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Systems of Brownian particles with asymmetric collisions

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Abstract. We study systems of Brownian particles on the real line which interact by splitting the local times of collisions among themselves in an asymmetric manner. We prove strong existence and uniqueness of such processes and identify them with the collections of ordered processes in a Brownian particle system, in which the drift coëfficients, the diffusion coëfficients, and the collision local times for the individual particles are assigned according to their ranks. These Brownian systems can be viewed as generalizations of those arising in first-order models for equity markets in the context of stochastic portfolio theory, and are able to correct for several shortcomings of such models while being equally amenable to computations. We also show that, in addition to being of interest in their own right, such systems of Brownian particles arise as universal scaling limits of systems of jump processes on the integer lattice with local interactions. A key step in the proof is the analysis of a generalization of Skorokhod maps which include "local times" at the intersection of faces of the nonnegative orthant. The result extends the convergence of the totally asymmetric simple exclusion process (TASEP) to its continuous analogue. Finally, we identify those among the Brownian particle systems which have a probabilistic structure of determinantal type.

Résumé. Nous étudions des systèmes de particules browniennes sur l'axe réel qui interagissent de façon asymétrique en fonction de leurs temps locaux de collision. Nous prouvons l'existence et l'unicité au sens fort de tels processus et les identifions avec un système de particules constitué d'une famille de processus browniens ordonnés où les coefficients de dérive et de diffusion, ainsi que les temps locaux de collision entre les particules, dépendent de leurs rangs. Ces systèmes browniens peuvent être compris comme des généralisations de processus stochastiques issus des marchés boursiers et de la gestion de portefeuilles. Nous pouvons pallier certaines lacunes de ces modèles tout en conservant la possibilité d'effectuer des calculs explicites. Nous montrons aussi, qu'en plus de leur intérêt intrinsèque, de tels systèmes de particules browniennes apparaissent comme des limites universelles de processus de sauts sur un réseau avec des interactions locales. Une étape clef dans la preuve est l'analyse d'une généralisation des applications de Skorokhod qui inclut les temps locaux à l'intersection des faces de l'orthant positif. Le résultat généralise la convergence du processus d'exclusion simple totalement asymétrique (TASEP) vers sa version continue. Finalement, nous identifions parmi les systèmes de particules browniennes ceux qui possèdent une structure probabiliste déterminantale.

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1. Introduction

Systems of Brownian particles with various types of interactions have been studied widely, and for some time. Recently, Brownian particles with electrostatic repulsion (Dyson's Brownian motion, see for instance Section 4.3 in [1]) have played a central rôle in the understanding of the universality properties of large Hermitian random matrices with independent entries (Wigner matrices, see the survey [9] and the references there). In addition, systems of Brownian particles interacting through their ranks, which were originally introduced in the context of the piecewise-linear filtering problem and in the resulting study of diffusions with piecewise-constant characteristics (see [4]), have been of great importance in the study of large equity markets within stochastic portfolio theory (see [10,12]). Finally, systems of Brownian particles interacting by stickiness have been recently introduced as continuous analogues of a certain random evolution for the distribution of mass on the integer lattice (see [19] and the references there).

As a starting point towards the formulation of our setup, we recall from Section 13.1 in [12] that the ordered processes in a system of Brownian particles interacting through their ranks are given by independent Brownian motions, which have constant drift and diffusion coëfficients and collide in a *symmetric* fashion; that is, each collision local time is split equally between the two colliding particles. In contrast, the particles in the process introduced by Warren in [34] evolve as independent Brownian motions with constant drift and diffusion coëfficients, and collide in a *totally asymmetric* manner; that is, the collision local time is assigned entirely to one of the two colliding particles.

1.1. *The continuum setup*

The dynamics of (1.1) below give the formal description of a system of ordered Brownian particles on the line, which move as independent Brownian motions with constant drift and diffusion coëfficients and collide *asymmetrically*; that is, the collision local times are apportioned unequally, in a manner that depends on the ranks of the particles involved in the collisions.

Consider a continuous, n-dimensional semimartingale $R(\cdot) = (R_1(\cdot), R_2(\cdot), \ldots, R_n(\cdot))$ with values in the Weyl chamber $\mathbb{W}^n = \{(r_1, r_2, \ldots, r_n)\colon \infty > r_1 \ge r_2 \ge \cdots \ge r_n > -\infty\}$, and with dynamics of the form

$$R_k(t) = R_k(0) + b_k t + \sigma_k \boldsymbol{\beta}_k(t) + q_k^- \Lambda^{(k,k+1)}(t) - q_k^+ \Lambda^{(k-1,k)}(t), \quad 0 \le t < \infty \tag{1.1}$$

for $k = 1, 2, \ldots, n$. Here the drifts $b_1, b_2, \ldots, b_n$ are given real numbers; the dispersions $\sigma_1, \sigma_2, \ldots, \sigma_n$ are given *strictly positive* real numbers; the collision parameters $q_1^\pm, q_2^\pm, \ldots, q_n^\pm$ are given *strictly positive* real numbers satisfying

$$q_k^- + q_{k+1}^+ = 1, \quad k = 1, 2, \ldots, n-1; \tag{1.2}$$

and the processes $\boldsymbol{\beta}_1(\cdot), \boldsymbol{\beta}_2(\cdot), \ldots, \boldsymbol{\beta}_n(\cdot)$ are independent standard Brownian motions. On the other hand, for each $k = 1, 2, \ldots, n-1$ the process

$$\Lambda^{(k,k+1)}(\cdot) \equiv L^{R_k - R_{k+1}}(\cdot; 0) \tag{1.3}$$

is the right-sided local time accumulated at the origin by the nonnegative semimartingale $R_k(\cdot) - R_{k+1}(\cdot)$; we set $\Lambda^{(0,1)}(\cdot) \equiv \Lambda^{(n,n+1)}(\cdot) \equiv 0$. The "regulating" rôle of these local times in (1.1) is to make sure the resulting process $R(\cdot)$ takes values in the wedge $\mathbb{W}^n$ at all times. This process can thus be regarded as Brownian motion with reflection on the faces of the polyhedral domain $\mathbb{W}^n$, in the sense of Harrison and Williams (cf. [17,35]).

We now discuss how the processes in (1.1) can be seen as describing the order statistics in Brownian particle systems, in which the particles are allowed to exchange their ranks. As we explain below, the latter can be used as models for the logarithmic capitalizations in large equity markets, and generalize the so-called "first order models" of stochastic portfolio theory. Consider an n-dimensional process $X(\cdot) = (X_1(\cdot), X_2(\cdot), \ldots, X_n(\cdot))$ that satisfies the system of stochastic differential equations

$$\begin{aligned}
\mathrm{d}X_i(t) = {} & \sum_{k=1}^n \mathbf{1}_{\{r_{X(t)}(X_i(t))=k\}} b_k \,\mathrm{d}t + \sum_{k=1}^n \mathbf{1}_{\{r_{X(t)}(X_i(t))=k\}} \sigma_k \,\mathrm{d}W_i(t) \\
& + \sum_{k=1}^n \mathbf{1}_{\{r_{X(t)}(X_i(t))=k\}} \big(q_k^- - (1/2)\big) \,\mathrm{d}\Lambda^{(k,k+1)}(t) \\
& - \sum_{k=1}^n \mathbf{1}_{\{r_{X(t)}(X_i(t))=k\}} \big(q_k^+ - (1/2)\big) \,\mathrm{d}\Lambda^{(k-1,k)}(t).
\end{aligned} \tag{1.4}$$

Here $W_1(\cdot), W_2(\cdot), \dots, W_n(\cdot)$ are independent standard Brownian motions; the rank function $(x, x_i) \mapsto r_x(x_i)$ gives the rank of the coordinate x_i among the coordinates $x_1, x_2, \dots, x_n$ of the vector $x = (x_1, x_2, \dots, x_n)$ with lexicographic resolution of ties (for example, $r_x(x_i) = i$, $i = 1, 2, \dots, n$ when $x = (1, 1, \dots, 1)$); whereas we write $R_1^X(\cdot) \geq R_2^X(\cdot) \geq \cdots \geq R_n^X(\cdot)$ for the descending order statistics of $X_1(\cdot), X_2(\cdot), \dots, X_n(\cdot)$, and denote by

$$\Lambda^{(k,\ell)}(\cdot) \equiv L^{R_k^X - R_\ell^X}(\cdot; 0), \quad \ell \geq k+1 \tag{1.5}$$

the local time accumulated at the origin by the nonnegative semimartingale $R_k^X(\cdot) - R_\ell^X(\cdot)$.

We shall discuss in detail the solvability of the system (1.4) in Theorem 5 below. For now, let us assume that a weak solution to this system exists and satisfies

$$\mathcal{L}\text{eb}\big(\big\{t \geq 0 | \exists 1 \leq i < j \leq n\colon\ X_i(t) = X_j(t)\big\}\big) = 0 \tag{1.6}$$

almost surely, where $\mathcal{L}$eb denotes the Lebesgue measure on $[0, \infty)$. Then with the notation $\mathcal{N}_k(t)$ for the set of indices of particles at the location of the kth ranked particle at time t, the Banner and Ghomrasni [3] formula

$$\mathrm{d}R_k^X(t) = \sum_{i=1}^n \mathbf{1}_{\{r_{X(t)}(X_i(t))=k\}}\, \mathrm{d}X_i(t) + \frac{1}{|\mathcal{N}_k(t)|}\left(\sum_{\ell=k+1}^n \mathrm{d}\Lambda^{(k,\ell)}(t) - \sum_{\ell=1}^{k-1} \mathrm{d}\Lambda^{(\ell,k)}(t)\right) \tag{1.7}$$

(cf. Theorem 2.3 in [3]) shows that the process of spacings

$$Z^X(\cdot) := \big(R_1^X(\cdot) - R_2^X(\cdot), R_2^X(\cdot) - R_3^X(\cdot), \dots, R_{n-1}^X(\cdot) - R_n^X(\cdot)\big), \tag{1.8}$$

when away from the boundary of the nonnegative orthant $(\mathbb{R}_+)^{n-1}$, moves according to the multidimensional process $((b_1 - b_2)t + \sigma_1 \boldsymbol{\beta}_1^X(t) - \sigma_2 \boldsymbol{\beta}_2^X(t), \dots, (b_{n-1} - b_n)t + \sigma_{n-1}\boldsymbol{\beta}_{n-1}^X(t) - \sigma_n \boldsymbol{\beta}_n^X(t))$, $t \geq 0$. Here the processes

$$\boldsymbol{\beta}_k^X(\cdot) := \sum_{i=1}^n \int_0^\cdot \mathbf{1}_{\{r_{X(t)}(X_i(t))=k\}}\, \mathrm{d}W_i(t), \quad k = 1, 2, \dots, n \tag{1.9}$$

are independent standard Brownian motions, by virtue of the P. Lévy theorem (see Section 3 in [2] for a very similar derivation). But on the strength of Lemma 1 below, which generalizes the boundary property of reflected Brownian motion established by Reiman and Williams in [30], the triple- or higher-order collision local times $\Lambda^{(k,\ell)}(\cdot)$ vanish for all $\ell \geq k+2$, so the expression in the Banner–Ghomrasni [3] formula (1.7) simplifies to

$$\begin{aligned}
\mathrm{d}R_k^X(t) &= \sum_{i=1}^n \mathbf{1}_{\{r_{X(t)}(X_i(t))=k\}}\, \mathrm{d}X_i(t) + \frac{1}{2}\,\mathrm{d}\Lambda^{(k,k+1)}(t) - \frac{1}{2}\,\mathrm{d}\Lambda^{(k-1,k)}(t) \\
&= \sum_{i=1}^n \mathbf{1}_{\{r_{X(t)}(X_i(t))=k\}} \sum_{\ell=1}^n \mathbf{1}_{\{r_{X(t)}(X_i(t))=\ell\}} \big(b_\ell\, \mathrm{d}t + \sigma_\ell\, \mathrm{d}W_i(t)\big) \\
&\quad + \sum_{i=1}^n \mathbf{1}_{\{r_{X(t)}(X_i(t))=k\}} \sum_{\ell=1}^n \mathbf{1}_{\{r_{X(t)}(X_i(t))=\ell\}} \big(q_\ell^- - (1/2)\big)\, \mathrm{d}\Lambda^{(\ell,\ell+1)}(t) \\
&\quad - \sum_{i=1}^n \mathbf{1}_{\{r_{X(t)}(X_i(t))=k\}} \sum_{\ell=1}^n \mathbf{1}_{\{r_{X(t)}(X_i(t))=\ell\}} \big(q_\ell^+ - (1/2)\big)\, \mathrm{d}\Lambda^{(\ell-1,\ell)}(t) \\
&\quad + \frac{1}{2}\,\mathrm{d}\Lambda^{(k,k+1)}(t) - \frac{1}{2}\,\mathrm{d}\Lambda^{(k-1,k)}(t), \quad k = 1, 2, \dots, n.
\end{aligned}$$

Evaluating the sums of this expression over i and ℓ and recalling the notation of (1.9), we get the dynamics (1.1) for the descending order statistics $R_1^X, R_2^X, \dots, R_n^X$.

1.2. *Interpretation and ramifications*

We shall think of the processes $X_1(\cdot), X_2(\cdot), \dots, X_n(\cdot)$ as representing the positions of a finite collection of Brownian particles, whose drift and dispersion coëfficients are assigned according to the ranks occupied by the particles when ordered from right to left. When the particles collide, they interact asymmetrically with their nearest neighbors through the collision local times at the origin, in the specific manner of (1.4) and with the notation of (1.5). We shall speak of the $\mathbb{R}^n$-valued semimartingale $X(\cdot) = (X_1(\cdot), \dots, X_n(\cdot))$ as the process of "names" (positions of individual particles), and of the components of the $\mathbb{W}^n$-valued semimartingale $R(\cdot) = (R_1(\cdot), \dots, R_n(\cdot))$ as the associated "ranked processes" (descending order statistics).

If we denote by $\mathfrak{r}_t^X(i)$ the rank occupied by particle i within the configuration of particles $X_1(t), X_2(t), \dots, X_n(t)$ at time t, we can write the system of equations (1.4) in the very informal, yet suggestive and slightly more compact form

$$\begin{aligned} \mathrm{d}X_i(t) &= b_{\mathfrak{r}_t^X(i)}\,\mathrm{d}t + \sigma_{\mathfrak{r}_t^X(i)}\,\mathrm{d}W_i(t) + \big(q^-_{\mathfrak{r}_t^X(i)} - (1/2)\big)\,\mathrm{d}\Lambda^{(\mathfrak{r}_t^X(i), \mathfrak{r}_t^X(i)+1)}(t) \\ &\quad - \big(q^+_{\mathfrak{r}_t^X(i)} - (1/2)\big)\,\mathrm{d}\Lambda^{(\mathfrak{r}_t^X(i)-1, \mathfrak{r}_t^X(i))}(t). \end{aligned} \tag{1.10}$$

To wit: at any given time t, every particle i gets assigned drift and dispersion parameters according to its current rank $\mathfrak{r}_t^X(i)$, and feels an upward (respectively, downward) local-time-like pressure, or "drag," when colliding with the particle right below it (respectively, right above it) in proportion to $q^{\mp}_{\mathfrak{r}_t^X(i)} - (1/2)$.

As we show in Section 3, Brownian particle systems of the type (1.1) arise as universal scaling limits for systems of jump processes on the integer lattice with local interactions. Consider on the integer lattice a system of n particles moving according to (possibly asymmetric) continuous time simple jump processes, which are independent as long as the particles are located at n different sites. When two or more particles land at the same site (we will refer to such events as "collisions"), the jump rates of the particles change in a manner that preserves the order of the particles, but is allowed to depend on the identities of the particles involved in the collision. In particular, if one adds 1 to the spacings between any two consecutive particles, one obtains a particle system known as an *exclusion process with speed change* (see Section VIII.6.1 in [26] for an extensive list of references on the latter). Special cases of such particle systems include the well-known asymmetric simple exclusion process (ASEP) and the gradient type exclusion processes with speed change studied by Funaki et al. in [13].

As we explain in Section 3, all such particle systems converge under a diffusive rescaling of time and space to the solution of a stochastic equation of the type (1.1). As a special case this result includes the convergence of the totally asymmetric simple exclusion process (TASEP) to its continuous analogue (Brownian TASEP) introduced in [34] and thereby recovers a result from [14] (see also the introduction of [27] for a discussion of this connection and its relatives in the setting of random polymers at positive temperature). The proof of the convergence result necessitates a detailed study of Skorokhod maps that transform noise to processes constrained to stay in the nonnegative orthant, but might involve "local time" push from the intersection of multiple faces of the orthant. Along the way, we generalize the boundary property of reflected Brownian motion established in [30] (see Lemma 1 below) and the invariance principle for reflected Brownian motion of [36] (see Proposition 9 below).

1.3. *Some special cases*

It is instructive to compare the system (1.4) in the two-dimensional case $n = 2$ with the systems of equations studied by Fernholz, Ichiba and Karatzas in [11] (see the systems of equations (1.2), (1.3) and (4.13), (4.14) of that paper). As one can see by comparing the coëfficients of the local time terms, in the case $n = 2$ the system (1.4) is a special case of the system of equations (4.13) and (4.14) in [11], namely

$$\begin{aligned} \mathrm{d}X_1(t) &= \mathbf{1}_{\{X_1(t) \ge X_2(t)\}}\big(b_1\,\mathrm{d}t + \sigma_1\,\mathrm{d}W_1(t)\big) + \mathbf{1}_{\{X_1(t) < X_2(t)\}}\big(b_2\,\mathrm{d}t + \sigma_2\,\mathrm{d}W_2(t)\big) + \kappa\,\mathrm{d}L^{|X_1 - X_2|}(t), \\ \mathrm{d}X_2(t) &= \mathbf{1}_{\{X_1(t) < X_2(t)\}}\big(b_1\,\mathrm{d}t + \sigma_1\,\mathrm{d}W_1(t)\big) + \mathbf{1}_{\{X_1(t) \ge X_2(t)\}}\big(b_2\,\mathrm{d}t + \sigma_2\,\mathrm{d}W_2(t)\big) + \kappa\,\mathrm{d}L^{|X_1 - X_2|}(t) \end{aligned}$$

with $\kappa = q_1^- - (1/2) = (1/2) - q_2^+$. The inequalities in the indicators reflect the convention on lexicographic resolution of ties we referred to earlier.

