Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2016, Vol. 52, No. 1, 448–482
DOI: 10.1214/14-AIHP645

ANNALES
DE L'INSTITUT
HENRI
POINCARÉ
PROBABILITÉS
ET STATISTIQUES
www.imstat.org/aihp

Weak extinction versus global exponential growth of total mass for superdiffusions

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Received 17 January 2013; revised 17 September 2014; accepted 17 September 2014

Abstract. Consider a superdiffusion X on $\mathbb{R}^d$ corresponding to the semi-linear operator $\mathcal{A}(u) = Lu + \beta u - ku^2$, where L is a second order elliptic operator, $\beta(\cdot)$ is in the Kato class, and $k(\cdot) \geq 0$ is bounded on compact subsets of $\mathbb{R}^d$ and is positive on a set of positive Lebesgue measure.

The main purpose of this paper is to complement the results obtained in (*Ann. Probab.* **32** (2004) 78–99), in the following sense. Let λ_∞ be the L^∞-growth bound of the semigroup corresponding to the Schrödinger-type operator $L + \beta$. If $\lambda_\infty \neq 0$, then we prove that, in some sense, the exponential growth/decay rate of $\|X_t\|$, the total mass of X_t, is λ_∞. We also describe the limiting behavior of $\exp(-\lambda_\infty t)\|X_t\|$, as $t \to \infty$. This should be compared to the result in (*Ann. Probab.* **32** (2004) 78–99), which says that the generalized principal eigenvalue λ_2 of the operator gives the rate of *local* growth when it is positive, and implies local extinction otherwise. It is easy to show that $\lambda_\infty \geq \lambda_2$, and we discuss cases when $\lambda_\infty > \lambda_2$ and when $\lambda_\infty = \lambda_2$.

When $\lambda_\infty = 0$, and under some conditions on β, we give a necessary and sufficient condition for the superdiffusion X to exhibit weak extinction. We show that the branching intensity k affects weak extinction; this should be compared to the known result that k does not affect weak *local* extinction. (The latter depends on the sign of λ_2 only, and it turns out to be equivalent to local extinction.)

Résumé. Soit une superdiffusion X sur $\mathbb{R}^d$ correspondant à l'opérateur semi-linéaire $\mathcal{A}(u) = Lu + \beta u - ku^2$, où L est lui-même un opérateur éliptique du second ordre, $\beta(\cdot)$ est dans la classe de Kato, et $k(\cdot) \geq 0$ est borné sur les compacts de $\mathbb{R}^d$ et est positif sur un ensemble de mesure de Lebesgue positive.

L'objectif principal de cet article est de compléter les résultats obtenus dans (*Ann. Probab.* **32** (2004) 78–99), dans le sens suivant. Soit λ_∞ la borne L^∞ de croissance du semigroupe correspondant à l'opérateur $L + \beta$ de type Schrödinger. Si $\lambda_\infty \neq 0$, nous prouvons alors que – dans un certain sens – le taux exponentiel de croissance/décroissance de la masse totale $\|X_t\|$, est λ_∞. Nous décrivons également le comportement limite de $\exp(-\lambda_\infty t)\|X_t\|$, quand $t \to \infty$, sous cette même hypothèse. Ces résultats sont à comparer avec ceux obtenus dans (*Ann. Probab.* **32** (2004) 78–99), où il est démontré que la valeur propre principale généralisée λ_2 de l'opérateur donne le taux de croissance *locale* quand elle est positive et qu'il y a extinction locale quand ce n'est pas le cas. Il est aisé de montrer que $\lambda_\infty \geq \lambda_2$, et nous discutons les cas $\lambda_\infty > \lambda_2$ et $\lambda_\infty = \lambda_2$.

Quand $\lambda_\infty = 0$, et sous certaines conditions portant sur β, nous obtenons une condition nécessaire et suffisante pour que la superdiffusion X s'éteigne faiblement. Nous montrons que l'intensité de branchement k affecte l'extinction faible; alors qu'il est connu que k n'affecte pas l'extinction faible *locale*. (Celle-ci dépendant uniquement du signe de λ_2 et est équivalente à l'extinction locale.)

[1]The research of this author is supported by NSFC (Grant No. 10971003, 11271030 and 11128101) and Specialized Research Fund for the Doctoral Program of Higher Education. Corresponding author.
[2]Research supported in part by a grant from the Simons Foundation (208236).

MSC: Primary 60J80; secondary 60G57; 60J45

Keywords: Superdiffusion; Superprocess; Measure-valued process; Gauge theorem; Kato class; Growth bound; Principal eigenvalue; h-transform; Weak extinction; Total mass

1. Introduction

1.1. *Model*

For any positive integer i and $\eta \in (0, 1]$, let $C^{i,\eta}(\mathbb{R}^d)$ denote the space of i times continuously differentiable functions with all their ith order derivatives belonging to $C^\eta(\mathbb{R}^d)$. (Here $C^\eta(\mathbb{R}^d)$ denotes the usual Hölder space.) For any $x \in \mathbb{R}^d$, we will use $\{\xi_t, \Pi_x, t \ge 0\}$ to denote the L-diffusion with $\Pi_x(\xi_0 = x) = 1$, where

$$L := \frac{1}{2}\nabla \cdot a\nabla + b \cdot \nabla \quad \text{on } \mathbb{R}^d,$$

and a, b satisfy the following

(1) the symmetric matrix $a = \{a_{i,j}\}$ satisfies

$$A_1|v|^2 \le \sum_{i,j=1}^{d} a_{i,j}(x)v_i v_j \le A_2|v|^2, \quad \text{for all } v \in \mathbb{R}^d \text{ and } x \in \mathbb{R}^d,$$

with some $A_1, A_2 > 0$, and $a_{i,j} \in C^{1,\eta}$, $i, j = 1, \dots, d$, for some η in $(0, 1]$;

(2) the coefficients b_i, $i = 1, \dots, d$, are measurable functions satisfying

$$\sum_{i=1}^{d}\bigl|b_i(x)\bigr| \le C\bigl(1 + |x|\bigr), \quad \text{for all } x \in \mathbb{R}^d,$$

with some $C > 0$;

(3) there exists a differentiable function $Q : \mathbb{R}^d \to \mathbb{R}$ such that $b = a\nabla Q$.

Remark 1.1. *Under* (1)–(3) *above, the diffusion process ξ is conservative on $\mathbb{R}^d$. That is,*

$$\Pi_x\bigl(\xi_t \in \mathbb{R}^d, \forall t > 0\bigr) = 1,$$

for all $x \in \mathbb{R}^d$; equivalently, the semigroup corresponding to ξ leaves the function $f \equiv 1$ invariant. For a proof, see, for instance, [32], *Theorem* 10.2.2. *It is well known that ξ has a transition density $p(t, x, y)$ with respect to the Lebesgue measure.*

Define

$$m(x) = \mathrm{e}^{2Q(x)}, \quad x \in \mathbb{R}^d. \tag{1.1}$$

Then ξ is an m-symmetric Markov process, that is, the semigroup of ξ in $L^2(\mathbb{R}^d, m(x)\,\mathrm{d}x)$ is symmetric in the sense that for any $t > 0$ and $f, g \in L^2(\mathbb{R}^d, m(x)\,\mathrm{d}x)$,

$$\int_{\mathbb{R}^d} f(x)\Pi_x g(\xi_t)m(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} g(x)\Pi_x f(\xi_t)m(x)\,\mathrm{d}x.$$

If $C_c^\infty(\mathbb{R}^d)$ denotes the space of infinitely differentiable functions with compact support, then the Dirichlet form $(\mathcal{E}, D(\mathcal{E}))$ of ξ in $L^2(\mathbb{R}^d, m(x)\,\mathrm{d}x)$ is the closure of the form given by

$$\mathcal{E}(u, v) = \frac{1}{2}\int_{\mathbb{R}^d}(\nabla u a\nabla v)\exp(2Q)\,\mathrm{d}x, \quad u, v \in C_c^\infty\bigl(\mathbb{R}^d\bigr).$$

For any measurable space $(E, \mathcal{B})$, we denote by $M(E)$ the set of all finite measures on $\mathcal{B}$, equipped with the weak topology. We denote by $\mathcal{M}$ the Borel σ-field on $M(E)$, and so $\mathcal{M}$ is generated by all the functions $f_B(\mu) = \mu(B)$ with $B \in \mathcal{B}$. The space of finite measures with compact support will be denoted by $M_c(E)$. The expression $\langle f, \mu \rangle$ stands for the integral of f with respect to μ.

With β belonging to a certain Kato class (see Definition 1.2) and k being locally bounded from above and nonnegative, we will define the fundamental quantity λ_2 in (1.4) and show that $\lambda_2 < \infty$. We will use $(\{X_t\}_{t \geq 0}; \mathbb{P}_\mu, \mu \in M(\mathbb{R}^d))$ to denote the *superprocess* (a measure-valued Markov process) with $\mathbb{P}_\mu(X_0 = \mu) = 1$, corresponding to the semilinear elliptic operator $\mathcal{A}(u) := Lu + \beta u - ku^2$ on $\mathbb{R}^d$. For the precise definition, see Definition 1.3 below. As we will see in Theorem 1.3, the superprocess is well defined.

1.2. *Motivation*

The main purpose of this paper is to complement the results obtained in [11]. In particular, we study the growth/decay rate of the total mass of X and weak extinction[3] of X. Whereas in [11], the local behavior of the mass has been shown to be intimately related to the generalized principal eigenvalue λ_2 corresponding to the semigroup, here we will show that the global behavior of the mass is linked to another important quantity λ_∞, the L^∞-growth bound for the semigroup.

1.3. *Known results*

We first recall some definitions from Engländer and Kyprianou [11].

Definition 1.1. *Fix a nonzero $\mu \in M(\mathbb{R}^d)$ with compact support.*

(i) *We say that X exhibits local extinction under $\mathbb{P}_\mu$ if for every bounded Borel set $B \subset \mathbb{R}^d$, there exists a random time τ_B such that*

$$\mathbb{P}_\mu(\tau_B < \infty) = 1 \quad \textit{and} \quad \mathbb{P}_\mu\big(X_t(B) = 0 \textit{ for all } t \geq \tau_B\big) = 1.$$

(ii) *We say that X exhibits weak local extinction under $\mathbb{P}_\mu$ if for every bounded Borel set $B \subset \mathbb{R}^d$, $\mathbb{P}_\mu(\lim_{t\to\infty} X_t(B) = 0) = 1$.*

(iii) *We say that X exhibits extinction under $\mathbb{P}_\mu$ if there exists a stopping time τ such that*

$$\mathbb{P}_\mu(\tau < \infty) = 1 \quad \textit{and} \quad \mathbb{P}_\mu\big(X_t(\mathbb{R}^d) = 0 \textit{ for all } t \geq \tau\big) = 1.$$

(iv) *We say that X exhibits weak extinction under $\mathbb{P}_\mu$ if $\mathbb{P}_\mu(\lim_{t\to\infty} X_t(\mathbb{R}^d) = 0) = 1$.*

Let λ_2 be the growth bound of the semigroup in $L^2(\mathbb{R}^d, m)$ corresponding to the operator $L + \beta$ (see (1.4) and (1.5)). In [27], Pinsky gave a criterion for the local extinction of X under the assumption that β is Hölder continuous, namely, he proved that X exhibits local extinction if and only if $\lambda_2 \leq 0$. In particular, local extinction does not depend on the starting measure μ or the branching intensity k, but it does depend on L and β. (Note that, in regions where $\beta > 0$, β can be considered as mass creation, whereas in regions where $\beta < 0$, β can be considered as mass annihilation.) Since local extinction depends on the sign of λ_2, therefore, heuristically, it depends on the competition between the outward speed of particles and the mass creation. The main tools of [27] are PDE techniques.

In [11], Engländer and Kyprianou presented probabilistic (martingale and spine) arguments for the fact that $\lambda_2 \leq 0$ implies weak local extinction under $\mathbb{P}_\mu$ for any $\mu \in M(\mathbb{R}^d)$ with compact support, while $\lambda_2 > 0$ implies that, for any $\lambda < \lambda_2$ and any nonempty relatively compact open set B,

$$\mathbb{P}_\mu\Big(\limsup_{t\to\infty} \mathrm{e}^{-\lambda t} X_t(B) = \infty\Big) > 0$$

holds for any nonzero initial measure μ.

[3] Some authors prefer to say that X ‘extinguishes.’

Putting things together, one concludes that in this case *local extinction is in fact equivalent to weak local extinction* and there is a dichotomy in the sense that the process either exhibits local extinction (when $\lambda_2 \le 0$), or there is local exponential growth with positive probability (when $\lambda_2 > 0$).

We will see that, on the other hand, extinction and weak extinction are different in general. The intuition behind this is that the total mass $\|X_t\|$ may stay positive but decay to zero, *while drifting out* (local extinction) and on its way obeying changing branching laws. (For a concrete example see Example 2.3.) This could not be achieved in a fixed compact region with fixed branching coefficients.

Hence, weak extinction without extinction contrasts with the case without spatial motion (continuous state branching process), where such a phenomenon requires a branching mechanism which does not satisfy the 'Grey property' [18].

In [11] branching diffusions were studied besides superdiffusions, by using spine and martingale methods. (Note that for branching diffusions, weak (local) extinction and (local) extinction are obviously the same, because the local/total mass is an integer.) The main results concerned local extinction and local growth, and it was already noted that the growth rate of the total mass may exceed λ_2 (see [11], Remark 4).

1.4. *Our main results*

It is important to point out that weak extinction, unlike local extinction, depends on the branching intensity k as well (see the $\lambda_\infty = 0$ case below). We will prove that the exponential growth rate of the total mass is λ_∞, defined by (1.8). More precisely, there are three cases:

1. If mass creation is large enough so that $\lambda_\infty > 0$, then the total mass of X tends to infinity exponentially with rate $\lambda_\infty > 0$, with positive probability. (Note that extinction always has a positive probability.)
2. If annihilation is strong enough so that $\lambda_\infty < 0$, then the total mass of X tends to zero exponentially with rate $\lambda_\infty < 0$, a.s., even under survival. (See Example 2.3 for a super-Brownian motion, where $\lambda_\infty < 0$, but the process survives with positive probability. Interestingly, as we will see in that example, having a small k term makes extinction avoidable, while it cannot prevent weak extinction.)
3. If $\lambda_\infty = 0$, then weak extinction depends on k.

Concerning the third case, under some further conditions on β, we will give a necessary and sufficient condition for X to exhibit weak extinction (see Remark 1.13).

Applying our findings to the super-Brownian ($L = \frac{1}{2}\Delta$) case will yield some interesting results; see Section 2.3.

In all the work mentioned above, β is assumed to be Hölder continuous. In this paper, we relax this condition by using results of [2,4,16,17,34] on Schrödinger operators. The results of this paper are new even under the assumption that β is Hölder-continuous. Furthermore, even under the Hölder-continuity assumption, the arguments of this paper can not be simplified by much.

Before we give the main results of this paper, let us introduce some definitions and notation.

Definition 1.2 (Kato class). *A measurable function q on $\mathbb{R}^d$ is said to be in the Kato class $\mathbf{K}(\xi)$ if*

$$\lim_{t\downarrow 0} \sup_{x\in\mathbb{R}^d} \Pi_x\left(\int_0^t \left|q(\xi_s)\right| \mathrm{d}s\right) = 0.$$

It is easy to see that any bounded function is in the Kato class $\mathbf{K}(\xi)$. For any $q \in \mathbf{K}(\xi)$, denote

$$e_q(t) := \exp\left(\int_0^t q(\xi_u)\,\mathrm{d}u\right), \tag{1.2}$$

and define

$$e_q(\infty) := \exp\left(\int_0^\infty q(\xi_s)\,\mathrm{d}s\right), \tag{1.3}$$

whenever the integral on the righthand side makes sense.

Assumption 1.1. *In the remainder of this article, we will always assume that* $\beta \in \mathbf{K}(\xi)$.

