

BAKRY-ÉMERY CURVATURE-DIMENSION CONDITION AND RIEMANNIAN RICCI CURVATURE BOUNDS

BY LUIGI AMBROSIO^{1,2}, NICOLA GIGLI AND GIUSEPPE SAVARÉ²

*Scuola Normale Superiore, Pisa, Université Pierre et Marie Curie–Paris 6
and Università di Pavia*

The aim of the present paper is to bridge the gap between the Bakry–Émery and the Lott–Sturm–Villani approaches to provide synthetic and abstract notions of lower Ricci curvature bounds.

We start from a strongly local Dirichlet form \mathcal{E} admitting a *Carré du champ* Γ in a Polish measure space (X, \mathfrak{m}) and a canonical distance $d_{\mathcal{E}}$ that induces the original topology of X . We first characterize the distinguished class of *Riemannian Energy measure spaces*, where \mathcal{E} coincides with the Cheeger energy induced by $d_{\mathcal{E}}$ and where every function f with $\Gamma(f) \leq 1$ admits a continuous representative.

In such a class, we show that if \mathcal{E} satisfies a suitable weak form of the *Bakry–Émery curvature dimension condition* $\text{BE}(K, \infty)$ then the metric measure space (X, d, \mathfrak{m}) satisfies the Riemannian Ricci curvature bound $\text{RCD}(K, \infty)$ according to [*Duke Math. J.* **163** (2014) 1405–1490], thus showing the equivalence of the two notions.

Two applications are then proved: the tensorization property for Riemannian Energy spaces satisfying the Bakry–Émery $\text{BE}(K, N)$ condition (and thus the corresponding one for $\text{RCD}(K, \infty)$ spaces without assuming non-branching) and the stability of $\text{BE}(K, N)$ with respect to Sturm–Gromov–Hausdorff convergence.

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1. Introduction. Besides its obvious geometric relevance, spaces with Ricci curvature bounded from below play an important role in many probabilistic and analytic investigations that reveal various deep connections between different fields.

Starting from the celebrated paper by Bakry–Émery [12], the curvature-dimension condition based on the Γ_2 -criterion in Dirichlet spaces provides crucial tools for proving refined estimates on Markov semigroups and many functional inequalities, of Poincaré, Log-Sobolev, Talagrand and concentration type (see, e.g., [8, 11, 13, 36–38]).

This general functional-analytic approach is also well suited to deal with genuinely infinite dimensional settings with applications to Wiener measure on the paths of Brownian motion with values in a Riemannian manifold, as in, for example, [14]. In fact, Ricci curvature also arises in Bismut-type formula [30] and its applications to gradient estimates [9, 10], and to the construction of couplings between Brownian motions [32, 35].

The importance of curvature bounds in the framework of optimal transport has been deeply analyzed in [21, 45, 56]. These and other important results led Sturm [53, 54] and Lott–Villani [41] to introduce a new synthetic notion of the curvature-dimension condition, in the general framework of metric-measure spaces.

The aim of the present paper is to bridge the gap between the Bakry–Émery and the Lott–Sturm–Villani approaches to provide synthetic and abstract notions of lower Ricci curvature bounds. In order to make this statement more precise, let us briefly review the main points of both settings.

The Bakry–Émery condition $\text{BE}(K, N)$: Dirichlet forms and Γ -calculus. The first approach is based on the functional Γ -calculus developed by Bakry–Émery since [12]; see [11, 15].

A possible starting point is a local and symmetric Dirichlet form \mathcal{E} on the measure space $(X, \mathcal{B}, \mathfrak{m})$ with dense domain $D(\mathcal{E}) \subset L^2(X, \mathfrak{m})$, and the associated Markov semigroup $(\mathbb{P}_t)_{t \geq 0}$ on $L^2(X, \mathfrak{m})$ with generator $\Delta_{\mathcal{E}}$ (general references are [18, 25, 42]). In a suitable algebra \mathcal{A} of functions dense in the domain $D(\Delta_{\mathcal{E}})$ of $\Delta_{\mathcal{E}}$ one introduces the *Carré du champ*

$$\Gamma(f, g) := \frac{1}{2}(\Delta_{\mathcal{E}}(fg) - f \Delta_{\mathcal{E}}g - g \Delta_{\mathcal{E}}f), \quad f, g \in \mathcal{A},$$

related to \mathcal{E} by the local representation formula

$$\mathcal{E}(f, g) = \int_X \Gamma(f, g) \, dm \quad \text{for every } f, g \in \mathcal{A}.$$

One also assumes that $\Delta_{\mathcal{E}}$ is a diffusion operator, that is, with the notation $\Gamma(f) := \Gamma(f, f)$, it holds

$$\Delta_{\mathcal{E}}\phi(f) = \phi'(f)\Delta_{\mathcal{E}}f + \phi''(f)\Gamma(f)$$

for every $f \in \mathcal{A}$, $\phi \in C^2(\mathbb{R})$ with bounded derivatives.

The model example is provided by a smooth Riemannian manifold (\mathbb{M}^d, g) endowed with the measure $m := e^{-V} \text{Vol}_g$ for a given smooth potential $V : \mathbb{M}^d \rightarrow \mathbb{R}$. In this case, one typically chooses $\mathcal{A} = C_c^\infty(\mathbb{M}^d)$ and

$$\mathcal{E}(f, g) = \int_{\mathbb{M}^d} \langle \nabla f, \nabla g \rangle_g \, dm \quad \text{so that}$$

$$\Gamma(f) = |\nabla f|_g^2 \quad \text{and} \quad \Delta_{\mathcal{E}} = \Delta_g - \langle \nabla V, \nabla \cdot \rangle_g,$$

where Δ_g is the usual Laplace–Beltrami operator on \mathbb{M} . This fundamental example shows that Γ carries the metric information of \mathbb{M}^d , since one can recover the Riemannian distance d_g in \mathbb{M}^d by the formula

$$(1.1) \quad d_g(x, y) = \sup\{\psi(y) - \psi(x) : \psi \in \mathcal{A}, \Gamma(\psi) \leq 1\}, \quad x, y \in \mathbb{M}^d.$$

A further iteration yields the Γ_2 operator, defined by

$$(1.2) \quad 2\Gamma_2(f) = \Delta_{\mathcal{E}}\Gamma(f) - 2\Gamma(f, \Delta_{\mathcal{E}}f), \quad f \in \mathcal{A}.$$

In the above example, Bochner’s formula yields

$$\Gamma_2(f) = \|\text{Hess}_g f\|_g^2 + (\text{Ric}_g + \text{Hess}_g V)(\nabla f, \nabla f),$$

and one obtains the fundamental inequality

$$(1.3) \quad \Gamma_2(f) \geq K\Gamma(f) + \frac{1}{N}(\Delta_{\mathcal{E}}f)^2 \quad \text{for every } f \in \mathcal{A},$$

if the quadratic form associated to the tensor $\text{Ric}_g + \text{Hess}_g V$ is bounded from below by $Kg + \frac{1}{N-d}\nabla V \otimes \nabla V$ for some $K \in \mathbb{R}$ and $N > d$. When $V \equiv 0$, it is possible to show that (\mathbb{M}^d, g) has Ricci curvature bounded from below by K iff (1.3) is satisfied for $N \geq d$.

It is then natural to use (1.3) as a definition of curvature-dimension bounds even in the abstract setting: it is the so-called *Bakry–Émery curvature-dimension condition*, that we denote here by $\text{BE}(K, N)$.

One of the most remarkable applications of (1.3) is provided by pointwise gradient estimates for the Markov semigroup (see, e.g., [11, 15] for relevant and deep applications). Considering here only the case $N = \infty$, (1.3) yields

$$(1.4) \quad \Gamma(P_t f) \leq e^{-2Kt} P_t(\Gamma(f)) \quad \text{for every } f \in \mathcal{A},$$

a property that is essentially equivalent to $\text{BE}(K, \infty)$ (we refer to [57] for other formulations of $\text{BE}(K, N)$ for Riemannian manifolds, see also the next Section 2.2) and involves only first order “differential” operators.

Up to the choice of an appropriate functional setting [in particular, the algebra \mathcal{A} and the distance d associated to Γ as in (1.1) play a crucial role], Γ -calculus and curvature-dimension inequalities provide a very powerful tool to establish many functional inequalities and geometric properties, often in sharp form.

Lower Ricci curvature bounds by optimal transport: The $\text{CD}(K, \infty)$ condition. A completely different approach to lower Ricci bounds has been recently proposed by Sturm [53, 54] and Lott–Villani [41]: here, the abstract setting is provided by metric measure spaces (X, d, m) , where (X, d) is a separable, complete and length metric space and m is a nonnegative σ -finite Borel measure. Just for simplicity, in this Introduction we also assume $m(X) < \infty$, but the theory covers the case of a measure satisfying the exponential growth condition $m(B_r(x)) \leq M \exp(cr^2)$ for some constants $M, c \geq 0$.

The Lott–Sturm–Villani theory (LSV in the following) is based on the notion of displacement interpolation [43], a powerful tool of optimal transportation that allows one to extend the notion of geodesic interpolation from the state space X to the space of Borel probability measures $\mathcal{P}_2(X)$ with finite quadratic moment. Considering here only the case $N = \infty$, a metric measure space (X, d, m) satisfies the LSV lower Ricci curvature bound $\text{CD}(K, \infty)$ if the relative entropy functional

$$(1.5) \quad \text{Ent}_m(\mu) := \int_X f \log f \, dm, \quad \mu = f m,$$

is displacement K -convex in the Wasserstein space $(\mathcal{P}_2(X), W_2)$ (see [3, 55] and Section 3.1 below). This definition is consistent with the Riemannian case [56], and thus equivalent to $\text{BE}(K, \infty)$ in such a smooth framework.

Differently from the Bakry–Émery’s approach, the LSV theory does not originally involve energy functionals or Markov semigroups, but it is intimately connected to the metric d (through the notion of displacement interpolation) and to the measure m [through the entropy functional (1.5)]. Besides many useful geometric and functional applications of this notion [26, 40, 48], one of its strongest features is its stability under measured Gromov–Hausdorff convergence [24], also in the weaker transport-formulation proposed by Sturm [54].

Starting from the $\text{CD}(K, \infty)$ assumption, one can then construct an evolution semigroup $(H_t)_{t \geq 0}$ on the convex subset of $\mathcal{P}_2(X)$ given by probability measures with finite entropy [26]: it is the metric gradient flow of the entropy functional in $\mathcal{P}_2(X)$ [3]. Since also Finsler geometries (as in the flat case of \mathbb{R}^d endowed with a non-Euclidean norm) can satisfy the $\text{CD}(K, \infty)$ condition, one cannot hope in such a general setting that H_t are linear operators. Still, $(H_t)_{t \geq 0}$ can be extended to a continuous semigroup of contractions in $L^2(X, m)$ (and in any $L^p(X, m)$ -space), which can also be characterized as the $L^2(X, m)$ -gradient flow $(P_t)_{t \geq 0}$ of

a convex and 2-homogeneous functional, the Cheeger energy [6, 20], Section 4.1, Remark 4.7,

$$(1.6) \quad \begin{aligned} & \text{Ch}(f) \\ & := \inf \left\{ \liminf_{n \rightarrow \infty} \frac{1}{2} \int_X |Df_n|^2 \, dm : f_n \in \text{Lip}_b(X), f_n \rightarrow f \text{ in } L^2(X, m) \right\} \end{aligned}$$

[here, $\text{Lip}_b(X)$ denotes the space of Lipschitz and bounded real functions defined in X and $|Df|$ is the local Lipschitz constant, or slope, of the Lipschitz function f , see Section 3.1].

The remarkable identification between $(H_t)_{t \geq 0}$ and $(P_t)_{t \geq 0}$ has been first proposed and proved in Euclidean spaces by a seminal paper of Jordan–Kinderlehrer–Otto [31] and then extended to Riemannian manifolds [23, 55], Hilbert spaces [7], Finsler spaces [44], Alexandrov spaces [28] and eventually to $\text{CD}(K, \infty)$ metric measure spaces [6].

Spaces with Riemannian Ricci curvature bounded from below: The $\text{RCD}(K, \infty)$ condition. Having the energy functional (1.6) and the contraction semigroup $(P_t)_{t \geq 0}$ at our disposal, it is then natural to investigate when LSV spaces satisfy $\text{BE}(K, \infty)$. In order to attack this question, one has of course to clarify when the Cheeger energy (1.6) is a Dirichlet (thus *quadratic*) form on $L^2(X, m)$ [or, equivalently, when $(P_t)_{t \geq 0}$ is a semigroup of *linear* operators] and when this property is also stable under Sturm–Gromov–Hausdorff convergence.

One of the most important results of [5] (see also [2] for general σ -finite measures) is that $\text{CD}(K, \infty)$ spaces with a quadratic Cheeger energy can be equivalently characterized as those metric measure spaces where there exists the Wasserstein gradient flow $(H_t)_{t \geq 0}$ of the entropy functional (1.5) in the EVI_K -sense. This condition means that for all initial data $\mu \in \mathcal{P}_2(X)$ with $\text{supp } \mu \subset \text{supp } m$ there exists a locally Lipschitz curve $t \mapsto H_t \mu \in \mathcal{P}_2(X)$ satisfying the evolution variational inequality:

$$(1.7) \quad \frac{d}{dt} \frac{W_2^2(H_t \mu, \nu)}{2} + \frac{K}{2} W_2^2(H_t \mu, \nu) + \text{Ent}_m(H_t \mu) \leq \text{Ent}_m(\nu)$$

for a.e. $t \in (0, \infty)$

for all $\nu \in \mathcal{P}_2(X)$ with $\text{Ent}_m(\nu) < \infty$.

Such a condition is denoted by $\text{RCD}(K, \infty)$ and it is stronger than $\text{CD}(K, \infty)$, since the existence of an EVI_K flow solving (1.7) yields both the geodesic K -convexity of the entropy functional Ent_m [22] and the linearity of $(H_t)_{t \geq 0}$ [5], Theorem 5.1, but it is still stable under Sturm–Gromov–Hausdorff convergence. When it is satisfied, the metric measure space (X, d, m) is called in [5] a space with *Riemannian Ricci curvature bounded from below by K* .

In $\text{RCD}(K, \infty)$ -spaces the Cheeger energy is associated to a strongly local Dirichlet form $\mathcal{E}_{\text{Ch}}(f, f) := 2\text{Ch}(f)$ admitting a Carré du champ Γ . With the calculus tools developed in [5], it can be proved that Γ has a further equivalent repre-

sentation $\Gamma(f) = |Df|_w^2$ in terms of the *minimal weak gradient* $|Df|_w$ of f . The latter is the element of minimal L^2 -norm among all the possible weak limits of $|Df_n|$ in the definition (1.6).

It follows that \mathcal{E}_{Ch} can also be expressed by $\mathcal{E}_{\text{Ch}}(f, f) = \int_X |Df|_w^2 \, dm$ and the set of Lipschitz functions f with $\int_X |Df|^2 \, dm < \infty$ is strongly dense in the domain of \mathcal{E}_{Ch} . In fact, the Dirichlet form \mathcal{E}_{Ch} enjoys a further upper-regularity property, common to every Cheeger energy ([4], Section 8.3):

(a) for every $f \in D(\mathcal{E})$ there exist $f_n \in D(\mathcal{E}) \cap C_b(X)$ and upper semicontinuous bounded functions $g_n : X \rightarrow \mathbb{R}$ such that

$$\Gamma(f_n) \leq g_n^2, \quad f_n \rightarrow f \text{ in } L^2(X, m), \quad \limsup_{n \rightarrow \infty} \int_X g_n^2 \, dm \leq \mathcal{E}(f, f).$$

Here and in the following, $C_b(X)$ denotes the space of continuous and bounded real functions defined on X .

From $\text{RCD}(K, \infty)$ to $\text{BE}(K, \infty)$. The previous properties of the Cheeger energy show that the investigation of Bakry–Émery curvature bounds makes perfectly sense in $\text{RCD}(K, \infty)$ spaces. One of the main results of [5] connecting these two approaches shows in fact that $\text{RCD}(K, \infty)$ yields $\text{BE}(K, \infty)$ in the gradient formulation (1.4) for every $f \in D(\mathcal{E}_{\text{Ch}})$.

In fact, an even more refined result holds ([5], Theorem 6.2), since it is possible to control the slope of $P_t f$ in terms of the minimal weak gradient of f

$$|DP_t f|^2 \leq e^{-2Kt} P_t(|Df|_w^2) \quad \text{whenever } f \in D(\mathcal{E}_{\text{Ch}}), |Df|_w \in L^\infty(X, m),$$

an estimate that has two useful geometric-analytic consequences:

(b) d coincides with the intrinsic distance associated to the Dirichlet form \mathcal{E}_{Ch} (introduced in Biroli–Mosco [16], see also [51, 52] and [50]), namely

$$d(x, y) = \sup\{\psi(y) - \psi(x) : \psi \in D(\mathcal{E}_{\text{Ch}}) \cap C_b(X), \Gamma(\psi) \leq 1\}, \quad x, y \in X.$$

(c) Every function $\psi \in D(\mathcal{E}_{\text{Ch}})$ with $\Gamma(\psi) \leq 1$ m-a.e. admits a continuous (in fact 1-Lipschitz) representative $\tilde{\psi}$.

From $\text{BE}(K, \infty)$ to $\text{RCD}(K, \infty)$. In the present paper, we provide necessary and sufficient conditions for the validity of the converse implication, that is, $\text{BE}(K, \infty) \Rightarrow \text{RCD}(K, \infty)$.

In order to state this result in a precise way, one has first to clarify how the metric structure should be related to the Dirichlet one. Notice that this problem is much easier from the point of view of the metric measure setting, since one has the canonical way (1.6) to construct the Cheeger energy.

Since we tried to avoid any local compactness assumptions on X as well as doubling or Poincaré conditions on m , we used the previous structural properties (a), (b), (c) as a guide to find a reasonable set of assumptions for our theory; notice that they are in any case necessary conditions to get a $\text{RCD}(K, \infty)$ space.

We thus start from a strongly local and symmetric Dirichlet form \mathcal{E} on a Polish topological space (X, τ) endowed with its Borel σ -algebra and a finite (for the scope of this introduction) Borel measure m . In the algebra $\mathbb{V}_\infty := D(\mathcal{E}) \cap L^\infty(X, m)$ we consider the subspace \mathbb{G}_∞ of functions f admitting a Carré du champ $\Gamma(f) \in L^1(X, m)$: they are characterized by the identity

$$(1.8) \quad \mathcal{E}(f, f\varphi) - \frac{1}{2}\mathcal{E}(f^2, \varphi) = \int_X \Gamma(f)\varphi \, dm \quad \text{for every } \varphi \in \mathbb{V}_\infty.$$

We can therefore introduce the intrinsic distance $d_\mathcal{E}$ as in (b)

$$(1.9) \quad d_\mathcal{E}(x, y) := \sup\{\psi(y) - \psi(x) : \psi \in \mathbb{G}_\infty \cap C(X), \Gamma(\psi) \leq 1\}, \quad x, y \in X,$$

and, following the standard approach, we will assume that $d_\mathcal{E}$ is a complete distance on X and the topology induced by $d_\mathcal{E}$ coincides with τ .

In this way, we end up with *Energy measure spaces* $(X, \tau, m, \mathcal{E})$ and in this setting we prove in Theorem 3.12 that $\mathcal{E} \leq \mathcal{E}_{\text{Ch}}$, where \mathcal{E}_{Ch} is the Cheeger energy associated to $d_\mathcal{E}$; moreover, Theorem 3.14 shows that $\mathcal{E} = \mathcal{E}_{\text{Ch}}$ if and only if (a) holds (see [33], Section 5, for a similar result in the case of doubling spaces satisfying a local Poincaré condition and for interesting examples where \mathcal{E}_{Ch} is not quadratic and $\mathcal{E} \neq \mathcal{E}_{\text{Ch}}$). It is also worth mentioning (Theorem 3.10) that for this class of spaces $(X, d_\mathcal{E})$ is always a length metric space, a result previously known in a locally compact framework [50, 52].

The Bakry-Émery condition $\text{BE}(K, \infty)$ can then be stated in a weak integral form (strongly inspired by [11, 15, 57]) just involving the Markov semigroup $(P_t)_{t \geq 0}$ [see (2.33) of Corollary 2.3 and (2.23), (2.24) for relevant definitions] by asking that the differential inequality

$$(1.10) \quad \frac{\partial^2}{\partial s^2} \int_X (P_{t-s}f)^2 P_s \varphi \, dm \geq 2K \frac{\partial}{\partial s} \int_X (P_{t-s}f)^2 P_s \varphi \, dm, \quad 0 < s < t,$$

is fulfilled for any $f \in L^2(X, m)$ and any nonnegative $\varphi \in L^2 \cap L^\infty(X, m)$. Notice that in the case $K = 0$ (1.10) is equivalent to the convexity in $(0, t)$ of the map $s \mapsto \int_X (P_{t-s}f)^2 P_s \varphi \, dm$.

If we also assume that $\text{BE}(K, \infty)$ holds, it turns out that (c) is in fact equivalent to a weak-Feller condition on the semigroup $(P_t)_{t \geq 0}$, namely P_t maps $\text{Lip}_b(X)$ in $C_b(X)$. Moreover, (c) implies the upper-regularity (a) of \mathcal{E} and the fact that every $f \in D(\mathcal{E}) \cap L^\infty(X, m)$ admits a Carré du champ Γ satisfying (1.8).

Independently of $\text{BE}(K, \infty)$, when properties (a) and (c) are satisfied, we call $(X, \tau, m, \mathcal{E})$ a *Riemannian Energy measure space*, since these space seem appropriate nonsmooth versions of Riemannian manifolds. It is also worth mentioning that in this class of spaces $\text{BE}(K, \infty)$ is equivalent to an (exponential) contraction property for the semigroup $(H_t)_{t \geq 0}$ with respect to the Wasserstein distance W_2 (see Corollary 3.18), in analogy with [34].

Our main equivalence result, Theorem 4.17, shows that a $\text{BE}(K, \infty)$ Riemannian Energy measure space satisfies the $\text{RCD}(K, \infty)$ condition: thus, in view of the converse implication proved in [5], $\text{BE}(K, \infty)$ is essentially equivalent to $\text{RCD}(K, \infty)$. A more precise formulation of our result, in the simplified case when the measure m is finite, is the following.

THEOREM 1.1 (Main result). *Let (X, τ) be a Polish space and let m be a finite Borel measure in X . Let $\mathcal{E} : L^2(X, m) \rightarrow [0, \infty]$ be a strongly local, symmetric Dirichlet form generating a mass preserving Markov semigroup $(P_t)_{t \geq 0}$ in $L^2(X, m)$, let $d_{\mathcal{E}}$ be the intrinsic distance defined by (1.9) and assume that:*

- (i) $d_{\mathcal{E}}$ is a complete distance on X inducing the topology τ and any function $f \in \mathbb{G}_{\infty}$ with $\Gamma(f) \leq 1$ admits a continuous representative;
- (ii) the Bakry–Émery $\text{BE}(K, \infty)$ condition (1.10) is fulfilled by $(P_t)_{t \geq 0}$.

Then $(X, d_{\mathcal{E}}, m)$ is a $\text{RCD}(K, \infty)$ space.

We believe that this equivalence result, between the “Eulerian” formalism of the Bakry–Émery $\text{BE}(K, \infty)$ theory and the “Lagrangian” formalism of the $\text{CD}(K, \infty)$ theory, is conceptually important and that it could be a first step for a better understanding of curvature conditions in metric measure spaces. Also, this equivalence is technically useful. Indeed, in the last section of this paper, we prove the tensorization of $\text{BE}(K, N)$ spaces. Then, in the case $N = \infty$, we can use the implication from $\text{BE}(K, \infty)$ to $\text{RCD}(K, \infty)$ to read this property in terms of tensorization of $\text{RCD}(K, \infty)$ spaces: this was previously known (see [5]) only under an a priori nonbranching assumption on the base spaces [notice that the $\text{CD}(K, N)$ theory, even with $N = \infty$, suffers at this moment the same limitation]. On the other hand, we use the implication from $\text{RCD}(K, \infty)$ to $\text{BE}(K, \infty)$, (1.11) below and the strong stability properties which follow by the EVI_K formulation to provide stability of the $\text{BE}(K, N)$ condition under a very weak convergence, the Sturm–Gromov–Hausdorff convergence.

Plan of the paper. Section 2 collects notation and preliminary results on Dirichlet forms, Markov semigroups and functional Γ -calculus, following the presentation of [18], which avoids any topological assumption. Particular attention is devoted to various formulations of the $\text{BE}(K, N)$ condition: they are discussed in Section 2.2, trying to present an intrinsic approach that does not rely on the introduction of a distinguished algebra of functions \mathcal{A} and extra assumptions on the Dirichlet form \mathcal{E} , besides locality. In its weak formulation [see (2.33) of Corollary 2.3 and (2.23), (2.24)],

$$(1.11) \quad \begin{aligned} \frac{1}{4} \frac{\partial^2}{\partial s^2} \int_X (P_{t-s} f)^2 P_s \varphi \, dm &\geq \frac{K}{2} \frac{\partial}{\partial s} \int_X (P_{t-s} f)^2 P_s \varphi \, dm \\ &+ \frac{1}{N} \int_X (\Delta_{\mathcal{E}} P_{t-s} f)^2 P_s \varphi \, dm, \end{aligned}$$

which is well suited to study stability issues, $BE(K, N)$ does not even need a densely defined Carré du Champ Γ , because only the semigroup $(P_t)_{t \geq 0}$ is involved.

Section 3 is devoted to study the interaction between energy and metric structures. A few metric concepts are recalled in Section 3.1, whereas Section 3.2 shows how to construct a dual semigroup $(H_t)_{t \geq 0}$ in the space of probability measures $\mathcal{P}(X)$ under suitable Lipschitz estimates on $(P_t)_{t \geq 0}$. By using refined properties of the Hopf–Lax semigroup, we also extend some of the duality results proved by Kuwada [34] to general complete and separable metric measure spaces, avoiding any doubling or Poincaré condition.

Section 3.3 presents a careful analysis of the intrinsic distance $d_{\mathcal{E}}$ (1.9) associated to a Dirichlet form and of Energy measure structures $(X, \tau, m, \mathcal{E})$. We will thoroughly discuss the relations between the Dirichlet form \mathcal{E} and the Cheeger energy Ch induced by a distance d , possibly different from the intrinsic distance $d_{\mathcal{E}}$ and we will obtain a precise characterization of the distinguished case when $d = d_{\mathcal{E}}$ and $\mathcal{E} = 2Ch$: here, conditions (a), (b) play a crucial role.

A further investigation when $BE(K, \infty)$ is also assumed is carried out in Section 3.4, leading to the class of *Riemannian Energy measure spaces*.

Section 4 contains the proof of the main equivalence result, Theorem 1.1, between $BE(K, \infty)$ and $RCD(K, \infty)$. Apart the basic estimates of Section 4.1, the argument is split into two main steps: Section 4.2 proves a first $L \log L$ regularization estimate for the semigroup $(H_t)_{t \geq 0}$, starting from arbitrary measures in $\mathcal{P}_2(X)$ (here, we follow the approach of [57]). Section 4.3 contains the crucial action estimates to prove the EVI_K inequality (1.7). Even if the strategy of the proof has been partly inspired by the geometric heuristics discussed in [22] (where the Eulerian approach of [46] to contractivity of gradient flows has been extended to cover also convexity and evolutions in the EVI_K sense) this part is completely new and it uses in a subtle way all the refined technical issues discussed in the previous sections of the paper.

In the last Section 5, we discuss the above mentioned applications of the equivalence between $BE(K, \infty)$ and $RCD(K, \infty)$.

2. Dirichlet forms, Markov semigroups, Γ -calculus.

2.1. *Dirichlet forms and Γ -calculus.* Let (X, \mathcal{B}) be a measurable space, let $m: \mathcal{B} \rightarrow [0, \infty]$ σ -additive and let $L^p(X, m)$ be the Lebesgue spaces (for notational simplicity, we omit the dependence on \mathcal{B}). Possibly enlarging \mathcal{B} and extending m we assume that \mathcal{B} is m -complete. In the next Sections 3 and 4, we will typically consider the case when \mathcal{B} is the m -completion of the Borel σ -algebra generated by a Polish topology τ on X .

In all of this paper, we will assume that

$$(2.1) \quad \mathcal{E}: L^2(X, m) \rightarrow [0, \infty] \text{ is a strongly local, symmetric Dirichlet form generating a Markov semigroup } (P_t)_{t \geq 0} \text{ in } L^2(X, m).$$

Let us briefly recall the precise meaning of this statement.

