Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2016, Vol. 52, No. 2, 898–914
DOI: 10.1214/14-AIHP659

ANNALES DE L'INSTITUT HENRI POINCARÉ
PROBABILITÉS ET STATISTIQUES
www.imstat.org/aihp

Functional inequalities for convolution probability measures

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Received 5 February 2014; revised 22 October 2014; accepted 24 October 2014

Abstract. Let μ and ν be two probability measures on $\mathbb{R}^d$, where $\mu(\mathrm{d}x) = \frac{\mathrm{e}^{-V(x)}\,\mathrm{d}x}{\int_{\mathbb{R}^d} \mathrm{e}^{-V(x)}\,\mathrm{d}x}$ for some $V \in C^1(\mathbb{R}^d)$. Explicit sufficient conditions on V and ν are presented such that $\mu * \nu$ satisfies the log-Sobolev, Poincaré and super Poincaré inequalities. In particular, if $V(x) = \lambda|x|^2$ for some $\lambda > 0$ and $\nu(\mathrm{e}^{\lambda\theta|\cdot|^2}) < \infty$ for some $\theta > 1$, then $\mu * \nu$ satisfies the log-Sobolev inequality. This improves and extends the recent results on the log-Sobolev inequality derived in (*J. Funct. Anal.* **265** (2013) 1064–1083) for convolutions of the Gaussian measure and compactly supported probability measures. On the other hand, it is well known that the log-Sobolev inequality for $\mu * \nu$ implies $\nu(\mathrm{e}^{\varepsilon|\cdot|^2}) < \infty$ for some $\varepsilon > 0$.

Résumé. Soit μ et ν deux mesures de probabilité sur $\mathbb{R}^d$, où $\mu(\mathrm{d}x) = \frac{\mathrm{e}^{-V(x)}\,\mathrm{d}x}{\int_{\mathbb{R}^d} \mathrm{e}^{-V(x)}\,\mathrm{d}x}$ avec $V \in C^1(\mathbb{R}^d)$. Des conditions explicites suffisantes sur V et ν sont présentées telles que $\mu * \nu$ satisfait des inégalités de Sobolev logarithmique, de Poincaré et de super-Poincaré. En particulier, si $V(x) = \lambda|x|^2$ pour quelque $\lambda > 0$ et $\nu(\mathrm{e}^{\lambda\theta|\cdot|^2}) < \infty$ avec $\theta > 1$, alors $\mu * \nu$ satisfait l'inégalité de Sobolev logarithmique. Cela améliore et étend des résultats récents sur l'inégalité de Sobolev logarithmique obtenus dans (*J. Funct. Anal.* **265** (2013) 1064–1083) pour des convolutions de la mesure de Gauss et des mesures de probabilité à support compact. D'autre part, il est bien connu que l'inégalité de Sobolev logarithmique pour $\mu * \nu$ implique $\nu(\mathrm{e}^{\varepsilon|\cdot|^2}) < \infty$ pour quelque $\varepsilon > 0$.

MSC: 60J75; 47G20; 60G52

Keywords: Log-Sobolev inequality; Poincaré inequality; Super Poincaré inequality; Convolution

1. Introduction

Functional inequalities of Dirichlet forms are powerful tools in characterizing the properties of Markov semigroups and their generators, see, e.g., [19] and references within. To establish functional inequalities for less explicit or less regular probability measures, one regards the measures as perturbations from better ones satisfying the underlying functional inequalities. For a probability measure μ on $\mathbb{R}^d$, the perturbation to μ can be made in the following two senses. The first type perturbation is in the sense of exponential potential: the perturbation of μ by a potential W is given by $\mu_W(\mathrm{d}x) := \frac{\mathrm{e}^{W(x)}\mu(\mathrm{d}x)}{\mu(\mathrm{e}^W)}$, for which functional inequalities have been studied in many papers, see [2,5,10] and references within. Another kind of perturbation is in the sense of independent sum of random variables: the perturbation of μ by a probability measure ν on $\mathbb{R}^d$ is given by their convolution

$$(\mu * \nu)(A) := \int_{\mathbb{R}^d} 1_A(x+y)\mu(\mathrm{d}x)\nu(\mathrm{d}y).$$

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Functional inequalities for the latter case is not yet well investigated, and the study is useful in characterizing distribution properties of random variables under independent perturbations, see, e.g., [20], Section 3, for an application in the study of random matrices.

In general, let μ and ν be two probability measures on $\mathbb{R}^d$. A straightforward result on functional inequalities of $\mu * \nu$ can be derived from the sub-additivity property; that is, if both μ and ν satisfy a type of functional inequality, $\mu * \nu$ will satisfy the same type inequality. In this paper, we will consider the Poincaré inequality and the super Poincaré inequality. We say that a probability measure μ satisfies the Poincaré inequality with constant $C > 0$, if

$$\mu(f^2) \le C\mu(|\nabla f|^2) + \mu(f)^2, \quad f \in C_b^1(\mathbb{R}^d). \tag{1.1}$$

We say that μ satisfies the super Poincaré inequality with $\beta : (0, \infty) \to (0, \infty)$, if

$$\mu(f^2) \le r\mu(|\nabla f|^2) + \beta(r)\mu(|f|)^2, \quad r > 0, f \in C_b^1(\mathbb{R}^d). \tag{1.2}$$

It is shown in [16], Corollary 3.3, or [17], Corollary 1.3, that the super Poincaré inequality holds with $\beta(r) = \mathrm{e}^{c/r}$ for some constant $c > 0$ if and only if the following Gross log-Sobolev inequality (see [12]) holds for some constant $C > 0$:

$$\mu(f^2 \log f^2) \le C\mu(|\nabla f|^2), \quad f \in C_b^1(\mathbb{R}^d), \mu(f^2) = 1. \tag{1.3}$$

Proposition 1.1. *Let μ and ν be two probability measures on $\mathbb{R}^d$.*

(1) *If μ and ν satisfy the Poincaré (resp. log-Sobolev) inequality with constants C_1 and $C_2 > 0$ respectively, then $\mu * \nu$ satisfies the same inequality with constant $C = C_1 + C_2$.*
(2) *If μ and ν satisfy the super Poincaré inequality with β_1 and β_2 respectively, then $\mu * \nu$ satisfies the super Poincaré inequality with*

$$\beta(r) := \inf\{\beta_1(r_1)\beta_2(r_2) \colon r_1, r_2 > 0, r_1 + r_2\beta_1(r_1) \le r\}, \quad r > 0.$$

Since the proof of this result is almost trivial by using functional inequalities for product measures (cf. [9], Corollary 13), we simply omit it. Due to Proposition 1.1, in this paper the perturbation measure ν may not satisfy the Poincaré inequality, it is in particular the case if the support of ν is disconnected.

Recently, when μ is the Gaussian measure with variance matrix δI for some $\delta > 0$, it is proved in [20] that $\mu * \nu$ satisfies the log-Sobolev inequality if ν has a compact support and either $d = 1$ or $\delta > 2R^2 d$, where R is the radius of a ball containing the support of ν, see [20], Theorem 2 and Theorem 17. The first purpose of this paper is to extend this result to more general μ and to drop the restriction $\delta > 2R^2 d$ for high dimensions. The main tool used in [20] is the Hardy type criterion for the log-Sobolev inequality due to [6], which is qualitatively sharp in dimension one. In this paper we will use a perturbation result of [2] and a Lyapunov type criterion introduced in [8] to derive more general and better results. In particular, as a consequence of Corollary 2.2 below, we have the following result where the compact support condition of ν is relaxed by an exponential integrability condition. We would like to indicate that the exponential integrability condition $\nu(\mathrm{e}^{\varepsilon|\cdot|^2}) < \infty$ for some $\varepsilon > 0$ is also necessary for $\mu * \nu$ to satisfy the log-Sobolev inequality. Indeed, it is well known that the log-Sobolev inequality for $\mu * \nu$ implies $(\mu * \nu)(\mathrm{e}^{c|\cdot|^2}) < \infty$ for some $c > 0$, so that $\nu(\mathrm{e}^{\varepsilon|\cdot|^2}) < \infty$ for $\varepsilon \in (0, c)$. However, it is not clear whether "$\theta > 1$" in the following result is sharp or not.

