Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2016, Vol. 52, No. 1, 16–46
DOI: 10.1214/14-AIHP632

ANNALES DE L'INSTITUT HENRI POINCARÉ
PROBABILITÉS ET STATISTIQUES
www.imstat.org/aihp

Determinantal point processes in the plane from products of random matrices[1]

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Received 21 October 2013; revised 22 April 2014; accepted 30 June 2014

Abstract. We show that the density of eigenvalues for three classes of random matrix ensembles is determinantal. First we derive the density of eigenvalues of product of k independent $n \times n$ matrices with i.i.d. complex Gaussian entries with a few of matrices being inverted. In second example we calculate the same for (compatible) product of rectangular matrices with i.i.d. Gaussian entries and in last example we calculate for product of independent truncated unitary random matrices. We derive exact expressions for limiting expected empirical spectral distributions of above mentioned ensembles.

Résumé. Nous montrons que la densité des valeurs propres pour trois classes d'ensembles de matrices aléatoires a une forme déterminantale. D'abord nous dérivons la densité des valeurs propres de produits de k matrices $n \times n$ indépendantes avec entrées i.i.d. gaussiennes avec certaines matrices inversées. Dans le deuxième exemple, nous calculons la même densité pour des produits compatibles de matrices rectangulaires avec entrées i.i.d. gaussiennes et dans le dernier exemple pour des produits de matrices unitaires tronquées aléatoires et indépendantes. Nous dérivons des expressions exactes pour les limites des distributions spectrales de ces exemples.

MSC: 60B20; 60G55; 15A18; 15B99

Keywords: Determinantal point process; Eigenvalues; Empirical spectral distribution; Limiting spectral distribution; Haar measure; QR decomposition; Random matrix; RQ decomposition; Generalized Schur decomposition; Unitary matrix; Wedge product

1. Introduction and main results

A *point process* χ on a locally compact Polish space Λ is a random integer-valued positive radon measure on Λ. If χ almost surely assigns at most measure 1 to singletons, it is a *simple* point process. Let μ be a Radon measure on Λ. If there exist functions $\rho_k : \Lambda^k \to [0, \infty)$ for $k \geq 1$, such that for any family of mutually disjoint subjects $D_1, \dots, D_k$ of Λ,

$$\mathbb{E}\left[\prod_{i=1}^{k} \chi(D_i)\right] = \int_{\prod_i D_i} \rho_k(x_1, \dots, x_k)\, \mathrm{d}\mu(x_1) \cdots \mathrm{d}\mu(x_k),$$

[1]Partially supported by UGC (under SAP-DSA Phase IV).
[2]Supported by CSIR-SPM fellowship, CSIR, Government of India.
[3]Supported in part by INSPIRE fellowship, DST, Government of India and VIMSS fellowship.

then $\rho_k, k \geq 1$ are called *k-point correlation functions* or *joint intensities* of a point process χ with respect to μ. In addition, we shall require that $\rho_k(x_1, \dots, x_k)$ vanish if $x_i = x_j$ for some $i \neq j$.

Let $\mathbb{K}(x, y) : \Lambda^2 \to \mathbb{C}$ be a measurable function. A point process χ on Λ is said to be a *determinantal process* with kernel $\mathbb{K}$ if it is simple and its joint intensities with respect to the measure μ satisfy

$$\rho_k(x_1, \dots, x_k) = \det\bigl(\mathbb{K}(x_i, x_j)\bigr)_{1 \leq i, j \leq k},$$

for every $k \geq 1$ and $x_1, \dots, x_k \in \Lambda$. For a detailed discussion on determinantal point processes, we refer the reader to [3,13,14] and [20].

In this article we show that the eigenvalues of certain classes of random matrix ensembles form determinantal point processes on the complex plane. In particular, we have obtained the density of the eigenvalues of these matrix ensembles and using that we have shown that they are determinantal.

Ginibre [9] introduced three ensembles of matrices with i.i.d. real, complex and quaternion Gaussian entries respectively without imposing a Hermitian condition. He showed that the eigenvalues of an $n \times n$ matrix with i.i.d. standard complex Gaussian entries form a determinantal process on the complex plane. Krishnapur [16] showed that the eigenvalues of $A^{-1}B$ (known as *spherical ensemble*) form a determinantal point process on the complex plane when A and B are independent random matrices with i.i.d. standard complex Gaussian entries. Akemann and Burda [1] derived the eigenvalue density for the product of k independent $n \times n$ matrices with i.i.d. complex Gaussian entries. In this case the joint probability distribution of the eigenvalues of the product matrix is found to be given by a determinantal point process as in the case of Ginibre [9], but with a complicated weight given by a Meijer G-function depending on k. Their derivation hinges on the generalized Schur decomposition for square matrices and the method of orthogonal polynomials. They computed all eigenvalue density correlation functions exactly for finite n and fixed k. A similar kind of study has been done on product of independent square matrices with quaternion Gaussian entries in [15].

Now following the works in [1] and [16], it is a natural question to ask, what can be said about the eigenvalues of product of k independent Ginibre matrices when a few of them are inverted? In particular, do the eigenvalues of $A = A_1^{\varepsilon_1} A_2^{\varepsilon_2} \cdots A_k^{\varepsilon_k}$ form a determinantal point process, where each ε_i is $+1$ or -1 and $A_1, A_2, \dots, A_k$ are independent matrices with i.i.d. standard complex Gaussian entries? The answer is yes and the following theorem, our first result, answers it precisely.

Theorem 1. *Let $A_1, A_2, \dots, A_k$ be independent $n \times n$ random matrices with i.i.d. standard complex Gaussian entries. Then the set of eigenvalues of $A = A_1^{\varepsilon_1} A_2^{\varepsilon_2} \cdots A_k^{\varepsilon_k}$, where each ε_i is $+1$ or -1, form a determinantal point process with kernel*

$$\mathbb{K}_n(z, w) = \sqrt{\omega(z)\omega(w)} \sum_{r=0}^{n-1} \frac{(z\overline{w})^r}{(2\pi)^k (r!)^p ((n-r-1)!)^{k-p}}$$

with respect to Lebesgue measure on $\mathbb{C}$, where $p = \#\{i\colon \varepsilon_i = 1, 1 \leq i \leq k\}$ and $\omega(z)$ is a weight function with

$$|\mathrm{d}z|^2 \omega(z) = \int_{x_1^{\varepsilon_1} \cdots x_k^{\varepsilon_k} = z} \mathrm{e}^{-\sum_{j=1}^k |x_j|^2} \prod_{j=1}^k |x_j|^{(1-\varepsilon_j)(n-1)} \prod_{j=1}^k |\mathrm{d}x_j|^2 \tag{1}$$

and $|\mathrm{d}z|^2$ is the Lebesgue measure on complex plane, which is same as $\mathrm{Re}(\mathrm{d}z)\,\mathrm{Im}(\mathrm{d}z)$. Equivalently, the vector of eigenvalues of A has density proportional to

$$\prod_{\ell=1}^n \omega(z_\ell) \prod_{i<j}^n |z_i - z_j|^2$$

with respect to Lebesgue measure on $\mathbb{C}^n$.

We write

$$|\mathrm{d}z|^2\omega(z) = \int_{h(x_1,x_2,\dots,x_k)=z} g(x_1,x_2,\dots,x_k)|\mathrm{d}x_1|^2|\mathrm{d}x_2|^2\cdots|\mathrm{d}x_k|^2$$

if

$$\int f(z)\omega(z)|\mathrm{d}z|^2 = \int f\big(h(x_1,x_2,\dots,x_k)\big)g(x_1,x_2,\dots,x_k)|\mathrm{d}x_1|^2|\mathrm{d}x_2|^2\cdots|\mathrm{d}x_k|^2$$

for all $f:\mathbb{C}\to\mathbb{C}$ integrable function.

In our next result, we deal with the eigenvalues of product of k independent rectangular matrices with i.i.d. complex Gaussian entries. Osborn [19] derived the density of the eigenvalues of product of two rectangular matrices. From there it follows that the eigenvalues of product of two rectangular matrices form a determinantal point process on the complex plane. We generalise this result for product of k rectangular matrices.

Theorem 2. *Let $n_1,n_2,\dots,n_{k+1}$ be $k+1$ positive integers such that $n_1=n_{k+1}=\min\{n_1,n_2,\dots,n_k\}$ and $A_1,A_2,\dots,A_k$ be independent rectangular matrices of dimension $n_i\times n_{i+1}$ for $i=1,2,\dots,k$, with i.i.d. standard complex Gaussian entries. Then the eigenvalues $z_1,z_2,\dots,z_{n_1}$ of $A=A_1A_2\cdots A_k$ form a determinantal point process on the complex plane with kernel*

$$\mathbb{K}_n(z,w) = \sqrt{\omega(z)\omega(w)}\sum_{r=0}^{n-1}\frac{(z\overline{w})^r}{(2\pi)^k\prod_{j=1}^k(n_j-n_1+r)!}$$

with respect to Lebesgue measure on $\mathbb{C}$, where $\omega(z)$ is a weight function with

$$|\mathrm{d}z|^2\omega(z) = \int_{x_1\cdots x_k=z} \mathrm{e}^{-\sum_{j=1}^k|x_j|^2}\prod_{j=1}^k|x_j|^{2(n_j-n_1)}\prod_{j=1}^k|\mathrm{d}x_j|^2.$$

Equivalently, the vector of eigenvalues of $A=A_1A_2\cdots A_k$ has density proportional to

$$\prod_{\ell=1}^{n_1}\omega(z_\ell)\prod_{i<j}^{n_1}|z_i-z_j|^2$$

with respect to Lebesgue measure on $\mathbb{C}^{n_1}$.

In a recent work [2], it was shown that the squares of singular values of product of rectangular matrices with i.i.d. complex Gaussian entries also form a determinantal point process.

In [6], Dyson introduced *circular unitary ensemble*, which is the set of eigenvalues of a random unitary matrix sampled from the Haar measure on the set of all $n\times n$ unitary matrices, $\mathcal{U}(n)$ and showed that this ensemble forms a determinantal point process on S^1. Życzkowski and Sommers [22] generalised the result of Dyson [6]. Let U be a matrix drawn from the Haar distribution on $\mathcal{U}(n)$. Życzkowski and Sommers showed in [22] that the eigenvalues of the left uppermost $m\times m$ block of U (where $m<n$) form a determinantal point process on $\mathbb{D}=\{z\in\mathbb{C}\colon |z|\le 1\}$. They found the exact distribution of the eigenvalues and from there it follows that they form a determinantal point process. In our last example, we generalise the result of [22]. We show that the eigenvalues of product of truncated unitary matrices and inverses of truncated unitary matrices are also determinantal. The following theorem states it precisely.

Theorem 3. *Let $U_1,U_2,\dots,U_k$ be k independent Haar distributed unitary matrices of dimension $n_i\times n_i$ for $i=1,2,\dots,k$ respectively, where $m\le n_i$ and $A_1,A_2,\dots,A_k$ be $m\times m$ left uppermost blocks of $U_1,U_2,\dots,U_k$ respectively. Then the eigenvalues $z_1,z_2,\dots,z_m$ of $A=A_1^{\varepsilon_1}A_2^{\varepsilon_2}\cdots A_k^{\varepsilon_k}$, where each ε_i is $+1$ or -1 form a determinantal point process on the complex plane with kernel*

$$\mathbb{K}_n(z,w) = \sqrt{\omega(z)\omega(w)}\sum_{r=0}^{n-1}\frac{(z\overline{w})^r}{(2\pi)^kC_r}$$

with respect to Lebesgue measure on $\mathbb{C}$, *where*

$$C_r = \prod_{i=1}^{k} \mathrm{Beta}\left(m\frac{1-\varepsilon_i}{2} + \frac{1+\varepsilon_i}{2} + r\varepsilon_i, n_j - m \right)$$

and $\omega(z)$ *is a weight function with*

$$|\mathrm{d}z|^2\omega(z) = \int_{x_1^{\varepsilon_1}\cdots x_k^{\varepsilon_k}=z} \prod_{j=1}^{k} \left(1-|x_j|^2\right)^{n_j-m-1} |x_j|^{(m-1)(1-\varepsilon_j)} \mathbf{1}_{\{|x_j|\le 1\}}(x_j)|\mathrm{d}x_j|^2.$$

Equivalently, the vector of eigenvalues of $A = A_1^{\varepsilon_1} A_2^{\varepsilon_2} \cdots A_k^{\varepsilon_k}$ *has density proportional to*

$$\prod_{\ell=1}^{m} \omega(z_\ell) \prod_{1\le i<j\le m} |z_i - z_j|^2$$

with respect to Lebesgue measure on $\mathbb{C}^m$.