A *symmetric Dirichlet form* \mathcal{E} is a $L^2(X, \mathfrak{m})$ -lower semicontinuous quadratic form satisfying the Markov property

$$(2.2) \quad \mathcal{E}(\eta \circ f) \leq \mathcal{E}(f) \quad \text{for every normal contraction } \eta : \mathbb{R} \rightarrow \mathbb{R},$$

that is, a 1-Lipschitz map satisfying $\eta(0) = 0$. We refer to [18, 25] for equivalent formulations of (2.2). We also define

$$\mathbb{V} := D(\mathcal{E}) = \{f \in L^2(X, \mathfrak{m}) : \mathcal{E}(f) < \infty\}, \quad \mathbb{V}_\infty := D(\mathcal{E}) \cap L^\infty(X, \mathfrak{m}).$$

We also assume that \mathbb{V} is dense in $L^2(X, \mathfrak{m})$.

We still denote by $\mathcal{E}(\cdot, \cdot) : \mathbb{V} \rightarrow \mathbb{R}$ the associated continuous and symmetric bilinear form

$$\mathcal{E}(f, g) := \frac{1}{4}(\mathcal{E}(f + g) - \mathcal{E}(f - g)).$$

We will assume *strong locality* of \mathcal{E} , namely

$$\forall f, g \in \mathbb{V} : \mathcal{E}(f, g) = 0 \quad \text{if } (f + a)g = 0 \text{ m-a.e. in } X \text{ for some } a \in \mathbb{R}.$$

It is possible to prove (see, e.g., [18], Proposition 2.3.2) that \mathbb{V}_∞ is an algebra with respect to pointwise multiplication, so that for every $f \in \mathbb{V}_\infty$ the linear form on \mathbb{V}_∞

$$(2.3) \quad \mathbf{\Gamma}[f; \varphi] := \mathcal{E}(f, f\varphi) - \frac{1}{2}\mathcal{E}(f^2, \varphi), \quad \varphi \in \mathbb{V}_\infty,$$

is well defined and for every normal contraction $\eta : \mathbb{R} \rightarrow \mathbb{R}$ it satisfies [18], Proposition 2.3.3,

$$(2.4) \quad 0 \leq \mathbf{\Gamma}[\eta \circ f; \varphi] \leq \mathbf{\Gamma}[f; \varphi] \leq \|\varphi\|_\infty \mathcal{E}(f) \quad \text{for every } f, \varphi \in \mathbb{V}_\infty, \varphi \geq 0.$$

Equation (2.4) shows that for every nonnegative $\varphi \in \mathbb{V}_\infty$ $f \mapsto \mathbf{\Gamma}[f; \varphi]$ is a quadratic form in \mathbb{V}_∞ which satisfies the Markov property and can be extended by continuity to \mathbb{V} . We call \mathbb{G} the set of functions $f \in \mathbb{V}$ such that the linear form $\varphi \mapsto \mathbf{\Gamma}[f; \varphi]$ can be represented by an absolutely continuous measure w.r.t. \mathfrak{m} with density $\Gamma(f) \in L^1_+(X, \mathfrak{m})$:

$$(2.5) \quad f \in \mathbb{G} \iff \mathbf{\Gamma}[f; \varphi] = \int_X \Gamma(f)\varphi \, d\mathfrak{m} \quad \text{for every } \varphi \in \mathbb{V}_\infty.$$

Since \mathcal{E} is strongly local, [18], Theorem 6.1.1, yields the representation formula

$$(2.6) \quad \mathcal{E}(f, f) = \int_X \Gamma(f) \, d\mathfrak{m} \quad \text{for every } f \in \mathbb{G}.$$

It is not difficult to check that \mathbb{G} is a closed vector subspace of \mathbb{V} , the restriction of \mathcal{E} to \mathbb{G} is still a strongly local Dirichlet form admitting the *Carré du champ* Γ defined by (2.5) (see, e.g., [18], Definition 4.1.2): Γ is a quadratic continuous map defined in \mathbb{G} with values in $L^1_+(X, \mathfrak{m})$. We will see in the next Section 2.2 that

if \mathcal{E} satisfies the $BE(K, \infty)$ condition, then \mathbb{G} coincides with \mathbb{V} and \mathcal{E} admits a functional Γ -calculus on the whole space \mathbb{V} .

Since we are going to use Γ -calculus techniques, we use the Γ notation also for the symmetric, bilinear and continuous map

$$\Gamma(f, g) := \frac{1}{4}(\Gamma(f + g) - \Gamma(f - g)) \in L^1(X, \mathfrak{m}), \quad f, g \in \mathbb{G},$$

which, thanks to (2.6), represents the bilinear form \mathcal{E} by the formula

$$\mathcal{E}(f, g) = \int_X \Gamma(f, g) \, d\mathfrak{m} \quad \text{for every } f, g \in \mathbb{G}.$$

Because of Markovianity and locality $\Gamma(\cdot, \cdot)$ satisfies the chain rule [18], Corollary 7.1.2,

$$(2.7) \quad \Gamma(\eta(f), g) = \eta'(f)\Gamma(f, g) \quad \text{for every } f, g \in \mathbb{G}, \eta \in \text{Lip}(\mathbb{R}), \eta(0) = 0,$$

and the Leibniz rule:

$$\Gamma(fg, h) = f\Gamma(g, h) + g\Gamma(f, h) \quad \text{for every } f, g, h \in \mathbb{G}_\infty := \mathbb{G} \cap L^\infty(X, \mathfrak{m}).$$

Notice that by [18], Theorem 7.1.1, (2.7) is well defined since for every Borel set $N \subset \mathbb{R}$ (as the set where ϕ is not differentiable)

$$(2.8) \quad \mathcal{L}^1(N) = 0 \implies \Gamma(f) = 0 \quad \text{m-a.e. on } f^{-1}(N).$$

Among the most useful consequences of (2.8) and (2.7) that we will repeatedly use in the sequel, we recall that for every $f, g \in \mathbb{G}$

$$\Gamma(f - g) = 0 \quad \text{m-a.e. on } \{f = g\},$$

and the following identities hold m-a.e.:

$$(2.9) \quad \begin{aligned} \Gamma(f \wedge g) &= \begin{cases} \Gamma(f), & \text{on } \{f \leq g\}, \\ \Gamma(g), & \text{on } \{f \geq g\}, \end{cases} \\ \Gamma(f \vee g) &= \begin{cases} \Gamma(f), & \text{on } \{f \geq g\}, \\ \Gamma(g), & \text{on } \{f \leq g\}. \end{cases} \end{aligned}$$

We conclude this section by stating the following lower semicontinuity result along a sequence $(f_n)_n \subset \mathbb{G}$ converging to $f \in \mathbb{G}$:

$$(2.10) \quad \begin{aligned} f_n \rightharpoonup f, \quad \sqrt{\Gamma(f_n)} \rightharpoonup G & \quad \text{in } L^2(X, \mathfrak{m}) \\ \implies \Gamma(f) \leq G^2 & \quad \text{m-a.e. in } X. \end{aligned}$$

It can be easily proved by using Mazur’s lemma and the m-a.e. convexity of $f \mapsto \sqrt{\Gamma(f)}$, namely

$$\sqrt{\Gamma((1-t)f + tg)} \leq (1-t)\sqrt{\Gamma(f)} + t\sqrt{\Gamma(g)} \quad \text{m-a.e. in } X, \text{ for all } t \in [0, 1],$$

which follows since Γ is quadratic and nonnegative.

The Markov semigroup and its generator. The Dirichlet form \mathcal{E} induces a densely defined self-adjoint operator $\Delta_{\mathcal{E}} : D(\Delta_{\mathcal{E}}) \subset \mathbb{V} \rightarrow L^2(X, \mathfrak{m})$ defined by the integration by parts formula $\mathcal{E}(f, g) = - \int_X g \Delta_{\mathcal{E}} f \, d\mathfrak{m}$ for all $g \in \mathbb{V}$.

When $\mathbb{G} = \mathbb{V}$, the operator $\Delta_{\mathcal{E}}$ is of “diffusion” type, since it satisfies the following chain rule for every $\eta \in C^2(\mathbb{R})$ with $\eta(0) = 0$ and bounded first and second derivatives (see [18], Corollary 6.1.4, and the next (2.16)): if $f \in D(\Delta_{\mathcal{E}})$ with $\Gamma(f) \in L^2(X, \mathfrak{m})$ then $\eta(f) \in D(\Delta_{\mathcal{E}})$ with

$$(2.11) \quad \Delta_{\mathcal{E}}\eta(f) = \eta'(f)\Delta_{\mathcal{E}}f + \eta''(f)\Gamma(f).$$

The heat flow P_t associated to \mathcal{E} is well defined starting from any initial condition $f \in L^2(X, \mathfrak{m})$. Recall that in this framework the heat flow $(P_t)_{t \geq 0}$ is an analytic Markov semigroup and $f_t = P_t f$ can be characterized as the unique C^1 map $f : (0, \infty) \rightarrow L^2(X, \mathfrak{m})$, with values in $D(\Delta_{\mathcal{E}})$, satisfying

$$\begin{cases} \frac{d}{dt} f_t = \Delta_{\mathcal{E}} f_t, & \text{for } t \in (0, \infty), \\ \lim_{t \downarrow 0} f_t = f, & \text{in } L^2(X, \mathfrak{m}). \end{cases}$$

Because of this, $\Delta_{\mathcal{E}}$ can equivalently be characterized in terms of the strong convergence $(P_t f - f)/t \rightarrow \Delta_{\mathcal{E}} f$ in $L^2(X, \mathfrak{m})$ as $t \downarrow 0$.

One useful consequence of the Markov property is the L^p contraction of $(P_t)_{t \geq 0}$ from $L^p \cap L^2$ to $L^p \cap L^2$. Because of the density of $L^p \cap L^2$ in L^p when $p \in [1, \infty)$, this allows to extend uniquely P_t to a strongly continuous semigroup of linear contractions in $L^p(X, \mathfrak{m})$, $p \in [1, \infty)$, for which we retain the same notation. Furthermore, $(P_t)_{t \geq 0}$ is sub-Markovian (cf. [18], Proposition 3.2.1), since it preserves one-sided essential bounds, namely $f \leq C$ (resp., $f \geq C$) \mathfrak{m} -a.e. in X for some $C \geq 0$ (resp., $C \leq 0$) implies $P_t f \leq C$ (resp., $P_t f \geq C$) \mathfrak{m} -a.e. in X for all $t \geq 0$.

We will mainly be concerned with the mass-preserving case, that is,

$$(2.12) \quad \int_X P_t f \, d\mathfrak{m} = \int_X f \, d\mathfrak{m} \quad \text{for every } f \in L^1(X, \mathfrak{m}),$$

a property which is equivalent to $1 \in D(\mathcal{E})$ when $\mathfrak{m}(X) < \infty$. In the next session (see Theorem 3.14), we will discuss a metric framework, which will imply (2.12).

The semigroup $(P_t)_{t \geq 0}$ can also be extended by duality to a weakly*-continuous semigroup of contractions in $L^\infty(X, \mathfrak{m})$, so that

$$\int_X P_t f \varphi \, d\mathfrak{m} = \int_X f P_t \varphi \, d\mathfrak{m} \quad \text{for every } f \in L^\infty(X, \mathfrak{m}), \varphi \in L^1(X, \mathfrak{m}).$$

It is easy to show that if $f_n \in L^\infty(X, \mathfrak{m})$ weakly* converge to f in $L^\infty(X, \mathfrak{m})$ then $P_t f_n \xrightarrow{*} P_t f$ in $L^\infty(X, \mathfrak{m})$.

The generator of the semigroup in $L^1(X, \mathfrak{m})$. Sometimes it will also be useful to consider the generator $\Delta_{\mathcal{E}}^{(1)} : D(\Delta_{\mathcal{E}}^{(1)}) \subset L^1(X, \mathfrak{m}) \rightarrow L^1(X, \mathfrak{m})$ of $(P_t)_{t \geq 0}$ in

$L^1(X, m)$ ([47], Section 1.1):

$$(2.13) \quad \begin{aligned} f \in D(\Delta_{\mathcal{E}}^{(1)}), \quad \Delta_{\mathcal{E}}^{(1)} f = g \\ \Leftrightarrow \lim_{t \downarrow 0} \frac{1}{t} (\mathbf{P}_t f - f) = g \quad \text{strongly in } L^1(X, m). \end{aligned}$$

Thanks to (2.13) and L^1 contractivity it is easy to check that

$$(2.14) \quad \begin{aligned} f \in D(\Delta_{\mathcal{E}}^{(1)}) \\ \Rightarrow \mathbf{P}_t f \in D(\Delta_{\mathcal{E}}^{(1)}), \quad \Delta_{\mathcal{E}}^{(1)} \mathbf{P}_t f = \mathbf{P}_t \Delta_{\mathcal{E}}^{(1)} f \quad \text{for all } t \geq 0, \end{aligned}$$

and, when (2.12) holds,

$$\int_X \Delta_{\mathcal{E}}^{(1)} f \, dm = 0 \quad \text{for every } f \in D(\Delta_{\mathcal{E}}^{(1)}).$$

The operator $\Delta_{\mathcal{E}}^{(1)}$ is m -accretive and coincides with the smallest closed extension of $\Delta_{\mathcal{E}}$ to $L^1(X, m)$: ([18], Proposition 2.4.2):

$$(2.15) \quad \begin{aligned} g = \Delta_{\mathcal{E}}^{(1)} f \\ \Leftrightarrow \begin{cases} \exists f_n \in D(\Delta_{\mathcal{E}}) \cap L^1(X, m) \text{ with } g_n = \Delta_{\mathcal{E}} f_n \in L^1(X, m) : \\ f_n \rightarrow f, g_n \rightarrow g \text{ strongly in } L^1(X, m). \end{cases} \end{aligned}$$

Whenever $f \in D(\Delta_{\mathcal{E}}^{(1)}) \cap L^2(X, m)$ and $\Delta_{\mathcal{E}}^{(1)} f \in L^2(X, m)$, one can recover $f \in D(\Delta_{\mathcal{E}})$ by (2.14), the integral formula $\mathbf{P}_t f - f = \int_0^t \mathbf{P}_r \Delta_{\mathcal{E}}^{(1)} f \, dr$ and the contraction property of $(\mathbf{P}_t)_{t \geq 0}$ in every $L^p(X, m)$, thus obtaining

$$(2.16) \quad \begin{aligned} f \in D(\Delta_{\mathcal{E}}^{(1)}) \cap L^2(X, m), \quad \Delta_{\mathcal{E}}^{(1)} f \in L^2(X, m) \\ \Rightarrow f \in D(\Delta_{\mathcal{E}}), \quad \Delta_{\mathcal{E}} f = \Delta_{\mathcal{E}}^{(1)} f. \end{aligned}$$

Semigroup mollification. A useful tool to prove the above formula is given by the mollified semigroup: we fix a

$$(2.17) \quad \text{nonnegative kernel } \kappa \in C_c^\infty(0, \infty) \text{ with } \int_0^\infty \kappa(r) \, dr = 1,$$

and for every $f \in L^p(X, m)$, $p \in [1, \infty]$, we set

$$(2.18) \quad \mathfrak{h}^\varepsilon f := \frac{1}{\varepsilon} \int_0^\infty \mathbf{P}_r f \kappa(r/\varepsilon) \, dr, \quad \varepsilon > 0,$$

where the integral should be intended in the Bochner sense whenever $p < \infty$ and by taking the duality with arbitrary $\varphi \in L^1(X, m)$ when $p = \infty$.

Since $\Delta_{\mathcal{E}}$ is the generator of $(\mathbf{P}_t)_{t \geq 0}$ in $L^2(X, m)$, it is not difficult to check ([47], Proof of Theorem 2.7), that if $f \in L^2 \cap L^p(X, m)$ for some $p \in [1, \infty]$ then

$$-\Delta_{\mathcal{E}}(\mathfrak{h}^\varepsilon f) = \frac{1}{\varepsilon^2} \int_0^\infty \mathbf{P}_r f \kappa'(r/\varepsilon) \, dr \in L^2 \cap L^p(X, m).$$

Since $\Delta_{\mathcal{E}}^{(1)}$ is the generator of $(P_t)_{t \geq 0}$ in $L^1(X, m)$, the same property holds for $\Delta_{\mathcal{E}}^{(1)}$ if $f \in L^1(X, m)$:

$$(2.19) \quad -\Delta_{\mathcal{E}}^{(1)}(h^\varepsilon f) = \frac{1}{\varepsilon^2} \int_0^\infty P_r f \kappa'(r/\varepsilon) dr \in L^1(X, m).$$

2.2. *On the functional Bakry–Émery condition.* We will collect in this section various equivalent characterizations of the Bakry–Émery condition $BE(K, N)$ given in (1.3) for the Γ_2 operator (1.2). We have been strongly inspired by [11, 15, 57]: even if the essential estimates are well known, here we will take a particular care in establishing all the results in a weak form, under the minimal regularity assumptions on the functions involved. We consider here the case of finite dimension as well, despite the fact that the next Sections 3 and 4 will be essentially confined to the case $N = \infty$. Applications of $BE(K, N)$ with $N < \infty$ will be considered in the last Section 5.

Let us denote by $\Gamma : (\mathbb{V}_\infty)^3 \rightarrow \mathbb{R}$ the multilinear map

$$\Gamma[f, g; \varphi] := \frac{1}{2}(\mathcal{E}(f, g\varphi) + \mathcal{E}(g, f\varphi) - \mathcal{E}(fg, \varphi)), \quad \Gamma[f; \varphi] = \Gamma[f, f; \varphi].$$

Recalling (2.4), one can easily prove the uniform continuity property

$$(2.20) \quad \begin{aligned} f_n, \varphi_n \in \mathbb{V}_\infty, \quad f_n \rightarrow f, \quad \varphi_n \rightarrow \varphi \quad \text{in } \mathbb{V}, \quad \sup_n \|\varphi_n\|_\infty < \infty \\ \Rightarrow \exists \lim_{n \rightarrow \infty} \Gamma[f_n; \varphi_n] \in \mathbb{R}, \end{aligned}$$

which allows to extend Γ to a real multilinear map defined in $\mathbb{V} \times \mathbb{V} \times \mathbb{V}_\infty$, for which we retain the same notation. The extension Γ satisfies

$$(2.21) \quad \Gamma[f, g; \varphi] = \int_X \Gamma(f, g)\varphi \, dm \quad \text{if } f, g \in \mathbb{G}.$$

We also set

$$(2.22) \quad \Gamma_2[f; \varphi] := \frac{1}{2}\Gamma[f; \Delta_{\mathcal{E}}\varphi] - \Gamma[f, \Delta_{\mathcal{E}}f; \varphi], \quad (f, \varphi) \in D(\Gamma_2),$$

where

$$D(\Gamma_2) := \{(f, \varphi) \in D(\Delta_{\mathcal{E}}) \times D(\Delta_{\mathcal{E}}) : \Delta_{\mathcal{E}}f \in \mathbb{V}, \varphi, \Delta_{\mathcal{E}}\varphi \in L^\infty(X, m)\}.$$

As for (2.21), we have

$$\Gamma_2[f; \varphi] = \int_X \left(\frac{1}{2}\Gamma(f)\Delta_{\mathcal{E}}\varphi - \Gamma(f, \Delta_{\mathcal{E}}f)\varphi \right) dm$$

if $(f, \varphi) \in D(\Gamma_2), f, \Delta_{\mathcal{E}}f \in \mathbb{G}$.

Since $(P_t)_{t \geq 0}$ is an analytic semigroup in $L^2(X, m)$, for a given $f \in L^2(X, m)$ and $\varphi \in L^2 \cap L^\infty(X, m)$, we can consider the functions

$$(2.23) \quad A_t[f; \varphi](s) := \frac{1}{2} \int_X (P_{t-s}f)^2 P_s\varphi \, dm, \quad t > 0, s \in [0, t],$$

$$(2.24) \quad A_t^\Delta[f; \varphi](s) := \frac{1}{2} \int_X (\Delta_\mathcal{E} P_{t-s} f)^2 P_s \varphi \, dm, \quad t > 0, s \in [0, t),$$

$$B_t[f; \varphi](s) := \Gamma[P_{t-s} f; P_s \varphi], \quad t > 0, s \in (0, t),$$

and, whenever $\Delta_\mathcal{E} \varphi \in L^2 \cap L^\infty(X, m)$,

$$C_t[f; \varphi](s) := \Gamma_2[P_{t-s} f; P_s \varphi], \quad t > 0, s \in [0, t).$$

Notice that whenever $\Delta_\mathcal{E} f \in L^2(X, m)$

$$A_t^\Delta[f; \varphi](s) = A_t[\Delta_\mathcal{E} f; \varphi](s), \quad t > 0, s \in [0, t).$$

LEMMA 2.1. *For every $f \in L^2(X, m)$, $\varphi \in L^2 \cap L^\infty(X, m)$ and every $t > 0$, we have:*

- (i) *the function $s \mapsto A_t[f; \varphi](s)$ belongs to $C^0([0, t]) \cap C^1((0, t))$;*
- (ii) *the function $s \mapsto A_t^\Delta[f; \varphi](s)$ belongs to $C^0([0, t])$;*
- (iii) *the function $s \mapsto B_t[f; \varphi](s)$ belongs to $C^0((0, t))$ and*

$$(2.25) \quad \frac{\partial}{\partial s} A_t[f; \varphi](s) = B_t[f; \varphi](s) \quad \text{for every } s \in (0, t).$$

Equation (2.25) and the regularity of A and B extend to $s = t$ if $f \in \mathbb{V}$ and to $s = 0$ if $\varphi \in \mathbb{V}_\infty$.

(iv) *If φ is nonnegative, $s \mapsto A_t[f; \varphi](s)$ and $s \mapsto A_t^\Delta[f; \varphi](s)$ are nondecreasing.*

(v) *If $\Delta_\mathcal{E} \varphi \in L^2 \cap L^\infty(X, m)$ then C belongs to $C^0([0, t])$, B belongs to $C^1([0, t])$, and*

$$(2.26) \quad \frac{\partial}{\partial s} B_t[f; \varphi](s) = 2C_t[f; \varphi](s) \quad \text{for every } s \in [0, t).$$

In particular, $A \in C^2([0, t])$.

PROOF. The continuity of A is easy to check, since $s \mapsto (P_{t-s} f)^2$ is strongly continuous with values in $L^1(X, m)$ and $s \mapsto P_s \varphi$ is weakly* continuous in $L^\infty(X, m)$. Analogously, the continuity of B follows from the fact that $s \mapsto P_{t-s} f$ is a continuous curve in \mathbb{V} whenever $s \in [0, t)$ thanks to the regularizing effect of the heat flow and (2.20). The continuity of C follows by a similar argument, recalling the definition (2.22) and the fact that the curves $s \mapsto \Delta_\mathcal{E} P_{t-s} f$ and $s \mapsto \Delta_\mathcal{E} P_s \varphi$ are continuous with values in \mathbb{V} in the interval $[0, t)$.

In order to prove (2.25) and (2.26), let us first assume that $\varphi \in D(\Delta_\mathcal{E})$ with $\Delta_\mathcal{E} \varphi \in L^\infty(X, m)$ and $f \in L^2 \cap L^\infty(X, m)$. Since

$$\lim_{h \rightarrow 0} \frac{P_{t-(s+h)} f - P_{t-s} f}{h} = -\Delta_\mathcal{E} P_{t-s} f \quad \text{strongly in } \mathbb{V} \text{ for } s \in [0, t),$$

$$\lim_{h \rightarrow 0} \frac{P_{s+h} \varphi - P_s \varphi}{h} = \Delta_\mathcal{E} P_s \varphi \quad \text{weakly* in } L^\infty(X, m) \text{ for } s \in [0, t),$$

we easily get

$$\begin{aligned} \frac{\partial}{\partial s} A_t[f; \varphi](s) &= \int_X \left(-P_{t-s}f \Delta_{\mathcal{E}} P_{t-s}f P_s\varphi + \frac{1}{2} (P_{t-s}f)^2 \Delta_{\mathcal{E}} P_s\varphi \right) dm \\ &= \mathcal{E}(P_{t-s}f, P_{t-s}f P_s\varphi) - \frac{1}{2} \mathcal{E}((P_{t-s}f)^2, P_s\varphi) = B_t[f; \varphi](s), \end{aligned}$$

by the very definition (2.3) of Γ , since $P_{t-s}f$ is essentially bounded and therefore $(P_{t-s}f)^2 \in \mathbb{V}_\infty$. A similar computation yields (2.26).

In order to extend the validity of (2.25) and (2.26) to general $f \in L^2(X, m)$, we approximate f by truncation, setting $f_n := -n \vee f \wedge n$, $n \in \mathbb{N}$, and we pass to the limit in the integrated form

$$A_t[f_n; \varphi](s_2) - A_t[f_n; \varphi](s_1) = \int_{s_1}^{s_2} B_t[f_n; \varphi](s) ds \quad \text{for every } 0 \leq s_1 < s_2 < t,$$

observing that $P_{t-s}f_n$ converge strongly to $P_{t-s}f$ in \mathbb{V} as $n \rightarrow \infty$ for every $s \in [0, t)$, so that (2.20) yields the pointwise convergence of the integrands in the previous identity. A similar argument holds for (2.26), since $\Delta_{\mathcal{E}} P_t f_n$ converges strongly to $\Delta_{\mathcal{E}} P_t f$ in \mathbb{V} .

Eventually, we extend (2.25) to arbitrary $\varphi \in L^2 \cap L^\infty(X, m)$ by approximating φ with $h^\varepsilon \varphi$ given by (2.18), (2.17). It is not difficult to check that $P_s(h^\varepsilon \varphi) \rightarrow P_s \varphi$ in \mathbb{V} as $\varepsilon \downarrow 0$ with uniform L^∞ bound if $s > 0$ (and also when $s = 0$ if $\varphi \in \mathbb{V}_\infty$). \square

LEMMA 2.2. *Let us consider functions $\mathbf{a} \in C^1([0, t])$, $\mathbf{g} \in C^0([0, t])$ and a parameter $\nu \geq 0$. The following properties are equivalent:*

(i) \mathbf{a}, \mathbf{g} satisfy the differential inequality

$$\mathbf{a}'' \geq 2K\mathbf{a}' + \nu\mathbf{g} \quad \text{in } \mathcal{D}'(0, t),$$

and pointwise in $[0, t)$, whenever $\mathbf{a} \in C^2([0, t))$.

(ii) \mathbf{a}', \mathbf{g} satisfy the differential inequality

$$\frac{d}{ds} (e^{-2Ks} \mathbf{a}'(s)) \geq \nu e^{-2Ks} \mathbf{g}(s) \quad \text{in } \mathcal{D}'(0, t).$$

(iii) For every $0 \leq s_1 < s_2 < t$ and every test function $\zeta \in C^2([s_1, s_2])$, we have

$$(2.27) \quad \int_{s_1}^{s_2} \mathbf{a}(\zeta'' + 2K\zeta') ds + [\mathbf{a}'\zeta]_{s_1}^{s_2} - [\mathbf{a}(\zeta' + 2K\zeta)]_{s_1}^{s_2} \geq \nu \int_{s_1}^{s_2} \mathbf{g}\zeta ds.$$

(iv) For every $0 \leq s_1 < s_2$, we have

$$(2.28) \quad e^{-2K(s_2-s_1)} \mathbf{a}'(s_2) \geq \mathbf{a}'(s_1) + \nu \int_{s_1}^{s_2} e^{-2K(s-s_1)} \mathbf{g}(s) ds.$$

The *proof* is straightforward; we only notice that (2.27) holds also for $s_1 = 0$ since $\mathbf{a} \in C^1([0, t])$.

The inequality (2.27) has two useful consequences that we make explicit in terms of the functions I_K and $I_{K,2}$ defined by

$$(2.29) \quad \begin{aligned} I_K(t) &= \int_0^t e^{Ks} \, ds = \frac{e^{Kt} - 1}{K}, \\ I_{K,2}(t) &= \int_0^t I_K(s) \, ds = \frac{e^{Kt} - Kt - 1}{K^2}, \end{aligned}$$

with the obvious definition for $K = 0$: $I_0(t) = t, I_{0,2}(t) = t^2/2$.

Choosing $s_1 = 0, s_2 = \tau$ and

$$\zeta(s) := I_{2K}(\tau - s) = \frac{e^{2K(\tau-s)} - 1}{2K} \quad \text{so that } \zeta' + 2K\zeta = -1, \zeta(\tau) = 0,$$

we obtain

$$(2.30) \quad I_{2K}(\tau)\mathbf{a}'(0) + \nu \int_0^\tau I_{2K}(\tau - s)g(s) \, ds \leq \mathbf{a}(\tau) - \mathbf{a}(0)$$

for every $\tau \in [0, t)$.

Choosing

$$\zeta(s) := I_{-2K}(s) = \frac{1 - e^{-2Ks}}{2K} \quad \text{so that } \zeta' + 2K\zeta = 1, \zeta(0) = 0,$$

we obtain

$$(2.31) \quad \mathbf{a}(\tau) - \mathbf{a}(0) + \nu \int_0^\tau I_{-2K}(s)g(s) \, ds \leq \mathbf{a}'(\tau)I_{-2K}(\tau)$$

for every $\tau \in [0, t)$.

