] \xrightarrow{N} 0$ uniformly for $\mathbf{x} \in \mathbb{Z}^d$. Since

$$\mathcal{F}_N^{uN^d}(B) \supset \mathcal{F}_N^{\tau_\rho}(B) \setminus \{\tau_\rho > uN^d\},$$

it is thus enough to show $\mathbf{P_x}[\mathcal{F}_N^{\tau_\rho}(B)] \xrightarrow{N} 1$ uniformly for $\mathbf{x} \in \mathbb{Z}^d$.

A ρ-dense B-itinerary (defined in Section 5.1) is essentially a product space of $\rho\|B\|^{d-2}$ SRWs conditioned to cross B^7 from top to bottom. A B-boundary-connected-path $F(t)$ is a map from $\mathbb{Z}^{\geq 0}$ to 2^{B^7} with certain properties (defined before Lemma 5.3). By Lemma 5.3 there is a ρ-dense B-itinerary H (independent of $\mathbf{x}$) and a B-boundary-connected-path $F(t)$ (dependent on $\mathbf{x}$) such that

$$\mathbf{P_x}\big[\mathcal{F}_N^{\tau_\rho}(B)\big] \geq \mathbb{P}_H\big[\{\mathcal{R} \cup F(t) : t \geq 0\} \subset \mathcal{G}_k^\rho\big].$$

By Corollary 6.3 below, the RHS approaches one as N tends to infinity uniformly for ρ-dense $B(\lceil N/10 \rceil)$-itineraries [and independently of $F(t)$]. $\square$

In the below lemma, we use a dimensional constant $p_d < 1$ from Corollary 4.2.

LEMMA 6.2. *Let $B = B(n)$, fix $j > 0$ and $\rho > 0$. Let $C_1(\rho, j)$ be such that for any $\rho\Lambda$-dense B-itinerary h, B-boundary-connected-path $f(t)$, and $n > C_1$ we have*

$$\mathbb{P}_h\big[\{\mathcal{R} \cup f(t) : t \geq 0\} \subset \mathcal{G}_{j-1}^\rho\big] > p_d.$$

Then for all $p < 1$ there is a $C_2(p, \rho)$ such that for any ρ-dense B-itinerary H, any B-boundary-connected-path $F(t)$ and all $n > C_2$

$$\mathbb{P}_H\big[\{\mathcal{R} \cup F(t) : t \geq 0\} \subset \mathcal{G}_j^\rho\big] > p.$$

PROOF. Fix a ρ-dense $B(n)$-itinerary H and a B-boundary-connected-path $F(t)$. Let $\sigma_j = |\sigma(B)|^{1/d}$ and let

$$\mathcal{S} = \big\{b \in \sigma(B) : \forall t \geq 0, b \text{ is } \big(\mathcal{R} \cup F(t), j-1, \rho\Lambda\big)\text{-good}\big\}.$$

Observe that if $\Delta\mathcal{S} \in \mathcal{P}(\sigma_j)$ and $\{\mathcal{R} \cup F(t) : t \geq 0\} \subset \mathcal{G}_0^\rho$, this implies that $\{\mathcal{R} \cup F(t) : t \geq 0\} \subset \mathcal{G}_j^\rho$.

See Definition 5.5 for the definition of $\mathcal{D}^{\sigma}_{\Lambda\rho}$, which is roughly, the event that each $b \in \sigma(B)$ is traversed top to bottom at least $\Lambda\rho\|b\|^{d-2}$ times. By Theorem 5.8, for any $q > 0$ there is a $C_1(q, \rho)$ such that for all $n > C_1$,

$$\mathbb{P}_H[\mathcal{D}^{\sigma}_{\Lambda\rho}] > 1 - q.$$

By Theorem 5.12, for any $q > 0$ and all $n > C_2(q, \rho)$,

$$\mathbb{P}_H[\{\mathcal{R} \cup F(t) : t \geq 0\} \subset \mathcal{G}^{\rho}_0 | \mathcal{D}^{\sigma}_{\Lambda\rho}] > 1 - q.$$

Thus, if we also prove that for all $n > C_3(q)$

$$\mathbb{P}_H[\Delta\mathcal{S} \in \mathcal{P}(\sigma_j) | \mathcal{D}^{\sigma}_{\Lambda\rho}] > 1 - q, \tag{28}$$

then for $q < (1 - p)/4$ and $n > C_1(q, \rho) \vee C_2(q, \rho) \vee C_3(q)$

$$\mathbb{P}_H[\{\mathcal{R} \cup F(t) : t \geq 0\} \subset \mathcal{G}^{\rho}_j] > p > 1 - 4q,$$

and we are done.

Let $F_j \subset B(\sigma_j)$ and let $b \in \sigma(B)$. Write $\mathcal{F}_j(F_j, b)$ for the event that $\Delta(\mathcal{S} \setminus B_{\Delta}(b, 20)) = F_j$. By Corollary 4.2 (a consequence of the main theorem in [10]), to prove (28) for all $n > C(p)$, it is enough to show that for any such F_j for which $\mathbb{P}_H[\mathcal{D}^{\sigma}_{\Lambda\rho}, \mathcal{F}_j] > 0$,

$$\mathbb{P}_H[b \in \mathcal{S} | \mathcal{D}^{\sigma}_{\Lambda\rho}, \mathcal{F}_j] > p_d. \tag{29}$$

Since $\{b \in \mathcal{S}\}$ is a function of $(\mathcal{R} \cup F(t)) \cap b^7$, by Lemma 5.6, (29) follows from our assumption. $\square$

COROLLARY 6.3. *Fix $k > 0$ and $p < 1$. Let $B = B(n)$, let H be a ϱ-dense B-itinerary, $\varrho > 0$ and let $F(t)$ be a B-boundary-connected-path. Then for all $n > C(\varrho, k, p)$*

$$\mathbb{P}_H[\{\mathcal{R} \cup f(t) : t \geq 0\} \subset \mathcal{G}^{\varrho}_k] > p.$$

PROOF. W.l.o.g. $p > p_d$. Let $b = b(s^{(k)}(n))$. By Theorem 5.12 for any $\varrho\Lambda^k$-dense b-itinerary h, b-boundary-connected-path $f(t)$, and whenever $s^{(k)}(n) > C(1 - p, \rho)$

$$\mathbb{P}_h[\{\mathcal{R} \cup f(t) : t \geq 0\} \in \mathcal{G}^{\varrho\Lambda^k}_0] > p.$$

Iterate Lemma 6.2 with above p from $j = 1, \rho = \varrho\Lambda^{k-1}$ to $j = k, \rho = \varrho$ to finish. $\square$

7. Random interlacements. In this section, we prove Theorem 2.3. Notation and definition of random interlacements appear in Appendix C.

DEFINITION 7.1. Denote by $\omega_{u',u}(\mathrm{Top} \to \mathrm{Bot})$ the set of trajectories $w \in \mathrm{Supp}(\omega_{u',u}(W^*_{\mathrm{Top}}))$ such that the first exit position of w from B^7 is in $\mathrm{Bot} \subset \partial B^7$.

DEFINITION 7.2. Let $u_\rho = \inf\{u > 0 : |\omega_u(\mathrm{Top} \to \mathrm{Bot})| > \rho N^{d-2}\}$.

LEMMA 7.3. *u_ρ is finite $\mathbb{P}$ a.s.*

PROOF. Denote by $p(N) = \mathbb{P}[|\omega_1(\mathrm{Top} \to \mathrm{Bot})| \geq 1]$. Then for every N, $p(N) > 0$. By independence between $\omega_{u,u'}$ and $\omega_{v,v'}$ for $u < u' \leq v \leq v'$, we obtain by the Borel–Cantelli lemma that

$$\begin{aligned}
&\mathbb{P}\bigl[\exists k, \bigl|\omega_k(\mathrm{Top} \to \mathrm{Bot})\bigr| \geq \rho N^{d-2}\bigr] \\
&\quad \geq \mathbb{P}\Bigl[\limsup_{i \to \infty} \bigl|\omega_{i,i+1}(\mathrm{Top} \to \mathrm{Bot})\bigr| \geq 1\Bigr] = 1.
\end{aligned}$$ □

We now prove the equivalent of Lemma 5.2.

LEMMA 7.4. *For every $u > 0$, there is a $\rho(u) > 0$ such that*

$$\lim_{N \to \infty} \mathbb{P}[u_\rho < u] = 1.$$

PROOF. First, $|\omega_u(W^*_{\mathrm{Top}})|$ is Poisson($u\mathrm{Cap}(\mathrm{Top})$) distributed. There exists a dimension dependent constant c'_d such that, $\mathrm{cap}(\mathrm{Top}) = c'_d N^{d-2}$ (see [9] Proposition 6.5.2). By the invariance principle, there is a dimension dependent constant $c_d > 0$, such that $\min_{x \in \mathrm{Top}} \mathbf{P}^x[X_{\tau_{B^7}} \in \mathrm{Bot}] \geq c_d$, for large enough N. Thus, $|\omega_u(\mathrm{Top} \to \mathrm{Bot})|$ stochastically dominates a Poisson($uc_d N^{d-2}$) distribution. Now take any $\rho < uc_d$, and by Chebyshev's inequality we obtain that

$$\mathbb{P}\bigl[\bigl|\omega_u(\mathrm{Top} \to \mathrm{Bot})\bigr| < \rho N^{d-2}\bigr] \underset{N \to \infty}{\to} 0,$$

which concludes the lemma. □

LEMMA 7.5. *For $N > 0$ large enough fix a box $b = b(\lceil N/10 \rceil)$, $\rho > 0$, $u > u_\rho$. Then for any $\mathcal{A} \subset 2^{b^7}$ there is a ρ-dense b-itinerary $H = H(\mathcal{A})$ and a b-boundary-connected-path $F(t) = F(\mathbf{x}, t)$ such that*

$$\mathbb{P}[\mathcal{I}^u \cap b^7 \in \mathcal{A}] \geq \mathbb{P}_H\bigl[\{\mathcal{R} \cup F(t) : t \geq 0\} \in \mathcal{A}\bigr].$$