Theorem 1.2. *Let $V = \lambda|\cdot|^2$ for some constant $\lambda > 0$, and $\mu(\mathrm{d}x) = \frac{\mathrm{e}^{-V(x)}\,\mathrm{d}x}{\int_{\mathbb{R}^d} \mathrm{e}^{-V(x)}\,\mathrm{d}x}$ be a probability measure on $\mathbb{R}^d$. Then for any probability measure ν on $\mathbb{R}^d$ with $\nu(\mathrm{e}^{\lambda\theta|\cdot|^2}) < \infty$ for some constant $\theta > 1$, the log-Sobolev inequality*

$$(\mu * \nu)(f^2 \log f^2) \le C(\mu * \nu)(|\nabla f|^2), \quad f \in C_b^1(\mathbb{R}^d), (\mu * \nu)(f^2) = 1$$

holds for some constant $C > 0$.

According to the above-mentioned results in [20], one may wish to prove that the log-Sobolev inequality is stable under convolution with compactly supported probability measures; i.e., if μ satisfies the log-Sobolev inequality, then so does $\mu * \nu$ for a probability measure ν having compact support. This is however not true, a simple counterexample is that $\mu = \delta_0$, the Dirac measure at point 0, which obviously satisfies the log-Sobolev inequality, but $\mu * \nu = \nu$ does not have to satisfy the log-Sobolev inequality even if ν is compactly supported. Thus, to ensure that $\mu * \nu$ satisfies the log-Sobolev inequality for any compactly supported probability measure ν, one needs additional assumptions on μ. Moreover, since when $\lambda \to \infty$, the Gaussian measure μ in Theorem 1.2 converges to δ_0, this counterexample also fits to the assertion of Theorem 1.2 that for large λ we need a stronger concentration condition on ν.

In the remainder of this paper, let $\mu(\mathrm{d}x) = \mathrm{e}^{-V(x)}\,\mathrm{d}x$ be a probability measure on $\mathbb{R}^d$ such that $V \in C^1(\mathbb{R}^d)$. For a probability measure ν on $\mathbb{R}^d$, we define

$$p_\nu(x) = \int_{\mathbb{R}^d} \mathrm{e}^{-V(x-z)}\nu(\mathrm{d}z), \qquad V_\nu(x) = -\log p_\nu(x), \quad x \in \mathbb{R}^d.$$

Then

$$(\mu * \nu)(\mathrm{d}x) = p_\nu(x)\,\mathrm{d}x = \mathrm{e}^{-V_\nu(x)}\,\mathrm{d}x. \tag{1.4}$$

Moreover, let

$$\nu_x(\mathrm{d}z) = \frac{1}{p_\nu(x)}\mathrm{e}^{-V(x-z)}\nu(\mathrm{d}z), \quad x \in \mathbb{R}^d.$$

In the following three sections, we will investigate the log-Sobolev inequality, Poincaré and super Poincaré inequalities for $\mu * \nu$ respectively.

As a complement to the present paper, Cheng and Zhang investigated the weak Poincaré inequality in [11] for convolution probability measures, by using the Lyapunov type conditions as we did in Sections 3 and 4 for the Poincaré and super Poincaré inequalities respectively.

2. Log-Sobolev inequality

In this section we will use two different arguments to study the log-Sobolev inequality for $\mu * \nu$. One is the perturbation argument due to [1,2], and the other is the Lyapunov criterion presented in [8].

2.1. *Perturbation argument*

Theorem 2.1. *Assume that the log-Sobolev inequality* (1.3) *holds for μ with some constant $C > 0$. If $V \in C^1(\mathbb{R}^d)$ such that*

$$\Phi_\nu(x) := \int_{\mathbb{R}^d} \big(\nabla \mathrm{e}^{-V}\big)(x-z)\nu(\mathrm{d}z), \quad x \in \mathbb{R}^d$$

is well-defined and continuous, and there exists a constant $\delta > 1$ such that

$$\int_{\mathbb{R}^d} \exp\left\{\frac{\delta C}{4}\left(\int_{\mathbb{R}^d} \big|\nabla V(x) - \nabla V(x-z)\big|\nu_x(\mathrm{d}z)\right)^2\right\}\mu(\mathrm{d}x) < \infty, \tag{2.1}$$

then $\mu * \nu$ also satisfies the log-Sobolev inequality, i.e., for some constant $C' > 0$,

$$(\mu * \nu)\big(f^2 \log f^2\big) \le C'(\mu * \nu)\big(|\nabla f|^2\big), \quad f \in C_b^1\big(\mathbb{R}^d\big), (\mu * \nu)\big(f^2\big) = 1.$$

Obviously, $\Phi_\nu \in C(\mathbb{R}^d; \mathbb{R}^d)$ holds if either ν has compact support or $\nabla \mathrm{e}^{-V}$ is bounded. Moreover, (2.2) below holds for bounded Hess_V and compactly supported ν. So, the following direct consequence of Theorem 2.1 improves the above-mentioned main results in [20]. Indeed, this corollary implies Theorem 1.2.

Corollary 2.2. *Assume that* (1.3) *holds and* Φ_ν *is well defined and continuous. If* $V \in C^2(\mathbb{R}^d)$ *with bounded* Hess_V *such that*

$$\int_{\mathbb{R}^d} \exp\left\{\frac{\delta C}{4}\|\mathrm{Hess}_V\|_\infty^2 \left(\int_{\mathbb{R}^d} |z|\nu_x(\mathrm{d}z)\right)^2\right\}\mu(\mathrm{d}x) < \infty \tag{2.2}$$

holds for some constant $\delta > 1$, *then* $\mu * \nu$ *satisfies the log-Sobolev inequality.*

Before presenting the proof of Theorem 2.1, we first prove Theorem 1.2 using Corollary 2.2.

Proof of Theorem 1.2. Let $Z = \int_{\mathbb{R}^d} \mathrm{e}^{-\lambda|x|^2}\,\mathrm{d}x$. Since, in the framework of Corollary 2.2, $V(x) = \lambda|x|^2 + \log Z$, we have $\|\mathrm{Hess}_V\|_\infty^2 = 4\lambda^2$ and (1.3) holds for $C = \frac{1}{\lambda}$. Moreover, since $\theta > 1$, there exists a constant $\varepsilon \in (0,1)$ such that $\delta := \theta - \frac{\varepsilon}{1-\varepsilon} > 1$. So, by the Jensen inequality

$$\begin{aligned} I &:= \int_{\mathbb{R}^d} \exp\left\{\frac{\delta C}{4}\|\mathrm{Hess}_V\|_\infty^2 \nu_x\big(|\cdot|\big)^2\right\}\mu(\mathrm{d}x) \le \int_{\mathbb{R}^d} \mathrm{e}^{\delta\lambda\nu_x(|\cdot|^2)}\mu(\mathrm{d}x) \\ &\le \int_{\mathbb{R}^d\times\mathbb{R}^d} \mathrm{e}^{\delta\lambda|z|^2}\nu_x(\mathrm{d}z)\mu(\mathrm{d}x) = \int_{\mathbb{R}^d} \frac{\int_{\mathbb{R}^d} \mathrm{e}^{\lambda\delta|z|^2-\lambda|x-z|^2}\nu(\mathrm{d}z)}{\int_{\mathbb{R}^d} \mathrm{e}^{-\lambda|x-z|^2}\nu(\mathrm{d}z)}\mu(\mathrm{d}x). \end{aligned} \tag{2.3}$$

Take $R > 0$ such that $\nu(B(0,R)) \ge \frac{1}{2}$. We have

$$\int_{\mathbb{R}^d} \mathrm{e}^{-\lambda|x-z|^2}\nu(\mathrm{d}z) \ge \int_{B(0,R)} \mathrm{e}^{-\lambda R^2 - 2\lambda R|x| - \lambda|x|^2}\nu(\mathrm{d}z) \ge \frac{1}{2}\mathrm{e}^{-\lambda R^2 - 2\lambda R|x| - \lambda|x|^2}.$$

Moreover, for the above $\varepsilon \in (0,1)$ we have

$$-|x-z|^2 \le 2|x|\cdot|z| - |x|^2 - |z|^2 \le -\varepsilon|x|^2 + \frac{\varepsilon}{1-\varepsilon}|z|^2.$$

Combining this with (2.3), we obtain

$$\begin{aligned} I &\le \frac{2\mathrm{e}^{\lambda R^2}}{Z}\int_{\mathbb{R}^d\times\mathbb{R}^d} \mathrm{e}^{\lambda\delta|z|^2 - \lambda|x-z|^2 + 2\lambda R|x|}\nu(\mathrm{d}z)\,\mathrm{d}x \\ &\le \frac{2\mathrm{e}^{\lambda R^2}}{Z}\int_{\mathbb{R}^d\times\mathbb{R}^d} \mathrm{e}^{\lambda\delta|z|^2 - \lambda\varepsilon|x|^2 + (\lambda\varepsilon)/(1-\varepsilon)|z|^2 + 2\lambda R|x|}\,\mathrm{d}x\,\nu(\mathrm{d}z) \\ &= \frac{2\mathrm{e}^{\lambda R^2}}{Z}\int_{\mathbb{R}^d\times\mathbb{R}^d} \mathrm{e}^{\lambda\theta|z|^2 - \lambda\varepsilon|x|^2 + 2\lambda R|x|}\,\mathrm{d}x\,\nu(\mathrm{d}z) < \infty. \end{aligned}$$

Then the proof is finished by Corollary 2.2. □

To prove Theorem 2.1, we introduce the following perturbation result due to [2], Lemma 3.1, and [1], Lemma 4.1.