COROLLARY 2.3. *Let \mathcal{E} be a Dirichlet form in $L^2(X, \mathbf{m})$ as in (2.1), and let $K \in \mathbb{R}$ and $\nu \geq 0$. The following conditions are equivalent:*

(i) *For every $(f, \varphi) \in D(\Gamma_2)$, with $\varphi \geq 0$, we have*

$$\Gamma_2[f; \varphi] \geq K\Gamma[f; \varphi] + \nu \int_X (\Delta \mathcal{E} f)^2 \varphi \, d\mathbf{m}.$$

(ii) *For every $f \in L^2(X, \mathbf{m})$ and every nonnegative $\varphi \in D(\Delta \mathcal{E}) \cap L^\infty(X, \mathbf{m})$ with $\Delta \mathcal{E} \varphi \in L^\infty(X, \mathbf{m})$, we have*

$$(2.32) \quad \mathbf{C}_t[f; \varphi](s) \geq K\mathbf{B}_t[f; \varphi](s) + 2\nu\mathbf{A}_t^\Delta[f; \varphi](s) \quad \text{for every } 0 \leq s < t.$$

(iii) For every $f \in L^2(X, \mathfrak{m})$, every nonnegative $\varphi \in L^2 \cap L^\infty(X, \mathfrak{m})$ and $t > 0$

$$(2.33) \quad \frac{\partial^2}{\partial s^2} \mathbf{A}_t[f; \varphi](s) \geq 2K \frac{\partial}{\partial s} \mathbf{A}_t[f; \varphi](s) + 4\nu \mathbf{A}_t^\Delta[f; \varphi](s)$$

in the sense of distribution of $\mathcal{D}'(0, t)$ (or, equivalently, the inequality (2.33) holds pointwise in $[0, t)$ for every nonnegative $\varphi \in L^2 \cap L^\infty(X, \mathfrak{m})$ with $\Delta_\varepsilon \varphi \in L^2 \cap L^\infty(X, \mathfrak{m})$).

(iv) For every $f \in L^2(X, \mathfrak{m})$ and $t > 0$, we have $\mathbf{P}_t f \in \mathbb{G}$ and

$$(2.34) \quad \mathbf{I}_{2K}(t) \Gamma(\mathbf{P}_t f) + 2\nu \mathbf{I}_{2K,2}(t) (\Delta_\varepsilon \mathbf{P}_t f)^2 \leq \frac{1}{2} \mathbf{P}_t(f^2) - \frac{1}{2} (\mathbf{P}_t f)^2$$

m-a.e. in X .

(v) $\mathbb{G} = \mathbb{V}$ and for every $f \in \mathbb{V}$

$$\frac{1}{2} \mathbf{P}_t(f^2) - \frac{1}{2} (\mathbf{P}_t f)^2 + 2\nu \mathbf{I}_{-2K,2}(t) (\Delta_\varepsilon \mathbf{P}_t f)^2 \leq \mathbf{I}_{-2K,2}(t) \mathbf{P}_t \Gamma(f) \quad \text{m-a.e. in } X.$$

(vi) \mathbb{G} is dense in $L^2(X, \mathfrak{m})$ and for every $f \in \mathbb{G}$ and $t > 0$ $\mathbf{P}_t f$ belongs to \mathbb{G} with

$$(2.35) \quad \Gamma(\mathbf{P}_t f) + 2\nu \mathbf{I}_{-2K}(t) (\Delta_\varepsilon \mathbf{P}_t f)^2 \leq e^{-2Kt} \mathbf{P}_t \Gamma(f) \quad \text{m-a.e. in } X.$$

If one of these equivalent properties holds, then $\mathbb{G} = \mathbb{V}$ (i.e., \mathcal{E} admits the Carré du Champ Γ in \mathbb{V}).

PROOF. The implication (i) \Rightarrow (ii) is obvious, choosing $s = 0$. The converse implication is also true under the regularity assumption of (i): it is sufficient to pass to the limit in (2.32) as $s \uparrow t$ and then as $t \downarrow 0$.

(ii) \Rightarrow (iii) follows by (2.26) when $\Delta_\varepsilon \varphi \in L^2 \cap L^\infty(X, \mathfrak{m})$; the general case follows by approximation by the same argument we used in the proof of Lemma 2.1.

(iii) \Rightarrow (iv): by applying (2.30) (with obvious notation) we get

$$\mathbf{I}_{2K}(t) \Gamma[\mathbf{P}_t f; \varphi] + 2\nu \mathbf{I}_{2K,2}(t) \int_X (\Delta_\varepsilon \mathbf{P}_t f)^2 \varphi \, d\mathfrak{m} \leq \frac{1}{2} \int_X (\mathbf{P}_t(f^2) - (\mathbf{P}_t f)^2) \varphi \, d\mathfrak{m}$$

for every nonnegative $\varphi \in \mathbb{V}_\infty$. Thus, setting $h := \mathbf{P}_t(f^2) - (\mathbf{P}_t f)^2 \in L^1_+(X, \mathfrak{m})$, the linear functional ℓ on \mathbb{V}_∞ defined by $\ell(\varphi) := \mathbf{I}_{2K}(t) \Gamma[\mathbf{P}_t f; \varphi]$ satisfies

$$(2.36) \quad 0 \leq \ell(\varphi) \leq \int_X h \varphi \, d\mathfrak{m} \quad \text{for every } \varphi \in \mathbb{V}_\infty, \varphi \geq 0.$$

Since \mathbb{V}_∞ is a lattice of functions generating \mathcal{B} [because \mathcal{B} is complete and \mathbb{V}_∞ is dense in $L^2(X, \mathfrak{m})$] satisfying the Stone property $\varphi \in \mathbb{V}_\infty \Rightarrow \varphi \wedge 1 \in \mathbb{V}_\infty$ and clearly (2.36) yields $\ell(\varphi_n) \rightarrow 0$ whenever $(\varphi_n)_{n \geq 0} \subset \mathbb{V}_\infty$ is a sequence of functions pointwise decreasing to 0, Daniell construction ([17], Theorem 7.8.7), and the Radon–Nikodym theorem yields $\Gamma[\mathbf{P}_t f; \varphi] = \int_X g \varphi \, d\mathfrak{m}$ for some $g \in L^1_+(X, \mathfrak{m})$, so that $\mathbf{P}_t f \in \mathbb{G}$ and (2.34) holds.

This argument also shows that \mathbb{G} is invariant under the action of $(P_t)_{t \geq 0}$ and dense in $L^2(X, \mathfrak{m})$. A standard approximation argument yields the density in \mathbb{V} (see, e.g., [5], Lemma 4.9) and, therefore, $\mathbb{G} = \mathbb{V}$ (since \mathbb{G} is closed in \mathbb{V} ; see also [18], Proposition 4.1.3).

Analogously, (iii) \Rightarrow (v) follows by (2.31), while (iii) \Rightarrow (vi) follows by (2.28).

Let us now show that (vi) \Rightarrow (iii). Since \mathbb{G} is dense in $L^2(X, \mathfrak{m})$ and invariant with respect to $(P_t)_{t \geq 0}$, we already observed that $\mathbb{G} = \mathbb{V}$. Let us now write (2.35) with $h > 0$ instead of t and with $f := P_{t-s}v$ for some $0 < h < s < t$. Multiplying by $P_{s-h}\varphi$ and integrating with respect to \mathfrak{m} , we obtain

$$B_t[v; \varphi](s - h) + 4\nu I_{-2K}(h)A_t^\Delta[v; \varphi](s - h) \leq e^{-2Kh}B_t[v; \varphi](s).$$

It is not restrictive to assume $\Delta_{\mathcal{E}}\varphi \in L^2 \cap L^\infty(X, \mathfrak{m})$, so that B is of class C^1 in $(0, t)$. We subtract $B_t[v; \varphi](s)$ from both sides of the inequality, we divide by $h > 0$ and let $h \downarrow 0$ obtaining

$$\frac{\partial}{\partial s}B_t[v; \varphi](s) - 2KB_t[v; \varphi](s) \geq 4\nu A_t^\Delta[v; \varphi](s),$$

that is, (2.33).

To show that (iv) \Rightarrow (iii), we first write (2.34) at $t = h > 0$ in the form

$$I_{2K,2}(h)(K\Gamma(P_h f) + 2\nu(\Delta_{\mathcal{E}}P_h f)^2) \leq \frac{1}{2}P_h(f^2) - \frac{1}{2}(P_h f)^2 - h\Gamma(P_h f),$$

obtaining by subtracting $h\Gamma(P_h f)$ from both sides of the inequality. Then we choose $f = P_{t-s-h}v$ and we multiply the inequality by $P_s\varphi$, with $\varphi \in L^2 \cap L^\infty(X, \mathfrak{m})$ nonnegative and $\Delta_{\mathcal{E}}\varphi \in L^2 \cap L^\infty(X, \mathfrak{m})$. We obtain

$$\begin{aligned} I_{2K,2}(h)(2KB_t[v; \varphi](s) + 4\nu A_t^\Delta[v; \varphi](s)) \\ \leq A_t[v; \varphi](s + h) - A_t[v; \varphi](s) - hB_t[v; \varphi](s). \end{aligned}$$

Since A is of class C^2 and $A' = B$, dividing by $h^2 > 0$ and passing to the limit as $h \downarrow 0$ a simple Taylor expansion yields

$$\frac{1}{2} \frac{\partial^2}{\partial s^2}A_t[v; \varphi](s) \geq \frac{1}{2}(2KB_t[v; \varphi](s) + 4\nu A_t^\Delta[v; \varphi](s)).$$

A similar argument shows the last implication (v) \Rightarrow (iii). \square

DEFINITION 2.4 [The condition $BE(K, N)$]. Let $K \in \mathbb{R}$ and $\nu \geq 0$. We say that a Dirichlet form \mathcal{E} in $L^2(X, \mathfrak{m})$ as in (2.1) satisfies a functional $BE(K, N)$ condition if one of the equivalent properties in Corollary 2.3 holds with $N := 1/\nu$.

Notice that

$$BE(K, N) \Rightarrow BE(K', N') \quad \text{for every } K' \leq K, N' \geq N,$$

in particular $BE(K, N) \Rightarrow BE(K, \infty)$.

REMARK 2.5 (Carré du Champ in the case $N = \infty$). If a strongly local Dirichlet form \mathcal{E} satisfies $\text{BE}(K, \infty)$ for some $K \in \mathbb{R}$, then it admits a Carré du Champ Γ on \mathbb{V} , that is, $\mathbb{G} = \mathbb{V}$, by (v) of Corollary 2.3; moreover, the spaces

$$(2.37) \quad \begin{aligned} \mathbb{V}_\infty^1 &:= \{\varphi \in \mathbb{V}_\infty : \Gamma(\varphi) \in L^\infty(X, \mathfrak{m})\}, \\ \mathbb{V}_\infty^2 &:= \{\varphi \in \mathbb{V}_\infty^1 : \Delta_{\mathcal{E}}\varphi \in L^\infty(X, \mathfrak{m})\}, \end{aligned}$$

are dense in \mathbb{V} : in fact (2.35) shows that they are invariant under the action of $(P_t)_{t \geq 0}$ and combined with (2.34) [and possibly combined with a further mollification as in (2.18) in the case of \mathbb{V}_∞^2] it also shows that any element of $L^2 \cap L^\infty(X, \mathfrak{m})$ belongs to their closure w.r.t. the $L^2(X, \mathfrak{m})$ norm. The invariance and the standard approximation argument of, for example, [5], Lemma 4.9, yield the density in \mathbb{V} .

3. Energy metric measure structures. In this section, besides the standing assumptions we made on \mathcal{E} , we shall study the relation between the measure/energetic structure of X and an additional metric structure. Our main object will be the canonical distance $d_{\mathcal{E}}$ associated to the Dirichlet form \mathcal{E} , that we will introduce and study in the next Section 3.3. Before doing that, we will recall the metric notions that will be useful in the following. Since many properties will just depend of a few general compatibility conditions between the metric and the energetic structure, we will try to enucleate such a conditions and state the related theorems in full generality.

Our first condition just refers to the measure \mathfrak{m} and a distance d and it does not involve the Dirichlet form \mathcal{E} :

CONDITION (MD: Measure-Distance interaction). d is a distance on X such that:

- (MD.a) (X, d) is a complete and separable metric space, \mathcal{B} coincides with the completion of the Borel σ -algebra of (X, d) with respect to \mathfrak{m} , and $\text{supp}(\mathfrak{m}) = X$;
- (MD.b) $\mathfrak{m}(B_r(x)) < \infty$ for every $x \in X, r > 0$.

Besides the finiteness condition (MD.b), we will often assume a further exponential growth condition on the measures of the balls of (X, d) , namely that there exist $x_0 \in X, M > 0$ and $c \geq 0$ such that

$$(MD.exp) \quad \mathfrak{m}(B_r(x_0)) \leq M \exp(cr^2) \quad \text{for every } r \geq 0.$$

In this case, we will collectively refer to the above conditions (MD) and (MD.exp) as (MD+exp).

3.1. *Metric notions.* In this section, we recall a few basic definitions and results which are related to a metric measure space (X, d, m) satisfying (MD).

Absolutely continuous curves, Lipschitz functions and slopes. $AC^p([a, b]; X)$, $1 \leq p \leq \infty$, is the collection of all the absolutely continuous curves $\gamma : [a, b] \rightarrow X$ with finite p -energy: $\gamma \in AC^p([a, b]; X)$ if there exists $v \in L^p(a, b)$ such that

$$(3.1) \quad d(\gamma(s), \gamma(t)) \leq \int_s^t v(r) \, dr \quad \text{for every } a \leq s \leq t \leq b.$$

The metric velocity of γ , defined by

$$|\dot{\gamma}|(r) := \lim_{h \rightarrow 0} \frac{d(\gamma(r+h), \gamma(r))}{|h|},$$

exists for \mathcal{L}^1 -a.e. $r \in (a, b)$, belongs to $L^p(a, b)$, and provides the minimal function v , up to \mathcal{L}^1 -negligible sets, such that (3.1) holds. The length of an absolutely continuous curve γ is then defined by $\int_a^b |\dot{\gamma}|(r) \, dr$.

We say that (X, d) is a *length space* if for every $x_0, x_1 \in X$

$$(3.2) \quad d(x_0, x_1) = \inf \left\{ \int_0^1 |\dot{\gamma}|(r) \, dr : \gamma \in AC([0, 1]; X), \gamma(i) = x_i \right\}.$$

We denote by $Lip(X)$ the space of all Lipschitz functions $\varphi : X \rightarrow \mathbb{R}$, by $Lip_b(X)$ the subspace of bounded functions and by $Lip^1(X)$ the subspace of functions with Lipschitz constant less than 1.

Every Lipschitz function φ is absolutely continuous along any absolutely continuous curve; we say that a bounded Borel function $g : X \rightarrow [0, \infty)$ is an *upper gradient* of $\varphi \in Lip(X)$ if for any curve $\gamma \in AC([a, b]; X)$ the absolutely continuous map $\varphi \circ \gamma$ satisfies

$$(3.3) \quad \left| \frac{d}{dt} \varphi(\gamma(t)) \right| \leq g(\gamma(t)) |\dot{\gamma}|(t) \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in (a, b).$$

Among the upper gradients of a function $\varphi \in Lip(X)$, its *slopes* and its *local Lipschitz constant* play a crucial role: they are defined by 0 at every isolated point and by

$$\begin{aligned} |D^\pm \varphi|(x) &:= \limsup_{y \rightarrow x} \frac{(\varphi(y) - \varphi(x))_\pm}{d(y, x)}, & |D\varphi|(x) &:= \limsup_{y \rightarrow x} \frac{|\varphi(y) - \varphi(x)|}{d(y, x)}, \\ |D^* \varphi|(x) &:= \limsup_{\substack{y, z \rightarrow x \\ y \neq z}} \frac{|\varphi(y) - \varphi(z)|}{d(y, z)}, \end{aligned}$$

at every accumulation point $x \in X$. Whenever (X, d) is a length space we have

$$(3.4) \quad |D^* \varphi|(x) = \limsup_{y \rightarrow x} |D\varphi|(y), \quad Lip(\varphi) = \sup_{x \in X} |D\varphi|(x) = \sup_{x \in X} |D^* \varphi|(x).$$

In fact, (3.3) written for $g := |D\varphi|$ and the length condition (3.2) easily yield

$$|\varphi(y) - \varphi(z)| \leq d(y, z) \sup_{B_{2r}(x)} |D\varphi| \quad \text{if } y, z \in B_r(x)$$

and provide the inequality $|D^*\varphi| \leq \limsup_{y \rightarrow x} |D\varphi|(y)$. The proof of the converse inequality is trivial and a similar argument shows the last identity in (3.4).

The Hopf–Lax evolution formula. Let us suppose that (X, d) is a metric space; the Hopf–Lax evolution map $Q_t : C_b(X) \rightarrow C_b(X)$, $t \geq 0$, is defined by

$$(3.5) \quad Q_t f(x) := \inf_{y \in X} f(y) + \frac{d^2(y, x)}{2t}, \quad Q_0 f(x) = f(x).$$

We introduce as in [6], Section 3, the maps

$$D^+(x, t) := \sup_{(y_n)} \limsup_n d(x, y_n), \quad D^-(x, t) := \inf_{(y_n)} \liminf_n d(x, y_n),$$

where the supremum and the infimum run among minimizing sequences for (3.5). We recall that D^+ and D^- are respectively upper and lower semicontinuous, non-decreasing w.r.t. s , and that $D^+(x, r) \leq D^-(x, s) \leq D^+(x, s)$ whenever $0 < r < s$. These properties imply $D^-(x, s) = \sup_{r < s} D^+(x, r)$. We shall need the inequality

$$(3.6) \quad Q_{s'} f(x) - Q_s f(x) \leq \frac{(D^+(x, s))^2}{2} \left(\frac{1}{s'} - \frac{1}{s} \right), \quad s' > s,$$

as well as the pointwise properties

$$(3.7) \quad -\frac{d^\pm}{ds} Q_s f(x) = \frac{(D^\pm(x, s))^2}{2s^2}, \quad |DQ_s f|(x) \leq \frac{D^+(x, s)}{s},$$

(these are proved in Proposition 3.3 and Proposition 3.4 of [6]). Since

$$d(y, x) > 2s \operatorname{Lip}(f) \quad \Rightarrow \quad f(y) + \frac{d^2(x, y)}{2s} > f(x) \geq Q_s f(x),$$

we immediately find $D^+(x, s) \leq 2s \operatorname{Lip}(f)$.

Since by (3.6) the map $s \mapsto Q_s f(x)$ is locally Lipschitz in $(0, \infty)$, integrating the first identity of (3.7) in the interval (ε, t) , $\varepsilon > 0$, and then letting $\varepsilon \downarrow 0$ we get

$$f(x) - Q_t f(x) = \int_0^t \frac{(D^+(x, s))^2}{2s^2} ds = \frac{t}{2} \int_0^1 \left(\frac{D^+(x, tr)}{tr} \right)^2 dr.$$

Combining the above identity with the formula expressing the descending slope (see [3], Lemma 3.1.5)

$$|D^- f|^2(x) = 2 \limsup_{t \downarrow 0} \frac{f(x) - Q_t f(x)}{t},$$

we end up with

$$(3.8) \quad |Df|^2(x) \geq |D^- f|^2(x) = \limsup_{t \downarrow 0} \int_0^1 \left(\frac{D^+(x, tr)}{tr} \right)^2 dr.$$

When (X, d) is a length space $(Q_t)_{t \geq 0}$ is a semigroup and we have the refined identity [6], Theorem 3.6,

$$(3.9) \quad \frac{d^+}{ds} Q_s f(x) = -\frac{1}{2} |DQ_s f|^2(x) = -\frac{(D^+(x, s))^2}{2s^2}.$$

In addition, (3.7) and the length property of X yield the a priori bounds

$$(3.10) \quad \text{Lip}(Q_s f) \leq 2\text{Lip}(f), \quad \text{Lip}(Q_s f(x)) \leq 2[\text{Lip}(f)]^2.$$

The Cheeger energy. The Cheeger energy of a function $f \in L^2(X, m)$ is defined as

$$\text{Ch}(f) := \inf \left\{ \liminf_{n \rightarrow \infty} \frac{1}{2} \int_X |Df_n|^2 dm : f_n \in \text{Lip}_b(X), f_n \rightarrow f \text{ in } L^2(X, m) \right\}.$$

If $f \in L^2(X, m)$ with $\text{Ch}(f) < \infty$, then there exists a unique function $|Df|_w \in L^2(X, m)$, called *minimal weak gradient of f* , satisfying the two conditions

$$\begin{aligned} \text{Lip}_b(X) \cap L^2(X, m) \ni f_n \rightarrow f, \quad |Df_n| \rightarrow G \quad \text{in } L^2(X, m) \\ \Rightarrow |Df|_w \leq G, \end{aligned}$$

$$\text{Ch}(f) = \frac{1}{2} \int_X |Df|_w^2 dm.$$

In the next Section 3.3, we will also use a more refined approximation result, replacing $|Df_n|$ with $|D^* f_n|$ in the approximation, proved in [4], Section 8.3 (see also [1] for a more detailed proof): for every $f \in L^2(X, m)$ with $\text{Ch}(f) < \infty$ there exist $f_n \in \text{Lip}_b(X) \cap L^2(X, m)$ such that

$$(3.11) \quad f_n \rightarrow f, \quad |D^* f_n| \rightarrow |Df|_w \quad \text{strongly in } L^2(X, m).$$

Wasserstein distances. The metric structure allows us to introduce the corresponding spaces $\mathcal{P}(X)$ of Borel probability measures and $\mathcal{P}_p(X)$ of Borel probability measures with finite p th moment, namely $\mu \in \mathcal{P}_p(X)$ if

$$\int_X d^p(x, x_0) d\mu(x) < \infty \quad \text{for some, and then for all, } x_0 \in X.$$

The L^p -Wasserstein transport (extended) distance W_p on $\mathcal{P}(X)$ is defined by

$$(3.12) \quad \begin{aligned} &W_p^p(\mu_1, \mu_2) \\ &:= \inf \left\{ \int_{X \times X} d^p(x_1, x_2) d\mu(x_1, x_2) : \mu \in \mathcal{P}(X \times X), \pi_{\sharp}^i \mu = \mu_i \right\}, \end{aligned}$$

where $\pi^i : X \times X \ni (x_1, x_2) \rightarrow x_i$ is the coordinate map and for a Borel measure $\mu \in \mathcal{P}(Y)$ on a metric space Y and every Borel map $\mathbf{r} : Y \rightarrow X$, the push-forward measure $\mathbf{r}_{\sharp} \mu \in \mathcal{P}(X)$ is defined by

$$\mathbf{r}_{\sharp} \mu(B) := \mu(\mathbf{r}^{-1}(B)) \quad \text{for every Borel set } B \subset X.$$

In particular, the competing measures $\mu \in \mathcal{P}(X \times X)$ in (3.12) have marginals μ_1 and μ_2 , respectively.

We also introduce a family of bounded distances on $\mathcal{P}(X)$ associated to a continuous, concave and bounded modulus of continuity $\beta : [0, \infty) \rightarrow [0, \infty)$, with $0 = \beta(0) < \beta(r)$ for every $r > 0$.

As in (3.12), we set

$$W_{(\beta)}(\mu_1, \mu_2) := \inf \left\{ \int_{X^2} \beta(d(x_1, x_2)) d\mu(x_1, x_2) : \mu \in \mathcal{P}(X \times X), \pi_{\#}^i \mu = \mu_i \right\}.$$

$W_{(\beta)}$ is thus the L^1 -Wasserstein distance induced by the bounded distance $d_{\beta}(x_1, x_2) := \beta(d(x_1, x_2))$. $(\mathcal{P}(X), W_{(\beta)})$ is then a complete and separable metric space, whose topology coincides with the topology of weak convergence of probability measures.

Entropy and RCD(K, ∞) spaces. In the following, we will fix $x_0 \in X, z > 0, c \geq 0$ such that

$$(3.13) \quad \tilde{m} = \frac{1}{z} e^{-V^2} m \in \mathcal{P}(X) \quad \text{with } V(x) := \sqrt{cd}(x, x_0).$$

Notice that in the case $m(X) < \infty$ we can always take $V \equiv c = 0$ with $z = m(X)$. When $m(X) = \infty$, the possibility to choose $x_0 \in X, z > 0, c \geq 0$ satisfying (3.13) follows from (MD.exp) [possibly with a different constant c ; it is in fact equivalent to (MD.exp)]. If $n : \mathcal{B} \rightarrow [0, \infty]$ is σ -additive, the *relative entropy* $\text{Ent}_n(\rho)$ of a probability measure $\rho : \mathcal{B} \rightarrow [0, 1]$ with respect to n is defined by

$$\text{Ent}_n(\rho) := \begin{cases} \int_X f \log f \, dn, & \text{if } \rho = f n; \\ +\infty, & \text{otherwise.} \end{cases}$$

The expression makes sense if n is a probability measure, and thanks to Jensen’s inequality defines a nonnegative functional. More generally, we recall (see [6], Lemma 7.2, for the simple proof) that, when $n = m$ and (3.13) hold, the formula above makes sense on measures $\rho = f m \in \mathcal{P}_2(X)$ thanks to the fact that the negative part of $f \log f$ is m -integrable. Precisely, defining $\tilde{m} \in \mathcal{P}(X)$ as in (3.13) above, the following formula for the change of reference measure will be useful to estimate the negative part of $\text{Ent}_m(\rho)$:

$$\text{Ent}_m(\rho) = \text{Ent}_{\tilde{m}}(\rho) - \int_X V^2(x) d\rho(x) - \log z.$$

DEFINITION 3.1 [RCD(K, ∞) spaces]. Let (X, d, m) be a metric measure space satisfying (MD+exp) and the length property (3.2). We say that (X, d, m) has Riemannian curvature bounded from below by $K \in \mathbb{R}$ if for all $\rho \in \mathcal{P}_2(X)$

there exists a solution $(H_t \rho)_{t \geq 0} \subset \mathcal{P}_2(X)$ of the EVI_K -differential inequality starting from ρ , namely $H_t \rho \rightarrow \rho$ as $t \downarrow 0$ and (denoting by $\frac{d^+}{dt}$ the upper right derivative)

$$(3.14) \quad \frac{d^+}{dt} \frac{W_2^2(H_t \rho, \nu)}{2} + \frac{K}{2} W_2^2(H_t \rho, \nu) + \text{Ent}_m(H_t \rho) \leq \text{Ent}_m(\nu)$$

for every $t \in (0, \infty)$

for all $\nu \in \mathcal{P}_2(X)$ with $\text{Ent}_m(\nu) < \infty$.

As we already quoted in the **Introduction**, among the properties of $\text{RCD}(K, \infty)$ spaces proved in [5] we recall that the Cheeger energy

(QCh) Ch is quadratic, that is, $\text{Ch}(f) = \frac{1}{2} \mathcal{E}_{\text{Ch}}(f)$ for a Dirichlet form \mathcal{E}_{Ch} as in (2.1), with $|Df|_w^2 = \Gamma(f)$ for every $f \in D(\text{Ch}) = \mathbb{V}$,

(in particular $\mathbb{G} = \mathbb{V}$ and \mathcal{E}_{Ch} admits the Carré du Champ Γ in \mathbb{V}) and \mathcal{E}_{Ch} satisfies the $\text{BE}(K, \infty)$ condition. A further crucial property will be recalled in Section 3.3 below; see Condition (ED) and Remark 3.8.

3.2. *The dual semigroup and its contractivity properties.* In this section, we study the contractivity property of the dual semigroup of $(P_t)_{t \geq 0}$ in the spaces of Borel probability measures.

Thus, \mathcal{E} is a strongly local Dirichlet form as in (2.1), $(P_t)_{t \geq 0}$ satisfies the mass-preserving property (2.12) and d is a distance on X satisfying condition (MD) [assumption (MD.exp) is not needed here].

We see how, under the mild contractivity property

$$(3.15) \quad P_t f \in \text{Lip}_b(X) \quad \text{and} \quad \text{Lip}(P_t f) \leq C(t) \text{Lip}(f)$$

for all $f \in \text{Lip}_b(X) \cap L^2(X, m)$,

with C bounded on all intervals $[0, T]$, $T > 0$, a dual semigroup H_t in $\mathcal{P}(X)$ can be defined, satisfying the contractivity property (3.19) below w.r.t. $W_{(\beta)}$ and to W_1 . This yields also the fact that P_t has a (unique) pointwise defined version \tilde{P}_t , canonically defined also on bounded Borel functions, and mapping $C_b(X)$ to $C_b(X)$ [we will always identify $P_t f$ with $\tilde{P}_t f$ whenever $f \in C_b(X)$]. Then we shall prove, following the lines of [34], that in length metric spaces the pointwise Bakry-Émery-like assumption

$$(3.16) \quad |DP_t f|^2(x) \leq C^2(t) \tilde{P}_t |Df|^2(x)$$

for all $x \in X$, $f \in \text{Lip}_b(X) \cap L^2(X, m)$,

with C bounded on all intervals $[0, T]$, $T > 0$, provides contractivity of H_t even w.r.t. W_2 . Notice that formally (3.16) implies (3.15), but one has to take into account that (3.16) involves a pointwise defined version of the semigroup, which might depend on (3.15).