Proofs of the theorems rely greatly on RQ decomposition, Schur decomposition of a matrix and generalization of Schur decomposition to product of rectangular matrices. We describe these decomposition in the next section. We also use QR decomposition of a matrix in the proof of some lemmas stated in Section 5. We discuss QR decomposition before proving those lemmas in the Appendix.

We organize this paper as follows. In Section 2, we define notation and describe basic facts about wedge product, Schur and RQ decompositions. Then we introduce notion of generalized Schur decomposition for product of rectangular matrices. We state a few lemmas which are used in the proof of above three theorems. We prove these lemmas in Appendix A.1. In Sections 3 and 4, we prove Theorems 1 and 2, respectively. In Section 5, we state three lemmas and using them we prove Theorem 3. Proofs of these lemmas are given in Appendix A.2.

In Section 6, we identify the limit of expected empirical distribution of these matrix ensembles. In particular, in Theorem 19, we calculate the limiting expected empirical distribution of square radial part of eigenvalues of product of Ginibre and inverse Ginibre matrices. Since one point correlation function of the corresponding point process, which gives the expected empirical spectral distribution, does not depend on the angular part of the eigenvalues, the limiting distribution of the radial part identifies the limiting spectral distribution completely. In Theorem 21, we calculate the limit of expected empirical distribution of square radial part of eigenvalues of product of rectangular matrices with independent complex Gaussian entries. We have a simple explicit expression for the limit in terms of uniform distribution. Finally in Theorem 22 we calculate the same for product of truncated unitary matrices.

2. Notation and tools

In this section we describe notation and basic facts about wedge product, Schur decomposition and RQ decomposition. We also develop a new technique to decompose a finite collection of rectangular matrices based on Schur and RQ decompositions.

We denote the differential of a complex vector x as

$$\mathrm{D}x = (\mathrm{d}x_1, \mathrm{d}x_2, \dots, \mathrm{d}x_n) \tag{2}$$

and define

$$|\mathrm{D}x| := \bigwedge_{i=1}^{n} |\mathrm{d}x_i|^2,$$

where

$$|\mathrm{d}x_i|^2 = \mathrm{d}x_i \wedge \overline{\mathrm{d}x_i}$$

and $\mathrm{d}x_i \wedge \mathrm{d}x_j$ denotes the wedge product between differentials $\mathrm{d}x_i$ and $\mathrm{d}x_j$. For a complex matrix M, we denote

$$|\mathrm{D}M| := \bigwedge_{i,j} \left(\mathrm{d}M(i,j) \wedge \mathrm{d}\overline{M(i,j)}\right). \tag{3}$$

Here wedge product is taken only for the non-zero variables of matrix M.

If $\mathrm{d}y_j = \sum_{k=1}^{n} a_{j,k}\, \mathrm{d}x_k$, for $1 \le j \le n$, then using the alternating property of wedge product, $\mathrm{d}x \wedge \mathrm{d}y = -\mathrm{d}y \wedge \mathrm{d}x$, it is easy to see that

$$\mathrm{d}y_1 \wedge \mathrm{d}y_2 \wedge \cdots \wedge \mathrm{d}y_n = \det(a_{j,k})_{j,k \le n}\, \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \cdots \wedge \mathrm{d}x_n. \tag{4}$$

Let U be a $n \times n$ unitary matrix with columns $u_1, u_2, \dots, u_n$ and with non-negative real diagonal entries. Then the Haar measure $|\mathrm{d}H(U)|$ on the set of such U can be expressed in terms of wedge product as follows

$$\left|\mathrm{d}H(U)\right| = \bigwedge_{i=1}^{n} \bigwedge_{j=i+1}^{n} \left|u_j^* \mathrm{D}u_i\right|^2. \tag{5}$$

Let $U = [u_1\ u_2\ \cdots\ u_m]$ be a $n \times m$ matrix with orthonormal columns and non-negative real diagonal entries. Let $V = [u_{m+1}\ u_{m+2}\ \cdots\ u_n]$ be a $n \times (n-m)$ matrix (function of U) such that $[U\ V]$ is unitary. Then Haar measure on the set of such U can be expressed as

$$\left|\mathrm{d}H(U)\right| = \bigwedge_{i=1}^{m} \bigwedge_{j=i+1}^{n} \left|u_j^* \mathrm{D}u_i\right|^2. \tag{6}$$

For a detailed discussion on Haar measure, see [8,17].

Schur decomposition. If M is a $n \times n$ matrix with complex entries, then there exist a $n \times n$ unitary matrix U with non-negative real diagonal entries, a strictly upper triangular matrix T and a diagonal matrix Z with diagonal elements in decreasing lexicographic order ($z \ge w$ if $\Re(z) > \Re(w)$ or $\Re(z) = \Re(w)$ and $\Im(z) \ge \Im(w)$) such that

$$M = U(Z+T)U^*. \tag{7}$$

This decomposition is unique if Z has distinct diagonal entries and U has positive diagonal entries. The set of all $n \times n$ complex matrices which admit unique Schur decomposition has full Lebesgue measure. We call this set of matrices as $\mathcal{M}$. The Lebesgue measure on $M \in \mathcal{M}$ can be written in terms of Lebesgue measure on Z, T and Haar measure on U as follows

$$|\mathrm{D}M| = \prod_{i<j} |z_i - z_j|^2 \left|\mathrm{d}H(U)\right| |\mathrm{D}Z| |\mathrm{D}T|, \tag{8}$$

where $z_1, z_2, \dots, z_n$ are the diagonal entries of Z, $|\mathrm{D}M|$, $|\mathrm{D}Z|$, $|\mathrm{D}T|$ are as defined in (3) and $|\mathrm{d}H(U)|$ is as defined in (5). For proof of Schur decomposition (7), see [10] and for proof of (8), see (6.3.5) on p. 104 of [14].

RQ-decomposition. If M is a $m \times n$ matrix with complex entries with $m \le n$, then there exist a $m \times m$ upper triangular matrix S and a $m \times n$ matrix U^* with orthonormal rows and non-negative real diagonal elements such that

$$M = SU^*. \tag{9}$$

This can be done by applying Gram–Schmidt orthogonalization process to the rows of M from bottom to top and fixing the argument of diagonal entries of U^* to be zero. If S is invertible and U has positive real diagonal entries, then this decomposition is unique. Observe that if S is invertible then M has full rank. The set of $m \times n$ complex matrices which admit unique RQ-decomposition has full measure. For a detailed proof of (9), see [10]. In the next lemma we express Lebesgue measure on M in terms of Lebesgue measure on S and Haar measure on U.

Lemma 4. *Let M be a $m \times n$ $(m \leq n)$ complex matrix of rank m and decomposable uniquely as SU^*. Then*

$$|\mathrm{D}M| = \prod_{i=1}^{m} |S_{i,i}|^{2(n-m+i)-1} |\mathrm{D}S| \big|\mathrm{d}H(U)\big|, \tag{10}$$

where $|\mathrm{d}H(U)|$ is as defined in (6).

We prove this lemma in Appendix A.1. For a detailed discussion on Schur and RQ decompositions, we refer the reader to [7,8,17] and [21].

2.1. *Generalized Schur decomposition of rectangular matrices*

In the following proposition we decompose a finite collection of rectangular matrices using the idea of Schur decomposition and RQ decomposition.

Proposition 5. *Let $n_1, n_2, \ldots, n_{k+1}$ be $k+1$ integers such that $n_{k+1} = n_1 = \min\{n_1, n_2, \ldots, n_k\}$ and A_i be a rectangular matrix of size $n_i \times n_{i+1}$ for $i = 1, 2, \ldots, k$. Then there exist k upper triangular matrices $S_1, S_2, \ldots, S_k$ of size $n_1 \times n_1$ with diagonal entries of $S_1 S_2 \cdots S_k$ in decreasing lexicographic order, and $[U_i \;\; V_i]$, $i = 1, 2, \ldots, k$ unitary matrices with U_i's having non-negative real diagonal entries, and $B_2, B_3, \ldots, B_k$ rectangular matrices with suitable dimensions, such that*

$$\begin{aligned} A_1 &= U_1 S_1 U_2^*, \\ A_2 &= U_2 S_2 U_3^* + V_2 B_2, \\ &\vdots \\ A_{k-1} &= U_{k-1} S_{k-1} U_k^* + V_{k-1} B_{k-1}, \\ A_k &= U_k S_k U_1^* + V_k B_k. \end{aligned} \tag{11}$$

We refer the decomposition in (11) *as generalized Schur decomposition.*

Proof. The first step in computing generalized Schur decomposition (11) is to apply the Schur decomposition to the product of $A_1, A_2, \ldots, A_k$ and using that, we have

$$A_1 A_2 \cdots A_k = U_1 T U_1^*, \tag{12}$$

where U_1 is $n_1 \times n_1$ unitary matrix with non-negative real diagonal entries and T is upper triangular matrix with diagonal entries in decreasing lexicographic order.

Next, by sequential application of RQ-decomposition starting from $i = 1$, we have $U_i^* A_i = S_i U_{i+1}^*$ for $i = 1, 2, \ldots, k-1$ where S_i is $n_1 \times n_1$ upper triangular matrix , U_{i+1}^* is $n_1 \times n_{i+1}$ matrix with orthonormal rows and with non-negative real diagonal entries. S_k be any $n_1 \times n_1$ upper triangular matrix such that $S_1 S_2 \cdots S_{k-1} S_k = T$.

Now we construct V_i uniquely from U_i such that $[U_i \;\; V_i]$ is a $n_i \times n_i$ unitary matrix for $i = 2, 3, \ldots, k$.

Let $e_1, e_2, \ldots, e_{n_i}$ be standard basis for $\mathbb{C}^{n_i}$. Let e_{i_1} be the basis vector with least index, not belonging to the subspace spanned by the columns of U_i. Now take projection of e_{i_1} onto orthogonal complement of the subspace spanned by the columns of U_i and normalize it, call it v_1. To choose v_1 uniquely, we require that the first non-zero component of it be positive. We take v_1 as the first column of V_i.

Let e_{i_2} be the basis vector with least index, not belonging to the subspace spanned of columns of U_i and v_1. Now the unit vector v_2, obtained by normalizing the projection of e_{i_2} onto the orthogonal complement of the subspace spanned by the columns of U_i and v_1, is chosen as second column of V_i. Again, to choose v_2 uniquely we require that the first non-zero component of it be positive. Proceeding this way, we get required V_i. Also observe that $B_i = V_i^* A_i$. □

Remark 6. *The above decomposition will be unique if Schur decomposition in (12) and RQ decompositions in subsequent steps are unique i.e., $S_1S_2\cdots S_k$ has distinct diagonal entries, U_i's have positive real diagonal entries and S_i's are invertible. For fixed k, the set of k tuple $(A_1, A_2, \dots, A_k)$ of matrices for which generalized Schur decomposition is unique has full measure.*

In the next lemma, we express the Lebesgue measure on $(A_1, A_2, \dots, A_k)$ in terms of Lebesgue measure on $S_1, S_2, \dots, S_{k-1}, S_k, B_2, B_3, \dots, B_k$ and Haar measure on $U_1, \dots, U_k$.

Lemma 7. *Let A_i be a rectangular matrix of size $n_i \times n_{i+1}$ for $i = 1, 2, \dots, k$, where $n_{k+1} = n_1 = \min\{n_1, n_2, \dots, n_k\}$. If A_i's are uniquely decomposable as in Proposition 5, then*

$$\prod_{i=1}^{k} |\mathrm{D}A_i| = \prod_{1\le i<j\le n_1} |z_i - z_j|^2 \prod_{i=1}^{k} \left|\det(S_i)\right|^{2(n_{i+1}-n_1)} \left|\mathrm{d}H(U_i)\right| |\mathrm{D}S_i| |\mathrm{D}B_i|, \tag{13}$$

where $z_1, z_2, \dots, z_{n_1}$ are the eigenvalues of $A_1A_2\cdots A_k$ and $\mathrm{d}H(U_i)$ is as defined in (6).

We refer (13) as Jacobian determinant formula corresponding to generalized Schur decomposition of rectangular matrices.

Remark 8. *Observe that if $A_1, A_2, \dots, A_k$ are square matrices of size $n_1 \times n_1$, then (13) takes the following form*

$$\prod_{i=1}^{k} |\mathrm{D}A_i| = \prod_{1\le i<j\le n_1} |z_i - z_j|^2 \prod_{i=1}^{k} \left|\mathrm{d}H(U_i)\right| |\mathrm{D}S_i|. \tag{14}$$

Generalized Schur decomposition and the change of measure are the main ingredients in the proofs of our theorems. Proof of Lemma 7 is given in Appendix A.1.