Lemma 2.3. *Assume that the probability measure* $\mu(\mathrm{d}x) = \mathrm{e}^{-V(x)}\,\mathrm{d}x$ *satisfies the log-Sobolev inequality* (1.3) *with some constant* $C > 0$. *Let* $\mu_{V_0}(\mathrm{d}x) = \mathrm{e}^{-V_0(x)}\,\mathrm{d}x$ *be a probability measure on* $\mathbb{R}^d$. *If* $F := \frac{1}{2}(V - V_0) \in C^1(\mathbb{R}^d)$ *such that*

$$\int_{\mathbb{R}^d} \exp\big(\delta C|\nabla F|^2\big)\,\mathrm{d}\mu < \infty, \tag{2.4}$$

holds for some constant $\delta > 1$, *then the defective log-Sobolev inequality*

$$\mu_{V_0}\big(f^2\log f^2\big) \le C_1\mu_{V_0}\big(|\nabla f|^2\big) + C_2, \quad f \in C_b^1\big(\mathbb{R}^d\big), \mu_{V_0}\big(f^2\big) = 1, \tag{2.5}$$

holds for some constants $C_1, C_2 > 0$.

Proof of Theorem 2.1. Since by (1.4) we have $(\mu * \nu)(\mathrm{d}x) = \mathrm{e}^{-V_\nu(x)}\,\mathrm{d}x$, to apply Lemma 2.3 we take $V_0 = V_\nu$, so that

$$F(x) = \frac{1}{2}\big(V(x) - V_0(x)\big) = \frac{1}{2}\log\int_{\mathbb{R}^d}\mathrm{e}^{V(x)-V(x-z)}\nu(\mathrm{d}z).$$

Since Φ_ν is locally bounded, for any $x \in \mathbb{R}^d$ we have

$$\lim_{y\to 0}\big(p_\nu(x+y) - p_\nu(x)\big) = \lim_{y\to 0}\int_0^1\big\langle y, \Phi_\nu(x+sy)\big\rangle\,\mathrm{d}s = 0.$$

So, $p_\nu \in C(\mathbb{R}^d)$. Then the continuity of Φ_ν implies that

$$\Psi(x) := \int_{\mathbb{R}^d}(\nabla V)(x-z)\nu_x(\mathrm{d}z) = -\frac{\Phi_\nu(x)}{p_\nu(x)}$$

is continuous in x as well. Therefore, for any $x, v \in \mathbb{R}^d$,

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}\frac{F(x+\varepsilon v) - F(x)}{\varepsilon} &= \lim_{\varepsilon\downarrow 0}\frac{1}{2\varepsilon}\int_0^\varepsilon\big\langle v, \nabla V(x+sv) - \Psi(x+sv)\big\rangle\,\mathrm{d}s \\ &= \frac{1}{2}\big\langle v, \nabla V(x) - \Psi(x)\big\rangle.\end{aligned}$$

Thus, by the continuity of Ψ and ∇V we conclude that $F \in C^1(\mathbb{R}^d)$ and

$$\big|\nabla F(x)\big|^2 = \frac{1}{4}\big|\nabla V(x) - \Psi(x)\big|^2 \le \frac{1}{4}\bigg(\int_{\mathbb{R}^d}\big|\nabla V(x) - \nabla V(x-z)\big|\nu_x(\mathrm{d}z)\bigg)^2.$$

Combining this with (2.1), we are able to apply Lemma 2.3 to derive the defective log-Sobolev inequality for $\mu * \nu$. Moreover, the form

$$\mathscr{E}(f,g) := \int_{\mathbb{R}^d}\langle\nabla f, \nabla g\rangle\,\mathrm{d}(\mu * \nu), \qquad f, g \in C_b^1\big(\mathbb{R}^d\big)$$

is closable in $L^2(\mu * \nu)$, and its closure is a symmetric, conservative, irreducible Dirichlet form. Thus, according to [18], Corollary 1.3 (see also [14], Theorem 1), the defective log-Sobolev inequality implies the desired log-Sobolev inequality. Then the proof is finished. □

To see that Corollary 2.2 has a broad range of application beyond [20], Theorem 2, and Proposition 1.1(1) for the log-Sobolev inequality, we present below an example where the support of ν is unbounded and disconnected.

Example 2.4. *Let $d = 1$, $V(x) = \frac{1}{2}\log\pi + x^2$ and*

$$\nu(\mathrm{d}z) = \frac{1}{\gamma}\sum_{i\in\mathbb{Z}}\mathrm{e}^{-\lambda i^2}\delta_i(\mathrm{d}z), \qquad \gamma := \sum_{i\in\mathbb{Z}}\mathrm{e}^{-\lambda i^2},$$

where δ_i is the Dirac measure at point i and $\lambda > 0$. Then $\mu * \nu$ satisfies the log-Sobolev inequality.

Proof. For the present V it is well known from [12] that the log-Sobolev inequality (1.3) holds with $C = 1$. On the other hand, it is easy to see that for any $i \in \mathbb{Z}$, $x \in \mathbb{R}$ and $\lambda > 0$, we have

$$|x-i|^2 + \lambda i^2 = (1+\lambda)\bigg(i - \frac{x}{\lambda+1}\bigg)^2 + \frac{\lambda x^2}{1+\lambda}. \tag{2.6}$$

Let $\tilde{p}(x) = \sum_{i\in\mathbb{Z}} \mathrm{e}^{-(1+\lambda)(i-x/(1+\lambda))^2}$. Then

$$\begin{aligned}
\nu_x(\mathrm{d}z) &= \frac{1}{\gamma(x)} \sum_{i\in\mathbb{Z}} \mathrm{e}^{-|x-i|^2-\lambda i^2} \delta_i(\mathrm{d}z) \\
&= \frac{1}{\tilde{p}(x)} \sum_{i\in\mathbb{Z}} \mathrm{e}^{-(1+\lambda)(i-x/(1+\lambda))^2} \delta_i(\mathrm{d}z),
\end{aligned} \tag{2.7}$$

where $\gamma(x) = \sum_{i\in\mathbb{Z}} \mathrm{e}^{-|x-i|^2-\lambda i^2}$. So,

$$\begin{aligned}
\int_{\mathbb{R}^d} |z|\nu_x(\mathrm{d}z) &= \frac{1}{\tilde{p}(x)} \sum_{i\in\mathbb{Z}} |i|\mathrm{e}^{-(1+\lambda)(i-x/(1+\lambda))^2} \\
&\le \frac{|x|}{1+\lambda} + \frac{1}{\tilde{p}(x)} \sum_{i\in\mathbb{Z}} \left| i - \frac{x}{1+\lambda} \right| \mathrm{e}^{-(1+\lambda)(i-x/(1+\lambda))^2} \\
&\le \frac{|x|}{1+\lambda} + c, \quad x \in \mathbb{R}
\end{aligned}$$

holds for

$$c := \sup_{x\in[0,1+\lambda]} \frac{1}{\tilde{p}(x)} \sum_{i\in\mathbb{Z}} \left| i - \frac{x}{1+\lambda} \right| \mathrm{e}^{-(1+\lambda)(i-x/(1+\lambda))^2} < \infty \tag{2.8}$$

since the underlying function is periodic with a period $[0, 1+\lambda]$. Noting that $C = 1$ and $\|\mathrm{Hess}_V\|^2 = 4$, we conclude from this that condition (2.2) holds for $\delta \in (1, 1+\lambda)$. Then the proof is finished by Corollary 2.2. $\square$

Finally, the following example shows that Theorem 2.1 may also work for unbounded Hess_V.