PROOF. Since $u > u_\rho$ we know $|\omega_u(\text{Top} \to \text{Bot})| > \rho N^{d-2}$. Order the trajectories in $\omega_u(\text{Top} \to \text{Bot})$ by some arbitrary but fixed method. For every $1 \le i \le \rho N^{d-2}$ and trajectory $w_i \in \omega_u(\text{Top} \to \text{Bot})$ denote by $a(i) \in \text{Top}$, the starting point of w_i and by $z(i) \in \text{Bot}$, the exit point. For every $1 \le i \le \rho N^{d-2}$ let $\eta_i = (a(i), z(i))$, and $H = (\eta_1, \dots, \eta_{\lceil \rho N^{d-2} \rceil})$. For all $t > 0$, let $F(t) = \bigcup_{w \in \operatorname{Supp}(\omega_u(W^*_{b^7}))} \operatorname{range}(w) \cap b^7 \setminus \bigcup_{i=1}^{\rho N^{d-2}} \operatorname{range}(w_i)$. Then

$$\mathbb{P}[\operatorname{range}(\omega(W^*_{b^7})) \cap b^7 \subset \mathcal{A}] = \mathbb{P}_H[\{\mathcal{R} \cup F(t) : t \ge 0\} \subset \mathcal{A}]. \qquad \square$$

THEOREM 7.6. *For every $k \in \mathbb{N}$ and $u, \rho > 0$, there exists a constant $\alpha(k, \rho)$ such that*

$$\mathbb{P}[\mathcal{I}^u \notin \mathcal{G}_k^\rho(n)] \le \frac{\alpha}{n^2}.$$

PROOF. The proof follows Theorem 6.1 without the union on top level boxes. $\square$

We now prove the bound on the heat kernel of random interlacements.

THEOREM 7.7. *Let $u > 0$ and let X_n be a random walk on the graph $\mathcal{I}^u$. For large enough N, if $\mathcal{I}^u \in \mathcal{G}_k^\rho(N)$ and $0 \in \mathcal{I}^u$, there exists a constant $C(k, \rho)$ such that*

$$\mathbf{P}_0^u[X_N = 0] \le \frac{C(k, \rho) \log^{(k-1)}(N)}{N^{d/2}}.$$

PROOF. Let $\varepsilon > 0$. By [13] (Theorem 2), there exists a constant $\tilde{c}$ such that if for some $\varepsilon > 0$

$$\tilde{c} \int_1^{4/\varepsilon} \frac{dr}{r\phi^2(r)} \le n, \tag{30}$$

then $\mathbf{P}_0[X_n = 0] \le \varepsilon$. In order to bound $\mathbf{P}_0[X_n = 0]$ it is enough to consider the isoperimetric constant of sets inside $B(n)$. Indeed consider a new graph $\tilde{\mathcal{I}}^u$ which is the same as $\mathcal{I}^u$ inside $B(n)$ but all the edges are open outside $B(n)$. Since a random walk cannot leave $B(n)$ before time n, it is enough to prove the theorem for the graph $\tilde{\mathcal{I}}^u$. Next, we prove an isoperimetric inequality for the graph $\tilde{\mathcal{I}}^u$. For every set $A \subset \mathbb{Z}^d$, such that $|A| > n^{1/3}$, if $A \cap B(n) = \phi$ then by the isoperimetric inequality of $\mathbb{Z}^d$, $|\partial A| \ge |A|^{(d-1)/d}$. If $A \cap B(n) \ne \phi$ and $|A \cap B(n)^c| \ge \frac{1}{2}|A|$, by the triangle inequality and isoperimetric inequality of $\mathbb{Z}^d$,

$$|\partial A| \ge |\partial(A \cap B(n)^c)| \ge (|A \cap B(n)^c|)^{(d-1)/d} \ge \left(\frac{1}{2}|A|\right)^{(d-1)/d}.$$

If $A\cap B(n)\neq\phi$ and $|A\cap B(n)|>\frac{1}{2}|A|$, since $|\partial A\cap B^c|\geq|A\cap\partial B|$ (a straight line between two points is the shortest path)

$$|\partial A|\geq\bigl|\partial(A\cap B)\bigr|>c(k,\rho)\bigl(\tfrac{1}{2}|A|\bigr)^{(d-1)/d}\bigl(s^{(l)}(n)\bigr)^{d-1}.$$

If $\phi(r)$ is realized by a set of size smaller than $N^{1/3}$, then $\phi(r)\geq N^{-1/3}$. By Theorem 3.3,

$$\begin{aligned}\int_1^{4/\varepsilon}\frac{dr}{r\phi^2(r)}&=\int_1^{4/\varepsilon}\frac{dr}{r(1/N^{2/3})}+\int_{N^{1/3}}^{4/\varepsilon}\frac{dr}{r\hat{\phi}^2(r)}\\&=N^{2/3}\log\biggl(\frac{4}{\varepsilon}\biggr)+\int_{N^{1/3}}^{4/\varepsilon}\frac{c(k,\rho)^{-1}(s^{(k)}(N))^{2d-2}\,dr}{rr^{-2/d}}\\&=N^{2/3}\log\biggl(\frac{4}{\varepsilon}\biggr)+\frac{c'(s^{(k)}(N))^{2d-2}}{\varepsilon^{d/2}}.\end{aligned}\tag{31}$$

Thus, if $\varepsilon\geq\frac{c''(s^{(k)}(N))^{2d-2}}{N^{d/2}}$, $\mathbf{P}_0[X_N=0]\leq\frac{c''(s^{(k)}(N))^{2d-2}}{N^{d/2}}\leq\frac{c'''\log^{k-1}(N)}{N^{d/2}}$. $\square$

The proof of Theorem 2.3 follows from Theorems 7.7 and 7.6.

APPENDIX A

Recall the notation from Section 5.2 and let $\tau_0=0$, and for $\mathbf{z}(i)\in G, m_i\in\mathbb{N}$ where $i\geq 1$ recursively define

$$\tau_i=\tau_i\bigl(\{\mathbf{z}(i)\},\{m_i\}\bigr)=\inf\bigl\{t>\tau_{i-1}+m_{i-1}:S(t)=\mathbf{z}(i)\bigr\}.$$

PROPOSITION A.1. *Fix $n\in\mathbb{N}$, $s_0,\dots,s_n,g_1,\dots,g_n\in\mathbf{S}(G)$ and $C_1,\dots,C_n\subset\mathbf{S}(G)$. Set $\mathbf{z}(i)=s_i(0),m_i=\|s_i\|,\mathbf{a}(i)=s_i(m_i)$. Define the events $\mathcal{A}=\bigcap_{i=0}^n S(\tau_i,\tau_i+m_i)=s_i,\mathcal{B}(A_1,\dots,A_n)=\bigcap_{i=1}^n S(\tau_{i-1}+m_{i-1},\tau_i)\in A_i$. Writing $\mathcal{B}_g$ for $\mathcal{B}(\{g_1\},\dots,\{g_n\})$ and $\mathcal{B}_C$ for $\mathcal{B}(C_1,\dots,C_n)$ and assuming $\mathbf{P}_{\mathbf{z}(0)}[\mathcal{A},\mathcal{B}_C]>0$ we have*

$$\mathbf{P}_{\mathbf{z}(0)}[\mathcal{A},\mathcal{B}_g|\mathcal{A},\mathcal{B}_C]=\prod_{i=1}^n\mathbf{P}_{\mathbf{a}(i)}\bigl[S(0,\tau_{\mathbf{z}(i)})=g_i|S(0,\tau_{\mathbf{z}(i)})\in C_i\bigr].$$

PROOF. See Figure 6 for an illustration. Observe that if for some i, $g_i\notin C_i$, then both sides are 0, thus we assume $g_i\in C_i$.

$$\mathbf{P}_{\mathbf{z}(0)}[\mathcal{A},\mathcal{B}_g|\mathcal{A},\mathcal{B}_C]=\mathbf{P}_{\mathbf{z}(0)}[\mathcal{A},\mathcal{B}_g]/\mathbf{P}_{\mathbf{z}(0)}[\mathcal{A},\mathcal{B}_C].$$

Let $W_1,\dots,W_n\subset\mathbf{S}(G)$ be with the property that for each $1\leq i\leq n$ and $w\in W_i$, $w=(\mathbf{a}(i),\mathbf{v}_1,\dots,\mathbf{v}_k,\mathbf{z}(i))\in\mathbf{S}(G)$, $\mathbf{v}_j\neq\mathbf{z}(i)$ $\forall 1\leq j\leq k$. For each

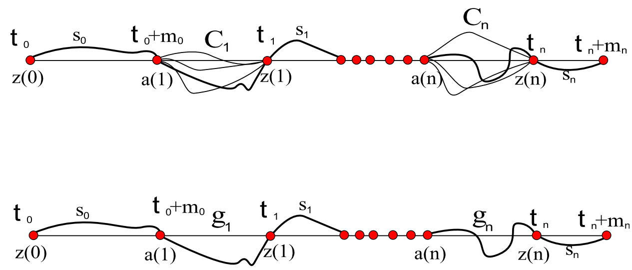

FIG. 6. *Scheme of $\mathcal{A}, \mathcal{B}_C$ on top and of $\mathcal{A}, \mathcal{B}_g$ on bottom.*

$(w_1, \dots, w_n) \in W_1 \times \cdots \times W_n$, we decompose $\mathcal{A}, \mathcal{B}(\{w_1\}, \dots, \{w_n\})$ according to the Markov property and sum to get

$$\begin{aligned} &\mathbf{P}_{\mathbf{z}(0)}[\mathcal{A}, \mathcal{B}(W_1, \dots, W_n)] \\ &\quad = \left(\prod_{i=1}^{n} \mathbf{P}_{\mathbf{z}(i-1)}[S(0, m_{i-1}) = s_{i-1}]\mathbf{P}_{\mathbf{a}(i)}[S(0, \tau_{\mathbf{z}(i)}) \in W_i]\right) \\ &\quad\quad \times \mathbf{P}_{\mathbf{z}(n)}[S(0, m_n) = s_n]. \end{aligned} \tag{32}$$