3. Product of Ginibre matrices and inverse Ginibre matrices

In this section we prove Theorem 1. We begin with some remarks on this theorem.

Remark 9. (i) *If $k = 2$, $\varepsilon_1 = -1$ and $\varepsilon_2 = 1$, then from (1) we get that*

$$|\mathrm{d}z|^2\omega(z) = \int_{x_2/x_1=z} \mathrm{e}^{-(|x_1|^2+|x_2|^2)} |x_1|^{2(n-1)} |\mathrm{d}x_1|^2 |\mathrm{d}x_2|^2 = c\frac{|\mathrm{d}z|^2}{(1+|z|^2)^{(n+1)}},$$

with some constant c. Hence the density of the eigenvalues of $A_1^{-1}A_2$ is proportional to

$$\prod_{i=1}^{n} \frac{1}{(1+|z_i|^2)^{n+1}} \prod_{i<j} |z_i - z_j|^2.$$

From the above expression it is clear that the eigenvalues of $A_1^{-1}A_2$ form a determinantal point process in a complex plane. This result was proved by Krishnapur in [16] using a change of variable which differs from generalized Schur decomposition.

(ii) *If $\varepsilon_i = 1$ for $i = 1, 2, \dots, k$, then by Theorem 1 it follows that the eigenvalues of $A_1A_2\cdots A_k$ form a determinantal point process. This result is due to Akemann and Burda [1]. The key ingredient of their paper is Jacobian* *determinant computation for generalized Schur decomposition of square matrices. It is done by using Jacobian determinant for Schur decomposition of a larger square matrix, which contains* $A_1, A_2, \dots, A_k$ *as sub blocks in positions* $(1,2), (2,3), \dots, (k-1,k)$ *and* $(k,1)$, *respectively and the rest being zero matrices. Whereas, we present a different way of computing Jacobian determinant for generalized Schur decomposition of rectangular matrices (square matrices follow as a special case) by breaking it into many simpler decompositions whose Jacobian's can be computed very easily.*

Remark 10. *Like in* [1], *by using Mellin transform, one can see that weight function* $\omega(z)$ *in Theorem* 1 *can be written as*

$$\omega(z) = (2\pi)^{k-1} G^{p,k-p}_{k-p,p}\left[\begin{matrix} (-n,-n,\dots,-n)_{k-p} \\ (0,0,\dots,0)_p \end{matrix} \middle| |z|^2 \right],$$

where $p = \#\{i\colon \varepsilon_i = 1, 1 \le i \le k\}$ *and the symbol* $G^{nm}_{pq}(\cdots|z)$ *denotes Meijer's G-function. For a detailed discussion on Meijer's G-function, see* [4], [12].

Proof of Theorem 1. The density of $(A_1, A_2, \dots, A_k)$ is proportional to

$$\prod_{\ell=1}^{k} \mathrm{e}^{-\mathrm{tr}(A_\ell A_\ell^*)} |\mathrm{D}A_\ell|.$$

Actually, here the proportional constant is $\frac{1}{\pi^{kn^2}}$, but to make life less painful for ourselves, we shall omit constants in every step to follow. Since we are dealing with probability measures, the constants can be recovered at the end by finding normalization constants.

Now by generalized Schur-decomposition (11), we have

$$A_i^{\varepsilon_i} = U_i S_i^{\varepsilon_i} U_{i+1}^* \quad \text{for } i = 1, 2, \dots, k,$$

where $S_1, S_2, \dots, S_k$ are upper triangular matrices and $U_1, U_2, \dots, U_k$ are unitary matrices with $U_{k+1} = U_1$. Let the diagonal entries of S_i be $(x_{i1}, x_{i2}, \dots, x_{in})$. One can see that eigenvalues $z_1, z_2, \dots, z_n$ of $A = A_1^{\varepsilon_1} A_2^{\varepsilon_2} \cdots A_k^{\varepsilon_k}$ are given by

$$z_j = \prod_{i=1}^{k} x_{ij}^{\varepsilon_i}, \quad j = 1, 2, \dots, n.$$

Now, by using Jacobian determinant formula (14) for generalised Schur decomposition of square matrices, we get

$$\prod_{i=1}^{k} \left|\mathrm{D}A_i^{\varepsilon_i}\right| = \left|\Delta(\underline{z})\right|^2 \prod_{i=1}^{k} \left|\mathrm{D}S_i^{\varepsilon_i}\right| \left|\mathrm{d}H(U_i)\right|, \tag{15}$$

where $\Delta(\underline{z}) = \prod_{i<j}(z_i - z_j)$. For any square complex matrix A and complex vector x, using (4), it is easy to see that

$$\left|(\mathrm{D}x)A\right| = \left|A(\mathrm{D}x)^t\right| = \left|\det(A)\right|^2 |\mathrm{D}x|, \tag{16}$$

where $\mathrm{D}x$ is as defined in (2). Since

$$\left|\mathrm{D}A^{-1}\right| = \left|A^{-1}(\mathrm{D}A)A^{-1}\right|,$$

using (16), we have

$$\left|\mathrm{D}A_i^{\varepsilon_i}\right| = \left|\det(A_i)\right|^{4n((\varepsilon_i-1)/2)} |\mathrm{D}A_i|. \tag{17}$$

By similar calculation for upper triangular matrices S_i, we get

$$\left|\mathrm{D}S_i^{\varepsilon_i}\right| = \left|\det(S_i)\right|^{2(n+1)((\varepsilon_i-1)/2)}|\mathrm{D}S_i|. \tag{18}$$

Now using (15), (17) and (18), and since $|\det(S_i)| = |\det(A_i)|$, we get

$$\prod_{i=1}^k |\mathrm{D}A_i| = \left|\Delta(\underline{z})\right|^2 \prod_{i=1}^k \left|\det(S_i)\right|^{(1-\varepsilon_i)(n-1)}|\mathrm{D}S_i|\left|\mathrm{d}H(U_i)\right|. \tag{19}$$

The density of $A_1, A_2, \ldots, A_k$ can be written in new variables as

$$\left|\Delta(\underline{z})\right|^2 \prod_{i=1}^k \mathrm{e}^{-\operatorname{tr}(S_i S_i^*)}\left|\det(S_i)\right|^{(1-\varepsilon_i)(n-1)}|\mathrm{D}S_i|\left|\mathrm{d}H(U_i)\right|.$$

By integrating out the non-diagonal entries of $S_1, S_2, \ldots, S_k$, we get the density of diagonal entries of $S_1, S_2, \ldots, S_k$ to be proportional to

$$\left|\Delta(\underline{z})\right|^2 \prod_{i=1}^k \prod_{j=1}^n \mathrm{e}^{-|S_i(j,j)|^2}\left|S_i(j,j)\right|^{(1-\varepsilon_i)(n-1)}\left|\mathrm{d}S_i(j,j)\right|^2.$$

Hence the density of the vector $(Z_1, Z_2, \ldots, Z_n)$ of eigenvalues of A is proportional to

$$\prod_{\ell=1}^n \omega(z_\ell)\prod_{i<j}^n |z_i - z_j|^2$$

with a weight function

$$|\mathrm{d}z|^2\omega(z) = \int_{x_1^{\varepsilon_1}\cdots x_k^{\varepsilon_k}=z} \mathrm{e}^{-\sum_{j=1}^k |x_j|^2}\prod_{j=1}^k |x_j|^{(1-\varepsilon_j)(n-1)}\prod_{j=1}^k |\mathrm{d}x_j|^2. \tag{20}$$

Using this density formula and Fact 11, we show that the eigenvalues form a determinantal point process on the complex plane.

Fact 11. *Suppose $\{\varphi_k\}_{k=1}^n$ is an orthogonal set in $L^2(\Lambda)$. Then there exists a determinantal point process with kernel $\mathbb{K}(z,w) = \sum_{k=1}^n \varphi_k(z)\overline{\varphi_k(w)}$ with respect to Lebesgue measure on Λ.*

For proof of this fact we refer the reader to Lemma 4.5.1 of [14]. First observe that

$$\prod_{i<j} |z_i - z_j| = \left|\det\left(z_i^{j-1}\right)_{1\le i,j\le n}\right| = \left|\det\left(\phi_j(z_i)\right)_{1\le i,j\le n}\right|,$$

where ϕ_j are monic polynomials of degree $(j-1)$. So

$$\prod_{i<j} |z_i - z_j|^2 = \det\left[\left(\phi_j(z_i)\right)_{i,j}\left(\overline{\phi_i(z_j)}\right)_{i,j}\right] = \det\left[\sum_{k=1}^n \phi_k(z_i)\overline{\phi_k(z_j)}\right]_{i,j}.$$

If we choose ϕ_j, $1 \le j \le n$ to be orthonormal polynomials with respect to measure $f(z)|\mathrm{d}z|^2$, then

$$\mathbb{K}(z_i, z_j) = \sqrt{f(z_i)f(z_j)}\sum_{k=1}^n \phi_k(z_i)\overline{\phi_k(z_j)}$$

will be our required kernel for the correlation functions with respect to the Lebesgue measure on $\mathbb{C}$, that is, for $1 \le k \le n$, k-point correlation function will be given by $\det(\mathbb{K}(z_i, z_j))_{1\le i,j\le k}$.

Observe that, the weight function $\omega(z)$ in (20) is angle independent, that is, $\omega(z)$ is function of $|z|$ only. It implies that monic polynomials $P_i(z) = z^i$ are orthogonal with respect to this weight functions and we have

$$\begin{aligned}\int z^a \big(\overline{z}^b\big)\omega(z)|\mathrm{d}z|^2 &= \prod_{j=1}^{k} \int (x_j)^{\varepsilon_j a} (\overline{x_j})^{\varepsilon_j b} \mathrm{e}^{-|x_j|^2} |x_j|^{(1-\varepsilon_j)(n-1)} |\mathrm{d}x_j|^2 \\ &= \delta_{ab} (2\pi)^k (a!)^p \big((n-a-1)!\big)^{k-p}.\end{aligned}$$

Therefore the kernel is given by

$$\mathbb{K}_n(z, w) = \sqrt{\omega(z)\omega(w)} \sum_{r=0}^{n-1} \frac{(z\overline{w})^r}{(2\pi)^k (r!)^p ((n-r-1)!)^{k-p}}.$$

Hence the eigenvalues of A form a determinantal point process with above kernel with respect to Lebesgue measure on $\mathbb{C}$. □

4. Product of rectangular matrices

In this section we prove Theorem 2 borrowing some results from Section 2. Before that we make some remarks on the assumption and on the weight function of Theorem 2.

Remark 12. *The condition $n_1 = \min\{n_1, n_2, \ldots, n_k\}$ in Theorem 2 is taken for simplicity. Since we want to calculate density of non-zero eigenvalues of product of (compatible) rectangular matrices $A_1 A_2 \cdots A_k$ and the set of non-zero eigenvalues of $A_1 A_2 \cdots A_k$ remains unaltered for any rotational combination of $A_1, A_2, \ldots, A_k$. So the set of non-zero eigenvalues of $A_1 A_2 \cdots A_k$ is less or equal to $\min\{n_1, n_2, \ldots, n_k\}$. Therefore, we can assume that n_1 is the minimum among $n_1, n_2, \ldots, n_k$.*

Remark 13. *Using Mellin transform, the weight function $\omega(z)$ in Theorem 2 can be written as*

$$\omega(z) = (2\pi)^{k-1} G^{k,0}_{0,k}\left[\begin{matrix} - \\ (n_1 - n_1, n_2 - n_1, \ldots, n_k - n_1)\end{matrix} \,\middle|\, |z|^2\right],$$

where $G^{k,0}_{0,k}$ denotes Meijer's G-function.