One may define a semigroup $\{P_t^\beta\}_{t\geq 0}$ on $L^p(\mathbb{R}^d, m)$, for any $p \in [1,\infty]$, by

$$P_t^\beta f(x) := \Pi_x\big[e_\beta(t) f(\xi_t)\big].$$

For any $p \in [1,\infty]$, $\|\cdot\|_p$ stands for the norm in $L^p(\mathbb{R}^d, m)$, while $\|\cdot\|_{p,p}$ stands for the operator norm from $L^p(\mathbb{R}^d, m)$ to $L^p(\mathbb{R}^d, m)$. It follows from [5], Theorem 3.10, that, for any $t > 0$ and $p \in [1,\infty)$, $\|P_t^\beta\|_{p,p} \leq \|P_t^\beta\|_{\infty,\infty} \leq \mathrm{e}^{c_1 t + c_2}$ for some constants c_1, c_2, and that $\{P_t^\beta\}_{t\geq 0}$ is a strongly continuous semigroup in $L^p(\mathbb{R}^d, m)$ for any $1 \leq p < \infty$. We define

$$\lambda_2(\beta) := \lim_{t\to\infty} \frac{1}{t} \log \big\| P_t^\beta \big\|_{2,2}. \tag{1.4}$$

Remark 1.2 (Probabilistic representation). *In fact, the following probabilistic characterization holds* (*see Appendix* B):

$$\lambda_2(\beta) = \sup_{A \subset\subset \mathbb{R}^d} \lim_{t\to\infty} \frac{1}{t} \log \sup_{x\in A} \Pi_x\big(e_\beta(t); \tau_A > t\big). \tag{1.5}$$

(*Here* $A \subset\subset \mathbb{R}^d$ *means that* A *is a bounded set in* $\mathbb{R}^d$.) *In particular,* $\lambda_2(0)$ *is the 'rate of escape from compacts' for the diffusion* ξ. *In general, when* β *is Hölder-continuous,* $\lambda_2(\beta)$ *coincides with the so-called generalized principal eigenvalue of* $L + \beta$ *defined in* [26]. *In our symmetric setting however, for such a* β, *the situation is even simpler*: $\lambda_2(\beta)$ *is the supremum of the* L^2*-spectrum for the self-adjoint realization of the symmetric operator* $L + \beta$ *on* $\mathbb{R}^d$, *obtained via the Friedrichs extension theorem.* (*See* [26], *Chapter* 4, *especially Proposition* 4.10.1 *there, for more explanation.*)

Now we recall the definition of an (L, β, k)-superprocess. For background material on superprocesses, see [6,8–10, 22].

Definition 1.3 (***(L, β, k)-superprocess***). *An* (L, β, k)*-superprocess is a measure-valued Markov process* $(\{X_t\}_{t\geq 0}; \mathbb{P}_\mu, \mu \in M(\mathbb{R}^d))$ *such that* $\mathbb{P}_\mu(X_0 = \mu) = 1$, *and for any bounded Borel* $f \geq 0$ *on* $\mathbb{R}^d$, *one has*

$$\mathbb{P}_\mu \exp\langle -f, X_t\rangle = \exp\big\langle -u(t,\cdot), \mu\big\rangle, \tag{1.6}$$

where u *is the minimal nonnegative solution to*

$$u(t,x) + \Pi_x \int_0^t k(\xi_s)\big(u(t-s,\xi_s)\big)^2 \,\mathrm{d}s - \Pi_x \int_0^t \beta(\xi_s) u(t-s,\xi_s)\,\mathrm{d}s = \Pi_x f(\xi_t). \tag{1.7}$$

We will also say that $(\{X_t\}_{t\geq 0}; \mathbb{P}_\mu, \mu \in M(\mathbb{R}^d))$ is the superprocess 'corresponding to the semi-linear elliptic operator $\mathcal{A}(u) := Lu + \beta u - ku^2$ on $\mathbb{R}^d$.'

Theorem 1.3 (Existence). *Suppose that* $\beta \in \mathbf{K}(\xi)$ *and* $k \geq 0$ *is locally bounded. Then the* (L, β, k)*-superprocess exists.*

Remark 1.4 (Minimality and uniqueness). *Under our general condition on* k, *we do not claim the uniqueness of the solution to the cumulant equation* (1.7). *In the Appendix* A, *we will construct a minimal solution instead. If, however,* $k \in \mathbf{K}(\xi)$ *holds as well, then the solution is unique, see Remark* A.1.

Right after the construction of the superprocess, one of course would like to know what regularity properties of the paths one can assume.

Theorem 1.5 (Path regularity). *Assume that $\beta \in \mathbf{K}(\xi)$ and is bounded from above, and $k \geq 0$ is locally bounded. Then the superprocess constructed in Theorem 1.3 has a version which possesses càdlàg paths (that is, right continuous paths with left limits, in the weak topology of measures).*

The proofs of Theorems 1.3 and 1.5 are relegated to Appendix A.

Throughout this paper, the following assumption will be in force:

Assumption 1.2 (Regularity assumption). *The superprocess X has càdlàg paths.*

Remark 1.6. *Note that, by Theorem 1.5, the condition that β is bounded from above is a sufficient condition for the existence of a regular version of X. What we need in the rest of this paper is the existence of a regular version of X. With Assumption 1.2 in force, we do not need to assume that β is bounded from above in the rest of this paper.*

Returning now to the analytic tools needed, another very important quantity besides λ_2, is given in the following definition.

Definition 1.4 (L^∞-growth bound). *Define*

$$\lambda_\infty(\beta) := \lim_{t\to\infty} \frac{1}{t} \log \| P_t^\beta \|_{\infty,\infty} = \lim_{t\to\infty} \frac{1}{t} \log \sup_{x\in\mathbb{R}^d} \Pi_x e_\beta(t). \tag{1.8}$$

We call $\lambda_\infty = \lambda_\infty(\beta)$ the L^∞-growth bound.

It follows from (1.5) and (1.8) that $\lambda_\infty(\beta) \geq \lambda_2(\beta)$. In fact, $\lambda_\infty(\beta) = \lambda_2(\beta)$ and $\lambda_\infty(\beta) > \lambda_2(\beta)$ are both possible. For conditions under which $\lambda_\infty(\beta) = \lambda_2(\beta)$, we refer to Chen [3], Section 4, and the references therein. We will give some examples of $\lambda_\infty(\beta) > \lambda_2(\beta)$ in Section 2.

For simplicity, we will write $\lambda_2(\beta)$ as λ_2, and $\lambda_\infty(\beta)$ as λ_∞ when the potential β is fixed.

The following notion is of fundamental importance.

Definition 1.5 (Gauge function). *For any $\beta \in \mathbf{K}(\xi)$, we define*

$$g_\beta(x) := \Pi_x\big(e_\beta(\infty)\big), \quad x \in \mathbb{R}^d, \tag{1.9}$$

when the right hand side is well defined. The function g_β, called the gauge function, is very useful in studying the potential theory of the Schrödinger-type operator $L + \beta$.

We are now ready to state the main results of this paper, the first of which treats the 'over-scaling' and 'under-scaling' of the total mass $\|X_t\| := \langle 1, X_t\rangle$.

Theorem 1.7 (Over- and under-scaling). *Let $\mu \in M(\mathbb{R}^d)$ be nonzero.*

(1) *For any $\lambda > \lambda_\infty$,*

$$\mathbb{P}_\mu\Big(\lim_{t\to\infty} \mathrm{e}^{-\lambda t}\|X_t\| = 0\Big) = 1. \tag{1.10}$$

In particular, if $\lambda_\infty < 0$, then X suffers weak extinction.

(2) *Assume that k is bounded. If $\lambda_\infty > 0$ and*

$$\liminf_{t\to\infty} \frac{\Pi_x e_\beta(t)}{\sup_{y\in\mathbb{R}^d} \Pi_y e_\beta(t)} > 0 \quad \text{for all } x \in \mathbb{R}^d \tag{1.11}$$

holds, then for any $\lambda < \lambda_\infty$,

$$\mathbb{P}_\mu\Big(\limsup_{t\to\infty} \mathrm{e}^{-\lambda t}\|X_t\| = \infty\Big) > 0. \tag{1.12}$$

Condition (1.11) is rather restrictive. It is certainly satisfied when β is a constant. Using Lemma 2.4 below, one can come up with many examples of nonconstant functions satisfying this condition.

The next two theorems give some insight as to what happens when the scaling of the total mass is exactly at λ_∞. Obviously, the conditions in the next two theorems are not optimal. We plan to establish more general versions of these two theorems in an upcoming paper.

Theorem 1.8 (Scaling at λ_∞). *Let $\mu \in M(\mathbb{R}^d)$ be nonzero.*

(1) *Assume that $\lambda_\infty > 0$ and that (1.11) holds. If*

$$\lim_{t\to\infty} \Pi_x e_{\beta-\lambda_\infty}(t) = \infty \quad \text{for all } x \in \mathbb{R}^d, \tag{1.13}$$

then

$$\mathbb{P}_\mu\Big(\limsup_{t\to\infty} \mathrm{e}^{-\lambda_\infty t}\|X_t\| = \infty\Big) > 0. \tag{1.14}$$

(2) *If $g_{\beta-\lambda_\infty}(x) \equiv 0$ in $\mathbb{R}^d$ and*

$$\sup_{x\in\mathbb{R}^d} \Pi_x\Big(\sup_{t\ge 0} e_{\beta-\lambda_\infty}(t)\Big) < \infty, \tag{1.15}$$

then

$$\mathbb{P}_\mu\Big(\liminf_{t\to\infty} \mathrm{e}^{-\lambda_\infty t}\|X_t\| = 0\Big) = 1. \tag{1.16}$$

If, in addition, $\beta \le 0$ on $\mathbb{R}^d$, then the superprocess suffers weak extinction.

Remark 1.9. *Assuming $g_{\beta-\lambda_\infty} \equiv \infty$ would automatically imply (1.13).*

Unlike in the previous two results, the next two involve the coefficient k as well.

The result below relates scaling and positive solutions (in the sense of distributions) of $(L+\beta-\lambda_\infty)h = 0$. Recall that a function h is a solution to $(L+\beta)h = 0$ in the sense of distributions if the generalized derivative ∇h is locally L^2-integrable with respect to $m(x)\,\mathrm{d}x$ and for any $\varphi \in C_c^\infty(\mathbb{R}^d)$,

$$\frac{1}{2}\int_{\mathbb{R}^d} (\nabla h a \nabla\varphi)\exp(2Q)\,\mathrm{d}x - \int_{\mathbb{R}^d} h(x)\varphi(x)\beta(x)\,\mathrm{d}x = 0.$$

Theorem 1.10. *Assume that there is a bounded solution $h > 0$ of $(L+\beta-\lambda_\infty)h = 0$ in $\mathbb{R}^d$ in the sense of distributions. If there exists an $x_0 \in \mathbb{R}^d$ such that*

$$\Pi_{x_0}\int_0^\infty e_{\beta-2\lambda_\infty}(s)k(\xi_s)\,\mathrm{d}s < \infty, \tag{1.17}$$

then $\lim_{t\to\infty} \mathrm{e}^{-\lambda_\infty t}\langle h, X_t\rangle$ exists $\mathbb{P}_\mu$-a.s. and in $L^2(\mathbb{P}_\mu)$, and $\mathbb{P}_\mu(\|X_t\| > 0, \forall t > 0) > 0$ for all nonzero measures $\mu \in M_c(\mathbb{R}^d)$. If, in addition, h satisfies that

$$\inf_{x\in\mathbb{R}^d} h(x) > 0, \tag{1.18}$$

then the scaling at λ_∞ is the correct one in the sense that for every nonzero $\mu \in M_c(\mathbb{R}^d)$,

$$\mathbb{P}_\mu\Big(\limsup_{t\to\infty} \mathrm{e}^{-\lambda_\infty t}\|X_t\| < \infty\Big) = 1 \tag{1.19}$$

and

$$\mathbb{P}_\mu\Big(\liminf_{t\to\infty} \mathrm{e}^{-\lambda_\infty t}\|X_t\| > 0\Big) > 0. \tag{1.20}$$

Remark 1.11 (On condition (1.17)). *Assume that the coefficients are smooth and $h > 0$ is a strong solution in Theorem 1.10. From the fact that the operator $(L + \beta - \lambda_\infty)^h$ defined by*

$$(L + \beta - \lambda_\infty)^h u(x) = \frac{1}{h(x)}(L + \beta - \lambda_\infty)(uh)(x)$$

has no potential (zeroth order) part, it follows that

$$\Pi_{x_0} e_\beta(s) h(\xi_s) \le \mathrm{e}^{\lambda_\infty s} h(x_0).$$

Thus, if $k \le Ch$, then

$$\Pi_{x_0} \int_0^\infty e_{\beta - 2\lambda_\infty}(s) k(\xi_s)\,\mathrm{d}s \le C_1 \int_0^\infty \mathrm{e}^{-\lambda_\infty s}\,\mathrm{d}s.$$

Consequently if $\lambda_\infty > 0$ and k/h is bounded from above (in particular, if $k \in C_c(\mathbb{R}^d)$), then condition (1.17) is automatically satisfied.

Similarly, if $f > 0$ solves $(L - \lambda_2(0)) f = 0$ (such a positive harmonic function always exists if L has smooth coefficients), then

$$\Pi_{x_0} f(\xi_s) \le \mathrm{e}^{\lambda_2(0)s} f(x_0).$$

Suppose now that $\beta \equiv B$, where B is an arbitrary constant. Since ξ is conservative, $\lambda_\infty = B$. So, if $k \le Cf$ (in particular, if $k \in C_c(\mathbb{R}^d)$), then

$$\Pi_{x_0} \int_0^\infty e_{\beta - 2\lambda_\infty}(s) k(\xi_s)\,\mathrm{d}s \le C_1 \int_0^\infty \mathrm{e}^{(-B + \lambda_2(0))s}\,\mathrm{d}s. \tag{1.21}$$

If $B > 0$, then the integral on the righthand side of (1.21) is always finite (since $\lambda_2(0) \le 0$), and so condition (1.17) is automatically satisfied.

If $B \le 0$, it is still satisfied as long as $|B| < |\lambda_2(0)|$, that is, when the motion is sufficiently transient. To give a concrete example, consider an 'outward' Ornstein–Uhlenbeck process, with parameter $\gamma > 0$, corresponding to the operator

$$L = \frac{1}{2}\Delta + \gamma x \cdot \nabla \quad \text{on } \mathbb{R}^d.$$

Since $\lambda_2 = -\gamma d$, what we need is $0 < B + \gamma d$.

We now present a partial converse to Theorem 1.10. To state this result, we need to introduce another function class. We note that the Kato class $\mathbf{K}$ introduced in Definition 1.2 was defined by a local condition, while the class $\mathbf{K}_\infty$ introduced below is defined by a *global* condition.

Definition 1.6 (The class $\mathbf{K}_\infty(\xi)$). *Assume that ξ is transient. A function $q \in \mathbf{K}(\xi)$ is said to be in the class $\mathbf{K}_\infty(\xi)$ if for any $\varepsilon > 0$ there exist a compact set K and a constant $\delta > 0$ such that for any subset A of K with $m(A) < \delta$,*

$$\sup_{x \in \mathbb{R}^d} \int_{(\mathbb{R}^d \setminus K) \cup A} \widetilde{G}(x, y) \big| q(y) \big| m(y)\,\mathrm{d}y < \varepsilon, \tag{1.22}$$

where m is the function defined in (1.1) and $\widetilde{G}(x, y)$ is the Green function corresponding to ξ with respect to $m(x)\,\mathrm{d}x$ in $\mathbb{R}^d$.

The class $\mathbf{K}_\infty(\xi)$ was first introduced in [2,4]. When ξ is transient and $\beta \in \mathbf{K}_\infty(\xi)$, we have $\lambda_\infty \ge 0$. In fact, it follows from [4], Proposition 2.1, that $\Pi_x(\int_0^\infty |\beta|(\xi_s)\,\mathrm{d}s)$ is bounded in $\mathbb{R}^d$. Let M be the upper bound. By Jensen's inequality, we have

$$\Pi_x e_\beta(t) \ge \exp\left(-\Pi_x \int_0^\infty |\beta|(\xi_s)\,\mathrm{d}s\right) \ge \mathrm{e}^{-M}, \tag{1.23}$$

which implies that

$$\frac{1}{t}\log \sup_{x\in\mathbb{R}^d} \Pi_x e_\beta(t) \geq -M/t.$$

Thus by definition,

$$\lambda_\infty = \lim_{t\to\infty} \sup_{x\in\mathbb{R}^d} \frac{1}{t}\log \Pi_x e_\beta(t) \geq 0.$$

Note that (1.23) implies that $g_\beta \geq \mathrm{e}^{-M}$. It follows from the gauge theorem (see [4], Theorem 2.2, or [2], Theorem 2.6) that, if ξ is transient and $\beta \in \mathbf{K}_\infty(\xi)$, then g_β is either bounded or identically infinite. It follows from [2], Corollary 2.9, that the boundedness of g_β implies that $\sup_{x\in\mathbb{R}^d} \Pi_x(\sup_{t\geq 0} e_\beta(t)) < \infty$ for every $x \in \mathbb{R}^d$, and hence $\lambda_\infty(\beta) = 0$.

Recall that a function f on $\mathbb{R}^d$ is said to be *radial* if there exists some function $\widetilde{f}$ on $[0,\infty)$ such that $f(x) = \widetilde{f}(|x|)$ for all $x \in \mathbb{R}^d$.