Since we assume $\mathbf{P}_{\mathbf{z}(0)}[\mathcal{A}, \mathcal{B}_C] > 0$, we have that each C_i consists of paths with the constraints above. Using (32) with $W_i = C_i$ and $W_i = \{g_i\}$ we get

$$\mathbf{P}_{\mathbf{z}(0)}[\mathcal{A}, \mathcal{B}_g | \mathcal{A}, \mathcal{B}_C] = \prod_{i=1}^{n} \left(\mathbf{P}_{\mathbf{a}(i)}[S(0, \tau_{\mathbf{z}(i)}) = g_i]/\mathbf{P}_{\mathbf{a}(i)}[S(0, \tau_{\mathbf{z}(i)}) \in C_i]\right),$$

and are done. $\square$

PROPOSITION A.2. *Let $\mathcal{X}, \mathcal{Y}$ be events in some probability space, and let $\{\mathcal{Y}_\alpha\}_{\alpha \in I}$ be a partition of $\mathcal{Y}$ where $\forall \alpha \in I, \Pr[\mathcal{Y}_\alpha] > 0$. Then for some $\gamma, \Gamma \in I$,*

$$\Pr[\mathcal{X} | \mathcal{Y}_\gamma] \le \Pr[\mathcal{X} | \mathcal{Y}] \le \Pr[\mathcal{X} | \mathcal{Y}_\Gamma].$$

PROOF. Follows from the identity,

$$\Pr[\mathcal{X} | \mathcal{Y}] = \sum_{\alpha \in I} \Pr[\mathcal{X} | \mathcal{Y}_\alpha] \Pr[\mathcal{Y}_\alpha] / \Pr[\mathcal{Y}].$$

$\square$

Recall $\mathcal{J}_B[b]$ from Definition 5.5.

LEMMA A.3. *Let $B = B(n)$, let $b \in \sigma(B)$, where we write $m = \|b\|$. There is a $c_\Lambda(d) > 0$, independent of n, such that for any $\mathbf{a} \in A(B), \mathbf{z} \in Z(B)$ and all large n,*

$$\mathbf{P}_{\mathbf{a}}[\mathcal{J}_B[b] | \tau_{\partial B^7} = \tau_{\mathbf{z}}] > c_\Lambda \left(\frac{m}{n}\right)^{d-2}.$$

PROOF. For $\mathbf{y} = (y_1, \dots, y_d) \in \mathbb{Z}^d$, define $\mathcal{B}(\mathbf{y})$ to be the event $S(\cdot)$ hits b at $\mathbf{y}$ and then the first coordinate of $S(\cdot)$ hits $y_1 + 4m$ and then hits $y_1 - 8m$, while the maximal change in the other coordinates is less than m. Let $\tau_{\mathcal{B}}(\mathbf{y}) = \inf\{t \geq 0 : S(0,t) \subset \mathcal{B}(\mathbf{y})\}$ be the first time the occurrence of $\mathcal{B}(\mathbf{y})$ is implied by $S(t)$. By Proposition A.5 for some $\mathbf{x} \in b$,

$$\begin{aligned}\mathbf{P_a}\bigl[\tau_b, \tau_{\mathcal{B}}\bigl(S(\tau_b)\bigr) < \tau_{\partial B^7} | \tau_{\partial B^7} = \tau_{\mathbf{z}}\bigr] \\ \geq \mathbf{P_a}[\tau_b < \tau_{\partial B^7} | \tau_{\partial B^7} = \tau_{\mathbf{z}}]\mathbf{P_x}\bigl[\tau_{\mathcal{B}}(\mathbf{x}) < \tau_{\partial B^7} | \tau_{\partial B^7} = \tau_{\mathbf{z}}\bigr].\end{aligned}$$

Using Lemma A.4 together with Corollary A.11, we have

$$\mathbf{P_a}[\tau_b < \tau_{\partial B^7} | \tau_{\partial B^7} = \tau_{\mathbf{z}}] > c_1 \left(\frac{m}{n}\right)^{d-2}.$$

Since $\{\tau_{\mathcal{B}}(\mathbf{x}) < \tau_{\partial B^7}\} \subset \mathcal{J}_B[b]$, we are done if we show for some $c_2(d) > 0$

$$\mathbf{P_x}\bigl[\tau_{\mathcal{B}}(\mathbf{x}) < \tau_{\partial B^7} | \tau_{\partial B^7} = \tau_{\mathbf{z}}\bigr] > c_2. \tag{33}$$

Partitioning over $S(\tau_{\mathcal{B}}(\mathbf{x})) \in b^{10}$ and using the Markov property, we have for some $\mathbf{y} \in b^{10}$,

$$\begin{aligned}\mathbf{P_x}\bigl[\tau_{\mathcal{B}}(\mathbf{x}) < \tau_{\partial B^7} | \tau_{\partial B^7} = \tau_{\mathbf{z}}\bigr]\mathbf{P_x}[\tau_{\partial B^7} = \tau_{\mathbf{z}}] \\ \geq \mathbf{P_x}\bigl[\tau_{\mathcal{B}}(\mathbf{x}) < \tau_{\partial B^7}\bigr]\mathbf{P_y}[\tau_{\partial B^7} = \tau_{\mathbf{z}}].\end{aligned} \tag{34}$$

By the invariance principle, $\mathbf{P_x}[\tau_{\mathcal{B}}(\mathbf{x}) < \tau_{\partial B^7}]$ is bounded away from zero by a dimensional constant independent of $\mathbf{x}$. Since $\mathbf{x}, \mathbf{y}$ from (34) are contained in B^6 for all large n, we use Lemma A.10 to get (33). $\square$

In the lemma below, we look at the number vertices hit in an interior set $M \subset B^6$ by a B-traversal, and lower bound the probability for this number to be small in terms of $|M|$.

LEMMA A.4. *Let $B = B(n)$, let $M \subset B^6$, set $\mathbf{a} \in B^7$ and $\mathbf{z} \in Z(B)$. Let $X(M) = |\{\mathbf{v} \in M : \tau_{\mathbf{v}} < \tau_{\partial B^7}\}|$ and let $\mu_X = \mu_X(\mathbf{a}, \mathbf{z}) = \mathbf{E_a}[X(M) | \tau_{\partial B^7} = \tau_{\mathbf{z}}]$. There is a $c_1(d) > 0$, independent of n, $\mathbf{a}$ and $\mathbf{z}$, such that for all large n*

$$\mathbf{P_a}\bigl[X(M) \geq \tfrac{1}{2}\mu_X | \tau_{\partial B^7} = \tau_{\mathbf{z}}\bigr] > c_1 \mu_X |M|^{-2/d}.$$

Thus, if $\mathbf{P_a}[\tau_{\mathbf{v}} < \tau_{\partial B^7} | \tau_{\partial B^7} = \tau_{\mathbf{z}}] > f(n)$ for every $\mathbf{v} \in M$, then since $X(M) = \sum_{\mathbf{v} \in M} \mathbf{1}_{\{\tau_{\mathbf{v}} < \tau_{\partial B^7}\}}$, we have

$$\mathbf{P_a}\bigl[X(M) \geq \tfrac{1}{2}\mu_X | \tau_{\partial B^7} = \tau_{\mathbf{z}}\bigr] > c_1 |M|^{1-2/d} f(n).$$

PROOF. Write μ_{X^2} for $\mathbb{E}_{\mathbf{a}}[X^2 | \tau_{\partial B^7} = \tau_{\mathbf{z}}]$. By the Paley–Zygmund inequality, $\mathbf{P_a}[X \geq \frac{1}{2}\mu_X] \geq \frac{1}{4}\mu_X^2 / \mu_{X^2}$, so enough to show

$$\mu_{X^2} < C m^2 \mu_X,$$

where $m = |M|^{1/d}$.

By linearity,

$$\mu_{X^2} = \sum_{\mathbf{v}\in M}\sum_{\mathbf{w}\in M} \mathbf{P_a}[\tau_{\mathbf{w}}, \tau_{\mathbf{v}} < \tau_{\partial B^7} | \tau_{\partial B^7} = \tau_{\mathbf{z}}].$$

For two vertices $\mathbf{x}, \mathbf{y}$, let $\tau_{\mathbf{v},\mathbf{w}} = \inf\{t \geq \tau_{\mathbf{x}} : S(t) = \mathbf{y}\}$. By a union bound

$$\begin{aligned}\mathbf{P_a}[\tau_{\mathbf{w}}, \tau_{\mathbf{v}} < \tau_{\partial B^7} | \tau_{\partial B^7} = \tau_{\mathbf{z}}] &\leq \mathbf{P_a}[\tau_{\mathbf{w}} \leq \tau_{\mathbf{w},\mathbf{v}} < \tau_{\partial B^7} | \tau_{\partial B^7} = \tau_{\mathbf{z}}]\\ &\quad + \mathbf{P_a}[\tau_{\mathbf{v}} \leq \tau_{\mathbf{v},\mathbf{w}} < \tau_{\partial B^7} | \tau_{\partial B^7} = \tau_{\mathbf{z}}].\end{aligned}$$

By Bayes theorem and the Markov property,

$$\begin{aligned}&\mathbf{P_a}[\tau_{\mathbf{w}} \leq \tau_{\mathbf{w},\mathbf{v}} < \tau_{\partial B^7} | \tau_{\partial B^7} = \tau_{\mathbf{z}}]\\ &\quad = \frac{\mathbf{P_a}[\tau_{\mathbf{w}} < \tau_{\partial B^7} | \tau_{\partial B^7} = \tau_{\mathbf{z}}]\mathbf{P_a}[\tau_{\mathbf{w}} \leq \tau_{\mathbf{w},\mathbf{v}} < \tau_{\partial B^7} = \tau_{\mathbf{z}} | \tau_{\mathbf{w}} < \tau_{\partial B^7}]}{\mathbf{P_a}[\tau_{\partial B^7} = \tau_{\mathbf{z}} | \tau_{\mathbf{w}} < \tau_{\partial B^7}]}\\ &\quad = \mathbf{P_a}[\tau_{\mathbf{w}} < \tau_{\partial B^7} | \tau_{\partial B^7} = \tau_{\mathbf{z}}]\mathbf{P_w}[\tau_{\mathbf{v}} < \tau_{\partial B^7} | \tau_{\partial B^7} = \tau_{\mathbf{z}}].\end{aligned}$$