In particular, with $\Upsilon(\cdot) = X_1(\cdot) - X_2(\cdot)$, $\Xi(\cdot) = X_1(\cdot) + X_2(\cdot)$, $\lambda_1 = b_1 - b_2$ and $\lambda_2 = b_1 + b_2$, we have $R_1(\cdot) - R_2(\cdot) = |\Upsilon(\cdot)|$, and the processes

$$W(t) = \Upsilon(t) - \lambda_1 \int_0^t \operatorname{sgn}\bigl(\Upsilon(s)\bigr)\,\mathrm{d}s, \qquad V(t) = \Xi(t) - \lambda_2 t - 2\kappa L^{|\Upsilon|}(t), \quad 0 \le t < \infty$$

are now Brownian motions with diffusion coëfficients $\sqrt{\sigma_1^2 + \sigma_2^2}$ and with covariation $\langle W, V\rangle(\cdot) = (\sigma_1^2 - \sigma_2^2) \times \int_0^{\cdot} \operatorname{sgn}(\Upsilon(t))\,\mathrm{d}t$.

Let us also remark that the solution to the above two-dimensional system can be realized as the solution of the system of equations (1.2) and (1.3) in [11] with

$$\kappa = q_1^- - \frac{1}{2} = \frac{1}{2} - q_2^+ = \frac{1-\eta_1}{2} = \frac{1-\eta_2}{2} = \frac{1-\zeta_1}{2} = \frac{1-\zeta_2}{2}, \tag{1.11}$$

and with $\eta_1, \eta_2, \zeta_1, \zeta_2$ the parameters introduced in [11]. In particular, the condition (1.5) in [11], which is necessary and sufficient for the well-posedness of the system of equations (1.2) and (1.3) in [11], is here fulfilled.

• Now, suppose that we have, formally at least,

$$b_1 = \cdots = b_n = 0, \qquad \sigma_1 = \cdots = \sigma_n = 1 \quad \text{and} \quad q_1^- = \cdots = q_n^- = 0, \qquad q_1^+ = \cdots = q_n^+ = 1.$$

In this case one can think of the particles as having masses that decrease from right to left, so that the mass of each particle is negligible compared to the mass of its right neighbor. To wit, whenever a particle collides with its right neighbor, it is reflected off this considerably "heavier" particle. In this situation, the vector of ranked processes $R(\cdot)$ is given by the continuous version of the *Totally Asymmetric Simple Exclusion Process* (TASEP). We refer the reader to Section 4 in [34] for some of the properties of this process, and to [14] for its appearance as the scaling limit of TASEP.

• Let us also note that the special case

$$q_k^\pm = \frac{1}{2}, \quad k = 1, 2, \ldots, n, \tag{1.12}$$

in which all local times disappear from (1.4), (1.10) and get equal weights in (1.1), and *all collisions of ranked particles are symmetric*, was studied in detail by Ichiba, Karatzas and Shkolnikov in [22]; in this case, *individual* particles collide with each other without feeling any local time drag from their nearest neighbors.

1.4. *Outline*

The rest of the paper is organized as follows. In Section 2.1 we show the strong existence and uniqueness of the solution to the system of equations (1.1) for the "ranks." Subsequently, in Section 2.2 we prove the appropriate generalization of the boundary property of reflected Brownian motion established in [30] to processes involving local times accumulated on lower-dimensional faces of the boundary. The attainability of lower-dimensional parts of the boundary is explored in Section 2.3.

These results are combined in Section 2.4 to prove the strong existence and uniqueness of the solution to the system of equations (1.4) for the "names," under the explicit conditions on nonattainability of lower-dimensional boundaries derived in Section 2.3. Then, in Section 2.5 we view the solution to the system of equations (1.4) as a model for the logarithmic capitalizations in large equity markets, and analyze the resulting capital distributions.

Section 3 identifies the processes in (1.1) as universal scaling limits of systems of one-dimensional continuous time jump processes with local interactions, which include many well-known interacting particle systems such as the asymmetric exclusion process and various exclusion processes with speed change. Finally, in Section 4 we characterize the sets of collision parameters for which the solution to (1.1) has a probabilistic structure of determinantal type. These results generalize those of [34], on Brownian particle systems with totally asymmetric collisions.

1.5. *Comparison with existing literature*

Some of our results are significant extensions of existing results in a well-explored field. For the benefit of the reader the following list compares the former to the latter.

(i) The result in Section 2.2 is a generalization of Theorem 7.7 in [5] with a shorter proof. Our result allows for arbitrary constant drift vectors, diffusion matrices, and directions of reflection. The simplicity of our argument is mainly due to a very natural assumption (2.8) which we believe to be a tool of independent interest. More significantly, while the results in [5] are restricted to having zero local time on lower-dimensional parts of the boundary, we also analyze the attainability of the latter. In the context of [5] it is not obvious whether the reflected Brownian motion can hit the lower-dimensional parts of the boundary at positive times, whereas this is certainly possible in our more general context.

(ii) The existence or absence of triple collisions is a recurrent theme in this paper. The results of Section 2.3 greatly extend those of [4,22] and [32] (within our setup, the first two only address the case (1.12), whereas the last one only addresses the case of rank-independent parameters).

(iii) Part of Section 3 is devoted to an extension of the invariance principle of [36] (see Proposition 9 and its proof). The class of discrete processes for which we derive the scaling limits admits the possibility of triple and higher order collisions. These present the main difficulty in our analysis. Some results in a similar direction can be found in [24], which allows for generic domains with piecewise smooth boundaries. But, in the case of the orthant, the results in [24] assume very special reflection directions on lower-dimensional parts of the boundary and, in particular, do not allow for the kind of reflections resulting from triple collisions of particles in our setup.

2. Analysis of the continuous process

2.1. *Ranks*

We start with the construction of the vector $R(\cdot) = (R_1(\cdot), \dots, R_n(\cdot))$ of ranked semimartingales as in (1.1). We note first that, due to the positivity of the coëfficients $q_1^{\pm}, q_2^{\pm}, \dots, q_n^{\pm}$, there exist positive constants $c_1, c_2, \dots, c_n$ such that the process $\sum_{i=1}^n c_i R_i(t)$, $t \ge 0$ is a Brownian motion with drift, that is, the contribution of the local times to its dynamics vanishes. This observation allows us to construct the ranked processes $R_1(\cdot), R_2(\cdot), \dots, R_n(\cdot)$ using the following procedure: first, we define the auxiliary Brownian motion $\widetilde{R}(\cdot) = (\widetilde{R}_1(\cdot), \widetilde{R}_2(\cdot), \dots, \widetilde{R}_n(\cdot))$, for which

$$\mathrm{d}\left(\sum_{k=1}^n c_k \widetilde{R}_k(t)\right) = \left(\sum_{k=1}^n c_k b_k\right) \mathrm{d}t + \sum_{k=1}^n c_k \sigma_k \, \mathrm{d}\boldsymbol{\beta}_k(t)$$

and

$$\mathrm{d}\big(\widetilde{R}_k(t) - \widetilde{R}_{k+1}(t)\big) = (b_k - b_{k+1})\, \mathrm{d}t + \sigma_k \, \mathrm{d}\boldsymbol{\beta}_k(t) - \sigma_{k+1} \, \mathrm{d}\boldsymbol{\beta}_{k+1}(t)$$

for $k = 1, 2, \dots, n-1$. Next, we introduce the process

$$Y(\cdot) := \left(\sum_{k=1}^n c_k \widetilde{R}_k(\cdot), \widetilde{R}_1(\cdot) - \widetilde{R}_2(\cdot), \dots, \widetilde{R}_{n-1}(\cdot) - \widetilde{R}_n(\cdot)\right)$$

and apply the Harrison–Reiman [16] version of the Skorokhod reflection map Ψ^{HR} for the orthant $(\mathbb{R}_+)^{n-1}$ to the last $(n-1)$ components of $Y(\cdot)$, using the reflection matrix

$$\mathcal{R} = \mathbf{I}_{n-1} - \mathcal{Q}, \quad \text{where} \quad \mathcal{Q} := \begin{pmatrix} 0 & q_2^- & 0 & 0 \\ q_2^+ & 0 & q_3^- & 0 \\ 0 & \ddots & \ddots & \ddots \\ 0 & 0 & q_{n-1}^+ & 0 \end{pmatrix} \tag{2.1}$$

and $\mathbf{I}_{n-1}$ is the unit $(n-1) \times (n-1)$ matrix.

The main observation here, is that *the spectral radius of the matrix $\mathcal{Q}$ is strictly less than* 1. Indeed, the transpose $\mathcal{Q}'$ is an irreducible substochastic matrix, and can be made into a stochastic matrix by adding an absorbing point (here and throughout this paper, the superscript $'$ denotes matrix transposition). Hence, by virtue of the Perron–Frobenius Theorem, the spectral radius of $\mathcal{Q}'$, and hence also of $\mathcal{Q}$, is strictly less than 1.

All in all, we see that Theorem 1 of [16] is applicable here. We can now complete the definition of the process $R(\cdot) = (R_1(\cdot), \dots, R_n(\cdot))$ by imposing

$$\sum_{k=1}^{n} c_k R_k(\cdot) = \sum_{k=1}^{n} c_k \widetilde{R}_k(\cdot),$$
$$\big(R_1(\cdot) - R_2(\cdot), \dots, R_{n-1}(\cdot) - R_n(\cdot)\big) = \Psi^{\mathrm{HR}}\big(\widetilde{R}_1(\cdot) - \widetilde{R}_2(\cdot), \dots, \widetilde{R}_{n-1}(\cdot) - \widetilde{R}_n(\cdot)\big).$$

Next, let us observe that the process of spacings

$$Z(\cdot) := \big(R_1(\cdot) - R_2(\cdot), R_2(\cdot) - R_3(\cdot), \dots, R_{n-1}(\cdot) - R_n(\cdot)\big) \tag{2.2}$$

for the process $R(\cdot)$ we just constructed in the manner of (1.1), is a reflected Brownian motion (RBM) in the nonnegative orthant $(\mathbb{R}_+)^{n-1}$, with drift vector $(b_1 - b_2, \dots, b_{n-1} - b_n)$, covariance matrix

$$\mathcal{A} = \begin{pmatrix} \sigma_1^2 + \sigma_2^2 & -\sigma_2^2 & 0 & 0 \\ -\sigma_2^2 & \sigma_2^2 + \sigma_3^2 & -\sigma_3^2 & 0 \\ 0 & \ddots & \ddots & \ddots \\ 0 & 0 & -\sigma_{n-1}^2 & \sigma_{n-1}^2 + \sigma_n^2 \end{pmatrix}, \tag{2.3}$$

and reflection matrix $\mathcal{R} = \mathbf{I}_{n-1} - \mathcal{Q}$ given as in (2.1). In particular, the components $Z_k(\cdot) := R_k(\cdot) - R_{k+1}(\cdot)$, $k = 1, \dots, n-1$ of this vector process are continuous, nonnegative semimartingales

$$\begin{aligned} Z_k(t) &= Z_k(0) + (b_k - b_{k+1})t + \sigma_k \boldsymbol{\beta}_k(t) - \sigma_{k+1}\boldsymbol{\beta}_{k+1}(t) \\ &\quad - q_k^+ \Lambda^{(k-1,k)}(t) - q_{k+1}^- \Lambda^{(k+1,k+2)}(t) + \Lambda^{(k,k+1)}(t), \quad 0 \le t < \infty \end{aligned}$$

with quadratic variations $\langle Z_k\rangle(t) = (\sigma_k^2 + \sigma_{k+1}^2)t$, thus also

$$\big(\sigma_k^2 + \sigma_{k+1}^2\big) \int_0^\infty \mathbf{1}_{\{Z_k(t)=0\}}\,\mathrm{d}t = \int_0^\infty \mathbf{1}_{\{Z_k(t)=0\}}\,\mathrm{d}\langle Z_k\rangle(t) = 0, \quad \text{a.s.} \tag{2.4}$$

(cf. Exercise 3.7.10, p. 225 in Karatzas and Shreve [25]). The continuous, adapted and nondecreasing processes $\Lambda^{(k,k+1)}(\cdot)$ satisfy the "flat off" condition

$$\int_0^\infty \mathbf{1}_{\{Z_k(t)>0\}}\,\mathrm{d}\Lambda^{(k,k+1)}(t) = 0, \quad \text{a.s.,} \tag{2.5}$$

and we set $\Lambda^{(0,1)}(\cdot) \equiv \Lambda^{(n,n+1)}(\cdot) \equiv 0$. Now the local time accumulated at the origin by this nonnegative semimartingale is

$$\begin{aligned} L^{R_k - R_{k+1}}(\cdot) = L^{Z_k}(\cdot) &= \int_0^\cdot \mathbf{1}_{\{Z_k(t)=0\}}\,\mathrm{d}Z_k(t) \\ &= \int_0^\cdot \mathbf{1}_{\{Z_k(t)=0\}}\big[(b_k - b_{k+1})\,\mathrm{d}t + \mathrm{d}\Lambda^{(k,k+1)}(t) - q_k^+\,\mathrm{d}\Lambda^{(k-1,k)}(t) - q_{k+1}^-\,\mathrm{d}\Lambda^{(k+1,k+2)}(t)\big] \end{aligned}$$

(Karatzas and Shreve [25], p. 223). In this last expression, the first (Lebesgue) integral vanishes because of (2.4), whereas the second (Stieltjes) integral is equal to $\Lambda^{(k,k+1)}(\cdot)$ on account of (2.5). As for the third and fourth (Stieltjes) integrals, they also vanish, because

$$\int_0^\cdot \mathbf{1}_{\{Z_k(t)=0\}}\,\mathrm{d}\Lambda^{(k-1,k)}(t) = \int_0^\cdot \mathbf{1}_{\{Z_k(t)=0=Z_{k-1}(t)\}}\,\mathrm{d}\Lambda^{(k-1,k)}(t) \equiv 0$$

and

$$\int_0^{\cdot} \mathbf{1}_{\{Z_k(t)=0\}}\,\mathrm{d}\Lambda^{(k+1,k+2)}(t) = \int_0^{\cdot} \mathbf{1}_{\{Z_k(t)=0=Z_{k+1}(t)\}}\,\mathrm{d}\Lambda^{(k+1,k+2)}(t) \equiv 0$$

hold on account of (2.5) and of Theorem 1, in Reiman and Williams [30]. We conclude from all this $L^{R_k-R_{k+1}}(\cdot) \equiv \Lambda^{(k,k+1)}(\cdot)$, to wit, the identification of (1.3).

Finally, the strong uniqueness for the system of equations (1.1), (1.3) follows from the uniqueness of the solution to the multi-dimensional Skorokhod reflection problem in Harrison and Reiman [16], Reiman [29]; see Theorem 1 of [16].

2.2. *A boundary property of reflected Brownian motion*

Throughout this paper we shall rely frequently on a generalization of a boundary property of reflected Brownian motion which was established in [30] and is of interest in its own right.

Consider a continuous semimartingale $Q(\cdot)$ taking values in the orthant $(\mathbb{R}_+)^{n-1}$ and satisfying

$$Q(\cdot) = B(\cdot) + \sum_{k=1}^{n-1} \mathfrak{R}^{(k)}\mathfrak{Y}^{(k)}(\cdot), \tag{2.6}$$

where $B(\cdot)$ is an $(n-1)$-dimensional Brownian motion with a constant drift vector and a constant, nondegenerate diffusion matrix. Here, for each $k \in \{1, 2, \dots, n-1\}$ and with

$$m = \binom{n-1}{k},$$

$\mathfrak{R}^{(k)}$ is an $(n-1) \times m$ matrix with real entries; whereas $\mathfrak{Y}^{(k)}(\cdot)$ is a continuous $(\mathbb{R}_+)^m$-valued process, whose components are indexed by the sets $J \subset \{1, 2, \dots, n-1\}$ with k elements, start at $\mathfrak{Y}^{(k)}(0) = 0$, are nondecreasing and satisfy

$$\int_0^{\infty} \sum_{j\in J} \mathbf{1}_{\{Q_j(t)>0\}}\,\mathrm{d}\mathfrak{Y}_J^{(k)}(t) = 0, \quad J \subset \{1, 2, \dots, n-1\}, \quad |J| = k, \tag{2.7}$$

$$\forall 0 \le s < t: \quad \mathfrak{Y}_{J_2}^{(|J_2|)}(t) - \mathfrak{Y}_{J_2}^{(|J_2|)}(s) \le \mathfrak{Y}_{J_1}^{(|J_1|)}(t) - \mathfrak{Y}_{J_1}^{(|J_1|)}(s), \quad \text{for any} \quad J_1 \subset J_2 \subset \{1, 2, \dots, n-1\}. \tag{2.8}$$

We recall from [30] the following notion: the matrix $\mathfrak{R}^{(1)}$ is called *completely-$\mathcal{S}$*, if there exists a vector $\lambda \in [0, \infty)^{n-1}$ such that $\mathfrak{R}^{(1)}\lambda \in [1, \infty)^{n-1}$, and the same property holds for every principal submatrix of $\mathfrak{R}^{(1)}$. It is well-known that matrices of Harrison–Reiman type [16], in particular the matrix $\mathfrak{R}^{(1)}$ of (2.1), are completely-$\mathcal{S}$ (see e.g. the discussion on p. 88 of [30]). In the following lemma we show that the completely-$\mathcal{S}$ property allows to identify the semimartingale $Q(\cdot)$ of (2.4) with a reflected Brownian motion in the sense of [17,35].

Lemma 1. *Let $Q(\cdot)$ be a process as in (2.6), and suppose that the matrix $\mathfrak{R}^{(1)}$ is completely-$\mathcal{S}$. Then all processes $\mathfrak{Y}^{(k)}(\cdot)$, $k = 2, 3, \dots, n-1$ are identically zero.*

In particular, $Q(\cdot)$ is then a reflected Brownian motion in the orthant $(\mathbb{R}_+)^{n-1}$ with reflection matrix $\mathfrak{R}^{(1)}$.