Theorem 1.12 (Weak extinction in the radial case). *Assume that k and β are radial functions, and L is radial (i.e., $a_{i,j}$, $i, j = 1, 2, \ldots, d$, and Q are radial functions). Assume that ξ is transient, $\beta \in \mathbf{K}_\infty(\xi)$, and that $g_\beta(x)$ is not identically infinite (which implies that g_β is bounded and hence $\lambda_\infty = 0$). If*

$$\Pi_x\left[\int_0^\infty e_\beta(s)k(\xi_s)\,\mathrm{d}s\right] = \infty \quad \text{for all } x \in \mathbb{R}^d, \tag{1.24}$$

then for every $\mu \in M(\mathbb{R}^d)$,

$$\mathbb{P}_\mu\Big(\lim_{t\to\infty} \|X_t\| = 0\Big) = 1. \tag{1.25}$$

Remark 1.13. *In particular, if ξ is transient, $\beta \in \mathbf{K}_\infty(\xi)$ and g_β is not identically infinite, then g_β is a solution of $(L+\beta)u = 0$ in the distribution sense, and is bounded between two positive numbers (see the paragraphs after (1.23)). In this case, Theorem 1.10 and Theorem 1.12 imply that condition (1.24) is a necessary and sufficient condition for X to exhibit weak extinction.*

In Section 2 we will give some examples for which the conditions of our theorems are satisfied. The assumption that k, β, L are radial in Theorem 1.12 is rather restrictive. We expect that an appropriate version of Theorem 1.12 will be valid in the nonradial case too; we plan to address this problem in an upcoming project.

1.5. *Outline*

The rest of the paper is organized as follows. In the next section we illustrate our results with examples. In the two sections following the examples, we provide the proofs. Those proofs utilize some known results from Gauge Theory, as well as probabilistic techniques. We presume that the probabilistic audience likely to read this article would prefer to see the (largely probabilistic) proofs of the results without first being halted by a lengthy read about the technicalities of Gauge Theory. Therefore, in order to make the material presented easier to digest, we relegate those technical lemmas into Appendix B. In the same vein, to make the paper less overwhelmed by technical details at the beginning, we defer the proof of path regularity to Appendix A. The reader may consider, of course, to read the appendices right after reading the main results.

2. Examples

2.1. *Some superdiffusions with $\lambda_\infty > \lambda_2$*

We start with an example in one dimension and with constant mass creation.

Example 2.1. *Consider the elliptic operator*

$$L = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} - b_0\frac{\mathrm{d}}{\mathrm{d}x}$$

on $\mathbb{R}$, *where* $b_0 > 0$ *is a constant. Then the diffusion corresponding to* L *is conservative and transient. It is easy to see that the corresponding generalized principal eigenvalue is* $\lambda_2(0) = -b_0^2/2$. *Let the potential* β *be a nonnegative constant. We have* $\lambda_2(\beta) = \beta - b_0^2/2$ *and* $\lambda_\infty(\beta) = \beta$. *The Green function of* ξ *is* $G(x, y) = \frac{2\pi}{b_0}\exp(-2b_0(x-y)^+)$. *Note that* $L - \beta + \lambda_\infty(\beta) = L$.

For the large time behavior of X *the following hold.*

(i) *According to* [27], *Theorem* 7 *and Example* 1, X *exhibits local extinction if and only if* $\beta \in [0, b_0^2/2]$. *Furthermore, when* $\beta \in (b_0^2/2, \infty)$, X *does not exhibit local extinction, and the exponential expected growth rate of the local mass is* $(\beta - b_0^2/2)$. *More precisely, for any continuous function* g *on* $\mathbb{R}$ *with compact support and any nonzero* $\mu \in M_c(\mathbb{R})$, *one has*

$$\lim_{t\to\infty} \mathrm{e}^{\rho t}\mathbb{P}_\mu\langle g, X_t\rangle = \begin{cases} 0, & \varrho \le -(\beta - b_0^2/2), \\ +\infty, & \varrho > -(\beta - b_0^2/2). \end{cases}$$

In fact, by [11], *the local mass grows exponentially with positive probability, that is, not just in expectation.*

(ii) *If* $\beta > 0$, *since* $\Pi_x e_\beta(t) = \mathrm{e}^{\beta t}$ *for all* $x \in \mathbb{R}$ *and* $t \ge 0$, (1.11) *is satisfied. Thus by Theorem* 1.7, *we have that, for any* $\lambda > \beta$,

$$\mathbb{P}_\mu\Big(\liminf_{t\to\infty} \mathrm{e}^{-\lambda t}\|X_t\| = 0\Big) = 1,$$

and that if k *is bounded, then, for any* $\lambda < \beta$,

$$\mathbb{P}_\mu\Big(\limsup_{t\to\infty} \mathrm{e}^{-\lambda t}\|X_t\| = \infty\Big) > 0.$$

(iii) *Since* $u \equiv 1$ *solves* $Lu = 0$, *by Theorem* 1.10, *if there exists an* $x_0 \in \mathbb{R}$ *such that*

$$\Pi_{x_0}\int_0^\infty \mathrm{e}^{-\beta s}k(\xi_s)\,\mathrm{d}s < \infty, \tag{2.1}$$

then for any nonzero $\mu \in M_c(\mathbb{R}^d)$, *the limit* $\lim_{t\to\infty}\exp(-\beta t)\|X_t\|$ *exists* $\mathbb{P}_\mu$*-a.s. and in* $L^2(\mathbb{P}_\mu)$, *and*

$$0 < \mathbb{P}_\mu\Big(\Big[\lim_{t\to\infty}\exp(-\beta t)\|X_t\|\Big]^2\Big) < \infty.$$

Hence,

$$\mathbb{P}_\mu\Big(\lim_{t\to\infty}\exp(-\beta t)\|X_t\| = 0\Big) < 1,$$

and

$$\mathbb{P}_\mu\Big(\lim_{t\to\infty}\exp(-\beta t)\|X_t\| = \infty\Big) = 0.$$

(iv) *Since* L *is radial, by Theorem* 1.12 *we have that in the case of critical branching* ($\beta = 0$), *if*

$$\int_{-\infty}^x \exp\big(-b_0(x-y)\big)k(y)\,\mathrm{d}y + \int_x^\infty k(y)\,\mathrm{d}y = \infty, \quad x \in \mathbb{R}, \tag{2.2}$$

then

$$\mathbb{P}_\mu\Big(\lim_{t\to\infty}\|X_t\| = 0\Big) = 1.$$

In summary,

(a) *If* $\beta > 0$, *the exponential growth rate of the total mass is* β.
(b) *If* $\beta = 0$, *weak extinction depends on the branching rate function* k: *the superprocess exhibits weak extinction if and only if* (2.2) *holds*.

In the next example the motion component is a multidimensional 'outward Ornstein–Uhlenbeck' process.

Example 2.2. *Consider the elliptic operator*

$$L = \frac{1}{2}\Delta + \gamma x \cdot \nabla \quad \text{on } \mathbb{R}^d,$$

where $d \geq 1$ *and* $\gamma > 0$. *Then the diffusion corresponding to* L *is conservative and transient*, *and* $\lambda_2(0) = -\gamma d$. *Let the potential* β *be a positive constant*. *Then* $\lambda_2(\beta) = \beta - \gamma d$, *and* $\lambda_\infty(\beta) = \beta$.

(i) X *exhibits local extinction if and only if* $\beta \in [0, \gamma d]$. *If* $\beta \in (\gamma d, \infty)$, *then* X *does not exhibit local extinction*, *and the exponential growth rate of the local mass is* $\beta - \gamma d$. *More precisely*, *for any continuous function* g *on* $\mathbb{R}^d$ *with compact support*,

$$\lim_{t\to\infty} \mathrm{e}^{(\beta-\gamma d)t}\langle g, X_t\rangle = N_\mu \int_{\mathbb{R}^d} g(x)\exp\bigl(-\gamma|x|^2/2\bigr)\,\mathrm{d}x, \quad \text{in } \mathbb{P}_\mu\text{-probability}$$

for some random variable N_μ *with mean* $\int_{\mathbb{R}^d}\exp(-\gamma|x|^2/2)\mu(\mathrm{d}x)$, *whenever there exists a* $K > 0$ *such that*

$$k(x) \leq K\exp\bigl(\gamma|x|^2/2\bigr), \quad \text{for all } x \in \mathbb{R}^d,$$

and the starting measure $\mu = X_0$ *satisfies*

$$\int_{\mathbb{R}^d}\exp\bigl(-\gamma|x|^2/2\bigr)\mu(\mathrm{d}x) < \infty.$$

See [14], *Theorem* 1, *and* [13], *Example* 23.

(ii) *By Theorem* 1.7, *we have that*, *for any* $\lambda > \beta$,

$$\mathbb{P}_\mu\Bigl(\liminf_{t\to\infty}\mathrm{e}^{-\lambda t}\|X_t\| = 0\Bigr) = 1,$$

and that if k *is bounded in* $\mathbb{R}^d$, *then*, *for any* $\lambda < \beta$,

$$\mathbb{P}_\mu\Bigl(\limsup_{t\to\infty}\mathrm{e}^{-\lambda t}\|X_t\| = \infty\Bigr) > 0.$$

(iii) *Obviously*, $u \equiv 1$ *is a bounded solution to* $Lu = 0$, *and by Theorem* 1.10 *and its proof*, *we have that if the branching rate* k *satisfies*

$$\Pi_x\int_0^\infty \mathrm{e}^{-\beta s}k(\xi_s)\,\mathrm{d}s < \infty, \quad x \in \mathbb{R}^d,$$

then for any nonzero $\mu \in M_c(\mathbb{R}^d)$, *there exists* $\lim_{t\to\infty}\exp(-\beta t)\|X_t\|$ $\mathbb{P}_\mu$-*a.s.*, *and*

$$\mathbb{P}_\mu\Bigl[\lim_{t\to\infty}\exp(-\beta t)\|X_t\|\Bigr]^2 \in (0,\infty).$$

Hence,

$$\mathbb{P}_\mu\Bigl(\lim_{t\to\infty}\exp(-\beta t)\|X_t\| = 0\Bigr) < 1,$$

and

$$\mathbb{P}_\mu\Bigl(\lim_{t\to\infty}\exp(-\beta t)\|X_t\| = \infty\Bigr) = 0.$$

2.2. *Extinction and weak extinction*

Next is an example illustrating the difference between extinction and weak extinction. The superprocess X below exhibits local extinction and also weak extinction, nevertheless, it survives with positive probability.

Example 2.3 (Weak and also local extinction, but survival). *Let $B, \varepsilon > 0$ and consider the super-Brownian motion in $\mathbb{R}$ with $\beta(x) \equiv -B$ and $k(x) = \exp[\mp\sqrt{2(B+\varepsilon)}x]$, that is, let X correspond to the semi-linear elliptic operator $\mathcal{A}$, where*

$$\mathcal{A}(u) := \frac{1}{2}\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} - Bu - \exp\big[\mp\sqrt{2(B+\varepsilon)}x\big]u^2.$$

By Theorem 1.7, X suffers weak extinction: for any $\delta > 0$,

$$\lim_{t\to 0} \mathrm{e}^{(B-\delta)t} \|X_t\| = 0.$$

Also, clearly, $\lambda_2 = -B$, yielding that X also exhibits local extinction.

Now we are going to show that, despite the above, the process X survives with positive probability, that is

$$\mathbb{P}_\mu\big(\|X_t\| > 0, \forall t > 0\big) > 0,$$

for any nonzero $\mu \in M(\mathbb{R}^d)$. In order to do this, we will use the definition and basic properties of h-transforms and weighted superprocesses. These can be found in Section 2 of [12].

The function $h(x) := \mathrm{e}^{\pm\sqrt{2(B+\varepsilon)}x}$ transforms the operator $\mathcal{A}$ into $\mathcal{A}^h$, where

$$\mathcal{A}^h(u) := \frac{1}{h}\mathcal{A}(hu) = \frac{1}{2}\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} \pm \sqrt{2(B+\varepsilon)}\frac{\mathrm{d}u}{\mathrm{d}x} + \varepsilon u - u^2.$$

(Note that $h''/2 - (B+\varepsilon)h = 0$.) The superprocess X^h corresponding to $\mathcal{A}^h$ is in fact the same as the original process X, weighted by the function h, and consequently, survival (with positive probability) is invariant under h-transforms. But X^h has a conservative motion component and constant branching mechanism, which is supercritical, and therefore X^h survives with positive probability; the same is then true for X.

2.3. *The super-Brownian motion case*

In this subsection we focus on the special case when the underlying motion process is a Brownian motion, that is, when $L = \Delta/2$; in the remainder of this section we will always assume that this is the case. In this case $\beta \in \mathbf{K}(\xi)$ if and only if

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^d} \int_{|y-x|<r} u(x-y)\big|\beta(y)\big|\,\mathrm{d}y = 0,$$

where u is the function defined by

$$u(x) := \begin{cases} |x|^{2-d}, & d \geq 3, \\ \log|x|^{-1}, & d = 2, \\ |x|, & d = 1. \end{cases} \tag{2.3}$$

When $d \geq 3$, $\mathbf{K}_\infty(\xi)$ coincides with the class $\mathbf{K}_d^\infty$ defined in [34]. We recall the definition of the class $\mathbf{K}_d^\infty$ defined in [16,17] in the case $d \leq 2$.

Definition 2.1 (The classes $\mathbf{K}_1^\infty(\xi)$ and $\mathbf{K}_2^\infty(\xi)$). *Let $L = \Delta/2$.*

(1) *If $d = 1$, a function $q \in \mathbf{K}(\xi)$ is said to be in the class $\mathbf{K}_1^\infty(\xi)$ if*

$$\int_{|y|\geq 1} \left|yq(y)\right| \mathrm{d}y < \infty.$$

(2) *If $d = 2$, a function $q \in \mathbf{K}(\xi)$ is said to be in the class $\mathbf{K}_2^\infty(\xi)$ if*

$$\int_{|y|\geq 1} \log\bigl(|y|\bigr)\left|q(y)\right| \mathrm{d}y < \infty.$$

2.3.1. *The $d \geq 3$ case*

Recall that we have proved, in the paragraph below Definition 1.6, that for any $\beta \in \mathbf{K}_\infty(\xi)$ we have $\lambda_\infty(\beta) \geq 0$. The following definition is from [29].

Definition 2.2 (Criticality in terms of λ_∞). *Let $L = \Delta/2$ and $\beta \in \mathbf{K}_\infty(\xi)$. Then β is said to be*

(a) *supercritical iff $\lambda_\infty(\beta) > 0$,*
(b) *critical iff $\lambda_\infty(\beta) = 0$ and for any nontrivial nonnegative continuous function q of compact support, $\lambda_\infty(\beta + q) > 0$.*
(c) *subcritical iff it neither supercritical nor critical.*

Note. *The reader should not confuse the above properties of the function β with the (local) criticality (or sub- or supercriticality) of the branching, which simply refer to the sign of β (in certain regions).*

The following result relates the above definition to the solutions of

$$(L + \beta)u = 0, \tag{2.4}$$

and is due to [34].

Lemma 2.1. *Let $L = \Delta/2$, $\beta \in \mathbf{K}_\infty(\xi)$ and $d \geq 3$. Then the following conditions are equivalent:*

(a) *β is subcritical.*
(b) *$g_\beta(x) \equiv \Pi_x e_\beta(\infty)$ is bounded in $\mathbb{R}^d$.*
(c) *There exists a solution u to (2.4) with $\inf_{x\in\mathbb{R}^d} u(x) > 0$.*
(d) *There exists a solution u to (2.4) with $0 < \inf_{x\in\mathbb{R}^d} u(x) \leq \sup_{x\in\mathbb{R}^d} u(x) < \infty$.*

Moreover, if β is subcritical, then (2.4) has a unique (up to constant multiples) positive bounded solution and the solution must be of the form $cg_\beta(x)$ for some $c > 0$.

However, if β is critical, then there is no positive solution to (2.4) which is bounded away from zero. Pinchover [25] proved the following result (see [25], Lemma 2.7).

Lemma 2.2. *Let $L = \Delta/2$, $\beta \in \mathbf{K}_\infty(\xi)$ and $d \geq 3$. If β is critical, then there is an $h > 0$ satisfying (2.4) on $\mathbb{R}^d$ and such that*

$$h \sim c_d|x|^{2-d}, \quad \text{as } |x| \to \infty, \tag{2.5}$$

where c_d is a positive constant depending only on d.

It is easy to check that, for any $p > d/2$, $\beta \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ implies that $\beta \in \mathbf{K}_\infty(\xi)$. In this special case, the following result shows that h can be obtained as a large time asymptotic limit of the Schrödinger semigroup (see [29], Theorem 3.1).