Again by the Markov property,

$$\mathbf{P_w}[\tau_{\mathbf{v}} < \tau_{\partial B^7} | \tau_{\partial B^7} = \tau_{\mathbf{z}}] = \frac{\mathbf{P_v}[\tau_{\partial B^7} = \tau_{\mathbf{z}}]\mathbf{P_w}[\tau_{\mathbf{v}} < \tau_{\partial B^7}]}{\mathbf{P_w}[\tau_{\partial B^7} = \tau_{\mathbf{z}}]}.$$

So by Lemma A.10, since $\mathbf{v}, \mathbf{w} \in M \subset B^6$,

$$\mathbf{P_w}[\tau_{\mathbf{v}} < \tau_{\partial B^7} | \tau_{\partial B^7} = \tau_{\mathbf{z}}] < C\mathbf{P_w}[\tau_{\mathbf{v}} < \tau_{\partial B^7}] < C\mathbf{P_w}[\tau_{\mathbf{v}} < \infty].$$

Thus, by symmetry,

$$\mu_{X^2} \leq 2\sum_{\mathbf{w}\in M} \mathbf{P_a}[\tau_{\mathbf{w}} < \tau_{\partial B^7} | \tau_{\partial B^7} = \tau_{\mathbf{z}}] \sum_{\mathbf{v}\in M} C\mathbf{P_w}[\tau_{\mathbf{v}} < \infty].$$

By Markov's inequality, $\mathbf{P_w}[\tau_{\mathbf{v}} < \infty] < G(\mathbf{w}, \mathbf{v})$ where $G(\cdot,\cdot)$ is the Green's function of a simple random walk on $\mathbb{Z}^d$. Standard estimates for $G(\cdot,\cdot)$ (see, e.g., Theorem 1.5.4 in [8]) give that $\mathbf{P_w}[\tau_{\mathbf{v}} < \infty] < C(d)\|\mathbf{w}-\mathbf{v}\|_2^{2-d}$, and thus

$$\sum_{\mathbf{v}\in M} \mathbf{P_w}[\tau_{\mathbf{v}} < \infty] < \sum_{\mathbf{v}\in M} \|\mathbf{w}-\mathbf{v}\|_2^{2-d}.$$

For some $\hat{c}(d) < \infty$, and all $r > 0$, a ball of radius $\hat{c}r$ around the origin contains at least r^d vertices in $\mathbb{Z}^d$. Since the RHS above can only be increased by moving a vertex in M closer to $\mathbf{w}$, we have

$$\sum_{\mathbf{v}\in M} \mathbf{P_w}[\tau_{\mathbf{v}} < \infty] < \hat{C}\sum_{r=1}^{\hat{c}m} r^{d-1}r^{2-d} < Cm^2 = C|M|^{2/d}.$$

Since $\mu_X = \sum_{\mathbf{w}\in M} \mathbf{P_a}[\tau_{\mathbf{w}} < \tau_{\partial B^7} | \tau_{\partial B^7} = \tau_{\mathbf{z}}]$, we are done. $\square$

Let $\tau_0 : (\mathbb{Z}^d)^{\mathbb{Z}^{\geq 0}} \to \mathbb{Z}^{\geq 0}$ be a stopping time for the random walk $S(t)$. We denote by τ_0^t the stopping time on the t-time shifted sequences, that is, $\tau_0^t(\mathbf{a}_0, \mathbf{a}_1, \ldots) = \tau_0(\mathbf{a}_t, \mathbf{a}_{t+1}, \ldots) + t$. We call τ_0 a *simple stopping time* if $\tau_0^t \geq \tau_0$ for every $t \geq 0$.

PROPOSITION A.5. *Let $B = B(n)$, set $\mathbf{a} \in B^7$ and $\mathbf{z} \in Z(B)$. Let τ_1, τ_2 be simple stopping times* (*see above*). *Then there exists a* $\mathbf{x}$ *satisfying* $\mathbf{P_a}[S(\tau_1) = \mathbf{x}, \tau_1 < \tau_{\partial B^7} | \tau_{\partial B^7} = \tau_\mathbf{z}] > 0$ *such that*

$$\mathbf{P_a}[\tau_1, \tau_2 \le \tau_{\partial B^7} | \tau_{\partial B^7} = \tau_\mathbf{z}] \ge \mathbf{P_a}[\tau_1 < \tau_{\partial B^7} | \tau_{\partial B^7} = \tau_\mathbf{z}]\mathbf{P_x}[\tau_2 \le \tau_{\partial B^7} | \tau_{\partial B^7} = \tau_\mathbf{z}].$$

PROOF. Let

$$\pi_\mathbf{y} = \mathbf{P_a}\big[S(\tau_1) = \mathbf{y}, \tau_1 < \tau_{\partial B^7} | \tau_{\partial B^7} = \tau_\mathbf{z}\big].$$

For $\mathbf{y}$ satisfying $\pi_\mathbf{y} > 0$, we have by Bayes

$$\begin{aligned}&\mathbf{P_a}\big[\tau_1, \tau_2 \le \tau_{\partial B^7}, S(\tau_1) = \mathbf{y} | \tau_{\partial B^7} = \tau_\mathbf{z}\big] \\ &\quad = \frac{\pi_\mathbf{y}\mathbf{P_a}[\tau_1, \tau_2 \le \tau_{\partial B^7} = \tau_\mathbf{z} | S(\tau_1) = \mathbf{y}, \tau_1 < \tau_{\partial B^7}]}{\mathbf{P_a}[\tau_{\partial B^7} = \tau_\mathbf{z} | S(\tau_1) = \mathbf{y}, \tau_1 < \tau_{\partial B^7}]}.\end{aligned} \tag{35}$$

Since τ_2 is a simple stopping time,

$$\begin{aligned}&\mathbf{P_a}\big[\tau_1, \tau_2 \le \tau_{\partial B^7} = \tau_\mathbf{z} | S(\tau_1) = \mathbf{y}, \tau_1 < \tau_{\partial B^7}\big] \\ &\quad \ge \mathbf{P_a}\big[\tau_1 \le \tau_2^{\tau_1} \le \tau_{\partial B^7} = \tau_\mathbf{z} | S(\tau_1) = \mathbf{y}, \tau_1 < \tau_{\partial B^7}\big].\end{aligned}$$

Plugging the above into (35) and using the strong Markov property, we get

$$\mathbf{P_a}\big[\tau_1, \tau_2 \le \tau_{\partial B^7}, S(\tau_1) = \mathbf{y} | \tau_{\partial B^7} = \tau_\mathbf{z}\big] \ge \pi_\mathbf{y}\mathbf{P_y}[\tau_2 \le \tau_{\partial B^7} | \tau_{\partial B^7} = \tau_\mathbf{z}].$$

Let $\mathbf{x} \in \{\mathbf{y} : \pi_\mathbf{y} > 0\}$ be the vertex for which $\mathbf{P_x}[\tau_2 \le \tau_{\partial B^7} | \tau_{\partial B^7} = \tau_\mathbf{z}]$ is minimal. Summing both sides over $\{\mathbf{y} : \pi_\mathbf{y} > 0\}$, we are done. □

We quote the Harnack principle for $\mathbb{Z}^d$ from Theorem 1.7.6 in [8].

PROPOSITION A.6. *Let U be a compact subset of $\mathbb{R}^d$ contained in a connected open set V. Then there exists a $c = c(U, V) < \infty$ such that if $A_n = nU \cap \mathbb{Z}^d$, $D_n = nV \cap \mathbb{Z}^d$, and $f : D_n \cup \partial D_n \to [0, \infty)$ is harmonic in D_n, then*

$$f(x) \le cf(y), x, y \in A_n.$$

LEMMA A.7. *Let $B = B(n)$ and let F be the union of all hyperplanes in $\mathbb{Z}^d$ that intersect B^6 and are parallel to $Z(B)$. There is a $C > 0$ such that for any $\mathbf{y} \in Z(B) \cup A^+(B)$ and any $\mathbf{v} \in F \cap B^7$,*

$$\mathbf{P_v}[\tau_{\partial B^7} = \tau_\mathbf{y}] < Cn^{1-d}.$$

PROOF. We prove for $\mathbf{y} \in Z(B)$. The proof $A^+(B)$ is the same so we omit it. Let H be the infinite hyperplane in $\mathbb{Z}^d$ that contains $Z(B)$, and let H_0 a parallel hyperplane, which is the component of $\partial_{B^7} F$ closer to $Z(B)$. Let $h(\mathbf{y})$ be the l_1-closest vertex to $\mathbf{y}$ in H_0. By vertex transitivity, there is a function $g(n)$ such that for any $\mathbf{y} \in Z(B)$, $\mathbf{P}_{h(\mathbf{y})}[\tau_H = \tau_\mathbf{y}] = g(n)$. Observe that $\mathbf{P}_{(\cdot)}[\tau_H = \tau_\mathbf{y}]$ is a nonnegative harmonic function in the component of $\mathbb{Z}^d \setminus H$ containing H_0, so by the Harnack principle for $\mathbb{Z}^d$ (Proposition A.6), for some $c > 0$, any $\mathbf{v} \in F \cap B^7, \mathbf{y} \in Z(B)$ satisfies

$$c\mathbf{P}_{\mathbf{v}}[\tau_H = \tau_{\mathbf{y}}] > \mathbf{P}_{h(\mathbf{y})}[\tau_H = \tau_{\mathbf{y}}] = g(n).$$

Summing both sides over $\mathbf{y} \in Z(B)$, we get

$$g(n) < Cn^{1-d}.$$

Since $\{\tau_{\partial B^7} = \tau_{\mathbf{y}}\} \subset \{\tau_H = \tau_{\mathbf{y}}\}$, another application of the Harnack principle finishes the proof. $\square$