Proof. The proof is similar to that of Reiman and Williams in Theorem 1 of [30]. In view of Girsanov's Theorem, it suffices to consider the case that the drift vector of $B(\cdot)$ is equal to zero. We consider the functions $\phi_\varepsilon(\cdot)$, $\varepsilon \in (0, 1)$ defined in the proof of Lemma 4 in [30]:

$$\phi_\varepsilon(x) = \begin{cases} \frac{1}{3-n}\int_\varepsilon^1 r^{n-3}((x+r\alpha)'\mathcal{A}^{-1}(x+r\alpha))^{(3-n)/2}\,\mathrm{d}r, & \text{if } n \ge 4, \\ \frac{1}{2}\int_\varepsilon^1 \log((x+r\alpha)'\mathcal{A}^{-1}(x+r\alpha))\,\mathrm{d}r, & \text{if } n = 3. \end{cases} \tag{2.9}$$

Here $\mathcal{A}$ is the diffusion matrix of (2.3), $\alpha = \mathcal{A}\lambda$, and $\lambda \in [0,\infty)^{n-1}$ is as in the definition of the completely-$\mathcal{S}$ property for $(\mathfrak{R}^{(1)})'$ (note that the completely-$\mathcal{S}$ property is preserved under transposition, see Lemma 3 in [30]). The functions $\phi_\varepsilon(\cdot)$, $\varepsilon \in (0,1)$ are harmonic for the generator of $B(\cdot)$ and bounded on compact subsets of $(\mathbb{R}_+)^{n-1}$, uniformly over $\varepsilon \in (0,1)$. We claim that, for any $k = 1,2,\dots,n$ and any column v of the matrix $\mathfrak{R}^{(k)}$, there is a constant $C_k < \infty$ depending only on the matrices $\mathfrak{R}$, $\mathfrak{R}^{(k)}$ and the diffusion matrix of $B(\cdot)$, such that

$$\forall \varepsilon > 0, x \in (\mathbb{R}_+)^{n-1}: \quad v \cdot \nabla\phi_\varepsilon(x) \ge -C_k. \tag{2.10}$$

Indeed, one can argue exactly as on p. 94 of [30] in the derivation of the bound (24) there. Now, define the stopping times

$$\tau_m = \min\left(m, \inf\left\{t \ge 0\colon\ \|Q(t)\| + \sum_{k=1}^{n-1}\|\mathfrak{Y}^{(k)}(t)\| \ge m\right\}\right), \quad m \in \mathbb{N},$$

where we wrote $\|\cdot\|$ for the Euclidean norm. Applying Itô's formula to the semimartingale $Q(\cdot)$, and recalling that the functions $\phi_\varepsilon(\cdot)$ are harmonic with respect to the generator of $B(\cdot)$, one obtains for all $\varepsilon \in (0,1)$ and $m \in \mathbb{N}$ the identity

$$\phi_\varepsilon\bigl(Q(\tau_m)\bigr) - \phi_\varepsilon\bigl(Q(0)\bigr) = \int_0^{\tau_m} \nabla\phi_\varepsilon\bigl(Q(t)\bigr)\cdot \mathrm{d}B(t) + \sum_{k=1}^{n-1}\sum_{J\in\mathfrak{J}_k}\int_0^{\tau_m} v_J \cdot \nabla\phi_\varepsilon\bigl(Q(t)\bigr)\,\mathrm{d}\mathfrak{Y}_J^{(k)}(t).$$

Here $\mathfrak{J}_k$ stands for the set of all subsets of $\{1,2,\dots,n-1\}$ with k elements, and v_J denotes the Jth column of $\mathfrak{R}^{(|J|)}$. Finally, taking the expectation on both sides and using the bound (18) in [30] and the bound (2.10) above, one ends up with

$$\begin{aligned}
&\mathbb{E}\bigl[\phi_\varepsilon\bigl(Q(\tau_m)\bigr)\bigr] - \mathbb{E}\bigl[\phi_\varepsilon\bigl(Q(0)\bigr)\bigr] \\
&\quad \ge -(\log\varepsilon + 1)\sum_{j=1}^{n-1} c_j \mathbb{E}\left[\int_0^{\tau_m} \mathbf{1}_{\{\|Q(t)\|<\varepsilon\beta_j\}}\,\mathrm{d}\mathfrak{Y}_j^{(1)}(t)\right] - \sum_{k=1}^{n-1} C_k \sum_{J\in\mathfrak{J}_k}\mathbb{E}\bigl[\mathfrak{Y}_J^{(k)}(\tau_m)\bigr],
\end{aligned}$$

where the positive constants c_j, β_j, $j = 1,2,\dots,n-1$ are defined as in [30] and the constants C_k, $k \in \{1,2,\dots,n-1\}$ are as in (2.10). Dividing both sides of the latter inequality by $(\log\varepsilon + 1)$ and taking the limit $\varepsilon \downarrow 0$ gives

$$\lim_{\varepsilon\downarrow 0}\sum_{j=1}^{n-1} c_j \mathbb{E}\left[\int_0^{\tau_m} \mathbf{1}_{\{\|Q(t)\|<\varepsilon\beta_j\}}\,\mathrm{d}\mathfrak{Y}_j^{(1)}(t)\right] \le 0.$$

Thus, by Fatou's Lemma and the nonnegativity of the integrand, one can conclude

$$\int_0^{\tau_m} \mathbf{1}_{\{Q(t)=0\}}\,\mathrm{d}\mathfrak{Y}_j^{(1)}(t) = 0, \quad j = 1,2,\dots,n-1,$$

with probability one for every $m \in \mathbb{N}$, and therefore also

$$\int_0^{\infty} \mathbf{1}_{\{Q(t)=0\}}\,\mathrm{d}\mathfrak{Y}_j^{(1)}(t) = 0, \quad j = 1,2,\dots,n-1, \tag{2.11}$$

with probability one. Finally, the backward induction argument of Lemma 5 in [30] allows us to strengthen this statement to

$$\forall j_0 \in J \subset \{1,2,\dots,n-1\}: \qquad \int_0^{\infty} \mathbf{1}_{\{Q_j(t)=0,\, j\in J\}}\,\mathrm{d}\mathfrak{Y}_{j_0}^{(1)}(t) = 0, \tag{2.12}$$

almost surely. In view of (2.7) and (2.8), this finishes the proof. □

2.3. *Absence of triple collisions*

This subsection, and the one that follows, are devoted to the construction of the solution to the system of stochastic equations (1.4), when one of the following two conditions holds:

(A) $(1-q_k^-)\sigma_k^2 \geq q_k^- \sigma_{k+1}^2, \qquad \big(1-q_k^+\big)\sigma_k^2 \geq q_k^+ \sigma_{k-1}^2, \quad k=2,3,\dots,n-1.$

(B) $q_k^- = q_k^+ = \left(1+\dfrac{\sigma_{k-1}^2+\sigma_{k+1}^2}{2\sigma_k^2}\right)^{-1}, \quad k=2,3,\dots,n-1.$

As we show below, each of these conditions prevents collisions of three or more particles. It is not hard to see that neither of these two conditions implies the other.

When all the collision parameters are equal to $1/2$, as in (1.12), condition (B) mandates that the graph of the variances-by-rank $k \mapsto \sigma_k^2$ be linear; thus, under condition (B), Proposition 3 below generalizes the results in [21] on the absence of triple collisions to situations where particles feel local-time-like drag from their immediate neighbors, when they collide with each other.

We start with a result ruling out triple collisions in the case $n=3$ under the condition (2.13) below; in this three-dimensional case, condition (2.13) is weaker than each of the conditions (A) and (B).

Proposition 2. *Suppose that $n=3$ and $R_1(0)-R_3(0)>0$.*

(i) *If the condition*

$$2\sigma_2^2 \geq q_2^-\big(\sigma_2^2+\sigma_3^2\big)+q_2^+\big(\sigma_1^2+\sigma_2^2\big) \tag{2.13}$$

holds, then we have

$$\mathbb{P}\big(\exists t\geq 0:\ R_1(t)=R_2(t)=R_3(t)\big)=0. \tag{2.14}$$

(ii) *Conversely, if* (2.14) *holds, then* (2.13) *holds as well.*

Proof. We consider the reflected Brownian motion $(R_1(\cdot)-R_2(\cdot), R_2(\cdot)-R_3(\cdot))$ in the nonnegative quadrant, with the reflection matrix $\mathcal{R}=\mathbf{I}_2-\mathcal{Q}$ of (2.1) and the covariance matrix $\mathcal{A}$ of (2.3) with $n=3$. Moreover, let $\mathcal{O}$ be the 2×2 orthogonal matrix such that $\mathcal{L}:=\mathcal{O}'\mathcal{A}\mathcal{O}$ is a diagonal matrix. Then the process

$$\mathcal{L}^{-1}\mathcal{O}\big(R_1(\cdot)-R_2(\cdot), R_2(\cdot)-R_3(\cdot)\big)' \tag{2.15}$$

is a reflected Brownian motion in a wedge, in the sense of Varadhan and Williams in [33]. Letting

$$\mathcal{D}:=\operatorname{diag}(\mathcal{A}) \tag{2.16}$$

be the diagonal matrix whose diagonal entries coincide with those of $\mathcal{A}$, and following the computations in Section 3.2.1 of [21], one concludes that the normal vectors to the two sides of the wedge are given by the columns of the matrix

$$\overline{\mathcal{N}}:=\mathcal{L}^{1/2}\mathcal{O}\mathcal{D}^{-1/2}; \tag{2.17}$$

whereas the reflection matrix of the new reflected Brownian motion in the sense of [33] is given by

$$\overline{\mathcal{Q}}:=\mathcal{L}^{-1/2}\mathcal{O}\mathcal{R}\mathcal{D}^{1/2}-\overline{\mathcal{N}}.$$

Furthermore, as observed in the proof of Lemma 3.2 in [21], *the corner of the wedge is attainable by the new reflected Brownian motion, if and only if the sum of the two off-diagonal entries of the matrix $\overline{\mathcal{N}}'\overline{\mathcal{Q}}$ is nonnegative*. Next, we note that

$$\overline{\mathcal{N}}'\overline{\mathcal{Q}}=\mathcal{D}^{-1/2}\mathcal{R}\mathcal{D}^{1/2}-\overline{\mathcal{N}}'\overline{\mathcal{N}}. \tag{2.18}$$

Moreover, both off-diagonal entries of $\overline{\mathcal{N}}'\overline{\mathcal{N}}$ are given by the negative cosine of the angle between the two sides of the wedge, which can be computed to be

$$-\frac{(\mathcal{A}^{-1}\mathfrak{e}_1)'\mathfrak{e}_2}{((\mathcal{A}^{-1}\mathfrak{e}_1)'\mathfrak{e}_1)^{1/2}((\mathcal{A}^{-1}\mathfrak{e}_2)'\mathfrak{e}_2)^{1/2}},$$

where $\mathfrak{e}_1, \mathfrak{e}_2$ is the canonical basis of $\mathbb{R}^2$. Putting everything together, we can compute the sum of the two off-diagonal entries of the matrix $\overline{\mathcal{N}}'\overline{\mathcal{Q}}$ as

$$-q_2^- \frac{\sqrt{\sigma_2^2+\sigma_3^2}}{\sqrt{\sigma_1^2+\sigma_2^2}} - q_2^+ \frac{\sqrt{\sigma_1^2+\sigma_2^2}}{\sqrt{\sigma_2^2+\sigma_3^2}} + \frac{2\sigma_2^2}{\sqrt{(\sigma_1^2+\sigma_2^2)(\sigma_2^2+\sigma_3^2)}}.$$

Simplifying this expression, we obtain the condition (2.13) of the proposition. □

Now, we turn our attention to the case of general $n \geq 3$.

Proposition 3. *Suppose that $n \geq 3$, and that either condition* (A) *or condition* (B) *holds. Then no triple collisions are possible, that is,*

$$\mathbb{P}\bigl(\exists t \geq 0, 1 \leq i < j < k \leq n\colon\ R_i(t) = R_j(t) = R_k(t)\bigr) = 0. \tag{2.19}$$

The proof relies on an inductive argument and the following lemma.

Lemma 4. *Let $n \geq 3$. Then, condition* (A) *is equivalent to the following condition:*

$$\forall 1 \leq j, k \leq n-1, j \neq k: \quad \bigl[\mathcal{A}^{-1}(\mathbf{I}_{n-1} - \mathcal{Q})\bigr]_{jk} \geq 0. \tag{2.20}$$

Proof. We start by recalling that the entries of $\mathcal{A}^{-1}$ can be computed by the formula

$$\mathcal{A}^{-1}_{jk} = (-1)^{j+k}\frac{\det(\mathcal{A}^{j,k})}{\det(\mathcal{A})}, \quad 1 \leq j, k \leq n-1, \tag{2.21}$$

where $\mathcal{A}^{j,k}$ is the $(n-2) \times (n-2)$ submatrix of the symmetric matrix $\mathcal{A}$ in (2.3), that one obtains by removing the jth row and the kth column from $\mathcal{A}$. Next, we introduce the notation

$$\Pi_{k_1}^{k_2} := \left(\prod_{k=k_1}^{k_2} \sigma_k^2\right)\left(\sum_{k=k_1}^{k_2} \frac{1}{\sigma_k^2}\right), \quad 1 \leq k_1 \leq k_2 \leq n, \tag{2.22}$$

and claim that the determinants $\det(\mathcal{A}^{j,k})$, $1 \leq j \leq k \leq n-1$ are given by

$$(-1)^{j+k}\det\bigl(\mathcal{A}^{j,k}\bigr) = \sigma_1^2\sigma_2^2\cdots\sigma_n^2\left(\sum_{\ell=1}^{j}\frac{1}{\sigma_\ell^2}\right)\left(\sum_{\ell=k+1}^{n}\frac{1}{\sigma_\ell^2}\right) = \Pi_1^j\left(\prod_{\ell=j+1}^{k}\sigma_\ell^2\right)\Pi_{k+1}^n. \tag{2.23}$$

This claim can be verified easily, using induction over n and distinguishing the cases $k \geq j+2$, $k = j+1$ and $k = j$. Therefore, for any fixed $1 \leq j < k \leq n-1$, the inequality

$$\bigl[\mathcal{A}^{-1}(\mathbf{I}_{n-1} - \mathcal{Q})\bigr]_{jk} \geq 0 \tag{2.24}$$

of condition (2.20) can be rewritten in the equivalent form

$$\Pi_1^j\left(\prod_{\ell=j+1}^{k}\sigma_\ell^2\right)\Pi_{k+1}^n \geq q_k^-\Pi_1^j\left(\prod_{\ell=j+1}^{k-1}\sigma_\ell^2\right)\Pi_k^n + \bigl(1-q_k^-\bigr)\Pi_1^j\left(\prod_{\ell=j+1}^{k+1}\sigma_\ell^2\right)\Pi_{k+2}^n. \tag{2.25}$$

In view of the strict positivity of the variances, the inequality (2.25) simplifies to

$$\sigma_k^2 \Pi_{k+1}^n \geq q_k^- \Pi_k^n + \left(1-q_k^-\right)\sigma_k^2\sigma_{k+1}^2\Pi_{k+2}^n. \tag{2.26}$$

Now, we note the following relations among Π_k^n, Π_{k+1}^n and Π_{k+2}^n:

$$\Pi_{k+1}^n = \Pi_{k+2}^n\sigma_{k+1}^2 + \sigma_{k+2}^2\sigma_{k+3}^2\cdots\sigma_n^2, \tag{2.27}$$

$$\Pi_k^n = \Pi_{k+2}^n\sigma_k^2\sigma_{k+1}^2 + \left(\sigma_{k+1}^2+\sigma_k^2\right)\sigma_{k+2}^2\sigma_{k+3}^2\cdots\sigma_n^2. \tag{2.28}$$

Plugging these into (2.26) and simplifying further, we end up with

$$\left(1-q_k^-\right)\sigma_k^2 \geq q_k^-\sigma_{k+1}^2. \tag{2.29}$$

An analogous computation, now with $1 \leq k < j \leq n-1$, shows that the inequality (2.24) in this case is equivalent to

$$\left(1-q_j^+\right)\sigma_j^2 \geq q_j^+\sigma_{j-1}^2. \tag{2.30}$$

We conclude that condition (2.20) is equivalent to condition (A). □

Proof of Proposition 3. We note first that, by the same application of Girsanov's Theorem as in Section 2.2 of [21], or as in the proof of Lemmata 6 and 7 in [22], we need only consider the case $b_1 = b_2 = \cdots = b_n = 0$ in (1.1).

• We start by assuming that condition (A) is satisfied, and proceed by induction over $n \geq 3$. For $n = 3$, very simple computation shows that condition (A) implies condition (2.13). The statement of the proposition for $n = 3$ follows then directly from Proposition 2.

Next, we assume that $n \geq 4$, and that the statement of the proposition holds under condition (A) for all $\nu = 3, 4, \ldots, n-1$ (the "induction hypothesis"). Introducing the stopping times

$$\tau_\delta = \inf\Big\{t \geq 0\colon \max_{k=1,2,\ldots,n-1}\big(R_k(t) - R_{k+1}(t)\big) \leq \delta\Big\}, \quad \delta > 0, \tag{2.31}$$

we claim that

$$\mathbb{P}\big(\exists 0 \leq t < \tau_\delta, 1 \leq i < j < k \leq n\colon R_i(t) = R_j(t) = R_k(t)\big) = 0, \quad \delta > 0. \tag{2.32}$$

Indeed, for any fixed $\delta > 0$, the time interval $[0, \tau_\delta)$ can be written as

$$[0, \tau_\delta) = \bigcup_{m=0}^{M}\left[\tau_\delta^m, \tau_\delta^{m+1}\right) \tag{2.33}$$

for some $M \in \mathbb{N} \cup \{\infty\}$ and with stopping times $0 = \tau_\delta^0 < \tau_\delta^1 < \cdots$ satisfying

$$\exists k = k(m) \in \{1, 2, \ldots, n-1\}\colon \quad R_k(t) - R_{k+1}(t) \geq \delta, \quad t \in \left[\tau_\delta^m, \tau_\delta^{m+1}\right). \tag{2.34}$$

Moreover, on each of the time intervals $[\tau_\delta^m, \tau_\delta^{m+1})$, $0 \leq m < M$, the system (1.1) splits into two subsystems of the same type which evolve independently, conditional on $R_1(\tau_\delta^m), R_2(\tau_\delta^m), \ldots, R_n(\tau_\delta^m)$. Hence, (2.32) is a consequence of the induction hypothesis.