Proof of Theorem 2. Density of $A_1, A_2, \ldots, A_k$ is proportional to

$$\prod_{i=1}^{k} \mathrm{e}^{-\mathrm{tr}(A_i A_i^*)} |\mathrm{D}A_i|,$$

where $|\mathrm{D}A_i|$ is as defined in (3). Now using the decomposition as discussed in Proposition 5 and also using (13), the density of $A_1, A_2, \ldots, A_k$ can be written in new variables as

$$\big|\Delta(Z)\big|^2 \prod_{i=1}^{k} \mathrm{e}^{-\mathrm{tr}(S_i S_i^* + B_i B_i^*)} \big|\det(S_i)\big|^{2(n_{i+1}-n_1)} |\mathrm{D}S_i| \big|\mathrm{d}H(U_i)\big| |\mathrm{D}B_i|,$$

where B_i, S_i are as in (11) respectively and

$$\Delta(Z) = \prod_{i<j}^{n_1} (z_i - z_j) \quad \text{and} \quad z_j = \prod_{i=1}^{k} S_i(j, j) \quad \text{for } j = 1, 2, \ldots, n_1.$$

We take $B_1 = 0$ and $|\mathrm{D}B_1| = 1$. By integrating out the variables in $B_2, \dots, B_k, U_1, U_2, \dots, U_k$ and the non-diagonal entries of $S_1, S_2, \dots, S_k$, we get the density of diagonal entries of $S_1, S_2, \dots, S_k$ to be proportional to

$$\left|\Delta(Z)\right|^2 \prod_{i=1}^{k} \prod_{j=1}^{n_1} \mathrm{e}^{-|S_i(j,j)|^2} \left|S_i(j,j)\right|^{2(n_{i+1}-n_1)} \left|\mathrm{d}S_i(j,j)\right|^2.$$

Hence the density of $z_1, z_2, \dots, z_{n_1}$ is proportional to

$$\prod_{\ell=1}^{n_1} \omega(z_\ell) \prod_{i<j}^{n_1} |z_i - z_j|^2$$

with a weight function

$$|\mathrm{d}z|^2 \omega(z) = \int_{z_1 \cdots z_k = z} \mathrm{e}^{-\sum_{j=1}^{k} |z_j|^2} \prod_{j=1}^{k} |z_j|^{2(n_j - n_1)} \prod_{j=1}^{k} |\mathrm{d}z_j|^2.$$

Using this density formula we show that eigenvalues of A form a determinantal point process. Note that, the weight function $\omega(z)$ is angle independent and hence, the monic polynomials $P_i(z) = z^i$ are orthogonal with respect to this weight function. Now we have

$$\begin{aligned}\int z^a \left(\overline{z}^b\right) \omega(z) |\mathrm{d}z|^2 &= \prod_{j=1}^{k} \int (x_j)^a \left(\overline{x_j}\right)^b \mathrm{e}^{-|x_j|^2} |x_j|^{2(n_j - n_1)} |\mathrm{d}x_j|^2 \\ &= \delta_{ab} (2\pi)^k \prod_{j=1}^{k} (n_j - n_1 + a)!\end{aligned}$$

and the corresponding kernel of orthogonal polynomials is given by

$$\mathbb{K}_n(z, w) = \sqrt{\omega(z)\omega(w)} \sum_{r=0}^{n-1} \frac{(z\overline{w})^r}{(2\pi)^k \prod_{j=1}^{k} (n_j - n_1 + r)!}.$$

Hence by Fact 11, the eigenvalues of A form a determinantal point process with above kernel $\mathbb{K}(z, w)$ with respect to Lebesgue measure on $\mathbb{C}$. □

5. Product of truncated unitary matrices

We begin this section with some remarks on Theorem 3. In particular we present some results that it generalises.

Remark 14. (i) *If $k = 1$ and $\varepsilon_1 = 1$, then Theorem 3 says that the eigenvalues $z_1, z_2, \dots, z_m$ of $A_{m\times m}$ left-upper block of Haar distributed unitary matrix*

$$U_{n\times n} = \begin{bmatrix} A_{m\times m} & B_{m\times(n-m)} \\ C_{(n-m)\times m} & D_{(n-m)\times(n-m)} \end{bmatrix}$$

form a determinantal point process with density proportional to

$$\prod_{1\le j<k\le m} |z_j - z_k|^2 \prod_{i=1}^{m} \left(1 - |z_i|^2\right)^{n-m-1} \mathbf{1}_{\{|z_i|\le 1\}}(z_i). \tag{21}$$

This special case was proved by Życzkowski and Sommers in [22]. *But our way of proof of Theorem* 3 *is completely different from their proof.*

(ii) *For simplicity we have taken $m \times m$ left-upper blocks of matrices. But we can take any $m \times m$ blocks of matrices, because their probability distributions are similar.*

Remark 15. *Using Mellin transform, the weight function $\omega(z)$ in Theorem* 3 *can be written as*

$$(2\pi)^{k-1}\prod_{j=1}^{k}\Gamma(n_j-m)G^{p,k-p}_{k,k}\left[\begin{matrix}(-m,\dots,-m,n_1-m,\dots,n_p-m)_k\\(0,\dots,0,-n_{p+1},\dots,-n_k)_k\end{matrix}\,\middle|\,|z|^2\right],$$

when $\varepsilon_i = 1$ for $i = 1,\dots,p$ and $\varepsilon_i = -1$ for $p+1,\dots,k$.

Before proving Theorem 3, we need to introduce some basic notation and facts. Let $\mathcal{M}_n$ be the space of all $n \times n$ complex matrices equipped with Euclidean norm, $\|M\| = \sqrt{\operatorname{tr}(M^*M)}$.

Weyl chamber. This is a subset of $\mathbb{C}^n$ and is defined as

$$\mathcal{W}_n := \bigl\{(z_1,z_2,\dots,z_n):\ z_1 \ge z_2 \ge \cdots \ge z_n\bigr\} \subset \mathbb{C}^n,$$

where $z \ge w$ if $\Re(z) > \Re(w)$ or $\Re(z) = \Re(w)$ and $\Im(z) \ge \Im(w)$. The metric on Weyl chamber is given by

$$\|\underline{z}-\underline{w}\|_{\mathcal{W}} = \min_{\sigma}\sqrt{\sum_{i=1}^{n}|z_i - w_{\sigma i}|^2},$$

minimum is taken over all permutations of $\{1,2,\dots,n\}$. Weyl chamber with this metric is a polish space. We take the space of eigenvalues of $n \times n$ matrices as Weyl chamber through the following map $\Phi_n : (\mathcal{M}_n, \|\cdot\|) \to (\mathcal{W}_n, \|\cdot\|_{\mathcal{W}})$ which is defined as

$$\Phi_n(M) = (z_1,z_2,\dots,z_n), \tag{22}$$

where $z_1 \ge z_2 \ge \cdots \ge z_n$ are the eigenvalues of M. The map Φ_n is a continuous map. This can be seen from the fact that roots of complex polynomial are continuous functions of its coefficients and eigenvalues are roots of characteristic polynomial whose coefficients are continuous functions of matrix entries. Note that the map

$$\Psi_{n,m} : (\mathcal{M}_n, \|\cdot\|) \to (\mathcal{M}_m, \|\cdot\|) \quad (n \ge m), \tag{23}$$

taking every matrix to its $m \times m$ left uppermost block is also continuous.

The vector of ordered eigenvalues $\underline{Z} = (z_1,z_2,\dots,z_m)$ of $m \times m$ left uppermost block of a $n \times n$ random matrix defines a measure μ on $\mathcal{W}_m$. In other words $\underline{Z}$ is $\mathcal{W}_m$-valued random variable distributed according to μ. Suppose there exists a function $p(z_1,z_2,\dots,z_m)$ such that expectation of any complex valued bounded continuous function f on $(\mathcal{W}_m, \|\cdot\|_{\mathcal{W}})$ is given by

$$\mathbf{E}\bigl[f(\underline{Z})\bigr] = \int_{\mathcal{W}_m} f(\underline{z})p(\underline{z})|\mathrm{d}\underline{z}|^2 = \int_{\mathbb{C}^m}\frac{1}{m!}f(\underline{z})p(\underline{z})|\mathrm{d}\underline{z}|^2, \tag{24}$$

where p and f are extended to $\mathbb{C}^m$ by defining

$$f(z_1,z_2,\dots,z_m) = f(z_{(1)},z_{(2)},\dots,z_{(m)}), \qquad p(z_1,z_2,\dots,z_m) = p(z_{(1)},z_{(2)},\dots,z_{(m)}),$$

where $\{z_1,z_2,\dots,z_m\} = \{z_{(1)},z_{(2)},\dots,z_{(m)}\}$ and $z_{(1)} \ge z_{(2)} \ge \cdots \ge z_{(n)}$ and $|\mathrm{d}\underline{z}|^2$ is Lebesgue measure on $\mathbb{C}^m$. Then $p(\underline{z})$ gives the joint probability density of eigenvalues $\underline{Z}$. Also note that the set of symmetric continuous functions on $\mathbb{C}^m$ are in natural bijection with set of continuous functions on $(\mathcal{W}_m, \|\cdot\|_{\mathcal{W}})$.

Haar measure. Let $\mathcal{U}(n)$ be the space of all $n \times n$ unitary matrices. It is a manifold of dimension n^2 in $\mathbb{R}^{2n^2}$. Haar measure on $\mathcal{U}(n)$ is normalized volume measure on manifold $\mathcal{U}(n)$ which is denoted by $H_{\mathcal{U}(n)}$. Define

$$\mathcal{N}_{m,n} := \{Y \in \mathcal{M}_n\colon\ Y_{i,j} = 0, 1 \le j < i \le m\}, \tag{25}$$

to be the set of all $n \times n$ matrices with zeros in the lower triangle of left upper most $m \times m$ block and

$$\mathcal{V}_{m,n} := \mathcal{N}_{m,n} \cap \mathcal{U}(n). \tag{26}$$

But we suppress subscripts m, n from $\mathcal{V}_{m,n}$ and $\mathcal{N}_{m,n}$ when there is no confusion. Let $H_{\mathcal{V}}$ be the normalized volume measure on manifold $\mathcal{V}_{m,n}$. Now we need the following lemmas to prove Theorem 3. We prove these lemmas in Appendix A.2. The next lemma approximates Haar measure on unitary group by Lebesgue measure on a its open neighbourhood in $\mathcal{M}_n$.

Lemma 16. *Let $f : \mathcal{M}_n \to \mathbb{C}$ be a continuous function. Then*

$$\int f(U)\left|\mathrm{d}H_{\mathcal{U}(n)}(U)\right| = \lim_{\varepsilon\to 0} \frac{\int_{\|X^*X-I\|<\varepsilon} f(X)|\mathrm{D}X|}{\int_{\|X^*X-I\|<\varepsilon} |\mathrm{D}X|},$$

where $|\mathrm{D}X|$ and $\|\cdot\|$ denote differential element of volume measure and Euclidean norm on $\mathcal{M}_n$, manifold of $n \times n$ complex matrices respectively and $H_{\mathcal{U}(n)}$ is the normalized volume measure on manifold $\mathcal{U}(n)$.

Lemma 17 approximates volume measure on $\mathcal{V}_{m,n}$ by Lebesgue measure on its open neighbourhood in $\mathcal{N}_{m,n}$.

Lemma 17. *Let $f : \mathcal{M}_n \to \mathbb{C}$ be a continuous function. Then*

$$\int f(V)\left|\mathrm{d}H_{\mathcal{V}}(V)\right| = \lim_{\varepsilon\to 0} \frac{\int_{\|X^*X-I\|<\varepsilon} f(X)|\mathrm{D}X|}{\int_{\|X^*X-I\|<\varepsilon} |\mathrm{D}X|},$$

where $|\mathrm{D}X|$ and $\|\cdot\|$ denote differential element of volume measure and Euclidean norm on $\mathcal{N}_{m,n}$ respectively and $H_{\mathcal{V}}$ is the normalized volume measure on manifold $\mathcal{V}_{n,m}$.