Lemma 2.3. *Let $L = \Delta/2$, $\beta \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ and $d \geq 3$. If β is critical, then*

$$\lim_{t\to\infty} f(t)^{-1} \sup_{x\in\mathbb{R}^d} \Pi_x\big[e_\beta(t)\big] = C, \tag{2.6}$$

and

$$\lim_{t\to\infty} f(t)^{-1} \Pi_x\big[e_\beta(t)\big] = h(x), \quad \forall x \in \mathbb{R}^d, \tag{2.7}$$

where C is a positive constant, $h > 0$ is bounded and solves (2.4) (general theory implies, in the critical case, the existence of such a solution) and

$$f(t) = \begin{cases} t, & d \geq 5, \\ t/(\log t), & d = 4, \\ t^{1/2} & d = 3. \end{cases} \tag{2.8}$$

Lemma 2.4. *Let $L = \Delta/2$ and $d \geq 3$. If $\lambda_\infty(\beta) > 0$ and $\beta - \lambda_\infty \in L^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$, then conditions (1.11) and (1.13) are satisfied.*

Proof. Note that

$$g_\beta(t) = \sup_{x\in\mathbb{R}^d} \Pi_x e_\beta(t) = \mathrm{e}^{\lambda_\infty t} \sup_{x\in\mathbb{R}^d} \Pi_x e_{\beta-\lambda_\infty}(t).$$

By Lemma 2.3 we have

$$g_\beta(t) \sim C\mathrm{e}^{\lambda_\infty t} f(t), \quad \text{as } t \to \infty$$

with $f(t)$ defined by (2.8), and

$$\lim_{t\to\infty} g_\beta^{-1}(t)\Pi_x e_\beta(t) = \frac{1}{C} \lim_{t\to\infty} f^{-1}(t)\Pi_x e_{\beta-\lambda_\infty}(t) > 0,$$

which means that conditions (1.11) and (1.13) are satisfied. □

This subsection shows that there are many examples of β satisfying the conditions of Theorems 1.7–1.8(1).

2.3.2. *The $d \leq 2$ case*

The purpose of this subsection is to show that the assumptions of Theorem 1.8(2) are satisfied for some super-Brownian motions in $\mathbb{R}^d$ with $d \leq 2$.

The following lemma is due to [16,17].

Lemma 2.5. *Let $d \leq 2$, $L = \Delta/2$ and $\beta \in \mathbf{K}_d^\infty(\xi)$. The following conditions are equivalent.*

(a) *β is critical.*
(b) *There exists a positive bounded solution to (2.4).*

Moreover, if β is critical, then the positive bounded solution h to (2.4) is unique (up to constant multiples), and h possesses the following representation:

$$h(x) = \begin{cases} h(0)\lim_{r\downarrow 0} \Pi_x e_\beta(T_{B(0,r)}), & d = 2, \\ h(0)\Pi_x e_\beta(T_0), & d = 1, \end{cases}$$

where for every open set B, $T_B = \inf\{t > 0; \xi_t \in B\}$ denotes the first hitting time of B, and $T_0 = T_{\{0\}}$ denotes the first hitting time of ξ at the point 0. Moreover, h is bounded away from zero.

It follows from the previous lemma that, in the case $d \le 2$, if $\lambda_\infty(\beta) > 0$, $\beta - \lambda_\infty(\beta) \in \mathbf{K}_d^\infty$ and $\beta - \lambda_\infty(\beta)$ is critical, then the assumption (1.18) of Theorem 1.10 is satisfied.

Remark 2.6. *Let $d \le 2$ and $L = \Delta/2$. Murata proved the following result (see* [24], *Theorem* 4.1): *If $\beta \sim |x|^{-\rho}$ ($\rho > 4$) as $|x| \to \infty$ (obviously $\beta \in \mathbf{K}_d^\infty$) and β is subcritical, then there exists a positive solution h to* (2.4) *such that*

$$h(x) = \begin{cases} (2\pi)^{-1} \log \frac{|x|}{2} + \mathcal{O}(1), & \text{for } d = 2, \\ (2\pi^{1/4})^{-1}|x| + \mathcal{O}(1), & \text{for } d = 1, \end{cases}$$

as $|x| \to \infty$.

Thus if $d \le 2$, $L = \Delta/2$, $\beta - \lambda \in \mathbf{K}_d^\infty$ and $\beta - \lambda$ is subcritical, then there is no positive bounded solution to $(L + \beta - \lambda)h = 0$. In order to deal with the subcritical case, we need to develop some results on Schrödinger semigroups. We believe that these results are also of independent interest.

Lemma 2.7. *Let $d \le 2$, $L = \Delta/2$ and $\beta \in \mathbf{K}_d^\infty$. If $\lambda_\infty(\beta) = 0$, then*

$$\sup_{t \ge 0} \sup_{x \in \mathbb{R}^d} \Pi_x e_\beta(t) < \infty. \tag{2.9}$$

Proof. Since $\lambda_\infty(\beta) = 0$, β is either critical or subcritical. For the subcritical case we will prove a stronger result later, see Lemma 2.9 below. Thus, we now assume that β is critical. Then Lemma 2.5 asserts that there exists a bounded solution ψ to (2.4) such that $\psi > 0$ and $\sup_{x \in \mathbb{R}^d} \psi^{-1}(x) < \infty$. We then have

$$\begin{aligned} \Pi_x e_\beta(t) &= \Pi_x\big(e_\beta(t)\big(\psi^{-1}\psi\big)(\xi_t)\big) \\ &\le \Big(\sup_{x \in \mathbb{R}^d} \psi^{-1}(x)\Big)\Pi_x\big(e_\beta(t)\psi(\xi_t)\big) \\ &= \Big(\sup_{x \in \mathbb{R}^d} \psi^{-1}(x)\Big)\psi(x) \\ &\le \sup_{x \in \mathbb{R}^d} \psi(x) \Big/ \inf_{x \in \mathbb{R}^d} \psi(x) < \infty. \end{aligned}$$

This proves (2.9). □

Remark 2.8. *Murata (see* [24], *Corollary* 1.6) *proved the above result for $d = 2$ under the condition that $\beta \sim |x|^{-\rho}$ ($\rho > 4$) as $|x| \to \infty$, which implies that $\beta \in \mathbf{K}_2^\infty$. Our proof above goes along the line given in the proof of* [24], *Corollary* 1.6(ii).

If β is subcritical, we have the following stronger result.

Lemma 2.9. *Let $d \le 2$, $L = \Delta/2$ and $\beta \in \mathbf{K}_d^\infty$. If β is subcritical, then*

$$\sup_{x \in \mathbb{R}^d} \Pi_x \sup_{0 \le t \le \infty} e_\beta(t) < \infty. \tag{2.10}$$

Proof. We first prove the result for dimension $d = 2$. For $r > 0$ we denote the open ball of radius r with center at the origin and its open exterior by

$$B_r = \big\{x \in \mathbb{R}^d, |x| < r\big\}; \qquad B_r^* = \big\{x \in \mathbb{R}^d, |x| > r\big\}.$$

According to [17], Proposition 2.2, there exists an $r_0 > 0$ such that for all $r \ge r_0$ and $x \in B_r^*$,

$$\Pi_x e_{\beta^+}(\tau_{B_r^*}) \le 2, \qquad \mathrm{e}^{-1/2} \le \Pi_x e_\beta(\tau_{B_r^*}) \le 2. \tag{2.11}$$

Choose r_0 large enough such that $\mathrm{supp}(\mu) \subset B_{r_0}$. We fix two real numbers r and R with $R > r \geq r_0$. Since β is subcritical, by [16], Theorem 2.1,

$$\Pi_x e_\beta(\tau_{B_R}) < \infty, \quad \forall x \in B_R.$$

We define

$$S = \tau_{B_R} + \tau_{B_r^*} \circ \theta_{\tau_{B_R}}.$$

Put

$$S_0 = 0; \qquad S_n = S_{n-1} + S \circ \theta_{S_{n-1}}, \quad n \geq 1.$$

In particular, $S_1 = S$. For any $f \in C(\partial B_r)$, we define

$$(A_S f)(x) = \Pi_x\big(e_\beta(S) f(\xi_S)\big), \quad x \in \partial B_r.$$

Note that

$$A_S^n f(x) = \Pi_x\big[e_\beta(S_n) f(S_n)\big], \quad x \in \partial B_r.$$

The spectral radius of A_S is defined by

$$\widetilde{\lambda}(\beta) := \lim_{n\to\infty} \|A_S^n\|^{1/n}.$$

It follows from [17], Theorem 2.4, that $\widetilde{\lambda}(\beta) < 1$. Thus there exists $\delta > 0$ such that $\widetilde{\lambda}(\beta) + \delta < 1$, and sufficiently large n such that, $\|A_S^n\| \leq (\widetilde{\lambda}(q) + \delta)^n$. Therefore we have

$$\sum_{n=0}^{\infty} \sup_{x\in\mathbb{R}^d} \big|A_S^n 1(x)\big| = \sum_{n=0}^{\infty} \sup_{x\in\mathbb{R}^d} \Pi_x e_\beta(S_n) < \infty. \tag{2.12}$$

By the strong Markov property applied at τ_{B_R}, along with the simple fact that $\int_0^t e_{\beta^+}(s)\beta^+(s)\,\mathrm{d}s = e_{\beta^+}(t) - 1$, and finally by(2.11), we have

$$\begin{aligned}
\Pi_x \int_0^S e_\beta(t)\beta^+(t)\,\mathrm{d}t &= \Pi_x \int_0^{\tau_{B_R}} e_\beta(t)\beta^+(t)\,\mathrm{d}t + \Pi_x\bigg[\Pi_{\xi_{\tau_{B_R}}} \int_0^{\tau_{B_r^*}} e_\beta(t)\beta^+(t)\,\mathrm{d}t\bigg] \\
&\leq \Pi_x \int_0^{\tau_{B_R}} e_\beta(t)\beta^+(t)\,\mathrm{d}t + \Pi_x\bigg[\Pi_{\xi_{\tau_{B_R}}} \int_0^{\tau_{B_r^*}} e_{\beta^+}(t)\beta^+(t)\,\mathrm{d}t\bigg] \\
&= \Pi_x \int_0^{\tau_{B_R}} e_\beta(t)\beta^+(t)\,\mathrm{d}t + \Pi_x\big[\Pi_{\xi_{\tau_{B_R}}} e_{\beta^+}(\tau_{B_r^*})\big] - 1 \\
&\leq \Pi_x \int_0^{\tau_{B_R}} e_\beta(t)\beta^+(t)\,\mathrm{d}t + 1.
\end{aligned}$$

Let ξ^{B_R} denote the Brownian motion killed upon exiting B_R. Since β is subcritical, the function $x \to \Pi_x e_\beta(\tau_{B_R})$ is bounded on B_R. It follows from [2], Theorem 2.8, that

$$\sup_{x\in B_R} \Pi_x \int_0^{\tau_{B_R}} e_\beta(t)\beta^+(t)\,\mathrm{d}t < \infty.$$

Thus

$$C := \sup_{x\in\partial B_r} \Pi_x \int_0^S e_\beta(t)\beta^+(t)\,\mathrm{d}t < \infty. \tag{2.13}$$

By the strong Markov property, applied at S_n, and by (2.12), and (2.13), we have

$$\begin{aligned}\sup_{x\in\mathbb{R}^d} \Pi_x \int_0^\infty e_\beta(t)\beta^+(t)\,\mathrm{d}t &\le \sum_{n=0}^\infty \sup_{x\in\mathbb{R}^d} \Pi_x\left[\int_{S_n}^{S_{n+1}} e_\beta(t)\beta^+(t)\,\mathrm{d}t\right] \\ &= \sum_{n=0}^\infty \sup_{x\in\mathbb{R}^d} \Pi_x\left[e_\beta(S_n)\Pi_{\xi_{S_n}} \int_0^S e_\beta(t)\beta^+(t)\,\mathrm{d}t\right] \\ &\le C\sum_{n=0}^\infty \sup_{x\in\mathbb{R}^d} \Pi_x e_\beta(S_n) < \infty. \end{aligned} \tag{2.14}$$

Observe that

$$e_\beta(t) = 1 + \int_0^t e_\beta(s)\beta(s)\,\mathrm{d}s \le 1 + \int_0^t e_\beta(s)\beta^+(s)\,\mathrm{d}s,$$

and so

$$\sup_{0\le t\le\infty} e_\beta(t) \le 1 + \int_0^\infty e_\beta(s)\beta^+(s)\,\mathrm{d}s.$$

Using (2.14) we get (2.10) and we finish the proof for dimension $d = 2$.

Now let $d = 1$. Define

$$u(a,b) := \Pi_x e_\beta(T_b), \quad a, b \in \mathbb{R}^1,$$

where T_b is the first hitting time of ξ at the point b. By [16], Theorem 4.8, $u(a,b)u(b,a) < 1$ for any $a, b \in \mathbb{R}^1$. For any $x \in \mathbb{R}^1$, define

$$S_x = T_{x+1} + T_x \circ \theta_{T_{x+1}}.$$

Then

$$\Pi_x e_\beta(S_x) = u(x, x+1)u(x+1, x) < 1.$$

Repeating the above proof for $d = 2$ with S replaced by S_x we can similarly obtain (2.10) for $d = 1$. We omit the details. $\square$

Lemma 2.10. *Let $d \le 2$, $L = \Delta/2$ and $\beta \in \mathbf{K}_d^\infty$. If β is subcritical, then*

$$\lim_{t\to\infty} \Pi_x e_\beta(t) = \Pi_x e_\beta(\infty) \equiv 0 \quad \text{in } \mathbb{R}^d. \tag{2.15}$$

Proof. By (2.10) and by dominated convergence, it suffices to show

$$\Pi_x e_\beta(\infty) = 0, \quad \forall x \in \mathbb{R}^d. \tag{2.16}$$

We continue to use the notations in the proof of Lemma 2.9. We first prove (2.16) for dimension $d = 2$. Using the strong Markov property of ξ, applied at τ_{B_R}, and Fatou's lemma, we get

$$\begin{aligned}\Pi_0 e_\beta(\infty) &= \Pi_0\big[e_\beta(\xi_{\tau_{B_r}})\Pi_{\xi_{\tau_{B_r}}} e_\beta(\infty)\big] \\ &\le \Pi_0\Big[e_\beta(\tau_{B_r}) \lim_{n\to\infty}\big|\big(A_S^n\big)1(\xi_{\tau_{B_r}})\big|\Big] \\ &\le \big[\Pi_0 e_\beta(\tau_{B_r})\big] \lim_{n\to\infty}\big\|A_S^n\big\| \\ &\le \big[\Pi_0 e_\beta(\tau_{B_r})\big] \lim_{n\to\infty}\big(\widetilde{\lambda}(\beta)+\delta\big)^n = 0. \end{aligned}$$

Thus by Lemma B.7 in the Appendix B, $\Pi_x e_\beta(\infty) \equiv 0$ in $\mathbb{R}^2$.

Now let $d = 1$. For any $x \in \mathbb{R}$, let S_x be defined as in the proof of Lemma 2.9. By the strong Markov property of ξ applied at S_x, we have, for any $x \in \mathbb{R}$,

$$\Pi_x e_\beta(\infty) = \Pi_x e_\beta(S_x)\Pi_x e_\beta(\infty).$$

Since $\Pi_x e_\beta(S_x) = u(x, x+1)u(x+1, x) < 1$, the above equality yields $\Pi_x e_\beta(\infty) = 0$ for every $x \in \mathbb{R}$. □

It follows from the two results above that, if $d \leq 2$, $L = \Delta/2$, $\lambda_\infty(\beta) > 0$, $\beta - \lambda_\infty(\beta) \in \mathbf{K}_d^\infty$ and $\beta - \lambda_\infty(\beta)$ is subcritical, then the assumptions of Theorem 1.8(2) are satisfied.

2.4. *Compactly supported mass annihilation*

We conclude with two simple examples which satisfy the assumptions of Theorem 1.8(2). In both cases we consider compactly supported mass annihilation terms.

We start with a two-dimensional example.