COROLLARY A.8. *Let $B = B(n)$. There is a $C > 0$ such that for any $\mathbf{a} \in A(B), \mathbf{z} \in Z(B)$*

$$\mathbf{P}_{\mathbf{a}}[\tau_{\mathbf{z}} = \tau_{\partial B^7}] < Cn^{-d}.$$

PROOF. Using the notation of Lemma A.7, by the Markov property F

$$\mathbf{P}_{\mathbf{a}}[\tau_{\mathbf{z}} = \tau_{\partial B^7}] = \sum_{\mathbf{x}} \mathbf{P}_{\mathbf{a}}[\tau_F < \tau_{\partial B^7}, S(\tau_F) = \mathbf{x}]\mathbf{P}_{\mathbf{x}}[\tau_{\mathbf{z}} = \tau_{\partial B^7}].$$

The right term is uniformly bounded by Cn^{1-d} by Lemma A.7. Summing over $\mathbf{x}$, the event $\{\tau_F < \tau_{\partial B^7}\}$ implies that a one dimensional random walk starting at 1 hits n before hitting 0, an event of probability n^{-1}. $\square$

PROPOSITION A.9. *Let $B = B(n)$. There is a $c(d) > 0$ such that for any $A \subset \mathbb{Z}^d$ and $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^d \setminus A$*

$$c < \frac{\mathbf{P}_{\mathbf{v}}[\tau_{\mathbf{w}} < \tau_A]}{\mathbf{P}_{\mathbf{w}}[\tau_{\mathbf{v}} < \tau_A]} < c^{-1}.$$

PROOF. Write $G_A(\mathbf{v}, \mathbf{w})$ for the Green's function of a random walk killed on hitting A, that is, the expected number of visits to $\mathbf{w}$ for a walk starting at $\mathbf{v}$ before it hits A. By elementary Markov theory, we have symmetry of Green's function, $G_A(\mathbf{v}, \mathbf{w}) = G_A(\mathbf{w}, \mathbf{v})$ and the following identity:

$$\mathbf{P}_{\mathbf{v}}[\tau_{\mathbf{w}} < \tau_{\partial B^7}]G_A(\mathbf{w}, \mathbf{w}) = \mathbf{P}_{\mathbf{w}}[\tau_{\mathbf{v}} < \tau_{\partial B^7}]G_A(\mathbf{v}, \mathbf{v}).$$

For any $\mathbf{v} \in \mathbb{Z}^d \setminus A$, $G_A(\mathbf{v}, \mathbf{v}) \geq 1$, and is bounded above by the reciprocal of the probability a simple random walk never returns to $\mathbf{v}$, which by transience in $d > 2$, is a finite dimensional constant. $\square$

LEMMA A.10. *Let $B = B(n)$. There is a $c > 0$ such that for any $\mathbf{a} \in A(B), \mathbf{z} \in Z(B)$ and any $\mathbf{x} \in B^6$,*

$$cn^{1-d} < \mathbf{P}_{\mathbf{a}}[\tau_{\mathbf{x}} < \tau_{\partial B^7}], \qquad \mathbf{P}_{\mathbf{x}}[\tau_{\mathbf{z}} = \tau_{\partial B^7}] < c^{-1}n^{1-d}. \tag{36}$$

PROOF. Let $D_{\mathbf{x}}(r) = \{\mathbf{v} \in \mathbb{Z}^d : \|\mathbf{v}-\mathbf{x}\|_2 \le r\}$. Lemma 1.7.4 in [8] tells us there is a $c_1(d) > 0$ such that for any $\mathbf{r} \in \partial D_{\mathbf{0}}(r)$,

$$\mathbf{P}_{\mathbf{0}}[\tau_{\partial D_{\mathbf{0}}(r)} = \tau_{\mathbf{r}}] > c_1 n^{1-d}. \tag{37}$$

Fix $\mathbf{y} \in A^+(B)$. Then there is a $\mathbf{v} \in B^6$ such that $\mathbf{y} \in \partial D_{\mathbf{v}}(n)$, $D_{\mathbf{v}}(n) \subset B^7$. Since $\{\tau_{\partial D_{\mathbf{v}}(n)} = \tau_{\mathbf{y}}\}$ implies $\{\tau_{\partial B^7} = \tau_{\mathbf{y}}\}$, we get that

$$\mathbf{P}_{\mathbf{v}}[\tau_{\partial B^7} = \tau_{\mathbf{y}}] > c_1 n^{1-d}. \tag{38}$$

The probability to exit B^7 at $\mathbf{y}$ is a nonnegative harmonic function in B^7. Thus, by the Harnack principle for $\mathbb{Z}^d$ (Proposition A.6), and since c_1 is independent of $\mathbf{y}$, the above is true for any $\mathbf{v} \in B^6$ and any $\mathbf{y} \in A^+(B)$ with an appropriate constant $c_2 > 0$ replacing c_1.

The same argument proves the lower bound in (36) for $\mathbf{P}_{\mathbf{x}}[\tau_{\mathbf{z}} = \tau_{\partial B^7}]$.

Next, by Proposition A.9 we have $\mathbf{P}_{\mathbf{a}}[\tau_{\mathbf{x}} < \tau_{\partial B^7}] > c\mathbf{P}_{\mathbf{x}}[\tau_{\mathbf{a}} < \tau_{\partial B^7}]$. Let $\mathbf{a}^+$ be $\mathbf{a}$'s neighbor in $A^+(B)$. Since $\{\tau_{\mathbf{a}} < \tau_{\partial B^7}\} \supset \{\tau_{\mathbf{a}^+} = \tau_{\partial B^7}\}$ and by (38), we get

$$\mathbf{P}_{\mathbf{x}}[\tau_{\mathbf{a}} < \tau_{\partial B^7}] \ge \mathbf{P}_{\mathbf{x}}[\tau_{\mathbf{a}^+} = \tau_{\partial B^7}] > cn^{1-d},$$

which proves the lower bound in (36) for $\mathbf{P}_{\mathbf{a}}[\tau_{\mathbf{x}} < \tau_{\partial B^7}]$ as well.

The upper bound for $\mathbf{P}_{\mathbf{x}}[\tau_{\mathbf{z}} = \tau_{\partial B^7}]$ is immediate from Lemma A.7. To prove for $\mathbf{P}_{\mathbf{a}}[\tau_{\mathbf{x}} < \tau_{\partial B^7}]$, we first use the lemma to get

$$\mathbf{P}_{\mathbf{x}}[\tau_{\mathbf{a}^+} = \tau_{\partial B^7}] < Cn^{1-d},$$

which implies the bound for $\mathbf{P}_{\mathbf{x}}[\tau_{\mathbf{a}} < \tau_{\partial B^7}]$, since by the Markov property, the probability for exiting B^7 one step after hitting $\mathbf{a}$ for the first time is

$$\mathbf{P}_{\mathbf{x}}[\tau_{\mathbf{a}} < \tau_{\partial B^7}] \cdot \frac{1}{2d} \le \mathbf{P}_{\mathbf{x}}[\tau_{\mathbf{a}^+} = \tau_{\partial B^7}].$$

Using Proposition A.9 again, we get the bound with a new factor for $\mathbf{P}_{\mathbf{a}}[\tau_{\mathbf{x}} < \tau_{\partial B^7}]$. ☐

COROLLARY A.11. *Let $B = B(n)$. There is a $c > 0$ such that for any $\mathbf{a} \in A(B) \cup B^6$, $\mathbf{z} \in Z(B)$ and any $\mathbf{x} \in B^6$,*

$$\mathbf{P}_{\mathbf{a}}[\tau_{\mathbf{x}} < \tau_{\partial B^7} | \tau_{\partial B^7} = \tau_{\mathbf{z}}] > cn^{2-d}. \tag{39}$$

PROOF. By the Markov property,

$$\begin{aligned}\mathbf{P}_{\mathbf{a}}[\tau_{\mathbf{x}} < \tau_{\partial B^7} | \tau_{\partial B^7} = \tau_{\mathbf{z}}]\mathbf{P}_{\mathbf{a}}[\tau_{\partial B^7} = \tau_{\mathbf{z}}] &= \mathbf{P}_{\mathbf{a}}[\tau_{\partial B^7} = \tau_{\mathbf{z}} | \tau_{\mathbf{x}} < \tau_{\partial B^7}]\mathbf{P}_{\mathbf{a}}[\tau_{\mathbf{x}} < \tau_{\partial B^7}] \\ &= \mathbf{P}_{\mathbf{x}}[\tau_{\partial B^7} = \tau_{\mathbf{z}}]\mathbf{P}_{\mathbf{a}}[\tau_{\mathbf{x}} < \tau_{\partial B^7}].\end{aligned}$$

If $\mathbf{a} \in A(B)$, then Lemma A.10 and Corollary A.8 give the bound. If $\mathbf{a} \in B^6$, then Lemma A.10 gives us the LHS is greater than $c\mathbf{P}_{\mathbf{a}}[\tau_{\mathbf{x}} < \tau_{\partial B^7}]$.