Lemma 18. *The probability density of the vector of diagonal elements $(Z_1, Z_2, \dots, Z_m)$ of $m \times m$ left uppermost block of $n \times n$ random matrix distributed according to probability measure $H_{\mathcal{V}}$ is proportional to*

$$\prod_{i=1}^{m} \left(1 - |z_i|^2\right)^{n-m-1} \mathbf{1}_{\{|z_i|\le 1\}}(z_i).$$

Proof of Theorem 3. For the sake of simplicity, let $n_i = n$ for $i = 1, 2, \dots, k$. Let $z_1 \ge z_2 \ge \cdots \ge z_m$ be the eigenvalues of $A = A_1^{\varepsilon_1} A_2^{\varepsilon_2} \cdots A_k^{\varepsilon_k}$ where A_i be the left uppermost $m \times m$ block of U_i and $U_1, U_2, \dots, U_k$ be $n \times n$ independent Haar distributed unitary matrices, $\varepsilon_i = 1$ or -1. We denote the vector of eigenvalues of A by

$$\underline{Z} = (z_1, z_2, \dots, z_m).$$

We use (24) to find the joint probability density of eigenvalues of A. Let f be any bounded continuous function of $\underline{Z}$. In computation of expectation of $f(\underline{Z})$, we approximate Haar measure on direct product of k unitary groups by normalised Lebesgue measure on direct product of their open neighbourhoods in $\mathcal{M}(n)$. By Lemma 16, we have

$$\begin{aligned}
\mathbf{E}\left[f(\underline{Z})\right] &= \int f(\underline{z}) \prod_{i=1}^{k} \left|\mathrm{d}H_{\mathcal{U}_i(n)}(U_i)\right| \\
&= \lim_{\varepsilon_i\to 0} \frac{\int_{\bigcap_{i=1}^k \|X_i^*X_i-I\|<\varepsilon_i} f(\underline{z})|\mathrm{D}X_1||\mathrm{D}X_2|\cdots|\mathrm{D}X_k|}{\int_{\bigcap_{i=1}^k \|X_i^*X_i-I\|<\varepsilon_i} |\mathrm{D}X_1||\mathrm{D}X_2|\cdots|\mathrm{D}X_k|}.
\end{aligned}$$

Limit is taken for all ε_i one by one. For $1 \le i \le k$, let

$$X_i = \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix}.$$

We apply generalised Schur decomposition to $m \times m$ left uppermost blocks of those k matrices (with powers ε_i). Now by generalized Schur-decomposition, we have

$$A_i^{\varepsilon_i} = S_i T_i^{\varepsilon_i} S_{i+1}^*, \quad i = 1, 2, \dots, k, \text{ and } k+1 = 1,$$

where $T_1, T_2, \dots, T_k$ are upper triangular matrices and $S_1, S_2, \dots, S_k$ are unitary matrices with $S_{k+1} = S_1$. Let the diagonal entries of T_i be $(x_{i1}, x_{i2}, \dots, x_{im})$. We denote $\{x_{ij} \colon i = 1, 2, \dots, k,\ j = 1, 2, \dots, m\}$ by $\underline{x}$. Now

$$X_i = \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix} = \begin{bmatrix} S_{i+(1-\varepsilon_i)/2} & 0 \\ 0 & I \end{bmatrix} Y_i \begin{bmatrix} S_{i+(1+\varepsilon_i)/2}^* & 0 \\ 0 & I \end{bmatrix}$$

and

$$Y_i = \begin{bmatrix} T_i & \tilde{B}_i \\ \tilde{C}_i & \tilde{D}_i \end{bmatrix}.$$

One can see that eigenvalues $z_1, z_2, \dots, z_m$ of $A = A_1^{\varepsilon_1} A_2^{\varepsilon_2} \cdots A_k^{\varepsilon_k}$ are given by

$$z_j = \prod_{i=1}^{k} x_{ij}^{\varepsilon_i}, \quad j = 1, 2, \dots, m. \tag{27}$$

Now, by using Jacobian determinant formula (14) for generalised Schur decomposition of square matrices, we get

$$\prod_{i=1}^{k} \left|\mathrm{D}A_i^{\varepsilon_i}\right| = \left|\Delta(\underline{z})\right|^2 \prod_{i=1}^{k} \left|\mathrm{D}T_i^{\varepsilon_i}\right| \prod_{i=1}^{k} \left|\mathrm{d}H(S_i)\right|,$$

where $\Delta(\underline{z}) = \prod_{i<j}(z_i - z_j)$. Since

$$\left|\mathrm{D}A_i^{\varepsilon_i}\right| = \left|\det(A_i)\right|^{4m((\varepsilon_i-1)/2)} |\mathrm{D}A_i|, \qquad \left|\mathrm{D}T_i^{\varepsilon_i}\right| = \left|\det(T_i)\right|^{2(m+1)((\varepsilon_i-1)/2)} |\mathrm{D}T_i|$$

and $|\det(T_i)| = |\det(A_i)|$, we get

$$\prod_{i=1}^{k} |\mathrm{D}A_i| = \left|\Delta(\underline{z})\right|^2 L(\underline{x}) \prod_{i=1}^{k} |\mathrm{D}T_i| \left|\mathrm{d}H(S_i)\right|, \tag{28}$$

where

$$L(\underline{x}) = \prod_{i=1}^{k} \left|\det(T_i)\right|^{(1-\varepsilon_i)(m-1)}.$$

We integrate out the unitary matrix variables that come from this generalised Schur decomposition. Then by de-approximating, we get back to integration on direct product of k copies of $\mathcal{V}_{m,n}$. Using (28) and Lemma 17, we get

$$\begin{aligned}
\mathbf{E}\big[f(\underline{Z})\big] &= \lim_{\varepsilon_i \to 0} \frac{\int_{\bigcap_{i=1}^{k} \|Y_i{}^* Y_i - I\| < \varepsilon_i} |\Delta(\underline{z})|^2 f(\underline{x}) L(\underline{x}) |\mathrm{D}Y_1||\mathrm{D}Y_2| \cdots |\mathrm{D}Y_k|}{\int_{\bigcap_{i=1}^{k} \|Y_i{}^* Y_i - I\| < \varepsilon_i} |\Delta(\underline{z})|^2 L(\underline{x}) |\mathrm{D}Y_1||\mathrm{D}Y_2| \cdots |\mathrm{D}Y_k|} \\
&= C \int \left|\Delta(\underline{z})\right|^2 f(\underline{x}) L(\underline{x}) \prod_{i=1}^{k} \left|\mathrm{d}H_{\mathcal{V}_i}(V_i)\right|,
\end{aligned} \tag{29}$$

where $\mathcal{V}_i = \mathcal{N}_{m,n} \cap \mathcal{U}_i(n)$ and $C^{-1} = \int |\Delta(\underline{z})|^2 L(\underline{x}) \prod_{i=1}^k |\mathrm{d}H_{\mathcal{V}_i}(V_i)|$. We want to integrate out all the variables except the first m diagonal entries of each V_i. We do that by applying Lemma 18 to each V_i and get joint probability density of these diagonal variables $(x_{i1}, x_{i2}, \dots, x_{im})$. So we end up with joint probability density of $\underline{x}$ and that is proportional to

$$\prod_{\ell=1}^m \prod_{j=1}^k \left(1-|x_{j\ell}|^2\right)^{n-m-1} |x_{j\ell}|^{(m-1)(1-\varepsilon_j)} \mathbf{1}_{\{|x_{j\ell}|\le 1\}}(x_{j\ell}) |\mathrm{d}x_{j\ell}|^2 \prod_{1\le i<j\le m} |z_i - z_j|^2.$$

Now using (27), from the above we get that the joint probability density of $\underline{Z}$ is proportional to

$$\prod_{\ell=1}^m \omega(z_\ell) \prod_{1\le i<j\le m} |z_i - z_j|^2$$

with a weight function

$$|\mathrm{d}z|^2 \omega(z) = \int_{x_1^{\varepsilon_1}\cdots x_k^{\varepsilon_k}=z} \prod_{j=1}^k \left(1-|x_j|^2\right)^{n-m-1} |x_j|^{(m-1)(1-\varepsilon_j)} \mathbf{1}_{\{|x_j|\le 1\}}(x_j) |\mathrm{d}x_j|^2.$$

This completes the proof of the theorem when all n_i are equal. If n_i are not all equal, we will have in (29), $\mathcal{V}_i = \mathcal{N}_{m,n_i} \cap \mathcal{U}(n_i)$. After integrating out all unwanted variables, we will have joint probability density of $\underline{Z}$ proportional to

$$\prod_{\ell=1}^m \omega(z_\ell) \prod_{1\le i<j\le m} |z_i - z_j|^2$$

with a weight function

$$|\mathrm{d}z|^2 \omega(z) = \int_{x_1^{\varepsilon_1}\cdots x_k^{\varepsilon_k}=z} \prod_{j=1}^k \left(1-|x_j|^2\right)^{n_j-m-1} |x_j|^{(m-1)(1-\varepsilon_j)} \mathbf{1}_{\{|x_j|\le 1\}}(x_j) |\mathrm{d}x_j|^2.$$

Now using this density formula we show that the eigenvalues of A form a determinantal point process on the complex plane. Since the weight function $\omega(z)$ is angle independent, the monic polynomials $P_i(z) = z^i$ are orthogonal with respect to this $\omega(z)$. Then we have

$$\begin{aligned}
&\int z^a \left(\overline{z}^b\right) \omega(z) |\mathrm{d}z|^2 \\
&\quad = \prod_{j=1}^k \int (x_j)^{\varepsilon_j a} (\overline{x_j})^{\varepsilon_j b} \left(1-|x_j|^2\right)^{n_j-m-1} |x_j|^{(m-1)(1-\varepsilon_j)} \mathbf{1}_{\{|x_j|\le 1\}}(x_j) |\mathrm{d}x_j|^2 \\
&\quad = \delta_{ab} (2\pi)^k \underbrace{\prod_{j=1}^k \mathrm{Beta}\left(m \cdot \frac{1-\varepsilon_j}{2} + \frac{1+\varepsilon_j}{2} + a \cdot \varepsilon_j, n_j - m\right)}_{C_a}.
\end{aligned}$$

Corresponding kernel of orthogonal polynomials is given by

$$\mathbb{K}(z,w) = \sqrt{\omega(z)\omega(w)} \sum_{r=0}^{n-1} \frac{(z\overline{w})^r}{(2\pi)^k C_r}.$$

Hence by Fact 11 the eigenvalues of A form a determinantal point process with above kernel $\mathbb{K}(z,w)$ with respect to Lebesgue measure on $\mathbb{C}$. □

6. Limiting spectral distributions

In this section we calculate the expected limiting spectral distribution of product of Ginibre and inverse of Ginibre matrices, product of compatible rectangular matrices and product of truncated unitary matrices. For result on limit of empirical spectral distribution of product of independent square matrices with independent entries, we refer the reader to [11,18].

Theorem 19. *Let $A_1, A_2, \dots, A_k$ be independent $n \times n$ random matrices with i.i.d. standard complex Gaussian entries. Then the limiting expected empirical distribution of square radial part of eigenvalues of*

$$\left(\frac{A_1}{\sqrt{n}}\right)^{\varepsilon_1}\left(\frac{A_2}{\sqrt{n}}\right)^{\varepsilon_2}\cdots\left(\frac{A_k}{\sqrt{n}}\right)^{\varepsilon_k},$$

where each ε_i is either 1 or -1, is the same as the distribution of

$$U^p\left(\frac{1}{1-U}\right)^{k-p},$$

where U is a random variable distributed uniformly on $[0,1]$ and $p = \#\{\varepsilon_i \colon \varepsilon_i = 1\}$.

Proof. Let $A = A_1^{\varepsilon_1}A_2^{\varepsilon_2}\cdots A_k^{\varepsilon_k}$ and $p = \#\{\varepsilon_i \colon \varepsilon_i = 1\}$. We showed in Theorem 1 that the eigenvalues of A form a determinantal point process with kernel

$$\mathbb{K}_n(x,y) = \sqrt{\omega(x)\omega(y)}\sum_{a=0}^{n-1}\frac{(x\overline{y})^a}{(2\pi)^k(a!)^p((n-a-1)!)^{k-p}},$$

where $\omega(z)$ is given by

$$|\mathrm{d}z|^2\omega(z) = \int_{x_1^{\varepsilon_1}\cdots x_k^{\varepsilon_k}=z} \mathrm{e}^{-\sum_{j=1}^k |x_j|^2}\prod_{j=1}^k |x_j|^{(1-\varepsilon_i)(n-1)}\prod_{j=1}^k |\mathrm{d}x_j|^2.$$

Then the scaled one-point correlation function $\frac{1}{n}\mathbb{K}_n(z,z)$ gives the density of the expected empirical spectral distribution of A where

$$\mathbb{K}_n(z,z) = \omega(z)\sum_{a=0}^{n-1}\frac{|z|^{2a}}{(2\pi)^k(a!)^p((n-a-1)!)^{k-p}}.$$

Let $(X_{n,1}, X_{n,2}, \dots, X_{n,k})$ be random variables with joint probability density

$$\frac{1}{n}\mathrm{e}^{-\sum_{j=1}^k |x_j|^2}\prod_{j=1}^k |x_j|^{(1-\varepsilon_j)(n-1)}\sum_{a=0}^{n-1}\frac{|x_1^{\varepsilon_1}x_2^{\varepsilon_2}\cdots x_k^{\varepsilon_k}|^{2a}}{(2\pi)^k(a!)^p((n-a-1)!)^{k-p}}.$$

Then $\frac{1}{n}\mathbb{K}_n(z,z)$ is the density of the random variable $X_{n,1}^{\varepsilon_1}X_{n,2}^{\varepsilon_2}\cdots X_{n,k}^{\varepsilon_k}$. Now the density of expected empirical spectral distribution of

$$\left(\frac{A_1}{\sqrt{n}}\right)^{\varepsilon_1}\left(\frac{A_2}{\sqrt{n}}\right)^{\varepsilon_2}\cdots\left(\frac{A_k}{\sqrt{n}}\right)^{\varepsilon_k}$$

is the density of random variable

$$\left(\frac{X_{n,1}}{\sqrt{n}}\right)^{\varepsilon_1}\left(\frac{X_{n,2}}{\sqrt{n}}\right)^{\varepsilon_2}\cdots\left(\frac{X_{n,k}}{\sqrt{n}}\right)^{\varepsilon_k}.$$

