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# Smoothing effect of rough differential equations driven by fractional Brownian motions

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**Abstract.** In this work we study the smoothing effect of rough differential equations driven by a fractional Brownian motion with parameter $H > 1/4$. The regularization estimates we obtain generalize to the fractional Brownian motion previous results by Kusuoka and Stroock.

**Résumé.** Dans ce travail nous étudions l'effet de régularisation pour des équations différentielles stochastiques conduites par un mouvement brownien fractionnaire de paramètre $H > 1/2$. Les estimées obtenues généralisent des estimées obtenues précédemment par Kusuoka et Stroock.

*MSC:* 60

*Keywords:* Rough paths; Smoothing effect; Fractional Brownian motion

## 1. Introduction

In this paper, we study stochastic differential equations driven by a fractional Brownian motion with Hurst parameter $H \in (1/4, 1)$. More precisely, let us consider the equation

$$X_t^x = x + \sum_{i=1}^{d} \int_0^t V_i\left(X_s^x\right) \mathrm{d}B_s^i, \tag{1.1}$$

where the vector fields $V_1, \ldots, V_d$ are $C^\infty$-bounded vector fields on $\mathbb{R}^n$ and where $B$ is a continuous $\mathbb{R}^d$-valued centered Gaussian process with covariance

$$\mathbb{E}(B_s \otimes B_t) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

The parameter $H$ is the so-called Hurst parameter of the fractional Brownian motion. It quantifies the sample path regularity of $B$ since a straightforward application of the Kolmogorov continuity theorem implies that the paths of $B$ are almost surely locally Hölder of index $H - \varepsilon$ for $0 < \varepsilon < H$. When $H = 1/2$, $B$ is a Brownian motion. Fractional Brownian and equations driven by it appear as a natural model in biology and physics (see for instance [10,21,22]).

If $H > 1/2$, then the paths of $B$ are regular enough and the equation (1.1) is understood in the sense of Young. Existence and uniqueness of solutions are well-known in that case (see [16,19,23]). When $1/4 < H \leq 1/2$, it can be shown (see [7]) that $B$ can canonically be lifted as a geometric $p$-rough path with $p > 1/H$. As a consequence, rough paths theory (see [8,17]) can be used to give a sense to what is solution of equation (1.1). In the case $H = 1/2$, this notion of solution coincides with the solution of the corresponding Stratonovitch stochastic differential equation.

In the past few years, the study of the regularity of the law of $X_t^x$ has generated a great amount of work. In [2], the authors prove, in the regular case $H > 1/2$, that if the vector fields $V_1, \dots, V_d$ satisfy the classical Hörmander's bracket generating condition, then for $t > 0$, the random variable $X_t^x$ admits a smooth density with respect to the Lebesgue measure. In [4], the authors prove, in the case $H > 1/4$, and under the same assumption on the vector fields, the existence of the density. The smoothness of this density is proved in [9] for $H > 1/3$, conditioned on the integrability of the Jacobian of such systems which is established in [6]. Finally, smoothness of the density function in the case $H > 1/4$ is proved in [5].

The regularity of the law of $X_t^x$ is intimately related to the regularization properties of the operator:

$$P_t f(x) = \mathbb{E}\big(f\big(X_t^x\big)\big),$$

that is defined for a Borel and bounded function $f$. It should be noted that when $H \neq 1/2$, $(P_t)_{t\geq 0}$ is not a semigroup and that there is no direct connection with the theory of partial differential equations unless the vector fields $V_1, \dots, V_d$ commute (see [1] for further discussion on that aspect). By regularization property of $P_t$, we mean that $P_t$ has a "smoothing" effect on the function $f$ in the sense that all the $V_i$'s directional derivatives of $P_t f$, for every $t > 0$, can be controlled in terms of the sup-norm of $f$ only. In the Brownian motion case, that is if $H = 1/2$, the regularization property of $P_t$ has been extensively studied and explicitly quantified by Kusuoka and Stroock [12–14] and Kusuoka [11]. In particular, in his work [11], Kusuoka introduces the UFG condition on the vector fields (this is our Assumption 3.1) and proves that if this condition is satisfied, then the following theorem holds:

**Theorem 1.1 (Brownian motion case, Kusuoka [11]).** *Assume that the vector fields $V_1, \dots, V_d$ satisfy Kusuoka's UFG condition* (*see Assumption* 3.1). *Let $x \in \mathbb{R}^n$. For any integer $k \geq 1$ and $0 \leq i_1, \dots, i_k \leq d$, there exists a constant $C > 0$* (*depending on $x$*) *such that for every $C^\infty$ bounded function $f$ and $t \in (0, 1]$,*

$$\big|V_{i_1} \cdots V_{i_k} P_t f(x)\big| \leq C t^{-k/2} \|f\|_\infty.$$

The main purpose of the present paper is to generalize this statement to any $H \in (1/4, 1)$. More precisely, we prove the following theorem:

**Theorem 1.2 (Fractional Brownian motion case).** *Assume $H \in (1/4, 1)$ and that the vector fields $V_1, \dots, V_d$ satisfy Kusuoka's UFG condition. Let $x \in \mathbb{R}^n$. For any integer $k \geq 1$ and $0 \leq i_1, \dots, i_k \leq d$, there exists a constant $C > 0$* (*depending on $x$*) *such that for every $C^\infty$ bounded function $f$ and $t \in (0, 1]$,*

$$\big|V_{i_1} \cdots V_{i_k} P_t f(x)\big| \leq C t^{-Hk} \|f\|_\infty.$$

Our result is obviously an extension of Kusuoka's result, since it encompasses the case $H = 1/2$. It is interesting to observe that the framework given by the most recent developments in rough paths theory (see in particular [5,6,9]) actually simplifies Kusuoka's approach and, in our opinion, provides an overall simpler and clearer proof of his result which originally built on delicate estimates proved in [12–14].

We should also mention that Theorem 1.2 was already proved by two of the authors in the regular case $H > 1/2$ and under a strong ellipticity assumption on the vector fields, see [3]. The rough setting and the more general UFG assumption on the vector fields make the proof of Theorem 1.2 much more difficult.

The paper is organized as follows. In Section 2, we give the necessary background on Malliavin calculus that will be needed throughout the paper. In Section 3, we show how the integration by part technique of Kusuoka–Stroock [14] and Kusuoka [11] can essentially be adapted to the fractional Brownian motion case after suitable changes. Let us however observe that we obtain the correct order in $t$ by using a rescaling argument on the vector fields $V_i$'s instead of analyzing the small time behavior of the estimates of Section 2.

Section 4 is devoted to the proof of the main technical estimates that are needed to justify the integration by parts performed in Section 3. In a sense, it is the heart of our contribution. In the Brownian motion case, similar estimates are obtained in [11,13,14], but the proof of those heavily relies on Markov and martingale methods. We prove here that such estimates may be obtained in a more general setting by using quantitative rough paths versions of Norris' type lemma (see [2] and [9]) which are based on interpolation inequalities and small ball probability estimates for fractional Brownian motions (see [15]).

## 2. Stochastic differential driven by fractional Brownian motions

In this preliminary section, we present the tools about the stochastic analysis of the fractional Brownian motion that are needed for the remainder of the paper.

### 2.1. *Fractional Brownian motion*

A fractional Brownian motion $B=(B^1,\dots,B^d)$ is a $d$-dimensional centered Gaussian process, whose covariance is given by

$$R(t,s) := \mathbb{E}\big(B_s^j B_t^j\big) = \frac{1}{2}\big(s^{2H}+t^{2H}-|t-s|^{2H}\big), \quad \text{for } s,t\in[0,1] \text{ and } j=1,\dots,d.$$

In particular it can be shown, by a standard application of Kolmogorov's criterion, that $B$ admits a continuous version whose paths are $\gamma$-Hölder continuous for any $\gamma < H$.

Let $\mathcal{E}$ be the space of $\mathbb{R}^d$-valued step functions on $[0,1]$, and $\mathcal{H}$ the closure of $\mathcal{E}$ for the scalar product:

$$\big\langle(\mathbf{1}_{[0,t_1]},\dots,\mathbf{1}_{[0,t_d]}),(\mathbf{1}_{[0,s_1]},\dots,\mathbf{1}_{[0,s_d]})\big\rangle_{\mathcal{H}} = \sum_{i=1}^d R(t_i,s_i).$$

When $H>\frac{1}{2}$ it can be shown that $\mathbf{L}^{1/H}([0,1],\mathbb{R}^d)\subset\mathcal{H}$, and that for $\phi,\psi\in\mathbf{L}^{1/H}([0,1],\mathbb{R}^d)$, we have

$$\langle\phi,\psi\rangle_{\mathcal{H}} = H(2H-1)\int_0^1\int_0^1 |s-t|^{2H-2}\big\langle\phi(s),\psi(t)\big\rangle_{\mathbb{R}^d}\,\mathrm{d}s\,\mathrm{d}t.$$

The following interpolation inequality that was proved in [2], will be an essential tool in our analysis. For every $\gamma > H-\frac{1}{2}$, there exists a constant $C$ such that for every continuous function $f\in\mathcal{H}$,

$$\|f\|_{\mathcal{H}} \geq C\frac{\|f\|_\infty^{3+1/\gamma}}{\|f\|_\gamma^{2+1/\gamma}}, \tag{2.1}$$

where

$$\|f\|_\gamma = \sup_{0\leq s<t\leq 1}\frac{\|f(t)-f(s)\|}{|t-s|^\gamma} + \|f\|_\infty,$$

is the usual Hölder norm.