A crucial point here is that we want to avoid doubling or local Poincaré assumptions on the metric measure space. For the aim of this section, we introduce the following notation:

(3.17) \mathcal{Z} is the collection of probability densities $f \in L^1_+(X, \mathfrak{m})$, \mathcal{K} is the set of nonnegative bounded Borel functions $f : X \rightarrow \mathbb{R}$ with bounded support.

PROPOSITION 3.2. *Let \mathcal{E} and $(\mathbf{P}_t)_{t \geq 0}$ be as in (2.1) and (2.12) and let \mathfrak{d} be a distance on X satisfying the condition (MD). If (3.15) holds, then:*

(i) *The mapping $H_t(f\mathfrak{m}) := (\mathbf{P}_t f)\mathfrak{m}$, $f \in \mathcal{Z}$, uniquely extends to a $W_{(\beta)}$ -Lipschitz map $H_t : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$ satisfying for every $\mu, \nu \in \mathcal{P}(X)$*

$$(3.18) \quad W_{(\beta)}(H_t\mu, H_t\nu) \leq (C(t) \vee 1)W_{(\beta)}(\mu, \nu),$$

$$(3.19) \quad W_1(H_t\mu, H_t\nu) \leq C(t)W_1(\mu, \nu),$$

with $C(t)$ given by (3.15).

(ii) *Defining $\tilde{\mathbf{P}}_t f(x) := \int_X f \, dH_t\delta_x$ on bounded or nonnegative Borel functions, $\tilde{\mathbf{P}}_t$ is everywhere defined and maps $C_b(X)$ to $C_b(X)$. Moreover, $\tilde{\mathbf{P}}_t$ is a version of \mathbf{P}_t for all Borel functions f with $\int_X |f| \, d\mathfrak{m} < \infty$, namely $\tilde{\mathbf{P}}_t f(x)$ is defined and $\mathbf{P}_t f(x) = \tilde{\mathbf{P}}_t f(x)$ for \mathfrak{m} -a.e. $x \in X$. In particular, $\tilde{\mathbf{P}}_t f$ is \mathfrak{m} -a.e. defined for every Borel function semiintegrable w.r.t. \mathfrak{m} .*

(iii) H_t is dual to $\tilde{\mathbf{P}}_t$ in the following sense:

$$(3.20) \quad \int_X f \, dH_t\mu = \int_X \tilde{\mathbf{P}}_t f \, d\mu$$

for all $f : X \rightarrow \mathbb{R}$ bounded Borel, $\mu \in \mathcal{P}(X)$.

(iv) *For every $f \in C_b(X)$ and $x \in X$, we have $\lim_{t \downarrow 0} \tilde{\mathbf{P}}_t f(x) = f(x)$. In particular, for every $\mu \in \mathcal{P}(X)$ the map $t \mapsto H_t\mu$ is weakly continuous in $\mathcal{P}(X)$.*

PROOF. The concavity of β yields that β is subadditive, so that \mathfrak{d}_β is a distance. Let us first prove that \mathbf{P}_t maps \mathfrak{d}_β -Lipschitz functions in \mathfrak{d}_β -Lipschitz functions.

We use the envelope representation

$$\beta(r) = \inf_{(a,b) \in \mathbf{B}} a + br, \quad \mathbf{B} = \{(a, b) \in [0, \infty)^2 : \beta(s) \leq a + bs \text{ for every } s \geq 0\},$$

and the fact that a function $\varphi : X \rightarrow \mathbb{R}$ is ℓ -Lipschitz with respect to a distance \mathfrak{d} on X if and only if

$$\varphi(x) \leq R_\ell\varphi(x) := \inf_{y \in X} \varphi(y) + \ell\mathfrak{d}(x, y) \quad \text{for every } x \in X.$$

It is easy to check that if φ is bounded, then $R_\ell\varphi$ is bounded and satisfies

$$(3.21) \quad \inf_X \varphi \leq R_\ell\varphi(x) \leq \varphi(x) \quad \text{for every } x \in X, \ell \geq 0.$$

Furthermore, if φ has also bounded support then $R_\ell\varphi$ has bounded support as well, so that R_ℓ maps \mathcal{K} in \mathcal{K} . The contractivity property (3.15) then yields for every $\varphi \in \text{Lip}_b(X) \cap L^2(X, \mathfrak{m})$ with $\text{Lip}(\varphi) \leq b$

$$P_t\varphi \leq R_{C(t)b}(P_t\varphi).$$

Let us now suppose that $\varphi \in \mathcal{K}$ is d_β -Lipschitz, with Lipschitz constant less than 1, so that for every $(a, b) \in \mathbb{B}$

$$\varphi(x) \leq \inf_{y \in X} \varphi(y) + \beta(d(x, y)) \leq \inf_{y \in X} \varphi(y) + a + bd(x, y) = a + R_b\varphi(x).$$

Since $(P_t)_{t \geq 0}$ is order preserving, we get for $\varphi \in \mathcal{K}$

$$P_t\varphi \leq a + P_t(R_b\varphi) \leq a + R_{C(t)b}(P_t(R_b\varphi)) \leq a + R_{C(t)b}(P_t\varphi),$$

where we used the right inequality of (3.21) and the fact that $\text{Lip}(R_b\varphi) \leq b$. It follows that for every $x, y \in X$ and every $(a, b) \in \mathbb{B}$

$$P_t\varphi(x) \leq P_t\varphi(y) + a + C(t)bd(x, y),$$

that is,

$$P_t\varphi(x) - P_t\varphi(y) \leq \beta(C(t)d(x, y)).$$

By Kantorovich duality, for $f, g \in \mathcal{Z}$ we get

$$\begin{aligned} W_{(\beta)}(P_t f \mathfrak{m}, P_t g \mathfrak{m}) &= \sup \left\{ \int_X \varphi(P_t f - P_t g) \, d\mathfrak{m} : \varphi \in \mathcal{K}, \text{Lip}_{d_\beta}(\varphi) \leq 1 \right\} \\ &= \sup \left\{ \int_X P_t \varphi(f - g) \, d\mathfrak{m} : \varphi \in \mathcal{K}, \text{Lip}_{d_\beta}(\varphi) \leq 1 \right\} \\ &\leq (C(t) \vee 1) W_{(\beta)}(f \mathfrak{m}, g \mathfrak{m}). \end{aligned}$$

Hence, (3.18) holds when $\mu = f \mathfrak{m}, \nu = g \mathfrak{m}$. By the density of $\{f \mathfrak{m} : f \in \mathcal{Z}\}$ in $\mathcal{P}(X)$ w.r.t. $W_{(\beta)}$ we get (3.18) for arbitrary $\mu, \nu \in \mathcal{P}(X)$. A similar argument yields (3.19).

(ii) Continuity of $x \mapsto \tilde{P}_t f(x)$ when $f \in C_b(X)$ follows directly by the continuity of $x \mapsto H_t \delta_x$. The fact that $\tilde{P}_t f$ is a version of P_t when f is Borel and \mathfrak{m} -integrable is a simple consequence of the fact that P_t is self-adjoint; see [5] for details.

(iii) When $f, g \in C_b(X) \cap L^2(X, \mathfrak{m})$ and $\mu = g \mathfrak{m}$, the identity (3.20) reduces to the fact that P_t is selfadjoint. The general case can be easily achieved using a monotone class argument.

(iv) In the case of $\varphi \in \text{Lip}_b(X) \cap L^2(X, \mathfrak{m})$ it is easy to prove that $\tilde{P}_t \varphi(x) \rightarrow \varphi(x)$ for all $x \in X$ as $t \downarrow 0$, since $\tilde{P}_t \varphi$ are equi-Lipschitz, converge in $L^2(X, \mathfrak{m})$ to φ and $\text{supp } \mathfrak{m} = X$. By (3.20), it follows that

$$\lim_{t \downarrow 0} \int_X \varphi \, dH_t \mu = \lim_{t \downarrow 0} \int_X \tilde{P}_t \varphi \, d\mu = \int_X \varphi \, d\mu \quad \text{for every } \varphi \in \text{Lip}_b(X) \cap L^2(X, \mathfrak{m}).$$

By a density argument, we obtain that the same holds on $\text{Lip}_b(X)$, so that $t \mapsto H_t \mu$ is weakly continuous. Since $\tilde{P}_t f(x) = \int_X f \, dH_t \delta_x$, we conclude that $\tilde{P}_t f(x) \rightarrow f(x)$ for arbitrary $f \in C_b(X)$. \square

Writing $\mu = \int_X \delta_x \, d\mu(x)$ and recalling the definition of \tilde{P}_t , we can also write (3.20) in the form

$$(3.22) \quad H_t \mu = \int_X H_t \delta_x \, d\mu(x) \quad \forall \mu \in \mathcal{P}(X).$$

In order to prove that (3.16) yields the contractivity property

$$(W_2\text{-cont}) \quad W_2(H_t \mu, H_t \nu) \leq C(t) W_2(\mu, \nu) \quad \text{for every } \mu, \nu \in \mathcal{P}(X), t \geq 0,$$

we need the following auxiliary results.

LEMMA 3.3. *Assume that $(\mu_n) \subset \mathcal{P}(X)$ weakly converges to $\mu \in \mathcal{P}(X)$, and that f_n are equibounded Borel functions satisfying*

$$\limsup_{n \rightarrow \infty} f_n(x_n) \leq f(x) \quad \text{whenever } x_n \rightarrow x$$

for some Borel function f . Then $\limsup_n \int_X f_n \, d\mu_n \leq \int_X f \, d\mu$.

PROOF. Possibly adding a constant, we can assume that all functions f_n are nonnegative. For all integers k and $t > 0$, it holds

$$\mu \left(\overline{\bigcup_{m=k}^{\infty} \{f_m > t\}} \right) \geq \limsup_{n \rightarrow \infty} \mu_n \left(\overline{\bigcup_{m=k}^{\infty} \{f_m > t\}} \right) \geq \limsup_{n \rightarrow \infty} \mu_n(\{f_n > t\}).$$

Taking the intersection of the sets in the left-hand side and noticing that it is contained, by assumption, in $\{f \geq t\}$, we get $\limsup_n \mu_n(\{f_n > t\}) \leq \mu(\{f \geq t\})$. By Cavalieri’s formula and Fatou’s lemma, we conclude. \square

LEMMA 3.4. *Assume (3.15), (3.16) and the length property (3.2). For all $f \in \text{Lip}_b(X)$ nonnegative and with bounded support, $Q_t f$ is Lipschitz, nonnegative with bounded support and it holds*

$$|P_t Q_1 f(x) - P_t f(y)| \leq \frac{1}{2} C^2(t) d^2(x, y) \quad \text{for every } t \geq 0, x, y \in X.$$

PROOF. It is immediate to check that $Q_s f(x) = 0$ if $f(x) = 0$, so that the support of all functions $Q_s f$, $s \in [0, 1]$, are contained in a given ball and $Q_s f$ are also equi-bounded.

The stated inequality is trivial for $t = 0$, so assume $t > 0$. By (3.9), for every $s > 0$, setting $r_k := s - 1/k \uparrow s$, the sequence $r_k^2 |D Q_{r_k} f|^2(x)$ monotonically converges to the function $D^-(x, s)$; we can thus pass to the limit in the upper gradient inequality [which is a consequence of (3.16)]

$$|P_t Q_{r_k} f(\gamma_1) - P_t Q_{r_k} f(\gamma_0)| \leq C(t) \int_0^1 \sqrt{\tilde{P}_t |D Q_{r_k} f|^2(\gamma_s)} |\dot{\gamma}_s| \, ds$$

to get that the function $C(t)G_s$, with $G_s(x) := s^{-1}\sqrt{\bar{P}_t(D^-(x, s)^2)}$, is an upper gradient for $P_t(Q_s f)$. Moreover, combining (3.6) and (3.5), we obtain

$$(3.23) \quad \begin{aligned} \limsup_{h \downarrow 0} \frac{Q_{s+h}f(x_h) - Q_s f(x_h)}{h} &\leq \frac{1}{2s^2} \limsup_{h \downarrow 0} -(D^+(x_h, s))^2 \\ &\leq -\frac{1}{2s^2} (D^-(x, s))^2 \end{aligned}$$

along an arbitrary sequence $x_h \rightarrow x$.

Let γ be a Lipschitz curve with $\gamma_1 = x$ and $\gamma_0 = y$. We interpolate with a parameter $s \in [0, 1]$, setting $g(s) := P_t Q_s f(\gamma_s)$. Using (3.10) and (3.15), we obtain that g is absolutely continuous in $[0, 1]$, so that we need only to estimate $g'(s)$. For $h > 0$, we write

$$\frac{g(s+h) - g(s)}{h} = \int_X \frac{Q_{s+h}f - Q_s f}{h} dH_t \delta_{\gamma_{s+h}} + \frac{P_t Q_s f(\gamma_{s+h}) - P_t Q_s f(\gamma_s)}{h}$$

and estimate the two terms separately. The first term can be estimated as follows:

$$(3.24) \quad \begin{aligned} \limsup_{h \downarrow 0} \int_X \frac{Q_{s+h}f - Q_s f}{h} dH_t \delta_{\gamma_{s+h}} &\leq -\frac{1}{2s^2} \int_X D^-(\cdot, s)^2 dH_t \delta_{\gamma_s} \\ &= -\frac{1}{2} G_s^2(\gamma_s). \end{aligned}$$

Here, we applied Lemma 3.3 with $f_h(x) = (Q_{s+h}f(x) - Q_s f(x))/h$, $\mu_h = H_t \delta_{\gamma_{s+h}}$ and $\mu = H_t \delta_{\gamma_s}$, taking (3.23) into account.

The second term can be estimated as follows. By the upper gradient property of $C(t)G_s$ for $P_t(Q_s f)$, we get

$$(3.25) \quad \limsup_{h \downarrow 0} \frac{|P_t Q_s f(\gamma_{s+h}) - P_t Q_s f(\gamma_s)|}{h} \leq G_s(\gamma_s) C(t) |\dot{\gamma}_s|$$

for a.e. $s \in (0, 1)$, more precisely at any Lebesgue point of $|\dot{\gamma}|$ and of $s \mapsto G_s(\gamma_s)$. Combining (3.24) and (3.25) and using the Young inequality, we get $|P_t Q_1 f(x) - P_t f(y)| \leq C^2(t) \frac{1}{2} \int_0^1 |\dot{\gamma}_s|^2 ds$. Minimizing with respect to γ gives the result. \square

THEOREM 3.5. *Let \mathcal{E} and $(P_t)_{t \geq 0}$ be as in (2.1) and (2.12), and let d be a distance on X under the assumptions (MD) and (3.2). Then (3.15) and (3.16) are satisfied by $(P_t)_{t \geq 0}$ if and only if $(W_2\text{-cont})$ holds.*

PROOF. We only prove the W_2 contraction assuming that (3.15) and (3.16) hold, since the converse implication have been already proved in [34] (see also [5], Theorem 6.2) and it does not play any role in this paper.

We first notice that Kantorovich duality provides the identity

$$\frac{1}{2} W_2^2(H_t \delta_x, H_t \delta_y) = \sup |P_t Q_1 f(x) - P_t f(y)|,$$

where the supremum runs in the class of bounded, nonnegative Lipschitz functions f with bounded support. Therefore, Lemma 3.4 gives

$$(3.26) \quad W_2^2(H_t \delta_x, H_t \delta_y) \leq C^2(t) d^2(x, y).$$

Now, given $\mu, \nu \in \mathcal{P}(X)$ with $W_2(\mu, \nu) < \infty$ and a corresponding optimal plan γ , we may use a measurable selection theorem (see, e.g., [17], Theorem 6.9.2) to select in a γ -measurable way optimal plans γ_{xy} from $H_t \delta_x$ to $H_t \delta_y$. Then we define

$$\gamma_0 := \int_{X \times X} \gamma_{xy} d\gamma(x, y),$$

and notice that, because of (3.22), γ_0 is an admissible plan from $H_t \mu$ to $H_t \nu$. Since (3.26) provides the inequality $\int d^2 d\gamma_0 \leq C^2(t) \int d^2 d\gamma$, we conclude. \square

3.3. *Energy measure spaces.* In this section, we want to study more carefully the interaction between the energy and the metric structures, particularly in the case when the initial structure is not provided by a distance, but rather by a Dirichlet form \mathcal{E} .

Given a Dirichlet form \mathcal{E} in $L^2(X, m)$ as in (2.1), assume that \mathcal{B} is the m -completion of the Borel σ -algebra of (X, τ) , where τ is a given topology in X . Then, under these structural assumptions, we define a first set of “locally 1-Lipschitz” functions as follows:

$$\mathcal{L} := \{\psi \in \mathbb{G} : \Gamma(\psi) \leq 1 \text{ m-a.e. in } X\}, \quad \mathcal{L}_C := \mathcal{L} \cap C(X).$$

With this notion at hand, we can generate canonically the intrinsic (possibly infinite) pseudo-distance [16]:

$$d_{\mathcal{E}}(x_1, x_2) := \sup_{\psi \in \mathcal{L}_C} |\psi(x_2) - \psi(x_1)| \quad \text{for every } x_1, x_2 \in X.$$

We also introduce 1-Lipschitz truncation functions $S_k \in C^1(\mathbb{R})$, $k > 0$, defined by

$$(3.27) \quad S_k(r) := kS(r/k) \quad \text{with } S(r) = \begin{cases} 1, & \text{if } |r| \leq 1, \\ 0, & \text{if } |r| \geq 3, \end{cases} \quad |S'(r)| \leq 1.$$

We have now all the ingredients to define the following structure.

DEFINITION 3.6 (Energy measure space). Let (X, τ) be a Polish space, let m be a Borel measure with full support, let \mathcal{B} be the m -completion of the Borel σ -algebra and let \mathcal{E} be a Dirichlet form in $L^2(X, m)$ satisfying (2.1) of Section 2.1. We say that $(X, \tau, m, \mathcal{E})$ is a Energy measure space if:

(a) There exists a function

$$(3.28) \quad \theta \in C(X), \theta \geq 0, \text{ such that } \theta_k := S_k \circ \theta \text{ belongs to } \mathcal{L} \text{ for every } k > 0.$$

(b) $d_{\mathcal{E}}$ is a finite distance in X which induces the topology τ and $(X, d_{\mathcal{E}})$ is complete.

Notice that if $m(X) < \infty$ and $1 \in D(\mathcal{E})$ then (a) is always satisfied by choosing $\theta \equiv 0$. In the general case, condition (a) is strictly related to the finiteness property of the measure of balls (MD.b). In fact, we shall see in Theorem 3.9 that $(X, d_{\mathcal{E}}, m)$ satisfies the measure distance condition (MD).

REMARK 3.7 (Completeness and length property). Whenever $d_{\mathcal{E}}$ induces the topology τ [and thus $(X, d_{\mathcal{E}})$ is a separable space], completeness is not a restrictive assumption, since it can always be obtained by taking the abstract completion \bar{X} of X with respect to $d_{\mathcal{E}}$. Since (X, τ) is a Polish space, X can be identified with a Borel subset of \bar{X} ([17], Theorem 6.8.6), and m can be easily extended to a Borel measure \bar{m} on \bar{X} by setting $\bar{m}(B) := m(B \cap X)$; in particular $\bar{X} \setminus X$ is \bar{m} -negligible and \mathcal{E} can be considered as a Dirichlet form on $L^2(\bar{X}, \bar{m})$ as well. Finally, once completeness is assumed, the length property is a consequence of the definition of the intrinsic distance $d_{\mathcal{E}}$; see [50, 52] in the locally compact case and the next Corollary 3.10 in the general case.

In many cases, τ is already induced by a distance d satisfying the compatibility condition (MD), so that we are actually dealing with a structure (X, d, m, \mathcal{E}) . In this situation, it is natural to investigate under which assumptions the identity $d = d_{\mathcal{E}}$ holds: this in particular guarantees that $(X, \tau, m, \mathcal{E})$ is an Energy measure space according to Definition 3.6. In the following remark, we examine the case when \mathcal{E} is canonically generated starting from d and m , and then we investigate possibly more general situations.

REMARK 3.8 (The case of a quadratic Cheeger energy). Let (X, d, m) be a metric measure space satisfying (MD) and let us assume that the Cheeger energy is quadratic (i.e., (X, d, m) is infinitesimally Hilbertian according to [27]), $\mathcal{E}_{Ch} := 2Ch$. Then it is clear that any 1-Lipschitz function $f \in L^2(X, m)$ belongs to \mathcal{L}_C , hence $d_{\mathcal{E}} \geq d$. It follows that $d = d_{\mathcal{E}}$ if and only if every continuous function $f \in D(Ch)$ with $|Df|_w \leq 1$ m -a.e. in X is 1-Lipschitz w.r.t. d . In particular, this is the case of $RCD(K, \infty)$ spaces.

If $X = [0, 1]$ endowed with the Lebesgue measure and the Euclidean distance, and if $m = \sum_n 2^{-n} \delta_{q_n}$, where (q_n) is an enumeration of $\mathbb{Q} \cap [0, 1]$, then it is easy to check that $Ch \equiv 0$ (see [6] for details), hence $d_{\mathcal{E}}(x, y) = \infty$ whenever $x \neq y$.

If d is a distance on X satisfying (MD), in order to provide links between the Dirichlet form \mathcal{E} and the distance d we introduce the following condition.

CONDITION (ED: Energy-Distance interaction). d is a distance on X such that:

- (ED.a) every function $\psi \in \mathcal{L}_C$ is 1-Lipschitz with respect to d ;
- (ED.b) every function $\psi \in Lip(X, d)$ with $|D\psi| \leq 1$ and bounded support belongs to \mathcal{L}_C .

THEOREM 3.9. *If $(X, \tau, \mathfrak{m}, \mathcal{E})$ is an Energy measure space according to Definition 3.6 then the canonical distance $\mathfrak{d}_{\mathcal{E}}$ satisfies conditions (MD), (ED).*

Conversely, if \mathcal{E} is a Dirichlet form as in (2.1) and \mathfrak{d} is a distance on $X \times X$ inducing the topology τ and satisfying conditions (MD), (ED), then $(X, \tau, \mathfrak{m}, \mathcal{E})$ is an Energy measure space and

$$(3.29) \quad \mathfrak{d}(x_1, x_2) = \mathfrak{d}_{\mathcal{E}}(x_1, x_2) \quad \text{for every } x_1, x_2 \in X.$$

PROOF. Let us first assume that $(X, \tau, \mathfrak{m}, \mathcal{E})$ is an Energy measure space. (MD.a) is immediate since $\mathfrak{d}_{\mathcal{E}}$ is complete by assumption and τ is separable. (ED.a) is also a direct consequence of the definition of \mathcal{E} , since

$$(3.30) \quad |\psi(x_2) - \psi(x_1)| \leq \mathfrak{d}_{\mathcal{E}}(x_1, x_2) \quad \text{for every } \psi \in \mathcal{L}_C, x_1, x_2 \in X.$$

Let us now prove (MD.b) and (ED.b).

We first observe that the function θ of (3.28) is bounded on each ball $B_r(y)$, $y \in X$ and $r > 0$, otherwise we could find a sequence of points $y_k \in B_r(y)$, $k \in \mathbb{N}$, such that $\theta(y_k) \geq 3k$ and, therefore, $\theta_k(y) - \theta_k(y_k) \geq k$ whenever $\theta(y) \leq k$. This contradicts the fact that θ_k is 1-Lipschitz by (3.30). As a consequence, for every $y \in X$ and $r > 0$ there exists $k_{y,r} \in \mathbb{N}$ such that

$$\theta_k(x) \equiv k \quad \text{for every } x \in B_r(y), k \geq k_{y,r}.$$

In particular, since $\theta_k \in L^2(X, \mathfrak{m})$, we get that all the sets $B_r(y)$ with $y \in X$ and $r > 0$ have finite measure, so that (MD.b) holds.

We observe that by the separability of $X \times X$ we can find a countable family $(\psi_n) \subset \mathcal{L}_C$ such that

$$\mathfrak{d}_{\mathcal{E}}(x_1, x_2) = \sup_n |\psi_n(x_2) - \psi_n(x_1)| \quad \text{for every } x_1, x_2 \in X.$$

We set

$$\mathfrak{d}_{k,N}(x_1, x_2) := \left(\sup_{n \leq N} |\psi_n(x_2) - \psi_n(x_1)| \right) \wedge \theta_k(x_2),$$

observing that for every $y \in X$ the map $x \mapsto \mathfrak{d}_{k,N}(y, x)$ belongs to \mathcal{L}_C . Passing to the limit as $N \rightarrow \infty$, it is easy to check that $\mathfrak{d}_{k,N}(y, \cdot) \rightarrow \mathfrak{d}_k(y, \cdot) = \mathfrak{d}_{\mathcal{E}}(y, \cdot) \wedge \theta_k$ pointwise in X and, therefore, in $L^2(X, \mathfrak{m})$, since $\theta_k \in L^2(X, \mathfrak{m})$. We deduce that $\mathfrak{d}_k(y, \cdot) \in \mathcal{L}$ for every $y \in X$ and $k \in \mathbb{N}$.

Let us now prove that every map $f \in \text{Lip}(X, \mathfrak{d}_{\mathcal{E}})$ with $|Df| \leq 1$ and bounded support belongs to \mathcal{L} ; it is not restrictive to assume f nonnegative. Since $(X, \mathfrak{d}_{\mathcal{E}})$ is a length metric space (see Theorem 3.10 below), f is 1-Lipschitz and it is easy to check that, setting $f_k = f \wedge \theta_k$, it holds

$$\begin{aligned} f_k(x) &= \left(\inf_{z \in X} (f(z) + \mathfrak{d}_{\mathcal{E}}(z, x)) \right) \wedge \theta_k(x) = \inf_{z \in X} ((f_k(z) + \mathfrak{d}_{\mathcal{E}}(z, x)) \wedge \theta_k(x)) \\ &= \left(\inf_{z \in X} (f_k(z) + \mathfrak{d}_{\mathcal{E}}(z, x) \wedge \theta_k(x)) \right) \wedge \theta_k(x) \\ &= \left(\inf_{z \in X} (f_k(z) + \mathfrak{d}_k(z, x)) \right) \wedge \theta_k(x). \end{aligned}$$

Let (z_i) be a countable dense set of X . The functions

$$f_{k,n}(x) := \left(\min_{1 \leq i \leq n} f_k(z_i) + d_k(z_i, x) \right) \wedge \theta_k(x)$$

belong to \mathcal{L} , are nonincreasing with respect to n , and satisfy $0 \leq f_{k,n} \leq \theta_k$. Since

$$\{x \in X : \theta_k(x) > 0\} \subset \{x \in X : \theta_{3k}(x) = 3k\},$$

we easily see that $m(\text{supp}(\theta_k)) < \infty$. Passing to the limit as $n \uparrow \infty$, since $z \mapsto d_k(z, x)$ is continuous, they converge monotonically to

$$\left(\inf_{z \in X} (f_k(z) + d_k(z, x)) \right) \wedge \theta_k(x) = f_k(x),$$

and their energy is uniformly bounded by $m(\text{supp}(\theta_k))$. This shows that $f_k \in \mathcal{L}$. Eventually, letting $k \uparrow \infty$ and recalling that $\text{supp}(f_k) \subset \text{supp}(f)$ and $m(\text{supp}(f)) < \infty$ by (MD.a), we obtain $f \in \mathcal{L}$.