Since the joint probability density of $(X_{n,1}, X_{n,2}, \dots, X_{n,k})$ is rotational invariant, we calculate the density only for radial part of

$$\left(\frac{X_{n,1}}{\sqrt{n}}\right)^{\varepsilon_1}\left(\frac{X_{n,2}}{\sqrt{n}}\right)^{\varepsilon_2}\cdots\left(\frac{X_{n,k}}{\sqrt{n}}\right)^{\varepsilon_k} =: Z_n.$$

Now we have

$$\begin{aligned}|Z_n|^2 &= \left(\frac{|X_{n,1}|^2}{n}\right)^{\varepsilon_1}\left(\frac{|X_{n,2}|^2}{n}\right)^{\varepsilon_2}\cdots\left(\frac{|X_{n,k}|^2}{n}\right)^{\varepsilon_k}\\ &= \left(\frac{R_{n,1}}{n}\right)^{\varepsilon_1}\left(\frac{R_{n,2}}{n}\right)^{\varepsilon_2}\cdots\left(\frac{R_{n,k}}{n}\right)^{\varepsilon_k}, \quad \text{say.}\end{aligned}$$

The joint probability density of $(R_{n,1}, R_{n,2}, \dots, R_{n,k})$ is proportional to

$$\frac{1}{n}\mathrm{e}^{-\sum_{j=1}^{k} r_{n,j}}\prod_{j=1}^{k}|r_{n,j}|^{(1-\varepsilon_j)(n-1)/2}\sum_{a=0}^{n-1}\frac{(r_{n,1}^{\varepsilon_1}r_{n,2}^{\varepsilon_2}\cdots r_{n,k}^{\varepsilon_k})^a}{(a!)^p((n-a-1)!)^{k-p}}$$

and the density $f(r)$ of $R_{n,j}$ is given by

$$f(r) = \frac{1}{n}\mathrm{e}^{-r}\sum_{a=0}^{n-1}\frac{r^a}{a!}, \quad 0 < r < \infty.$$

So the density of $\frac{R_{n,j}}{n}$

$$\mathrm{e}^{-nr}\sum_{a=0}^{n-1}\frac{(nr)^a}{a!} = \mathbf{P}\big[\mathrm{Pois}(nr) \le n-1\big] \to \begin{cases}1 & \text{if } 0 < r < 1,\\ 0 & \text{otherwise,}\end{cases}$$

as $n \to \infty$, where $\mathrm{Pois}(\lambda)$ $(\lambda > 0)$ is a discrete random variable with probability mass function

$$\mathbf{P}\big(\mathrm{Pois}(\lambda) = k\big) = \frac{\lambda^k \mathrm{e}^{-\lambda}}{k!} \quad \text{for } k \ge 0.$$

Hence we have

$$\frac{R_{n,j}}{n} \xrightarrow{\mathcal{D}} U \quad \text{as } n \to \infty, \tag{30}$$

where U is a random variable distributed uniformly on [0, 1]. The joint density of $R_{n,j}, R_{n,k}$ is

$$\frac{1}{n}\mathrm{e}^{-(x+y)}\sum_{a=0}^{n-1}\frac{(xy)^a}{a!a!}$$

if both $\varepsilon_j, \varepsilon_k$ are either $+1$ or -1. Then

$$\begin{aligned}&\mathbf{E}\big[|R_{n,j} - R_{n,k}|^2\big]\\ &\quad= \int_0^\infty\int_0^\infty \frac{1}{n}\mathrm{e}^{-(x+y)}(x-y)^2\sum_{a=0}^{n-1}\frac{(xy)^a}{a!}\,\mathrm{d}x\,\mathrm{d}y\\ &\quad= \frac{2}{n}\int_0^\infty\int_0^\infty \mathrm{e}^{-(x+y)}\sum_{a=0}^{n-1}\frac{x^{a+2}y^a}{a!a!}\,\mathrm{d}x\,\mathrm{d}y - \frac{2}{n}\int_0^\infty\int_0^\infty \mathrm{e}^{-(x+y)}\sum_{a=0}^{n-1}\frac{x^{a+1}y^{a+1}}{a!a!}\,\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{n}\left[\sum_{a=0}^{n-1}\left\{(a+2)(a+1)-(a+1)^2\right\}\right] \\
&= n+1.
\end{aligned}
$$

Therefore

$$
\left(\frac{R_{n,j}}{n}-\frac{R_{n,k}}{n}\right)\stackrel{L^2}{\to} 0 \quad \text{as } n\to\infty. \tag{31}
$$

If $\varepsilon_j=1, \varepsilon_k=-1$, then the joint density of $R_{n,j}, R_{n,k}$ is

$$
\frac{1}{n}\mathrm{e}^{-(x+y)}\sum_{a=0}^{n-1}\frac{x^a y^{n-1-a}}{a!(n-1-a)!}.
$$

Therefore we have

$$
\begin{aligned}
&\mathbf{E}\left[|R_{n,j}+R_{n,k}-n|^2\right] \\
&= \frac{1}{n}\int_0^\infty\int_0^\infty \mathrm{e}^{-(x+y)}(x+y-n)^2\sum_{a=0}^{n-1}\frac{x^a y^{(n-a-1)}}{a!(n-a-1)!}\,\mathrm{d}x\,\mathrm{d}y \\
&= \frac{2}{n}\int_0^\infty \mathrm{e}^{-x}\sum_{a=0}^{n-1}\frac{x^{a+2}}{a!}\,\mathrm{d}x - 4\int_0^\infty \mathrm{e}^{-x}\sum_{a=0}^{n-1}\frac{x^{a+1}}{a!}\,\mathrm{d}x \\
&\quad + \frac{2}{n}\int_0^\infty\int_0^\infty \mathrm{e}^{-(x+y)}\sum_{a=0}^{n-1}\frac{x^{a+1}y^{n-a}}{a!(n-a-1)!}\,\mathrm{d}x\,\mathrm{d}y + n^2 \\
&= \frac{2}{n}\sum_{a=0}^{n-1}(a+2)(a+1) - 4\sum_{a=0}^{n-1}(a+1) + \frac{2}{n}\sum_{a=0}^{n-1}(a+1)(n-a) + n^2 \\
&= \frac{2}{n}\sum_{a=0}^{n-1}(n+2)(a+1) - 4\sum_{a=0}^{n-1}(a+1) + n^2 \\
&= (n+2)(n+1) - 2n(n+1) + n^2 = n+2
\end{aligned}
$$

and hence

$$
\left(\frac{R_{n,j}}{n}+\frac{R_{n,k}}{n}-1\right)\stackrel{L^2}{\to} 0 \quad \text{as } n\to\infty. \tag{32}
$$

Now combining (30), (31) and (32), we get

$$
\left(\frac{R_{n,1}}{n}\right)^{\varepsilon_1}\left(\frac{R_{n,2}}{n}\right)^{\varepsilon_2}\cdots\left(\frac{R_{n,k}}{n}\right)^{\varepsilon_k}\stackrel{\mathcal{D}}{\to} U^p\left[\frac{1}{1-U}\right]^{k-p},
$$

where $p=\#\{\varepsilon_i\colon\ \varepsilon_i=1\}$. □

Remark 20. *If $k=1$ and $\varepsilon=1$, then it follows from Theorem 19 that the expected limiting spectral distribution of properly scaled Ginibre matrix is well known circular law. If $k=2$ with $\varepsilon_1=-1, \varepsilon_2=1$, we get the expected limiting spectral distribution of spherial ensemble.*

In the following theorem we describe the limiting distribution of the radial part of the eigenvalues of product of rectangular matrices. Limit of empirical spectral distribution of product of independent rectangular matrices has been derived in [5], but the limiting density is obtained in terms of M-transform. However, we have a simple explicit expression in terms of uniform distribution for the limit.

Theorem 21. *Let $A_1, A_2, \ldots, A_k$ be independent rectangular matrices of dimension $n_i \times n_{i+1}$ for $i = 1, 2, \ldots, k$, with $n_{k+1} = n_1 = \min\{n_1, n_2, \ldots, n_k\}$, and with i.i.d. standard complex Gaussian entries. If $\frac{n_j}{n_1} \to \alpha_j$ as $n_1 \to \infty$ for $j = 2, 3, \ldots, k$, then the limiting expected empirical distribution of square radial part of eigenvalues of $\frac{A_1}{\sqrt{n}}\frac{A_2}{\sqrt{n}}\cdots\frac{A_k}{\sqrt{n}}$ is the same as the distribution of following random variable*

$$U(U - 1 + \alpha_2)\cdots(U - 1 + \alpha_k),$$

where U is a uniform random variable on $[0, 1]$.

Proof. We have shown in Theorem 2 that the eigenvalues of $A_1 A_2 \cdots A_k$ form a determinantal point process with kernel

$$\mathbb{K}_n(x, y) = \sqrt{\omega(x)\omega(y)}\sum_{a=0}^{n-1}\frac{(x\overline{y})^a}{(2\pi)^k\prod_{j=1}^k(n_j - n_1 + a)!},$$

where $\omega(z)$ is the weight function, given by

$$|\mathrm{d}z|^2\omega(z) = \int_{x_1 x_2 \ldots x_k = z} \mathrm{e}^{-\sum_{j=1}^k |x_j|^2}\prod_{j=1}^k |x_j|^{2(n_j - n_1)}\prod_{j=1}^k |\mathrm{d}x_j|^2.$$

Then the scaled one-point correlation function $\frac{1}{n}\mathbb{K}_n(z, z)$ gives the density of the expected empirical spectral distribution of $A_1 A_2 \cdots A_k$. Let $n_1 = n$ and $(X_{n,1}, X_{n,2}, \ldots, X_{n,k})$ be random variables with joint probability density

$$\frac{1}{n}\mathrm{e}^{-\sum_{j=1}^k |x_j|^2}\prod_{j=1}^k |x_j|^{2(n_j - n)}\sum_{a=0}^{n-1}\frac{|x_1 x_2 \cdots x_k|^{2a}}{(2\pi)^k\prod_{j=1}^k(n_j - n + a)!}.$$

Then the density of $(X_{n,1}X_{n,2}\cdots X_{n,k})$ is given by $\frac{1}{n}\mathbb{K}_n(z, z)$. Now the density of expected empirical spectral distribution of $\frac{A_1}{\sqrt{n}}\frac{A_2}{\sqrt{n}}\cdots\frac{A_k}{\sqrt{n}}$ is the same as the density of the random variable $Z_n = \frac{X_{n,1}}{\sqrt{n}}\frac{X_{n,2}}{\sqrt{n}}\cdots\frac{X_{n,k}}{\sqrt{n}}$. Clearly, the joint probability density of $(X_{n,1}, X_{n,2}, \ldots, X_{n,k})$ is rotational invariant. So we calculate the density of square of radial part of Z_n. We have

$$|Z_n|^2 = \frac{|X_{n,1}|^2}{n}\frac{|X_{n,2}|^2}{n}\cdots\frac{|X_{n,k}|^2}{n} = \frac{R_{n,1}}{n}\frac{R_{n,2}}{n}\cdots\frac{R_{n,k}}{n}, \quad \text{say.}$$

The joint probability density of $(R_{n,1}, R_{n,2}, \ldots, R_{n,k})$ is

$$\frac{1}{n}\mathrm{e}^{-\sum_{j=1}^k r_j}\prod_{j=1}^k r_j^{(n_j - n)}\sum_{a=0}^{n-1}\frac{(r_1 r_2 \cdots r_k)^a}{\prod_{j=1}^k(n_j - n + a)!}.$$

Now by routine calculation it can be shown that

$$\frac{R_{n,1}}{n} \xrightarrow{\mathcal{D}} U \quad \text{as } n \to \infty, \tag{33}$$

$$\mathbf{E}(R_{n,1} - R_{n,j}) = (n - n_j) \quad \text{for } j = 2, 3, \ldots, k, \tag{34}$$

$$\mathbf{E}\big[(R_{n,1} - R_{n,j})^2\big] = (n - n_j)^2 + n_j + 1 \quad \text{for } j = 2, 3, \ldots, k, \tag{35}$$

where U is a uniform random variable on [0, 1]. By (34) and (35), we have

$$\frac{R_{n,1}}{n} - \frac{R_{n,j}}{n} - (1-\alpha_j) \xrightarrow{L^2} 0 \quad \text{for } j = 2, 3, \dots, k. \tag{36}$$

Therefore by (33) and (36), we conclude that the limiting distribution of $|Z_n|^2$ is the same as the distribution of the following random variable

$$U(U-1+\alpha_2)\cdots(U-1+\alpha_k).$$

This completes proof of the theorem. □

In the following theorem we describe the limiting distribution of radial part of eigenvalues of product of truncated unitary matrices.