When $\frac{1}{4}<H<\frac{1}{2}$ one has

$$\mathcal{H}\subset\mathbf{L}^2\big([0,1]\big)$$

and the following interpolation inequality classically holds for every $f\in\mathcal{H}$,

$$\|f\|_{\mathcal{H}}\geq C\|f\|_{L^2}.$$

Let us also mention the following inequality that will be useful to bound from below the $L^2$ norm by the supremum norm and the Hölder norm

$$\|f\|_\infty \leq 2\max\big\{\|f\|_{L^2},\|f\|_{L^2}^{2\gamma/(2\gamma+1)}\|f\|_\gamma^{1/(2\gamma+1)}\big\}.$$

We point out that such inequality was already used in connection with the space $\mathcal{H}$ in [9].

## 2.2. *Malliavin calculus*

Let us remind the basic framework of Malliavin calculus (see [18] for further details). A real valued random variable $F$ is then said to be cylindrical if it can be written, for a given $n \geq 1$, as

$$F = f\left(\int_0^1 \langle h_s^1, \mathrm{d}B_s\rangle, \ldots, \int_0^1 \langle h_s^n, \mathrm{d}B_s\rangle\right),$$

where $h^i \in \mathcal{H}$ and $f:\mathbb{R}^n \to \mathbb{R}$ is a $C^\infty$-bounded function. The set of cylindrical random variables is denoted $\mathcal{S}$. The Malliavin derivative is then defined as follows: for $F \in \mathcal{S}$, the derivative of $F$ is the $\mathbb{R}^d$-valued stochastic process $(\mathbf{D}_t F)_{0\leq t\leq 1}$ given by

$$\mathbf{D}_t F = \sum_{i=1}^n h^i(t)\frac{\partial f}{\partial x_i}\left(\int_0^1 \langle h_s^1, \mathrm{d}B_s\rangle, \ldots, \int_0^1 \langle h_s^n, \mathrm{d}B_s\rangle\right).$$

More generally, we can introduce iterated derivatives. If $F \in \mathcal{S}$, we set

$$\mathbf{D}^k_{t_1,\ldots,t_k} F = \mathbf{D}_{t_1}\cdots\mathbf{D}_{t_k} F.$$

For any $p \geq 1$, it can be checked that the operator $\mathbf{D}^k$ is closable from $\mathcal{S}$ into $\mathbf{L}^p(\Omega)$. We will denote by $\mathbb{D}^{k,p}$ the domain of this closure, that is closure of the class of cylindrical random variables with respect to the norm

$$\|F\|_{k,p} = \left(\mathbb{E}\big(|F|^p\big) + \sum_{j=1}^k \mathbb{E}\big(\|\mathbf{D}^j F\|^p_{\mathcal{H}^{\otimes j}}\big)\right)^{1/p}$$

and

$$\mathbb{D}^\infty = \bigcap_{p\geq 1}\bigcap_{k\geq 1}\mathbb{D}^{k,p}.$$

For $p > 1$ we can consider the divergence operator $\delta$ which is defined as the adjoint of $\mathbf{D}$ defined on $\mathbf{L}^p(\Omega)$. It is characterized by the duality formula:

$$\mathbb{E}(F\delta u) = \mathbb{E}\big(\langle \mathbf{D}F, u\rangle_{\mathcal{H}}\big), \quad F \in \mathbb{D}^{1,p}.$$

It is proved in [18], Proposition 1.5.7 that $\delta$ is continuous from $\mathbb{D}^{1,p}$ into $\mathbf{L}^p(\Omega)$.

## 2.3. *Stochastic differential equations driven by fractional Brownian motions*

In this paper, we will consider the following kind of equation:

$$X_t^x = x + \sum_{i=1}^d \int_0^t V_i\big(X_s^x\big)\,\mathrm{d}B_s^i, \tag{2.2}$$

where the vector fields $V_1, \ldots, V_d$ are $C^\infty$ bounded vector fields on $\mathbb{R}^n$ and where $B$ is a fractional Brownian motion with parameter $H \in (1/4, 1)$.

If $H > 1/2$. The equation (2.2) is understood in Young's sense, but if $H \in (1/3, 1/2]$, we need to understand the equation in the sense of rough paths theory (see, e.g., [7,8]). In both cases, the $C^\infty$ boundedness of the vector fields is more than enough to ensure the existence and uniqueness of solutions.

Once equation (2.2) is solved, the vector $X_t^x$ is a typical example of random variable which can be differentiated in the sense of Malliavin. It is classical that one can express this Malliavin derivative in terms of the first variation process $J$ of the equation, which is defined by the relation $J^{ij}_{0\to t} = \partial_{x_j} X^{x,i}_t$. Setting $\partial V_j$ for the Jacobian of $V_j$ seen as a function from $\mathbb{R}^n$ to $\mathbb{R}^n$, it is well known that $J$ is the unique solution to the linear equation

$$J_{0\to t} = I + \sum_{j=1}^{d} \int_0^t \partial V_j\big(X^x_s\big) J_{0\to s}\, \mathrm{d}B^j_s, \tag{2.3}$$

and that the following results hold true (see [4] and [20] for further details):

**Proposition 2.1.** *Let $X^x$ be the solution to equation (2.2). Then for every $i = 1, \dots, n$ and $t > 0$, and $x \in \mathbb{R}^n$, we have $X^{x,i}_t \in \mathbb{D}^\infty$ and*

$$\mathbf{D}^j_s X^x_t = J_{s\to t} V_j(X_s), \quad j = 1, \dots, d, 0 \le s \le t,$$

*where $\mathbf{D}^j_s X^{x,i}_t$ is the $j$th component of $\mathbf{D}_s X^{x,i}_t$, $J_{0\to t} = \partial_x X^x_t$ and $J_{s\to t} = J_{0\to t} J^{-1}_{0\to s}$.*

We finally mention the recent result [6], which gives a useful estimate for moments of the Jacobian of rough differential equations driven by Gaussian processes.

**Proposition 2.2.** *Let $p > 1/H$. For any $n \ge 0$,*

$$\mathbb{E}\big(\|J\|^n_{p\text{-var};[0,1]}\big) < +\infty, \tag{2.4}$$

*where $\|\cdot\|_{p\text{-var};[0,1]}$ denotes the $p$-variation norm on the interval* [0, 1].

## 3. Integration by parts formula and regularization

Let us consider vector fields $V_1, \dots, V_d$ on $\mathbb{R}^n$. Let $\mathcal{A} = \{\varnothing\} \cup \bigcup_{k=1}^\infty \{1, 2, \dots, d\}^k$ and $\mathcal{A}_1 = A \setminus \{\varnothing\}$. We say that $I \in \mathcal{A}$ is a word of length $k$ if $I = (i_1, \dots, i_k)$ and we write $|I| = k$. If $I = \varnothing$, then we denote $|I| = 0$. For any integer $l \ge 1$, we denote by $\mathcal{A}(l)$ the set $\{I \in \mathcal{A}; |I| \le l\}$ and by $\mathcal{A}_1(l)$ the set $\{I \in \mathcal{A}_1; |I| \le l\}$. We also define an operation $*$ on $\mathcal{A}$ by $I * J = (i_1, \dots, i_k, j_1, \dots, j_l)$ for $I = (i_1, \dots, i_k)$ and $J = (j_1, \dots, j_l)$ in $\mathcal{A}$. We define vector fields $V_{[I]}$ inductively by

$$V_{[j]} = V_j, \qquad V_{[I*j]} = [V_{[I]}, V_j], \quad j = 1, \dots, d.$$

Throughout this paper, we will make the following assumptions on the vector fields.

***Assumption 3.1.***

1. *The $V_i$'s are bounded smooth vector fields on $\mathbb{R}^n$ with bounded derivatives at any order.*
2. *There exists an integer $l \ge 1$ and $\omega^J_I \in C^\infty_b(\mathbb{R}^n, \mathbb{R})$ such that for any $x \in \mathbb{R}^n$*

$$V_{[I]}(x) = \sum_{J \in \mathcal{A}(l)} \omega^J_I(x) V_{[J]}(x), \quad I \in \mathcal{A}_1. \tag{3.1}$$

The second condition was introduced by S. Kusuoka in [11]. It holds for a system of vector fields that satisfy a uniform strong Hörmander's bracket generating condition, but observe that in order that Assumption 3.1, (3.1), holds, it is not even necessary that the bracket generating condition holds. As a consequence $X^{\epsilon,x}$ below may be degenerate in the sense of Malliavin, and this is actually one of the main difficulties we have to overcome in our analysis.