The converse implication is easier: it is immediate to check that (ED.a) is equivalent to

$$(3.31) \quad d(x_1, x_2) \geq d_{\mathcal{E}}(x_1, x_2) \quad \text{for every } x_1, x_2 \in X;$$

if (ED.b) holds and balls have finite measure according to (MD.b), we have $x \mapsto T_k(d(y, x)) \in \mathcal{L}$ for every $y \in X$, where $T_k(r) := r \wedge S_k(r)$. Since

$$d(x_1, x_2) = T_k(d(x_2, x_1)) - T_k(d(x_2, x_2)) \quad \text{whenever } k > d(x_1, x_2),$$

we easily get the converse inequality to (3.31) and, therefore, (3.29) and property (b) of Definition (3.6). In order to get also (a), it is sufficient to take $\theta(x) := d(x, x_0)$ for an arbitrary $x_0 \in X$. \square

THEOREM 3.10 (Length property of $d_{\mathcal{E}}$). *If $(X, \tau, m, \mathcal{E})$ is an Energy measure space, then $(X, d_{\mathcal{E}})$ is a length metric space, that is, it also satisfies (3.2).*

PROOF. We follow the same argument as in [50, 52]. Since $(X, d_{\mathcal{E}})$ is complete, it is well known (see, e.g., [19], Theorem 2.4.16) that the length condition is equivalent to show that for every couple of points $x_0, x_1 \in X$ and $\varepsilon \in (0, r)$ with $r := d_{\mathcal{E}}(x_0, x_1)$ there exists an ε -midpoint $y \in X$ such that

$$d_{\mathcal{E}}(y, x_i) < \frac{r}{2} + \varepsilon, \quad i = 0, 1.$$

We argue by contradiction assuming that $B_{r/2+\varepsilon}(x_0) \cap B_{r/2+\varepsilon}(x_1) = \emptyset$ for some $\varepsilon \in (0, r)$ and we introduce the function

$$\psi(x) := \left(\frac{1}{2}(r + \varepsilon) - d_{\mathcal{E}}(x, x_0) \right)_+ - \left(\frac{1}{2}(r + \varepsilon) - d_{\mathcal{E}}(x, x_1) \right)_+.$$

ψ is Lipschitz, has bounded support and it is easy to check that $\Gamma(\psi) \leq 1$ m-a.e. in X , since $B_{r/2+\varepsilon}(x_0)$ and $B_{r/2+\varepsilon}(x_1)$ are disjoint. It turns out that it is 1-Lipschitz by (3.30). On the other hand, $\psi(x_0) - \psi(x_1) = r + \varepsilon > d_{\mathcal{E}}(x_0, x_1)$. \square

We now examine some additional properties of Energy measure spaces.

PROPOSITION 3.11. *Let $(X, \tau, m, \mathcal{E})$ be an Energy measure space. Let $f \in \mathbb{G} \cap C_b(X)$ and let $\zeta : X \rightarrow [0, \infty)$ be a bounded upper semicontinuous function such that $\Gamma(f) \leq \zeta^2$ m-a.e. in X . Then f is Lipschitz (with respect to the induced distance $d_\mathcal{E}$) and $|D^* f| \leq \zeta$. In particular, ζ is an upper gradient of f .*

PROOF. We know that (ED) holds with $d = d_\mathcal{E}$, by the previous theorem. Since ζ is bounded, f is Lipschitz by (ED.a). We fix $x \in X$ and for every $\varepsilon > 0$ we set $G_\varepsilon := \sup_{B_{3\varepsilon}(x)} \zeta$. The Lipschitz function

$$\psi_\varepsilon(y) := [|f(y) - f(x)| \vee (G_\varepsilon d(y, x))] \wedge (G_\varepsilon S_\varepsilon(d(x, y)))$$

belongs to \mathbb{V}_∞ and $\sqrt{\Gamma(\psi)} \leq \max\{\zeta, G_\varepsilon\}$, since $|S'_\varepsilon| \leq 1$; moreover, $\psi_\varepsilon(y) = 0$ if $d(y, x) \geq 3\varepsilon$, so that $\sqrt{\Gamma(\psi_\varepsilon)} \leq G_\varepsilon$ m-a.e. in X . It follows that ψ_ε is G_ε -Lipschitz and $\psi_\varepsilon(y) \leq G_\varepsilon d(y, x)$ for every $y \in X$ since $\psi_\varepsilon(x) = 0$, so that $G_\varepsilon S_\varepsilon(d(x, y)) = G_\varepsilon \varepsilon$ for $d(x, y) < \varepsilon$ gives

$$|Df(x)| \leq \limsup_{y \rightarrow x} \frac{|f(y) - f(x)|}{d(y, x)} \leq \limsup_{y \rightarrow x} \frac{\psi_\varepsilon(y)}{d(y, x)} \leq G_\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary and $\lim_{\varepsilon \downarrow 0} G_\varepsilon = \zeta(x)$ we obtain $|Df(x)| \leq \zeta(x)$. Since ζ is upper semicontinuous and X is a length space, we also get $|D^* f| \leq \zeta$. \square

The following result provides a first inequality between \mathcal{E} and Ch, in the case when a priori the distances d and $d_\mathcal{E}$ are different, and we assume only (ED.b).

THEOREM 3.12. *\mathcal{E} be a Dirichlet form in $L^2(X, m)$ satisfying (2.1) of Section 2.1 and let d be a distance on X satisfying condition (MD). Then condition (ED.b) is satisfied if and only if for every function $f \in \text{Lip}(X, d)$ with bounded support we have*

$$(3.32) \quad f \in \mathbb{G}, \quad |Df|^2 \geq \Gamma(f) \quad \text{m-a.e. in } X.$$

In addition, if (MD) and (ED.b) hold, we have

$$(3.33) \quad 2\text{Ch}(g) \geq \mathcal{E}(g) \quad \text{for every } g \in L^2(X, m),$$

$$(3.34) \quad D(\text{Ch}) \subset \mathbb{G} \subset \mathbb{V}, \quad |Dg|_w^2 \geq \Gamma(g) \quad \text{for every } g \in D(\text{Ch}).$$

PROOF. The implication (3.32) \Rightarrow (ED.b) is trivial; let us consider the converse one.

Since the statement is local, possibly replacing f with $0 \vee (f + c) \wedge S_k(d(x_0, \cdot))$ with $c = 0 \vee \sup_{B_{3k}(x_0)}(-f)$ (notice that f is bounded) and k large enough, we can assume that f is nonnegative and $f \leq S_k(d(x_0, \cdot))$.

Recall the Hopf–Lax formula (3.5) for the map Q_t ; if (z_i) is a countable dense subset of X we define

$$(3.35) \quad \begin{aligned} Q_t^n f(x) &= \left(\min_{1 \leq i \leq n} f(z_i) + \frac{1}{2t} d^2(z_i, x) \right), \\ Q_t^{n,k} f(x) &= Q_t^n f(x) \wedge 2S_k(d(x_0, x)), \end{aligned}$$

and we set $I_n(x) = \{i \in \{1, \dots, n\} : z_i \text{ minimizes (3.35)}\}$. By the density of (z_n) , it is clear that $Q_t^{n,k} f \downarrow Q_t f \wedge 2S_k(d(x_0, \cdot)) = Q_t f$, because $Q_t f \leq f \leq S_k(d(x_0, \cdot))$. Therefore, if $z_n(x) \in \{z_i : i \in I_n(x)\}$, it turns out that $(z_n(x))$ is a minimizing sequence for $Q_t f(x)$, namely

$$\frac{1}{2t} d^2(x, z_n(x)) + f(z_n(x)) \rightarrow Q_t f(x) \quad \text{as } n \rightarrow \infty.$$

The very definition of $D^+(t, x)$ then gives

$$(3.36) \quad \limsup_{n \rightarrow \infty} \frac{1}{t} d(x, z_n(x)) \leq D^+(t, x).$$

The locality property, the fact that $(f(z_i) + d^2(z_i, \cdot)/2t) \wedge S_k(d(x_0, \cdot)) \in \mathbb{V}$ and the obvious bound

$$d^2(z_i, x) \leq 4kt \quad \text{if } i \in I_n(x), x \in \{Q_t^n f \leq 2S_k(d(x_0, \cdot))\},$$

yield

$$\sqrt{\Gamma(Q_t^{n,k} f)(x)} \leq \frac{1}{t} \max_{i \in I_n(x)} d(x, z_i) \leq 2\sqrt{\frac{k}{t}}$$

for m-a.e. $x \in \{Q_t^n f \leq 2S_k(d(x_0, \cdot))\}$. If we define $z_n(x)$ as a value z_j that realizes the maximum for $d(z_i, x)$ as $i \in I_n(x)$, the previous formula yields

$$(3.37) \quad \sqrt{\Gamma(Q_t^{n,k} f)(x)} \leq \frac{1}{t} d(x, z_n(x)) \leq 2\sqrt{\frac{k}{t}}$$

for m-a.e. $x \in \{Q_t^n f \leq 2S_k(d(x_0, \cdot))\}$,

so that

$$\Gamma(Q_t^{n,k} f) \leq 4\left(1 \vee \frac{k}{t}\right) \quad \text{m-a.e. in } X,$$

$$\Gamma(Q_t^{n,k} f) = 0 \quad \text{m-a.e. in } X \setminus B_{3k}(x_0).$$

Since $Q_t^{n,k} f$ and $\Gamma(Q_t^{n,k} f)$ are uniformly bounded and supported in a bounded set, and $Q_t^{n,k} f$ pointwise converges to $Q_t f$ as $n \rightarrow \infty$, considering any weak limit point G of $\sqrt{\Gamma(Q_t^{n,k} f)}$ in $L^2(X, m)$ we obtain by (2.10), (3.36) and (3.37)

$$(3.38) \quad \Gamma(Q_t f)(x) \leq G^2(x) \leq \frac{(D^+(x, t))^2}{t^2}$$

for m-a.e. $x \in \{Q_t f < 2S_k(d(x_0, \cdot))\}$.

Since, by the inequalities $Q_t f \leq f \leq S_k(d(x_0, \cdot))$, $Q_t f$ vanishes on $\{Q_t f \geq 2S_k(d(x_0, \cdot))\}$, the inequality above holds m-a.e. in X .

Since f is Lipschitz, it follows that $D^+(x, t)/t$ is uniformly bounded. Integrating (3.8) on an arbitrary bounded Borel set A and applying Fatou’s lemma, from (3.38) we get

$$\begin{aligned} \int_A |Df|^2 \, dm &\geq \int_A \limsup_{t \downarrow 0} \int_0^1 \left(\frac{D^+(x, tr)}{tr} \right)^2 \, dr \, dm(x) \\ &\geq \limsup_{t \downarrow 0} \int_0^1 \int_A \left(\frac{D^+(x, tr)}{tr} \right)^2 \, dm(x) \, dr \\ &\geq \limsup_{t \downarrow 0} \int_0^1 \int_A \Gamma(Q_{tr}f)(x) \, dm(x) \, dr \\ &\geq \int_0^1 \liminf_{t \downarrow 0} \left(\int_A \Gamma(Q_{tr}f) \, dm \right) \, dr \geq \int_A \Gamma(f) \, dm, \end{aligned}$$

where in the last inequality we applied (2.10) once more. Since A is arbitrary we conclude.

In order to prove (3.33), it is not restrictive to assume $\text{Ch}(g) < \infty$. By the very definition of the Cheeger energy, we can then find a sequence of functions $f_n \in \text{Lip}_b(X) \cap L^2(X, m)$ converging to g in $L^2(X, m)$ with $\lim_{n \rightarrow \infty} \int_X |Df_n|^2 \, dm = 2\text{Ch}(g)$.

By replacing f_n with $g_n(x) = f_n(x)S_1(d(x, x_0)/n)$, we can even obtain a sequence of Lipschitz functions with bounded support. (3.32) and the lower semi-continuity of \mathcal{E} then provide (3.33) and (3.34). \square

In order to conclude our analysis of the relations between \mathcal{E} and Ch for Energy measure spaces $(X, \tau, m, \mathcal{E})$, we introduce a further property.

DEFINITION 3.13 (Upper regularity). Let $(X, \tau, m, \mathcal{E})$ be an Energy measure space. We say that the Dirichlet form \mathcal{E} is upper-regular if for every f in a dense subset of \mathbb{V} there exist $f_n \in \mathbb{G} \cap C_b(X)$ converging strongly to f in $L^2(X, m)$ and $g_n : X \rightarrow \mathbb{R}$ bounded and upper semicontinuous such that

$$(3.39) \quad \sqrt{\Gamma(f_n)} \leq g_n \quad \text{m-a.e.}, \quad \limsup_{n \rightarrow \infty} \int_X g_n^2 \, dm \leq \mathcal{E}(f).$$

THEOREM 3.14. Let $(X, \tau, m, \mathcal{E})$ be an Energy measure space. Then the Cheeger energy associated to $(X, d_{\mathcal{E}}, m)$ coincides with \mathcal{E} , that is,

$$(3.40) \quad \mathcal{E}(f) = 2\text{Ch}(f) \quad \text{for every } f \in L^2(X, m),$$

if and only if \mathcal{E} is upper-regular. In this case $\mathbb{G} = \mathbb{V}$, \mathcal{E} admits a Carré du Champ Γ and

$$(3.41) \quad \Gamma(f) = |Df|_w^2 \quad \text{m-a.e. in } X \text{ for every } f \in \mathbb{V}.$$

In particular, the space $\mathbb{V} \cap \text{Lip}_b(X, d_{\mathcal{E}})$ is dense in \mathbb{V} . If moreover (MD.exp) holds, then $(P_t)_{t \geq 0}$ satisfies the mass preserving property (2.12).

PROOF. Since Ch is always upper-regular by (3.11), the condition is clearly necessary. In order to prove its sufficiency, by (3.33) of Theorem 3.12 we have just to prove that every $f \in \mathbb{V}$ satisfies the inequality $2\text{Ch}(f) \leq \mathcal{E}(f)$. If f_n, g_n are sequences as in (3.39), Proposition 3.11 yields that f_n are Lipschitz and

$$|Df_n| \leq g_n, \quad \text{Ch}(f_n) \leq \frac{1}{2} \int_X |Df_n|^2 \, dm \leq \frac{1}{2} \int_X g_n^2 \, dm.$$

Passing to the limit as $n \rightarrow \infty$, we obtain the desired inequality thanks to the lower-semicontinuity of Ch in $L^2(X, m)$. The last statement of the Theorem follows by [6], Theorem 4.20. \square

3.4. *Riemannian Energy measure spaces and the $\text{BE}(K, \infty)$ condition.* In this section, we will discuss various consequences of the Energy measure space axiomatization in combination with $\text{BE}(K, \infty)$.

Taking into account the previous section, the Bakry-Émery condition $\text{BE}(K, N)$ as stated in Definition 2.4 makes perfectly sense for a Energy measure space $(X, \tau, m, \mathcal{E})$. In the next result, we will show that under a weak-Feller property on the semigroup $(P_t)_{t \geq 0}$ we gain upper-regularity of \mathcal{E} , the identifications $\mathcal{E} = 2\text{Ch}$ of Theorem 3.14 and $\mathcal{L} = \mathcal{L}_C$.

THEOREM 3.15. *Let $(X, \tau, m, \mathcal{E})$ be a Energy measure space satisfying the $\text{BE}(K, \infty)$ condition. Then its Markov semigroup $(P_t)_{t \geq 0}$ satisfies the weak-Feller property*

$$\begin{aligned} \text{(w-Feller)} \quad & f \in \text{Lip}_b(X, d_{\mathcal{E}}) \text{ with bounded support,} \quad |Df| \leq 1 \\ & \Rightarrow P_t f \in C_b(X) \quad \forall t \geq 0 \end{aligned}$$

if and only if

$$(3.42) \quad \mathcal{L} = \mathcal{L}_C,$$

that is, if every function $f \in \mathcal{L}$ admits a continuous representative. In this case, \mathcal{E} is upper-regular and, as a consequence, (3.40), (3.41) hold.

PROOF. The implication (3.42) \Rightarrow (w-Feller) is easy, since for any $f \in \text{Lip}_b(X, d)$ with $|Df| \leq 1$ and bounded support, the Bakry-Émery condition $\text{BE}(K, \infty)$ [i.e., (2.35) with $\nu = 0$] and the bound $\Gamma(f) \leq 1$ given by (ED.b) yields $e^{Kt} P_t f \in \mathcal{L} = \mathcal{L}_C$.

Now we prove the converse implication, from (w-Feller) to (3.42). The Bakry-Émery condition $\text{BE}(K, \infty)$ in conjunction with (ED.b) and (w-Feller) yield $e^{Kt} P_t f \in \mathcal{L}_C$ for every $f \in \text{Lip}_b(X, d)$ with $|Df| \leq 1$ and bounded support and $t > 0$, thus in particular $e^{Kt} P_t f$ is 1-Lipschitz by (ED.a). Let us now fix $f \in \mathcal{L} \cap L^\infty(X, m)$ and let us consider a sequence of uniformly bounded functions $f_n \in \text{Lip}_b(X) \cap L^2(X, m)$ with bounded support converging to f in $L^2(X, m)$. By the previous step, we know that $P_t f_n \in \text{Lip}_b(X)$ and the estimate (2.34) shows

that $\Gamma(P_t f_n) \leq C/t$ for a constant C independent of n . (ED) then shows that $\text{Lip}(P_t f_n) \leq C/t$; passing to the limit as $n \rightarrow \infty$, we can find a subsequence $n_k \rightarrow \infty$ such that $\lim_k P_t f_{n_k}(x) = P_t f(x)$ for every $x \in X \setminus \mathcal{N}$, with $m(\mathcal{N}) = 0$. Since $P_t f_n$ are uniformly Lipschitz functions, also $P_t f$ is Lipschitz in $X \setminus \mathcal{N}$ so that it admits a Lipschitz representative \tilde{f}_t in X .

On the other hand, $\text{BE}(K, \infty)$ and (ED.a) show that $\text{Lip}(f_t) \leq e^{-Kt}$. Passing to the limit along a suitable sequence $t_n \downarrow 0$ and repeating the previous argument we obtain that f admits a Lipschitz representative.

Let us prove now that \mathcal{E} is upper regular, by checking (3.39) for every f in the space \mathbb{V}_∞^1 of (2.37), which is dense in \mathbb{V} . Observe that the estimate (2.34), (3.42) and (ED.a) yield that, for every $t > 0$ and every $g \in L^2 \cap L^\infty(X, m)$, the function $P_t g$ admits a Lipschitz, thus continuous, and bounded representative g_t . Choosing in particular $g := \sqrt{\Gamma(f)}$, we obtain by (2.35)

$$\Gamma(P_t f) \leq e^{-2Kt} g_t^2, \quad P_t f \rightarrow f \text{ in } L^2(X, m),$$

$$\lim_{t \downarrow 0} \int_X e^{-2Kt} g_t^2 \, dm = \int_X g^2 \, dm = \mathcal{E}(f). \quad \square$$

According to the previous theorem we introduce the natural, and smaller, class of Energy measure spaces $(X, \tau, m, \mathcal{E})$, still with no curvature bound, but well adapted to the Bakry–Émery condition. In such a class, that we call *Riemannian Energy measure spaces*, the Dirichlet form \mathcal{E} coincides with the Cheeger energy Ch associated to the intrinsic distance $d_\mathcal{E}$ and every function in \mathcal{L} admits a continuous (thus 1-Lipschitz, by the Energy measure space axiomatization) representative.

DEFINITION 3.16 (Riemannian Energy measure spaces). $(X, \tau, m, \mathcal{E})$ is a Riemannian Energy measure space if the following properties hold:

- (a) $(X, \tau, m, \mathcal{E})$ is a Energy measure space;
- (b) \mathcal{E} is upper regular according to Definition 3.13;
- (c) every function in \mathcal{L} admits a continuous representative.

The next theorem presents various equivalent characterizations of Riemannian Energy measure spaces in connection with $\text{BE}(K, \infty)$.

THEOREM 3.17. *The following conditions are equivalent:*

- (i) $(X, \tau, m, \mathcal{E})$ is a Riemannian Energy measure space satisfying $\text{BE}(K, \infty)$.
- (ii) $(X, \tau, m, \mathcal{E})$ is a Energy measure space satisfying (w-Feller) and $\text{BE}(K, \infty)$.
- (iii) $(X, \tau, m, \mathcal{E})$ is a Energy measure space satisfying $\mathcal{L} = \mathcal{L}_C$ and $\text{BE}(K, \infty)$.
- (iv) $(X, \tau, m, \mathcal{E})$ is a Energy measure space with \mathcal{E} upper regular, and for every function $f \in L^2(X, m) \cap \text{Lip}_b(X, d_\mathcal{E})$ with $|Df| \in L^2(X, m)$

$$(3.43) \quad P_t f \in \text{Lip}(X, d_\mathcal{E}), \quad |DP_t f|^2 \leq e^{-2Kt} P_t(|Df|^2) \quad m\text{-a.e. in } X.$$

(v) \mathcal{E} is a Dirichlet form in $L^2(X, \mathfrak{m})$ as in (2.1), there exists a length distance d on X inducing the topology τ and satisfying conditions (MD), (ED.b), and for every $f \in \mathcal{L}_C \cap L^\infty(X)$ and $t > 0$

$$(3.44) \quad P_t f \in \text{Lip}_b(X, d), \quad |DP_t f|^2 \leq e^{-2Kt} P_t \Gamma(f) \quad \mathfrak{m}\text{-a.e. in } X.$$

If one of the above equivalent conditions holds with (MD.exp), then (3.15) holds with $C(t) = e^{-Kt}$, the semigroups $(\tilde{P}_t)_{t \geq 0}$ and $(H_t)_{t \geq 0}$ are well defined according to Proposition 3.2, $H_t(\mu) \ll \mathfrak{m}$ for every $t > 0$ and $\mu \in \mathcal{P}(X)$, and the strong Feller property

$$(S\text{-Feller}) \quad \tilde{P}_t \text{ maps } L^2 \cap L^\infty(X, \mathfrak{m}) \text{ into } \text{Lip}_b(X, d_\mathcal{E})$$

holds with

$$(3.45) \quad |D\tilde{P}_t f|^2 = \Gamma(P_t f) \quad \mathfrak{m}\text{-a.e. in } X \text{ for every } t > 0, f \in L^2 \cap L^\infty(X, \mathfrak{m}).$$

Eventually, defining $I_{2K}(t)$ as in (2.29), there holds

$$(3.46) \quad 2I_{2K}(t)|D\tilde{P}_t f|^2 \leq \tilde{P}_t f^2 \quad \text{for every } t \in (0, \infty), f \in L^\infty(X, \mathfrak{m}),$$

and in particular

$$(3.47) \quad \sqrt{2I_{2K}(t)} \text{Lip}(P_t f) \leq \|f\|_{L^\infty(X, \mathfrak{m})} \quad \text{for every } t \in (0, \infty).$$

PROOF. The equivalence (i) \Leftrightarrow (ii) \Leftrightarrow (iii) is just the statement of Theorem 3.15.

(v) \Rightarrow (ii) and (v) \Rightarrow (iv). Thanks to (ED.b), we immediately get (w-Feller). Equation (3.44) also yields the density of \mathbb{V}_∞ in \mathbb{V} and, thanks to (3.32), condition (2.35) for $\text{BE}(K, \infty)$. Since $(P_t)_{t \geq 0}$ is order preserving, (3.32) and (v) yield (ED.a): if $f \in \mathcal{L}_C$ then (3.44) and the length property yield $P_t f$ Lipschitz with constant less than e^{-Kt} . Since along a suitable vanishing sequence $t_n \downarrow 0$ $P_{t_n} f \rightarrow f$ \mathfrak{m} -a.e. as $n \rightarrow \infty$, arguing as in the proof of Theorem 3.15 it is easy to check that f is 1-Lipschitz. We can thus apply Theorem 3.17 to get that $(X, \tau, \mathfrak{m}, \mathcal{E})$ is an Energy measure space with $d \equiv d_\mathcal{E}$. Since (ii) in particular shows that \mathcal{E} is upper-regular, we proved that (v) \Rightarrow (iv) as well.

(iv) \Rightarrow (ii). By the density of $\mathbb{V} \cap \text{Lip}_b(X, d_\mathcal{E})$ in $D(\text{Ch}) = \mathbb{V}$ stated in Theorem 3.14 and the upper bound (3.32) we get (2.35) which is one of the equivalent characterizations of $\text{BE}(K, \infty)$. Moreover, (3.43) clearly yields (w-Feller).

(i) \Rightarrow (v) with the choice $d := d_\mathcal{E}$ and (3.46). Let us observe that the estimate (2.34) and the property $\mathcal{L} \subset \text{Lip}(X)$ yield that for every $t > 0$ and every function $f \in L^2 \cap L^\infty(X, \mathfrak{m})$ $P_t f$ admits a Lipschitz representative satisfying (3.47). Moreover, if f is also Lipschitz, (2.35) yields the estimate (3.15) with $C(t) := e^{-Kt}$. We can then apply proposition 3.2 and conclude that when $f \in C_b(X) \cap L^2(X, \mathfrak{m})$ the Lipschitz representative of $P_t f$ coincides with $\tilde{P}_t f$. Since by definition

$$\tilde{P}_t f(x) = \int_X f dH_t \delta_x \quad \text{for all Borel } f \text{ bounded from below}$$

we can use a monotone class argument to prove the identification of \tilde{P}_t with the continuous version of P_t in the general case of bounded, Borel and square integrable functions. Notice that we use (3.47) to convert monotone equi-bounded convergence of f_n into pointwise convergence on X of (the continuous representative of) $P_t f_n$, $t > 0$.

Another immediate application of (3.47) is the absolute continuity of $H_t \mu$ w.r.t. m for all $\mu \in \mathcal{P}(X)$ and $t > 0$. Indeed, if A is a Borel and m -negligible set, then $\tilde{P}_t \chi_A$ is identically null (being equal to $P_t \chi_A$, hence continuous, and null m -a.e. in X), hence (3.20) gives $H_t \mu(A) = 0$. As a consequence, we can also compute $\tilde{P}_t f(x)$ for m -measurable functions f , provided f is semi-integrable with respect to $H_t \delta_x$. If now $f \in \mathcal{L}$, (2.35) then yields

$$\Gamma(P_t f) \leq e^{-2Kt} \tilde{P}_t \Gamma(f) \quad m\text{-a.e. in } X,$$

and Proposition 3.11 yields (3.44) since $\tilde{P}_t \Gamma(f)$ is continuous and bounded. A similar argument shows (3.46), starting from (2.34).

Let us eventually prove (3.45). Since the inequality \geq is true by assumption, let us see why (3.44) provides the converse one: we start from

$$|D\tilde{P}_t f|^2 \leq e^{-2K\varepsilon} \tilde{P}_\varepsilon(\Gamma(P_{t-\varepsilon} f)) \quad \text{for every } \varepsilon \in (0, t).$$

Recalling that $\Gamma(P_{t-\varepsilon} f)$ converges strongly in $L^2(X, m)$ to $\Gamma(P_t f)$ as $\varepsilon \downarrow 0$, we get (3.45). \square

Recalling Theorem 3.17, Theorem 3.5, the characterization (3.43), and the notation (3.17), we immediately have

COROLLARY 3.18. *Let $(X, \tau, m, \mathcal{E})$ be a Energy measure space satisfying the upper-regularity property (3.39) (in particular a Riemannian one) and (MD.exp).*

Then $BE(K, \infty)$ holds if and only if the semigroup $(P_t)_{t \geq 0}$ satisfies the contraction property

$$(3.48) \quad W_2((P_t f)m, (P_t g)m) \leq e^{-Kt} W_2(fm, gm)$$

for every probability densities $f, g \in L^1_+(X, m)$.

4. Proof of the equivalence result. In this section, we assume that $(X, \tau, m, \mathcal{E})$ is a Riemannian Energy measure space satisfying (MD.exp) (relative to $d_{\mathcal{E}}$) and the $BE(K, \infty)$ condition, as discussed in Section 3.4. In particular, all results of the previous sections on existence of the dual semigroup $(H_t)_{t \geq 0}$, its W_2 -contractivity and regularizing properties of $(P_t)_{t \geq 0}$ are applicable. Furthermore, by Theorem 3.14, the Dirichlet form setup described in Section 2 and the metric setup described in Section 3.1 are completely equivalent. In particular, we can apply the results of [6].

4.1. *Entropy, Fisher information and moment estimates.* Let us first recall that the Fisher information functional $F: L^1_+(X, m) \rightarrow [0, \infty]$ is defined by

$$F(f) := 4\mathcal{E}(\sqrt{f}), \quad \sqrt{f} \in \mathbb{V},$$

set equal to $+\infty$ if $\sqrt{f} \in L^2(X, m) \setminus \mathbb{V}$. Since $f_n \rightarrow f$ in $L^1_+(X, m)$ implies $\sqrt{f_n} \rightarrow \sqrt{f}$ in $L^2(X, m)$, F is L^1 -lower semicontinuous.

PROPOSITION 4.1. *Let $f \in L^1_+(X, m)$. Then $\sqrt{f} \in \mathbb{V}$ if and only if $f_N := \min\{f, N\} \in \mathbb{V}$ for all N and $\int_{\{f>0\}} \Gamma(f)/f \, dm < \infty$. If this is the case,*

$$F(f) = \int_{\{f>0\}} \frac{\Gamma(f)}{f} \, dm.$$

In addition, F is convex in $L^1_+(X, m)$.

We refer to [6], Lemma 4.10, for the proof, observing only that $\Gamma(f)$ is well defined m -a.e. in the class of functions satisfying $f_N := \min\{f, N\} \in \mathbb{V}$ for all N thanks to the locality property (2.9). By applying the results of [6], we can prove that $(H_t)_{t \geq 0}$ is a continuous semigroup in $\mathcal{P}_2(X)$ and we can calculate the dissipation rate of the entropy functional along it. Some of the results below are very simple in the case $m(X) < \infty$, where the function V in (3.13) reduces to a constant.