For $r > 0$, let $\mathfrak{b}^r = \{\mathbf{x} \in \mathbb{R}^d : \forall i, 1 \le i \le d, |x_i| < r/2\}$ and for $\mathbf{y} \in \mathbb{R}^d$ let $\mathfrak{d}(\mathbf{y}, r) = \{\mathbf{x} \in \mathbb{R}^d : \|\mathbf{x} - \mathbf{y}\|_2 < r\}$. Choose $K(d)$ points $\mathbf{y}_1, \dots, \mathbf{y}_K \in \mathfrak{b}^6$ such that

$\mathfrak{b}^6 \subset \bigcup_{i=1}^K \mathfrak{d}(\mathbf{y}_i, 0.1) \subset \mathfrak{b}^{6.1}$. Let $D_i^\alpha(n) = \mathfrak{d}(n\mathbf{y}_i, \alpha n) \cap \mathbb{Z}^d$. Then for $\alpha \geq 0.1$ and all n

$$B^6 \subset \bigcup_{i=1}^K D_i^\alpha \subset B^{6+\alpha}.$$

Let $p_{\mathbf{a}}(\mathbf{x}) = \mathbf{P_a}[\tau_{\mathbf{x}} < \tau_{\partial B^7}]$. To show $p_{\mathbf{a}}(\mathbf{x}) > cn^{2-d}$ uniformly in $\mathbf{x}, \mathbf{a} \in B^6$, it is enough to show, w.l.o.g., that there is a $c_1 > 0$ such that for any $\mathbf{x} \in D_1^{0.1}$, $\mathbf{a} \in B^6$, $p_{\mathbf{a}}(\mathbf{x}) > c_1 n^{2-d}$. Since $p_{\mathbf{a}}(\mathbf{x})$ is harmonic as a function of $\mathbf{a}$ in $B^7 \setminus \{\mathbf{x}\}$, by the maximum (minimum) principle,

$$\min_{\mathbf{a} \in D_1^{0.2} \setminus \{\mathbf{x}\}} p_{\mathbf{a}}(\mathbf{x}) \geq \min_{\mathbf{a} \in \partial D_1^{0.2} \cup \{\mathbf{x}\}} p_{\mathbf{a}}(\mathbf{x}).$$

Since $p_{\mathbf{x}}(\mathbf{x}) = 1$, and $\partial D_1^{0.2} \subset B^{6.5}$, it is thus enough to lower bound $p_{\mathbf{a}}(\mathbf{x})$ for $\mathbf{a} \in B^{6.5} \setminus D_1^{0.2}$. Since $p_{\mathbf{a}}(\mathbf{x})$ is harmonic and positive in $B^7 \setminus D_1^{0.1}$, by the Harnack principle for $\mathbb{Z}^d$ (Proposition A.6), there is a $c_2(d) > 0$ such that for any $\mathbf{a}, \mathbf{b} \in B^{6.5} \setminus D_1^{0.2}$

$$p_{\mathbf{b}}(\mathbf{x}) \geq c_2 p_{\mathbf{a}}(\mathbf{x}).$$

Thus, it is enough to bound for some fixed $\mathbf{a} \in \partial D_1^{0.2} \cap B^6$. Let $D_* = \mathfrak{d}(\mathbf{a}, 0.6n) \cap \mathbb{Z}^d$ and note that $\mathbf{x} \in D_* \subset B^7$, implying $p_{\mathbf{a}}(\mathbf{x}) \geq \mathbf{P_a}[\tau_{\mathbf{x}} < \tau_{\partial D_*}]$. By Proposition 1.5.9 in [8], since $\mathbf{x} \in \mathfrak{d}(\mathbf{a}, 0.4n) \cap \mathbb{Z}^d$, $\mathbf{P_x}[\tau_{\mathbf{a}} < \tau_{\partial D_*}] \geq cn^{2-d}$, and by Proposition A.9 we are done. $\square$

APPENDIX B: DISTANCE BOUND

In this section, we prove the following theorem.

THEOREM B.1. *Let $\omega_0 \subset \mathcal{T}(N)$. If $\omega_0 \subset \mathcal{T}(N)$ is (N, k, ρ)-good (see Section 2.5) where $k \geq 1$, $\rho > 0$, then there is a $C(k, \rho) < \infty$ such that for all large N and any two vertices $\mathbf{x}, \mathbf{y} \in \omega_0$*

$$d_{\omega_0}(\mathbf{x}, \mathbf{y}) < C d_{\mathcal{T}}(\mathbf{x}, \mathbf{y}) \log^{(k-1)} N + C(\log N)^{4d+2},$$

where $\log^{(m)} N$ is $\log(\cdot)$ iterated m times of N.

We start by reducing from the torus to top-level.boxes. To prove the theorem, it is enough to show that there exists a $C(k, \rho) < \infty$ such that for all large n, any $\omega \in \mathcal{G}_k^\rho(n)$ and any $\mathbf{x}, \mathbf{y} \in \omega \cap b^5(n)$ satisfy

$$d_\omega(\mathbf{x}, \mathbf{y}) < C d_{b^7(n)}(\mathbf{x}, \mathbf{y}) \log^{(k-1)} n + C(\log n)^{4d+2}. \tag{40}$$

Note that while ω_0 is a subgraph of $\mathcal{T}$ as far as graph distance, we require (40) to hold for ω as a subgraph of $\mathbb{Z}^d$ (no wrap around). To see why this is enough, let

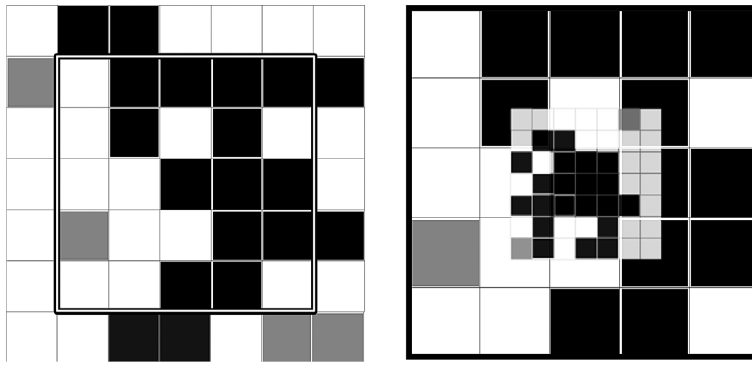

FIG. 7. *On the left is a schematic example of $B^7(n)$ where the black boxes represent $\beta_k = \mathbf{C}_{k-1}(B(n))$ and the gray ones $\mathcal{S}(B, k-1) \setminus \beta_k$. On the right is a blowup of the framed region on the left where the small black boxes are a part of β_{k-1}.*

$\mathbf{x}, \mathbf{y} \in \mathcal{T}(N)$ and set $n = \lceil N/10 \rceil$. First assume there is a top-level box $b_*(\mathbf{a}, n)$ and $\hat{\mathbf{x}}, \hat{\mathbf{y}} \in b_*^3$ such that $\mathbf{x} = \Pi_N(\hat{\mathbf{x}}), \mathbf{y} = \Pi_N(\hat{\mathbf{y}})$. Let $\omega = \Pi_N^{-1}(\omega_0) \cap b_*^7$. Note $d_\omega(\hat{\mathbf{x}}, \hat{\mathbf{y}}) \geq d_{\omega_0}(\mathbf{x}, \mathbf{y})$ but since $\|b_*^3\| < N/2, d_{b_*^7}(\hat{\mathbf{x}}, \hat{\mathbf{y}}) = d_{\mathcal{T}}(\mathbf{x}, \mathbf{y})$. By (40), since b_* is $(\Pi_N^{-1}(\omega_0), k, \rho)$-good by definition, we are done. If no such b_* exists, then by our construction of top-level boxes, $d_{\mathcal{T}}(\mathbf{x}, \mathbf{y}) > n$. Let $b_{\mathbf{x}}, b_{\mathbf{y}}$ be the top-level boxes such that $\mathbf{x} \in b_{\mathbf{x}}, \mathbf{y} \in b_{\mathbf{y}}$. We can make a Δ-connected path of top-level boxes from $\mathbf{x}$ to $\mathbf{y}$ of length at most $10d$. Since b_1, b_2 that are Δ-neighbors satisfy that $b_1 \subset b_2^5$, by Remark 2.6, (40) implies the theorem.

To simplify notation, we fix k, ρ, n and $\omega \in \mathcal{G}_k^\rho(n)$ for the remainder of the section. We write $\mathcal{G}_i$ (resp., i-good) for $\mathcal{G}_i^{\rho\Lambda^{k-i}}$ [resp., $(\omega, i, \rho\Lambda^{k-i})$-good].

We now utilize the recursive goodness properties of ω to extract a single connected cluster of ω which is a power of log ω-distance from its complement in ω and is "nicely" embedded in $\mathbb{Z}^d$. Given an $(i+1)$-good box B where $0 \leq i < k$, we write

$$\mathcal{S}(B, i) = \{b \in \sigma(B) : b \text{ is } i\text{-good}\},$$

and let $\sigma_B = \|\Delta(\sigma(B))\| = |\sigma(B)|^{1/d}$. Since B is $(i+1)$-good, by definition we have that $\Delta\mathcal{S}(B, i) \in \mathcal{P}(\sigma_B)$. Thus, there exists a good cluster $\mathcal{C}(\Delta(\mathcal{S}(B, i)))$ satisfying Percolation properties 1, 2, 3 (see Section 2.3). Let $\mathbf{C}_i(B) \subset \sigma(B)$ be the set for which $(\Delta\mathbf{C}_i(B)) = \mathcal{C}(\Delta(\mathcal{S}(B, i)))$. For $i = 0, \dots, k$ let us define $\beta_i = \beta_i(\omega, n)$. Set $\beta_k = \{B(n)\}$ and for $i = k-1, \dots, 0$ recursively define $\beta_i = \{b \in \mathbf{C}_i(B) : B \in \beta_{i+1}\}$. See Figure 7 for a schematic illustration.

Let $n_j = s^{(k-j)}(n)$. Thus, for $b \in \beta_j$ we have $\|b\| = n_j$ and also $|\sigma(b)|^{1/d} < 6n_j/n_{j-1}$ for all large n. Note that by Percolation property 1, $\{\beta_j(n)\}_{j=0}^k$ are nonempty for all large n. Roughly, $\bigcup \beta_0$ is the nicely embedded cluster referred to above. Its precise properties follow.

Given an $(i+1)$-good box B, let $\mathbf{C}_i^5(B) = \{b \in \mathbf{C}_i(B) : b \cap B^5 \neq \varnothing\}$.

LEMMA B.2. *Set $b_k = B(n)$. There is a $C(k)$ such that for any $\mathbf{x}_k \in b_k^5 \cap \omega$, there are boxes $\{b_i\}_{i=0}^{k-1}$ satisfying*: (i) $b_i \in \mathbf{C}_i^5(b_{i+1}) \subset \beta_i$, *and* (ii) *there is a* $\mathbf{x}_0 \in \omega \cap b_0$ *such that*

$$d_\omega(\mathbf{x}_k, \mathbf{x}_0) < C(k)(\log n)^{4d+2}.$$

PROOF. We use backward induction. For $1 \le j \le k$, we prove that if $\mathbf{x}_j \in B_j^5 \cap \omega$ where $B_j \in \beta_j$, then there is a $b_{j-1} \in \mathbf{C}_{j-1}^5(B_j) \subset \beta_{j-1}$ and a $\mathbf{x}_{j-1} \in b_{j-1}$ satisfying

$$d_\omega(\mathbf{x}_j, \mathbf{x}_{j-1}) < c(d) n_{j-1}^d (\log n_j)^2. \tag{41}$$

Since the conditions of the lemma provide us with an initial $\mathbf{x}_k \in B^5(n)$ where by definition $B(n) \in \beta_k$, the bound on $d_\omega(\mathbf{x}_k, \mathbf{x}_0)$ is proved by connecting $\mathbf{x}_k, \mathbf{x}_{k-1}, \dots, \mathbf{x}_0$.