### 3.1. *Integration by parts formula*

Let us consider the following rescaled differential equations, which depend on the parameter $\epsilon > 0$:

$$\begin{aligned} X_t^{\epsilon,x} &= x + \sum_{i=1}^d \int_0^t V_i^\epsilon\big(X_s^{\epsilon,x}\big)\,\mathrm{d}B_s^i \\ &= x + \sum_{i=1}^d \int_0^t \epsilon^H V_i\big(X_s^{\epsilon,x}\big)\,\mathrm{d}B_s^i. \end{aligned} \tag{3.2}$$

Clearly, the rescaled vector fields $V_i^\epsilon$ are defined as $V_i^\epsilon(x) = \epsilon^H V_i(x)$. More generally, for any $I \in \mathcal{A}_1(l)$, we denote $V_{[I]}^\epsilon(x) = \epsilon^{|I|H} V_{[I]}(x)$. Note that:

$$\begin{aligned} V_{[I]}^\epsilon(x) &= \epsilon^{|I|H} V_{[I]}(x) \\ &= \sum_{J\in\mathcal{A}_1(l)} \epsilon^{|I|H} \omega_I^J(x) V_{[J]}(x) \\ &= \sum_{J\in\mathcal{A}_1(l)} \epsilon^{(|I|-|J|)H} \omega_I^J(x) V_{[J]}^\epsilon(x) \\ &= \sum_{J\in\mathcal{A}_1(l)} \omega_I^{J,\epsilon}(x) V_{[J]}^\epsilon(x), \end{aligned}$$

where $\omega_I^{J,\epsilon}(x) = \epsilon^{(|I|-|J|)H}\omega_I^J(x)$.

It is known that for any $\epsilon \in (0,1]$ and any $t > 0$, the map $\Phi_t^\epsilon(x) = X_t^{\epsilon,x} : \mathbb{R}^n \to \mathbb{R}^n$ is a flow of $C^\infty$ diffeomorphism (see [8]). We denote the Jacobian of $\Phi_t^\epsilon(x)$ by $J_{0\to t}^\epsilon = \frac{\partial X_t^{\epsilon,x}}{\partial x}$. As we mentioned it earlier, $J_{0\to t}^\epsilon$ and $(J_{0\to t}^\epsilon)^{-1}$ satisfy the following linear equations:

$$\mathrm{d}J_{0\to t}^\epsilon = \sum_{i=1}^d \partial V_i^\epsilon\big(X_t^{\epsilon,x}\big) J_{0\to t}^\epsilon\,\mathrm{d}B_t^i, \quad \text{with } J_0^\epsilon = I, \tag{3.3}$$

and

$$\mathrm{d}\big(J_{0\to t}^\epsilon\big)^{-1} = -\sum_{i=1}^d \big(J_{0\to t}^\epsilon\big)^{-1} \partial V_i^\epsilon\big(X_t^{\epsilon,x}\big)\,\mathrm{d}B_t^i, \quad \text{with } \big(J_0^\epsilon\big)^{-1} = I. \tag{3.4}$$

Let us introduce a linear system $\beta_I^{J,\epsilon}(t,x)$ that satisfies the following linear equations:

$$\begin{cases} \mathrm{d}\beta_I^{J,\epsilon}(t,x) = \sum_{j=1}^d \big(\sum_{K\in\mathcal{A}_1(l)} -\omega_{I*j}^{K,\epsilon}(X_t^{x,\epsilon})\beta_K^{J,\epsilon}(t,x)\big)\,\mathrm{d}B_t^j, \\ \beta_I^{J,\epsilon}(0,x) = \delta_I^J. \end{cases} \tag{3.5}$$

Our first result concerns the representation of the pullback of the vector fields $V_{[I]}^\epsilon(X_t^{\epsilon,x})$ in terms of the $\beta_I^{J,\epsilon}(t,x)$'s.

**Lemma 3.2.** *Fix* $\epsilon \in (0,1]$. *For any* $I \in \mathcal{A}_1(l)$, *we have*:

$$\big(J_{0\to t}^\epsilon\big)^{-1}\big(V_{[I]}^\epsilon\big(X_t^{\epsilon,x}\big)\big) = \sum_{J\in\mathcal{A}_1(l)} \beta_I^{J,\epsilon}(t,x) V_{[J]}^\epsilon(x).$$

**Proof.** To simplify the notation, let us denote

$$a_I^\epsilon(t,x) = \big(J_{0\to t}^\epsilon\big)^{-1}\big(V_{[I]}^\epsilon\big(X_t^{\epsilon,x}\big)\big),$$

and

$$b_I^\epsilon(t,x) = \sum_{J\in\mathcal{A}_1(l)} \beta_I^{J,\epsilon}(t,x) V_{[J]}^\epsilon(x).$$

By definition, we have $a_I^\epsilon(0,x) = b_I^\epsilon(0,x) = V_{[I]}^\epsilon(x)$. Next, we show that $a_I^\epsilon(t,x)$ and $b_I^\epsilon(t,x)$ satisfy the same differential equation, from which we can deduce that $a_I^\epsilon(t,x) = b_I^\epsilon(t,x)$. Indeed, by the change of variable formula, which can be used since the driving noise is described by a geometric rough path, we have:

$$\begin{aligned}
\mathrm{d}a_I^\epsilon(t,x) &= \mathrm{d}\big(J_{0\to t}^\epsilon\big)^{-1}\big(V_{[I]}^\epsilon\big(X^{\epsilon,x}\big)\big) \\
&= \sum_{j=1}^d (-1)\big(J_{0\to t}^\epsilon\big)^{-1}\big[V_{[I]}^\epsilon, V_j^\epsilon\big]\big(X_t^{\epsilon,x}\big)(x)\,\mathrm{d}B_t^j \\
&= \sum_{j=1}^d \sum_{J\in\mathcal{A}_1(l)} -\omega_{I*j}^{J,\epsilon}\big(X_t^{\epsilon,x}\big)\big(J_{0\to t}^\epsilon\big)^{-1} V_{[J]}^\epsilon\big(X_t^{\epsilon,x}\big)\,\mathrm{d}B_t^j \\
&= \sum_{j=1}^d \sum_{J\in\mathcal{A}_1(l)} -\omega_{I*j}^{J,\epsilon}\big(X_t^{\epsilon,x}\big) a_J^\epsilon(t,x)\,\mathrm{d}B_t^j.
\end{aligned}$$

On the other hand, by the definition of $\beta_I^{J,\epsilon}(t,x)$, we have:

$$\begin{aligned}
\mathrm{d}b_I^\epsilon(t,x) &= \mathrm{d}\bigg(\sum_{K\in\mathcal{A}_1(l)} \beta_I^{K,\epsilon}(t,x) V_{[K]}^\epsilon(x)\bigg) \\
&= \sum_{K\in\mathcal{A}_1(l)} \mathrm{d}\beta_I^{K,\epsilon}(t,x) V_{[K]}^\epsilon(x) \\
&= \sum_{j=1}^d \sum_{J\in\mathcal{A}_1(l)} -\omega_{I*j}^{J,\epsilon}\big(X_t^{\epsilon,x}\big) \sum_{K\in\mathcal{A}_1(l)} \beta_J^{K,\epsilon}(t,x) V_{[K]}^\epsilon(x)\,\mathrm{d}B_t^j \\
&= \sum_{j=1}^d \sum_{J\in\mathcal{A}_1(l)} -\omega_{I*j}^{J,\epsilon}\big(X_t^{\epsilon,x}\big) b_J^\epsilon(t,x)\,\mathrm{d}B_t^j.
\end{aligned}$$

The result then follows by the uniqueness of solutions for rough linear equations. □

We now turn to the integration by parts formula and introduce the following notations: for any $J\in\mathcal{A}_1(l)$,

$$D^{(J)} f\big(X_t^{\epsilon,x}\big) = \big\langle \mathbf{D}.f\big(X_t^{x,\epsilon}\big), \beta^{J,\epsilon}(\cdot,x)1_{[0,t]}(\cdot)\big\rangle_{\mathcal{H}},$$

where we denote by $\beta^{J,\epsilon}(\cdot,x)$ the column vector $(\beta_i^{J,\epsilon}(\cdot,x))_{i=1,\dots,n}$. For any $I, J\in\mathcal{A}_1(l)$, we define

$$M_{I,J}^\epsilon(t,x) = \big\langle \beta^{I,\epsilon}(\cdot,x)1_{[0,t]}(\cdot), \beta^{J,\epsilon}(\cdot,x)1_{[0,t]}(\cdot)\big\rangle_{\mathcal{H}}.$$

In the following, we will only consider the case $t=1$ and we write $M_{I,J}^\epsilon(x)$ instead of $M_{I,J}^\epsilon(1,x)$.

The following theorem is the main technical difficulty of our work, its proof is rather long and intricate, so we postpone it to a later section, for the readability of the paper.