In view of (2.32), in order to show (2.19) and complete the induction argument, it suffices to show

$$\mathbb{P}\Big(\lim_{\delta\downarrow 0}\tau_\delta = \infty\Big) = 1. \tag{2.35}$$

To this end, we introduce the functions $F, G : (\mathbb{R}_+)^{n-1} \setminus \{0\} \to \mathbb{R}_+$ given by

$$F(z) := \big(G(z)\big)^{(3-n)/2}, \qquad G(z) := \big\langle \mathcal{A}^{-1}z, z\big\rangle. \tag{2.36}$$

Next, we apply the change of variable formula of Theorem 2 in [16] to the function F and the reflected Brownian motion with drift $Z(\cdot)$ formed by the spacings (see (2.2) and (1.8)), noting that the function F is harmonic with respect to the generator of the Brownian motion driving these spacings. We obtain the semimartingale decomposition

$$F\big(Z(\cdot)\big) = F\big(Z(0)\big) + M(\cdot) + V(\cdot),$$

where $M(\cdot)$ is a real-valued local martingale of the form $M(\cdot) = \sum_{k=1}^{n} \int_0^{\cdot} \xi_k(t)\,\mathrm{d}\boldsymbol{\beta}_k(t)$, and

$$\begin{aligned} V(\cdot) &= \sum_{\ell=1}^{n-1} \int_0^{\cdot} \big(\partial_{z_\ell} - q_\ell^+ \partial_{z_{\ell-1}} - q_\ell^- \partial_{z_{\ell+1}}\big) F\big(Z(t)\big)\,\mathrm{d}\Lambda^{(\ell,\ell+1)}(t) \\ &= \frac{3-n}{2} \int_0^{\cdot} \big(G\big(Z(t)\big)\big)^{(1-n)/2} \sum_{\ell,k=1}^{n-1} Z_k(t)\big(\mathcal{A}_{k\ell}^{-1} - q_\ell^+ \mathcal{A}_{k(\ell-1)}^{-1} - q_{\ell+1}^- \mathcal{A}_{k(\ell+1)}^{-1}\big)\,\mathrm{d}\Lambda^{(\ell,\ell+1)}(t), \end{aligned}$$

a process of finite variation on compact intervals. We also note that the processes $\xi_k(\cdot \wedge \tau_\delta)$ are all uniformly bounded, so the stopped local martingale $M(\cdot \wedge \tau_\delta)$ is in fact a martingale.

In view of Lemma 4, condition (A) ensures that, in this last expression for the finite variation process $V(\cdot)$, the coëfficients appearing in front of the local time processes are all nonpositive, so we conclude

$$\forall t \ge 0, \delta > 0: \quad \mathbb{E}\big[F\big(Z(t \wedge \tau_\delta)\big)\big] \le \mathbb{E}\big[F\big(Z(0)\big)\big]. \tag{2.37}$$

Moreover, if the limit $\lim_{\delta \downarrow 0} \tau_\delta$ were finite with positive probability, then for any given real number $c > 0$ there would exist real numbers $t > 0$ sufficiently large, and $\delta > 0$ sufficiently small, such that the value of the left-hand side in (2.37) would exceed c; but this would then contradict (2.37). We conclude that (2.35) holds, and thus the proposition is established under condition (A).

• Condition (B) simply paraphrases the skew-symmetry condition for the $(n-1)$-dimensional Brownian motion $Z(\cdot)$ with reflection on the faces of the nonnegative orthant in the sense of Harrison and Williams (cf. [17,18,35] as well as Section 2.5 below), and the result of Proposition 3 in this case can be found in Theorem 1.1(iii) of [35]. □

Remark 1. *In the course of the editorial review process for this paper, the result of Proposition* 3 *has been improved by A. Sarantsev in Theorem* 1.4 *of* [31]. *It is shown there that the requirement*

$$2\sigma_{k+1}^2 \ge q_{k+1}^-\big(\sigma_{k+1}^2 + \sigma_{k+2}^2\big) + q_{k+1}^+\big(\sigma_k^2 + \sigma_{k+1}^2\big), \quad k = 1, 2, \dots, n-2$$

(*that is*, *condition* (2.13) *applied to any three consecutive ranks*) *is both necessary and sufficient for the absence of triple collisions*.

2.4. *Names*

We can now combine the results of the previous two sections with those in [11], to construct a strong solution for the system of equations (1.4) subject to the condition (1.6), and to show that pathwise uniqueness holds for this system.

Theorem 5. *Suppose that either condition* (A) *or condition* (B) *is satisfied. Then the system* (1.4) *has a strong solution satisfying* (1.6), *and such a solution is pathwise unique.*

Proof. We start with the proof of strong existence, which proceeds by induction over n. As remarked in Section 1.3, for $n = 2$, the system of equations (1.4) is a special case of the system (4.13) and (4.14) in [11]. Therefore, we may deduce strong existence for $n = 2$ from Theorem 4.2 in [11]. Moreover, it is shown in that paper (see (6.13)–(6.15) in [11]) that the distributions of the random variables $X_1(t) - X_2(t)$, $t \ge 0$ have a density with respect to the Lebesgue measure on $\mathbb{R}$ for every $t \in (0, \infty)$, and therefore

$$\mathbb{E}\big[\mathcal{L}\mathrm{eb}\big(\big\{t \ge 0: X_1(t) = X_2(t)\big\}\big)\big] = 0 \tag{2.38}$$

by Fubini's theorem. Hence, $X_1(\cdot)$, $X_2(\cdot)$ satisfy (1.6).

We now consider $n \geq 3$, and assume that a strong solution to the system (1.4) satisfying (1.6) has already been constructed for all $\nu = 2, 3, \ldots, n-1$ and all choices of drift, dispersion and collision parameters obeying condition (A) or (B) (the "induction hypothesis"). We shall construct a strong solution $(X_1(\cdot), X_2(\cdot), \ldots, X_n(\cdot))$ of the system (1.4) consecutively on the random time intervals

$$\bigl[\eta^0_{2^{-p}}, \eta^1_{2^{-p}}\bigr), \bigl[\eta^1_{2^{-p}}, \eta^2_{2^{-p}}\bigr), \ldots, \bigl[\eta^{M(p)}_{2^{-p}}, \eta^{M(p)}_{2^{-p}}\bigr), \quad p \in \mathbb{N}, \tag{2.39}$$

where $\eta^m_{2^{-p}}$, $m = 0, 1, \ldots, M(p)$, $p \in \mathbb{N}$ are stopping times such that

$$\lim_{m \uparrow M(p)} \eta^m_{2^{-p}} = \inf\Bigl\{t \geq 0\text{: } \max_{k=1,\ldots,n-1} \bigl(R^X_k(t) - R^X_{k+1}(t)\bigr) \leq 2^{-p}\Bigr\} = \tau_{2^{-p}}, \quad p \in \mathbb{N}_0,$$

$$\forall p \in \mathbb{N}, m = 0, \ldots, M(p), \exists k = k(p,m)\text{:} \quad R^X_{k+1}(t) - R^X_k(t) \geq 2^{-p}, \quad t \in \bigl[\eta^m_{2^{-p}}, \eta^{m+1}_{2^{-p}}\bigr)$$

and $M(p) \in \mathbb{N} \cup \{\infty\}$, $p \in \mathbb{N}$. We have recalled here the notation of (2.31), and the fact that the process of ranks $(R^X_1(\cdot), R^X_2(\cdot), \ldots, R^X_n(\cdot))$ solves the system of equations (1.1) (recall the discussion in Section 1 for more details).

On each interval $[\eta^m_{2^{-p}}, \eta^{m+1}_{2^{-p}})$, we define $(X_1(\cdot), X_2(\cdot), \ldots, X_n(\cdot))$ by letting the processes $(X_1(\cdot), X_2(\cdot), \ldots, X_{k(p,m)}(\cdot))$ evolve as a strong solution of the $k(p,m)$-dimensional system corresponding to the first $k(p,m)$ equations in (1.4), started at the point $(X_1(\eta^m_{2^{-p}}), \ldots, X_{k(p,m)}(\eta^m_{2^{-p}}))$; and by letting $(X_{k(p,m)+1}(\cdot), \ldots, X_n(\cdot))$ evolve as a strong solution of the $(n - k(p,m))$-dimensional system corresponding to the last $(n - k(p,m))$ equations in (1.4), started at $(X_{k(p,m)+1}(\eta^m_{2^{-p}}), \ldots, X_n(\eta^m_{2^{-p}}))$. Note that the strong solutions to the lower-dimensional systems exist by the induction hypothesis. The resulting process $(X_1(\cdot), X_2(\cdot), \ldots, X_n(\cdot))$ is a strong solution to the system of equations (1.4), up to the random time

$$\lim_{p \uparrow \infty} \lim_{m \uparrow M(p)} \eta^m_{2^{-p}} = \lim_{p \uparrow \infty} \tau_{2^{-p}}. \tag{2.40}$$

We can conclude from Proposition 3 that the quantity in (2.40) must be equal to infinity with probability one. Thus, we have constructed a strong solution to (1.4) for all $t \in [0, \infty)$. Finally, it is clear that this solution satisfies (1.6) by the induction hypothesis and its construction.

To prove pathwise uniqueness, we argue again by induction over n. For $n = 2$, pathwise uniqueness is a consequence of Theorem 4.2 in [11]. Now, let $n \geq 3$ and assume that pathwise uniqueness holds for all $\nu = 2, 3, \ldots, n-1$ and all choices of drift, dispersion and collision parameters satisfying condition (A) or condition (B) (the new induction hypothesis). Next, suppose that $\widetilde{X}(\cdot) = (\widetilde{X}_1(\cdot), \widetilde{X}_2, \ldots, \widetilde{X}_n(\cdot))$ is another strong solution of (1.4) satisfying (1.6), defined on the same probability space as the strong solution $X(\cdot) = (X_1(\cdot), X_2(\cdot), \ldots, X_n(\cdot))$ constructed above. Considering the intervals in (2.39) consecutively and employing the induction hypothesis, we conclude that $\widetilde{X}(\cdot) = X(\cdot)$ must hold up to the time given by (2.40). However, as we have seen above, the latter must be infinite with probability one, by virtue of Proposition 3. This yields the desired pathwise uniqueness. □

Remark 2. *A careful reading of the proof of Theorem 5 shows that for any choice of the parameters the solution of the system (1.4), (1.6) is pathwise unique up to the first occurence of a triple collision. Whether or not pathwise uniqueness fails after such a collision is currently an open problem.*

2.5. *Skew-symmetry and invariant measures*

From Ichiba et al. (see equation (5.8) in [23]), we know that the reflected Brownian motion of spacings in (2.2) is skew-symmetric in the sense of Harrison and Williams (cf. [17,18]), if the following condition is satisfied

$$2(\mathcal{D} - \mathcal{A}) = \mathcal{Q}\mathcal{D} + \mathcal{D}\mathcal{Q} \tag{2.41}$$

with the notation $\mathcal{D} = \operatorname{diag}(\mathcal{A})$ of (2.16). Plugging in these equations the expressions for the matrices $\mathcal{A}$ and $\mathcal{Q}$ from (2.3) and (2.1), respectively, we can simplify this condition to

$$q^-_k = q^+_k = \left(1 + \frac{\sigma^2_{k-1} + \sigma^2_{k+1}}{2\sigma^2_k}\right)^{-1}, \quad k = 2, 3, \ldots, n-1, \tag{2.42}$$

that is, exactly the condition (B).

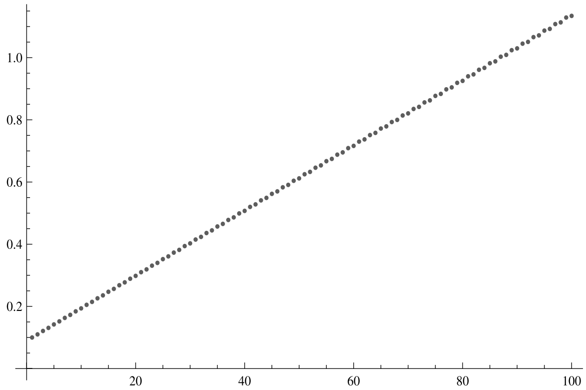

Fig. 1. Variances by rank.

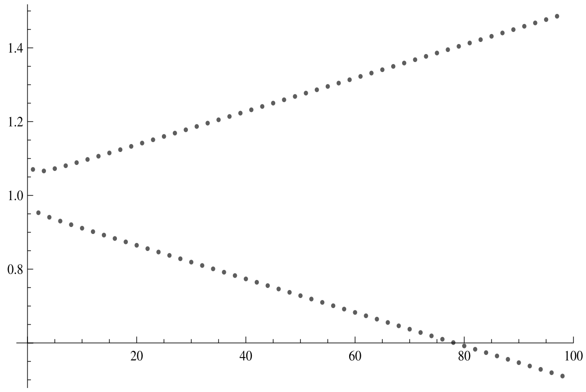

Fig. 2. Increments of log-variances with respect to log-rank.

In view of our assumption (1.2), the condition (2.42) amounts to the requirement

$$\frac{2\sigma_2^2}{\sigma_1^2+\sigma_3^2}=\frac{\sigma_2^2+\sigma_4^2}{2\sigma_3^2}=\frac{2\sigma_4^2}{\sigma_3^2+\sigma_5^2}=\cdots. \tag{2.43}$$

Figure 1 shows the variances σ_k^2, $k=1,2,\dots,n$, and Figure 2 the slopes

$$k\mapsto\frac{\log(\sigma_{k+1}^2-\sigma_1^2)-\log(\sigma_k^2-\sigma_1^2)}{\log k-\log(k-1)}$$

of the function $\log k\mapsto\log(\sigma_{k+1}^2-\sigma_1^2)$, for $n=100$ and a nonlinear choice of initial parameters σ_1^2, σ_2^2, σ_3^2, namely $\sigma_1^2=0.1$, $\sigma_2^2=0.11$, $\sigma_3^2=0.121$. One can see that these slopes can be made to deviate significantly from 1 even when one only slightly perturbs a linear specification (all the slopes would be equal to 1 in a skew-symmetric specification of a model with symmetric collisions: $q_k^\pm=1/2$, $k=1,2,\dots,n-1$).

Under the skew-symmetry condition (2.43), one can compute the invariant distribution of the spacings process (2.2) explicitly, using Lemma 3.6 in the dissertation of Ichiba [20]; one ends up with a product of exponential distributions with parameter vector

$$\gamma=2\big[\operatorname{diag}(\mathcal{A})\big]^{-1}\mathcal{R}^{-1}(b_2-b_1,b_3-b_2,\dots,b_n-b_{n-1})'. \tag{2.44}$$

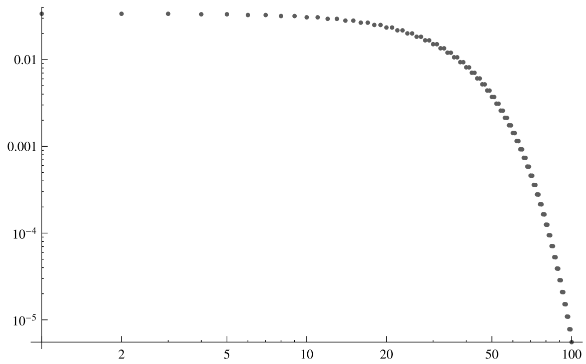

Fig. 3. Capital distribution curve.

Note that, by virtue of (1.2) and (2.42), the matrix $\mathcal{R}$ takes the form

$$\mathcal{R} = \mathbf{I}_{n-1} - \mathcal{Q} = \begin{pmatrix} 1 & -q & 0 & 0 & 0 \\ -q & 1 & -(1-q) & 0 & 0 \\ 0 & -(1-q) & 1 & -q & 0 \\ 0 & 0 & -q & 1 & \ddots \\ 0 & 0 & 0 & \ddots & \ddots \end{pmatrix}, \tag{2.45}$$

where

$$q := q_2^- = \left(1 + \frac{\sigma_1^2 + \sigma_3^2}{2\sigma_2^2}\right)^{-1}. \tag{2.46}$$

Remark 3. *An interesting special case is the specification*

$$b_1 = b_2 = \cdots = b_{n-1} = 0, b_n = \mathfrak{g} n \quad \textit{for some } \mathfrak{g} > 0. \tag{2.47}$$

Under this specification, the model of (1.4) *might be called a* q-Atlas model *by analogy with the term "Atlas model" introduced by Fernholz in* [10] *and studied further by Banner, Fernholz and Karatzas in* [2]. *Figure* 3 *shows on a log–log plot the capital distribution curve*

$$k \mapsto \frac{\mathrm{e}^{R_k(t)}}{\sum_{\ell=1}^n \mathrm{e}^{R_\ell(t)}}, \quad k = 1, 2, \ldots, 100 \tag{2.48}$$

for such a q-Atlas model with $n = 100$, $\mathfrak{g} = 1$, $\sigma_1^2 = 0.1$, $\sigma_2^2 = 0.11$, $\sigma_3^2 = 0.121$, *when the spacings process* $(R_1(\cdot) - R_2(\cdot), R_2(\cdot) - R_3(\cdot), \ldots, R_{n-1}(\cdot) - R_n(\cdot))$ *takes the mean value under its stationary distribution. By comparing with the plots of real-world capital disribution curves from U.S. equity market data of the Center of Research in Securities Prices* (*CRSP*) *at the University of Chicago* (*see Figure* 5.1 *on p.* 95 *of* [10]), *one sees that a q-Atlas model can capture the concave shape of the capital distribution curve, as well as its linear structure at the top.*

We also note that, since the mapping

$$(b_2 - b_1, b_3 - b_2, \ldots, b_n - b_{n-1}) \mapsto \gamma \tag{2.49}$$

is bijective, one can determine the drifts up to an additive constant by fitting the vector γ to the observed capital distribution curves, such as the ones in Figure 5.1 on p. 95 of [10].

3. Scaling limit of asymmetrically colliding random walks

Let us return to the processes corresponding to the system of stochastic equations (1.1), (1.3). Our objective in this section is to show that such processes with asymmetric local time components arise as scaling limits of random walks with asymmetric interactions upon "collision." We start with the following informal description.

Consider an n-dimensional continuous-time jump process $\Gamma(\cdot) = (\Gamma_1(\cdot), \dots, \Gamma_n(\cdot))$ on the wedge

$$\mathbb{H}_n = \{(z_1, \dots, z_n) \in \mathbb{Z}^n: z_1 \geq z_2 \geq \cdots \geq z_n\}.$$

The process registers the positions on the integer lattice of n particles that always maintain their order. We will label the particles by the elements of the set $[n] = \{1, 2, \dots, n\}$, where 1 refers to the rightmost particle and n refers to the leftmost.

The movement of particles can be described as follows. Consider nonnegative parameters $\{a_k, b_k, \theta_k^L, \theta_k^R, k \in [n]\}$. If there are no other particles at the current site of particle k, then this particle moves to the right at rate a_k and to the left at rate b_k, independently of every other particle. On the other hand, suppose that particles $k, k+1, \dots, k+\ell$ are currently at the same location (a phenomenon we call "collision"), and this is the maximum length of the tie in the sense that the $(k-1)$st and the $(k+\ell+1)$st particles are not at that site. Then, at rate $\theta_k^R a_k$, particle k jumps to the right; and, at rate $\theta_{k+\ell}^L b_{k+\ell}$, particle $(k+\ell)$ jumps to the left, while the particles in between do not move.