We assumed $B_j \in \beta_j$, so in particular, B_j is j-good. Let $b_* \in \sigma(B_j)$ be the subbox of B_j containing $\mathbf{x}_j$ and assume $b_* \notin \mathbf{C}_{j-1}^5(B_j)$ as otherwise we are done. Consider

$$\mathbf{B} = \{b \in B_\Delta(b_*, \log|\sigma(B_j)|) : b \cap B_j^5 \neq \varnothing\}.$$

Since $b_* \cap B_j^5 \neq \varnothing$ by assumption, $|\mathbf{B}| > \log^2 |\sigma(B_j)|$, and thus by Percolation property 2 (see Section 2.3), there is a $b_{j-1} \in \mathbf{B} \cap \mathbf{C}_{j-1}^5(B_j)$. Thus, there is a Δ-path $\mathbf{p} \subset \sigma(B_j)$ of length at most $d \log |\sigma(B_j)|$ starting at b_* and ending at b_{j-1}. By Remark 2.6 on $\mathcal{G}_0^\rho$ (see Section 2.4), for any Δ-neighboring boxes b_α, b_β in the path, $\omega \cap b_\alpha$ is connected to $\omega \cap b_\beta$ in $\omega \cap b_\alpha^+$. Choosing some $\mathbf{x}_{j-1} \in \omega \cap b_{j-1}$ and using the volume of $\bigcup_{b \in \mathbf{p}} b^7$ as a trivial distance bound, we get (41). □

For $0 \le j < k$, note that although $\|b_1\| = \|b_2\|$ for any $b_1, b_2 \in \beta_j$, since they can be subboxes of different B_1, B_2, b_1 is not in general an element of $\mathrm{sp}\{b_2\}$. Thus, for each $0 \le j \le k$, we add a graph structure to β_j by defining a neighbor relation ($\overset{5}{\sim}$) between boxes $b_1, b_2 \in \beta_j$. We define that $b_1 \overset{5}{\sim} b_2$ if and only if $b_1 \subset b_2^5$ and $b_2 \subset b_1^5$. Note this relation is reflexive, and that for $(j+1)$-good B and $b_1, b_2 \in \mathbf{C}_j(B)$, $d_{\beta_j}(b_1, b_2) \le d_\Delta(b_1, b_2)$. For the remainder of the section, any graph properties of β_j referred to, such as connectivity or distance, use the graph structure created by $\overset{5}{\sim}$.

LEMMA B.3. *There is a C_d such that for each $0 < j \le k$, if $B_1, B_2 \in \beta_j$ are $\overset{5}{\sim}$-connected, and we have $b_1 \in \mathbf{C}_{j-1}^5(B_1)$, $b_2 \in \mathbf{C}_{j-1}^5(B_2)$, then $d_{\beta_{j-1}}(b_1, b_2) < C_d(d_{\beta_j}(B_1, B_2) \vee 1) n_j / n_{j-1}$. In particular, b_1, b_2 are $\overset{5}{\sim}$-connected in β_{j-1}.*

PROOF. We prove the lemma for the special case of $B_1 \overset{5}{\sim} B_2$ [i.e., $d_{\beta_j}(B_1, B_2) \le 1$]. The general lemma follows by applying the neighbor case over a path in β_j realizing the $\overset{5}{\sim}$-distance between two fixed boxes. By definition, $\mathbf{C}_{j-1}(B_1)$ and $\mathbf{C}_{j-1}(B_2)$ are each Δ-connected sets, and thus $\overset{5}{\sim}$-connected. By Percolation property 3, for $i = 1, 2$ and any $b, b' \in \mathbf{C}^5_{j-1}(B_i)$, $d_{\mathbf{C}_{j-1}(B_i)}(b, b') < C_d n_j/n_{j-1}$. Since $\mathbf{C}_{j-1}(B_i) \subset \beta_{j-1}$ and $\overset{5}{\sim}$-distance is at most Δ-distance, to complete the proof it is enough to show existence of $\hat{b}_1 \in \mathbf{C}^5_{j-1}(B_1)$ and $\hat{b}_2 \in \mathbf{C}^5_{j-1}(B_2)$ such that $\hat{b}_1 \subset \hat{b}^5_2$ and $\hat{b}_2 \subset \hat{b}^5_1$. For $i = 1, 2$, let $\mathbf{D}_i = \{b \in \sigma(B_i) : b \subset B_2\}$ and let $\mathbf{E}_i = \mathbf{D}_i \cap \mathbf{C}_{j-1}(B_i)$. Let $D_i = \bigcup \mathbf{D}_i$ and let $E_i = \bigcup \mathbf{E}_i$. Since $E_i \subset D_i$, we have $B_2 \setminus E_i = (B_2 \setminus D_i) \cup (D_i \setminus E_i)$. By a volume bound, $|B_2 \setminus D_i| \le 2dn_{j-1}n_j^{d-1}$ and by Percolation property 1, $|\mathbf{D}_i \setminus \mathbf{E}_i| < 10^{-d}|\sigma(B_i)|$. Since $|\sigma(B_i)|^{1/d} < 6n_j/n_{j-1}$, this implies $|D_i \setminus E_i| < (0.6n_j)^d$. As $|B_2| = n_j^d$ and $d > 2$, we have by the bound on $|B_2 \setminus E_i|$ for $i = 1, 2$ that there is a $\mathbf{x} \in E_1 \cap E_2$. The containing boxes $\mathbf{x} \in \hat{b}_i \in \mathbf{C}_{j-1}(B_i)$ for $i = 1, 2$ are thus $\overset{5}{\sim}$-neighbors. $\square$

We now prove the theorem by showing there exists a $C(k, \rho) < \infty$ such that for any $\mathbf{x}, \mathbf{y} \in \omega \cap B^5(n)$, (40) holds for all large n.

PROOF OF THEOREM B.1. We demonstrate there is a path from $\mathbf{x}$ to $\mathbf{y}$ in ω shorter than the RHS of (40). Let $b_{\mathbf{x},k} = B(n)$ and apply Lemma B.2 to $\mathbf{x}$ to get boxes $\{b_{\mathbf{x},i}\}_{i=0}^{k-1}$ satisfying: (i) $b_{\mathbf{x},i} \in \mathbf{C}^5_i(b_{\mathbf{x},i+1}) \subset \beta_i$, and (ii) there is a $\mathbf{x}_0 \in \omega \cap b_{\mathbf{x},0}$ such that $d_\omega(\mathbf{x}, \mathbf{x}_0) < C(k)(\log n)^{4d+2}$. Observe that (i) implies $\mathbf{x}_0 \in b^6_{\mathbf{x},k-1}$ for all large n. Set $b_{\mathbf{y},k} = B(n)$ and apply the lemma to $\mathbf{y}$ as well to get $b_{\mathbf{y},i}$ and $\mathbf{y}_0$ with analogous properties.

By Lemma B.3, β_{k-1} is $\overset{5}{\sim}$-connected, and more specifically,

$$d_{\beta_{k-2}}(b_{\mathbf{x},k-2}, b_{\mathbf{y},k-2}) < C_d\big(d_{\beta_{k-1}}(b_{\mathbf{x},k-1}, b_{\mathbf{y},k-1}) \vee 1\big)\frac{n_{k-1}}{n_{k-2}}.$$

Iterating the lemma, we get

$$d_{\beta_0}(b_{\mathbf{x},0}, b_{\mathbf{y},0}) < C_d^{k-1}\big(d_{\beta_{k-1}}(b_{\mathbf{x},k-1}, b_{\mathbf{y},k-1}) \vee 1\big)\frac{n_{k-1}}{n_0}. \tag{42}$$

Since $b_{\mathbf{x},k-1}, b_{\mathbf{y},k-1} \in \mathbf{C}^5_{k-1}(B(n))$, by Percolation property 3

$$d_{\mathbf{C}_{k-1}(B(n))}(b_{\mathbf{x},k-1}, b_{\mathbf{y},k-1}) < c_a d_{\sigma(B(n))}(b_{\mathbf{x},k-1}, b_{\mathbf{y},k-1}) \vee c_a \log\frac{n_k}{n_{k-1}}. \tag{43}$$

Where both are defined, $\overset{5}{\sim}$-distance is at most Δ-distance, and thus we may replace $d_{\mathbf{C}_{k-1}(B(n))}(\cdot,\cdot)$ in (43) by $d_{\beta_{k-1}}(\cdot,\cdot)$. Since $n_{k-1} \cdot d_{\sigma(B(n))}(\cdot,\cdot)$ and $d_{B^7(n)}(\cdot,\cdot)$ are comparable, and using that $\mathbf{x}_0 \in b^6_{\mathbf{x},k-1}$, $\mathbf{y}_0 \in b^6_{\mathbf{y},k-1}$ we have

$$n_{k-1}d_{\beta_{k-1}}(b_{\mathbf{x},k-1}, b_{\mathbf{y},k-1}) < c'_a\big(d_{B^7(n)}(\mathbf{x}_0, \mathbf{y}_0) \vee n_{k-1}\log n_k\big).$$

Plugging this into (42), we get

$$d_{\beta_0}(b_{\mathbf{x},0}, b_{\mathbf{y},0}) < C(k)\big(d_{B^7(n)}(\mathbf{x}_0, \mathbf{y}_0) \vee \log^5 n\big)/n_0.$$

By properties of $\mathcal{G}_0^\rho$ (see Section 2.4), vertices in $\overset{5}{\sim}$-neighboring boxes in β_0 are connected in ω in a path which is at most twice the volume of one box, and thus we get

$$d_\omega(\mathbf{x}_0, \mathbf{y}_0) < C(k)\big(d_{B^7(n)}(\mathbf{x}_0, \mathbf{y}_0) \vee \log^5 n\big)(n_0)^{d-1}.$$