**Theorem 3.3.** *For every $x\in\mathbb{R}^n$, the matrix $(M_{I,J}^\epsilon(x))_{I,J\in\mathcal{A}_1(l)}$ is almost surely invertible. Moreover, for any $p\in(1,\infty)$,*

$$\sup_{\epsilon\in(0,1],x\in\mathbb{R}^n} \mathbb{E}\big(\big\|\big(M_{I,J}^\epsilon(x)\big)_{I,J\in\mathcal{A}_1(l)}\big\|^{-p}\big) < \infty.$$

With the theorem in hands, we can now state the basic integration by parts formula. In the sequel the notation $V^{\epsilon}_{[I]}[f(X^{\epsilon,x}_t)]$ should be understood as the differential operator $V^{\epsilon}_{[I]}$ acting on the function $x \to f(X^{\epsilon,x}_t)$. So, by the chain rule we have

$$V^{\epsilon}_{[I]}\big[f\big(X^{\epsilon,x}_t\big)\big] = \big\langle \nabla f\big(X^{\epsilon,x}_t\big), J^{\epsilon}_{0\to t} V^{\epsilon}_{[I]}(x)\big\rangle_{\mathbb{R}^n}.$$

This should not be confused with the notation $V^{\epsilon}_{[I]} f(X^{\epsilon,x}_t)$ which means that the function $V^{\epsilon}_{[I]} f$ is evaluated at $X^{\epsilon,x}_t$.

**Proposition 3.4.** *For any $f \in C^{\infty}_b(\mathbb{R}^n, \mathbb{R})$, $\epsilon \in (0, 1]$ and $x \in \mathbb{R}^n$, we have*

$$V^{\epsilon}_{[I]}\big[f\big(X^{\epsilon,x}_1\big)\big] = \sum_{J\in\mathcal{A}_1(l)} \big(M^{\epsilon}_{I,J}(x)\big)^{-1} D^{(J)} f\big(X^{\epsilon,x}_1\big).$$

**Proof.** First note that by the chain rule together with Lemma 3.2 we have:

$$\begin{aligned}
\mathbf{D}^j_t f\big(X^{\epsilon,x}_1\big) &= \big\langle \nabla f\big(X^{\epsilon,x}_1\big), \mathbf{D}^j_t X^{\epsilon,x}_1\big\rangle_{\mathbb{R}^n} \\
&= \big\langle \nabla f\big(X^{\epsilon,x}_1\big), J^{\epsilon}_{0\to 1}\big(J^{\epsilon}_{0\to t}\big)^{-1} V^{\epsilon}_j\big(X^{\epsilon,x}_s\big)\big\rangle_{\mathbb{R}^n} \\
&= \left\langle \nabla f\big(X^{\epsilon,x}_1\big), J^{\epsilon}_{0\to 1}\left(\sum_{I\in\mathcal{A}_1(l)} \beta^{I,\epsilon}_j(t,x) V^{\epsilon}_{[I]}(x)\right)\right\rangle_{\mathbb{R}^n} \\
&= \left\langle \nabla f\big(X^{\epsilon,x}_1\big), \sum_{I\in\mathcal{A}_1(l)} \beta^{I,\epsilon}_j(t,x) J^{\epsilon}_{0\to 1} V^{\epsilon}_{[I]}(x)\right\rangle_{\mathbb{R}^n} \\
&= \sum_{I\in\mathcal{A}_1(l)} \beta^{I,\epsilon}_j(t,x) V^{\epsilon}_{[I]}\big[f\big(X^{\epsilon,x}_1\big)\big].
\end{aligned}$$

Now for $J \in \mathcal{A}_1(l)$, by definition, we have:

$$\begin{aligned}
D^{(J)} f\big(X^{\epsilon,x}_1\big) &= \big\langle \mathbf{D}_{\cdot} f\big(X^{\epsilon,x}_1\big), \beta^{J,\epsilon}(\cdot,x)\big\rangle_{\mathcal{H}} \\
&= \left\langle \sum_{I\in\mathcal{A}_1(l)} \beta^{I,\epsilon}(\cdot,x) V^{\epsilon}_{[I]}\big[f\big(X^{\epsilon,x}_1\big)\big], \beta^{J,\epsilon}(\cdot,x)\right\rangle_{\mathcal{H}} \\
&= \sum_{I\in\mathcal{A}_1(l)} V^{\epsilon}_{[I]}\big[f\big(X^{\epsilon,x}_1\big)\big]\big\langle \beta^{I,\epsilon}(\cdot,x), \beta^{J,\epsilon}(\cdot,x)\big\rangle_{\mathcal{H}} \\
&= \sum_{I\in\mathcal{A}_1(l)} M^{\epsilon}_{I,J}(x) V^{\epsilon}_{[I]}\big[f\big(X^{\epsilon,x}_1\big)\big].
\end{aligned}$$

Hence we conclude

$$V^{\epsilon}_{[I]}\big[f\big(X^{\epsilon,x}_1\big)\big] = \sum_{J\in\mathcal{A}_1(l)} \big(M^{\epsilon}_{I,J}(x)\big)^{-1} D^{(J)} f\big(X^{\epsilon,x}_1\big).$$

□

Following Kusuoka [11], we set the following definition.

***Definition 3.5.*** *We denote by $\mathcal{K}$ the set of mappings $\Phi(\epsilon, x) : (0, 1] \times \mathbb{R}^n \to \mathbb{D}^{\infty}$ that satisfy the following conditions*:

1. *$\Phi(\epsilon, x)$ is smooth in $x$ and $\frac{\partial^{|\nu|}\Phi}{\partial x^{\nu}}(\epsilon, x)$ is continuous in $(\epsilon, x) \in (0, 1] \times \mathbb{R}^n$ with probability one for any muti-index $\nu$*;
2. *For any $k, p > 1$ and multi-index $\nu$ we have*:

$$\sup_{\epsilon\in(0,1]} \left\| \frac{\partial^{|\nu|}\Phi}{\partial^{\nu} x}(\epsilon, x)\right\|_{\mathbb{D}^{k,p}} < \infty.$$

**Lemma 3.6.** *For every $x \in \mathbb{R}^n$, we have*:

1. $\beta_I^{J,\epsilon}(1,x) \in \mathcal{K}$ *for any* $I, J \in \mathcal{A}_1(l)$;
2. $(M_{I,J}^{\epsilon}(x))^{-1} \in \mathcal{K}$ *for any* $I, J \in \mathcal{A}_1(l)$;
3. $\Psi_I(\epsilon,t,x) = \sum_{J\in\mathcal{A}_1(l)} \beta^{J,\epsilon}(t,x)(M_{I,J}^{\epsilon}(x))^{-1} \in \mathcal{K}$.

**Proof.** This is a direct consequence of Theorem 3.3 and of the fact that $\beta^{\epsilon}$ solves a linear system of equations (see also the Lemma 4.2 below). □

As a consequence of the integration by parts formula, we get then the following key result, which intuitively says that the adjoint of the vector field $V_{[I]}^{\epsilon}$ seen as an operator on the path space of the fractional Brownian motion maps $\mathcal{K}$ into itself.

**Proposition 3.7.** *Let $\Phi(\epsilon,x) \in \mathcal{K}$, then for any $I \in \mathcal{A}_1(l)$, there exists $T^*_{V_{[I]}^{\epsilon}}\Phi(\epsilon,x) \in \mathcal{K}$ such that*

$$\mathbb{E}\big(\Phi(\epsilon,x)V_{[I]}^{\epsilon}\big[f\big(X_1^{\epsilon,x}\big)\big]\big) = \mathbb{E}\big(f\big(X_1^{\epsilon,x}\big)T^*_{V_{[I]}^{\epsilon}}\Phi(\epsilon,x)\big).$$

**Proof.** We have

$$\begin{aligned}
\mathbb{E}\big(\Phi(\epsilon,x)V_{[I]}\big[f\big(X_1^{\epsilon,x}\big)\big]\big) &= \mathbb{E}\Bigg(\Phi(\epsilon,x)\sum_{J\in\mathcal{A}_1(l)}\big(M_{I,J}^{\epsilon}(x)\big)^{-1}D^{(J)}f\big(X_1^{\epsilon,x}\big)\Bigg)\\
&= \mathbb{E}\Bigg(\Phi(\epsilon,x)\sum_{J\in\mathcal{A}_1(l)}\big(M_{I,J}^{\epsilon}(x)\big)^{-1}\big\langle \mathbf{D}_{\cdot}f\big(X_1^{\epsilon,x}\big),\beta^{J,\epsilon}(\cdot,x)\big\rangle_{\mathcal{H}}\Bigg)\\
&= \mathbb{E}\Bigg(\Bigg\langle \mathbf{D}_{\cdot}f\big(X_1^{\epsilon,x}\big),\sum_{J\in\mathcal{A}_1(l)}\beta^{J,\epsilon}(\cdot,x)\big(M_{I,J}^{\epsilon}(x)\big)^{-1}\Phi(\epsilon,x)\Bigg\rangle_{\mathcal{H}}\Bigg)\\
&= \mathbb{E}\big(f\big(X_1^{\epsilon,x}\big)T^*_{V_{[I]}^{\epsilon}}\Phi(\epsilon,x)\big),
\end{aligned}$$

where

$$\begin{aligned}
T^*_{V_{[I]}^{\epsilon}}\Phi(\epsilon,x) &= \delta\Bigg(\sum_{J\in\mathcal{A}_1(l)}\beta^{J,\epsilon}(t,x)\big(M_{I,J}^{\epsilon}(x)\big)^{-1}\Phi(\epsilon,x)\Bigg)\\
&= \delta\big(\Psi_I(\epsilon,t,x)\Phi(\epsilon,x)\big).
\end{aligned}$$