We wish to take a diffusion limit of such systems. Before we go on, let us give a few examples which display the variety of behavior that can be expected from such particle systems.

Example 1. *Suppose each particle evolves according to a Poisson process with parameter a until a collision occurs. Thus every $a_k \equiv a$, every $b_k \equiv 0$, and the values of θ_k^L, $k \in [n]$ are irrelevant.*

Consider, first, $\theta_k^R \equiv 2$, $k \in [n]$. This essentially corresponds to the order statistics of a system of Poisson particles moving independently (the two systems differ in their behavior when triple or higher-order collisions of particles occur, but these events will be shown to prevail for an amount of time which is negligible in the diffusion limit). On the other hand, when each $\theta_k^R \equiv 1$, the higher particle blocks the movement of the lower one when they collide. The resulting particle system then evolves as the well-known TASEP process (except that in TASEP collisions occur when particles are at adjacent sites instead of at the same site).

Example 2. *Suppose the particles move according to i.i.d. simple symmetric random walks until collision. That is, $a_k \equiv b_k \equiv 1$, $k \in [n]$. Moreover, we take $\theta_k^L \equiv \theta_k^R \equiv \theta$, $k \in [n]$. Thus, when a particle is involved in a collision, this simply changes the rate of its next jump. The case $\theta = 2$ yields the ordered system of i.i.d. random walks. When $\theta \approx 0$, the particles get "sticky," while when θ is very large, the particles can be thought of as repelling one another.*

Example 3. *Consider, as before, particles moving according to i.i.d. simple symmetric random walks until collision. That is, $a_k \equiv b_k \equiv 1$, $k \in [n]$. Fix an $\ell \in [n]$, say $\ell = 2$. Choose the θ_2^L, θ_2^R parameters to be much larger than the rest of the collision parameters. Thus, whenever particle 1 or particle 3 collides with particle 2, the latter will almost immediately move away. Hence, particle 2 will remain* invisible *to its neighbors.*

Now, consider the triple collisions of particles 1, 2 and 3. Note that, with θ_2^L and θ_2^R being very large, such triple collisions will happen for about the same duration of time as collisions between particle 1 and particle 3 in the particle system obtained by removing particle 2. In other words, if one removes particle 2, this should not influence the behavior of the other particles in a significant way.

Example 4. *We let the particles evolve according to i.i.d. standard Poisson processes until collision, as in Example 1. However, now we set $\theta_2^R \equiv 0$, with all other collision parameters being positive. Then particle 2 and particle 3 freeze forever the first time they collide. Thereafter, particle 1 moves as a standard Poisson process independently from the rest of the particle system, while particles $4, 5, \dots, n$ eventually all coalesce at the site of particle 2.*

Physical heuristics: To develop a feel for these processes, let us discuss briefly the mechanics involved in these collisions. In classical models of collision (see [15]), the particles behave as hard billiard balls of infinitesimally small radius. This models *elastic collision* which we now explain.

Suppose two particles collide on the line. Particle 1, on the right, has mass m_1 and velocity u_1 right before the collision. Particle 2, coming from the left, has mass m_2 and velocity u_2. In elastic collisions the momentum and the kinetic energy are preserved before and after the collision. These two preserved quantities uniquely determine the velocities v_1 and v_2 of the two particles right after the collision, i.e.,

$$v_1 = \frac{u_1(m_1 - m_2) + 2m_2u_2}{m_1 + m_2}, \qquad v_2 = \frac{u_2(m_2 - m_1) + 2m_1u_1}{m_1 + m_2}.$$

Consider again our jump processes. We follow the heuristics of [15] and assume that particles jump together at discrete times. A similar but slightly more lengthy analysis can be done for continuous time, when we let one particle stay still and get hit by the other particle. We consider the Einsteinian viewpoint, that the n particles are bombarded on all sides by other small particles which lead to their random motions. Suppose particles k and $k + 1$ are adjacent and are of masses m_1 and m_2 respectively. We will think of the rate of jumps as the speed in the appropriate direction.

In the next small time interval, these two particles either do not collide, and move away from one another at velocity a_k and $-b_{k+1}$ respectively; or they collide, and $u_1 = -b_k$, $u_2 = a_{k+1}$. Thus, if the collision is elastic, we will observe that the velocity (total jump rate to the right) for particle k is

$$\delta_k := a_k + \frac{-b_k(m_1 - m_2) + 2m_2a_{k+1}}{m_1 + m_2},$$

and the total jump rate to the left for particle $(k + 1)$ is

$$\delta_{k+1} := -b_{k+1} + \frac{a_{k+1}(m_2 - m_1) - 2m_1b_k}{m_1 + m_2}.$$

Clearly, unless we impose specific constraints on the parameters, these are not equal to $\theta_k^R a_k$ and $-\theta_{k+1}^L b_{k+1}$, respectively. Hence, these collisions are not elastic in general.

Certain special cases are worth mentioning. Suppose all masses are equal to one. Then, under elastic collision, $\delta_k = a_k + a_{k+1}$ and $\delta_{k+1} = -b_k - b_{k+1}$. This is the case when particles exchange velocities and can be thought of, via relabeling, as crossing over. A specific choice of θ_k^R and θ_{k+1}^L would capture this scenario as in the case of $\theta_k^R = 2$ in Example 1 and $\theta = 2$ in Example 2 above.

On the other hand, suppose that all a_k and b_k are equal to 1. Then, under elastic collision, $\theta_k^R = \delta_k = 4m_2/(m_1 + m_2)$ and $\theta_{k+1}^L = \delta_{k+1} = 4m_1/(m_1 + m_2)$. In particular, if one of the quantities $\theta_k^R + \theta_{k+1}^L$, $k = 1, \dots, n-1$ is greater (or less) than 4, the mechanics generates excess energy (or absorbs energy) that cannot be explained by elastic collision.

Under suitable assumptions on the parameters of the model, we establish in Theorem 11 diffusion limits similar to (1.1) for these particle systems. There is an apparent paradox here. Consider that diffusion limit for $a_k \equiv b_k \equiv 1$, $k \in [n]$; somewhat surprisingly, the limit turns out to depend only on the ratio $\theta_k^R/\theta_{k+1}^L$. In other words, the diffusion limit is the same whether $\theta_k^R + \theta_{k+1}^L$ is equal to 4 or not. *The collisions among the limiting diffusion particles are always elastic*, and the quantities q_k^-'s and q_k^+'s can be thought of as the proportions of total mass shared by the colliding particles. This is a consequence of the fact that the occupation time of collisions has Lebesgue measure zero in the limit, as will be made clear in the proof.

To get a true inelastic limit, one has to let θ_k^R, θ_k^L go to zero suitably in the diffusion scaling. Such a regime has been studied in [28], in which case one obtains *sticky* colliding Brownian particles in the limit.

3.1. *The modified Skorokhod problem*

We start with a set of parameters: $a, b, (\lambda_k^L, \sigma_k^L, \theta_k^L, k \in [n]), (\lambda_k^R, \sigma_k^R, \theta_k^R, k \in [n])$. At this point we only assume that σ_k^L, σ_k^R are stricly positive and θ_k^L, θ_k^R are nonnegative for every k.

Taking a diffusion limit requires considering a sequence of interacting jump processes as described above. We shall generalize the setup by allowing nonexponential waiting times for the jumps. Let the sequence of interacting jump processes be indexed by $N > 0$. We fix a value of N and a probability space rich enough to support mutually independent sequences of i.i.d. random variables $(u_k^L(i), i \in \mathbb{N})$, $k \in [n]$ and $(u_k^R(i), i \in \mathbb{N})$, $k \in [n]$, all taking only positive values. These random variables denote the inter-jump times of the particles (the superscripts “L” and “R” standing for leftward and rightward jumps, respectively). Assume that, for any fixed $k \in [n]$,

$$\begin{aligned} \mathbb{E}\big[u_k^L(1)\big] &= \left(b + \frac{\lambda_k^L}{\sqrt{N}}\right)^{-1}, \qquad \operatorname{Var}\big[u_k^L(1)\big] = \big(\sigma_k^L\big)^2, \\ \mathbb{E}\big[u_k^R(1)\big] &= \left(a + \frac{\lambda_k^R}{\sqrt{N}}\right)^{-1}, \qquad \operatorname{Var}\big[u_k^R(1)\big] = \big(\sigma_k^R\big)^2. \end{aligned} \tag{3.1}$$

Next, we define the corresponding partial sum processes

$$U_k^L(0) = 0, \qquad U_k^L(j) = \sum_{i=1}^{j} u_k^L(i), \quad j \in \mathbb{N}, k \in [n],$$

$$U_k^R(0) = 0, \qquad U_k^R(j) = \sum_{i=1}^{j} u_k^R(i), \quad j \in \mathbb{N}, k \in [n],$$

and the corresponding renewal processes

$$S_k^L(t) = \max\big\{j \ge 0 \colon U_k^L(j) \le t\big\}, \quad t \ge 0, k \in [n],$$
$$S_k^R(t) = \max\big\{j \ge 0 \colon U_k^R(j) \le t\big\}, \quad t \ge 0, k \in [n].$$

Finally, we denote by $(\mathcal{F}_t, t \ge 0)$ the filtration generated by the processes $S_k^L(\cdot)$, $S_k^R(\cdot)$, $k \in [n]$. Informally, for each $k \in [n]$, the process $S_k^L(\cdot)$ (resp., $S_k^R(\cdot)$) records the leftward (respectively, rightward) jumps of the kth particle from the right, as long as this particle is not involved in a collision.

To describe the effect of collisions we shall use a stochastic time change. The following lemma encapsulates the idea that the leftward (respectively, rightward) movement for particle k can either be blocked, or proceed at a different rate, depending on whether there is a collision with particle $(k+1)$ (resp., with particle $(k-1)$). Such results are standard in the Queueing Theory literature (e.g., Section 2 in the seminal article [29] by Reiman), so we omit the proof.

Lemma 6. *For every $N \in \mathbb{N}$, and any $(\gamma_1, \gamma_2, \dots, \gamma_n) \in \mathbb{H}_n$, there exists a system of jump processes $\varGamma(\cdot) \equiv (\varGamma_k(\cdot), k \in [n])$ taking values in $\mathbb{H}_n$ and progressively measurable with respect to $(\mathcal{F}_t, t \ge 0)$, that satisfies the following set of equations pathwise*:

$$\begin{aligned} &\varGamma_k(t) = \gamma_k - S_k^L\big(T_k^L(t)\big) + S_k^R\big(T_k^R(t)\big), \quad \textit{where} \\ &T_k^L(t) = \int_0^t \mathbf{1}_{\{Q_{k-1}(s)>0\}}\,\mathrm{d}T_k(s) + \theta_k^L \int_0^t \mathbf{1}_{\{Q_{k-1}(s)=0\}}\,\mathrm{d}T_k(s), \quad \textit{and} \\ &T_k^R(t) = \int_0^t \mathbf{1}_{\{Q_k(s)>0\}}\,\mathrm{d}T_{k-1}(s) + \theta_k^R \int_0^t \mathbf{1}_{\{Q_k(s)=0\}}\,\mathrm{d}T_{k-1}(s), \end{aligned} \tag{3.2}$$

$k \in [n]$. We have denoted here by

$$Q(\cdot) \equiv \big(Q_k(\cdot), k \in [n-1]\big) \equiv \big(\varGamma_k(\cdot) - \varGamma_{k+1}(\cdot), k \in [n-1]\big) \tag{3.3}$$

the process of gaps, and have set

$$T_k(t) = \int_0^t \mathbf{1}_{\{Q_k(s)>0\}}\,\mathrm{d}s, \quad k = 0, 1, \dots, n,$$

with the convention $\mathbf{1}_{\{Q_0(\cdot)>0\}} \equiv \mathbf{1}_{\{Q_n(\cdot)>0\}} \equiv 1$.

It should be noted that, although we have suppressed the index N from the notation in the lemma above, this parameter determines the drifts of the coördinate processes. The key to passing to the scaling limit is to understand the time-changes involved. Our strategy is the following: (i) express the process of gaps as a Skorokhod map applied to a suitable "noise process"; (ii) show that the sequence of distributions of gaps is tight; (iii) and finally, show that tightness implies the convergence of the appropriately rescaled process $\Gamma(\cdot)$ to a semimartingale of the type described in (1.1).

We start by analyzing the process of gaps. To this end, we define the centered processes

$$\overline{S}_k^L(t) = S_k^L(t) - bt, \quad t \ge 0, \qquad \overline{S}_k^R(t) = S_k^R(t) - at, \quad t \ge 0$$

for all $k \in [n]$. We define also the following processes, which measure the time spent in the various collisions:

$$I_k(t) := t - T_k(t), \quad t \ge 0, k = 0, 1, \dots, n,$$
$$I_{k,k+1}(t) := \int_0^t \mathbf{1}_{\{Q_k(s)=Q_{k+1}(s)=0\}} \, \mathrm{d}s, \quad k = 0, 1, \dots, n-1.$$

With this notation, the gaps can be written as

$$\begin{aligned} Q_k(t) &= \gamma_k - \gamma_{k+1} - S_k^L\big(T_k^L(t)\big) + S_k^R\big(T_k^R(t)\big) + S_{k+1}^L\big(T_{k+1}^L(t)\big) - S_{k+1}^R\big(T_{k+1}^R(t)\big) \\ &= \gamma_k - \gamma_{k+1} - \overline{S}_k^L\big(T_k^L(t)\big) + \overline{S}_k^R\big(T_k^R(t)\big) + \overline{S}_{k+1}^L\big(T_{k+1}^L(t)\big) - \overline{S}_{k+1}^R\big(T_{k+1}^R(t)\big) \\ &\quad - b\big(T_k^L(t) - T_{k+1}^L(t)\big) + a\big(T_k^R(t) - T_{k+1}^R(t)\big), \quad k \in [n-1]. \end{aligned}$$

Moreover, for all $k \in [n]$ we have

$$\begin{aligned} T_k^L(t) &= T_k(t) + \big(\theta_k^L - 1\big) \int_0^t \mathbf{1}_{\{Q_{k-1}(s)=0,\, Q_k(s)>0\}} \, \mathrm{d}s \\ &= t - I_k(t) + \big(\theta_k^L - 1\big)\left(I_{k-1}(t) - \int_0^t \mathbf{1}_{\{Q_{k-1}(s)=Q_k(s)=0\}} \, \mathrm{d}s\right) \\ &= t - I_k(t) + \big(\theta_k^L - 1\big) I_{k-1}(t) - \big(\theta_k^L - 1\big) I_{k-1,k}(t), \quad t \ge 0 \end{aligned} \tag{3.4}$$

and

$$T_k^R(t) = t - I_{k-1}(t) + \big(\theta_k^R - 1\big) I_k(t) - \big(\theta_k^R - 1\big) I_{k-1,k}(t), \quad t \ge 0. \tag{3.5}$$

Next, we introduce the process $\overline{X}(\cdot) \equiv (\overline{X}_k(\cdot), k \in [n-1])$ via

$$\overline{X}_k(\cdot) := -\overline{S}_k^L\big(T_k^L(\cdot)\big) + \overline{S}_k^R\big(T_k^R(\cdot)\big) + \overline{S}_{k+1}^L\big(T_{k+1}^L(\cdot)\big) - \overline{S}_{k+1}^R\big(T_{k+1}^R(\cdot)\big);$$

this will play the rôle of "noise process," to which a Skorokhod map will be applied to obtain the process of gaps $Q(\cdot) \equiv (Q_k(\cdot), k \in [n-1])$.

Combining everything so far, we obtain the representation for these gaps

$$\begin{aligned} Q_k(t) &= \gamma_k - \gamma_{k+1} + \overline{X}_k(t) - b\big(I_{k+1}(t) - I_k(t)\big) - b\big(\theta_k^L - 1\big) I_{k-1}(t) + b\big(\theta_k^L - 1\big) I_{k-1,k}(t) \\ &\quad + b\big(\theta_{k+1}^L - 1\big) I_k(t) - b\big(\theta_{k+1}^L - 1\big) I_{k,k+1}(t) \\ &\quad + a\big(I_k(t) - I_{k-1}(t)\big) + a\big(\theta_k^R - 1\big) I_k(t) - a\big(\theta_k^R - 1\big) I_{k-1,k}(t) \\ &\quad - a\big(\theta_{k+1}^R - 1\big) I_{k+1}(t) + a\big(\theta_{k+1}^R - 1\big) I_{k,k+1}(t), \quad k \in [n]. \end{aligned} \tag{3.6}$$

Now, it is very convenient to write this equation in a more transparent, matrix–vector notation, so we introduce suitable "reflection matrices." Let $\mathfrak{R}$ be the $(n-1) \times (n-1)$ matrix such that, for every $k \in [n-1]$, all entries in the

kth row of $\mathfrak{R}$ are zero, except the $(k-1)$st, the kth and the $(k+1)$st. These are given by

$$\mathfrak{R}_{k,k-1} = -\frac{a + b(\theta_k^L - 1)}{a\theta_{k-1}^R + b\theta_k^L}, \qquad \mathfrak{R}_{k,k} = 1, \qquad \mathfrak{R}_{k,k+1} = -\frac{b + a(\theta_{k+1}^R - 1)}{a\theta_{k+1}^R + b\theta_{k+2}^L}. \tag{3.7}$$

Note that the columns of the matrix $\mathbf{I}_{n-1} - \mathfrak{R}$ add up to one. Similarly, define the $(n-1) \times (n-2)$ matrix $\widetilde{\mathfrak{R}}$ such that, for each $k \in [n-1]$, all entries in the kth row of $\widetilde{\mathfrak{R}}$ are zero, except for the $(k-1)$st and the kth, which are given as

$$\widetilde{\mathfrak{R}}_{k,k-1} = -a\big(\theta_k^R - 1\big) + b\big(\theta_k^L - 1\big), \qquad \widetilde{\mathfrak{R}}_{k,k} = a\big(\theta_{k+1}^R - 1\big) - b\big(\theta_{k+1}^L - 1\big). \tag{3.8}$$

Finally, we introduce the stochastic processes

$$\mathfrak{Y}_k(\cdot) \equiv \big(a\theta_k^R + b\theta_{k+1}^L\big) I_k(\cdot), \quad k \in [n-1], \qquad \widetilde{\mathfrak{Y}}_k(\cdot) \equiv I_{k,k+1}(\cdot), \quad k \in [n-2]. \tag{3.9}$$

With this notation, we have the following cleaner matrix–vector analogue of (3.6):

$$Q(\cdot) = Q(0) + \overline{X}(\cdot) + \mathfrak{R}\mathfrak{Y}(\cdot) + \widetilde{\mathfrak{R}}\widetilde{\mathfrak{Y}}(\cdot). \tag{3.10}$$

We shall refer to this equation as the *modified Skorokhod representation* for the gap processes in (3.3).