Example 2.5. *Let $V(x) = c + |x|^p$ with $p \in [2, 4)$ for some constant c such that $\mu(\mathrm{d}x) := \mathrm{e}^{-V(x)}\,\mathrm{d}x$ is a probability measure on $\mathbb{R}^d$. Let ν be a probability measure on $\mathbb{R}^d$ with compact support. Then $\mu * \nu$ satisfies the log-Sobolev inequality.*

Proof. Since $p \ge 2$, we have $V \in C^2(\mathbb{R}^d)$ and $\Phi_\nu \in C(\mathbb{R}^d, \mathbb{R}^d)$. Let $R = \sup\{|z|: z \in \operatorname{supp}\nu\}$. Then

$$\int_{\mathbb{R}^d} \left|\nabla V(x) - \nabla V(x-z)\right| \nu_x(\mathrm{d}z) \le R \sup_{z\in B(x,R)} \left|\mathrm{Hess}_V(z)\right| \le C(R)\bigl(1 + |x|^{p-2}\bigr)$$

holds for some constant $C(R) > 0$ and all $x \in \mathbb{R}^d$. Combining this with $2(p-2) < p$ implied by $p < 4$, we see that (2.1) holds. Then the proof is finished by Theorem 2.1. $\square$

We will see in Remark 4.1 below that the assertion in Example 2.5 remains true for $p \ge 4$. Indeed, when $p > 2$ the super Poincaré inequality presented in Example 4.4 below is stronger than the log-Sobolev inequality, see [16], Corollary 3.3.

2.2. *Lyapunov criterion*

Theorem 2.6. *Assume that $V \in C^2(\mathbb{R}^d)$ with bounded* Hess_V *such that*

$$\mathrm{Hess}_V \ge KI \text{ outside a compact set} \tag{2.9}$$

holds for some constant $K > 0$. Then $\mu * \nu$ satisfies the log-Sobolev inequality provided the following two conditions hold:

(C1) *There exists a constant* $c > 0$ *such that*

$$\nu_x\big(f^2\big) - \nu_x(f)^2 \le c\|\nabla f\|_\infty^2, \quad f \in C_b^1\big(\mathbb{R}^d\big), x \in \mathbb{R}^d.$$

(C2) $\limsup_{|x|\to\infty} \frac{\int_{\mathbb{R}^d} |\nabla V(-z)|\nu_x(\mathrm{d}z)}{|x|} < K$.

We believe that Theorems 2.1 and 2.6 are incomparable, since (2.9) is neither necessary for (1.3) to hold, nor provides explicit upper bound of C in (1.3) which is involved in condition (2.1) for Theorem 2.1. But it would be rather complicated to construct proper counterexamples confirming this observation.

The proof of Theorem 2.6 is based on the following Lyapunov type criterion due to [8], Theorem 1.2.

Lemma 2.7 [8]. *Let* $\mu_0(\mathrm{d}x) = \mathrm{e}^{-V_0(x)}\,\mathrm{d}x$ *be a probability measure on* $\mathbb{R}^d$ *for some* $V_0 \in C^2(\mathbb{R}^d)$. *Then* μ_0 *satisfies the log-Sobolev inequality provided the following two conditions hold*:

(i) *There exists a constant* $K_0 \in \mathbb{R}$ *such that* $\mathrm{Hess}_{V_0} \ge K_0 I$.
(ii) *There exists* $W \in C^2(\mathbb{R}^d)$ *with* $W \ge 1$ *such that*

$$\Delta W(x) - \langle \nabla V_0, \nabla W\rangle(x) \le \big(c_1 - c_2|x|^2\big)W(x), \quad x \in \mathbb{R}^d$$

holds for some constants $c_1, c_2 > 0$.

Proof of Theorem 2.6. By (1.4) and Lemma 2.7, it suffices to verify conditions (i) and (ii) for $V_0 = V_\nu := -\log p_\nu$.

(a) Proof of (i). By the boundedness of Hess_V and the condition (2.9), it is to see that $p_\nu \in C^2(\mathbb{R}^d)$ and for any $X \in \mathbb{R}^d$ with $|X| = 1$, we have

$$\mathrm{Hess}_{V_0}(X, X) = \frac{1}{p_\nu^2}\big((\nabla_X p)^2 - p_\nu \mathrm{Hess}_{p_\nu}(X, X)\big). \tag{2.10}$$

Moreover,

$$\nabla_X p_\nu(x) = -p_\nu(x)\int_{\mathbb{R}^d}\big(\nabla_X V(x-z)\big)\nu_x(\mathrm{d}z).$$

Then, letting $K_1 := \|\mathrm{Hess}_V\| < \infty$, we obtain

$$\begin{aligned}\mathrm{Hess}_{p_\nu}(X, X)(x) &= \int_{\mathbb{R}^d}\big(\big|\nabla_X V(x-z)\big|^2 - \mathrm{Hess}_V(X, X)(x-z)\big)\mathrm{e}^{-V(x-z)}\nu(\mathrm{d}z)\\ &\le p_\nu(x)\int_{\mathbb{R}^d}\big|\nabla_X V(x-z)\big|^2\nu_x(\mathrm{d}z) + K_1 p_\nu(x).\end{aligned}$$

Combining these with (2.10) and (C1), we conclude that

$$\begin{aligned}\mathrm{Hess}_{V_0}(X, X)(x) &\ge -K_1 - \int_{\mathbb{R}^d}\big(\nabla_X V(x-z)\big)^2\nu_x(\mathrm{d}z) + \bigg(\int_{\mathbb{R}^d}\nabla_X V(x-z)\nu_x(\mathrm{d}z)\bigg)^2\\ &\ge -K_1 - cK_1^2.\end{aligned}$$

Thus, (i) holds for $K_0 = -K_1 - cK_1^2$.

(b) Proof of (ii). Let $W(x) = \mathrm{e}^{\varepsilon|x|^2}$ for some constant $\varepsilon > 0$. Then

$$\frac{\Delta W - \langle \nabla V_0, \nabla W\rangle}{W}(x) = 2d\varepsilon + 4\varepsilon^2|x|^2 - \varepsilon\int_{\mathbb{R}^d}\big\langle x, \nabla V(x-z)\big\rangle\nu_x(\mathrm{d}z). \tag{2.11}$$

Since Hess_V is bounded and (2.9) holds, we know that $\int_{\mathbb{R}^d}\langle x, \nabla V(x-z)\rangle \nu_x(\mathrm{d}z)$ is well defined and locally bounded. By (2.9), there exists a constant $r_0 > 0$ such that $\mathrm{Hess}_V \geq KI$ holds on the set $\{|z| \geq r_0\}$. So, for $x \in \mathbb{R}^d$ with $|x| > 2r_0$,

$$\begin{aligned}\bigl\langle \nabla V(x-z) - \nabla V(-z), x\bigr\rangle &= |x|\int_0^{|x|} \mathrm{Hess}_V\biggl(\frac{x}{|x|}, \frac{x}{|x|}\biggr)\biggl(\frac{rx}{|x|} - z\biggr)\,\mathrm{d}r \\ &\geq K|x|^2 - K_1|x|\biggl|\biggl\{r \in \bigl[0, |x|\bigr]: \biggl|\frac{rx}{|x|} - z\biggr| \leq r_0\biggr\}\biggr| \\ &\geq K|x|^2 - 2K_1 r_0|x|.\end{aligned}$$

Combining this with (2.11) and (C2), and noting that

$$\bigl\langle x, \nabla V(x-z)\bigr\rangle \leq \bigl\langle \nabla V(x-z) - \nabla V(-z), x\bigr\rangle + |x| \cdot \bigl|\nabla V(-z)\bigr|,$$

we conclude that there exist constants $C_1, C_2 > 0$ such that

$$\frac{\Delta W - \langle \nabla V_0, \nabla W\rangle}{W}(x) \leq 2d\varepsilon + 4\varepsilon^2|x|^2 - \varepsilon C_1|x|^2 + \varepsilon C_2.$$

Taking $\varepsilon = \frac{C_1}{8}$, we prove (ii) for some constants $c_1, c_2 > 0$. □

Since when ν has compact support, we have

$$\nu_x\bigl(f^2\bigr) - \nu_x(f)^2 = \int_{\mathbb{R}^d\times\mathbb{R}^d} \bigl|f(z) - f(y)\bigr|^2 \nu_x(\mathrm{d}z)\nu_x(\mathrm{d}y) \leq R^2\|\nabla f\|_\infty^2,$$

where $R := \sup\{|z-y|: z, y \in \operatorname{supp}\nu\} < \infty$, and

$$\lim_{|x|\to\infty} \frac{\int_{\mathbb{R}^d} |\nabla V(-z)|\nu_x(\mathrm{d}z)}{|x|} \leq \lim_{|x|\to\infty} \frac{\sup_{\operatorname{supp}\nu}|\nabla V|}{|x|} = 0.$$

The following direct consequence of Theorem 2.6 improves the above mentioned results in [20] as well.