Theorem 22. *Let $U_1, U_2, \dots, U_k$ be k independent Haar distributed unitary matrices of dimension $n_i \times n_i$ for $i = 1, 2, \dots, k$ respectively and $A_1, A_2, \dots, A_k$ be $m \times m$ left uppermost blocks of $U_1, U_2, \dots, U_k$ respectively. If $\frac{n_i}{m} \to \alpha_i$ as $m \to \infty$ for $i = 1, 2, \dots, k$, then the limiting expected empirical distribution of square radial part of eigenvalues of $A_1^{\varepsilon_1} A_2^{\varepsilon_2} \cdots A_k^{\varepsilon_k}$ is the same as the distribution of following random variable*

$$\prod_{i=1}^{k} \left(\frac{(1-\varepsilon_i)/2 + \varepsilon_i U}{\alpha_i - (1+\varepsilon_i)/2 + \varepsilon_i U} \right)^{\varepsilon_i},$$

where U is a random variable uniformly distributed on [0, 1] *and each ε_i is $+1$ or -1.*

Proof. We have shown that the eigenvalues of $A_1^{\varepsilon_1} A_2^{\varepsilon_2} \cdots A_k^{\varepsilon_k}$ form a determinantal point process with kernel

$$\mathbb{K}_m(x, y) = \sqrt{\omega(x)\omega(y)} \sum_{a=0}^{m-1} \frac{(x\overline{y})^a}{(2\pi)^k C_a},$$

where $C_a = \prod_{\{j:\, \varepsilon_j = 1\}} B(a+1, n_j - m) \prod_{\{j:\, \varepsilon_j = -1\}} B(m-a, n_j - m)$ and $\omega(z)$ is the weight function, given by

$$|\mathrm{d}z|^2 \omega(z) = \int_{x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_k^{\varepsilon_k} = z} \prod_{j=1}^{k} \left(1 - |x_j|^2\right)^{(n_j - m - 1)} |x_j|^{(1-\varepsilon_j)(m-1)} \mathbf{1}_{|x_j| \le 1} \prod_{j=1}^{k} |\mathrm{d}x_j|^2.$$

Then the density of expected empirical spectral distribution of $A_1^{\varepsilon_1} A_2^{\varepsilon_2} \cdots A_k^{\varepsilon_k}$ is given by $\frac{1}{m}\mathbb{K}_m(z, z)$. Let $(X_{m,1}, X_{m,2}, \dots, X_{m,k})$ be random variables with joint probability density

$$\frac{1}{m} \prod_{j=1}^{k} \left(1 - |x_j|^2\right)^{(n_j - m - 1)} |x_j|^{(1-\varepsilon_j)(m-1)} \mathbf{1}_{|x_j| \le 1} \sum_{a=0}^{m-1} \frac{|x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_k^{\varepsilon_k}|^{2a}}{(2\pi)^k C_a}.$$

Then it is easy to see that the density of $Z_m = X_{m,1}^{\varepsilon_1} X_{m,2}^{\varepsilon_2} \cdots X_{m,k}^{\varepsilon_k}$ is also $\frac{1}{m}\mathbb{K}_m(z, z)$. Clearly, the joint probability density of $(X_{m,1}, X_{m,2}, \dots, X_{m,k})$ is rotational invariant. So we calculate the density for square radial part of Z_m. We have

$$|Z_m|^2 = |X_{m,1}|^{2\varepsilon_1} |X_{m,2}|^{2\varepsilon_2} \cdots |X_{m,k}|^{2\varepsilon_k} = R_{m,1}^{\varepsilon_1} R_{m,2}^{\varepsilon_2} \cdots R_{m,k}^{\varepsilon_k}, \quad \text{say.}$$

Now the joint probability density of $(R_{m,1}, R_{m,2}, \dots, R_{m,k})$ is

$$\frac{1}{m} \prod_{j=1}^{k} (1 - r_j)^{(n_j - m - 1)} r_j^{((1-\varepsilon_j)/2)(m-1)} \mathbf{1}_{0 < r_j \le 1} \sum_{a=0}^{m-1} \frac{|r_1^{\varepsilon_1} r_2^{\varepsilon_2} \cdots r_k^{\varepsilon_k}|^a}{C_a}.$$

For $\varepsilon_i = 1$, density of $R_{m,i}$ for $i = 1, 2, \dots, m$ is given by

$$\frac{1}{m}\sum_{a=0}^{m-1}\frac{(n_i-m+a)!}{a!(n_i-m-1)!}(1-r)^{(n_i-m-1)}r^a.$$

Therefore for any $\ell \in \mathbb{N}$, we have

$$\begin{aligned}
\mathbf{E}\big[R_{m,i}^{\ell}\big] &= \int_0^{\infty}\frac{1}{m}\sum_{a=0}^{m-1}\frac{(n_i-m+a)!}{a!(n_i-m-1)!}(1-r)^{(n_i-m-1)}r^{\ell+a} \\
&= \frac{1}{m}\sum_{a=0}^{m-1}\frac{(n_i-m+a)!}{a!(n_i-m-1)!}\frac{(\ell+a)!(n-m-1)!}{(n-m+\ell+r)!} \\
&= \frac{1}{m}\sum_{a=0}^{m-1}\frac{(a+\ell)(a+\ell-1)\cdots(r+1)}{(n_i-m+a+\ell)(n_i-m+a+\ell-1)\cdots(n_i-m+a+1)} \\
&\to \int_0^1\frac{x^{\ell}}{(\alpha_i-1+x)^{\ell}}\,\mathrm{d}x \quad \text{as } m\to\infty \\
&= \mathbf{E}\bigg[\bigg(\frac{U}{\alpha_i-1+U}\bigg)^{\ell}\bigg],
\end{aligned} \tag{37}$$

where U is uniform random variable on $[0, 1]$. By similar way it can be shown that if $\varepsilon_i = -1$, then for any $\ell \in \mathbb{N}$,

$$\mathbf{E}\big[R_{m,i}^{\ell}\big] \to \mathbf{E}\bigg[\bigg(\frac{1-U}{\alpha_i-U}\bigg)^{\ell}\bigg] \quad \text{as } m\to\infty. \tag{38}$$

If $\varepsilon_i = 1$ and $\varepsilon_j = 1$, then it is not hard to see that

$$\mathbf{E}\bigg[\frac{(\alpha_i-1)R_{m,i}}{1-R_{m,i}} - \frac{(\alpha_j-1)R_{m,j}}{1-R_{m,j}}\bigg]^2 \to 0 \quad \text{as } m\to\infty, \tag{39}$$

and if $\varepsilon_i = +1$ and $\varepsilon_j = -1$, then

$$\mathbf{E}\bigg[\frac{(\alpha_i-1)R_{m,i}}{1-R_{m,i}} + \frac{(\alpha_j-1)R_{m,j}}{1-R_{m,j}} - 1\bigg]^2 \to 0 \quad \text{as } m\to\infty. \tag{40}$$

Now by (37), (38), (39) and (40), it follows that the limiting distribution of $|Z_m|^2$ is the same as the distribution of the random variable

$$\prod_{i=1}^{k}\bigg(\frac{(1-\varepsilon_i)/2+\varepsilon_i U}{\alpha_i-(1+\varepsilon_i)/2+\varepsilon_i U}\bigg)^{\varepsilon_i}$$

and this completes the proof. □

Appendix

In this appendix we prove the lemmas stated in Sections 2 and 5. In Appendix A.1, we keep the proofs of Lemmas 4 and 7 stated in 2. In Appendix A.2, we prove the lemmas used in the proof of Theorem 3 in Section 5.

A.1. Proofs of Lemmas 4 and 7

Proof of Lemma 4. Recall that for any $m \times n$ complex matrix M of rank m can be written as

$$M = SU^*, \tag{41}$$

where S is a $m \times m$ upper triangular matrix and U^* has orthonormal rows with non-negative real diagonal entries. If all diagonal entries of U are positive, then this decomposition is unique. Now from (41), we get

$$\mathrm{d}M = S(\mathrm{d}U^*) + (\mathrm{d}S)U^*.$$

Let V be such that $[U \ \ V]$ is $n \times n$ unitary matrix.

$$\begin{aligned} \Lambda &:= (\mathrm{d}M)[U \quad V] = \big(S(\mathrm{d}U^*) + (\mathrm{d}S)U^*\big)[U \quad V] \\ &= S(\mathrm{d}U^*)[U \quad V] + \mathrm{d}S[I \quad 0] \\ &= S\Omega + [\mathrm{d}S \quad 0], \end{aligned} \tag{42}$$

where $\Omega := (\mathrm{d}U^*)[U \ \ V] = (\omega_{i,j})$ and $\Lambda = (\lambda_{i,j})$ are $m \times n$ matrices of one forms. Also observe that, the leftmost $m \times m$ block of Ω is skew-Hermitian.

Now we want to write the Lebesgue measure on M in terms of Lebesgue measure on S and Haar measure on U. For this we must find the Jacobian determinant for the change of variables from $\{\mathrm{d}M_{i,j}, \mathrm{d}\overline{M}_{i,j}, 1 \le i \le m, 1 \le j \le n\}$ to $\{\mathrm{d}S_{i,j}, 1 \le i, j \le m\}$ and Ω. Since for any fixed unitary matrix W, the transformation $M \to MW$ is unitary on the set of $m \times n$ complex matrices, we have

$$|\mathrm{D}M| = \bigwedge_{i,j} (\lambda_{i,j} \wedge \overline{\lambda}_{i,j}). \tag{43}$$

Thus we just have to find the Jacobian determinant for the change of variables from Λ to Ω, $\mathrm{d}S$ and their conjugates. We write (42) in the following way

$$\begin{aligned} \lambda_{i,j} &= \mathrm{d}S_{i,j} + \sum_{k=1}^{m} S_{i,k}\omega_{k,j} \\ &= \begin{cases} S_{i,i}\omega_{i,j} + [\sum_{k=i+1}^{m} S_{i,k}\omega_{k,j}] & \text{if } j < i \le m, \\ \mathrm{d}S_{i,i} + S_{i,i}\omega_{i,i} + [\sum_{k=i+1}^{m} S_{i,k}\omega_{k,j}] & \text{if } i = j, \\ \mathrm{d}S_{i,j} + [\sum_{k=i}^{m} S_{i,k}\omega_{k,j}] & \text{if } i < j \le m, \\ S_{i,i}\omega_{i,j} + [\sum_{k=i+1}^{m} S_{i,k}\omega_{k,j}] & \text{if } j > m. \end{cases} \end{aligned} \tag{44}$$

Now we arrange $\{\lambda_{i,j}, \overline{\lambda}_{i,j}\}$ in the ascending order given by the following relation

$$(i, j) \le (r, s) \quad \text{if } i > r \text{ or if } i = r \text{ and } j \le s.$$

Also observe that the expressions inside square brackets in (44) involve only those one-forms that have already appeared before in the given ordering of one-forms $\{\lambda_{i,j}, \overline{\lambda}_{i,j}\}$. Recall that the leftmost $m \times m$ block of Ω is skew-Hermitian, that is, $\omega_{i,j} = -\overline{\omega}_{j,i}$ for $i, j \le m$. Now taking wedge products of $\lambda_{i,j}$ in the above mentioned order and using the transformation rules given in (44), and with the help of last two observations, we get that

$$\begin{aligned} \bigwedge_{i,j} |\lambda_{i,j}|^2 &= \prod_{i=1}^{m} |S_{i,i}|^{2(n-m+i-1)} \bigwedge_{i} |\mathrm{d}S_{i,i} + S_{i,i}\omega_{i,i}|^2 \bigwedge_{i<j} |\mathrm{d}S_{i,j}|^2 \bigwedge_{i<j} |\omega_{i,j}|^2 \\ &= \prod_{i=1}^{m} |S_{i,i}|^{2(n-m+i-1)} \bigwedge_{i} |\mathrm{d}S_{i,i}|^2 \bigwedge_{i<j} |\mathrm{d}S_{i,j}|^2 \bigwedge_{i<j} |\omega_{i,j}|^2. \end{aligned} \tag{45}$$

We arrive at the last step in (45) by observing that $\omega_{k,k} \bigwedge_{i<j} |\omega_{i,j}|^2 = 0$ for any $k \le m$, because $\{U_{n\times m}\colon\ U^*U = I, U_{i,i} > 0\}$ is a smooth manifold of dimension $(2nm - m^2 - m)$ and its complement in $\{U_{n\times m}\colon\ U^*U = I, U_{i,i} \ge 0\}$ is of measure zero and $\omega_{k,k} \bigwedge_{i<j} |\omega_{i,j}|^2$ is an $(2nm - m^2 - m + 1)$-form.