LEMMA 4.2 (Estimates on moments, Fisher information, Entropy, metric derivative). *If $f \in L^2(X, m)$ is a probability density, for $\mu_t = P_t f \, m$ it holds*

$$(4.1) \quad |\dot{\mu}_t|^2 \leq F(P_t f) \quad \text{for } \mathcal{L}^1\text{-a.e. } t > 0.$$

If $f \in L^1(X, m)$ is a probability density, for every $T > 0$ there exists $C_T > 0$ such that

$$(4.2) \quad \int_0^T F(P_s f) \, ds + \int_0^T \int_X V^2 P_s f \, dm \, ds \leq C_T \left(\text{Ent}_m(f \, m) + \int_X V^2 f \, dm \right),$$

$$(4.3) \quad \text{Ent}_m(P_t f) \leq \text{Ent}_m(f \, m) \quad \forall t \geq 0.$$

Finally, for every $\bar{\mu} \in \mathcal{P}_2(X)$ the map $t \mapsto \mu_t := H_t \bar{\mu}$ is a continuous curve in $\mathcal{P}_2(X)$ with respect to W_2 .

PROOF. The estimate (4.1) follows by [6], Lemma 6.1, which can be applied here since the Dirichlet form \mathcal{E} coincides with the Cheeger energy (Theorem 3.14). The estimate (4.2) follows [6], Theorem 4.20, thanks to the integrability condition (3.13), when $f \in L^2(X, m)$. In the general case, it can be recovered by a truncation argument, using the lower semicontinuity of F . The estimate (4.3) can be obtained by a similar approximation argument from the detailed energy dissipation computation made in [6], Theorem 4.16(b) [the more general underlying fact

is that, independently of curvature assumptions, L^2 gradient flows are contained in W_2 gradient flows of the Entropy in the sense (weaker than EVI) of energy dissipation].

Concerning the continuity of the map $t \mapsto H_t \bar{\mu}$ with respect to W_2 for every $\bar{\mu} \in \mathcal{P}_2(X)$, it is a standard consequence of contractivity and existence of a dense set of initial conditions [namely, thanks to (4.1) and (4.2), the set $D(\text{Ent}_m) := \{\nu \in \mathcal{P}_2(X) : \text{Ent}_m(\nu) < \infty\}$] for which continuity holds up to $t = 0$. \square

Integration by parts for probability densities. We shall see now that assuming the Bakry–Émery $\text{BE}(K, \infty)$ condition, integration by parts formulae for the $\Delta_\varepsilon^{(1)}$ operator can be extended to probability densities with finite Fisher information, provided that the set of test functions φ is restricted to the spaces $\mathbb{V}_\infty^1, \mathbb{V}_\infty^2$ defined in (2.37). Recall that $\mathbb{V}_\infty^1, \mathbb{V}_\infty^2$ are dense in \mathbb{V} w.r.t. the strong topology and that

$$\Gamma(\varphi_\varepsilon - \varphi) \xrightarrow{*} 0 \quad \text{in } L^\infty(X, m) \text{ as } \varepsilon \downarrow 0 \text{ for every } \varphi \in \mathbb{V}_\infty^1,$$

where φ_ε are defined as in (2.18), since $\Gamma(\varphi_\varepsilon - \varphi)$ is uniformly bounded and converges to 0 in $L^1(X, m)$ (by the strong convergence of φ_ε to φ in \mathbb{V}).

In the sequel, we introduce an extension of the bilinear form $\Gamma(f, g)$, denoted $\tilde{\Gamma}(f, g)$, which is particularly appropriate to deal with probability densities f with finite Fisher information and test functions $\varphi \in \mathbb{V}_\infty^1$.

DEFINITION 4.3 [Extension of $\Gamma(f, \varphi)$]. Let $f = g^2 \in L^1_+(X, m)$ with $F(f) = 4\mathcal{E}(g) < \infty$ and $g \geq 0$. For all $\varphi \in \mathbb{V}_\infty^1$ we define

$$(4.4) \quad \tilde{\Gamma}(f, \varphi) := 2g\Gamma(g, \varphi).$$

The definition is well posed, consistent with the case when $f \in \mathbb{V}$ and it holds

$$(4.5) \quad \tilde{\Gamma}(f, \varphi) = \lim_{N \rightarrow \infty} \Gamma(f_N, \varphi) \quad \text{in } L^1(X, m).$$

Indeed, thanks to (2.7) and to the fact that if $F(f) < \infty$ then $f_N = (g_N)^2 \in \mathbb{V}$, where $g_N := g \wedge \sqrt{N}$; it follows that $\Gamma(f_N, \varphi) = 2g\chi_N\Gamma(g, \varphi)$, χ_N being the characteristic function of the set $\{f < N\}$ and

$$\begin{aligned} \int_X |\tilde{\Gamma}(f, \varphi) - \Gamma(f_N, \varphi)| \, dm &= 2 \int_X |1 - \chi_N| g \Gamma(g, \varphi) \, dm \\ &\leq \left(\|\Gamma(\varphi)\|_\infty F(f) \int_{\{f \geq N\}} f \, dm \right)^{1/2} \end{aligned}$$

thus proving the limit in (4.5). The same argument provides the estimate

$$(4.6) \quad \int_X \psi |\tilde{\Gamma}(f, \varphi)| \, dm \leq \sqrt{F(f)} \left(\int_X \psi^2 \Gamma(\varphi) f \, dm \right)^{1/2}, \quad \varphi \in \mathbb{V}_\infty^1, \psi \geq 0.$$

THEOREM 4.4 (Integration by parts of $\Delta_{\mathcal{E}}^{(1)}$). *If $\text{BE}(K, \infty)$ holds, then for every $f \in L^1_+(X, \mathfrak{m})$ with $F(f) < \infty$ we have*

$$(4.7) \quad \int_X \tilde{\Gamma}(f, \varphi) \, d\mathfrak{m} = - \int_X f \Delta_{\mathcal{E}} \varphi \, d\mathfrak{m} \quad \text{for every } \varphi \in \mathbb{V}^2_{\infty}.$$

In addition, if $f \in D(\Delta_{\mathcal{E}}^{(1)})$ it holds

$$(4.8) \quad \int_X \tilde{\Gamma}(f, \varphi) \, d\mathfrak{m} = - \int_X \varphi \Delta_{\mathcal{E}}^{(1)} f \, d\mathfrak{m} \quad \forall \varphi \in \mathbb{V}^1_{\infty}.$$

PROOF. Formula (4.7) follows by the limit formula in (4.5) simply integrating by parts before passing to the limit as $N \rightarrow \infty$. Assuming now $f \in D(\Delta_{\mathcal{E}}^{(1)})$ we have

$$\begin{aligned} - \int_X \Delta_{\mathcal{E}}^{(1)} f \varphi \, d\mathfrak{m} &= \lim_{t \downarrow 0} \frac{1}{t} \int_X (f - P_t f) \varphi \, d\mathfrak{m} = \lim_{t \downarrow 0} \frac{1}{t} \int_X f (\varphi - P_t \varphi) \, d\mathfrak{m} \\ &= \lim_{t \downarrow 0} \frac{1}{t} \int_0^t \int_X f \Delta_{\mathcal{E}} (P_s \varphi) \, d\mathfrak{m} \, ds \\ &= \lim_{t \downarrow 0} \frac{1}{t} \int_0^t \int_X \tilde{\Gamma}(f, P_s \varphi) \, d\mathfrak{m} \, ds = \int_X \tilde{\Gamma}(f, \varphi), \end{aligned}$$

where the last limit follows by (4.6) and the fact that

$$\Gamma\left(\frac{1}{t} \int_0^t P_s \varphi \, ds - \varphi\right) \leq \frac{1}{t} \int_0^t \Gamma(P_s \varphi - \varphi) \, ds \xrightarrow{*} 0 \quad \text{in } L^{\infty}(X, \mathfrak{m}) \text{ as } t \downarrow 0. \quad \square$$

4.2. *Log-Harnack and $L \log L$ estimates.*

LEMMA 4.5. *Let $\omega : [0, \infty) \rightarrow \mathbb{R}$ be a function of class C^2 , let $f \in \text{Lip}_b(X) \cap \mathbb{V}$ and let $\mu \in \mathcal{P}(X)$. The function*

$$G(s) := \int_X \omega(P_{t-s} f) \, dH_s \mu, \quad s \in [0, t],$$

belongs to $C^0([0, t]) \cap C^1(0, t)$ and for every $s \in (0, t)$ it holds

$$(4.9) \quad G'(s) = \int_X \omega''(P_{t-s} f) \Gamma(P_{t-s} f) \, dH_s \mu.$$

PROOF. Since $H_s \mu$ are all probability measures is not restrictive to assume $\omega(0) = 0$. Continuity of G is obvious, since $P_{t-s} f$ are equi-Lipschitz, equi-bounded and the semigroup H_t is weakly-continuous. Let us first consider the case $\mu = \zeta \mathfrak{m}$ with $\zeta \in L^1 \cap L^{\infty}(X, \mathfrak{m})$ [in particular $\zeta \in L^2(X, \mathfrak{m})$]. Setting $f_{t-s} := P_{t-s} f$ and $\zeta_s := P_s \zeta$, we observe that a.e. in the open interval $(0, t)$ the following properties hold:

- $s \mapsto \zeta_s$ is differentiable in $L^2(X, \mathfrak{m})$;

– $s \mapsto \omega(f_{t-s})$ is differentiable in $L^2(X, \mathfrak{m})$, with derivative $-\omega'(f_{t-s})\Delta_\varepsilon f_{t-s}$.

Therefore, the chain rule (2.11) gives

$$\begin{aligned} G'(s) &= - \int_X \omega'(f_{t-s})\zeta_s \Delta_\varepsilon f_{t-s} \, \mathrm{d}\mathfrak{m} + \int_X \omega(f_{t-s})\Delta_\varepsilon \zeta_s \, \mathrm{d}\mathfrak{m} \\ &= \int_X (\Gamma(\omega'(f_{t-s})\zeta_s, f_{t-s}) - \Gamma(\omega(f_{t-s}), \zeta_s)) \, \mathrm{d}\mathfrak{m} \\ &= \int_X \omega''(f_{t-s})\Gamma(f_{t-s})\zeta_s \, \mathrm{d}\mathfrak{m} \end{aligned}$$

for \mathcal{L}^1 -a.e. $s \in (0, t)$. But, since the right-hand side is continuous, the formula holds pointwise in $(0, t)$. The formula for arbitrary measures $\mu = \zeta \mathfrak{m} \in \mathcal{P}(X)$ then follows by monotone approximation (i.e., considering $\zeta_n = \min\{\zeta, n\}/c_n$ with $c_n \uparrow 1$ normalizing constants), by using the uniform L^∞ bound on $\omega''(f_{t-s})$ and on $\Gamma(f_{t-s})$; the formula (4.9) for G' still provides a continuous function since $\mathbb{H}_s \mu = \zeta_s \mathfrak{m}$ and $\zeta_s = \mathbb{P}_s \zeta$ is strongly continuous in $L^1(X, \mathfrak{m})$. Finally, if $\mu \in \mathcal{P}(X)$, for every $\varepsilon > 0$ and $s \in (\varepsilon, t - \varepsilon)$, setting $\mu_\varepsilon = \mathbb{H}_\varepsilon \mu$ we have

$$G(s) = \tilde{G}_\varepsilon(s - \varepsilon) \quad \text{where } \tilde{G}_\varepsilon(s) := \int_X \omega(\mathbb{P}_{(t-\varepsilon)-s} f) \, \mathrm{d}\mathbb{H}_s \mu_\varepsilon.$$

Since \tilde{G}_ε is a C^1 function in $(\varepsilon, t - \varepsilon)$ by the previous considerations and $\varepsilon > 0$ is arbitrary, we conclude that $G \in C^1(0, t)$ and for every $s \in (\varepsilon, t - \varepsilon)$ its derivative coincides with

$$\begin{aligned} \tilde{G}'_\varepsilon(s - \varepsilon) &= \int_X \omega''(\mathbb{P}_{t-\varepsilon-(s-\varepsilon)} f) \Gamma(\mathbb{P}_{t-\varepsilon-(s-\varepsilon)} f) \, \mathrm{d}\mathbb{H}_{s-\varepsilon} \mu_\varepsilon \\ &= \int_X \omega''(\mathbb{P}_{t-s} f) \Gamma(\mathbb{P}_{t-s} f) \, \mathrm{d}\mathbb{H}_s \mu. \end{aligned} \quad \square$$

In order to prove the $L \log L$ regularization, we use the next lemma, which follows by a careful adaptation to our more abstract context of a result by Wang [57], Theorem 1.1(6).

LEMMA 4.6 (Log-Harnack inequality). *For every nonnegative $f \in L^1(X, \mathfrak{m}) + L^\infty(X, \mathfrak{m})$, $t > 0$, $\varepsilon \in [0, 1]$, and $x, y \in X$ we have $\log(1 + f) \in L^1(X, \mathbb{H}_t \delta_y)$ with*

$$(4.10) \quad \tilde{\mathbb{P}}_t(\log(f + \varepsilon))(y) \leq \log(\tilde{\mathbb{P}}_t f(x) + \varepsilon) + \frac{\mathbf{d}^2(x, y)}{4\mathbb{I}_{2K}(t)}.$$

PROOF. In the following, we set $\omega_\varepsilon(r) := \log(r + \varepsilon)$, for $r \geq 0$ and $\varepsilon \in (0, 1]$. Let us first assume in addition that $f \in \text{Lip}_b(X) \cap L^1(X, \mathfrak{m}) \cap \mathbb{V}$, let $\gamma : [0, 1] \rightarrow X$ be a Lipschitz curve connecting x to y in X , and, recalling the definition (2.29) of \mathbb{I}_K , let

$$\tilde{\gamma}_r = \gamma_{\theta(r)} \quad \text{with } \theta(r) = \frac{\mathbb{I}_{2K}(r)}{\mathbb{I}_{2K}(t)}, r \in [0, t].$$

We set $f_{t-s} := P_{t-s}f$ and, for $r \in [0, t]$ and $s \in (0, t)$, we consider the functions

$$G(r, s) := \int_X \omega_\varepsilon(f_{t-s}) dH_s \delta_{\tilde{\gamma}_r} = F_s(\tilde{\gamma}_r) \quad \text{with } F_s(x) := \tilde{P}_s(\omega_\varepsilon(f_{t-s}))(x).$$

Notice that Lemma 4.5 with $\mu = \delta_{\tilde{\gamma}_r}$ ensures that for every $r \in [0, t]$ the function $s \mapsto G(r, s)$ is Lipschitz in $[0, t]$, with

$$(4.11) \quad \frac{\partial}{\partial s} G(r, s) = \int_X \omega''_\varepsilon(f_{t-s}) \Gamma(f_{t-s}) dH_s \delta_{\tilde{\gamma}_r} \quad \text{for every } s \in (0, t).$$

This gives immediately that $G(r, \cdot)$ are uniformly Lipschitz in $[0, t]$ for $r \in [0, t]$. On the other hand, since $\tilde{\gamma}$ is Lipschitz, the map $r \mapsto P_s \delta_{\tilde{\gamma}_r}$ is Lipschitz in $[0, t]$ with respect to the L^1 -Wasserstein distance W_1 uniformly w.r.t. $s \in [0, t]$. Hence, taking also the fact that $\omega(f_{t-s})$ are equi-Lipschitz into account, it follows that also the maps $G(\cdot, s)$ are Lipschitz in $[0, t]$, with Lipschitz constant uniform w.r.t. $s \in [0, t]$. These properties imply that the map $s \mapsto G(s, s)$ is Lipschitz in $[0, t]$.

Since the chain rule and (3.44) [which can be applied since we can subtract the constant $\omega_\varepsilon(0)$ from F_s without affecting the calculation of its slope] give

$$|DF_s|^2 \leq e^{-2Ks} P_s((\omega'_\varepsilon(f_{t-s}))^2 \Gamma(f_{t-s})),$$

we can use $\theta'(r) = e^{2Kr} / I_{2K}(t)$ to get the pointwise estimate

$$\begin{aligned} e^{K(s-r)} \limsup_{h \downarrow 0} \frac{|G(r+h, s) - G(r, s)|}{h} &\leq \sqrt{\theta'(r)} \frac{|\dot{\gamma}_{\theta(r)}|}{\sqrt{I_{2K}(t)}} e^{Ks} |DF_s|(\tilde{\gamma}_r) \\ &\leq \theta'(r) \frac{|\dot{\gamma}_{\theta(r)}|^2}{4I_{2K}(t)} + \int_X (\omega'_\varepsilon(f_{t-s}))^2 \Gamma(f_{t-s}) dP_s \delta_{\tilde{\gamma}_r}. \end{aligned}$$

Applying the calculus lemma [3], Lemma 4.3.4, and using the identity $\omega''_\varepsilon = -(\omega'_\varepsilon)^2$, the previous inequality with $r = s$ in combination with (4.11) gives

$$\begin{aligned} \frac{d}{ds} G(s, s) &\leq \lim_{h \downarrow 0} \frac{G(s, s-h) - G(s, s)}{h} + \limsup_{h \downarrow 0} \frac{|G(s+h, s) - G(s, s)|}{h} \\ &\leq \theta'(s) \frac{|\dot{\gamma}_{\theta(s)}|^2}{4I_{2K}(t)} \quad \text{for } \mathcal{L}^1\text{-a.e. } s \in (0, t). \end{aligned}$$

An integration in $(0, t)$ and a minimization w.r.t. γ yield

$$(4.12) \quad \int_X \omega_\varepsilon(f) dH_t \delta_y \leq \omega_\varepsilon(f_t)(x) + \frac{d^2(x, y)}{4I_{2K}(t)}.$$

If $f \in L^\infty(X, \mathfrak{m})$, we consider a uniformly bounded sequence (f_n) contained in $\text{Lip}_b(X, \mathfrak{m}) \cap L^1(X, \mathfrak{m}) \cap \mathbb{V}$ converging to f pointwise \mathfrak{m} -a.e. Since $\omega_\varepsilon \geq \log(\varepsilon)$ and $\tilde{P}_t f_n$ converges to $\tilde{P}_t f$ pointwise, Fatou's lemma yields (4.12) also in this

case. Finally, a truncation argument extends the validity of (4.12) and (4.10) to arbitrary nonnegative $f \in L^1(X, \mathfrak{m}) + L^\infty(X, \mathfrak{m})$. Passing to the limit as $\varepsilon \downarrow 0$, we get (4.10) also in the case $\varepsilon = 0$.

Finally, notice that (4.12) for $\varepsilon = 1$ and the fact that $\tilde{P}_t f(x)$ is finite for \mathfrak{m} -a.e. x yield the integrability of $\log(1 + f)$ w.r.t. $H_t \delta_y$. \square

In the sequel, we set

$$H_t \delta_y = u_t[y] \mathfrak{m} \quad \text{so that} \quad \tilde{P}_t f(y) = \int_X f u_t[y] \, d\mathfrak{m}$$

for every \mathfrak{m} -measurable and \mathfrak{m} -semi-integrable function f .

COROLLARY 4.7. *For every $t > 0$ and $y \in X$, we have*

$$(4.13) \quad \int_X u_t[y] \log(u_t[y]) \, d\mathfrak{m} \leq \log(u_{2t}[y](x)) + \frac{d^2(x, y)}{4I_{2K}(t)} \quad \text{for } \mathfrak{m}\text{-a.e. } x \in X.$$

In particular, when \mathfrak{m} is a probability measure,

$$u_{2t}[y](x) \geq \exp\left(-\frac{d^2(x, y)}{4I_{2K}(t)}\right) \quad \text{for } \mathfrak{m}\text{-a.e. } x \in X.$$

PROOF. Simply take $f = u_t[y]$ in (4.10) and notice that $\tilde{P}_t f(x) = u_{2t}[y](x)$ for \mathfrak{m} -a.e. $x \in X$ by the semigroup property. \square

In the next crucial result, we will show that (4.13) yields $\text{Ent}_{\mathfrak{m}}(H_t \mu) < \infty$ for every measure $\mu \in \mathcal{P}_2(X)$.

THEOREM 4.8 (L logL regularization). *Let $\mu \in \mathcal{P}_2(X)$ and let $f_t \in L^1(X, \mathfrak{m})$, $t > 0$, be the densities of $H_t \mu \in \mathcal{P}_2(X)$. Then*

$$(4.14) \quad \int_X f_t \log f_t \, d\mathfrak{m} \leq \frac{1}{2I_{2K}(t)} \left(r^2 + \int_X d^2(x, x_0) \, d\mu(x) \right) - \log(\mathfrak{m}(B_r(x_0)))$$

for every $x_0 \in X$ and $r, t > 0$.

PROOF. By approximation, it suffices to consider the case when $\mu = f \mathfrak{m}$ with $f \in L^2(X, \mathfrak{m})$. Let us fix $x_0 \in X$ and $r > 0$, set $z = \mathfrak{m}(B_r(x_0))$ and $\nu = z^{-1} \mathfrak{m} \llcorner B_r(x_0)$. Notice first that we have the pointwise inequality

$$\begin{aligned} \tilde{P}_t f(z) \log(\tilde{P}_t f(z)) &= \left(\int_X u_t[z] \, d\mu \right) \log \left(\int_X u_t[z] \, d\mu \right) \\ &\leq \int_X u_t[z](y) \log(u_t[z](y)) \, d\mu(y). \end{aligned}$$

Since $\tilde{P}_t f = f_t$ m-a.e. in X , integrating with respect to m and using the symmetry property of u_t , (4.13), and Jensen's inequality, we get

$$\begin{aligned} \int_X f_t \log f_t \, dm &\leq \int_X \left(\int_X u_t[y](z) \log u_t[y](z) \, dm(z) \right) d\mu(y) \\ &= \int_{X \times X} \left(\int_X u_t[y](z) \log u_t[y](z) \, dm(z) \right) dv(x) d\mu(y) \\ &\leq \int_{X \times X} \left(\log(u_{2t}[y](x)) + \frac{d^2(x, y)}{4I_{2K}(t)} \right) dv(x) d\mu(y) \\ &\leq \log \left(\int_X \int_X u_{2t}[y](x) \, dv(x) d\mu(y) \right) \\ &\quad + \frac{1}{2I_{2K}(t)} \left(r^2 + \int_X d^2(x, x_0) \, d\mu(x) \right) \\ &\leq \log \frac{1}{z} + \frac{1}{2I_{2K}(t)} \left(r^2 + \int_X d^2(x, x_0) \, d\mu(x) \right), \end{aligned}$$

where we used the inequality

$$\int_X u_{2t}[y](x) \, dv(x) = \frac{1}{z} \int_{B_r(x_0)} u_{2t}[y](x) \, dm(x) \leq \frac{1}{z} \quad \text{for every } y \in X. \quad \square$$

We conclude with a further regularization and an integration by parts formula for $\Delta_{\mathcal{E}}^{(1)}$ in a special case. Notice that thanks to the regularizing effect of H_t we can extend the mollification \mathfrak{h}^ε of the semigroup in (2.18) to measures $\mu \in \mathcal{P}_2(X)$, that is, we set

$$(4.15) \quad \mathfrak{h}^\varepsilon \mu := \frac{1}{\varepsilon} \int_0^\infty f_r \kappa(r/\varepsilon) \, dr, \quad f_r m = H_r \mu \quad \text{for } r > 0,$$

with κ as in (2.17), obtaining a map $\mathfrak{h}^\varepsilon : \mathcal{P}_2(X) \rightarrow D(\Delta_{\mathcal{E}}^{(1)})$ still satisfying (2.19) [via an elementary approximation based on the characterization (2.15) of $D(\Delta_{\mathcal{E}}^{(1)})$].

LEMMA 4.9. *Let $\tilde{\mu} \in \mathcal{P}_2(X)$ and let $f = \mathfrak{h}^\varepsilon \tilde{\mu}$ as in (4.15). Then for every $\varepsilon, T > 0$ there exists a constant $C(\varepsilon, T)$ such that*

$$(4.16) \quad F(P_t f) \leq C(\varepsilon, T) \left(1 + \int_X V^2 \, d\mu \right)$$

and, writing $\mu_t = P_t f m$,

$$(4.17) \quad |\dot{\mu}_t|^2 \leq C(\varepsilon, T) \left(1 + \int_X V^2 \, d\mu \right) \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in [0, T].$$

Moreover, for every bounded and nondecreasing Lipschitz function $\omega : [0, \infty) \rightarrow \mathbb{R}$ such that $\sup_r r\omega'(r) < \infty$, we have

$$(4.18) \quad \int_X \omega(f) \Delta_{\mathcal{E}}^{(1)} f \, dm + 4 \int_X f \omega'(f) \Gamma(\sqrt{f}) \, dm \leq 0.$$

PROOF. Combining (4.2), (4.14), the commutation identity $P_t \mathfrak{h}^\varepsilon = \mathfrak{h}^\varepsilon P_t$, and the convexity of F we get (4.16). We obtain immediately the Lipschitz estimate (4.17) from (4.16) and (4.1) when $f \in L^2(X, m)$. The general case follows by a truncation argument. Concerning (4.18), if $\tilde{\mu} = \tilde{f}m$ with $\tilde{f} \in L^2(X, m)$, then $f \in L^2(X, m)$, $\Delta_{\mathcal{E}}^{(1)} f = \Delta_{\mathcal{E}} f \in L^2(X, m)$ and the stated inequality is an equality, by the chain rule $\Gamma(f, \omega(f)) = \omega'(f)\Gamma(f) = 4f\omega'(f)\Gamma(\sqrt{f})$. In the general case, we approximate $\tilde{\mu}$ in $\mathcal{P}_2(X)$ by a sequence of measures $\tilde{\mu}_n = \tilde{f}_n m$ with $\tilde{f}_n \in L^2(X, m)$ and we consider $f_n = \mathfrak{h}^\varepsilon \mu_n$. By (2.19), we obtain that $\Delta_{\mathcal{E}}^{(1)} f_n \rightarrow \Delta_{\mathcal{E}}^{(1)} f$ in $L^1(X, m)$ while, setting $\phi(s) = \int_0^s \sqrt{r^2 \omega'(r^2)} \, dr$, the lower semicontinuity of $g \mapsto \int \Gamma(g) \, dm$ and the strong convergence of $\sqrt{f_n}$ to \sqrt{f} in $L^2(X, m)$ give

$$\begin{aligned} 4 \int_X f \omega'(f) \Gamma(\sqrt{f}) \, dm &= \int_X \Gamma(\phi(\sqrt{f})) \, dm \leq \liminf_{n \rightarrow \infty} \int_X \Gamma(\phi(\sqrt{f_n})) \, dm \\ &= \liminf_{n \rightarrow \infty} \int_X f_n \omega'(f_n) \Gamma(\sqrt{f_n}) \, dm. \quad \square \end{aligned}$$

Motivated by the regularity assumptions needed in the next section, we give the following definition.

DEFINITION 4.10 (Regular curve). Let $\rho_s = f_s m \in \mathcal{P}(X)$, $s \in [0, 1]$. We say that ρ is regular if:

- (a) $\rho \in AC^2([0, 1]; (\mathcal{P}_2(X), W_2))$;
- (b) $\text{Ent}_m(\rho_s)$ is bounded;
- (c) $f \in C^1([0, 1]; L^1(X, m))$;
- (d) There exists $\eta > 0$ such that for all $s \in [0, 1]$ the function f_s is representable in the form $\mathfrak{h}^\eta \tilde{f}_s$ for some $\tilde{f}_s \in L^1(X, m)$; in addition $\Delta_{\mathcal{E}}^{(1)} f \in C([0, 1]; L^1(X, m))$ and

$$(4.19) \quad \sup\{F(P_t f_s) : s \in [0, 1], t \in [0, T]\} < \infty \quad \forall T > 0.$$

In particular, if $\rho_s = f_s m$ is a regular curve, for every $T > 0$ there exist positive constants M_T, E_T, F_T such that

$$\int_X V^2 \, dH_t \rho_s \leq M_T, \quad \text{Ent}_m(P_t \rho_s) \leq E_T, \quad F(P_t f_s) \leq F_T, \quad s \in [0, 1], t \in [0, T].$$

PROPOSITION 4.11 (Approximation by regular curves). For all $\rho \in AC^2([0, 1]; (\mathcal{P}_2(X), W_2))$ there exist regular curves ρ^n such that $\rho_s^n \rightarrow \rho_s$ in $\mathcal{P}_2(X)$ for

all $s \in [0, 1]$ and

$$(4.20) \quad \limsup_n \int_0^1 |\dot{\rho}_s^n|^2 ds \leq \int_0^1 |\dot{\rho}_s|^2 ds.$$

Furthermore, we can build ρ^n in such a way that $\text{Ent}_m(\rho_0^n) \leq \text{Ent}_m(\rho_0)$.