Corollary 2.8. *Assume that $V \in C^2(\mathbb{R}^d)$ with bounded* Hess_V *such that* (2.9) *holds. Then $\mu * \nu$ satisfies the log-Sobolev inequality for any compactly supported probability measure ν.*

To show that Theorem 2.6 also has a range of application beyond Corollary 2.8 and Proposition 1.1(1) for the log-Sobolev inequality, we reprove Example 2.4 by using Theorem 2.6.

Proof of Example 2.4 using Theorem 2.6. Obviously, (2.9) holds for $K = 2$. Let

$$\tilde{\nu}_x = \frac{1}{\tilde{\gamma}(x)}\sum_{i\in\mathbb{Z}} \mathrm{e}^{-(1+\lambda)(i-x)^2}\delta_i, \qquad \tilde{\gamma}(x) = \sum_{i\in\mathbb{Z}} \mathrm{e}^{-(1+\lambda)(i-x)^2}.$$

By (2.6) we have $\tilde{\nu}_x = \nu_{(1+\lambda)x}$. Thus, we only need to verify conditions (C1) and (C2) for $\tilde{\nu}_x$ in place of ν_x.

(a) To prove condition (C1), we make use of a Hardy type inequality for birth–death processes with Dirichlet boundary introduced in [13]. Let $x \in \mathbb{R}$ be fixed. For any bounded function f on $\mathbb{Z}$, let $\tilde{f}(i) = f(i) - f(i_x)$, where $i_x := \sup\{i \in \mathbb{Z}: i \leq x\}$ is the integer part of x. Then

$$\tilde{\nu}_x\bigl(f^2\bigr) - \tilde{\nu}_x(f)^2 \leq \sum_{i=-\infty}^{i_x} \tilde{f}(i)^2\tilde{\nu}_x(i) + \sum_{i=i_x}^{\infty} \tilde{f}(i)^2\tilde{\nu}_x(i). \tag{2.12}$$

It is easy to see that there exists a constant $c>0$ independent of x such that for any $m \ge i_x > x-1$,

$$\sum_{i=i_x}^{m} \mathrm{e}^{(1+\lambda)(i-x)^2} \le c\mathrm{e}^{(1+\lambda)(m-x)^2}, \qquad \sum_{i=m+1}^{\infty} \mathrm{e}^{-(1+\lambda)(i-x)^2} \le c\mathrm{e}^{-(1+\lambda)(m+1-x)^2}.$$

Therefore,

$$\begin{aligned}
&\sup_{m\ge i_x} \left(\sum_{i=i_x}^{m} \mathrm{e}^{(1+\lambda)(i-x)^2}\right) \sum_{i=m+1}^{\infty} \mathrm{e}^{-(1+\lambda)(i-x)^2} \\
&\quad \le c^2 \mathrm{e}^{(1+\lambda)\{(m-x)^2-(m+1-x)^2\}} = c^2\mathrm{e}^{(1+\lambda)\{2(x-m)-1\}} \le c^2\mathrm{e}^{1+\lambda}.
\end{aligned}$$

By this and the Hardy inequality (see [19], Theorem 1.3.9), we have

$$\sum_{i=i_x}^{\infty} \tilde f(i)^2 \tilde\nu_x(i) \le 4c^2\mathrm{e}^{1+\lambda} \sum_{i=i_x}^{\infty} \bigl(f(i+1)-f(i)\bigr)^2 \tilde\nu_x(i).$$

Similarly,

$$\sum_{i=-\infty}^{i_x} \tilde f(i)^2 \tilde\nu_x(i) \le 4c^2\mathrm{e}^{1+\lambda} \sum_{i=-\infty}^{i_x} \bigl(f(i-1)-f(i)\bigr)^2 \tilde\nu_x(i).$$

Combining these with (2.12) we prove (C1) for $\tilde\nu_x$ and some constant $c>0$ (independent of $x\in\mathbb{R}$).

(b) Let $\tilde p(x) = \sum_{i\in\mathbb{Z}} \mathrm{e}^{-(1+\lambda)(i-x/(1+\lambda))^2}$. Noting that $\nabla V(z) = 2z$, by (2.7) we obtain

$$\begin{aligned}
\int_{\mathbb{R}^d} \bigl|\nabla V(-z)\bigr| \nu_x(\mathrm{d}z) &= \frac{2}{\tilde p(x)} \sum_{i\in\mathbb{Z}} |i| \mathrm{e}^{-(1+\lambda)(i-x/(1+\lambda))^2} \\
&\le \frac{2|x|}{1+\lambda} + \frac{2}{\tilde p(x)} \sum_{i\in\mathbb{Z}} \left| i - \frac{x}{1+\lambda}\right| \mathrm{e}^{-(1+\lambda)(i-x/(1+\lambda))^2} \\
&\le c + \frac{2|x|}{1+\lambda}
\end{aligned}$$

for $c>0$ in (2.8). Therefore,

$$\limsup_{|x|\to\infty} \frac{\int_{\mathbb{R}^d} |\nabla V(-z)|\nu_x(\mathrm{d}z)}{|x|} \le \frac{2}{1+\lambda} < 2 = K.$$

Thus, condition (C2) holds. □

At the end of this section, we present the following two remarks for perturbation argument and Lyapunov criteria to deal with convolution probability measures.

Remark 2.1. (1) *Both Theorems 2.1 and 2.6 are concerned with qualitative conditions ensuring the existence of the log-Sobolev inequality for convolution probability measures. It would be interesting to derive explicit estimates on the log-Sobolev constant, i.e., the smallest constant such that the log-Sobolev inequality holds. Recently, by using refining the conditions in Lemma 2.7, Zimmermann has estimated the log-Sobolev constant in [21] for the convolution of a Gaussian measure with a compactly supported measure (see [21], Theorem 10, for more details). Similar things can be done under the present general framework. However, as it is well known that estimates derived from perturbation arguments are in general less sharp, we will not go further in this direction and leave the quantitative estimates to a forthcoming paper by other means.*

(2) *As mentioned in Section 1, the convolution of probability measures refers to the sum of independent random variables. So, by induction we may use the Lyapunov criteria to investigate functional inequalities for multi-convolution measures. In this case it is interesting to study the behavior of the optimal constant (e.g., the log-Sobolev constant) as multiplicity goes to ∞. For this we need fine estimates on the constant in terms of the multiplicity, which is related to what we have discussed in Remark 2.1(1). Of course, for functional inequalities having the sub-additivity property, it is possible to derive multiplicity-free estimates on the optimal constant, see, e.g., the recent paper [15] for Beckner-type inequalities of convolution measures on the abstract Wiener space.*

3. Poincaré inequality

In the spirit of the proof of Theorem 2.6, in this section we study the Poincaré inequality for convolution measures using the Lyapunov conditions presented in [3,4]. One may also wish to use the following easy to check perturbation result on the Poincaré inequality corresponding to Lemma 2.3.

If the probability measure $\mu_V(\mathrm{d}x) = \mathrm{e}^{-V(x)}\,\mathrm{d}x$ satisfies the Poincaré inequality (1.1) with some constant $C > 0$, then for any $V_0 \in C^1(\mathbb{R}^d)$ such that $\int \mathrm{e}^{-V_0(x)}\,\mathrm{d}x = 1$ and $C\|\nabla(V - V_0)\|_\infty^2 < 2$, the probability measure $\mu_{V_0}(\mathrm{d}x) = \mathrm{e}^{-V_0(x)}\,\mathrm{d}x$ satisfies the Poincaré inequality (1.1) (with a different constant) as well.

Since the boundedness condition on $\nabla(V - V_0)$ is rather strong (for instance, it excludes Example 3.3(1) below for $p > 2$), here, and also in the next section for the super Poincaré inequality, we will use the Lyapunov criteria rather than this perturbation result. By combining the following Theorem 3.1 below with [3], Theorem 1.4, one may derive quantitative estimates on the Poincaré constant (or the spectral gap).