Example 2.4 **($d = 2$; constant annihilation in a compact set).** *Let ξ be planar Brownian motion, and $\beta(x) := -\alpha \mathbf{1}_K(x)$ with $\alpha > 0$ being a constant and $K \subset \mathbb{R}^2$ a compact set with nonempty interior.*

Proposition 2.11. *In this case weak extinction holds, whatever k is.*

Remark 2.12. *The point is that our result is true for any k. Indeed, it is easy to show that extinction holds when k is bounded from below (even with $\beta \equiv 0$).*

Proof of Proposition 2.11. It is well known that β is subcritical (see, e.g., [24], Theorem 1.4). By [1], Corollary 2, as $t \to \infty$,

$$\Pi_x\left[\exp\left(\int_0^t \beta(\xi_s)\,\mathrm{d}s\right)\right] \sim c(\log t)^{-1},$$

where $0 < c = c(x, K, \alpha)$. Therefore, for any $x \in \mathbb{R}^2$, $\lambda_\infty(\beta) \geq \lim_{t\to\infty} \frac{1}{t} \log \Pi_x e_\beta(t) = 0$. It is obvious that $\lambda_\infty(\beta) \leq 0$. Then $\lambda_\infty = 0$ and $g_{\beta-\lambda_\infty}(x) \equiv 0$. Clearly, (1.15) holds since $\beta \leq 0$. Using again that $\beta \leq 0$, we are done by part (2) of Theorem 1.8. □

Finally, we discuss an example in one-dimension.

Example 2.5 **($d = 1$; compactly supported mass annihilation).** *Let ξ be a Brownian motion in $\mathbb{R}$, and $\beta \leq 0$ a continuous function on $\mathbb{R}$ with compact support.*

Proposition 2.13. *Again, weak extinction holds, whatever k is.*

Proof. It is well known that β is subcritical (see [28]). By [33],

$$\lim_{t\to\infty} t^{-1/2} \int_0^t \beta(\xi_s)\,\mathrm{d}s = \eta \int_{-\infty}^{\infty} \beta(x)\,\mathrm{d}x, \tag{2.17}$$

in distribution, where η is a random variable with $\eta < 0$ a.s. This, along with Jensen's inequality, implies that, abbreviating $a := \int_{-\infty}^{\infty} \beta(x)\,\mathrm{d}x$,

$$\liminf_{t\to\infty}\left[\Pi_x \exp\left(\int_0^t \beta(\xi_s)\,\mathrm{d}s\right)\right]^{t^{-1/2}} \geq \lim_{t\to\infty} \Pi_x \exp\left(t^{-1/2}\int_0^t \beta(\xi_s)\,\mathrm{d}s\right) = \Pi_x \exp(a\eta).$$

Hence,

$$\liminf_{t\to\infty} t^{-1/2} \log\left[\Pi_x \exp\left(\int_0^t \beta(\xi_s)\,\mathrm{d}s\right)\right] \geq \log \Pi_x \exp(a\eta).$$

Thus, for $f(t) := t^{-1}\log \Pi_x \exp(\int_0^t \beta(\xi_s)\,\mathrm{d}s)$, we have $\liminf_{t\to\infty} f(t) \ge 0$. But $\beta \le 0$ implies that $\limsup_{t\to\infty} f(t) \le 0$, and so $\lambda_\infty = \lim_{t\to\infty} f(t) = 0$. By (2.17) (or, by the recurrence of ξ), $g_{\beta-\lambda_\infty}(x) = \Pi_x \exp(\int_0^\infty \beta(\xi_s)\,\mathrm{d}s) \equiv 0$. Again, $\beta \le 0$ implies (1.15), and we finish as in the proof of Proposition 2.11. □

3. Proofs of Theorems 1.7 and 1.8

For any nonzero $\mu \in M(\mathbb{R}^d)$, define

$$\Pi_\mu = \int_D \Pi_x \mu(\mathrm{d}x). \tag{3.1}$$

The following result is [8], Lemma 1.5.

Lemma 3.1. *The equation* (1.7) *is equivalent to*

$$u(t,x) + \Pi_x \int_0^t e_\beta(s) k(\xi_s)\big(u(t-s,\xi_s)\big)^2\,\mathrm{d}s = \Pi_x\big(e_\beta(t) f(\xi_t)\big). \tag{3.2}$$

Moreover, u is the minimal nonnegative solution to (1.7) *if and only if u is the minimal nonnegative solution to* (3.2).

Combining (1.6) and (3.2), we get the following expectation and variance formulae: for any bounded nonnegative function f on $\mathbb{R}^d$ and any nonzero $\mu \in M(\mathbb{R}^d)$,

$$\mathbb{P}_\mu\langle f, X_t\rangle = \Pi_\mu\big(f(\xi_t) e_\beta(t)\big) \tag{3.3}$$

and

$$\mathrm{Var}_\mu\langle f, X_t\rangle = \Pi_\mu\bigg(\int_0^t e_\beta(s) k(\xi_s) 2\big[\Pi_{\xi_s} e_\beta(t-s) f(\xi_{t-s})\big]^2\,\mathrm{d}s\bigg), \tag{3.4}$$

where Var_μ stands for variance under $\mathbb{P}_\mu$.

Lemma 3.2. *If* $\lambda_\infty > 0$, *then*

$$\liminf_{t\to\infty} \|P_t^\beta 1\|_\infty^{-1} \int_0^t \|P_s^\beta 1\|_\infty\,\mathrm{d}s < \infty. \tag{3.5}$$

Proof. For convenience, we denote $\|P_t^\beta 1\|_\infty$ by $h(t)$ in this proof. Suppose that the statement is false. Then

$$\lim_{t\to\infty} \frac{\int_0^t h(s)\,\mathrm{d}s}{h(t)} = \infty,$$

and so for any $K > 0$, there exists $T_K > 0$ such that for $t > T_K$,

$$\frac{\int_0^t h(s)\,\mathrm{d}s}{h(t)} > K,$$

i.e.,

$$h(t) < \frac{1}{K}\int_0^t h(s)\,\mathrm{d}s = \alpha + \frac{1}{K}\int_{T_K}^t h(s)\,\mathrm{d}s,$$

where $\alpha = \frac{1}{K}\int_0^{T_K} h(s)\,\mathrm{d}s$. By Gronwall's lemma, we get

$$h(t) \le \alpha\big(\mathrm{e}^{(t-T_2)/K} - 1\big).$$

However, if $\frac{1}{K} < \frac{\lambda_\infty}{2}$ ($K > \frac{2}{\lambda_\infty}$), then this contradicts the following easy consequence of the definition (1.8) of λ_∞:

$$\varlimsup_{t\to\infty} \frac{\log h(t)}{t} \geq \frac{\lambda_\infty}{2}.$$

This contradiction proves the lemma. □

3.1. *Proof of Theorem* 1.7

For the proof of the theorem, we will need the following slight generalization of Doob's maximal inequality for submartingales.

Lemma 3.3. *Assume that* $T \in (0,\infty)$, *and that the nonnegative*, *right continuous*, *adapted process* $(\{M_t\}_{0\leq t\leq T}, \{\mathcal{F}_t\}_{0\leq t\leq T}, \mathbf{P})$ *satisfies that there exists an* $a > 0$ *such that*

$$\mathbf{P}(M_t \mid \mathcal{F}_s) \geq aM_s, \quad 0 \leq s < t \leq T.$$

Then, *for every* $\alpha \in (0,\infty)$ *and* $0 \leq S \leq T$,

$$\mathbf{P}\Big(\sup_{t\in[0,S]} M_t \geq \alpha\Big) \leq (a\alpha)^{-1}\mathbf{P}(M_S).$$

Proof. Looking at the proof of Doob's inequality (see [31], Theorems 5.2.1 and 7.1.9, and their proofs), one can see that, when the submartingale property is replaced by our assumption, the whole proof goes through, except that now one has to include a factor a^{-1} on the right hand side. □

Proof of Theorem 1.7. (1) By a standard Borel–Cantelli argument, it suffices to prove that with an appropriate choice of $T > 0$, it is true that for any given $\varepsilon > 0$,

$$\sum_n \mathbb{P}_\mu\Big(\sup_{s\in[0,T]} \mathrm{e}^{-\lambda(nT+s)}\|X_{nT+s}\| > \varepsilon\Big) < \infty. \tag{3.6}$$

Pick

$$\gamma \geq -\lambda. \tag{3.7}$$

Then

$$\mathbb{P}_\mu\Big(\sup_{s\in[0,T]} \mathrm{e}^{-\lambda(nT+s)}\|X_{nT+s}\| > \varepsilon\Big) \leq \mathbb{P}_\mu\Big(\sup_{s\in[0,T]} \mathrm{e}^{\gamma(nT+s)}\|X_{nT+s}\| > \varepsilon\cdot \mathrm{e}^{(\lambda+\gamma)nT}\Big). \tag{3.8}$$

Let $M_t^{(n)} := \mathrm{e}^{\gamma(nT+t)}\|X_{nT+t}\|$ for $t \in [0,T]$. Pick a number $0 < a < 1$ and fix it. Let $\mathcal{F}_s^{(n)} := \sigma(X_{nT+r}\colon r \in [0,s])$. If we show that for a sufficiently small $T > 0$ and all $n \geq 1$, the process $\{M_t^{(n)}\}_{0\leq t\leq T}$ satisfies that for all $0 < s < t < T$,

$$\mathbb{P}_\mu\big(M_t^{(n)} \mid \mathcal{F}_s^{(n)}\big) \geq aM_s^{(n)}, \tag{3.9}$$

then, by using Lemma 3.3, we can continue (3.8) with

$$\begin{aligned}
\mathbb{P}_\mu\Big(\sup_{s\in[0,T]} \mathrm{e}^{-\lambda(nT+s)}\|X_{nT+s}\| > \varepsilon\Big) &\leq \frac{1}{a\varepsilon}\mathrm{e}^{-(\lambda+\gamma)nT}\mathbb{P}_\mu\big[\mathrm{e}^{\gamma(n+1)T}\|X_{(n+1)T}\|\big] \\
&= \frac{1}{a\varepsilon}\mathrm{e}^{(\lambda+\gamma)T}\mathrm{e}^{-\lambda(n+1)T}\mathbb{P}_\mu\|X_{(n+1)T}\| \\
&\leq \frac{\|\mu\|}{a\varepsilon}\mathrm{e}^{(\lambda+\gamma)T}\mathrm{e}^{-\lambda(n+1)T}\big\|P^\beta_{(n+1)T}1\big\|_\infty.
\end{aligned}$$

Since $\lambda > \lambda_\infty$ and $\|P^\beta_{(n+1)T}1\|_\infty = \exp(\lambda_\infty(n+1)T + \mathrm{o}(n))$ as $n\to\infty$, therefore (3.6) holds.

It remains to check (3.9). Let $0 < s < t < T$. Using the Markov and branching properties at time $nT + s$,

$$\begin{aligned}\mathbb{P}_\mu\big[M_t^{(n)} \mid \mathcal{F}_s^{(n)}\big] &= \mathbb{P}_{X_{nT+s}} \mathrm{e}^{\gamma(nT+t)} \|X_{t-s}\| = \big\langle \mathbb{P}_{\delta_x} \mathrm{e}^{\gamma(nT+t)} \|X_{t-s}\|, X_{nT+s}(\mathrm{d}x)\big\rangle \\ &= \big\langle \mathbb{P}_{\delta_x} \mathrm{e}^{\gamma(t-s)} \|X_{t-s}\|, \mathrm{e}^{\gamma(nT+s)} X_{nT+s}(\mathrm{d}x)\big\rangle. \end{aligned} \tag{3.10}$$

At this point we are going to determine T as follows. According to the assumption $\beta \in \mathbf{K}(\xi)$,

$$\lim_{t\downarrow 0} \sup_{x\in\mathbb{R}^d} \Pi_x \int_0^t |\beta|(\xi_s)\,\mathrm{d}s = 0.$$

Pick $T > 0$ such that

$$\gamma t + \Pi_x \int_0^t \beta(\xi_s)\,\mathrm{d}s \geq \log a,$$

for all $0 < t < T$ and all $x \in \mathbb{R}^d$. By Jensen's inequality,

$$\gamma t + \log \Pi_x \exp\left(\int_0^t \beta(\xi_s)\,\mathrm{d}s\right) \geq \log a,$$

and thus

$$\mathbb{P}_{\delta_x} \mathrm{e}^{\gamma t} \|X_t\| = \mathrm{e}^{\gamma t} \Pi_x \exp\left(\int_0^t \beta(\xi_s)\,\mathrm{d}s\right) \geq a$$

holds too, for all $0 < t < T$ and all $x \in \mathbb{R}^d$. Returning to (3.10), for $0 < s < t < T$,

$$\mathbb{P}_\mu\big[M_t^{(n)} \mid \mathcal{F}_s^{(n)}\big] \geq a\big\langle 1, \mathrm{e}^{\gamma(nT+s)} X_{nT+s}\big\rangle = a M_s^{(n)}, \quad \text{a.s.},$$

yielding (3.9).

(2) First note that to prove (1.12) it suffices to prove that there exists $c_0 > 0$ such that for all $K > 0$,

$$\mathbb{P}_\mu\Big(\limsup_{t\to\infty} \mathrm{e}^{-\lambda t} \|X_t\| \geq K\Big) \geq c_0. \tag{3.11}$$

Since

$$\Big\{\limsup_{t\to\infty} \mathrm{e}^{-\lambda t} \|X_t\| \geq K\Big\} \supseteq \limsup_{t\to\infty} \big\{\mathrm{e}^{-\lambda t} \|X_t\| \geq K\big\},$$

we have by the reverse Fatou lemma,

$$\begin{aligned}\mathbb{P}_\mu\Big(\limsup_{t\to\infty} \mathrm{e}^{-\lambda t} \|X_t\| \geq K\Big) &\geq \limsup_{t\to\infty} \mathbb{P}_\mu\big(\mathrm{e}^{-\lambda t} \|X_t\| \geq K\big) \\ &= \limsup_{t\to\infty} \mathbb{P}_\mu\big(\mathrm{e}^{-\lambda t} \|X_t\| - K \geq 0\big). \end{aligned} \tag{3.12}$$

The assumption $\lambda < \lambda_\infty$ implies that

$$\lim_{t\to\infty} \mathbb{P}_\mu\big(\mathrm{e}^{-\lambda t} \|X_t\|\big) = \lim_{t\to\infty} \mathrm{e}^{-\lambda t} \Pi_\mu e_\beta(t) = \infty. \tag{3.13}$$

Thus $\mathbb{P}_\mu \mathrm{e}^{-\lambda t} \|X_t\| > K$ for large t. It follows easily from the Cauchy–Schwarz inequality (see, for instance, [7], Chapter 1, Ex. 3.8) that for any nonnegative random variable Y with finite second moment, on a probability space $(\Omega, \mathcal{G}, P)$, and for any $a > 0$,

$$P(Y - a \geq 0) \geq \frac{(PY - a)^2}{P(Y^2)}.$$

Applying the above inequality ('Paley–Zygmund inequality') with $Y=\mathrm{e}^{-\lambda t}\|X_t\|$ and $a=K$, we get

$$\mathbb{P}_\mu\big(\mathrm{e}^{-\lambda t}\|X_t\|-K\geq 0\big)\geq \frac{(\mathbb{P}_\mu \mathrm{e}^{-\lambda t}\|X_t\|-K)^2}{\mathbb{P}_\mu(\mathrm{e}^{-\lambda t}\|X_t\|)^2}. \tag{3.14}$$

By (3.3) and (3.4), (3.12) and (3.14) yield

$$\begin{aligned}
&\mathbb{P}_\mu\Big(\limsup_{t\to\infty}\mathrm{e}^{-\lambda t}\|X_t\|\geq K\Big)\\
&\quad\geq \limsup_{t\to\infty}\frac{(\Pi_\mu \mathrm{e}^{-\lambda t}e_\beta(t)-K)^2}{(\Pi_\mu \mathrm{e}^{-\lambda t}e_\beta(t))^2+2\mathrm{e}^{-2\lambda t}\Pi_\mu\int_0^t e_\beta(s)k(\xi_s)[\Pi_{\xi_s}e_\beta(t-s)]^2\,\mathrm{d}s}\\
&\quad=\limsup_{t\to\infty}\Big(1-K\frac{\mathrm{e}^{\lambda t}}{\Pi_\mu e_\beta(t)}\Big)^2\Big(1+2\frac{\Pi_\mu(e_\beta(t)\int_0^t k(\xi_s)\Pi_{\xi_s}e_\beta(t-s)\,\mathrm{d}s)}{(\Pi_\mu e_\beta(t))^2}\Big)^{-1}\\
&\quad=\limsup_{t\to\infty}\Big(1+2\frac{\Pi_\mu(e_\beta(t)\int_0^t k(\xi_s)\Pi_{\xi_s}e_\beta(t-s)\,\mathrm{d}s)}{(\Pi_\mu e_\beta(t))^2}\Big)^{-1}.
\end{aligned}\tag{3.15}$$

Note that

$$\Pi_{\xi_s}e_\beta(t-s)\leq \big\|P^\beta_{(t-s)}1\big\|_\infty.$$

Thus we have

$$\begin{aligned}
\Pi_\mu\Big(e_\beta(t)\int_0^t k(\xi_s)\Pi_{\xi_s}e_\beta(t-s)\,\mathrm{d}s\Big)&\leq \|k\|_\infty\Pi_\mu e_\beta(t)\Big[\int_0^t\big\|P^\beta_{t-s}1\big\|_\infty\,\mathrm{d}s\Big]\\
&=\|k\|_\infty\Pi_\mu e_\beta(t)\Big[\int_0^t\big\|P^\beta_s 1\big\|_\infty\,\mathrm{d}s\Big].
\end{aligned}$$

So, we have for every $K>0$,

$$\mathbb{P}_\mu\Big(\limsup_{t\to\infty}\mathrm{e}^{-\lambda t}\|X_t\|\geq K\Big)\geq\Big(1+2\liminf_{t\to\infty}\frac{\|k\|_\infty\|P^\beta_t 1\|_\infty^{-1}\int_0^t\|P^\beta_s 1\|_\infty\,\mathrm{d}s}{\|P^\beta_t 1\|_\infty^{-1}\Pi_\mu e_\beta(t)}\Big)^{-1}. \tag{3.16}$$

We now consider the numerator and denominator of the right-hand side of (3.16) separately.

$$\liminf_{t\to\infty}\big\|P^\beta_t 1\big\|_\infty^{-1}\int_0^t\big\|P^\beta_s 1\big\|_\infty\,\mathrm{d}s<\infty.$$

By Fatou's lemma and (1.11),

$$\liminf_{t\to\infty}\big\|P^\beta_t 1\big\|_\infty^{-1}\Pi_\mu e_\beta(t)\geq\Big\langle\mu,\liminf_{t\to\infty}\big\|P^\beta_t 1\big\|_\infty^{-1}\Pi.e_\beta(t)\Big\rangle>0.$$

Now combining (3.16) and Lemma 3.2, we arrive at (3.11). □

3.2. *Proof of Theorem* 1.8

(1) Using Fatou's lemma and (1.13), we get

$$\liminf_{t\to\infty}\mathrm{e}^{-\lambda_\infty t}\Pi_\mu e_\beta(t)=\liminf_{t\to\infty}\Pi_\mu e_{\beta-\lambda_\infty}(t)\geq\Big\langle\liminf_{t\to\infty}\Pi.e_{\beta-\lambda_\infty}(t),\mu\Big\rangle=\infty,$$

which means that (3.13) holds with λ replaced by λ_∞. So the proof of Theorem 1.7(2) works with λ replaced by λ_∞.