Then, by using the continuity of the divergence $\delta : \mathbb{D}^{k+1} \to \mathbb{D}^k$ and Hölder's inequality we have:

$$\begin{aligned}
\big\|T^*_{V_{[I]}^{\epsilon}}\Phi(\epsilon,x)\big\|_{\mathbb{D}^{k,p}} &\le C_{k,p}\big\|\Psi_I(\epsilon,t,x)\Phi(\epsilon,x)\big\|_{\mathbb{D}^{k+1,p}}\\
&\le C_{k,p}\big\|\Psi_I(\epsilon,t,x)\big\|_{\mathbb{D}^{k+1,r}}\big\|\Phi(\epsilon,x)\big\|_{\mathbb{D}^{k+1,q}},
\end{aligned}$$

where $\frac{1}{r}+\frac{1}{q}=\frac{1}{p}$. □

### 3.2. *Regularization bounds*

Now we are ready to state our main theorem. Consider, as before, the equation:

$$X_t^x = x + \sum_{i=1}^{d}\int_0^t V_i\big(X_s^x\big)\,\mathrm{d}B_s^i, \tag{3.6}$$

where the vector fields $V_1, \dots, V_d$ are $C^{\infty}$ bounded vector fields on $\mathbb{R}^n$ that satisfy the UFG condition of Assumption 3.1, and where $B$ is a fractional Brownian motion with parameter $H \in (1/4, 1)$.

**Theorem 3.8.** *Let $x \in \mathbb{R}^n$ and $p \geq 1$. For any integer $k \geq 1$ and $I_1, \dots, I_k \in \mathcal{A}_1(l)$, there exists a constant $C > 0$ (depending on $x$) such that for every $C^\infty$ bounded function $f$,*

$$\left|V_{[I_1]} \cdots V_{[I_k]} P_t f(x)\right| \leq C t^{-(|I_1|+\cdots+|I_k|)H} \left(P_t f^p(x)\right)^{1/p}, \quad t \in (0,1].$$

**Proof.** Let $\epsilon = t$. By the fact that $X_\epsilon^x$ has the same distribution as $X_1^{\epsilon,x}$, we have:

$$\begin{aligned} V_{[I_1]} \cdots V_{[I_k]} P_t f(x) &= V_{[I_1]} \cdots V_{[I_k]}\left[\mathbb{E}\left(f\left(X_t^x\right)\right)\right] \\ &= V_{[I_1]} \cdots V_{[I_k]}\left[\mathbb{E}\left(f\left(X_\epsilon^x\right)\right)\right] \\ &= \epsilon^{-(|I_1|+\cdots+|I|_k)} V_{[I_1]}^\epsilon \cdots V_{[I_k]}^\epsilon\left[\mathbb{E}\left(f\left(X_1^{\epsilon,x}\right)\right)\right]. \end{aligned}$$

To prove the theorem, it is sufficient to show that there exists $\Phi(\epsilon, x) \in \mathcal{K}$ such that:

$$V_{[I_1]}^\epsilon \cdots V_{[I_k]}^\epsilon\left[\mathbb{E}\left(f\left(X_1^{\epsilon,x}\right)\right)\right] = \mathbb{E}\left(f\left(X_1^{\epsilon,x}\right)\Phi(\epsilon,x)\right). \tag{3.7}$$

And the result will follow by a simple application of Hölder's inequality. We prove the equation (3.7) by induction. When $k=1$, by Proposition 3.7, there exists $T^*_{V^\epsilon_{[I_1]}} 1(\epsilon,x) \in \mathcal{K}$. Now suppose the statement is true for $k=m$, then there exists $\Phi(\epsilon,x) \in \mathcal{K}$ and we have:

$$\begin{aligned} V_{[I_{m+1}]}^\epsilon V_{[I_m]}^\epsilon \cdots V_{[I_1]}^\epsilon\left[\mathbb{E}\left(f\left(X_1^{\epsilon,x}\right)\right)\right] &= V_{[I_{m+1}]}^\epsilon\left[\mathbb{E}\left(f\left(X_1^{\epsilon,x}\right)\Phi(\epsilon,x)\right)\right] \\ &= \mathbb{E}\left(\Phi(\epsilon,x) V_{[I_{m+1}]}^\epsilon\left[f\left(X_1^{\epsilon,x}\right)\right] + f\left(X_1^{\epsilon,x}\right) V_{[I_m]}^\epsilon \Phi(\epsilon,x)\right) \\ &= \mathbb{E}\left(f\left(X_1^{\epsilon,x}\right) T^*_{V^\epsilon_{[I_{m+1}]}} \Phi(\epsilon,x) + f\left(X_1^{\epsilon,x}\right) V_{[I_{m+1}]}^\epsilon \Phi(\epsilon,x)\right) \\ &= \mathbb{E}\left(f\left(X_1^{\epsilon,x}\right)\left(T^*_{V^\epsilon_{[I_{m+1}]}} \Phi(\epsilon,x) + V_{[I_{m+1}]}^\epsilon \Phi(\epsilon,x)\right)\right). \end{aligned}$$

Since by induction hypothesis we know $\Phi(\epsilon,x) \in \mathcal{K}$. Now by Proposition 3.7, we have that $(T^*_{V^\epsilon_{[I_{m+1}]}} \Phi(\epsilon,x) + V_{[I_{m+1}]}^\epsilon \Phi(\epsilon,x)) \in \mathcal{K}$ and this completes the proof. □

As a straightforward corollary of the previous result, we in particular deduce the following regularization result:

**Theorem 3.9.** *For any integer $k \geq 1$ and $I_1, \dots, I_k \in \mathcal{A}_1(l)$, there exists a constant $C > 0$ such that for every $C^\infty$ bounded function $f$,*

$$\left|V_{[I_1]} \cdots V_{[I_k]} P_t f(x)\right| \leq C t^{-(|I_1|+\cdots+|I_k|)H} \|f\|_\infty$$

*for any $t \in (0,1]$.*

## 4. Proof of Theorem 3.3

Our goal in this section is to prove Theorem 3.3 that we rewrite below for convenience:

**Theorem 4.1.** *For any $p \in (1, \infty)$,*

$$\sup_{\epsilon \in (0,1], x \in \mathbb{R}^n} \mathbb{E}\left(\left\|\left(M_{I,J}^\epsilon(x)\right)_{I,J \in \mathcal{A}_1(l)}\right\|^{-p}\right) < \infty.$$

The proof of the Theorem 4.1 is splitted in several steps and we will have to distinguish the cases $H > 1/2$ and $H \leq 1/2$. In the case $H = 1/2$, the result was proved by Kusuoka in [11]. The proof relies on delicate estimates that were obtained in [12–14]. The methods heavily use martingale techniques and the Markov property, and therefore cannot be adapted to our framework. Instead, we are going to use rough paths techniques that were developed in the past few years.

The following first lemma, which comes from a stochastic Taylor type expansion, gives the order of $\beta_I^{J,\epsilon}(t,x)$.