Theorem 3.1. *Let $\mu(\mathrm{d}x) = \mathrm{e}^{-V(x)}\,\mathrm{d}x$ be a probability measure on $\mathbb{R}^d$ and let ν be a probability measure on $\mathbb{R}^d$. Assume that Φ_ν in Theorem 2.1 is well-defined and continuous. Then $\mu * \nu$ satisfies the Poincaré inequality (1.1), if at least one of the following conditions holds*:

(1) $V \in C^1(\mathbb{R}^d)$ *such that* $\liminf_{|x|\to\infty} \frac{\int_{\mathbb{R}^d}\langle x, \nabla V(x-z)\rangle \nu_x(\mathrm{d}z)}{|x|} > 0.$

(2) $V \in C^2(\mathbb{R}^d)$ *such that* $\tilde{\Phi}_\nu(x) := \int_{\mathbb{R}^d}(\nabla^2 V)(x-z)\nu_x(\mathrm{d}z)$ *is well-defined and continuous in x, and there is a constant $\delta \in (0,1)$ such that*

$$\liminf_{|x|\to\infty} \int_{\mathbb{R}^d} \bigl(\delta\bigl|\nabla V(x-z)\bigr|^2 - \Delta V(x-z)\bigr)\nu_x(\mathrm{d}z) > 0.$$

Proof. Let $L_\nu = \Delta - \nabla V_\nu$. According to [4], Theorem 3.5, or [3], Theorem 1.4, $(\mu * \nu)(\mathrm{d}x) := \mathrm{e}^{-V_\nu(x)}\,\mathrm{d}x$ satisfies the Poincaré inequality if there exist a C^2-function $W \ge 1$ and some positive constants θ, b, R such that for all $x \in \mathbb{R}^d$,

$$L_\nu W(x) \le -\theta W(x) + b 1_{B(0,R)}(x). \tag{3.1}$$

In particular, by [3], Corollary 1.6, if either

$$\liminf_{|x|\to\infty} \frac{\langle \nabla V_\nu(x), x\rangle}{|x|} > 0, \tag{3.2}$$

or there is a constant $\delta \in (0,1)$ such that

$$\liminf_{|x|\to\infty} \bigl(\delta\bigl|\nabla V_\nu(x)\bigr|^2 - \Delta V_\nu(x)\bigr) > 0, \tag{3.3}$$

then the inequality (3.1) fulfills.

Now, as shown in the proof of Theorem 2.1 that the continuity of Φ_ν implies that $V_\nu \in C^1(\mathbb{R}^d)$ and

$$\bigl\langle \nabla V_\nu(x), x\bigr\rangle = \int_{\mathbb{R}^d} \bigl\langle \nabla V(x-z), x\bigr\rangle \nu_x(\mathrm{d}z).$$

Then condition (1) in Theorem 3.1 implies (3.2), and hence the Poincaré inequality for $\mu * \nu$.

On the other hand, repeating the argument leading to $F \in C^1(\mathbb{R}^d)$ in the proof of Theorem 2.1, we conclude that the continuity of Φ_ν and $\tilde{\Phi}_\nu$ implies $V_\nu \in C^2(\mathbb{R}^d)$ and

$$\left|\nabla V_\nu(x)\right|^2 = \left(\int_{\mathbb{R}^d} \nabla V(x-z)\nu_x(\mathrm{d}z)\right)^2,$$

$$\Delta V_\nu(x) = \left|\nabla V_\nu(x)\right|^2 + \int_{\mathbb{R}^d} \left\{\Delta V(x-z) - \left|\nabla V(x-z)\right|^2\right\}\nu_x(\mathrm{d}z).$$

Then for any $\delta \in (0,1)$,

$$\begin{aligned}\delta\left|\nabla V_\nu(x)\right|^2 - \Delta V_\nu(x) &= \int_{\mathbb{R}^d}\left(\left|\nabla V(x-z)\right|^2 - \Delta V(x-z)\right)\nu_x(\mathrm{d}z) - (1-\delta)\left|\nabla V_\nu(x)\right|^2 \\ &\ge \int_{\mathbb{R}^d}\left(\delta\left|\nabla V(x-z)\right|^2 - \Delta V(x-z)\right)\nu_x(\mathrm{d}z).\end{aligned}$$

Combining this with condition (2) in Theorem 3.1 we prove (3.3), and hence the Poincaré inequality for $\mu * \nu$. □

When the measure ν is compactly supported, we have the following consequence of Theorem 3.1.

Corollary 3.2. *Let ν be a probability measure on $\mathbb{R}^d$ with compact support such that $R := \sup\{|z|: z \in \operatorname{supp}\nu\} < \infty$. If either $V \in C^1(\mathbb{R}^d)$ with*

$$\liminf_{|x|\to\infty} \frac{\langle \nabla V(x), x\rangle - R|\nabla V(x)|}{|x|} > 0, \tag{3.4}$$

or $V \in C^2(\mathbb{R}^d)$ and there is a constant $\delta \in (0,1)$ such that

$$\liminf_{|x|\to\infty}\left(\delta\left|\nabla V(x)\right|^2 - \Delta V(x)\right) > 0, \tag{3.5}$$

then $\mu * \nu$ satisfies the Poincaré inequality.

Proof. Since the support of ν is compact, the continuity of Φ_ν when $V \in C^1(\mathbb{R}^d)$ and that of $\tilde{\Phi}_\nu$ when $V \in C^2(\mathbb{R}^d)$ are obvious. Below we prove conditions (1) and (2) in Theorem 3.1 using (3.4) and (3.5) respectively.

(a) By (3.4) we obtain

$$\begin{aligned}\int_{\mathbb{R}^d}\langle x, \nabla V(x-z)\rangle\nu_x(\mathrm{d}z) &= \int_{\mathbb{R}^d}\left(\langle x-z, \nabla V(x-z)\rangle + \langle z, \nabla V(x-z)\rangle\right)\nu_x(\mathrm{d}z) \\ &\ge \int_{\mathbb{R}^d}\left(\langle x-z, \nabla V(x-z)\rangle - R\left|\nabla V(x-z)\right|\right)\nu_x(\mathrm{d}z) \\ &\ge \int_{\mathbb{R}^d}\left(c_1|x-z| - c_2\right)\nu_x(\mathrm{d}z) \\ &\ge c_1\left(|x|-R\right)^+ - c_2\end{aligned}$$

for some constants $c_1, c_2 > 0$. Then condition (1) in Theorem 3.1 holds.

(b) According to (3.5), there are constants r_1, c_3 and $c_4 > 0$ such that for all $x \in \mathbb{R}^d$

$$\begin{aligned}&\int_{\mathbb{R}^d}\left(\delta\left|\nabla V(x-z)\right|^2 - \Delta V(x-z)\right)\nu_x(\mathrm{d}z) \\ &\quad\ge c_3\int_{\{|x-z|>r_1\}}\nu_x(\mathrm{d}z) - c_4\int_{\{|x-z|\le r_1\}}\nu_x(\mathrm{d}z).\end{aligned} \tag{3.6}$$

Since for $x \in \mathbb{R}^d$ with $|x| > R + r_1$ we have

$$\int_{\{|x-z|>r_1\}} \nu_x(\mathrm{d}z) \geq \int_{\{|z|\leq R\}} \nu_x(\mathrm{d}z) = 1$$

and

$$\int_{\{|x-z|\leq r_1\}} \nu_x(\mathrm{d}z) \leq \int_{\{|z|>R\}} \nu_x(\mathrm{d}z) = 0,$$

then (3.6) implies condition (2) in Theorem 3.1. □

Finally, we present the following examples to illustrate Theorem 3.1 and Corollary 3.2.

Example 3.3. (1) *Let $V(x) = c + |x|^p$ for some $p \geq 1$ and constant c such that $\mu(\mathrm{d}x) := \mathrm{e}^{-V(x)}\,\mathrm{d}x$ is a probability measure on $\mathbb{R}^d$. Then $\mu * \nu$ satisfies the Poincaré inequality for every compactly supported probability measure ν on $\mathbb{R}^d$.*

(2) *Let $d = 1$, $V(x) = c + \sqrt{1+x^2}$ and*

$$\nu(\mathrm{d}z) = \frac{1}{\gamma}\sum_{i\in\mathbb{Z}} \mathrm{e}^{-|i|}\delta_i(\mathrm{d}z), \qquad \gamma := \sum_{i\in\mathbb{Z}} \mathrm{e}^{-|i|},$$

where $c = \log\int_{\mathbb{R}} \mathrm{e}^{-\sqrt{1+x^2}}\,\mathrm{d}x$ and δ_i is the Dirac measure at point i. Then $\mu * \nu$ satisfies the Poincaré inequality.

Proof. Since when $p < 2$ the function $V(x) = c + |x|^p$ is not in C^2 at point 0, we take $\tilde{V} \in C^2(\mathbb{R}^d)$ such that $\tilde{V}(x) = V(x)$ for $|x| \geq 1$. Let $\tilde{\mu}(\mathrm{d}x) = \tilde{C}\mathrm{e}^{-\tilde{V}(x)}\,\mathrm{d}x$, where $\tilde{C} > 0$ is a constant such that $\tilde{\mu}$ is a probability measure. By the stability of Poincaré inequality under the bounded perturbations (e.g., see [9], Proposition 17), it suffices to prove that $\tilde{\mu} * \nu$ satisfies the Poincaré inequality.