(2) By (3.3), we have

$$\mathbb{P}_\mu\big[\exp(-\lambda_\infty t)\|X_t\|\big] = \Pi_\mu e_{\beta-\lambda_\infty}(t). \tag{3.17}$$

Letting $t \to \infty$ and using Fatou's lemma, we get

$$\mathbb{P}_\mu\Big(\liminf_{t\to\infty} \exp(-\lambda_\infty t)\|X_t\|\Big) \le \liminf_{t\to\infty} \Pi_\mu e_{\beta-\lambda_\infty}(t). \tag{3.18}$$

Note that $\Pi_\mu e_{\beta-\lambda_\infty}(t) = \langle \Pi_\cdot e_{\beta-\lambda_\infty}(t), \mu\rangle$. Using (1.15) and the assumption that $g_{\beta-\lambda_\infty} \equiv 0$ in $\mathbb{R}^d$, we get

$$\lim_{t\to\infty} \Pi_\mu e_{\beta-\lambda_\infty}(t) = \Big\langle \lim_{t\to\infty} \Pi_\cdot e_{\beta-\lambda_\infty}(t), \mu\Big\rangle = \langle g_{\beta-\lambda_\infty}, \mu\rangle = 0,$$

where in the first equality we used the fact $\Pi_\cdot e_{\beta-\lambda_\infty}(t) \le \sup_{x\in\mathbb{R}^d} \Pi_x(\sup_{t\ge 0} e_{\beta-\lambda_\infty}(t)) < \infty$, which follows from (1.15), and the fact that μ is finite measure, and in the second equality we used the fact $e_{\beta-\lambda_\infty}(t) \le \sup_{t\ge 0} e_{\beta-\lambda_\infty}(t) < \infty$ Π_x-a.s. for any $x \in \mathbb{R}^d$. Hence by (3.18) we get

$$\mathbb{P}_\mu\Big(\liminf_{t\to\infty} \exp(-\lambda_\infty t)\|X_t\| = 0\Big) = 1,$$

which implies (1.16).

Finally, when $\beta \le 0$, trivially $\lambda_\infty \le 0$; hence $\mathbb{P}_\mu(\liminf_{t\to\infty} \|X_t\| = 0) = 1$. On the other hand, $\|X\|$ is a supermartingale by the expectation formula and the branching Markov property, and thus, $\lim_{t\to\infty} \|X_t\|$ exists $\mathbb{P}_\mu$-a.s. Hence, we can improve the liminf to a limit.

4. Proofs of Theorems 1.10 and 1.12

4.1. *Proof of Theorem* 1.10

We start with a lemma.

Lemma 4.1. *Assume that $\beta \in \mathbf{K}(\xi)$ and that $h > 0$ is a bounded solution to*

$$(L + \beta - \lambda_\infty)h = 0 \quad \text{in } \mathbb{R}^d$$

in the sense of distributions. Let $\mu \in M(\mathbb{R}^d)$ be nonzero and $\mathcal{F}_t := \sigma\{X_r, r \le t\}$. Then the process $(\{e_{-\lambda_\infty}(t)\langle h, X_t\rangle\}_{t\ge 0}, \{\mathcal{F}_t\}_{t\ge 0}, \mathbb{P}_\mu)$ is a positive martingale.

Proof. Recall that $D_n = B(0, n)$ and τ_n is the first exit time of ξ from D_n. Since h is harmonic with respect to the operator $L + \beta - \lambda_\infty$, we have

$$h(x) = \Pi_x\big[e_{\beta-\lambda_\infty}(t \wedge \tau_n)h(\xi_{t\wedge\tau_n})\big], \quad \text{for every } n \ge 1 \text{ and } t \ge 0, \tag{4.1}$$

see the proof of [30], Lemma 2.1. Since h is bounded, bounded convergence yields

$$h(x) = \Pi_x\big[e_{\beta-\lambda_\infty}(t)h(\xi_t)\big], \quad \text{for every } t \ge 0. \tag{4.2}$$

By the branching and Markov properties, for $0 \le s < t$, we have

$$\begin{aligned}
&\mathbb{P}_\mu\big(e_{-\lambda_\infty}(t)\langle h, X_t\rangle | \mathcal{F}_s\big) \\
&\quad = e_{-\lambda_\infty}(t)\mathbb{P}_{X_s}\langle h, X_{t-s}\rangle \\
&\quad = e_{-\lambda_\infty}(t)\big\langle \Pi_\cdot\big(e_\beta(t-s)h(\xi_{t-s})\big), X_s\big\rangle \\
&\quad = e_{-\lambda_\infty}(t)\big\langle \Pi_\cdot\big(e_\beta(t-s)h(\xi_{t-s})\big), X_s\big\rangle \\
&\quad = e_{-\lambda_\infty}(s)\langle h, X_s\rangle,
\end{aligned} \tag{4.3}$$

proving the assertion. □

Proof of Theorem 1.10. Suppose $\mu \in M_c(\mathbb{R}^d)$. Since M^h defined by

$$M_t^h := \exp(-\lambda_\infty t)\langle h, X_t\rangle$$

is a nonnegative $\mathbb{P}_\mu$-martingale, $\lim_{t\to\infty} M_t^h$ exists and is also finite $\mathbb{P}_\mu$-a.s. By the martingale property, we have

$$\mathbb{P}_\mu M_t^h = \exp(-\lambda_\infty t)\Pi_\mu\big[e_\beta(t)h(\xi_t)\big] = \langle h, \mu\rangle.$$

It follows from (1.17) and Lemma B.8 in Appendix B that

$$\Pi_\mu\bigg[\int_0^\infty e_{\beta-2\lambda_\infty}(s)k(\xi_s)h^2(\xi_s)\,\mathrm{d}s\bigg] \le C^2\Pi_\mu\bigg[\int_0^\infty e_{\beta-2\lambda_\infty}(s)k(\xi_s)\,\mathrm{d}s\bigg] < \infty,$$

where C is a positive constant such that $h(x) \le C$ for all $x \in \mathbb{R}^d$. Thus by the variance formula (3.4) and by (4.1), we have

$$\begin{aligned}\mathbb{P}_\mu\big[M_t^h\big]^2 &= \langle h, \mu\rangle^2 + \exp(-2\lambda_\infty t)\Pi_\mu\bigg[\int_0^t e_\beta(s)k(\xi_s)\big[\Pi_{\xi_s}\big(e_\beta(t-s)h(\xi_{t-s})\big)\big]^2\,\mathrm{d}s\bigg]\\ &= \langle h, \mu\rangle^2 + \Pi_\mu\bigg[\int_0^t e_\beta(s)\exp(-2\lambda_\infty s)k(\xi_s)\big[\Pi_{\xi_s}\big(e_{\beta-\lambda_\infty}(t-s)h(\xi_{t-s})\big)\big]^2\,\mathrm{d}s\bigg]\\ &= \langle h, \mu\rangle^2 + \Pi_\mu\bigg[\int_0^t e_{\beta-2\lambda_\infty}(s)k(\xi_s)h^2(\xi_s)\,\mathrm{d}s\bigg].\end{aligned}$$

By the L^2-convergence theorem, M_t^h converges to some η in $L^2(\mathbb{P}_\mu)$. In particular,

$$0 < \mathbb{P}_\mu\eta^2 = \langle h, \mu\rangle^2 + \Pi_\mu\int_0^\infty e_{\beta-2\lambda_\infty}(s)k(\xi_s)h^2(\xi_s)\,\mathrm{d}s < \infty,$$

and therefore,

$$\mathbb{P}_\mu(\eta < \infty) = 1, \quad \text{and} \quad \mathbb{P}_\mu(\eta = 0) < 1. \tag{4.4}$$

It is obvious that $\mathbb{P}_\mu(\eta = 0) < 1$ implies that $\mathbb{P}_\mu(\|X_t\| > 0, \forall t > 0) > 0$.

If h satisfies (1.18), then (4.4) implies (1.19) and (1.20). □

Remark 4.2. *Theorem 1.10 says that, under condition (1.17), not only the Kesten–Stigum Theorem holds (i.e., the martingale $M_t^h = \mathrm{e}^{-\lambda_\infty t}\langle h, X_t\rangle$ converges in $L^1(\mathbb{P}_\mu)$ as $t \to \infty$), but it can be upgraded to convergence in $L^2(\mathbb{P}_\mu)$. We plan to find a necessary and sufficient condition in an upcoming paper.*

Using the 'spine' method developed in Engländer and Kyprianou [11], we can give an alternative proof of Theorem 1.10, but with the weaker conclusion that the martingale $M_t^h = \mathrm{e}^{-\lambda_\infty t}\langle h, X_t\rangle$ converges in $L^1(\mathbb{P}_\mu)$ as $t \to \infty$.

4.2. *Preparation for the proof of Theorem 1.12*

In the remainder of this section, we suppose $\lambda_\infty = 0$ and that $h > 0$ is a bounded solution to $(L + \beta)u = 0$ in $\mathbb{R}^d$ in the sense of distributions. For $c > 0$, put

$$u_{ch}(t, x) := -\log\mathbb{P}_{\delta_x}\exp\big(-c\langle h, X_t\rangle\big), \tag{4.5}$$

then $u_{ch}(t, x)$ is a solution of the following integral equation:

$$u_{ch}(t, x) + \Pi_x\int_0^t \big[k(\xi_r)\big(u_{ch}(t-r, \xi_r)\big)^2 - \beta(\xi_r)u_{ch}(t-r, \xi_r)\big]\,\mathrm{d}r = c\Pi_x h(\xi_t). \tag{4.6}$$

By Lemma 3.1, the above integral equation is equivalent to

$$u_{ch}(t,x)+\Pi_x\int_0^t e_\beta(r)k(\xi_r)\big[u_{ch}(t-r,\xi_r)\big]^2\,\mathrm{d}r=c\Pi_x\big[e_\beta(t)h(\xi_t)\big]. \tag{4.7}$$

Since h is a bounded positive solution to $(L+\beta)u=0$, we have

$$\Pi_x\big[e_\beta(t)h(\xi_t)\big]=h(x).$$

Thus (4.7) can be rewritten as

$$u_{ch}(t,x)+\Pi_x\left[\int_0^t e_\beta(r)k(\xi_r)\big[u_{ch}(t-r,\xi_r)\big]^2\,\mathrm{d}r\right]=ch(x). \tag{4.8}$$

In particular,

$$u_{ch}(t,x)\le ch(x). \tag{4.9}$$

Put

$$u_{ch}(x):=-\log\mathbb{P}_{\delta_x}\exp\Big(-c\lim_{t\to\infty}\langle h,X_t\rangle\Big). \tag{4.10}$$

By Lemma 4.1, under $\mathbb{P}_\mu$, $\exp(-c\langle h,X_t\rangle)$, $t\ge 0$ is a bounded submartingale. Thus $u_{ch}(t,x)$ is nonincreasing in t. Hence, by the dominated convergence theorem, for every $x\in\mathbb{R}^d$,

$$u_{ch}(t,x)\downarrow u_{ch}(x)\quad \text{as } t\uparrow\infty.$$

Note that if k and β are radial functions, and if L is radial, then $u_{ch}(\cdot)$ is a radial function, i.e.,

$$u_{ch}(x)=u_{ch}\big(\|x\|\big).$$

Lemma 4.3.

(1) *For any $x\in\mathbb{R}^d$ and $r>0$,*

$$u_{ch}(x)\le\Pi_x\big(u_{ch}(\xi_{\tau_{B(x,r)}})e_\beta(\tau_{B(x,r)})\big).$$

(2) *If L, k and β are radial, then*

$$u_{ch}(x)=u_{ch}\big(\|x\|\big)\le u_{ch}(R)\Pi_x\big(e_\beta(\tau_{B(0,R)})\big),\quad \|x\|<R. \tag{4.11}$$

Proof. (1) By the special Markov property, for every fixed $x\in\mathbb{R}^d$, one has

$$\begin{aligned}\exp\big(-u_{ch}(x)\big)&=\mathbb{P}_{\delta_x}\exp\Big(-c\lim_{t\to\infty}\langle h,X_t\rangle\Big)\\&=\mathbb{P}_{\delta_x}\Big(P_{X_{\tau_{B(x,r)}}}\exp\Big(-c\lim_{t\to\infty}\langle h,X_t\rangle\Big)\Big)\\&=\mathbb{P}_{\delta_x}\exp\langle -u_{ch},X_{\tau_{B(x,r)}}\rangle.\end{aligned}$$

By Jensen's inequality,

$$\exp\big(-u_{ch}(x)\big)\ge\exp\big(-\mathbb{P}_{\delta_x}\langle u_{ch},X_{\tau_{B(x,r)}}\rangle\big)=\exp\big[-\Pi_x\big(u_{ch}(\xi_{\tau_{B(x,r)}})e_\beta(\tau_{B(x,r)})\big)\big],$$

which implies $u_{ch}(x)\le\Pi_x(u_{ch}(\xi_{\tau_{B(x,r)}})e_\beta(\tau_{B(x,r)}))$.

(2) Similarly we have, for $x\in B(0,R)$, that

$$u_{ch}(x)\le u_{ch}(R)\Pi_x\big(e_\beta(\tau_{B(0,R)})\big).$$ □

Note that $u_{ch}(x)$ is increasing in c. Let

$$u_{ch}(x) \uparrow u_\infty(x) = -\log \mathbb{P}_{\delta_x}\Bigl(\lim_{t\to\infty} \langle h, X_t\rangle = 0\Bigr). \tag{4.12}$$

Lemma 4.4. *Either $u_\infty(x) \equiv 0$ or $u_\infty \in (0, \infty]$ in $\mathbb{R}^d$. That is, if*

$$E_h := \Bigl\{\lim_{t\to\infty} \langle h, X_t\rangle = 0\Bigr\},$$

then either $\mathbb{P}_{\delta_x}(E_h) = 1, \forall x \in \mathbb{R}^d$, or $\mathbb{P}_{\delta_x}(E_h) < 1, \forall x \in \mathbb{R}^d$.