**Lemma 4.2.** *Let $I, J \in \mathcal{A}_1(l)$ such that $|I| \le |J|$, then*

$$\beta_I^{J,\epsilon}(t,x) = \sum_{L\in\mathcal{A}} \delta_{I*L}^J (-1)^{|L|} B_t^L + \gamma_I^{\epsilon,J}(t,x),$$

*where*

$$\sup_{x\in\mathbb{R}^n} \mathbb{E}\Big[\Big(\sup_{t\in(0,1],\epsilon\in(0,1]} t^{-(l+1-|I|)H}\big|\gamma_I^{\epsilon,J}(t,x)\big|\Big)^p\Big] < \infty$$

*holds for any $p \ge 1$.*

**Proof.** Let us consider the Taylor expansion obtained by iterating the equation (3.5). Note that since

$$V_{[I]}(x) = \sum_{J\in\mathcal{A}_1(l)} \omega_I^J(x) V_{[J]}(x),$$

then we know that for any $\epsilon \in (0,1]$ and when $|I| \le l$, $\omega_I^{J,\epsilon} = \omega_I^J = \delta_I^J$. For any $I, J \in \mathcal{A}_1(l)$ with $|I| \le |J|$, we have:

$$\begin{aligned}
\beta_I^{J,\epsilon}(t,x) &= \delta_I^J + \sum_{j=1}^d \int_0^t \Big(\sum_{K\in\mathcal{A}_1(l)} -\omega_{I*j}^{K,\epsilon}\big(X_s^\epsilon\big)\beta_K^{J,\epsilon}(s,x)\Big)\,\mathrm{d}B_s^j \\
&= \delta_I^J + \sum_{j=1}^d \int_0^t (-1)\beta_{I*j}^{J,\epsilon}(s,x)\,\mathrm{d}B_s^j.
\end{aligned}$$

Now let us iterate this equation $l - |I| + 1$ times and we have:

$$\begin{aligned}
\beta_I^{J,\epsilon}(t,x) &= \delta_I^J + \sum_{l_1=1}^d \int_0^t (-1)\beta_{I*l_1}^{J,\epsilon}(s_1,x)\,\mathrm{d}B_{s_1}^{l_1} \\
&= \delta_I^J + \sum_{l_1=1}^d (-1)B^{l_1}\delta_{I*l_1}^J + \sum_{l_1,l_2=1}^d \int_0^t \int_0^{s_1} (-1)^2 \beta_{I*l_1*l_2}^{J,\epsilon}(s_2,x)\,\mathrm{d}B_{s_2}^{l_2}\,\mathrm{d}B_{s_1}^{l_1} \\
&\;\;\vdots \\
&= \sum_{L\in\mathcal{A}} \delta_{I*L}^J (-1)^{|L|} B_t^L \\
&\quad + \sum_{L,j}\sum_{K\in\mathcal{A}_1(l)} \int_0^t \cdots \int_0^{s_k} (-1)^{|L|+1} \omega_{I*L*j}^{K,\epsilon}\big(X_{s_{k+1}}^\epsilon\big)\beta_K^{J,\epsilon}(s_{k+1},x)\,\mathrm{d}B_{s_{k+1}}^j \cdots \mathrm{d}B_{s_1}^{l_1} \\
&= \sum_{L\in\mathcal{A}} \delta_{I*L}^J (-1)^{|L|} B_t^L + \gamma_I^{\epsilon,J}(t,x),
\end{aligned}$$

where $\gamma_I^{\epsilon,J}(t,x)$ denotes the remainder term. Now, as an application of Theorem 10.41 in [8] (see also [1]), there exists a random variable $C \in \mathbf{L}^p$ such that:

$$\big\|\gamma_I^{\epsilon,J}(t,x)\big\| \le C t^{(l-|I|+1)H} \sum_{L,j}\sum_{K\in\mathcal{A}_1(l)} \big\|\omega_{I*L*j}^{K,\epsilon}\big\|_{\mathrm{Lip}^{\gamma-1}},$$

where $\gamma > 1/H$ and $\|\cdot\|_{\mathrm{Lip}^{\gamma-1}}$ is the $\gamma-1$-Lipschitz norm. The result follows then easily. □

***Remark 4.3.*** *Note that*

$$\sum_{L\in\mathcal{A}} \delta^{J}_{I*L}(-1)^{|L|}B^L_t = \begin{cases} (-1)^{|K|}B^K_t, & \text{if } J = I*K \text{ for some } K\in\mathcal{A};\\ 0, & \text{otherwise}.\end{cases}$$

*Therefore, when $t\to 0$, the dominating term of $\beta^{\epsilon,J}_I(t,x)$ is of order* $\mathrm{O}(t^{H(|J|-|I|)})$.

The second main ingredient is the following small-ball probability for iterated integrals of the fractional Brownian motion.

**Lemma 4.4.** *For $m\geq 0$ and $p\geq 1$, there exists a constant $C_{H,d,p}>0$ such that for any small $\epsilon>0$*

$$\sup_{\sum a_I^2=1}\mathbb{P}\biggl(\biggl\|\sum_{I\in\mathcal{A}(m)} a_I B^I_t\biggr\|_{\infty,[0,1]}<\epsilon\biggr)\leq C_{H,n,p}\epsilon^p.$$

**Proof.** We first prove the statement when $H>1/2$. Note that when $m=0$, $\mathcal{A}(m)=\{\varnothing\}$ and $\|a_\varnothing\|=1$. The statement is true for any $\epsilon<1$. When $m=1$, $\mathcal{A}(m)=\{\varnothing,1,2,\dots,d\}$. Let $f(t)=a_\varnothing+\sum_{i=1}^d a_{\{i\}}B^i_t$. We first assume that $a_\varnothing=0$, then $f(t)=\sum_{i=1}^d a_{\{i\}}B^i_t$ has the same law as one dimensional fractional Brownian motion $B_t$. Then by Theorem 4.6 in [15] we have:

$$\mathbb{P}\bigl(\|f(t)\|_{\infty,[0,1]}<\epsilon\bigr)=\mathbb{P}\bigl(\|B_t\|_{\infty,[0.1]}<\epsilon\bigr)\leq C_{H,p}\epsilon^p.$$

Now if $a_\varnothing\neq 0$, since $f(0)=a_\varnothing$, we have:

$$\begin{aligned}\mathbb{P}\bigl(\|f(t)\|_{\infty,[0,1]}<\epsilon\bigr)&\leq \mathbb{P}\bigl(\|f(t)\|_{\infty,[0,1]}<\epsilon,|a_\varnothing|\geq\epsilon\bigr)+\mathbb{P}\bigl(\|f(t)\|_{\infty,[0,1]}<\epsilon,|a_\varnothing|<\epsilon\bigr)\\ &=\mathbb{P}\bigl(\|f(t)\|_{\infty,[0,1]}<\epsilon,|a_\varnothing|<\epsilon\bigr)\\ &\leq \mathbb{P}\Biggl(\Biggl\|\sum_{i=1}^d a_{\{i\}}B^i_t\Biggr\|_{\infty,[0,1]}-|a_\varnothing|<\epsilon,|a_\varnothing|<\epsilon\Biggr)\\ &\leq \mathbb{P}\Biggl(\Biggl\|\sum_{i=1}^d a_{\{i\}}B^i_t\Biggr\|_{\infty,[0,1]}<2\epsilon\Biggr)\\ &\leq \mathbb{P}\Biggl(\Biggl\|\sum_{i=1}^d \frac{a_{\{i\}}}{\sqrt{\sum a^2_{\{i\}}}}B^i_t\Biggr\|_{\infty,[0,1]}<\frac{2\epsilon}{\sqrt{\sum a^2_{\{i\}}}}\Biggr).\end{aligned}$$

Note that when $|a_\varnothing|<\epsilon$, we have $\sum_{i=1}^d a^2_{\{i\}}\geq 1-\epsilon^2$. Therefore when $\epsilon<\frac{\sqrt{3}}{2}$, we have

$$\begin{aligned}\mathbb{P}\bigl(\|f(t)\|_{\infty,[0,1]}<\epsilon\bigr)&\leq \mathbb{P}\Biggl(\Biggl\|\sum_{i=1}^d \frac{a_{\{i\}}}{\sqrt{\sum a^2_{\{i\}}}}B^i_t\Biggr\|_{\infty,[0,1]}<4\epsilon\Biggr)\\ &\leq C_{n,p}\epsilon^p,\end{aligned}$$

where the last inequality follow by the earlier case when $a_\varnothing=0$. Now we assume that the statement is true for every $k=0,1,\dots,m$. As in the case when $m=1$, we may assume that $a_\varnothing=0$. Let $f(t)=\sum_{I\in\mathcal{A}_1(m+1)}a_IB^I_t$ with the restriction $\sum_{I\in\mathcal{A}_1(m+1)}a_I^2=1$. Note that $B^I_t$'s are iterated integrals and we have $B^{I*j}_t=\int_0^t B^I_s\,\mathrm{d}B^j_s$. Therefore,

$$f(t)=\sum_{I\in\mathcal{A}_1(m+1)}a_IB^I_t=\sum_{j=1}^d\int_0^t\biggl(\sum_{J\in\mathcal{A}(m)}a_{J*j}B^J_t\biggr)\mathrm{d}B^j_t,$$

where $\sum_{j=1}^{d}\sum_{J\in\mathcal{A}(m)} a_{J*j}^2 = 1$. Now by Propostion 3.4 in [2], we have:

$$\mathbb{P}\big(\|f(t)\|_{\infty,[0,1]} < \epsilon\big) \le C_p\epsilon^p + \min_{j=1,\dots,n}\left\{\mathbb{P}\left(\left\|\sum_{J\in\mathcal{A}(m)} a_{J*j}B_t^J\right\|_{\infty,[0,1]} < \epsilon^q\right)\right\}.$$

Note that since $\sum_{j=1}^{d}\sum_{J\in\mathcal{A}(m)} a_{J*j}^2 = 1$, there exists $1\le k\le d$ such that $\sum_{J\in\mathcal{A}(m)} a_{J*k}^2 \ge \frac{1}{d}$. Therefore,

$$\begin{aligned}
\mathbb{P}\big(\|f(t)\|_{\infty,[0,1]} < \epsilon\big) &\le C_p\epsilon^p + \mathbb{P}\left(\left\|\sum_{J\in\mathcal{A}(m)} a_{J*k}B_t^J\right\|_{\infty,[0,1]} < \epsilon^q\right)\\
&\le C_p\epsilon^p + \mathbb{P}\left(\left\|\sum_{J\in\mathcal{A}(m)} \frac{a_{J*k}}{\sqrt{\sum a_{J*k}^2}}B_t^J\right\|_{\infty,[0,1]} < \frac{\epsilon^q}{\sqrt{\sum a_{J*k}^2}}\right)\\
&\le C_p\epsilon^p + \mathbb{P}\left(\left\|\sum_{J\in\mathcal{A}(m)} \frac{a_{J*k}}{\sqrt{\sum a_{J*k}^2}}B_t^J\right\|_{\infty,[0,1]} < \sqrt{d}\epsilon^q\right)\\
&\le C_{H,d,p}\epsilon^p,
\end{aligned}$$

where the last inequality follows by the induction hypothesis. When $a_\varnothing \ne 0$, we repeat the argument in case $m=1$.