In case (1) the assertion is a direct consequence of Corollary 3.2. So, we only have to verify condition (1) in Theorem 3.1 for case (2). For simplicity, we only verify for $x \to \infty$, i.e.,

$$\lim_{x\to\infty} \frac{\int_{\mathbb{R}} xV'(x-z)\nu_x(\mathrm{d}z)}{|x|} > 0. \tag{3.7}$$

Let i_x be the integer part of x, and $h_x = x - i_x$. Note that for any $x > 0$,

$$\begin{aligned}
\frac{\int_{\mathbb{R}} xV'(x-z)\nu_x(\mathrm{d}z)}{|x|} &= \int_{\mathbb{R}} V'(x-z)\nu_x(\mathrm{d}z) \\
&= \frac{\sum_{i\in\mathbb{Z}}(x-i)/\sqrt{1+(x-i)^2}\mathrm{e}^{-\sqrt{1+(x-i)^2}-|i|}}{\sum_{i\in\mathbb{Z}} \mathrm{e}^{-\sqrt{1+(x-i)^2}-|i|}} \\
&= \frac{\sum_{k\in\mathbb{Z}}(h_x+k)/\sqrt{1+(h_x+k)^2}\mathrm{e}^{-\sqrt{1+(h_x+k)^2}-|i_x-k|}}{\sum_{k\in\mathbb{Z}} \mathrm{e}^{-\sqrt{1+(h_x+k)^2}-|i_x-k|}} \\
&=: 1 - p_\nu(x)^{-1}\sum_{k\in\mathbb{Z}}(a_k b_k)(x),
\end{aligned} \tag{3.8}$$

where

$$a_k(x) := \frac{\sqrt{1+(h_x+k)^2}-(h_x+k)}{\sqrt{1+(h_x+k)^2}},$$

$$b_k(x) := \mathrm{e}^{-\sqrt{1+(h_x+k)^2}-|i_x-k|}, \qquad p_\nu(x) = \sum_{k\in\mathbb{Z}} b_k(x).$$

It is easy to see that

$$0 \le a_k(x) \le \begin{cases} (1+k^2)^{-1/2}, & k \ge 0, \\ 2, & k < 0. \end{cases}$$

Then for any $n \ge 1$,

$$\begin{aligned}\sum_{k\in\mathbb{Z}}(a_kb_k)(x) &= \sum_{k\le 0}(a_kb_k)(x) + \sum_{k=1}^{n}(a_kb_k)(x) + \sum_{k=n+1}^{\infty} a_kb_k(x) \\ &\le 2\sum_{k\le 0} b_k(x) + \sum_{k=1}^{n} b_k(x) + \frac{1}{n+1}\sum_{k=n+1}^{\infty} b_k(x).\end{aligned}$$

Thus, for any $x > 0$ and $1 \le n \le i_x$,

$$\sum_{k\le 0} b_k(x) \le \mathrm{e}^{-x} + \sum_{k=-\infty}^{-1} \mathrm{e}^{-(-k-h_x)-(i_x-k)} \le \left(2\mathrm{e}^2+1\right)\mathrm{e}^{-x},$$

$$\sum_{k=1}^{n} b_k(x) \le n\mathrm{e}^{-x}, \qquad p_\nu(x) \ge \sum_{k=1}^{i_x} b_k(x) \ge i_x\mathrm{e}^{-x-1}.$$

Then for any $n \ge 1$,

$$\limsup_{x\to\infty}\frac{1}{p_\nu(x)}\sum_{k\in\mathbb{Z}}(a_kb_k)(x) \le \lim_{x\to\infty}\left\{\frac{\mathrm{e}^{x+1}(2\mathrm{e}^2+1+n)\mathrm{e}^{-x}}{i_x} + \frac{1}{n+1}\right\} = \frac{1}{n+1}.$$

Letting $n \to \infty$ we obtain $\lim_{x\to\infty} p_\nu(x)^{-1}\sum_{k\in\mathbb{Z}}(a_kb_k)(x) = 0$. Combining this with (3.8) we prove (3.7). □

4. Super Poincaré inequality

In this section we extend the results in Section 3 for the super Poincaré inequality.

Theorem 4.1. *Let $\mu(\mathrm{d}x) = \mathrm{e}^{-V(x)}\,\mathrm{d}x$ be a probability measure on $\mathbb{R}^d$ and let ν be a probability measure on $\mathbb{R}^d$. Define*

$$\alpha(r,s) = \left(1+s^{-d/2}\right)\frac{(\sup_{|x|\le r}\mathrm{e}^{-V(x)})^{d/2+1}}{(\inf_{|x|\le r}\mathrm{e}^{-V(x)})^{d/2+2}}, \quad s, r > 0.$$

(1) *If $V \in C^1(\mathbb{R}^d)$ such that*

$$\liminf_{|x|\to\infty}\frac{\int_{\mathbb{R}^d}\langle x, \nabla V(x-z)\rangle\nu_x(\mathrm{d}z)}{|x|} = \infty, \tag{4.1}$$

then $\mu * \nu$ satisfies the super Poincaré inequality (1.2) with

$$\beta(r) = c\left(1+\alpha\left(\psi(2/r), r/2\right)\right)$$

for some constant $c > 0$, where

$$\psi(r) := \inf\left\{s > 0\colon \inf_{|x|\ge s}\frac{\int_{\mathbb{R}^d}\langle x, \nabla V(x-z)\rangle\nu_x(\mathrm{d}z)}{|x|} \ge r\right\} < \infty, \quad r > 0.$$

(2) *Suppose that* $V \in C^2(\mathbb{R}^d)$ *and there is a constant* $\delta \in (0,1)$ *such that*

$$\liminf_{|x|\to\infty} \int_{\mathbb{R}^d} \big(\delta \big|\nabla V(x-z)\big|^2 - \Delta V(x-z)\big)\nu_x(\mathrm{d}z) = \infty. \tag{4.2}$$

Then, $\mu * \nu$ *satisfies the super Poincaré inequality* (1.2) *with*

$$\beta(r) = c\big(1 + \alpha\big(\tilde{\psi}(2/r), r/2\big)\big)$$

for some constant $c > 0$, *where*

$$\tilde{\psi}(r) := \inf\left\{s > 0\colon \inf_{|x|\ge s} \int_{\mathbb{R}^d} \big(\delta\big|\nabla V(x-z)\big|^2 - \Delta V(x-z)\big)\nu_x(\mathrm{d}z) \ge r\right\} < \infty, \quad r > 0.$$

The proof of Theorem 4.1 is based on the following lemma.

Lemma 4.2. *Let* $\mu(\mathrm{d}x) = \mathrm{e}^{-V(x)}\,\mathrm{d}x$ *be a probability measure on* $\mathbb{R}^d$. *Assume that there are functions* $W \ge 1$, $\phi > 0$ *with* $\liminf_{|x|\to\infty} \phi(x) = \infty$ *and constants* $b, r_0 > 0$ *such that*

$$\frac{\Delta W - \langle \nabla W, \nabla V\rangle}{W} \le -\phi + b 1_{B(0,r_0)}.$$

Then, *the following super Poincaré inequality holds*

$$\mu_V\big(f^2\big) \le r\mu_V\big(|\nabla f|^2\big) + \beta(r)\mu_V\big(|f|\big)^2,$$

with

$$\beta(r) = c\big(1 + \alpha\big(\psi_\phi(2/r), r/2\big)\big), \quad r > 0$$

for some constant $c > 0$ *and*

$$\psi_\phi(r) := \inf\Big\{s > 0\colon \inf_{|x|\ge s} \phi(x) \ge r\Big\}.$$

Proof. It is well known that (e.g., see [7], Proposition 3.1) there exists a constant $C > 0$ such that for any $t, s > 0$ and $f \in C^1(\mathbb{R}^d)$,

$$\int_{B(0,t)} f^2(x)\,\mathrm{d}x \le s\int_{B(0,t)} \big|\nabla f(x)\big|^2\,\mathrm{d}x + C\big(1 + s^{-d/2}\big)\left(\int_{B(0,t)} |f|(x)\,\mathrm{d}x\right)^2.$$