3.2. *The diffusion limit*

To be able to pass to the diffusion limit, we make the following assumptions on the parameters.

Assumption 1. *Assume that $a > 0$, $b > 0$,*

(i) *the entries of $\mathfrak{R}$ satisfy*

$$\frac{a + b(\theta_k^L - 1)}{a\theta_{k-1}^R + b\theta_k^L} \in (0, \infty), \qquad \frac{b + a(\theta_{k+1}^R - 1)}{a\theta_{k+1}^R + b\theta_{k+2}^L} \in (0, \infty), \quad k \in [n-1];$$

(ii) *there is a $k_0 \in [n+1]$ such that the numbers $a(\theta_k^R - 1) - b(\theta_k^L - 1)$, $k \in [n]$ are nonpositive for all $k < k_0$ and nonnegative for all $k \geq k_0$;*

(iii) *for some $\varepsilon > 0$, we have*

$$\sup_{N\in\mathbb{N}} \max_{k\in[n]} \big(\mathbb{E}\big[u_k^L(1)^{2+\varepsilon}\big] + \mathbb{E}\big[u_k^R(1)^{2+\varepsilon}\big]\big) < \infty.$$

Part (i) of the assumption ensures that $\mathfrak{R}$ is a reflection matrix of Harrison–Reiman type; part (ii) turns out to be an appropriate compatibility condition on the matrices $\mathfrak{R}$ and $\widetilde{\mathfrak{R}}$ under which the contribution of triple and higher-order collisions becomes negligible in the diffusion limit; finally, part (iii) is a standard moment assumption which ensures that the random walks driving the particles converge to Brownian motions in the diffusion limit. A simple case, in which parts (i) and (ii) of Assumption 1 are satisfied, is given by $\theta_k^L = \theta_k^R \geq 1$, $k \in [n]$.

For each $m \in \mathbb{N}$, we write $D^m[0,\infty)$ for the space of right-continuous $\mathbb{R}^m$-valued functions on $[0,\infty)$ having left limits, endowed with the topology of uniform convergence on compact sets. On the strength of part (iii) of the above assumption, the following result follows from Theorem 14.6 in [6] (note that one can improve the topology in the conclusion of Lemma 7 to the locally uniform topology, by noting the path continuity of the limit process, modifying the paths of the jump processes to continuous, piecewise linear functions, and using the fact that the Skorokhod topology relativized to the space of continuous functions coincides with the locally uniform topology there).

Lemma 7. *The distribution of the process*

$$\left(\frac{1}{\sqrt{N}}\overline{S}_k^L(Nt), t \geq 0, \frac{1}{\sqrt{N}}\overline{S}_k^R(Nt), t \geq 0, k \in [n]\right)$$

converges weakly in $D^{2n}[0,\infty)$ to the law of the vector of independent processes $(Z_k^L(\cdot), Z_k^R(\cdot), k \in [n])$.

Here, for each $k \in [n]$, $Z_k^L(\cdot)$ is a Brownian motion with drift coëfficient λ_k^L and diffusion coëfficient $b^{3/2}\sigma_k^L$, while $Z_k^R(\cdot)$ is a Brownian motion with drift coëfficient λ_k^R and diffusion coëfficient $a^{3/2}\sigma_k^R$.

Next, we introduce the rescaled versions of the gap process by

$$Q^N(\cdot) \equiv \big(Q_k^N(\cdot), \quad k \in [n-1]\big) \equiv \left(\frac{1}{\sqrt{N}} Q_k(Nt), \quad t \geq 0, \quad k \in [n-1]\right)$$

and define $\mathfrak{Y}^N(\cdot)$, $\widetilde{\mathfrak{Y}}^N(\cdot)$ accordingly. Then,

$$Q^N(\cdot) = Q^N(0) + \overline{X}^N(\cdot) + \mathfrak{R}\mathfrak{Y}^N(\cdot) + \widetilde{\mathfrak{R}}\widetilde{\mathfrak{Y}}^N(\cdot), \tag{3.11}$$

where

$$\overline{X}_k^N(t) \equiv \frac{-1}{\sqrt{N}}\big[\overline{S}_k^L\big(T_k^L(Nt)\big) - \overline{S}_k^R\big(T_k^R(Nt)\big) - \overline{S}_{k+1}^L\big(T_{k+1}^L(Nt)\big) + \overline{S}_{k+1}^R\big(T_{k+1}^R(Nt)\big)\big], \tag{3.12}$$

$t \geq 0$, $k \in [n-1]$. We can state now our main limit theorem for the process of gaps.

Theorem 8. *Suppose Assumption 1 holds, and that $\lim_{N\to\infty} Q^N(0) = \xi(0)$ in distribution for some random vector $\xi(0)$. Let $\xi(\cdot)$ be a reflected Brownian motion in $(\mathbb{R}_+)^{n-1}$ with initial condition $\xi(0)$, drift vector*

$$\mathfrak{b} = \big(-\lambda_k^L + \lambda_k^R + \lambda_{k+1}^L - \lambda_{k+1}^R, k \in [n-1]\big), \tag{3.13}$$

diffusion matrix $\mathfrak{A} = (\mathfrak{A}_{k,\ell})_{1\leq k,\ell\leq n-1}$ given by

$$\mathfrak{A}_{k,l} = \begin{cases} -a^3(\sigma_{k+1}^R)^2 - b^3(\sigma_{k+1}^L)^2 & \text{if } \ell = k+1, k \in [n-2], \\ -a^3(\sigma_k^R)^2 - b^3(\sigma_k^L)^2 & \text{if } \ell = k-1, \ell \in [n-2], \\ a^3(\sigma_k^R)^2 + a^3(\sigma_{k+1}^R)^2 + b^3(\sigma_k^L)^2 + b^3(\sigma_{k+1}^L)^2 & \text{if } k = \ell \in [n-1], \\ 0 & \text{otherwise}, \end{cases} \tag{3.14}$$

and reflection matrix $\mathfrak{R}$ as in (3.7).

Then, the processes $Q^N(\cdot)$, $N > 0$ converge in distribution to $\xi(\cdot)$, as $N \to \infty$, in $D^{n-1}[0,\infty)$. Moreover, the processes

$$\big(\mathfrak{Y}_k^N(\cdot), \quad k \in [n-1]\big) = \left(\big(a\theta_k^R + b\theta_k^L\big)\int_0^{\cdot} \mathbf{1}_{\{Q_k^N(s)=0\}}\,\mathrm{d}s, \quad k \in [n-1]\right) \tag{3.15}$$

converge in distribution in $D^{n-1}[0,\infty)$ to the process of local times accumulated by $\xi(\cdot)$ on the respective faces of the orthant $(\mathbb{R}_+)^{n-1}$, and the processes

$$\big(\widetilde{\mathfrak{Y}}_k^N(\cdot), \quad k \in [n-2]\big) = \left(\int_0^{\cdot} \mathbf{1}_{\{Q_k^N(s)=Q_{k+1}^N(s)=0\}}\,\mathrm{d}s, \quad k \in [n-2]\right) \tag{3.16}$$

tend to zero in distribution in $D^{n-2}[0,\infty)$.

We remark at this point that the limit process can be viewed as the process of gaps for a semimartingale as in (1.1) upon the appropriate identification of parameters (see Theorem 11 below for the details).

The proof of Theorem 8 relies heavily on an extension of the invariance principle of Williams [36], which we introduce and prove in the following subsection.

3.3. *Oscillation estimates for modified Skorokhod maps*

In this subsection, we consider families $(Q^N(\cdot), \overline{X}^N(\cdot), \mathfrak{Y}^N(\cdot), \widetilde{\mathfrak{Y}}^N(\cdot))$, $N > 0$ of processes with right-continuous paths having left limits, which satisfy the following properties. The processes $Q^N(\cdot)$, $N > 0$ take values in the orthant $(\mathbb{R}_+)^{n-1}$; the components of the processes $\mathfrak{Y}^N(\cdot)$, $N > 0$ and $\widetilde{\mathfrak{Y}}^N(\cdot)$, $N > 0$ are nondecreasing and start at 0; the modified Skorokhod representation

$$Q^N(\cdot) = \overline{X}^N(\cdot) + \mathfrak{R}\mathfrak{Y}^N(\cdot) + \widetilde{\mathfrak{R}}\widetilde{\mathfrak{Y}}^N(\cdot) \tag{3.17}$$

holds for each $N > 0$, where $\mathfrak{R}$ and $\widetilde{\mathfrak{R}}$ are as in (3.7) and (3.8), respectively; and the following are true for a suitable constant $c < \infty$:

$$\int_0^\infty \mathbf{1}_{\{Q_k^N(s)>0\}}\,\mathrm{d}\mathfrak{Y}_k^N(s) = 0, \quad k \in [n-1], \tag{3.18}$$

$$\int_0^\infty (\mathbf{1}_{\{Q_k^N(s)>0\}} + \mathbf{1}_{\{Q_{k+1}^N(s)>0\}})\,\mathrm{d}\widetilde{\mathfrak{Y}}_k^N(s) = 0, \quad k \in [n-2], \tag{3.19}$$

$$\forall 0 \le s < t: \quad \widetilde{\mathfrak{Y}}_k^N(t) - \widetilde{\mathfrak{Y}}_k^N(s) \le c \min\bigl(\mathfrak{Y}_k^N(t) - \mathfrak{Y}_k^N(s), \mathfrak{Y}_{k+1}^N(t) - \mathfrak{Y}_{k+1}^N(s)\bigr). \tag{3.20}$$

In this situation, we have the following extension of the invariance principle of [36].

Proposition 9. *Under parts* (i) *and* (ii) *of Assumption 1, suppose that the initial conditions $Q^N(0) = X^N(0)$, $N > 0$ converge in distribution to a random vector $\xi(0)$. Then*:

(a) *If the family of processes $\overline{X}^N(\cdot)$, $N > 0$ is tight on $D^{n-1}[0,\infty)$, then the same is true for $(Q^N(\cdot), \overline{X}^N(\cdot), \mathfrak{Y}^N(\cdot), \widetilde{\mathfrak{Y}}^N(\cdot))$, $N > 0$ on $D^{4n-5}[0,\infty)$.*

(b) *In the situation of part* (a), *any limit point $(Q^\infty(\cdot), \overline{X}^\infty(\cdot), \mathfrak{Y}^\infty(\cdot), \widetilde{\mathfrak{Y}}^\infty(\cdot))$ with continuous paths satisfies*

  (i) $Q^\infty(t) = \overline{X}^\infty(t) + \mathfrak{R}\mathfrak{Y}^\infty(t) + \widetilde{\mathfrak{R}}\widetilde{\mathfrak{Y}}^\infty(t) \in (\mathbb{R}_+)^{n-1}$, $t \ge 0$.

  (ii) *The components of $\mathfrak{Y}^\infty(\cdot)$ and $\widetilde{\mathfrak{Y}}^\infty(\cdot)$ are nondecreasing, and satisfy $\mathfrak{Y}^\infty(0) = 0$, $\widetilde{\mathfrak{Y}}^\infty(0) = 0$ with probability* 1.

  (iii) *The following two identities hold with probability* 1:

$$\int_0^\infty \mathbf{1}_{\{Q_k^\infty(s)>0\}}\,\mathrm{d}\mathfrak{Y}_k^\infty(s) = 0, \quad k \in [n-1], \tag{3.21}$$

$$\int_0^\infty (\mathbf{1}_{\{Q_k^\infty(s)>0\}} + \mathbf{1}_{\{Q_{k+1}^\infty(s)>0\}})\,\mathrm{d}\widetilde{\mathfrak{Y}}_k^\infty(s) = 0, \quad k \in [n-2]. \tag{3.22}$$

The proof will follow the ideas in [36], with additional complications caused by the last summand on the right-hand side of (3.17). As there, for any $m \in \mathbb{N}$ and any function $f \in D^m[0,\infty)$, we introduce the notation

$$\mathrm{Osc}_{t_1}^{t_2}(f) := \sup_{t_1 \le s < t \le t_2} \max_{1 \le k \le m} \bigl|f_k(t) - f_k(s)\bigr|. \tag{3.23}$$

The proof of Proposition 9 is based on the following lemma.

Lemma 10. *Suppose that the functions $q(\cdot), \overline{x}(\cdot), \mathfrak{y}(\cdot) \in D^{n-1}[0,\infty)$, $\widetilde{\mathfrak{y}}(\cdot) \in D^{n-2}[0,\infty)$ fulfill the analogue*

$$q(\cdot) = \overline{x}(\cdot) + \mathfrak{R}\mathfrak{y}(\cdot) + \widetilde{\mathfrak{R}}\widetilde{\mathfrak{y}}(\cdot) \tag{3.24}$$

of (3.17). *Moreover, suppose that the function $q(\cdot)$ takes values in $(\mathbb{R}_+)^{n-1}$, $\mathfrak{y}(0) = 0$, $\widetilde{\mathfrak{y}}(0) = 0$, all components of $\mathfrak{y}(\cdot)$, $\widetilde{\mathfrak{y}}(\cdot)$ are nondecreasing and with a suitable constant $c < \infty$*

$$\int_0^\infty \mathbf{1}_{\{q_k(s)>0\}}\,\mathrm{d}\mathfrak{y}_k(s) = 0, \quad k \in [n-1], \tag{3.25}$$

$$\int_0^\infty (\mathbf{1}_{\{q_k(s)>0\}} + \mathbf{1}_{\{q_{k+1}(s)>0\}})\,\mathrm{d}\widetilde{\mathfrak{y}}_k(s) = 0, \quad k \in [n-2], \tag{3.26}$$

$$\forall 0 \le s < t: \quad \widetilde{\mathfrak{y}}_k(t) - \widetilde{\mathfrak{y}}_k(s) \le c \min\bigl(\mathfrak{y}_k(t) - \mathfrak{y}_k(s), \mathfrak{y}_{k+1}(t) - \mathfrak{y}_{k+1}(s)\bigr). \tag{3.27}$$

Then, there is a constant $C \in (0, \infty)$ depending only on $\mathfrak{R}$, $\widetilde{\mathfrak{R}}$ and c (but not on the particular functions $q(\cdot)$, $\overline{x}(\cdot)$, $\mathfrak{y}(\cdot)$, $\widetilde{\mathfrak{y}}(\cdot)$) such that

$$\forall 0 \le t_1 < t_2: \quad \operatorname{Osc}_{t_1}^{t_2}(q) + \operatorname{Osc}_{t_1}^{t_2}(\mathfrak{y}) + \operatorname{Osc}_{t_1}^{t_2}(\widetilde{\mathfrak{y}}) \le C \operatorname{Osc}_{t_1}^{t_2}(\overline{x}). \tag{3.28}$$

Proof. We proceed by induction over the dimension $n \ge 2$. For $n = 2$, the lemma is a direct consequence of Theorem 5.1 in [36], since $\widetilde{\mathfrak{y}}(\cdot) \equiv 0$ in this case. From now on, we take $n \ge 3$ and assume that the lemma holds for all $\nu = 2, \dots, n-1$. Consider first the case that there exists a $k \in [n-1]$ such that $\mathfrak{y}_k(t_2) = \mathfrak{y}_k(t_1)$. Then, (3.27) shows that $\widetilde{\mathfrak{y}}_k(t_2) = \widetilde{\mathfrak{y}}_k(t_1)$ and $\widetilde{\mathfrak{y}}_{k-1}(t_2) = \widetilde{\mathfrak{y}}_{k-1}(t_1)$. This allows to reduce the dimension of the problem by one and to apply the induction hypothesis to obtain (3.28) (we refer the reader to pp. 15 and 16 in [36] for a detailed version of such a dimension reduction argument).

We now claim the existence of a vector $\lambda = (\lambda_1, \dots, \lambda_n) \in (\mathbb{R}_+)^{n-1}$ such that $(\lambda'\mathfrak{R})_k \ge 1$ and $(\lambda'\widetilde{\mathfrak{R}})_k \ge 0$ for all $k \in [n-1]$, where λ' stands for the transpose of λ. Noting that the off-diagonal elements in every column of $\mathfrak{R}$ are negative and add up to -1 (see Assumption 1(i)), and that the entries of each column of $\widetilde{\mathfrak{R}}$ add up to 0 and satisfy Assumption 1(ii), we see that we may choose $\lambda_1, \lambda_2, \dots, \lambda_{n-1}$ as $f(a_1), f(a_2), \dots, f(a_{n-1})$ for appropriate numbers $0 = a_1 < a_2 < \cdots < a_{n-1} < 1$ and an appropriate strictly concave function $f : [0, 1] \to (0, \infty)$. The latter argument is the main difference between the proof of Lemma 10 and the proof of the oscillation inequality in [36] and is due to the presence of an additional reflection term in (3.24).

From this point one can proceed as in the proof of Theorem 5.1 in [36]. For the convenience of the reader, we give a summary of the argument here. We fix a λ as described above, subtract equation (3.24) at t_1 from equation (3.24) at t_2 and use the two inequalities λ satisfies to conclude

$$\forall 0 \le t_1 < t_2: \quad \operatorname{Osc}_{t_1}^{t_2}(\mathfrak{y}) \le \lambda'\big(q(t_2) - q(t_1)\big) + \left(\sum_{k=1}^{n-1} \lambda_k\right) \operatorname{Osc}_{t_1}^{t_2}(\overline{x}). \tag{3.29}$$

We refer to the derivation of the inequality (18) in [36] for a more detailed explanation of this step. Next, equating the oscillations of the functions on both sides of (3.24) and bounding the oscillation of the right-hand side of (3.24) from above by using (3.29) and (3.27), we find a constant $C < \infty$ such that

$$\forall 0 \le t_1 < t_2: \quad \operatorname{Osc}_{t_1}^{t_2}(q) \le C\Big(\operatorname{Osc}_{t_1}^{t_2}(\overline{x}) + \max_{k \in [n-1]} q_k(t_2)\Big). \tag{3.30}$$

Please see the derivation of the estimate (20) in [36] for more details on this step.