We pay a $C(\log n)^{4d+2}$ term to connect $\mathbf{x}$, $\mathbf{y}$ to $\mathbf{x}_0$, $\mathbf{y}_0$, respectively. This terms also absorbs the $(n_0)^{d-1} \log^5 n$ factor above. Since $(n_0)^{d-1}$ is $o(\log^{(k-1)} n)$, we are done. □

APPENDIX C: RANDOM INTERLACEMENTS NOTATION

We try to follow as much as possible the canonical notation of Alain-Sol Sznitman [17]. Let W and W_+ be the spaces of doubly infinite and infinite trajectories in $\mathbb{Z}^d$ that spend only a finite amount of time in finite subsets of $\mathbb{Z}^d$:

$$W = \Big\{\gamma : \mathbb{Z} \to \mathbb{Z}^d; \big|\gamma(n) - \gamma(n+1)\big| = 1, \forall n \in \mathbb{Z}; \lim_{n\to\pm\infty} \big|\gamma(n)\big| = \infty\Big\},$$

$$W_+ = \Big\{\gamma : \mathbb{N} \to \mathbb{Z}^d; \big|\gamma(n) - \gamma(n+1)\big| = 1, \forall n \in \mathbb{Z}; \lim_{n\to\infty} \big|\gamma(n)\big| = \infty\Big\}.$$

The canonical coordinates on W and W_+ will be denoted by $X_n, n \in \mathbb{Z}$ and $X_n, n \in \mathbb{N}$, respectively. Here, we use the convention that $\mathbb{N}$ includes 0. We endow W and W_+ with the sigma-algebras $\mathcal{W}$ and $\mathcal{W}_+$, respectively, which are generated by the canonical coordinates. For $\gamma \in W$, let range $(\gamma) = \gamma(\mathbb{Z})$. Furthermore, consider the space W^* of trajectories in W modulo time shift:

$$W^* = W/\sim \qquad \text{where } w \sim w' \iff w(\cdot) = w'(\cdot + k) \text{ for some } k \in \mathbb{Z}.$$

Let π^* be the canonical projection from W to W^*, and let $\mathcal{W}^*$ be the sigma-algebra on W^* given by $\{A \subset W^* : (\pi^*)^{-1}(A) \in \mathcal{W}\}$. Given $K \subset \mathbb{Z}^d$ and $\gamma \in W_+$, let $\tilde{H}_K(\gamma)$ denote the hitting time of K by γ:

$$\tilde{H}_K(\gamma) = \inf\{n \ge 1 : X_n(\gamma) \in K\}. \tag{44}$$

For $x \in \mathbb{Z}^d$, let P_x be the law on $(W_+, \mathcal{W}_+)$ corresponding to simple random walk started at x, and for $K \subset \mathbb{Z}^d$, let P_x^K be the law of simple random walk, conditioned on not hitting K. Define the equilibrium measure of K:

$$e_K(x) = \begin{cases} P_x[\tilde{H}_K = \infty], & x \in K, \\ 0, & x \notin K. \end{cases} \tag{45}$$

Define the capacity of a set $K \subset \mathbb{Z}^d$ as

$$\operatorname{cap}(K) = \sum_{x\in\mathbb{Z}^d} e_K(x). \tag{46}$$

Next, we define a Poisson point process on $W^* \times \mathbb{R}_+$. The intensity measure of the Poisson point process is given by the product of a certain measure ν and the Lebesque measure on $\mathbb{R}_+$. The measure ν was constructed by Sznitman in [17], and now we characterize it. For $K \subset \mathbb{Z}^d$, let W_K denote the set of trajectories in W that enter K. Let $W_K^* = \pi^*(W_K)$ be the set of trajectories in W^* that intersect K. Define Q_K to be the finite measure on W_K such that for $A, B \in \mathcal{W}_+$ and $x \in \mathbb{Z}^d$,

$$Q_K\big[(X_{-n})_{n\geq 0} \in A, X_0 = x, (X_n)_{n\geq 0} \in B\big] = P_x^K[A]e_K(x)P_x[B]. \tag{47}$$

The measure ν is the unique σ-finite measure such that

$$\mathbb{1}_{W_K^*}\nu = \pi^* \circ Q_K \qquad \forall K \subset \mathbb{Z}^d \text{ finite.} \tag{48}$$

The existence and uniqueness of the measure was proved in Theorem 1.1 of [17]. Consider the set of point measures in $W^* \times \mathbb{R}_+$:

$$\begin{aligned} \Omega = \Bigg\{&\omega = \sum_{i=1}^{\infty} \delta_{(w_i^*, u_i)}; w_i^* \in W^*, u_i \in \mathbb{R}_+, \\ &\omega\big(W_K^* \times [0, u]\big) < \infty, \text{ for every finite } K \subset \mathbb{Z}^d \text{ and } u \in \mathbb{R}_+\Bigg\}. \end{aligned} \tag{49}$$

Also consider the space of point measures on W^*:

$$\tilde{\Omega} = \Bigg\{\sigma = \sum_{i=1}^{\infty} \delta_{w_i^*}; w_i^* \in W^*, \sigma\big(W_K^*\big) < \infty, \text{ for every finite } K \subset \mathbb{Z}^d\Bigg\}. \tag{50}$$

For $u > u' \geq 0$, we define the mapping $\omega_{u',u}$ from Ω into $\tilde{\Omega}$ by

$$\omega_{u',u} = \sum_{i=1}^{\infty} \delta_{w_i^*}\mathbb{1}\{u' \leq u_i \leq u\} \qquad \text{for } \omega = \sum_{i=1}^{\infty} \delta_{(w_i^*, u_i)} \in \Omega. \tag{51}$$

If $u' = 0$, we write ω_u. On Ω we let $\mathbb{P}$ be the law of a Poisson point process with intensity measure given by $\nu(dw^*)\,dx$. Observe that under $\mathbb{P}$, the point process $\omega_{u,u'}$ is a Poisson point process on $\tilde{\Omega}$ with intensity measure $(u - u')\nu(dw^*)$. Given $\sigma \in \tilde{\Omega}$, we define

$$\mathcal{I}(\sigma) = \bigcup_{w^* \in \operatorname{supp}(\sigma)} \operatorname{range}(w^*). \tag{52}$$

For $0 \leq u' \leq u$, we define

$$\mathcal{I}^{u',u} = \mathcal{I}(\omega_{u',u}), \tag{53}$$

which we call the *random interlacement set* between levels u' and u. In case $u' = 0$, we write $\mathcal{I}^u$.

Finally, we can define the measure of the random walk described in Theorem 2.3. Let $\mathbb{P}_0^u[\cdot] = \mathbb{P}[\cdot|0 \in \mathcal{I}^u]$. For every $\mathcal{I}^u$ distributed according to $\mathbb{P}_0^u$, let $\mathbf{P}_0^u$ be the law of a SRW on $\mathcal{I}^u$ starting from 0.

APPENDIX D: INDEX OF SYMBOLS BY ORDER OF APPEARANCE

Symbol	Page	Definition
$\mathcal{T}(N,d)$	1593	d-dimensional torus.
Π_N	1593	For $x \in \mathbb{Z}^d$, $\mathcal{O}_N(x) = (x_1 \bmod N, \dots, x_d \bmod N)$.
$\mathcal{R}(t)$	1593	The range of SRW on the torus.
∂	1594	Outer vertex boundary.
∂^{in}	1594	Inner vertex boundary.
$B(\mathbf{x}, n)$	1594	$\{\mathbf{y} \in \mathbb{Z}^d : \forall i, 1 \le i \le d, -n/2 \le \mathbf{x}(i) - \mathbf{y}(i) < n/2\}$.
$\text{sp}\{B(\mathbf{x}, n)\}$	1595	$\{B(\mathbf{x} + \sum_i \mathbf{e}_i k_i n, n) : (k_1, \dots, k_d) \in \mathbb{Z}^d\}$.
Δ	1595	The isomorphism, $\Delta \colon \text{sp}\{B\} \to \mathbb{Z}^d$.
B^α	1595	For a box $B = B(x,n)$, $B^\alpha = B(x, \alpha n)$.
$s(n)$	1595	$\lceil \log n \rceil^4$.
$s^{(i)}(n)$	1595	$s(\cdot)$ iterated i times.
$\sigma(B(\mathbf{x}, n))$	1595	$\text{sp}\{b(\mathbf{x}, s(n))\} \cap \{b(\mathbf{y}, s(n)) : \mathbf{y} \in B(x, 5n + 3\lceil \log n \rceil^6)\}$.
$\mathcal{P}(n)$	1595	Percolation configurations.
$\mathcal{G}_k^\rho(n)$	1596	k-good configurations.
$\hat{\phi}(r)$	1601	$\inf\{\Phi_S : N^{1/3} < \pi(S) \le r \wedge (1 - 1/4d)\lvert\omega_0\rvert\}$.
Φ_S	1602	$\frac{Q(S,S^c)}{\pi(S)}$.
$\Phi(u)$	1602	$\inf\{\Phi_S : 0 < \pi(S) \le u \wedge \frac{1}{2}\}$.
Top, Bot	1607	Top and bottom projections of B^3 on B^7.
B-*traversal*	1607	An ordered pair $\eta = (a, z)$, $a \in \text{Top}$, $z \in \text{Bot}$.
B-*itinerary*	1607	An ordered sequence of B-*traversals*.
$\tau_\rho(b)$	1607	$\tau_\rho(b) = \gamma^+_{\lceil \rho n^{d-2} \rceil}$.
$\mathcal{D}^\rho_{\Lambda\rho}$	1610, 1617	Each $b \in \sigma(B)$ is traversed top to bottom at least $\Lambda\rho\lVert b\rVert^{d-2}$ times.
$\mathcal{F}_N^T(b, k, \rho)$	1615	The event $\{b \cap \mathcal{O}_N^{-1} \circ \mathcal{R}_N(t) \in \mathcal{G}_k^\rho(n) : \forall t \ge T\}$.

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