Therefore,

$$\begin{aligned}
\int_{B(0,t)} f^2(x)\mu_V(\mathrm{d}x) &\le \Big(\sup_{|x|\le t} \mathrm{e}^{-V(x)}\Big)\int_{B(0,t)} f^2(x)\,\mathrm{d}x \\
&\le s\frac{\sup_{|x|\le t} \mathrm{e}^{-V(x)}}{\inf_{|x|\le t} \mathrm{e}^{-V(x)}} \int_{B(0,t)} \big|\nabla f(x)\big|^2 \mu_V(\mathrm{d}x) \\
&\quad + C\big(1 + s^{-d/2}\big)\frac{\sup_{|x|\le t} \mathrm{e}^{-V(x)}}{(\inf_{|x|\le t} \mathrm{e}^{-V(x)})^2}\left(\int_{B(0,t)} |f|(x)\mu_V(\mathrm{d}x)\right)^2 \\
&\le s\frac{\sup_{|x|\le t} \mathrm{e}^{-V(x)}}{\inf_{|x|\le t} \mathrm{e}^{-V(x)}}\mu_V\big(|\nabla f|^2\big) + C\big(1 + s^{-d/2}\big)\frac{\sup_{|x|\le t} \mathrm{e}^{-V(x)}}{(\inf_{|x|\le t} \mathrm{e}^{-V(x)})^2}\mu_V\big(|f|\big)^2.
\end{aligned}$$

Taking $s = r\frac{\inf_{|x|\le t} \mathrm{e}^{-V(x)}}{\sup_{|x|\le t} \mathrm{e}^{-V(x)}}$ in the inequality above, we arrive at that for any $t, r > 0$ and $f \in C^1(\mathbb{R}^d)$,

$$\int_{B(0,t)} f^2(x)\mu_V(\mathrm{d}x) \le r\mu_V\big(|\nabla f|^2\big) + C\alpha(t,r)\mu_V\big(|f|\big)^2.$$

Thus, the proof is finished by [7], Theorem 2.10, and the fact that the function $\alpha(r,s)$ is increasing with respect to r and decreasing with respect to s. □

Proof of Theorem 4.1. As the same to the proof of Theorem 3.1, let $L_\nu = \Delta - \nabla V_\nu$.

In case (1), we consider a smooth function such that $W(x) = \mathrm{e}^{2|x|}$ for $|x| \ge 1$ and $W(x) \ge 1$ for all $x \in \mathbb{R}^d$. We have

$$\frac{L_\nu W(x)}{W(x)} \le -\frac{\langle x, \nabla V_\nu(x)\rangle}{|x|}1_{\{|x|\ge 1\}} + b1_{\{|x|\le 1\}}$$

for some constant $b > 0$. Then, the required assertion follows from Lemma 4.2 and the proof of Theorem 3.1(1).

In case (2), we consider a smooth function such that $W(x) = \mathrm{e}^{(1-\delta)V(x)}$ for $|x| \ge 1$ and $W(x) \ge 1$ for all $x \in \mathbb{R}^d$. Then,

$$\frac{L_\nu W(x)}{W(x)} \le -(1-\delta)\big(\Delta V(x) - \delta\big|\nabla V(x)\big|^2\big) + b1_{\{|x|\le 1\}}$$

for some constant $b > 0$. This along with Lemma 4.2 and the proof of Theorem 3.1(2) also yields the desired assertion. □

According to the proof of Corollary 3.2, when the measure ν has the compact support, we can obtain the following statement from Theorem 4.1.

Corollary 4.3. *Let ν be a probability measure on $\mathbb{R}^d$ with compact support such that $R := \sup\{|z|\colon z \in \operatorname{supp}\nu\} < \infty$.*

(1) *If*

$$\liminf_{|x|\to\infty} \frac{\langle\nabla V(x), x\rangle - R|\nabla V(x)|}{|x|} = \infty, \tag{4.3}$$

then $\mu * \nu$ satisfies the super Poincaré inequality (1.2) *with*

$$\beta(r) = c\big(1 + \alpha\big(\psi(2/r), r/2\big)\big)$$

for some constant $c > 0$, where

$$\psi(r) := \inf\left\{s > 0\colon \inf_{|x|\ge 2s} \frac{\langle\nabla V(x), x\rangle - R|\nabla V(x)|}{|x|} \ge r\right\}.$$

(2) *If there is a constant $\delta \in (0,1)$ such that*

$$\liminf_{|x|\to\infty}\big(\delta\big|\nabla V(x)\big|^2 - \Delta V(x)\big) = \infty, \tag{4.4}$$

then $\mu * \nu$ satisfies the super Poincaré inequality (1.2) *with*

$$\beta(r) = c\big(1 + \alpha\big(\tilde\psi(2/r), r/2\big)\big)$$

for some constant $c > 0$, where

$$\tilde\psi(r) := \inf\left\{s > 0\colon \inf_{|x|\ge 2s}\big(\delta\big|\nabla V(x)\big|^2 - \Delta V(x)\big) \ge r\right\}.$$

The proof of Corollary 4.3 is similar to that of Corollary 3.2, and is thus omitted. Finally, we consider the following example to illustrate Corollary 4.3.

Example 4.4. *Let $V(x) = c + |x|^p$ for some $p > 1$ and $c \in \mathbb{R}$ such that $\mu(\mathrm{d}x) := \mathrm{e}^{-V(x)}\,\mathrm{d}x$ is a probability measure on $\mathbb{R}^d$. Then for any compactly supported probability measure ν, there exists a constant $c > 0$ such that $\mu * \nu$ satisfies the super Poincaré inequality* (1.2) *with*

$$\beta(r) = \exp\bigl(cr^{-p/(2(p-1))}\bigr), \quad r > 0. \tag{4.5}$$

Proof. Since by [18], Corollary 1.2, the super Poincaré inequality implies the Poincaré inequality, we may take $\beta(r) = 1$ for large $r > 0$. So, it suffices to prove the assertion for small $r > 0$. As explained in the proof of Example 3.3 up to a bounded perturbation, we may simply assume that $V \in C^2(\mathbb{R}^d)$. For any $\delta \in (0, 1)$ and any $x \in \mathbb{R}^d$ with $|x|$ large enough,

$$\delta\bigl|\nabla V(x)\bigr|^2 - \Delta V(x) \ge \eta\bigl(V(x)\bigr),$$

where η is a non-decreasing function such that $\eta(r) = \delta r^{2(p-2)/p}$ for some constant $\delta > 0$ and all $r \ge 1$. So,

$$\tilde{\psi}(u) \le c_1\bigl(1 + u^{1/(2(p-2))}\bigr), \quad u > 0$$

holds for some constant $c_1 > 0$. Next, it is easy to see that

$$\alpha(r, s) \le c_2\bigl(1 + s^{-d/2}\bigr)\mathrm{e}^{c_2 r^p}, \quad s, r > 0$$

holds for some constant $c_2 > 0$. Therefore, the desired assertion for small $r > 0$ follows from Corollary 4.3(2). □

Remark 4.1. (1) *By letting $\nu = \delta_0$ we have $\mu = \mu * \nu$. So, Example 4.4 implies that μ satisfies the super Poincaré inequality with β given in* (4.5) *for some constant $c > 0$, and moreover, the inequality is stable under convolutions of compactly supported probability measures. It is easy to see from* [16], *Theorem* 6.2, *that the rate function β given in* (4.5) *is sharp, i.e., $\mu * \nu$ does not satisfy the super Poincaré inequality with β such that* $\lim_{r\downarrow 0} r^{p/(2(p-1))} \log\beta(r) = 0$.

(2) *On the other hand, however, if ν has worse concentration property, $\mu * \nu$ may only satisfy a weaker functional inequality. For instance, let μ be in Example 4.4 but $\nu(\mathrm{d}z) = C\mathrm{e}^{-|z|^q}\,\mathrm{d}z$ for some constant $q \in (1, p)$ and normalization constant $C > 0$. As explained in Remark 4.1*(1) *for q in place p we see that ν satisfies the super Poincaré inequality with*

$$\beta(r) = \exp\bigl(cr^{-q/(2(q-1))}\bigr), \quad r > 0 \tag{4.6}$$

for some constant $c > 0$. Combining this with the super Poincaré inequality for μ with β given in (4.5), *from Proposition 1.1 we conclude that $\mu * \nu$ also satisfies the super Poincaré inequality with β given in* (4.6) *for some* (*different*) *constant $c > 0$, which is sharp according to* [16], *Theorem* 6.2, *as explained above. However, it is less straightforward to verify this super Poincaré inequality for $\mu * \nu$ using Theorem 4.1 instead of Proposition 1.1.*

Acknowledgements

Supported in part by NNSFC (11131003, 11431014, 11522106), the 985 project, the Laboratory of Mathematical and Complex Systems, the JSPS postdoctoral fellowship (26·04021), NSF of Fujian Province (2015J01003), the Program for Nonlinear Analysis and Its Applications (IRTL1206), and Fujian Provincial Key Laboratory of Mathematical Analysis and its Applications (FJKLMAA). We would like to thank the referee for helpful comments.

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