Now we turn to the irregular case when $1/4 \le H \le 1/2$. For the base case $m=0$ or $m=1$, the same argument as in the regular case $H>1/2$ works. We just need the irregular version of the Norris lemma (see Theorem 5.6 in [5]) to run the induction. Assume that the statement is true for $k=0,1,\dots,m$. Let $f(t)=\sum_{I\in\mathcal{A}_1(m+1)} a_I B_t^I$ with the restriction $\sum_{I\in\mathcal{A}_1(m+1)} a_I^2 = 1$.

We have:

$$f(t) = \int_0^t A_s\,\mathrm{d}B_s,$$

where $B_t=(B_t^1,\dots,B_t^d)$ and $A_t=(\sum_{J\in\mathcal{A}(m)} a_{J*1}B_t^J,\dots,\sum_{J\in\mathcal{A}(m)} a_{J*d}B_t^J)$. We pick $1\le k\le d$ such that $\sum_{J\in\mathcal{A}(m)} a_{J*k}^2 \ge \frac{1}{d}$. Then by Theorem 5.6 in [5], we have:

$$\left\|\sum_{J\in\mathcal{A}(m)} a_{J*k}B_t^J\right\|_{\infty,[0,1]} \le MR^q\|f(t)\|_{\infty,[0,1]}^r.$$

Therefore we have:

$$\begin{aligned}
\mathbb{P}\big(\|f\|_{\infty,[0,1]} < \epsilon\big) &= \mathbb{P}\big(\|f\|_{\infty,[0,1]}^r < \epsilon^r\big)\\
&\le \mathbb{P}\left(\frac{\|\sum_{J\in\mathcal{A}(m)} a_{J*k}B_t^J\|_{\infty,[0,1]}}{MR^q} \le \epsilon^r\right)\\
&\le \mathbb{P}\left(\left\|\sum_{J\in\mathcal{A}(m)} a_{J*k}B_t^J\right\|_{\infty,[0,1]} \le \epsilon^{r/2}\right) + \mathbb{P}\big(MR^q \ge \epsilon^{-r/2}\big)\\
&\le \mathbb{P}\left(\left\|\sum_{J\in\mathcal{A}(m)} \frac{a_{J*k}}{\sqrt{\sum a_{J*k}^2}}B_t^J\right\|_{\infty,[0,1]} \le \sqrt{d}\epsilon^{r/2}\right) + C_p\epsilon^p\\
&\le C_{H,d,p}\epsilon^p.
\end{aligned}$$

The last inequality follows from the induction hypothesis and the fact that $R$ has finite moment of all orders. □

**Corollary 4.5.** *For any $m \geq 0$ and $p > 1$, we have*

$$\mathbb{E}\left[\inf\left\{\int_0^1 \left(\sum_{I \in \mathcal{A}(m)} a_I B_t^I\right)^2 \mathrm{d}t;\ \sum_{I \in \mathcal{A}(m)} a_I^2 = 1\right\}^{-p}\right] = C_{H,d,m,p} < \infty.$$

**Proof.** By Lemma 2.3.1 in [18], we only need to show that for any $\epsilon > 0$, there exists $C_p > 0$ such that

$$\sup_{\sum_{I \in \mathcal{A}(m)} a_I^2 = 1} \mathbb{P}\left(\int_0^1 \left(\sum_{I \in \mathcal{A}(m)} a_I B_t^I\right)^2 \mathrm{d}t < \epsilon\right) \leq C_p \epsilon^p.$$

Let us denote that $f(t) = \sum_{I \in (A)(m)} a_I B_t^I$. Then we have:

$$\mathbb{P}\left(\int_0^1 \left(\sum_{I \in \mathcal{A}(m)} a_I B_t^I\right)^2 \mathrm{d}t < \epsilon\right) = \mathbb{P}\left(\|f\|_{L^2}^2 < \epsilon\right) = \mathbb{P}\left(\|f\|_{L^2} < \sqrt{\epsilon}\right).$$

By using the interpolation inequality

$$\|f\|_\infty \leq 2 \max\left\{\|f\|_{L^2}, \|f\|_{L^2}^{2r/(2r+1)} \|f\|_r^{1/(2r+1)}\right\},$$

we obtain:

$$\left\{\|f\|_{L^2} < \sqrt{\epsilon}\right\} \subseteq \left\{\frac{\|f\|_\infty}{2} < \sqrt{\epsilon}, \|f\|_{L^2} > \|f\|_r\right\} \cup \left\{\left(\frac{\|f\|_\infty}{2\|f\|_r^{1/(2r+1)}}\right)^{(2r+1)/(2r)} < \sqrt{\epsilon}, \|f\|_{L^2} < \|f\|_r\right\}.$$

Therfore we have:

$$\begin{aligned}
&\mathbb{P}\left(\|f\|_{L^2,[0,1]} < \sqrt{\epsilon}\right) \\
&\quad \leq \mathbb{P}\left(\|f\|_{\infty,[0,1]} < 2\sqrt{\epsilon}\right) + \mathbb{P}\left(\|f\|_{\infty,[0,1]}^{(2r+1)/(2r)} < \epsilon^{1/4}\right) + \mathbb{P}\left(\left(2\|f\|_r^{1/(2r+1)}\right)^{(2r+1)/(2r)} > \epsilon^{-1/4}\right) \\
&\quad \leq \mathbb{P}\left(\|f\|_{\infty,[0,1]} < 2\sqrt{\epsilon}\right) + \mathbb{P}\left(\|f\|_{\infty,[0,1]} < \epsilon^{1/(4r+1)}\right) + \mathbb{P}\left(\|f\|_r > 2^{-2r-1}\epsilon^{-r/2}\right).
\end{aligned}$$

Hence the result follows by Lemma 4.4 and the fact that $\|f\|_r$ has finite moments of all orders. □

We can observe that thanks to Corollary 4.5, we have for and $m \geq 0$, $p > 1$ and $T, s > 0$,

$$\mathbb{E}\left[\inf\left\{\int_0^T \left(\sum_{I \in \mathcal{A}(m)} a_I B_t^I\right)^2 \mathrm{d}t;\ \sum_{I \in \mathcal{A}(m)} T^{2|I|H+1} a_I^2 \geq s\right\}^{-p}\right] = C_{H,d,m,p} s^{-p}.$$

**Lemma 4.6.** *Let $m \geq 0$ and $I \in \mathcal{A}(m)$, if $g_I^\epsilon : (0,1]^2 \times \Omega \to \mathbb{R}$ is a continuous process such that*:

$$A_p = \sup_{T \in (0,1], \epsilon \in (0,1]} \mathbb{E}\left[\left(T^{-(m+1)H-1/2}\left(\sum_{I \in \mathcal{A}(m)} \int_0^T \left(g_I^\epsilon(t)\right)^2 \mathrm{d}t\right)^{1/2}\right)^p\right] < \infty,$$

*then*

$$\mathbb{P}\left(\inf\left\{\left(\int_0^T \left(\sum_{I \in \mathcal{A}(m)} a_I \left(B_t^I + g_I^\epsilon(t)\right)\right)^2 \mathrm{d}t\right)^{1/2};\ \sum_{I \in \mathcal{A}(m)} T^{2|I|H+1} a_I^2 = 1\right\} \leq z^{-1}\right) \leq \left(4^p C_{H,d,m,p} + A_{2p}\right) z^{-pr}$$

*for any $T \in (0,1]$ and $z \geq 1$, $r = \frac{H}{(m+1/2)H+1/2}$.*

**Proof.** For any $T \in (0,1]$ and $y \geq 1$, we have

$$\begin{aligned}
&\left(\int_0^T \left(\sum_{I\in\mathcal{A}(m)} a_I\left(B_t^I + g_I^\epsilon(t)\right)\right)^2 \mathrm{d}t\right)^{1/2} \\
&\quad\geq \left(\int_0^{T/y} \left(\sum_{I\in\mathcal{A}(m)} a_I\left(B_t^I + g_I^\epsilon(t)\right)\right)^2 \mathrm{d}t\right)^{1/2} \\
&\quad\geq \left(\int_0^{T/y} \left(\sum_{I\in\mathcal{A}(m)} a_I B_t^I\right)^2 \mathrm{d}t\right)^{1/2} \\
&\qquad - \left(\sum_{I\in\mathcal{A}(m)} T^{2|I|H+1} a_I^2\right)^{1/2} \left(T^{-(2mH+1)} \sum_{I\in\mathcal{A}(m)} \int^{T/y} g_I^\epsilon(t)^2\, \mathrm{d}t\right)^{1/2}.
\end{aligned}$$