Proof. We first prove that if there exists a measurable set $A \subset \mathbb{R}^d$ with positive Lebesgue measure such that $u_\infty > 0$ on A, then $u_\infty(x) > 0$ for every $x \in \mathbb{R}^d$. Indeed, for every $x \in \mathbb{R}^d$,

$$\begin{aligned}
&\mathbb{P}_{\delta_x}\Bigl(\lim_{t\to\infty} \langle h, X_t\rangle = 0\Bigr)\\
&\quad = \mathbb{P}_{\delta_x}\Bigl(\mathbb{P}_{X(1)}\Bigl(\lim_{t\to\infty} \langle h, X_t\rangle = 0\Bigr)\Bigr)\\
&\quad = \mathbb{P}_{\delta_x} \exp\bigl\langle -u_\infty, X(1)\bigr\rangle.
\end{aligned} \tag{4.13}$$

Note that

$$\mathbb{P}_{\delta_x}\bigl\langle u_\infty, X(1)\bigr\rangle = \Pi_x\bigl(u_\infty(\xi_1) e_\beta(1)\bigr) > 0. \tag{4.14}$$

(4.13) implies that $\mathbb{P}_{\delta_x}(\lim_{t\to\infty}\langle h, X_t\rangle = 0) < 1$. Thus we have $u_\infty(x) > 0$.

Now we prove that if $u_\infty = 0$ almost everywhere, then $u_\infty \equiv 0$. By (4.14), we know that $\mathbb{P}_{\delta_x}\langle u_\infty, X(1)\rangle = 0$, and thus $\langle u_\infty, X(1)\rangle = 0$, $\mathbb{P}_{\delta_x}$-a.s. By (4.13),

$$\mathbb{P}_{\delta_x}\Bigl(\lim_{t\to\infty} \langle h, X_t\rangle = 0\Bigr) = 1.$$

Hence $u_\infty(x) = 0$ for every $x \in \mathbb{R}^d$. □

4.3. *Proof of Theorem* 1.12

Since $\beta \in \mathbf{K}_\infty(\xi)$, by the Gauge Theorem (see [4], Theorem 2.2, or [2], Theorem 2.6), the assumption that g_β is not identically infinite implies that g_β is bounded between two positive numbers. By [2], Corollary 2.16, we have

$$\Pi_x\Bigl[\sup_{0\le t\le\infty} e_\beta(t)\Bigr] < \infty, \quad \forall x \in \mathbb{R}^d.$$

By dominated convergence,

$$g_\beta(x) = \lim_{R\to\infty} \Pi_x\bigl(e_\beta(\tau_{B(0,R)})\bigr), \quad x \in \mathbb{R}^d.$$

Take $h = g_\beta$. We know that h is a bounded solution of $(L+\beta)u = 0$ and satisfies (1.18); by Lemma 4.4 we only need to prove that if for every $x \in \mathbb{R}^d$, $\mathbb{P}_{\delta_x}(\lim_{t\to\infty}\|X_t\| = 0) < 1$, then

$$\Pi_x \int_0^\infty e_\beta(s) k(\xi_s)\, \mathrm{d}s < \infty, \quad x \in \mathbb{R}^d. \tag{4.15}$$

First note that the assumption that $\mathbb{P}_{\delta_x}(\lim_{t\to\infty}\|X_t\| = 0) < 1, x \in \mathbb{R}^d$ implies that

$$u_{ch}(x) = -\log \mathbb{P}_{\delta_x} \exp\Bigl(-c \lim_{t\to\infty} \langle h, X_t\rangle\Bigr) > 0 \quad \text{for every } x \in \mathbb{R}^d.$$

Since $u_{ch}(s,x) \geq u_{ch}(x)$ for every $s \in [0,t]$ and $x \in \mathbb{R}^d$, by (4.8), we have

$$\Pi_x \int_0^t e_\beta(s) k(\xi_s) u_{ch}^2(\xi_s)\,\mathrm{d}s \leq ch(x), \quad x \in \mathbb{R}^d.$$

Letting $t \to \infty$, we get

$$\Pi_x \int_0^\infty e_\beta(s) k(\xi_s) u_{ch}^2(\xi_s)\,\mathrm{d}s \leq ch(x), \quad x \in \mathbb{R}^d,$$

which can be rewritten as

$$\int_{\mathbb{R}^d} G_\beta(x,y) k(y) u_{ch}^2(y) m(\mathrm{d}y) \leq ch(x), \quad x \in \mathbb{R}^d. \tag{4.16}$$

Letting $R \to \infty$ in (4.11), one gets

$$u_{ch}(x) \leq h(x) \liminf_{R\to\infty} u_{ch}(R).$$

Since $u_{ch}(x) > 0$ and $0 < h(x) < \infty$, we have $\liminf_{R\to\infty} u_{ch}(R) > 0$. Then (4.16) implies (4.15).

Appendix A: Construction and path regularity

Proof of Theorem 1.3. Let $D_n, n \geq 1$, be a sequence of smooth bounded domains such that $D_n \uparrow \mathbb{R}^d$. According to Dynkin [8], for each n, the $(L|_{D_n} - \beta^-, \beta^+ \wedge n, k)$-superdiffusion $(X_t^n, t \geq 0)$ exists, where $L|_{D_n}$ is the generator of the process ξ killed upon leaving D_n, and β^+ and β^- are the positive and negative parts of β, respectively. Also note that $(X_t^n, t \geq 0)$ can be regarded as an $(L|_{D_n}, \beta \wedge n, k)$-superdiffusion.

Let f be a positive bounded measurable function on $\mathbb{R}^d$. According to Dynkin [8], for each n, there exists a unique bounded solution u_n to the following integral equation:

$$u_n(t,x) + \Pi_x \int_0^{t\wedge\tau_n} \bigl[-\bigl(\beta(\xi_s) \wedge n\bigr) u_n(t-s,\xi_s) + k(\xi_s) u^2(t-s,\xi_s)\bigr]\,\mathrm{d}s = \Pi_x\bigl[f(\xi_t), t < \tau_n\bigr],$$

where τ_n is the first exit time of the diffusion ξ from D_n. We rewrite the above equation in the following form (according to a result similar to our Lemma 3.1):

$$\begin{aligned} u_n(t,x) &+ \Pi_x \int_0^{t\wedge\tau_n} e_{\beta^+\wedge n}(s)\bigl[\beta^-(\xi_s) u_n(\xi_s, t-s) + k(\xi_s) u^2(\xi_s, t-s)\bigr]\,\mathrm{d}s \\ &= \Pi_x\bigl[e_{\beta^+\wedge n}(t) f(\xi_t), t < \tau_n\bigr]. \end{aligned} \tag{A.1}$$

By the (weak) parabolic maximum principle (see [23], p. 128, for example), u_n is increasing. Let $u_n(t,x) \uparrow u(t,x)$ as $n \uparrow \infty$. Letting $n \to \infty$ in the above integral equation, we get

$$u(t,x) + \Pi_x \int_0^t e_{\beta^+}(s)\bigl[\beta^-(\xi_s) u(t-s,\xi_s) + k(\xi_s) u^2(t-s,\xi_s)\bigr]\,\mathrm{d}s = \Pi_x\bigl[e_{\beta^+}(t) f(\xi_t)\bigr]. \tag{A.2}$$

The assumption that β is in the Kato class implies that $u(t,x) \leq \Pi_x[e_{\beta^+}(t) f(\xi_t)] \leq \mathrm{e}^{c_1+c_2 t}$ for some positive constants.

To see the minimality of u, let v be an arbitrary nonnegative measurable solution to (A.2). By the (weak) parabolic maximum principle, $v|_{D_n} \geq u_n$ for all $n \geq 1$, and thus $v \geq u$ on $\mathbb{R}^d$.

Equation (A.2) can be rewritten as

$$u(t,x) + \Pi_x \int_0^t \bigl[-\beta(\xi_s) u(t-s,\xi_s) + k(\xi_s) u^2(t-s,\xi_s)\bigr]\,\mathrm{d}s = \Pi_x\bigl[f(\xi_t)\bigr]. \tag{A.3}$$

Then following the arguments in Appendix A of Engländer and Pinsky [9], we can get the existence of our superdiffusion. $\square$

Remark A.1. *If $k \in \mathbf{K}(\xi)$ as well, then using Gronwall's lemma, u is the unique solution (bounded on any finite interval) of the integral equation (A.3).*

Before turning to the proof Theorem 1.5, we remark that [22], Appendix A, explains some important concepts (e.g. Ray cone, Ray topology) we will be working with, and that [22], Chapter 5, discusses regularity properties of superdiffusions, using similar methods, albeit under different assumptions on the nonlinear operator.

For the proof we first need a lemma. The function f is called[4] *α-supermedian* relative to P_t^0 for $\alpha > 0$, if $e^{-\alpha t} P_t^0 f \le f$ for $t \ge 0$.

Lemma A.2. *Assume that $\beta \in \mathbf{K}(\xi)$ satisfies $\beta \le B$ for some constant $B > 0$, and f is α-supermedian relative to P_t^0 for some $\alpha > 0$. Then for every $\mu \in M(\mathbb{R}^d)$,*

(i) *$M_t := e^{-(B+\alpha)t}\langle f, X_t\rangle$ is a $\mathbb{P}_\mu$-supermartingale.*
(ii) *$\mathbb{P}_\mu(\sup_{0\le r\le t, r\in\mathbb{Q}}\langle 1, X_t\rangle < \infty$ for all $t > 0) = 1$.*

Proof. (i) It is easy to see that it suffices to check

$$\mathbb{P}_\nu(M_t) \le M_0 = \langle f, \nu\rangle, \quad t > 0, \forall \nu \in M(\mathbb{R}^d). \tag{A.4}$$

This is because for $0 \le s < t$, by the Markov property at time s,

$$\mathbb{P}_\mu\big(e^{-Bt}\langle f, X_t\rangle \mid \mathcal{F}_s\big) = \mathbb{P}_{X_s} M_{t-s} e^{-(B+\alpha)s} \le \langle f, X_s\rangle e^{-(B+\alpha)s} = M_s,$$

where in the last inequality above we used (A.4) with $\nu = X_s$. Using the assumption that f is α-supermedian, we obtain

$$\mathbb{P}_{\delta_x} M_t = e^{-(B+\alpha)t}\big(P_t^\beta f\big)(x) \le e^{-\alpha t} P_t^0 f(x) \le f(x).$$

Therefore (A.4) holds.

(ii) By the proof of Theorem 1.7, there are $a, \gamma > 0$ and a sufficiently small $T > 0$ such that $M_r := e^{\gamma t}\langle 1, X_r\rangle$ satisfies

$$\mathbb{P}_\mu[M_r \mid \mathcal{F}_s] \ge a M_s, \quad 0 \le s \le r \le T \text{ with } r, s \in \mathbb{Q}.$$

Then by Doob's inequality (Lemma 3.3 in discrete time),

$$\mathbb{P}_\mu\Big(\sup_{0\le r\le T,\, r\in\mathbb{Q}}\langle 1, X_r\rangle > K\Big) \le (aK)^{-1}\mathbb{P}_\mu M_t \le (aK)^{-1} e^{(\gamma+B)T}.$$

Letting $K \uparrow 0$, we see that for any fixed $t > 0$, $\mathbb{P}_\mu(\sup_{0\le r\le T, r\in\mathbb{Q}}\langle 1, X_r\rangle = \infty) = 0$. Since we can split $[0,\infty)$ to intervals of length T, the result of (ii) holds. □

Proof of Theorem 1.5. Let $(\overline{\mathbb{R}}^d, \overline{\mathcal{B}(\mathbb{R}^d)})$ be the Ray–Knight compactification of $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ associated with the semigroup $\{P_t^0\colon t \ge 0\}$ and a suitably chosen countable Ray cone (see the last paragraph on [15], p. 342), and let $M_r(\overline{\mathbb{R}}^d)$ be the space of finite measures on $\overline{\mathbb{R}}^d$ with the weak Ray topology. Suppose W is the space of right continuous paths from $[0,\infty)$ to $M_r(\mathbb{R}^d)$ with left limits in $M_r(\overline{\mathbb{R}}^d)$, where $M_r(\mathbb{R}^d)$ carries the relative topology inherited from $M_r(\overline{\mathbb{R}}^d)$. We write $\tilde{X} = (\tilde{X}_t, t \ge 0)$ for the coordinate process on W and put $\mathcal{G} = \sigma\{\tilde{X}_t; t \ge 0)$. Using the above lemma, the argument in the proof of [15], Theorem 2.11, is applicable to our setup, so for any given $\mu \in M(\mathbb{R}^d)$ there exists a unique probability measure $\mathbf{P}_\mu$ on $(W, \mathcal{G})$ such that $\mathbf{P}_\mu(\tilde{X}_0 = \mu) = 1$ and $(\tilde{X}_t, t \ge 0)$ under $\mathbf{P}_\mu$ has the same law as the superprocess X under $\mathbb{P}_\mu$.

As before, let $M(\mathbb{R}^d)$ denote the space of finite measures on $\mathbb{R}^d$ with the weak topology, induced by the mappings $\langle f, \tilde{X}_t\rangle$ as f runs through the bounded continuous functions on $\mathbb{R}^d$. (The Borel σ-algebras on $M_r(\mathbb{R}^d)$ and on $M(\mathbb{R}^d)$ both coincide with $\mathcal{M}$.) Since the diffusion process ξ is continuous, using the arguments of [15], Section 3, we have

[4] In [22] a slightly different terminology is followed.

that if f is a bounded continuous function on $\mathbb{R}^d$, then $\langle f, \tilde{X}_\cdot \rangle$ is right continuous on $[0, \infty)$ almost surely; and if $f(\xi_\cdot)$ has left limits on $[0, \infty)$ almost surely, then so does $\langle f, \tilde{X}_\cdot \rangle$. That is to say, $\tilde{X}$ is a càdlàg process on the state space $M(\mathbb{R}^d)$. □

Appendix B: Review on Feynman–Kac semigroups and Gauge Theory

Recall that β is in the Kato class $\mathbf{K}(\xi)$. In this appendix we present some background material on the Feynman–Kac semigroup. Recall from Section 1 that

$$P_t^\beta f(x) := \Pi_x\big[e_\beta(t) f(\xi_t)\big],$$

and that $\{P_t^\beta, t \geq 0\}$ is a strongly continuous semigroup on $L^p(\mathbb{R}^d, m)$ for $1 \leq p < \infty$.

For any domain $D \subset \mathbb{R}^d$ and $x \in D$, we will use $\delta_D(x)$ to denote the distance from x to D^c: $\delta_D(x) := \inf\{|x-y|: y \in D^c\}$. Let ξ^D be the subprocess of ξ killed upon exiting D. It is well known that ξ^D has a transition density $p_D(t, x, y)$ with respect to the Lebesgue measure. We will use $\{P_t^{\beta,D}, t \geq 0\}$ to denote the semigroup of ξ^D:

$$P_t^{\beta,D} f(x) := \Pi_x\big[e_\beta(t) f(\xi_t), t < \tau_D\big],$$

where

$$\tau_D = \inf\{t > 0: \xi_t \notin D\}.$$

When D^c is nonpolar, that is, when $\Pi_x(\tau_D < \infty)$ is not identically zero, ξ^D is transient. In this case, the function $G_D(x, y) := \int_0^\infty p_D(t, x, y)\,\mathrm{d}t$ is well defined and is called the Green's function of ξ^D with respect to the Lebesgue measure. Then $\widetilde{G}_D(x, y) := G_D(x, y)/m(y)$ is the Green's function of ξ^D with respect to $m(y)\,\mathrm{d}y$.

For any $n \geq 1$, put $D_n = B(0, n)$. We will use the shorthand $\xi^{(n)}$ to denote ξ^{D_n} and G_n to denote G_{D_n}. It follows from [19,21] that G_n is comparable to the Green's function of the killed Brownian motion in D_n. Therefore we have the following result.

Proposition B.1. *There exists $c_1 = c_1(n, d) > 1$ such that when $d \geq 3$,*

$$c_1^{-1}\left(1 \wedge \frac{\delta_B(x)\delta_B(y)}{|x-y|^2}\right) \leq G_B(x, y) \leq c_1 \frac{1}{|x-y|^{d-2}}\left(1 \wedge \frac{\delta_B(x)\delta_B(y)}{|x-y|^2}\right), \quad x, y \in B \tag{B.1}$$

for any ball $B \subset D_n$; when $d = 2$

$$c_1^{-1}\log\left(1 + \frac{\delta_B(x)\delta_B(y)}{|x-y|^2}\right) \leq G_B(x, y) \leq c_1 \log\left(1 + \frac{\delta_B(x)\delta_B(y)}{|x-y|^2}\right), \quad x, y \in B \tag{B.2}$$

for any ball $B \subset D_n$; and when $d = 1$

$$c_1^{-1}\big(\delta_B(x) \wedge \delta_B(y)\big) \leq G_B(x, y) \leq c_1\big(\delta_B(x) \wedge \delta_B(y)\big), \quad x, y \in B \tag{B.3}$$

for any ball $B \subset D_n$.