PROOF. First, we extend ρ by continuity and with constant values in $(-\infty, 0) \cup (1, \infty)$. Then we define $\rho_s^{n,1} := H_{\tau_n} \rho_s$, with $\tau_n^{-1} \in [n, 2n]$. By the continuity and contractivity properties of H_t , we see that $\rho^{n,1}$ fulfils (a) and (b) of the definition of regularity, the convergence requirement of the Lemma and (4.20). Indeed, obviously condition (a) is fulfilled, while we gain $\rho_s^{n,1} \ll m$ and $\sup_s \text{Ent}_m(\rho_s^{n,1}) < \infty$ by Theorem 4.8. In addition, (4.3) shows that $\text{Ent}_m(\rho_0^{n,1}) \leq \text{Ent}_m(\rho_0)$.

In order to achieve condition (c), we do a second regularization, by averaging w.r.t. the s variable: precisely, denoting by $f_s^{n,1}$ the densities of $\rho_s^{n,1}$, we set $\rho_s^{n,2} := f_s^{n,2} m$, where

$$f_s^{n,2} := \int_{\mathbb{R}} f_{s-s'}^{n,1} \chi_n(s') ds'$$

and $\chi_n \in C_c^\infty(\mathbb{R})$ are standard convolution kernels with support in $(0, \infty)$ convergent to the identity. By the convexity properties of squared Wasserstein distance and entropy, we see that the properties (a), (b) are retained and that the action in (4.20) does not increase. In addition, we clearly gain property (c) and, since $\rho^{n,1}$ is constant in $(-\infty, 0]$, this regularization does not increase the entropy at time 0.

In the last step, we mollify using the heat semigroup, setting $\rho_s^n := f_s^n m$, where $f_s^n = \mathfrak{h}^{\varepsilon_n} f_s^{n,2}$ and $\varepsilon_n \downarrow 0$. By the same reasons used for $\rho^{n,2}$, property (a) is retained by ρ^n and the action in (4.20), as well as the Entropy at any time s , do not increase [because the support of the kernel κ is contained in $(0, \infty)$]. In addition, (c) is retained as well since \mathfrak{h}^ε is a continuous linear map from $L^1(X, m)$ to $L^1(X, m)$. With this mollification, we gain property (d) from (2.19) and from (4.16), which provides the sup bound (4.19) on Fisher information. \square

4.3. *Action estimates.* This section contains the core of the arguments leading to the equivalence Theorem 4.17. We refer to [22] for the underlying geometric ideas in a smooth Riemannian context and the role of the Bochner identity. Here, we had to circumvent many technical difficulties related to regularity issues, to the lack of ultracontractivity properties of the semigroup $(P_t)_{t \geq 0}$ (i.e., regularization from L^1 to L^∞), and to the weak formulation of the Bakry-Émery condition.

Since we shall often consider regular curves $\rho \in \text{AC}^2([0, 1]; (\mathcal{P}_2(X), W_2))$ of measures representable in the form $\rho = f m$ with $f \in C^1([0, 1]; L^1(X, m))$, we shall denote by $\dot{f} \in C([0, 1]; L^1(X, m))$ the functional derivative in $L^1(X, m)$, retaining the notation $|\dot{\rho}_t|$ for the metric derivative w.r.t. W_2 .

We begin with a simple estimate of the oscillation of $s \mapsto \int_X \varphi d\rho_s$ along absolutely continuous or C^1 curves.

LEMMA 4.12. For all $\rho \in \text{AC}^2([0, 1]; (\mathcal{P}_2(X), W_2))$, it holds

$$(4.21) \quad \left| \int_X \varphi \, d\rho_1 - \int_X \varphi \, d\rho_0 \right| \leq \int_0^1 |\dot{\rho}_s| \left(\int_X |\text{D}\varphi|^2 \, d\rho_s \right)^{1/2} \, ds$$

for every $\varphi \in \mathbb{V}_\infty^1$.

If moreover $\rho = f \mathbf{m}$ with $f \in C^1([0, 1]; L^1(X, \mathbf{m}))$, for all $\varphi \in \mathbb{V}_\infty^1$ it holds

$$(4.22) \quad \left| \int_X \dot{f}_s \varphi \, d\mathbf{m} \right| \leq |\dot{\rho}_s| \left(\int_X |\text{D}\varphi|^2 \, d\rho_s \right)^{1/2} \quad \text{for } \mathcal{L}^1\text{-a.e. } s \in (0, 1).$$

PROOF. It is easy to check that (4.21) can be obtained using the representation of ρ_s given by Lisini’s theorem [39] (see [6], Lemma 5.15).

Choosing now a Lebesgue point \bar{s} both for $s \mapsto |\dot{\rho}_s|^2$ and $\int_X |\text{D}\varphi|^2 \, d\rho_s$, for all $a > 0$ we can pass to the limit as $h \downarrow 0$ in the inequality

$$\frac{1}{h} \left| \int_X \varphi \, d\rho_{\bar{s}+h} - \int_X \varphi \, d\rho_{\bar{s}} \right| \leq \frac{1}{2h} \int_{\bar{s}}^{\bar{s}+h} \left(a |\dot{\rho}_s|^2 + \frac{1}{a} \int_X |\text{D}\varphi|^2 \, d\rho_s \right) \, ds$$

and then minimize w.r.t. a , obtaining (4.22). \square

LEMMA 4.13. Let $\rho = f \mathbf{m} \in \text{AC}^2([0, 1]; (\mathcal{P}_2(X), W_2))$ be a regular curve according to Definition 4.10, and let $\vartheta : [0, 1] \rightarrow [0, 1]$ be a C^1 function with $\vartheta(i) = i, i = 0, 1$. Define

$$\rho_{s,t} := \text{H}_{st} \rho_{\vartheta(s)} = f_{s,t} \mathbf{m}, \quad s \in [0, t], t \geq 0.$$

Then, for every $t \geq 0$, the curve $s \mapsto \rho_{s,t}$ belongs to $\text{AC}^2([0, 1]; (\mathcal{P}_2(X), W_2))$ and $\text{F}(f_{s,t})$ is uniformly bounded. Moreover, for any $\varphi \in \text{Lip}_b(X)$ with bounded support, setting $\varphi_s := \text{Q}_s \varphi$, the map $s \mapsto \int_X \varphi_s \, d\rho_{s,t}$ is absolutely continuous in $[0, 1]$ and

$$(4.23) \quad \begin{aligned} \frac{d}{ds} \int_X \varphi_s \, d\rho_{s,t} &= \dot{\vartheta}(s) \int_X \dot{f}_{\vartheta(s)} \text{P}_{st} \varphi_s \, d\mathbf{m} - \frac{1}{2} \int_X |\text{D}\varphi_s|^2 \, d\rho_{s,t} \\ &\quad - t \int_X \tilde{\Gamma}(f_{s,t}, \varphi_s) \, d\mathbf{m} \end{aligned}$$

for \mathcal{L}^1 -a.e. $s \in (0, 1)$.

PROOF. We only consider the case $t > 0$ and we set $\tilde{\rho}_s := \rho_{\vartheta(s)} = \tilde{f}_s \mathbf{m}$. Notice that $f_{s,t} = \text{P}_{st} \tilde{f}_s$ and $\tilde{\rho}_s$ satisfies the same assumptions than ρ_s . Since H_t is a Wasserstein K -contraction

$$\begin{aligned} W_2(\rho_{s_0,t}, \rho_{s_1,t}) &\leq e^{-Ks_0t} W_2(\tilde{\rho}_{s_0}, \text{H}_{(s_1-s_0)t} \tilde{\rho}_{s_1}) \\ &\leq e^{-Ks_0t} (W_2(\tilde{\rho}_{s_0}, \tilde{\rho}_{s_1}) + W_2(\tilde{\rho}_{s_1}, \text{H}_{(s_1-s_0)t} \tilde{\rho}_{s_1})), \end{aligned}$$

and (4.17) and the regularity of ρ give

$$W_2(\tilde{\rho}_{s_1}, \mathbf{H}_{(s_1-s_0)t}\tilde{\rho}_{s_1}) \leq C(\rho, T)(s_1 - s_0)t \quad \text{whenever } (s_1 - s_0)t \leq T.$$

We conclude that $s \mapsto \rho_{s,t}$ belongs to $AC^2([0, 1]; (\mathcal{P}_2(X), W_2))$. Moreover, using the splitting

$$\begin{aligned} & \int_X \varphi_{s_1} d\rho_{s_1,t} - \int_X \varphi_{s_0} d\rho_{s_0,t} \\ &= \int_X \varphi_{s_1} d\rho_{s_1,t} - \int_X \varphi_{s_0} d\rho_{s_1,t} + \int_X \varphi_{s_0} d\rho_{s_1,t} - \int_X \varphi_{s_0} d\rho_{s_0,t} \\ &\leq \|\varphi_{s_1} - \varphi_{s_0}\|_\infty + \text{Lip}(\varphi_{s_0}) W_2(\tilde{\rho}_{s_1}, \tilde{\rho}_{s_0}) \end{aligned}$$

we immediately see that also $s \mapsto \int_X \varphi_s d\rho_{s,t}$ is absolutely continuous. In order to compute its derivative, we write

$$\begin{aligned} \int \varphi_{s+h} d\rho_{s+h,t} - \int_X \varphi_s d\rho_{s,t} &= \int \varphi_{s+h} d\rho_{s+h,t} - \int_X \varphi_s d\rho_{s+h,t} \\ &\quad + \int \mathbf{P}_{(s+h)t}\varphi_s d(\tilde{\rho}_{s+h} - \tilde{\rho}_s) \\ &\quad + \int (\mathbf{P}_{ht}\varphi_s - \varphi_s) d\mathbf{H}_{st}\tilde{\rho}_s. \end{aligned}$$

Now, the Hopf–Lax formula (3.9) and the strong convergence of $f_{s+h,t}$ to $f_{s,t}$ in $L^1(X, \mathbf{m})$ yield

$$\lim_{h \downarrow 0} \frac{1}{h} \left(\int \varphi_{s+h} d\rho_{s+h,t} - \int_X \varphi_s d\rho_{s+h,t} \right) = -\frac{1}{2} \int_X |\mathbf{D}\varphi_s|^2 d\rho_{s,t}.$$

The differentiability of ρ_s in $L^1(X, \mathbf{m})$ yields

$$h^{-1} \int \mathbf{P}_{(s+h)t}\varphi d(\tilde{\rho}_{s+h} - \tilde{\rho}_s) \rightarrow \dot{\vartheta}(s) \int_X \mathbf{P}_{st}\varphi f_{\dot{\vartheta}(s)} d\mathbf{m}.$$

Finally, the next lemma yields

$$h^{-1} \int (\mathbf{P}_{ht}\varphi_s - \varphi_s) d\mathbf{P}_{st}\rho_s \rightarrow -t \int_X \tilde{\Gamma}(f_{s,t}, \varphi_s) d\mathbf{m}. \quad \square$$

LEMMA 4.14. For all $\varphi \in \mathbb{V}_\infty^1$ and all $\rho = f\mathbf{m} \in \mathcal{P}(X)$ with $\mathbf{F}(f) < \infty$ it holds

$$\lim_{h \downarrow 0} \int_X \frac{\mathbf{P}_h\varphi - \varphi}{h} f d\mathbf{m} = - \int_X \tilde{\Gamma}(f, \varphi) d\mathbf{m}.$$

PROOF. We argue as in [5], Lemma 4.2, proving first that

$$(4.24) \quad \int_X \frac{\mathbf{P}_h\varphi - \varphi}{h} f d\mathbf{m} = - \int_0^1 \int_X \tilde{\Gamma}(f, \mathbf{P}_{rh}\varphi) d\mathbf{m} dr.$$

Notice first that, possibly approximating φ with the functions $\varphi^\varepsilon := \mathfrak{h}^\varepsilon \varphi$ whose Laplacian is in $L^\infty(X, \mathfrak{m})$, in the proof of (4.24) we can assume with no loss of generality that $\Delta_\varepsilon \varphi \in L^\infty(X, \mathfrak{m})$. Indeed, (4.6) and the strong convergence in Γ norm of $\mathbb{P}_{rh} \varphi^\varepsilon$ to $\mathbb{P}_{rh} \varphi$ ensure the dominated convergence of the integrals in the right-hand sides, while the convergence of the left-hand sides is obvious.

Assuming $\Delta_\varepsilon \varphi \in L^\infty(X, \mathfrak{m})$, since

$$\int_X \frac{\mathbb{P}_h \varphi - \varphi}{h} g \, d\mathfrak{m} = \int_0^1 \int_X g \Delta_\varepsilon \mathbb{P}_{rh} \varphi \, d\mathfrak{m} \, dr$$

for all $g \in L^2(X, \mathfrak{m})$ we can consider the truncated functions $g_N = \min\{f, N\}$ and pass to the limit as $N \rightarrow \infty$ to get that f satisfies the same identity. Since $\Delta_\varepsilon \mathbb{P}_{rh} \varphi = \mathbb{P}_{rh} \Delta_\varepsilon \varphi \in L^\infty(X, \mathfrak{m})$ we can use (4.7) to obtain (4.24).

Having established (4.24), the statement follows using once more (4.6) and the strong convergence of $\mathbb{P}_{rh} \varphi$ to φ in \mathbb{V} . \square

Under the same assumptions of Lemma 4.13, the same computation leading to (4.23) (actually with a simplification, due to the fact that φ is independent on s) and (4.8) give

$$\frac{d}{ds} \int_X \varphi \, d\rho_{s,t} = \int_X (\dot{\vartheta}(s) \mathbb{P}_{st} \dot{f}_{\vartheta(s)} + t \mathbb{P}_{st} \Delta_\varepsilon^{(1)} f_{\vartheta(s)}) \varphi \, d\mathfrak{m}$$

for \mathcal{L}^1 -a.e. $s \in (0, 1)$ and all $\varphi \in \mathbb{V}_\infty^1$. Here we used the fact that $\Delta_\varepsilon^{(1)} (\mathbb{P}_r g) = \mathbb{P}_r \Delta_\varepsilon^{(1)} g$ whenever $g \in D(\Delta_\varepsilon^{(1)})$. Since $\Delta_\varepsilon^{(1)} f \in C([0, 1]; L^1(X, \mathfrak{m}))$, the right-hand side is a continuous function of s , hence

$$(4.25) \quad \frac{d}{ds} \int_X \varphi \, d\rho_{s,t} = \int_X (\dot{\vartheta}_s \mathbb{P}_{st} \dot{f}_{\vartheta(s)} + t \mathbb{P}_{st} \Delta_\varepsilon^{(1)} f_{\vartheta(s)}) \varphi \, d\mathfrak{m} \quad \text{for every } s \in (0, 1).$$

For $\varepsilon > 0$, let us now consider the regularized entropy functionals

$$E_\varepsilon(\rho) := \int_X e_\varepsilon(f) \, d\mathfrak{m}$$

where $e'_\varepsilon(r) := \log(\varepsilon + r \wedge \varepsilon^{-1}) \in \text{Lip}([0, \infty))$, $e_\varepsilon(0) = 0$.

Since we will mainly consider functions f with finite Fisher information, we will also introduce the function

$$p_\varepsilon(r) := e'_\varepsilon(r^2) - \log \varepsilon = \log(\varepsilon + r^2 \wedge \varepsilon^{-1}) - \log \varepsilon.$$

Since p_ε is also Lipschitz and $p_\varepsilon(0) = 0$, we have

$$f \in L^1_+(X, \mathfrak{m}), \quad F(f) < \infty \quad \Rightarrow \quad e'_\varepsilon(f) - \log \varepsilon = p_\varepsilon(\sqrt{f}) \in \mathbb{V}.$$

LEMMA 4.15 (Derivative of E_ε). *With the same notation of Lemma 4.13, if ρ is regular and $t > 0$ we have for $g_{s,t}^\varepsilon := p_\varepsilon(\sqrt{f_{s,t}})$*

$$(4.26) \quad \begin{aligned} & E_\varepsilon(\rho_{1,t}) - E_\varepsilon(\rho_{0,t}) \\ & \leq \int_0^1 \left(-t \int_X f_{s,t} \Gamma(g_{s,t}^\varepsilon) \, dm + \dot{\vartheta}_s \int_X P_{st}(g_{s,t}^\varepsilon) \dot{f}_{\dot{\vartheta}(s)} \, dm \right) ds. \end{aligned}$$

PROOF. The weak differentiability of $s \mapsto f_{s,t}$ (namely, in duality with functions in \mathbb{V}_∞^1) given in (4.25) can, thanks to the continuity assumption made on $\Delta_{\mathcal{E}}^{(1)} f_s$, turned into strong $L^1(X, m)$ differentiability, so that

$$(4.27) \quad \frac{d}{ds} f_{s,t} = \dot{\vartheta}_s P_{st} \dot{f}_{\dot{\vartheta}(s)} + t P_{st} (\Delta_{\mathcal{E}}^{(1)} f_{\dot{\vartheta}(s)})$$

in $L^1(X, m)$, for all $s \in (0, 1)$.

Since e_ε is of class $C^{1,1}$, it is easy to check that this implies the absolute continuity of $s \mapsto E_\varepsilon(\rho_{s,t})$. In addition, the mean value theorem gives

$$\frac{d}{ds} E_\varepsilon(\rho_{s,t}) = \lim_{h \rightarrow 0} \int_X e'_\varepsilon(f_{s,t}) \frac{f_{s+h,t} - f_{s,t}}{h} \, dm \quad \forall s \in (0, 1).$$

Notice also that Lemma 4.9 with $f = f_{s,t}$ and $\omega = e'_\varepsilon - \log \varepsilon$ gives [since $e'_\varepsilon(f_{s,t}) - \log \varepsilon = p_\varepsilon(\sqrt{f_{s,t}})$ is nonnegative and integrable and $P_{st}(\Delta_{\mathcal{E}}^{(1)} f_{\dot{\vartheta}(s)}) = \Delta_{\mathcal{E}}^{(1)}(P_{st} f_{\dot{\vartheta}(s)})$ has null mean]

$$(4.28) \quad \int_X P_{st}(\Delta_{\mathcal{E}}^{(1)} f_{\dot{\vartheta}(s)}) e'_\varepsilon(f_{s,t}) \, dm \leq -4 \int_X f_{s,t} e''_\varepsilon(f_{s,t}) \Gamma(\sqrt{f_{s,t}}) \, dm.$$

Now we use (4.27), (4.28) and conclude

$$\begin{aligned} \frac{d}{ds} E_\varepsilon(\rho_{s,t}) & \leq -t \int_X 4 f_{s,t} e''_\varepsilon(f_{s,t}) \Gamma(\sqrt{f_{s,t}}) \, dm \\ & \quad + \dot{\vartheta}_s \int_X (e'_\varepsilon(f_{s,t}) - \log \varepsilon) P_{st} \dot{f}_{\dot{\vartheta}(s)} \, dm. \end{aligned}$$

On the other hand, since $4r e''_\varepsilon(r) \geq 4r^2 (e''_\varepsilon(r))^2 = r (p'_\varepsilon(\sqrt{r}))^2$, we get

$$-4 f_{s,t} e''_\varepsilon(f_{s,t}) \Gamma(\sqrt{f_{s,t}}) \leq -f_{s,t} (p'_\varepsilon(\sqrt{f_{s,t}}))^2 \Gamma(\sqrt{f_{s,t}}) = -f_{s,t} \Gamma(p_\varepsilon(\sqrt{f_{s,t}}))$$

and an integration with respect to s and the definition of $g_{s,t}^\varepsilon$ yield (4.26). \square

THEOREM 4.16 (Action and entropy estimate on regular curves). *Let $\rho_s = f_s m$ be a regular curve. Then, setting $\rho_{1,t} = H_t \rho_1$, it holds*

$$(4.29) \quad W_2^2(\rho_0, \rho_{1,t}) + 2t \text{Ent}_m(\rho_{1,t}) \leq R_K^2(t) \int_0^1 |\dot{\rho}_s|^2 \, ds + 2t \text{Ent}_m(\rho_0),$$

where

$$R_K(t) := \frac{t}{I_K(t)} = \frac{Kt}{e^{Kt} - 1} \quad \text{if } K \neq 0; \quad R_0(t) \equiv 1.$$

PROOF. Set $\rho_{s,t}, f_{s,t}$ as in Lemma 4.13, $p_\varepsilon(r) = e'_\varepsilon(r^2) - \log \varepsilon$, $g_{s,t}^\varepsilon = p_\varepsilon(\sqrt{f_{s,t}})$ as in Lemma 4.15, $q_\varepsilon(r) := \sqrt{r}(2 - \sqrt{r}p'_\varepsilon(\sqrt{r}))$, and $\varphi_s := Q_s\varphi$ for a Lipschitz function φ with bounded support.

Notice that by (4.4)

$$\tilde{\Gamma}(f_{s,t}, \varphi_s) = 2\sqrt{f_{s,t}}\Gamma(\sqrt{f_{s,t}}, \varphi_s) = f_{s,t}\Gamma(g_{s,t}^\varepsilon, \varphi_s) + q_\varepsilon(f_{s,t})\Gamma(\sqrt{f_{s,t}}, \varphi_s).$$

Applying (4.23), (4.26) in the weaker form

$$tE_\varepsilon(\rho_{1,t}) - tE_\varepsilon(\rho_{0,t}) \leq \int_0^1 \left(t\dot{\vartheta}_s \int_X P_{st}(g_{s,t}^\varepsilon) \dot{f}_{\vartheta(s)} \, d\mathbf{m} - \frac{t^2}{2} \int_X f_{s,t} \Gamma(g_{s,t}^\varepsilon) \, d\mathbf{m} \right) ds$$

and eventually the Young inequality $2xy \leq ax^2 + y^2/a$ in (4.22) with $a := \dot{\vartheta}_s e^{-2Kst}$, we obtain

$$\begin{aligned} & \int_X \varphi_1 \, d\rho_{1,t} - \int_X \varphi_0 \, d\rho_{0,t} + t(E_\varepsilon(\rho_{1,t}) - E_\varepsilon(\rho_{0,t})) \\ & \leq \int_0^1 \left(\dot{\vartheta}_s \int_X \dot{f}_{\vartheta(s)} P_{st}(\varphi_s + tg_{s,t}^\varepsilon) \, d\mathbf{m} - \frac{1}{2} \int_X (|\mathbf{D}\varphi_s|^2 + t^2\Gamma(g_{s,t}^\varepsilon)) \, d\rho_{s,t} \right. \\ & \quad \left. - t \int_X \Gamma(g_{s,t}^\varepsilon, \varphi_s) \, d\rho_{s,t} - t \int_X q_\varepsilon(f_{s,t})\Gamma(\sqrt{f_{s,t}}, \varphi_s) \, d\mathbf{m} \right) ds \\ & \leq \int_0^1 \left(\dot{\vartheta}_s \int_X \dot{f}_{\vartheta(s)} P_{st}(\varphi_s + tg_{s,t}^\varepsilon) \, d\mathbf{m} - \frac{1}{2} \int_X \Gamma(\varphi_s + tg_{s,t}^\varepsilon) \, d\rho_{s,t} \right. \\ & \quad \left. - t \int_X q_\varepsilon(f_{s,t})\Gamma(\sqrt{f_{s,t}}, \varphi_s) \, d\mathbf{m} \right) ds \\ & \leq \int_0^1 \left(\dot{\vartheta}_s \int_X \dot{f}_{\vartheta(s)} P_{st}(\varphi_s + tg_{s,t}^\varepsilon) \, d\mathbf{m} \right. \\ & \quad \left. - \frac{1}{2} e^{2Kst} \int_X \Gamma(P_{st}(\varphi_s + tg_{s,t}^\varepsilon)) \, d\rho_{\vartheta(s)} \right. \\ & \quad \left. + t \int_X |q_\varepsilon(f_{s,t})| |\Gamma(\sqrt{f_{s,t}}, \varphi_s)| \, d\mathbf{m} \right) ds \\ & \leq \int_0^1 \left(\frac{1}{2} (\dot{\vartheta}_s)^2 e^{-2Kst} |\dot{\rho}_s|^2 + \frac{t}{8} \delta F(\rho_{s,t}) + \frac{t}{2\delta} \int_X q_\varepsilon^2(f_{s,t}) |\mathbf{D}\varphi_s|^2 \, d\mathbf{m} \right) ds. \end{aligned}$$

Now we pass first to the limit as $\varepsilon \downarrow 0$, observing that $p'_\varepsilon(r) = 2r(\varepsilon + r^2)^{-1} \chi_{r^2 < \varepsilon^{-1}}$ gives

$$q_\varepsilon^2(r) = 4r \left(1 - \frac{r}{\varepsilon + r} \right)^2 \chi_{r < \varepsilon^{-1}} \leq 4r, \quad \lim_{\varepsilon \downarrow 0} q_\varepsilon^2(r) = 0,$$

and then as $\delta \downarrow 0$; choosing

$$\vartheta(s) := \frac{I_K(st)}{I_K(t)} \quad \text{so that} \quad \dot{\vartheta}(s) = R_K(t)e^{Kst},$$

we obtain

$$\int_X \varphi_1 d\rho_{1,t} - \int_X \varphi_0 d\rho_{0,t} + t(\text{Ent}_m(\rho_{1,t}) - \text{Ent}_m(\rho_{0,t})) \leq \frac{1}{2}R_K^2(t) \int_0^1 |\dot{\rho}_s|^2 ds.$$

Eventually we take the supremum with respect to φ , obtaining

$$\frac{1}{2}W_2^2(\rho_{1,t}, \rho_0) + t(\text{Ent}_m(\rho_{1,t}) - \text{Ent}_m(\rho_{0,t})) \leq \frac{1}{2}R_K^2(t) \int_0^1 |\dot{\rho}_s|^2 ds. \quad \square$$

THEOREM 4.17 [BE(K, ∞) is equivalent to RCD(K, ∞)]. *If $(X, \tau, m, \mathcal{E})$ is a Riemannian Energy measure space satisfying (MD.exp) relative to $d_{\mathcal{E}}$ and BE(K, ∞), then $(X, d_{\mathcal{E}}, m)$ is a RCD(K, ∞) space.*

Conversely, if (X, d, m) is a RCD(K, ∞) space then, denoting by τ the topology induced by d and by $\mathcal{E} = 2\text{Ch}$ the Cheeger energy, $(X, \tau, m, \mathcal{E})$ is a Riemannian Energy measure space satisfying $d_{\mathcal{E}} = d$, (MD.exp), and BE(K, ∞).

PROOF. We have to show that (3.14) holds with $H_t\rho$ precisely given by the dual semigroup. By a standard density argument (see [5], Proposition 2.21) suffices to show this property for all $\rho \in \mathcal{P}_2(X)$ with finite entropy. By the semigroup property and (4.3), it is sufficient to prove (3.14) at $t = 0$ for all $v \in \mathcal{P}_2(X)$ with finite entropy. For any $\rho \in \text{AC}^2([0, 1]; (\mathcal{P}_2(X), W_2))$ joining $\rho_0 := v$ to $\rho_1 := \rho$ we find regular curves ρ^n as in Proposition 4.11 with $\text{Ent}_m(\rho_0^n) \leq \text{Ent}_m(\rho_0)$ and apply the action estimate (4.29) to the curves $\rho_{s,t}^n = H_{st}\rho_s^n$ to obtain

$$W_2^2(H_t\rho_1^n, \rho_0^n) + 2t\text{Ent}_m(H_t\rho_1^n) \leq R_K^2(t) \int_0^1 |\dot{\rho}_s^n|^2 ds + 2t\text{Ent}_m(\rho^0).$$

We pass to the limit as $n \rightarrow \infty$ and use the lower semicontinuity of W_2 and of the entropy, to get

$$W_2^2(H_t\rho, v) + 2t\text{Ent}_m(P_t\rho) \leq R_K^2(t) \int_0^1 |\dot{\rho}_s|^2 ds + 2t\text{Ent}_m(v).$$

We can now minimize w.r.t. ρ and use the fact that $(\mathcal{P}_2(X), W_2)$ is a length space because $(X, d_{\mathcal{E}})$ is (this can be obtained starting from an optimal Kantorovich plan π , choosing in a π -measurable way a ε -optimal geodesic with constant speed as in the proof of Theorem 3.5), getting

$$W_2^2(H_t\rho, v) + 2t\text{Ent}_m(P_t\rho) \leq R_K^2(t)W_2^2(\rho, v) + 2t\text{Ent}_m(v).$$

After dividing by $t > 0$, letting $t \downarrow 0$ and using $R_K(t) = 1 - \frac{K}{2}t + o(t)$ we obtain (3.14).

The converse implication, from $\text{RCD}(K, \infty)$ to $\text{BE}(K, \infty)$ has been proved in [5], Section 6. \square

We conclude with an immediate application of the previous result to metric measure spaces: it follows by Theorem 3.17 and Corollary 3.18. Notice that for the Cheeger energy condition (ED.b) and upper-regularity are always true.