Now, we fix $0 \le t_1 < t_2$ and let $0 < K < \infty$ be a constant, whose value will be determined later. We distinguish between two cases:

(a) $q_k(t_1) > K \operatorname{Osc}_{t_1}^{t_2}(\overline{x})$ for some $k \in [n-1]$.

(b) $q_k(t_1) \le K \operatorname{Osc}_{t_1}^{t_2}(\overline{x})$ for all $k \in [n-1]$.

In case (a), let τ be the first time $t \in [t_1, t_2]$ that $q_k(t) = 0$ and set $\tau = \infty$ if the latter event does not occur. If $\tau = \infty$, then (3.28) holds by the argument in the first paragraph of this proof as a consequence of the induction hypothesis. Now, suppose that $\tau \ne \infty$. Then, the same argument based on the induction hypothesis shows that

$$\operatorname{Osc}_{t_1}^{\tau}(q) \le C \operatorname{Osc}_{t_1}^{\tau}(\overline{x}), \tag{3.31}$$

where we have increased the value of $C < \infty$ if necessary. Next, let $C < K < \infty$. Since we are in case (a), we have

$$q_k(\tau) \ge q_k(t_1) - \operatorname{Osc}_{t_1}^{\tau}(q) \ge (K - C) \operatorname{Osc}_{t_1}^{\tau}(\overline{x}) > 0, \tag{3.32}$$

which is a contradiction to $\tau \ne \infty$.

In case (b), we distinguish two possibilities:

(i) $q_k(t) \le K \operatorname{Osc}_{t_1}^{t_2}(\overline{x})$ for all $k \in [n-1]$ and $t \in [t_1, t_2]$.

(ii) $q_k(t) > K \operatorname{Osc}_{t_1}^{t_2}(\overline{x})$ for some $k \in [n-1]$ and $t \in [t_1, t_2]$.

In case (i), the inequality (3.28) follows from

$$\mathrm{Osc}_{t_1}^{t_2}(q) \le \sup_{t_1 \le t \le t_2} \max_{k \in [n-1]} q_k(t), \tag{3.33}$$

(3.29) and (3.27). In case (ii), we let τ be the first time $t \in [t_1, t_2]$ such that $q_k(t) > K \,\mathrm{Osc}_{t_1}^{t_2}(\overline{x})$ for some $k \in [n-1]$. Then, we split the time interval $[t_1, t_2]$ into $[t_1, \tau]$ and $[\tau, t_2]$, and argue as in case (i) on $[t_1, \tau]$ and as in case (a) on $[\tau, t_2]$. □

Proof of Proposition 9. First, we claim that part (a) of Proposition 9 is a consequence of the inequality (3.28) in Lemma 10. Indeed, the necessary and sufficient conditions (a) and (b) of Corollary 3.7.4 in [8] hold for any subsequence of $\overline{X}^N(\cdot)$, $N > 0$, and carry over to the same subsequence of $(Q^N(\cdot), \overline{X}^N(\cdot), \mathfrak{Y}^N(\cdot), \widetilde{\mathfrak{Y}}^N(\cdot))$, $N > 0$ via the inequality (3.28). For more details on the same argument, please see the proof of Theorem 4.1 in [36].

Now, we turn to the proof of part (b) of Proposition 9 and let $(Q^\infty(\cdot), \overline{X}^\infty(\cdot), \mathfrak{Y}^\infty(\cdot), \widetilde{\mathfrak{Y}}^\infty(\cdot))$ be a limit point as there. The properties (i) and (ii) for it can be seen by using the Skorokhod Representation Theorem in the form of Theorem 3.1.8 in [8] (noting the path continuity of the limit) for the subsequence of $(Q^N(\cdot), \overline{X}^N(\cdot), \mathfrak{Y}^N(\cdot), \widetilde{\mathfrak{Y}}^N(\cdot))$, $N > 0$, which converges to that limit point, and taking the almost sure limit on both sides of the identity (3.17) through this particular subsequence. To deduce property (iii), we let $g : [0, \infty) \to [0, 1]$ be a continuous function such that, for some $\delta > 0$, are here $g(a) = 0$ whenever $0 \le a \le \delta$, and $g(a) = 1$ whenever $a \ge 2\delta$. By arguing as in the second half of the proof of Theorem 4.1 in [36], we conclude that the quantities

$$\int_0^\infty g\big(Q_k^N(s)\big)\,\mathrm{d}\mathfrak{Y}_k^N(s) = 0, \quad k \in [n-1], \tag{3.34}$$

$$\int_0^\infty g\big(Q_k^N(s)\big) + g\big(Q_{k+1}^N(s)\big)\,\mathrm{d}\widetilde{\mathfrak{Y}}_k^N(s) = 0, \quad k \in [n-2] \tag{3.35}$$

converge to the quantities

$$\int_0^\infty g\big(Q_k^\infty(s)\big)\,\mathrm{d}\mathfrak{Y}_k^\infty(s), \quad k \in [n-1], \tag{3.36}$$

$$\int_0^\infty g\big(Q_k^\infty(s)\big) + g\big(Q_{k+1}^\infty(s)\big)\,\mathrm{d}\widetilde{\mathfrak{Y}}_k^\infty(s), \quad k \in [n-2] \tag{3.37}$$

when we pass to the limit through the same subsequence as before. Now, letting $\delta \downarrow 0$, we obtain the property (iii). □

We are now ready for the proof of Theorem 8.

Proof of Theorem 8. *Step* 1. Consider the family of processes $\overline{X}^N(\cdot)$, $N > 0$ in (3.12). The family of processes $(\frac{1}{\sqrt{N}}\overline{S}_k^L(Nt), t \ge 0, \frac{1}{\sqrt{N}}\overline{S}_k^R(Nt), t \ge 0, k \in [n])$, $N > 0$, without the time change, is tight by Lemma 7. Moreover, the time-changes in (3.12) are Lipschitz functions of time. That is, there exists a constant $\Theta < \infty$ such that

$$\forall 0 \le s < t: \quad \max\big(T_k^L(t) - T_k^L(s), T_k^R(t) - T_k^R(s)\big) \le \Theta(t - s), \quad k \in [n].$$

Hence, using the necessary and sufficient conditions for tightness of Corollary 3.7.4 in [8], the tightness of $\overline{X}^N(\cdot)$, $N > 0$ is easily verified. However, we still have to identify the limit points.

Step 2. At this stage, we can use Proposition 9 to conclude that the family $(Q^N(\cdot), \overline{X}^N(\cdot), \mathfrak{Y}^N(\cdot), \widetilde{\mathfrak{Y}}^N(\cdot))$, $N > 0$ is tight. Recall now that, for any $N > 0$, the processes $\mathfrak{Y}^N(\cdot)$ and $\widetilde{\mathfrak{Y}}^N(\cdot)$ can be expressed, as in (3.9), in terms of the times that the particles spend in collisions. The tightness of $\mathfrak{Y}^N(\cdot)$, $N > 0$ and $\widetilde{\mathfrak{Y}}^N(\cdot)$, $N > 0$ now shows that the processes

$$\frac{1}{N} I_k(Nt), \quad t \ge 0, k \in [n-1], \qquad \frac{1}{N} I_{k,k+1}(Nt), \quad t \ge 0, k \in [n-2] \tag{3.38}$$

all tend to zero in $D[0,\infty)$.

Step 3. Putting together the conclusion of Step 2, (3.4) and (3.5), we conclude that each of the processes

$$\frac{1}{N}T_k^L(Nt),\quad t\ge 0, k\in[n],\qquad \frac{1}{N}T_k^R(Nt),\quad t\ge 0, k\in[n]$$

converge to the process t, $t\ge 0$ in $D[0,\infty)$. With the help of the lemma on p. 151 in [6], preceding Theorem 14.4, we deduce that the joint distributions of

$$\left(\frac{1}{\sqrt{N}}\overline{S}_k^L\big(T_k^L(Nt)\big), t\ge 0,\quad \frac{1}{\sqrt{N}}\overline{S}_k^R\big(T_k^R(Nt)\big), t\ge 0, k\in[n]\right),$$

now *with the time-change*, converge on $D^{2n}[0,\infty)$ to the limiting distribution described in Lemma 7 (note that we can improve the topology used in [6] to the topology of uniform convergence on compacts by observing the path continuity of the limit process, modifying the paths of the jump processes to continuous, piecewise linear paths and using the fact that the Skorokhod topology relativized to the space of continuous functions coincides with the locally uniform topology there).

Step 4. Step 3 implies that, in the limit $N\to\infty$, the processes $\overline{X}^N(\cdot)$, $N>0$ converge vaguely in $D^{n-1}[0,\infty)$ to a multidimensional Brownian motion with drift and diffusion coëfficients as described in Theorem 8. To conclude the proof, we note that every limit point $(Q^\infty(\cdot),\overline{X}^\infty(\cdot),\mathfrak{Y}^\infty(\cdot),\widetilde{\mathfrak{Y}}^\infty(\cdot))$ of the collection $(Q^N(\cdot),\overline{X}^N(\cdot),\mathfrak{Y}^N(\cdot),\widetilde{\mathfrak{Y}}^N(\cdot))$, $N>0$ has continuous paths by (3.28) and the fact that the processes $\overline{X}^N(\cdot)$, $N>0$ converge to a process with continuous paths. Thus, by part (b) of Proposition 9, every such limit point satisfies the properties (i), (ii), (iii) there, with $\overline{X}^\infty(\cdot)$ being the multidimensional Brownian motion just described. Lastly, by Lemma 1, we can identify $Q^\infty(\cdot)$ with $\xi(\cdot)$, $\mathfrak{Y}^\infty(\cdot)$ with the boundary local times of $\xi(\cdot)$, and deduce that $\widetilde{\mathfrak{Y}}^\infty(\cdot)\equiv 0$. This completes the proof. □

Finally, we consider the limit of the entire collection of jump processes $(\Gamma_k(\cdot), k\in[n])$ (we refer the reader to the expression (3.2)). Let $\Gamma^N(\cdot)$ denote the vector of centered and scaled jump processes given, for every $k\in[n]$, by

$$\Gamma_k^N(t):=\frac{1}{\sqrt{N}}\Gamma_k(0)+\frac{1}{\sqrt{N}}\big[S_k^R\big(T_k^R(Nt)\big)-S_k^L\big(T_k^L(Nt)\big)\big]-(a-b)t\sqrt{N},\quad t\ge 0.$$

Theorem 11. *Suppose that Assumption 1 holds and that*

$$\lim_{N\to\infty}N^{-1/2}\Gamma_k(0)=R(0)\in\mathbb{W}^n$$

in distribution. Further, let $R(\cdot)=(R_1(\cdot),R_2(\cdot),\dots,R_n(\cdot))$ denote the continuous n-dimensional semimartingale taking values in $\mathbb{W}^n$ and satisfying

$$\begin{aligned}R_k(t)&=R_k(0)+\big(\lambda_k^R-\lambda_k^L\big)t+\big(a^3\big(\sigma_k^R\big)^2+b^3\big(\sigma_k^L\big)^2\big)^{1/2}\boldsymbol{\beta}_k(t)\\&\quad+\frac{a(\theta_k^R-1)+b}{a\theta_k^R+b\theta_{k+1}^L}\Lambda^{(k,k+1)}(t)-\frac{b(\theta_k^L-1)+a}{a\theta_{k-1}^R+b\theta_k^L}\Lambda^{(k-1,k)}(t),\quad 0\le t<\infty,\end{aligned}$$

$k\in[n]$ *with the same notation as in* (1.1), (1.3).

Then the processes $\Gamma^N(\cdot)$, $N>0$ converge in $D^n[0,\infty)$ to the process $R(\cdot)$ described above in the limit $N\to\infty$.

Proof. The main observation is that, for any fixed $N>0$, we have

$$\begin{aligned}\Gamma_k^N(t)&=\frac{1}{\sqrt{N}}\Gamma_k(0)+\frac{1}{\sqrt{N}}\big[\overline{S}_k^R\big(T_k^R(Nt)\big)-\overline{S}_k^L\big(T_k^L(Nt)\big)\big]\\&\quad+\frac{a(\theta_k^R-1)+b}{a\theta_k^R+b\theta_{k+1}^L}\mathfrak{Y}_k^N(t)-\frac{b(\theta_k^L-1)+a}{a\theta_{k-1}^R+b\theta_k^L}\mathfrak{Y}_{k-1}^N(t)\\&\quad+\big(-a\big(\theta_k^R-1\big)+b\big(\theta_k^L-1\big)\big)\widetilde{\mathfrak{Y}}_{k-1}^N(t),\quad t\ge 0,\end{aligned}\tag{3.39}$$

$k \in [n]$. The same steps as in the proof of Theorem 8 now show that the processes in the first line of (3.39) converge jointly, in distribution, to the components of a multidimensional Brownian motion with drift and diffusion coëfficients as in the statement of this theorem, the process $\mathfrak{Y}^N(\cdot)$ converges to the process of boundary local times of a reflected Brownian motion as in Theorem 8, whereas the process $\widetilde{\mathfrak{Y}}^N(\cdot)$ converges to zero. Therefore, the processes $\Gamma^N(\cdot)$, $N > 0$ must also converge in distribution, and one can identify the limit of the "noise part" with the appropriate multidimensional Brownian motion and the limit of the "local time part" with the local time part in the decomposition of the process $R(\cdot)$ in the statement of the theorem. This completes the argument. □

4. Additional determinantal structures

This last section studies conditions on the parameters $b_1, b_2, \dots, b_n, \sigma_1, \sigma_2, \dots, \sigma_n$ and $q_1^{\pm}, q_2^{\pm}, \dots, q_n^{\pm}$, under which the process $R(\cdot) = (R_1(\cdot), R_2(\cdot), \dots, R_n(\cdot))$ of ranks as in (1.1), (1.3) has a probabilistic structure of *determinantal type*, in the sense that its transition densities are of the generalized Karlin–McGregor form

$$p(t, r, \widetilde{r}) = \sum_{\boldsymbol{\sigma} \in \mathbb{S}_n} \kappa_{\boldsymbol{\sigma}} \prod_{k=1}^{n} f^{k,\boldsymbol{\sigma}(k)}(t, \widetilde{r}_{\boldsymbol{\sigma}(k)} - r_k). \tag{4.1}$$

Here $\mathbb{S}_n$ is the group of permutations of a set with n elements, whereas $\kappa_{\boldsymbol{\sigma}}$, $\boldsymbol{\sigma} \in \mathbb{S}_n$ and $f^{k,\ell}$, $1 \le k, \ell \le n$ are suitable nonzero real numbers and not identically vanishing functions, respectively.

For the purposes of this section, we extend the framework of Section 1.1 by allowing the collision parameters $q_1^{\pm}, q_2^{\pm}, \dots, q_n^{\pm}$ to take values in $[0, 1]$, albeit insisting on the condition (1.2) and assuming that, if $q_k^+ = 1$ holds for some $k = 2, 3, \dots, n-1$, then $q_\ell^- < 1$ for all $\ell \ge k$. The latter assumption means that there is no subsystem of consecutive particles such that its rightmost particle receives a push to the left with collision parameter 1 and its leftmost particle receives a push to the right with collision parameter 1. This prevents the process from getting stuck in a situation, where the particles of such a subsystem collide simultaneously with each other and with the right- and left-neighbors of the subsystem. In fact, the process would cease to exist beyond such an event, since the reflected Brownian motion comprised by the spacings corresponding to such a collision would violate the completely-$\mathcal{S}$ condition, and would therefore be unable to re-enter the orthant once it hits the corner.

We claim that, in this more general framework, a weak solution to (1.1), (1.3) *continues to exist and be unique in distribution*. This is a consequence of Theorem 1.3 in [7], which asserts the weak existence and uniqueness in distribution for semimartingale reflecting Brownian motions in polyhedral domains. Indeed, the polyhedral domain of interest to us here is the Weyl chamber $\mathbb{W}^n$; it has $(n-1)$ faces with normal vectors given by the rows of the matrix

$$\mathbf{N} = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}; \tag{4.2}$$

and the directions of reflection in our case are given by the columns of the matrix

$$\mathbf{R} = \begin{pmatrix} q_1^- & 0 & 0 & 0 \\ -q_2^+ & q_2^- & 0 & 0 \\ 0 & -q_3^+ & q_3^- & 0 \\ 0 & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & -q_n^+ \end{pmatrix}. \tag{4.3}$$

Moreover, the matrix

$$(\mathbf{NR})' = \begin{pmatrix} 1 & -q_2^+ & 0 & 0 \\ -q_2^- & 1 & -q_3^+ & 0 \\ 0 & \ddots & \ddots & \ddots \\ 0 & 0 & -q_{n-1}^- & 1 \end{pmatrix} \tag{4.4}$$

is completely-$\mathcal{S}$. Indeed, it is not hard to check that one can choose the vector λ in the definition of the completely-$\mathcal{S}$ property as $(f(x_1), f(x_2), \ldots, f(x_{n-1}))$ for a suitable concave function f and points $x_1, x_2, \ldots, x_{n-1}$, and then perform a similar construction for all principal submatrices of $(\mathbf{NR})'$. Since the completely-$\mathcal{S}$ property is preserved under transposition (see Lemma 3 in [30]), the matrix **NR** is completely-$\mathcal{S}$ as well. Consequently, Theorem 1.3 in [7] applies and guarantees the weak existence and uniqueness in distribution for a reflected Brownian motion in the Weyl chamber corresponding to (1.1). Finally, the identification (1.3) of the regulating processes as local time processes can be carried out as in Section 2.1, after noting that the vector of differences of consecutive coordinates in a reflected Brownian motion on $\mathbb{W}^n$ forms a reflected Brownian motion on the orthant $(\mathbb{R}_+)^{n-1}$.

The question about the existence of transition densities of the generalized Karlin–McGregor form (4.1) is motivated by the following two extreme cases:

(i) If one considers $b_1 = b_2 = \cdots = b_n$, $\sigma_1 = \sigma_2 = \cdots = \sigma_n$ and $q_1^{\pm} = q_2^{\pm} = \cdots = q_n^{\pm} = 1/2$, then the ranks evolve as the ordered system of n Brownian motions, each with drift b_1 and dispersion σ_1. Therefore, in this case

$$p(t, r, \widetilde{r}) = \sum_{\boldsymbol{\sigma} \in S_n} \prod_{k=1}^{n} \varphi_{b_1,\sigma_1}(t, \widetilde{r}_{\boldsymbol{\sigma}(k)} - r_k), \tag{4.5}$$

where $\varphi_{b_1,\sigma_1}(t, \cdot)$ denotes the Gaussian density with mean $b_1 t$ and variance $\sigma_1^2 t$.