Now let us pick $z = y^{(m+1/2)H+1/2}$, we have

$$\begin{aligned}
&\mathbb{P}\left(\inf\left\{\left(\int_0^T \left(\sum_{I\in\mathcal{A}(m)} a_I\left(B_t^I + g_I^\epsilon(t)\right)\right)^2 \mathrm{d}t\right)^{1/2}; \sum_{I\in\mathcal{A}(m)} T^{2|I|H+1} a_I^2 = 1\right\} \leq z^{-1}\right) \\
&\quad\leq \mathbb{P}\left(\inf\left\{\left(\int_0^{T/y} \left(\sum_{I\in\mathcal{A}(m)} a_I B_t^I\right)^2 \mathrm{d}t\right)^{1/2}; \sum_{I\in\mathcal{A}(m)} T^{2|I|H+1} a_I^2 = 1\right\} \leq 2z^{-1}\right) \\
&\qquad + \mathbb{P}\left(T^{-(2mH+1)/2} \left(\sum_{I\in\mathcal{A}(m)} \int^{T/y} \left(g_I^\epsilon(t)\right)^2 \mathrm{d}t\right)^{1/2} \geq z^{-1}\right) \\
&\quad\leq \mathbb{P}\left(\inf\left\{\int_0^{T/y} \left(\sum_{I\in\mathcal{A}(m)} a_I B_t^I\right)^2 \mathrm{d}t; \sum_{I\in\mathcal{A}(m)} (T/y)^{2|I|H+1} a_I^2 \geq y^{-(2mH+1)}\right\} \leq 4z^{-2}\right) \\
&\qquad + \mathbb{P}\left((T/y)^{-(m+1)H-1/2} \left(\sum_{I\in\mathcal{A}(m)} \int^{T/y} \left(g_I^\epsilon(t)\right)^2 \mathrm{d}t\right)^{1/2} \geq y^{(m+1)H+1/2} z^{-1}\right) \\
&\quad\leq \left(4z^{-2} y^{2mH+1}\right)^p C_{m,n,p} + \left(y^{-(m+1)H-1/2} z\right)^{2p} A_{2p} \\
&\quad\leq \left(4^p C_{H,d,m,p} + A_{2p}\right) y^{-Hp} \\
&\quad\leq \left(4^p C_{H,d,m,p} + A_{2p}\right) z^{-rp}.
\end{aligned}$$

□

Now, by applying the above lemma with $m = l - 1$ and Lemma 4.2, we obtain the following corollary:

**Corollary 4.7.** *For any $p \geq 1$ and $\delta > 0$, there exists a constant $C_p$ such that*

$$\mathbb{P}\left(\inf\left\{\sum_{I,J\in\mathcal{A}_1(l)} \int_0^t t^{-(|I|+|J|-2)H+1} a_I a_J \left\langle\beta^{I,\epsilon}(s,x), \beta^{J,\epsilon}(s,x)\right\rangle_{\mathbb{R}^d} \mathrm{d}s; \sum_{I\in\mathcal{A}_1(l)} |a_I|^2 = 1\right\} \leq \delta\right) \leq C_p \delta^p,$$

*for any $\epsilon \in (0,1]$ and any $x \in \mathbb{R}^n$.*

We are finally in position to finish the proof of Theorem 4.1. First, let us recall that $M_{I,J}^\epsilon(x) = \langle\beta^{I,\epsilon}(\cdot,x), \beta^{J,\epsilon}(\cdot,x)\rangle_{\mathcal{H}}$. We separate the case $1/4 < H \leq 1/2$ and $H > 1/2$, since we are using different interpolation inequalities for each case. When $1/4 < H \leq 1/2$, for any $a \in \mathbb{R}^{\mathcal{A}_1(l)}$ we have:

$$\begin{aligned}
\sum_{I,J\in\mathcal{A}_1(l)} a_I a_J M^{\epsilon}_{I,J}(x) &= \sum_{j=1}^{d} \Big\| \sum_{I\in\mathcal{A}_1(l)} a_I \beta^{I,\epsilon}_j(\cdot, x)\Big\|^2_{\mathcal{H}} \\
&\geq C_H \sum_{j=1}^{d} \int_0^1 \Big( \sum_{I\in\mathcal{A}_1(l)} a_I \beta^{I,\epsilon}_j(t,x)\Big)^2 \,\mathrm{d}t \\
&= C_H \sum_{I,J\in\mathcal{A}_1(l)} \int_0^1 a_I a_J \big\langle \beta^{I,\epsilon}(t,x), \beta^{J,\epsilon}(t,x)\big\rangle_{\mathbb{R}^d} \,\mathrm{d}t.
\end{aligned}$$

Therefore we conclude that:

$$\mathbb{P}\bigg(\inf\bigg\{\sum_{I,J\in\mathcal{A}_1(l)} a_I a_J M^{\epsilon}_{I,J}(x);\ \sum_{I\in\mathcal{A}_1(l)} |a_I|^2 = 1\bigg\} \leq \delta\bigg) \leq C_{p,H}\delta^p,$$

by applying the Corollary 4.7 above when $t = 1$. Now we turn to the case when $H > 1/2$. To simpify the notation, let us denote $f_j = \sum_{I\in\mathcal{A}_1(l)} a_I \beta^{I,\epsilon}_j(t,x)$. Applying the interpolation inequality (2.1) and note that $\|f_j\|_\infty \geq \|f_j\|_{L^2}$ on the interval [0, 1], we have:

$$\begin{aligned}
\sum_{I,J\in\mathcal{A}_1(l)} a_I a_J M^{\epsilon}_{I,J}(x) &= \sum_{j=1}^{d} \Big\| \sum_{I\in\mathcal{A}_1(l)} a_I \beta^{I,\epsilon}_j(\cdot, x)\Big\|^2_{\mathcal{H}} \\
&\geq C_H \sum_{j=1}^{d} \bigg( \frac{\|f_j\|^{3+1/\gamma}_{L^2}}{\|f_j\|^{2+1/\gamma}_{\gamma}}\bigg)^2 \\
&\geq \frac{C_H \sum_{j=1}^{d} \|f_j\|^{6+2/\gamma}_{L^2}}{\max_{j=1,\dots,d} \|f_j\|^{4+2/\gamma}_{\gamma}} \\
&\geq \frac{C_H d^{-2-1/\gamma} (\sum_{j=1}^{d} \|f_j\|^2_{L^2})^{3+1/\gamma}}{\max_{j=1,\dots,d} \|f_j\|^{4+2/\gamma}_{\gamma}} \\
&= \frac{C_{d,H} (\sum_{I,J\in\mathcal{A}_1(l)} \int_0^1 a_I a_J \langle \beta^{I,\epsilon}(t,x), \beta^{J,\epsilon}(t,x)\rangle_{\mathbb{R}^d} \,\mathrm{d}t)^{3+1/\gamma}}{\max_{j=1,\dots,d} \|f_j\|^{4+2/\gamma}_{\gamma}}.
\end{aligned}$$

Then we have:

$$\begin{aligned}
&\mathbb{P}\bigg(\inf\bigg\{\sum_{I,J\in\mathcal{A}_1(l)} a_I a_J M^{\epsilon}_{I,J}(x);\ \sum_{I\in\mathcal{A}_1(l)} |a_I|^2 = 1\bigg\} \leq \delta\bigg) \\
&\quad\leq \mathbb{P}\bigg(\inf\bigg\{\sum_{I,J\in\mathcal{A}_1(l)} \int_0^1 a_I a_J \big\langle \beta^{I,\epsilon}(t,x), \beta^{J,\epsilon}(t,x)\big\rangle_{\mathbb{R}^d} \,\mathrm{d}t;\ \sum_{I\in\mathcal{A}_1(l)} |a_I|^2 = 1\bigg\} \leq \bigg(\frac{\delta^{1/2}}{C_{d,H}}\bigg)^{1/(3+1/\gamma)}\bigg) \\
&\qquad + \mathbb{P}\bigg(\inf\bigg\{\max_{j=1,\dots,d} \|f_j\|^{4+2/\gamma}_{\gamma};\ \sum_{I\in\mathcal{A}_1(l)} |a_I|^2 = 1\bigg\} \geq \delta^{-1/2}\bigg).
\end{aligned}$$

The result then follows by chosing $t = 1$ in Corollary 4.7 and by the fact that $\|f_j\|_\gamma$ has finite moment of all orders.

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