COROLLARY 4.18. *Let (X, d, m) be a metric measure space satisfying (MD+exp) with a quadratic Cheeger energy Ch defining the Dirichlet form $\mathcal{E} = 2\text{Ch}$ as in (QCh). (X, d, m) is a $\text{RCD}(K, \infty)$ -space if (and only if) at least one of the following properties hold:*

(i) (X, d) is a length space and $(P_t)_{t \geq 0}$ satisfies property (3.44), that is, for every function $f \in D(\text{Ch})$ with $|Df|_w \leq 1$ and every $t > 0$,

$$(4.30) \quad P_t f \in \text{Lip}_b(X), \quad |DP_t f|^2 \leq e^{-2Kt} P_t(|Df|_w^2) \quad m\text{-a.e. in } X.$$

(ii) Conditions (ED.a), (w-Feller) (or $\mathcal{L} = \mathcal{L}_C$), and $\text{BE}(K, \infty)$ hold.

(iii) Condition (ED.a) holds and $(H_t)_{t \geq 0}$ satisfies the contraction property (3.48) [or $(P_t)_{t \geq 0}$ satisfies the Lipschitz bound (3.43)].

5. Applications of the equivalence result. In this section, we present two applications of our equivalence result: in one direction, we can use it to prove that the $\text{RCD}(K, \infty)$ condition is stable under tensorization, a property proved in [5] only under a nonbranching assumption on the base spaces. We will also prove the same property for Riemannian Energy measure spaces satisfying the $\text{BE}(K, N)$ condition, obtaining in particular the natural bound on the dimension of the product.

In the other direction, we shall prove a stability result for Riemannian Energy measure spaces satisfying a uniform $\text{BE}(K, N)$ condition under Sturm–Gromov–Hausdorff convergence.

5.1. Tensorization. Let $(X, d_X, m_X), (Y, d_Y, m_Y)$ be $\text{RCD}(K, \infty)$ metric measure spaces.

We may define a product space (Z, d, m) by

$$(5.1) \quad \begin{aligned} Z &:= X \times Y, & d((x, y), (x', y')) &:= \sqrt{d_X^2(x, x') + d_Y^2(y, y')}, \\ m &:= m_X \times m_Y. \end{aligned}$$

Notice that also (Z, d, m) satisfies the quantitative σ -finiteness condition (MD.exp).

Denoting by $\mathcal{E}^X, \mathcal{E}^Y$ the Dirichlet forms associated to the respective (quadratic) Cheeger energies with domains $\mathbb{V}^X, \mathbb{V}^Y$, we consider the Cartesian–Dirichlet form

$$(5.2) \quad \mathcal{E}(f) := \int_Y \mathcal{E}^X(f^y) dm_Y(y) + \int_X \mathcal{E}^Y(f^x) dm_X(x), \quad f \in L^2(Z, m),$$

where for every $f \in L^2(Z, m)$ and $z = (x, y) \in Z$ we set $f^x = f(x, \cdot)$, $f^y(\cdot) = f(\cdot, y)$. By [5], Theorem 6.18, the proper domain \mathbb{V} of \mathcal{E} in $L^2(Z, m)$ is the Hilbert space

$$\mathbb{V} := \{f \in L^2(Z, m) : f^x \in \mathbb{V}^Y \text{ for } m_X\text{-a.e. } x \in X, \\ f^y \in \mathbb{V}^X \text{ for } m_Y\text{-a.e. } y \in Y, |Df^x|_w(y), |Df^y|_w(x) \in L^2(Z, m)\}.$$

Furthermore, $\frac{1}{2}\mathcal{E}$ coincides with the Cheeger energy Ch in (Z, d, m) , and

$$(5.3) \quad |Df|_w^2(x, y) = \Gamma(f)(x, y) = |Df^x|_w^2(y) + |Df^y|_w^2(x) \\ \text{for } m\text{-a.e. } (x, y) \in Z.$$

Even though the result in [5] is stated for metric measure spaces with finite measure, the proof extends with no difficulty to the σ -finite case. Also, it is worthwhile to mention that the curvature assumption on the base spaces plays almost no role in the proof, it is only used to build, via the product semigroup, an operator with good regularization properties (specifically from L^∞ to C_b); see [5], Lemma 6.13.

It will be convenient, as in [5] and in the previous sections, to work with a pointwise defined version of the semigroups in the base spaces, namely

$$P_t^X u(x) := \int u(x') dH_t^X \delta_x(x'), \quad P_t^Y v(y) := \int v(y') dH_t^Y \delta_y(y')$$

for $u : X \rightarrow \mathbb{R}$ and $v : Y \rightarrow \mathbb{R}$ bounded Borel, where $(H_t^X)_{t \geq 0}$ and $(H_t^Y)_{t \geq 0}$ denote the Wasserstein semigroups on the base spaces. These pointwise defined semigroups also provide the continuous versions of (S-Feller) for the base spaces, see [5], Theorem 6.1(iii).

Since the heat flows are linear, the tensorization (5.3) implies a corresponding tensorization of the heat flows, namely for all $g : Z \rightarrow \mathbb{R}$ bounded and Borel, for m -a.e. $(x, y) \in Z$ the following identities hold:

$$(5.4) \quad P_t g(z) = \int_X P_t^Y g(x', \cdot)(y) dH_t^X \delta_x(x'), \\ P_t g(z) = \int_Y P_t^X g(\cdot, y')(x) dH_t^Y \delta_y(y').$$

With these ingredients at hand, we can now prove the main tensorization properties.

THEOREM 5.1. *With the above notation, if (X, d_X, m_X) and (Y, d_Y, m_Y) are $\text{RCD}(K, \infty)$ spaces then the space (Z, d, m) is $\text{RCD}(K, \infty)$ as well.*

PROOF. According to the characterization of $\text{RCD}(K, \infty)$ given in point (i) of Corollary 4.18, since the Cheeger energy in Z satisfies (QCh) by the above mentioned result of [5], it suffices to show that the length space property and (4.30) are stable under tensorization.

Stability of the length space property. This is simple to check, one obtains an almost minimizing geodesic $\gamma : [0, 1] \rightarrow Z$ combining almost minimizing geodesics on the base spaces with constant speed and parameterized on $[0, 1]$.

Stability of (4.30). Let us first notice that P_t maps bounded and Borel functions into continuous ones, thanks of any of the two identities in (5.4) and (S-Feller).

Let $f \in \text{Lip}_b(Z) \cap L^2(Z, m)$. Keeping y initially fixed, the second identity in (5.4) tells us that $x \mapsto P_t f(x, y) = (P_t f)^y(x)$ is the mean w.r.t. y' , weighted with $H_t^Y \delta_y$, of the functions $P_t^X f^{y'}(x)$. Hence, the convexity of the slope gives

$$|D(P_t f)^y|(x) \leq \int_Y |DP_t^X f^{y'}|(x) dH_t^Y \delta_y(y'),$$

where gradients are understood with respect to the first variable. We can thus use the Hölder inequality to get

$$(5.5) \quad |D(P_t f)^y|^2(x) \leq \int_Y |DP_t^X f^{y'}|^2(x) dH_t^Y \delta_y(y').$$

Now, for m_Y -a.e. $y' \in Y$ we apply (4.30) in the space X to the functions $f^{y'}$ and use Fubini's theorem to get

$$(5.6) \quad |DP_t^X f^{y'}|^2(x) \leq e^{-2Kt} P_t^X |Df^{y'}|_w^2(x) \quad \text{for } m_Y\text{-a.e. } y' \in Y.$$

Combining (5.5) and (5.6) and using once more (5.4) with $g(x, y) = |Df^y|_w^2(x)$ we get

$$|D(P_t f)^y|^2(x) \leq e^{-2Kt} \int_Y P_t^X |Df^{y'}|_w^2(x) dH_t \delta_y(y') = e^{-2Kt} P_t |Df^y|_w^2(x).$$

Repeating a similar argument with the first identity in (5.4) and adding the two inequalities, we obtain

$$|D(P_t f)^y|^2(x) + |D(P_t f)^x|^2(y) \leq e^{-2Kt} P_t |Df|_w^2(x, y).$$

We conclude that (4.30) holds in (Z, d, m) using the calculus lemma [5], Lemma 6.2, which provides the information that the square root of $|D(P_t f)^y|^2(x) + |D(P_t f)^x|^2(y)$ is an upper gradient of $P_t f$. It follows that $e^{-Kt} \sqrt{P_t |Df|_w^2}$ is an upper gradient as well; being continuous, it provides a pointwise upper bound for the slope. \square

Let us now consider the corresponding version of the tensorization theorem for Riemannian Energy measure spaces with a finite upper bound on the dimension.

THEOREM 5.2. *Let $(X, \tau_X, \mathcal{E}^X, m_X)$, $(Y, \tau_Y, \mathcal{E}^Y, m_Y)$ be Riemannian Energy measure spaces satisfying the Bakry–Émery conditions $\text{BE}(K, N_X)$ and $\text{BE}(K, N_Y)$ respectively, and let us consider the cartesian Dirichlet form \mathcal{E} defined by (5.2) on $Z = X \times Y$ endowed with the product topology $\tau = \tau_X \otimes \tau_Y$ and the product measure m as in (5.1).*

Then $(Z, \tau, \mathfrak{m}, \mathcal{E})$ is a Riemannian Energy measure space, it satisfies the Bakry-Émery condition $\text{BE}(K, N_X + N_Y)$ and the induced distance $d_{\mathcal{E}}$ on Z coincides with the product distance defined in (5.1).

PROOF. It is not restrictive to assume that $N_X, N_Y < \infty$; by the previous Theorem we already know that $(Z, \tau, \mathfrak{m}, \mathcal{E})$ is a Riemannian Energy measure space satisfying $\text{BE}(K, \infty)$, whose induced distance $d_{\mathcal{E}}$ is given by (5.1); we want to prove that (2.35) holds with $\nu_Z := \nu_X \nu_Y / (\nu_X + \nu_Y)$ where $\nu_X := N_X^{-1}$ and $\nu_Y := N_Y^{-1}$. We argue as in (5.6), observing that for \mathfrak{m}_Y -a.e. $y' \in Y$ (2.35) and (3.45) yield

$$|\text{DP}_t^X f^{y'}|_w^2 + 2\nu_X \text{I}_{2K,2}(t) (\Delta_X \text{P}_t^X f^{y'})^2 \leq e^{-2Kt} \text{P}_t^X |Df^{y'}|_w^2$$

for \mathfrak{m}_X -a.e. $x \in X$.

Integrating w.r.t. the measure $\text{H}_t^Y \delta_{y'}$ in y' and recalling (5.4) and (5.5), we get

$$(5.7) \quad |D(\text{P}_t f)^y|_w^2(x) + 2\nu_X \text{I}_{2K,2}(t) (\Delta_X (\text{P}_t f)^y)^2(x) \leq e^{-2Kt} \text{P}_t (|Df^y|_w^2)(x)$$

m-a.e. in Z , where we also used the Hölder inequality

$$\int_Y (\Delta_X \text{P}_t^X f^{y'})^2(x) \, d\text{H}_t \delta_{y'}(y') \geq \left(\int_Y \Delta_X \text{P}_t^X f^{y'}(x) \, d\text{H}_t \delta_{y'}(y') \right)^2 = (\Delta_X \text{P}_t f^y)^2(x).$$

By repeating a similar argument inverting the role of X and Y , we get

$$(5.8) \quad |D(\text{P}_t f)^x|_w^2(y) + 2\nu_Y \text{I}_{2K,2}(t) (\Delta_Y (\text{P}_t f)^x)^2(y) \leq e^{-2Kt} \text{P}_t (|Df^x|_w^2)(y)$$

m-a.e. in Z . Adding (5.7) and (5.8), and recalling the elementary inequality

$$\nu_X a^2 + \nu_Y b^2 \geq \frac{\nu_X \nu_Y}{\nu_X + \nu_Y} (a + b)^2 \quad \text{for every } a, b \geq 0,$$

we conclude thanks to the next simple lemma. \square

LEMMA 5.3. Assume that $f \in \mathbb{V}$ satisfies

$$(5.9) \quad (f^y, f^x) \in D(\Delta_X) \times D(\Delta_Y)$$

for m-a.e. $(x, y) \in Z, \Delta_X f^y, \Delta_Y f^x \in L^2(Z, \mathfrak{m})$.

Then $f \in D(\Delta_Z)$ and $\Delta_Z f(x, y) = \Delta_X f^y(x) + \Delta_Y f^x(y)$ for m-a.e. $(x, y) \in Z$.

PROOF. If (5.9) holds, Fubini's theorem and the very definition of Δ_X, Δ_Y yield for every $\varphi \in \mathbb{V}$

$$\begin{aligned} \mathcal{E}(f, \varphi) &= \int_Y \mathcal{E}^X(f^y, \varphi^y) \, d\mathfrak{m}_Y(y) + \int_X \mathcal{E}^Y(f^x, \varphi^x) \, d\mathfrak{m}_X(x) \\ &= - \int_Y \left(\int_X \Delta_X f^y \varphi^y \, d\mathfrak{m}_X \right) d\mathfrak{m}_Y - \int_X \left(\int_Y \Delta_Y f^x \varphi^x \, d\mathfrak{m}_Y \right) d\mathfrak{m}_X \\ &= - \int_Z (\Delta_X f^y + \Delta_Y f^x) \varphi \, d\mathfrak{m}. \end{aligned}$$

\square

5.2. *Sturm–Gromov–Hausdorff convergence and stability of the BE(K, N) condition.*

Preliminaries. Here and in the following we adopt the notation $\bar{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$.

Let us first recall an equivalent characterization of Sturm–Gromov–Hausdorff (SGH) convergence of a sequence (X_n, d_n, m_n) , $n \in \bar{\mathbb{N}}$, of metric measure spaces that is very well adapted to our aims; for the sake of simplicity, we restrict here to the case when $m_n \in \mathcal{P}_2(X_n)$. The general case of σ -finite measures satisfying (MD.exp) could be attacked by the techniques developed in [29], assuming that (MD.exp) holds uniformly along the sequence. We refer to [5], Section 2.3, [54], Section 3.1, for other definitions and important properties of SGH convergence.

DEFINITION 5.4 (SGH-convergence). Let (X_n, d_n, m_n) , $n \in \bar{\mathbb{N}}$, be complete and separable metric measure spaces with $m_n \in \mathcal{P}_2(X_n)$ for every $n \in \bar{\mathbb{N}}$. We say that (X_n, d_n, m_n) SGH-converge to $(X_\infty, d_\infty, m_\infty)$ as $n \rightarrow \infty$ if there exist a complete and separable metric space (X, d) and isometries $\iota_n : X_n \rightarrow X$, $n \in \bar{\mathbb{N}}$, such that $W_2((\iota_n)_\# m_n, (\iota_\infty)_\# m_\infty) \rightarrow 0$ as $n \rightarrow \infty$.

A more intrinsic approach would state SGH-convergence for the equivalence classes of metric-measure spaces induced by measure-preserving isometries: according to this point of view, two metric-measure spaces (X_1, d_1, m_1) and (X_2, d_2, m_2) are isomorphic if there exists an isometry $\iota : \text{supp}(m_1) \rightarrow X_2$ such that $m_2 = \iota_\# m_1$.

Here, we do not insist on this aspect, since owing to the Definition 5.4 we will always consider an effective realization of such a convergence provided by the space (X, d) and the system of isometries (ι_n) , $n \in \bar{\mathbb{N}}$. Moreover, by identifying X_n with $\iota_n(X_n)$ in X , and m_n with $(\iota_n)_\# m_n$ in $\mathcal{P}_2(X)$, it will not restrictive to assume that

$$(5.10) \quad \begin{aligned} & m_n \in \mathcal{P}_2(X), X_n \subset X, d_n \equiv d \text{ for every } n \in \bar{\mathbb{N}}, m_n \text{ are converging to } m_\infty \\ & \text{in } \mathcal{P}_2(X) \text{ as } n \rightarrow \infty; \end{aligned}$$

one has just to take care that in general m_n, m could be not fully supported.

REMARK 5.5. It is important to notice that, by construction, the Cheeger energy is invariant by isometries: considering, for example, the situation of Definition 5.4, if $\iota_\infty : X_\infty \rightarrow X$ is an isometric imbedding of (X_∞, d_∞) in a complete and separable metric space (X, d) , with $\tilde{m}_\infty := (\iota_\infty)_\# m_\infty$, the Cheeger energy $\frac{1}{2} \tilde{\mathcal{E}}_\infty$ associated to (X, d, \tilde{m}_∞) in $L^2(X, \tilde{m}_\infty)$ satisfies

$$\tilde{\mathcal{E}}_\infty(f) = \mathcal{E}_\infty(f \circ \iota_\infty) \quad \text{for every } f \in L^2(X, \tilde{m}_\infty).$$

Since the composition with ι_∞ provides an order preserving isomorphism between $L^2(X, \tilde{m}_\infty)$ and $L^2(X_\infty, m_\infty)$, it is immediate to check that $(X_\infty, \tau_\infty, m_\infty, \mathcal{E}_\infty)$ satisfies BE(K, N) if and only if $(X, \tau, \tilde{m}_\infty, \tilde{\mathcal{E}}_\infty)$ satisfies BE(K, N) as well (here τ is the topology induced by d in X).

We will also need a few results, strongly related to the theory of Young measures, concerning convergence for sequences of functions defined in L^2 -spaces associated to different measures (see, e.g., [3], Section 5.4). We first make precise this notion of convergence.

DEFINITION 5.6. Let (X, d) be a complete and separable metric space, let $(m_n) \subset \mathcal{P}_2(X), n \in \bar{\mathbb{N}}$, be converging in $\mathcal{P}_2(X)$, and consider a sequence of vector valued functions $f_n \in L^2(X, m_n; \mathbb{R}^k), n \in \bar{\mathbb{N}}, k \in \mathbb{N}$. We say that (f_n) converges to f_∞ as $n \rightarrow \infty$ if

$$(5.11) \quad (i \times f_n)_\# m_n \rightarrow (i \times f_\infty)_\# m_\infty \quad \text{in } \mathcal{P}_2(X \times \mathbb{R}^k).$$

We will use three properties stated in the next lemma.

LEMMA 5.7. Let $(X, d), (f_n)$ and $(m_n) \subset \mathcal{P}_2(X), n \in \bar{\mathbb{N}}$, as in Definition 5.6 above.

(i) (5.11) is equivalent to the convergence of each component $f_n^j, j = 1, \dots, k$, to f^j .

(ii) In the scalar case $k = 1$, if f_n satisfy

$$\lim_{n \rightarrow \infty} \int_X f_n \varphi \, dm_n = \int_X f \varphi \, dm \quad \text{for every } \varphi \in C_b(X),$$

$$\lim_{n \rightarrow \infty} \int_X f_n^2 \, dm_n = \int_X f^2 \, dm,$$

then f_n converges to f according to (5.11). The same conclusion holds if $f_n \in L^1_+(X, m_n) \cap L^\infty(X, m_n)$ are uniformly bounded probability densities satisfying

$$f_n m_n \rightharpoonup f m \quad \text{in } \mathcal{P}(X), \quad \lim_{n \rightarrow \infty} \int_X f_n \log f_n \, dm_n = \int_X f \log f \, dm.$$

(iii) Finally, if $r: \mathbb{R}^k \rightarrow \mathbb{R}^h$ is a continuous map with linear growth, and f_n converge to f according to (5.11) then $r \circ f_n$ converge to $r \circ f$.

PROOF. (i) follows by disintegration and the fact that a probability measure in \mathbb{R}^k is a Dirac mass if and only if its coordinate projections are Dirac masses; see, for example, [3], Lemma 5.3.2.

Property (ii) is a consequence of the fact that, for strictly convex functions, equality holds in Jensen’s inequality only when the measure is a Dirac mass. A detailed argument is presented in [3], Theorem 5.4.4 (the fact that the base space X is a general metric space instead of an Hilbert space is not relevant here).

The proof of (iii) is straightforward. \square

Stability of $BE(K, N)$ under SGH-convergence.

THEOREM 5.8. Let $(X_n, \tau_n, \mathcal{E}_n, m_n), n \in \mathbb{N}$, be Riemannian energy measure spaces satisfying $BE(K, N)$ with $m_n \in \mathcal{P}_2(X_n)$ and let us suppose that, denoting by d_n the corresponding distances $d_{\mathcal{E}_n}, (X_n, d_n, m_n)$ converge to $(X_\infty, d_\infty, m_\infty)$

in the Sturm–Gromov–Hausdorff sense of Definition 5.4. If $\frac{1}{2}\mathcal{E}_\infty$ is the Cheeger energy in the limit space and τ_∞ the topology induced by d_∞ , then $(X_\infty, \tau_\infty, m_\infty, \mathcal{E}_\infty)$ is a Riemannian Energy measure space satisfying $\text{BE}(K, N)$.

PROOF. According to the Definition 5.4 of SGH-convergence and taking Remark 5.5 into account, it is not restrictive to assume that (5.10) holds, so that all the spaces X_n are subsets of a fixed complete and separable metric space (X, d) , d_n are the restrictions of d on X_n , the isometries ι_n are just the inclusions maps, m_n can be identified with $(\iota_n)_\#m_n$ and can be considered as measures in $\mathcal{P}_2(X)$ converging to m_∞ , and the Cheeger energies \mathcal{E}_n are Dirichlet forms on $L^2(X, m_n)$ satisfying the $\text{BE}(K, N)$ condition.

The case $N = \infty$ follows by the identification Theorem 4.17 and [5], Theorem 6.10. In particular, the limit Cheeger energy associated to (X, d, m_∞) is a Dirichlet form that we call $\frac{1}{2}\mathcal{E}_\infty$ and the limit space endowed with the Cheeger energy and the topology τ induced by d is a Riemannian Energy measure space.

We can thus consider the case $N < \infty$. In order to show that $(X, \tau, m_\infty, \mathcal{E}_\infty)$ satisfies $\text{BE}(K, N)$ we will prove that the distributional characterization (2.33) of $\text{BE}(K, N)$ holds for every $f \in L^2(X, m_\infty)$ and nonnegative $\varphi \in L^\infty(X, m_\infty)$. By standard approximation, it is also not restrictive to assume $f \in L^\infty(X, m_\infty)$. Our argument consists in passing to the limit in the corresponding distributional inequality written for suitable approximating sequences in the spaces $(X, \tau, m_n, \mathcal{E}_n)$.

Let us thus denote by $(P_t^n)_{t \geq 0}$ the Markov semigroups in $L^2(X, m_n)$ with generators $\Delta_n := \Delta_{\mathcal{E}^n}$, $n \in \bar{\mathbb{N}}$. By [5], Lemma 6.12, and Lemma 5.7(ii), for every $f, \varphi \in L^\infty(X, m)$, φ nonnegative, we can find sequences $f_n, \varphi_n \in L^\infty(X, m_n)$, φ_n nonnegative, converging to f, φ according to Definition 5.6. We can also suppose that f_n, φ_n are uniformly bounded by some constant $C > 0$.

Applying Lemma 5.9 below, we get that for every $t \geq 0$ and $s \in [0, t]$ $P_{t-s}^n f_n$ converge to $P_{t-s}^\infty f$ and $P_s^n \varphi_n$ converge to $P_s^\infty \varphi$ as $n \rightarrow \infty$ according to Definition 5.6.

Applying Lemma 5.7(i) to the function $\mathbf{f}_n := (P_{t-s}^n f_n, P_s^n \varphi_n)$ and choosing the bounded and continuous test function $\psi(x, r_1, r_2) = r_1^2 r_2 S_C(r_2)$, $(x, r_1, r_2) \in X \times \mathbb{R}^2$ [with S_C defined as in (3.27)], we obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_X (P_{t-s}^n f_n)^2 P_s^n \varphi_n \, dm_n &= \lim_{n \rightarrow \infty} \int \psi \, d(\mathbf{i} \times \mathbf{f}_n)_\# m_n = \int \psi \, d(\mathbf{i} \times \mathbf{f})_\# m \\ &= \int_X (P_{t-s}^\infty f)^2 P_s^\infty \varphi \, dm. \end{aligned}$$

A similar argument yields

$$\lim_{n \rightarrow \infty} \int_X (\Delta_{\mathcal{E}^n} P_{t-s}^n f_n)^2 P_s^n \varphi_n \, dm_n = \int_X (\Delta_{\mathcal{E}} P_{t-s}^\infty f)^2 P_s^\infty \varphi \, dm$$

for every $t > 0, s \in [0, t)$. We can thus pass to the limit in the distributional inequality (2.33) written for f_n, φ_n . \square

LEMMA 5.9. *Let (X, d) be a complete and separable metric space, $m_n \in \mathcal{P}_2(X)$, $n \in \bar{\mathbb{N}}$, be a converging sequence such that (X, d, m_n) is a $\text{RCD}(K, \infty)$ space with Cheeger energy $\frac{1}{2}\mathcal{E}_n$, and let us denote by $(P_t^n)_{t \geq 0}$ the Markov semigroups in $L^2(X, m_n)$ with generators $\Delta_n := \Delta_{\mathcal{E}_n}$, $n \in \bar{\mathbb{N}}$.*

If $f_n \in L^\infty(X, m_n)$ converge to $f_\infty \in L^\infty(X, m_\infty)$ according to Definition 5.6, with uniformly bounded L^∞ norm, then $P_t^n f_n$ converge to $P_t^\infty f_\infty$ for every $t \geq 0$ and $\Delta_n P_t^n f_n$ converge to $\Delta_\infty P_t^\infty f_\infty$ as $n \rightarrow \infty$ for every $t > 0$.

PROOF. When f_n are probability densities the convergence of $P_t^n f_n$ follows by applying Lemma 5.7(ii) and the convergence results of [5], Theorem 6.11 [which shows that $P_t^n f_n m_n$ converges to $P_t^\infty f_\infty m_\infty$ in $\mathcal{P}_2(X)$] and [29] [which yields the convergence of the entropies $\text{Ent}_{m_n}(f_n m_n) \rightarrow \text{Ent}_m(f_\infty m_\infty)$].

The case $f_n \in L^1_+(X, m_n)$ can be easily reduced to the previous one by a rescaling, since $\int_X f_n dm_n \rightarrow \int_X f dm$ by (5.11) and $(P_t^n)_{t \geq 0}$ is mass preserving.

The general case can be proved by decomposing each f_n into the difference $f_n^+ - f_n^-$ of its positive and negative part, observing that f_n^\pm converge to f_∞^\pm thanks to Lemma 5.7(iii). Thus, by (i) it follows that $(P_t^n f_n^+, P_t^n f_n^-)$ converge to $(P_t^\infty f_\infty^+, P_t^\infty f_\infty^-)$ and a further application of (iii) yields the convergence result by the linearity of the semigroups.

In order to prove the convergence of $\Delta_n P_t^n f_n$ we still apply (ii) of Lemma 5.7: recall that P_t^n are analytic semigroups in $L^2(X, m_n)$, $\Delta_n P_t^n f_n = \frac{d}{dt} P_t^n f_n$, and the uniform estimates (see, e.g., [49], page 75, step 2)

$$(5.12) \quad t^j \left\| \frac{d^j}{dt^j} P_t^n f_n \right\|_{L^2(X, m_n)} \leq A_j \|f_n\|_{L^2(X, m_n)} \quad \text{for every } t > 0, n \in \mathbb{N}$$

hold with universal constants A_j for every $t > 0$. Since we just proved that for every $\varphi \in C_b(X)$ the sequence of functions $\zeta_n(t) := \int_X P_t^n f_n \varphi dm_n$ converge pointwise to the corresponding ζ as $n \rightarrow \infty$, (5.12) yields that

$$\lim_{n \rightarrow \infty} \zeta'_n(t) = \lim_{n \rightarrow \infty} \int_X \Delta_n P_t^n f_n \varphi dm_n = \zeta'(t) = \int_X \Delta_\infty P_t^\infty f_\infty \varphi dm$$

for every $t > 0$. The same argument holds for

$$t \mapsto \int_X (\Delta_n P_t^n f_n)^2 dm_n = \frac{1}{4} \frac{d^2}{dt^2} \int_X (P_t^n f_n)^2 dm_n. \quad \square$$

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L. AMBROSIO
SCUOLA NORMALE SUPERIORE
PIAZZA DEI CAVALIERI, 7
56126 PISA
ITALY
E-MAIL: l.ambrosio@sns.it

N. GIGLI
INSTITUT DE MATHÉMATIQUES DE JUSSIEU
CASE 247, 4 PLACE JUSSIEU
75252 PARIS CEDEX 05
FRANCE
E-MAIL: giglin@math.jussieu.fr

G. SAVARÉ
DIPARTIMENTO DI MATEMATICA
UNIVERSITÀ DI PAVIA
VIA FERRATA 1
27100 PAVIA
ITALY
E-MAIL: giuseppe.savare@unipv.it
URL: <http://www.imati.cnr.it/savare>