(ii) Now, consider the case of $b_1 = b_2 = \cdots = b_n = 0$, $\sigma_1 = \sigma_2 = \cdots = \sigma_n = 1$ and

$$q_k^+ = 1, \quad k = 1, 2, \ldots, n, \qquad q_k^- = 0, \quad k = 1, 2, \ldots, n. \tag{4.6}$$

Then, the process of ranks is given by the continuous version of the totally asymmetric simple exclusion process (TASEP) treated in detail by Warren in Section 4 of [34]. In this case, the transition probability densities are of the form

$$p(t, r, \widetilde{r}) = \sum_{\boldsymbol{\sigma} \in \mathbb{S}_n} (-1)^{\operatorname{sgn}(\boldsymbol{\sigma})} \prod_{k=1}^{n} \psi^{k,\boldsymbol{\sigma}(k)}(t, \widetilde{r}_{\boldsymbol{\sigma}(k)} - r_k) \tag{4.7}$$

for suitable functions $\psi^{k,\ell}$, $1 \le k, \ell \le n$ (see Proposition 8 in [34]) and with $\operatorname{sgn}(\boldsymbol{\sigma})$ standing for the signum of a permutation $\boldsymbol{\sigma} \in \mathbb{S}_n$.

The main result of this section is the following proposition.

Proposition 12. *Suppose that the nonnegative collision parameters $q_1^{\pm}, q_2^{\pm}, \ldots, q_n^{\pm}$ satisfy* (1.2) *and are such that, if $q_k^+ = 1$ holds for some $k = 2, 3, \ldots, n-1$, then $q_\ell^- < 1$ for all $\ell \ge k$. Suppose that in addition, for every $\varepsilon > 0$, the transition probability densities of the process in* (1.1) *belong to the function space $C_b((\varepsilon, \infty) \times \mathbb{W}^n \times \mathbb{W}^n)$, and are continuously differentiable in the first coördinate and twice continuously differentiable in the second coördinate with derivatives in $C_b((\varepsilon, \infty) \times \mathbb{W}^n \times \mathbb{W}^n)$.*

Then these transition densities are given by (4.1) *with suitable nonzero constants $\kappa_{\boldsymbol{\sigma}}$, $\boldsymbol{\sigma} \in \mathbb{S}_n$ and not identically vanishing functions $f_{k,\ell}$, $1 \le k, \ell \le n$, if and only if: $b_1 = b_2 = \cdots = b_n$, $\sigma_1 = \sigma_2 = \cdots = \sigma_n$ and either*

(i) *for all $k = 1, 2, \ldots, n-1$ one has $q_k^- = q_{k+1}^+ = 1/2$, or*

(ii) *there is a $k \in \{1, 2, \ldots, n\}$ such that $q_\ell^- = 1$, $q_{\ell+1}^+ = 0$ for all $\ell < k$ and $q_\ell^- = 0$, $q_{\ell+1}^+ = 1$ for all $\ell \ge k$.*

Moreover, in case (i) *one may choose $\kappa_{\boldsymbol{\sigma}} = 1$, $\boldsymbol{\sigma} \in \mathbb{S}_n$ and $f^{k,\ell} = \varphi_{b_1,\sigma_1}$, $1 \le k, \ell \le n$, and in case* (ii) *one may choose $\kappa_{\boldsymbol{\sigma}} = \operatorname{sgn}(\boldsymbol{\sigma})$, $\boldsymbol{\sigma} \in \mathbb{S}_n$ and the functions $f^{k,\ell}$, $1 \le k, \ell \le n$, according to the formulas* (4.24)–(4.26) *below.*

Proof. We pick an arbitrary compactly supported continuous function $g : \mathbb{W}^n \to \mathbb{R}$, an arbitrary real number $T > 0$, and consider the martingale

$$\int_{\mathbb{W}^n} p\big(T - t, R(t), \widetilde{r}\big) g(\widetilde{r})\, d\widetilde{r} =: F\big(T - t, R(t)\big), \quad 0 \le t \le T. \tag{4.8}$$

By differentiating inside the integral on the left side above, it follows that the function $F(t,r)$ is once continuously differentiable in the first coordinate and twice continuously differentiable in the second (the property $C^{1,2}$), with bounded derivatives continuously extending to the boundary of $\mathbb{W}^n$. By the Whitney extension theorem, one can consider $F(t,r)$ as the restriction of a function that is $C^{1,2}$ on the entire $\mathbb{R}\times\mathbb{R}^n$. This allows us to use Itô's formula on the martingale $F(T-\cdot, R(\cdot))$.

Thus, we apply Itô's formula to the right-hand side of (4.8) and differentiate under the integral sign on the left. Start the process R from an arbitrary point r in the interior of $\mathbb{W}^n$. From the martingale property of the process $F(\cdot, R(\cdot))$ it follows that the transition probability kernel satisfies the heat equation

$$\partial_t p(t,r,\widetilde{r}) = \frac{1}{2}\sum_{k=1}^{n}\sigma_k^2\partial_{r_k}^2 p(t,r,\widetilde{r}) + \sum_{k=1}^{n} b_k\partial_{r_k} p(t,r,\widetilde{r}) \tag{4.9}$$

in the interior of $\mathbb{W}^n$.

On the other hand, if we start the process R from a point r on a boundary hyperplane of $\mathbb{W}^n$, the time point 0 is in the support of the local time measure at that boundary. This is a consequence of the Skorokhod decomposition and the fact that the noise is Brownian. Thus, the martingale property of $F(\cdot, R(\cdot))$ for the semimartingale R satisfying (1.1) forces the following elastic boundary condition:

$$q_k^-\partial_{r_k} p(t,r,\widetilde{r}) - q_{k+1}^+\partial_{r_{k+1}} p(t,r,\widetilde{r}) = 0 \quad \text{whenever } r_k = r_{k+1}, \tag{4.10}$$

for $t>0$, $(r,\widetilde{r})\in(\mathbb{W}^n)^2$.

Substituting the expression of (4.1) into (4.10), we deduce

$$\begin{aligned}
&q_k^-\sum_{\boldsymbol{\sigma}\in\mathbb{S}_n}\kappa_{\boldsymbol{\sigma}}\prod_{\ell\neq k} f^{\ell,\boldsymbol{\sigma}(\ell)}(t,\widetilde{r}_{\boldsymbol{\sigma}(\ell)}-r_\ell)D_2 f^{k,\boldsymbol{\sigma}(k)}(t,\widetilde{r}_{\boldsymbol{\sigma}(k)}-r_k)\\
&\quad - q_{k+1}^+\sum_{\boldsymbol{\sigma}\in\mathbb{S}_n}\kappa_{\boldsymbol{\sigma}}\prod_{\ell\neq k+1} f^{\ell,\boldsymbol{\sigma}(\ell)}(t,\widetilde{r}_{\boldsymbol{\sigma}(\ell)}-r_\ell)D_2 f^{k+1,\boldsymbol{\sigma}(k+1)}(t,\widetilde{r}_{\boldsymbol{\sigma}(k+1)}-r_{k+1}) = 0
\end{aligned}$$

whenever $r_k = r_{k+1}$. Here, D_2 denotes differentiation with respect to the second argument. Plugging in r_k for r_{k+1} in this last expression, and grouping together functions that have the same arguments, results in

$$\begin{aligned}
0 &= q_k^-\kappa_{\boldsymbol{\sigma}} D_2 f^{k,\ell_1}(t,\widetilde{r}_{\ell_1}-r_k) f^{k+1,\ell_2}(t,\widetilde{r}_{\ell_2}-r_k)\\
&\quad + q_k^-\kappa_{\widetilde{\boldsymbol{\sigma}}} f^{k+1,\ell_1}(t,\widetilde{r}_{\ell_1}-r_k) D_2 f^{k,\ell_2}(t,\widetilde{r}_{\ell_2}-r_k)\\
&\quad - q_{k+1}^+\kappa_{\boldsymbol{\sigma}} f^{k,\ell_1}(t,\widetilde{r}_{\ell_1}-r_k) D_2 f^{k+1,\ell_2}(t,\widetilde{r}_{\ell_2}-r_k)\\
&\quad - q_{k+1}^+\kappa_{\widetilde{\boldsymbol{\sigma}}} D_2 f^{k+1,\ell_1}(t,\widetilde{r}_{\ell_1}-r_k) f^{k,\ell_2}(t,\widetilde{r}_{\ell_2}-r_k)
\end{aligned} \tag{4.11}$$

for all permutations $\boldsymbol{\sigma}$, $\widetilde{\boldsymbol{\sigma}}$ such that $\boldsymbol{\sigma}(k)=\ell_1$, $\boldsymbol{\sigma}(k+1)=\ell_2$ and $\widetilde{\boldsymbol{\sigma}}=(\ell_1\ell_2)\boldsymbol{\sigma}$. Here $(\ell_1\ell_2)$ is the permutation that transposes ℓ_1 and ℓ_2, and $(\ell_1\ell_2)\boldsymbol{\sigma}$ is the product of the two permutations in the sense of the usual group structure on $\mathbb{S}_n$.

Let us recall now (1.2), and take the Fourier transform with respect to the variables $\widetilde{r}_{\ell_1}-r_k$ (parameter a) and $\widetilde{r}_{\ell_2}-r_k$ (parameter b), to obtain equations of the form

$$\begin{aligned}
&q\kappa a G(t,a)H(t,b) + q\tilde{\kappa}K(t,a)bL(t,b)\\
&\quad = (1-q)\kappa G(t,a)bH(t,b) + (1-q)\tilde{\kappa}aK(t,a)L(t,b),
\end{aligned} \tag{4.12}$$

or equivalently

$$G(t,a)H(t,b)\left(\frac{-q\kappa a + (1-q)\kappa b}{q\tilde{\kappa}b - (1-q)\tilde{\kappa}a}\right) = K(t,a)L(t,b). \tag{4.13}$$

The only cases in which the fraction

$$\frac{-q\kappa a + (1-q)\kappa b}{q\tilde{\kappa} b - (1-q)\tilde{\kappa} a}$$

can be factored as the product of a function only of a and a function only of b, are given by $q = 1/2$, $q = 1$ and $q = 0$. Moreover, it follows that there exist nonzero constants $c_{k,\ell}$, $1 \le k, \ell \le n$ such that

$$f^{k,\ell} = c_{k,\ell} f^{k+1,\ell}, \quad \ell = 1, 2, \dots, n \quad \text{whenever } q_k^- = 1/2, \tag{4.14}$$

$$D_2 f^{k,\ell} = c_{k,\ell} f^{k+1,\ell}, \quad \ell = 1, 2, \dots, n \quad \text{whenever } q_k^- = 1, \tag{4.15}$$

$$f^{k,\ell} = c_{k,\ell} D_2 f^{k+1,\ell}, \quad \ell = 1, 2, \dots, n \quad \text{whenever } q_k^- = 0. \tag{4.16}$$

Substituting these formulas into (4.11) we conclude that

$$\kappa_{\widetilde{\boldsymbol{\sigma}}} = \begin{cases} \kappa_{\boldsymbol{\sigma}} \frac{c_{k,\ell_1}}{c_{k,\ell_2}} & \text{if } \widetilde{\boldsymbol{\sigma}} = (\ell_1\ell_2)\boldsymbol{\sigma}, \boldsymbol{\sigma}(k) = \ell_1, \boldsymbol{\sigma}(k+1) = \ell_2, q_k^- = q_{k+1}^+ = \frac{1}{2}, \\ -\kappa_{\boldsymbol{\sigma}} \frac{c_{k,\ell_1}}{c_{k,\ell_2}} & \text{if } \widetilde{\boldsymbol{\sigma}} = (\ell_1\ell_2)\boldsymbol{\sigma}, \boldsymbol{\sigma}(k) = \ell_1, \boldsymbol{\sigma}(k+1) = \ell_2, q_k^-, q_{k+1}^+ \in \{0, 1\}. \end{cases} \tag{4.17}$$

Now, suppose that there is a k such that $q_k^- = q_{k+1}^+ = \frac{1}{2}$ and $q_{k+1}^-, q_{k+2}^+ \in \{0, 1\}$ ("case (a)"), or $q_k^-, q_{k+1}^+ \in \{0, 1\}$ and $q_{k+1}^- = q_{k+2}^+ = \frac{1}{2}$ ("case (b)"). We consider a permutation $\boldsymbol{\sigma}$ such that $\boldsymbol{\sigma}(k) = \ell_1$, $\boldsymbol{\sigma}(k+1) = \ell_2$, $\boldsymbol{\sigma}(k+2) = \ell_3$ and apply the formula (4.17) repeatedly, to obtain

$$\kappa_{(\ell_1\ell_2)(\ell_1\ell_3)(\ell_2\ell_3)\boldsymbol{\sigma}} = \kappa_{\boldsymbol{\sigma}} \frac{c_{k+1,\ell_1} c_{k,\ell_1}}{c_{k+1,\ell_3} c_{k,\ell_3}}, \qquad \kappa_{(\ell_2\ell_3)(\ell_1\ell_3)(\ell_1\ell_2)\boldsymbol{\sigma}} = -\kappa_{\boldsymbol{\sigma}} \frac{c_{k,\ell_1} c_{k+1,\ell_1}}{c_{k,\ell_3} c_{k+1,\ell_3}} \tag{4.18}$$

in case (a), and

$$\kappa_{(\ell_1\ell_2)(\ell_1\ell_3)(\ell_2\ell_3)\boldsymbol{\sigma}} = -\kappa_{\boldsymbol{\sigma}} \frac{c_{k+1,\ell_1} c_{k,\ell_1}}{c_{k+1,\ell_3} c_{k,\ell_3}}, \qquad \kappa_{(\ell_2\ell_3)(\ell_1\ell_3)(\ell_1\ell_2)\boldsymbol{\sigma}} = \kappa_{\boldsymbol{\sigma}} \frac{c_{k,\ell_1} c_{k+1,\ell_1}}{c_{k,\ell_3} c_{k+1,\ell_3}} \tag{4.19}$$

in case (b). In both cases $(\ell_1\ell_2)(\ell_1\ell_3)(\ell_2\ell_3)\boldsymbol{\sigma} = (\ell_2\ell_3)(\ell_1\ell_3)(\ell_1\ell_2)\boldsymbol{\sigma}$ yields a contradiction to the assumption that the constants $\kappa_{\boldsymbol{\sigma}}$, $\boldsymbol{\sigma} \in \mathbb{S}_n$ and $c_{k,\ell}$, $1 \le k, \ell \le n$ are nonzero.

Next, we introduce the linear parabolic operators

$$\mathcal{R}_k = \partial_t - \frac{1}{2}\sigma_k^2 \partial_{r_k}^2 - b_k \partial_{r_k}, \quad k = 1, 2, \dots, n \tag{4.20}$$

and substitute the expression of (4.1) into (4.9), to obtain

$$0 = \sum_{\boldsymbol{\sigma} \in \mathbb{S}_n} \kappa_{\boldsymbol{\sigma}} \sum_{k=1}^{n} \left(\prod_{\ell \neq k} f^{\ell,\boldsymbol{\sigma}(\ell)}(t, \widetilde{r}_{\boldsymbol{\sigma}(\ell)} - r_\ell) \right) \mathcal{R}_k f^{k,\boldsymbol{\sigma}(k)}(t, \widetilde{r}_{\boldsymbol{\sigma}(k)} - r_k), \tag{4.21}$$

which shows

$$\mathcal{R}_k f^{k,\boldsymbol{\sigma}(k)} = 0, \quad k = 1, 2, \dots, n, \boldsymbol{\sigma} \in \mathbb{S}_n. \tag{4.22}$$

This observation and (4.14)–(4.16) imply $b_1 = b_2 = \cdots = b_n$ and $\sigma_1 = \sigma_2 = \cdots = \sigma_n$. The proof of the "only if" part is complete.

Conversely, suppose that the conditions on the drift, dispersion and collision parameters in the proposition are satisfied. If we are in the case

$$q_k^- = 1/2, \quad k = 1, 2, \dots, n, \tag{4.23}$$

then we just need to argue as in point (i) preceding the statement of the proposition, in order to finish the proof. In all other cases, we define $f^{k,\ell}$, $1 \le k, \ell \le n$ by the formulas

$$f^{k,k} = \varphi_{b_1,\sigma_1}, \tag{4.24}$$

$$f^{k+1,\ell} = D_2 f^{k,\ell}, \quad \text{whenever } q_k^- = 1, q_{k+1}^+ = 0, \tag{4.25}$$

$$f^{k+1,\ell}(t,y) = \int_{-\infty}^{y} f^{k,\ell}(t,z)\,\mathrm{d}z, \quad \text{whenever } q_k^- = 0, q_{k+1}^+ = 1. \tag{4.26}$$

We can express this state of affairs as follows: In order to determine the entry $f^{k,\ell}(t,\cdot)$ in (4.1) for $\ell > k$, we count the number $\mathfrak{u}$ of ones in $\{q_k^-, \dots, q_{\ell-1}^-\}$ and the number $\mathfrak{z}$ of zeros in $\{q_k^-, \dots, q_{\ell-1}^-\}$; and then compute $f^{k,\ell}(t,\cdot)$ by differentiating the Gaussian probability density function $\varphi_{b_1,\sigma_1}(t,\cdot)$ with respect to its second coördinate $\mathfrak{u}$ times, and integrating the result with respect to the second coördinate $\mathfrak{z}$ times. The entries $f^{k,\ell}(t,\cdot)$ for $\ell < k$ are computed similarly. From this point on, one can argue as in the proofs of Proposition 8 and Lemma 7 in [34] to deduce that the expression of (4.1) with these choices of $f^{k,\ell}$, $1 \le k, \ell \le n$ and $\kappa_{\boldsymbol{\sigma}} = \mathrm{sgn}(\boldsymbol{\sigma})$, $\boldsymbol{\sigma} \in \mathbb{S}_n$ gives the transition densities of the process $R(\cdot)$. In particular, a line-by-line repetition of the proof of Lemma 7 in Section 6 of [34] shows that for every continuous bounded function g on $\mathbb{W}^n$ vanishing in a neighborhood of the boundary of $\mathbb{W}^n$,

$$\lim_{t\to 0} \int_{\mathbb{W}^n} p(t,r,\widetilde{r})g(\widetilde{r})\,\mathrm{d}\widetilde{r} = g(r) \tag{4.27}$$

holds uniformly in $r \in \mathbb{W}^n$. $\square$

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