B.1. The 3G inequalities and the Martin kernel

Recall that u is defined by (2.3). Using (B.1)–(B.3), we can easily get the following.

Proposition B.2 (The 3G inequalities). *There exists $c = c(d, n)$ such that, when $d \geq 3$,*

$$\frac{G_B(x, y)G_B(y, z)}{G_B(x, z)} \leq c\big(u(x-y) + u(y-z)\big), \quad x, y, z \in B \tag{B.4}$$

for any ball $B \subset D_n$; when $d = 2$,

$$\frac{G_B(x,y)G_B(y,z)}{G_B(x,z)} \le c\big[\big(1 \vee u(x-y)\big) + \big(1 \vee u(y-z)\big)\big], \quad x, y, z \in B \tag{B.5}$$

for any ball $B \subset D_n$; and when $d = 1$,

$$\frac{G_B(x,y)G_B(y,z)}{G_B(x,z)} \le c, \quad x, y, z \in B \tag{B.6}$$

for any ball $B \subset D_n$.

Proof. The $d \ge 3$ case follows from [5], Theorem 6.5, the $d = 2$ case follows from [5], Theorem 6.15, while $d = 1$ follows from direct calculation. □

The three inequalities in Proposition B.2 are called *3G inequalities*. For any ball B and $x_0 \in B$, the Martin kernel $M_B(x,z)$, $(x,z) \in B \times \partial B$, based at x_0 is defined by

$$M_B(x,z) := \lim_{B \ni y \to z \in \partial B} \frac{G_B(x,y)}{G_B(x_0,y)}.$$

The base x_0 plays no essential role here. One then can easily deduce the following result from the 3G inequalities above.

Proposition B.3. *There exists $c = c(d,n) > 0$ such that, when $d \ge 3$,*

$$\frac{G_B(x,y)M_B(y,z)}{M_B(x,z)} \le c\big(u(x-y) + u(y-z)\big), \quad x, y \in B, z \in \partial B \tag{B.7}$$

for every ball $B \subset D_n$; when $d = 2$,

$$\frac{G_B(x,y)M_B(y,z)}{M_B(x,z)} \le c\big[\big(1 \vee u(x-y)\big) + \big(1 \vee u(y-z)\big)\big], \quad x, y \in B, z \in \partial B \tag{B.8}$$

for every ball $B \subset D_n$; when $d = 1$,

$$\frac{G_B(x,y)M_B(y,z)}{M_B(x,z)} \le c, \quad x, y \in B, z \in \partial B \tag{B.9}$$

for every ball $B \subset D_n$.

The following result is proved in [20,21].

Proposition B.4. *For any $n \ge 1$, there exist $c_i = c_i(n) > 1$, $i = 1, 2$, such that the transition density $p_t^{(n)}$ of $\xi^{(n)}$ with respect to the Lebesgue measure satisfies*

$$\begin{aligned} c_1^{-1} t^{-d/2}\left(1 \wedge \frac{\delta_n(x)}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_n(x)}{\sqrt{t}}\right)\mathrm{e}^{-c_2|x-y|^2/t} &\le p_t^{(n)}(x,y) \\ &\le c_1 t^{-d/2}\left(1 \wedge \frac{\delta_n(x)}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_n(x)}{\sqrt{t}}\right)\mathrm{e}^{-|x-y|^2/c_2 t} \end{aligned} \tag{B.10}$$

for all $(t,x,y) \in (0,1] \times D_n \times D_n$.

We then have the following result.

Proposition B.5. *If $\beta \in \mathbf{K}(\xi)$, then for any $n \geq 1$,*

$$\lim_{r\to 0}\sup_{x\in D_n}\int_{|y-x|<r} u(x-y)\big|\beta(y)\big|\,\mathrm{d}y = 0.$$

Proof. It follows from (B.10) that there exist constants $c_1, c_2 > 1$ such that for any $(t, x, y) \in (0, 1] \times D_n \times D_n$,

$$p_t^{(n+1)}(x,y) \geq c_1^{-1}\exp\left\{-\frac{c_2|x-y|^2}{t}\right\}.$$

Since

$$\int_0^t \Pi_x\big[\big|\beta(\xi_s)\big|\big]\,\mathrm{d}s \geq \int_0^t\int_{D_n} p_s^{(n+1)}(x,y)\big|\beta(y)\big|\,\mathrm{d}y\,\mathrm{d}s,$$

we can apply the arguments in the proof of [5], Lemma 3.5, and the first part of the proof of [5], Theorem 3.6, to get the conclusion of our proposition. □

B.2. Probabilistic representation of λ_2

The following result is a generalization of [26], Theorem 4.4.4, and it implies that (1.5) is valid when $\beta \in \mathbf{K}(\xi)$.

Proposition B.6 (Probabilistic representation of λ_2). *Let $\{D_n\}_{n\geq 1}$ be an increasing sequence of bounded domains with $D_n \uparrow \mathbb{R}^d$ as $n \to \infty$. If $\tau_n := \inf_{t\geq 0}\{t\colon \xi_t \notin D_n\}$, $n \geq 1$, then*

$$\lambda_2(\beta) = \sup_n \lim_{t\to\infty}\frac{1}{t}\log\sup_{x\in D_n}\Pi_x\big(e_\beta(t); t<\tau_n\big).$$

Proof. Let $P_t^{\beta,n}$ stand for P_t^{β,D_n} and let

$$\lambda_2^n := \lim_{t\to\infty}\frac{1}{t}\log\big\|P_t^{\beta,n}\big\|_{2,2},$$

where $\|P_t^{\beta,n}\|_{2,2}$ stands for the operator norm of $P_t^{\beta,n}$ from $L^2(D_n, m)$ to $L^2(D_n, m)$. It is well known (see, for instance, [3]) that

$$-\lambda_2(\beta) = \inf\left\{\frac{1}{2}\int_{\mathbb{R}^d}(\nabla f a\nabla f)\mathrm{e}^{2Q}\,\mathrm{d}x - \int_{\mathbb{R}^d} f^2\beta\mathrm{e}^{2Q}\,\mathrm{d}x\colon\ f\in C_c^\infty\big(\mathbb{R}^d\big), \|f\|_2 = 1\right\} \tag{B.11}$$

and

$$-\lambda_2^n(\beta) = \inf\left\{\frac{1}{2}\int_{\mathbb{R}^d}(\nabla f a\nabla f)\mathrm{e}^{2Q}\,\mathrm{d}x - \int_{\mathbb{R}^d} f^2\beta\mathrm{e}^{2Q}\,\mathrm{d}x\colon\ f\in C_c^\infty(D_n), \|f\|_2 = 1\right\}. \tag{B.12}$$

For any $n \geq 1$, by using (B.1)–(B.3) and Proposition B.5 we can easily see that $\beta \in \mathbf{K}_\infty(\xi^{(n)})$. (The definition of the Kato class $\mathbf{K}_\infty(\xi^{(n)})$ is similar to Definition 1.6; see [4] for details.) Thus it follows from [3], Theorem 2.3, that for any $n \geq 1$,

$$-\lambda_2^n(\beta) = \lim_{t\to\infty}\frac{1}{t}\log\sup_{x\in D_n} P_t^{\beta,n}1(x).$$

Since $\lambda_2^n(\beta) \to \lambda_2(\beta)$, combining the above with (B.11)–(B.12) yields the conclusion of our proposition. □

B.3. Properties of the gauge function

Recall that the gauge function g_β is defined in Definition 1.5. For any open set $D \subset \mathbb{R}^d$ and nonnegative measurable function f on ∂D, we define

$$g^D_{\beta,f}(x) := \Pi_x\big[e_\beta(\tau_D) f(\xi_{\tau_D}) 1_{\{\tau_D<\infty\}}\big], \quad x \in D.$$

The Harnack-type inequalities in the following result will be used later.

Lemma B.7.

(1) *For any open set $D \subset \mathbb{R}^d$ and nonnegative measurable function f on ∂D, if the function $g^D_{\beta,f}$ is not identically infinite on D, then for any compact set K, $g^D_{\beta,f}$ is bounded on K and there exists $A = A(D, K, \beta) > 1$, independent of f, such that*

$$\sup_{x\in K} g^D_{\beta,f}(x) \le A \inf_{x\in K} g^D_{\beta,f}(x). \tag{B.13}$$

Furthermore, $g^D_{\beta,f}$ is a continuous solution of $(L + \beta)h = 0$ in D in the sense of distributions.

(2) *If g_β is not identically infinite in $\mathbb{R}^d$, then for any compact set $K \subset \mathbb{R}^d$, $g_\beta(x)$ is bounded on K and there exists an $A = A(K, \beta) > 1$ such that*

$$\sup_{x\in K} g_\beta(x) \le A \inf_{x\in K} g_\beta(x). \tag{B.14}$$

Furthermore, g_β is a continuous solution of $(L + \beta)h = 0$ in $\mathbb{R}^d$ in the sense of distributions.

(3) *If g_β is not identically zero in $\mathbb{R}^d$, then $g_\beta(x) > 0$ for all $x \in \mathbb{R}^d$.*

Proof. (1) The proof follows the same line of arguments as that of [5], Theorem 5.18. Without loss of generality, we may and do assume that $K \subset B(0, n)$ and that there exists $x_1 \in K$ such that $g^D_{\beta,f}(x_1) < \infty$. Then, by the definition of $g^D_{\beta,f}$ and the strong Markov property, for any ball $B = B(x_1, r) \subset \overline{B(x_1, r)} \subset D$, we have

$$g^D_{\beta,f}(x_1) = \Pi_{x_1}\big[e_\beta(\tau_B) g^D_{\beta,f}(\xi_{\tau_B})\big].$$

By (B.7)–(B.9) and Proposition B.5, for any $\varepsilon > 0$, we can choose $r_0 = r_0(n, \beta) \in (0, 1]$ such that for any $r \in (0, r_0)$ and any $(x, z) \in B \times \partial B$:

$$\Pi^z_x \int_0^{\tau_B} e_{|\beta|}(t)\,\mathrm{d}t \le \frac{1}{2},$$

where Π^z_x stands for the law of the $M_B(\cdot, z)$-conditioned diffusion, i.e., the process such that for all bounded Borel function on B and $t > 0$,

$$\Pi^z_x\big[f(\xi_t)\big] = \frac{1}{M_B(x, z)} \Pi_x\big[f(\xi_t) M_B(\xi_t, z); t < \tau_B\big].$$

Repeating the argument of [5], Theorem 5.17, we get that

$$\frac{1}{2} \le \Pi^z_x e_\beta(\tau_B) \le 2.$$

Put $v(x, z) := \Pi^z_x e_\beta(\tau_B)$, then by [5], Proposition 5.12 (which is also valid for ξ by the same arguments contained in [5], Section 5.2) we have

$$g^D_{\beta,f}(x_1) = \int_{\partial B} v(x_1, z) K_B(x_1, z) g^D_{\beta,f}(z)\,\sigma(\mathrm{d}z),$$

where σ stands for the surface measure on ∂B and K_B is the Poisson kernel of B with respect to ξ. It follows from the Harnack inequality (applied to the harmonic functions of ξ) that there exists some $c > 1$ such that

$$\sup_{x\in B(x_1,r/2)} K_B(x,z) \le c \inf_{x\in B(x_1,r/2)} K_B(x,z), \quad \forall z \in \partial B.$$

Since, for $x \in B$,

$$g^D_{\beta,f}(x) = \int_{\partial B} v(x,z) K_B(x,z) g^D_{\beta,f}(z)\sigma(\mathrm{d}z),$$

therefore we have

$$\sup_{x\in B(x_1,r/2)} g^D_{\beta,f}(x) \le c \inf_{x\in B(x_1,r/2)} g^D_{\beta,f}(x). \tag{B.15}$$

Now (B.13) follows from a standard chain argument. In fact, for any compact subset K of D, there exist $r \in (0,1]$ and an integer $N > 1$ such that, for any $x, x' \in K$, there exists a subset $\{y_i\colon i = 1,\dots,l\}$, $1 \le l \le N$, with $\overline{B(y_i,r)} \subset D$, $i = 1,\dots,l$, and

$$|x - y_1| < \frac{r}{2}, \qquad |y_i - y_{i+1}| < \frac{r}{2}, \quad i = 1,\dots,l-1, \qquad |x' - y_l| < \frac{r}{2}.$$

Applying (B.15) repeatedly, we arrive at (B.13). The last assertion of (1) can be proved by repeating the argument of the Corollary to [5], Theorem 5.18, and we omit the details.

(2) The proof of (2) is similar to that of (1).

(3) The proof of this part is similar to that of [5], Proposition 8.10, and we omit the details. □

B.4. The operator G^β

For any $f \ge 0$ on $\mathbb{R}^d$, set

$$G^\beta f(x) := \Pi_x \int_0^\infty e_\beta(s) f(\xi_s)\,\mathrm{d}s. \tag{B.16}$$

$G^0 f$ will be denoted as Gf. The following result will be needed later.

Lemma B.8. *Suppose that $f \ge 0$ is locally bounded on $\mathbb{R}^d$. If there exists an $x_1 \in \mathbb{R}^d$ such that $G^\beta f(x_1) < \infty$, then $G^\beta f$ is locally bounded on $\mathbb{R}^d$.*

Proof. The proof is similar to that of the first part of Lemma B.7. For convenience, we put $\widetilde{f} := G^\beta f$ in this proof. Without the loss of generality, we may and do assume that the compact set K satisfies $K \subset B(0,n)$, and furthermore, that there exists an $x_1 \in K$ such that $\widetilde{f}(x_1) < \infty$. Let $v(x,z) := \Pi^z_x e_\beta(\tau_B)$. By the strong Markov property, for any $B = B(x_1,r)$, we have

$$\begin{aligned}\widetilde{f}(x_1) &= \Pi_{x_1}\int_0^{\tau_B} e_\beta(s) f(\xi_s)\,\mathrm{d}s + \Pi_{x_1}\left[e_\beta(\tau_B)\Pi_{\xi_{\tau_B}}\int_0^\infty e_\beta(s) f(\xi_s)\,\mathrm{d}s\right]\\ &= \Pi_{x_1}\int_0^{\tau_B} e_\beta(s) f(\xi_s)\,\mathrm{d}s + \int_{\partial B} v(x_1,z) K_B(x_1,z)\widetilde{f}(z)\sigma(\mathrm{d}z).\end{aligned} \tag{B.17}$$

By (B.7)–(B.9), Proposition B.5 and the argument of [5], Theorem 5.17, for any $\varepsilon > 0$, we can choose $r_0 = r_0(n,\beta) \in (0,1]$ such that for any $r \in (0,r_0)$ and any $(x,z) \in B \times \partial B$:

$$\frac{1}{2} \le \Pi^z_x\left[e_\beta(\tau_B)\right] \le \Pi^z_x\left[e_{|\beta|}(\tau_B)\right] \le 2; \qquad \Pi_x \tau_B^2 \le 2; \qquad \Pi_x\left[e_{2|\beta|}(\tau_B)\right] \le 2.$$

We then have

$$\widetilde{f}(x_1) \geq \frac{1}{2} \int_{\partial B} K_B(x_1, z) \widetilde{f}(z) \sigma(\mathrm{d}z)$$

and

$$\begin{aligned}
\widetilde{f}(x) &= \Pi_x \int_0^{\tau_B} e_\beta(s) f(\xi_s)\,\mathrm{d}s + \int_{\partial B} v(x,z) K_B(x,z) \widetilde{f}(z) \sigma(\mathrm{d}z) \\
&\leq C \Pi_x\big(\tau_B e_{|\beta|}(\tau_B)\big) + \int_{\partial B} v(x,z) K_B(x,z) \widetilde{f}(z) \sigma(\mathrm{d}z) \\
&\leq C\big[\Pi_x \tau_B^2\big]^{1/2}\big[\Pi_x\big[e_{2|\beta|}(\tau_B)\big]\big]^{1/2} + \int_{\partial B} v(x,z) K_B(x,z) \widetilde{f}(z) \sigma(\mathrm{d}z),
\end{aligned}$$

where C is the upper bound of f on B. It follows from the Harnack inequality (for harmonic functions of ξ) that there exists some $c > 1$ such that

$$\sup_{x \in B(x_1, r/2)} K_B(x,z)) \leq c \inf_{x \in B(x_1, r/2)} K_B(x,z).$$

Thus

$$\sup_{x \in B(x_1, r/2)} \widetilde{f}(x) \leq 2C + 4c \widetilde{f}(x_1).$$

Now the assertion of the lemma follows from a standard chain argument, as was done in the proof of Lemma B.7(1). □

Acknowledgements

The first author owes thanks to Zenghu Li for valuable discussions about path regularity questions, and to Peking University for their hospitality when visiting Y. Ren. We also thank the two referees for several helpful comments and suggestions on the first version of this paper.

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