Finally, using (43) and (45) we arrive at the following Jacobian determinant formula

$$|\mathrm{D}M| = \prod_{i=1}^{m} |S_{i,i}|^{2(n-m+i)-1} |\mathrm{D}S| \big|\mathrm{d}H(u)\big|.$$

This completes the proof of the lemma. □

Proof of Lemma 7. Let us recall from Proposition 5 that given k rectangular matrices $A_1, A_2, \dots, A_k$, there exist upper triangular square matrices $S_1, S_2, \dots, S_k$ with diagonal entries of $S_1S_2\cdots S_k$ in decreasing lexicographic order, and $[U_i\ \ V_i]$, $i = 1, 2, \dots, k$ unitary matrices with U_i's having non-negative real diagonal entries, and $B_2, B_3, \dots, B_k$ rectangular matrices with suitable dimensions, such that

$$\begin{aligned}
A_1 &= U_1 S_1 U_2^*,\\
A_2 &= U_2 S_2 U_3^* + V_2 B_2,\\
&\vdots\\
A_{k-1} &= U_{k-1} S_{k-1} U_k^* + V_{k-1} B_{k-1},\\
A_k &= U_k S_k U_1^* + V_k B_k.
\end{aligned}$$

Now, we apply the following transformations step by step to arrive at Jacobian determinant formula (13) for generalized Schur decomposition.

Step 1. We first transform

$$(A_1, A_2, \dots, A_k) \to (X_1, X_2, \dots, X_k),$$

where $X_i = A_i$ for $i = 1, 2, \dots, k-1$ and

$$X_k = \begin{bmatrix} A_1A_2\cdots A_{k-1} \\ V_k^* \end{bmatrix} A_k,$$

where V_k^* is $(n_k - n_1) \times n_k$ matrix with orthonormal rows. Also the rows of V_k^* are orthogonal to rows of $A_1A_2\cdots A_{k-1}$. It is easy to see that the Jacobian determinant formula for this transformation is given by

$$\prod_{i=1}^{k} |\mathrm{D}A_i| = \det\big((A_1A_2\cdots A_{k-1})(A_1A_2\cdots A_{k-1})^*\big)^{-n_1} \prod_{i=1}^{k} |\mathrm{D}X_i|. \tag{46}$$

Step 2. By applying Schur-decomposition to upper $n_1 \times n_1$ block of X_k we get

$$X_k = \begin{bmatrix} U_1 T U_1^* \\ B_k \end{bmatrix},$$

where $B_k = V_k^* A_k$. Using (8), the Lebesgue measure on X_k can be written in terms of U_1, T, B_k as follows

$$|\mathrm{D}X_k| = \big|\Delta(T)\big|^2 \big|\mathrm{d}H(U_1)\big| |\mathrm{D}T| |\mathrm{D}B_k|, \tag{47}$$

where $|\mathrm{d}H(U_1)|$ is Haar measure on $\mathcal{U}(n_1)/\mathcal{U}(1)^{n_1}$, $|\mathrm{D}T|$ is the Lebesgue measure on T and

$$\Delta(T) = \prod_{1\le i<j\le n_1} (T_{i,i} - T_{j,j}).$$

If we denote the eigenvalues of $A_1A_2\cdots A_k$ by $z_1, z_2, \ldots, z_{n_1}$, then $\Delta(T)$ is basically equal to

$$\Delta(T) = \prod_{1\le i<j\le n_1}(T_{i,i} - T_{j,j}) = \prod_{1\le i<j\le n_1}(z_i - z_j).$$

Step 3. Now we apply the following transformation

$$X_1 \to U_1^* X_1 = S_1 U_2^*,$$

where U_1 is as in Step 2 and second part of above equation is by RQ-decomposition of $U_1^*X_1$. U_2^* is $n_1 \times n_2$ matrix with orthonormal rows and non-negative real diagonal entries and S_1 is $n_1 \times n_1$ upper triangular matrix. We shall omit matrices X_1 for which $U_1^*X_1$ is not of full rank (this set is of measure zero). Now using (10), the Lebesgue measure on X_1 can be written in terms of U_2, S_1 as follows

$$|\mathrm{D}X_1| = J(S_1)\left|\mathrm{d}H(U_2)\right||\mathrm{D}S_1|, \tag{48}$$

where

$$J(S_1) = \prod_{i=1}^{n_1}\left|S_1(i,i)\right|^{2(n_2-n_1+i-1)}.$$

Step $i+2$ for $i = 2, 3, \ldots, k-1$. At $(i+2)$th step we apply the following transformation

$$X_i \to \begin{bmatrix} U_i^* \\ V_i^* \end{bmatrix} X_i = \begin{bmatrix} S_i U_{i+1}^* \\ B_i \end{bmatrix},$$

where U_i is as in Step $i+1$ and $[U_i \ \ V_i]$ is an unitary matrix. The second part of the above equation is obtained by RQ-decomposition of $U_i^*X_i$, where U_{i+1}^* is $n_1 \times n_{i+1}$ matrix with orthonormal rows and non-negative real diagonal entries, and S_i is $n_1 \times n_1$ upper triangular matrix. Also note that $B_i = V_i^*X_i = V_i^*A_i$ for $2 \le i \le k-1$. We shall omit matrices X_i for which $U_i^*X_i$ is not of full rank (this set is of measure zero). Now using (10), the Lebesgue measure on X_i can be written in terms of U_{i+1}, S_i, B_i as

$$|\mathrm{D}X_i| = J(S_i)\left|\mathrm{d}H(U_{i+1})\right||\mathrm{D}S_i||\mathrm{D}B_i|, \tag{49}$$

where

$$J(S_i) = \prod_{j=1}^{n_1}\left|S_i(j,j)\right|^{2(n_{i+1}-n_1+j-1)}.$$

Step $k+2$. Now we transform T to S_k as follows

$$T \to S_k := (S_1S_2\cdots S_{k-1})^{-1}T.$$

The Jacobian determinant formula for this transformation is given by

$$|\mathrm{D}T| = \prod_{i=1}^{k-1} L(S_i)|\mathrm{D}S_k|, \tag{50}$$

where

$$L(S_i) = \prod_{j=1}^{n_1}\left|S_i(j,j)\right|^{2(n_1-j+1)}.$$

Applying the above transformations in the given order, we can write Lebesgue measure on $(A_1, A_2, \dots, A_k)$ in terms of $U_1, U_2, \dots, U_k, S_1, S_2, \dots, S_k, B_2, B_3, \dots, B_k$. Also observe that

$$A_1 A_2 \cdots A_{k-1} = U_1 S_1 S_2 \cdots S_{k-1} U_k^*, \qquad U_k^* A_k = S_k U_1^*.$$

So

$$\det(A_1 A_2 \cdots A_{k-1})(A_1 A_2 \cdots A_{k-1})^* = \det(S_1 S_2 \cdots S_{k-1})^2. \tag{51}$$

Now combining (46) to (51), we get that

$$\begin{aligned}\prod_{i=1}^{k} |\mathrm{D}A_i| &= \left|\Delta(T)\right|^2 \prod_{i=1}^{k-1} J(S_i) L(S_i) \left|\det(S_i)\right|^{-2n_1} \prod_{i=1}^{k} \left|\mathrm{d}H(U_i)\right| |\mathrm{D}S_i| \prod_{i=2}^{k} |\mathrm{D}B_i| \\ &= \prod_{1 \le i < j \le n_1} |z_i - z_j|^2 \prod_{i=1}^{k} \left|\det(S_i)\right|^{2(n_{i+1}-n_1)} \left|\mathrm{d}H(U_i)\right| |\mathrm{D}S_i| |\mathrm{D}B_i|,\end{aligned} \tag{52}$$

where $z_1, z_2, \dots, z_{n_1}$ are the eigenvalues of $A_1 A_2 \cdots A_k$. □

A.2. *Proofs of Lemmas* 16, 17 *and* 18

In this subsection we prove the Lemmas 16, 17 and 18 stated in Section 5. To prove these lemmas we compute Jacobian determinant formula for QR-decomposition of matrices in $\mathcal{M}_n$ and $\mathcal{N}_{m,n}$.

Jacobian computation for QR-decomposition in $\boldsymbol{\mathcal{M}_n}$. QR-decomposition can be thought of as polar decomposition for matrices. Any matrix $M \in \mathcal{M}_n$ can be written as

$$M = QR,$$

where Q is unitary matrix and R is upper triangular matrix with non-negative diagonal entries. This can be done by applying Gram–Schmidt orthogonalization process to the columns of M from left to right. Then

$$M_j = \sum_{i=1}^{j} Q_i R_{i,j},$$

where M_j and Q_j are jth columns of M and Q respectively. We would like to write Lebesgue measure on M in terms of Haar measure on Q and Lebesgue measure on R. Since $M_1 = R_{1,1} Q_1$, so Lebesgue measure on M_1 is given by

$$|\mathrm{D}M_1| = R_{1,1}^{2n-1}\, \mathrm{d}R_{1,1}\, \mathrm{d}\sigma_{\mathbb{T}^n}(Q_1),$$

where $\mathrm{d}\sigma_{\mathbb{T}^n}$ denotes volume measure on unit sphere $\mathbb{T}^n$ in $\mathbb{C}^n$. Once Q_1 is fixed, any new column M_2 can be written as

$$M_2 = Q_1 R_{1,2} + Q_2 R_{2,2},$$

where Q_2 is unit vector orthogonal to Q_1 and $R_{2,2} \ge 0$. By unitary invariance of Lebesgue measure, Lebesgue measure on M_2 can be written as

$$|\mathrm{D}M_2| = R_{2,2}^{2n-3}\, \mathrm{d}R_{2,2} |\mathrm{d}R_{1,2}|^2\, \mathrm{d}\sigma_{\mathbb{T}^n \cap Q_1^\perp}(Q_2),$$

where $Q_1^\perp$ is the sub-space which is perpendicular to Q_1 and $\mathrm{d}\sigma_{\mathbb{T}^n \cap Q_1^\perp}$ denotes volume measure on manifold $\mathbb{T}^n \cap Q_1^\perp$. Continuing this way, Lebesgue measure on M_i,

$$|\mathrm{D}M_i| = R_{i,i}^{2(n-i+1)-1}\, \mathrm{d}R_{i,i} |\mathrm{d}R_{1,i}|^2 |\mathrm{d}R_{2,i}|^2 \cdots |\mathrm{d}R_{i-1,i}|^2\, \mathrm{d}\sigma_{\mathbb{T}^n \cap \{Q_1, Q_2, \dots, Q_{i-1}\}^\perp}(Q_i).$$

Therefore we have

$$|\mathrm{D}M| = \prod_{i=1}^{n} |\mathrm{D}M_i| = \left[\prod_{i=1}^{n} R_{i,i}^{2(n-i+1)-1}\,\mathrm{d}R_{i,i}\right]\left[\prod_{i<j} |\mathrm{d}R_{i,j}|^2\right]\left[\prod_{i=1}^{n} \mathrm{d}\sigma_{\mathbb{T}^n\cap\{Q_1,Q_2,\dots,Q_{i-1}\}^\perp}(Q_i)\right].$$

We can see that measure on Q given by

$$\left[\prod_{i=1}^{n} \mathrm{d}\sigma_{\mathbb{T}^n\cap\{Q_1,Q_2,\dots,Q_{i-1}\}^\perp}(Q_i)\right]$$

is Haar measure on unitary group $\mathcal{U}(n)$. So, finally we have

$$|\mathrm{D}M| = \left[\prod_{i=1}^{n} R_{i,i}^{2(n-i+1)-1}\,\mathrm{d}R_{i,i}\right]\left[\prod_{i<j} |\mathrm{d}R_{i,j}|^2\right]\left|\mathrm{d}H_{\mathcal{U}(n)}(Q)\right|. \tag{53}$$

Jacobian computation for QR-decomposition in $\mathcal{N}_{m,n}$. First recall from (25) that $\mathcal{N}_{m,n} = \{Y \in \mathcal{M}_n\colon\ Y_{i,j} = 0, 1 \le j < i \le m\}$ and $\mathcal{V}_{m,n} = \mathcal{N}_{m,n} \cap \mathcal{U}(n)$. Any matrix $M \in \mathcal{N}_{m,n}$ can be written as

$$M = QR,$$

where Q is unitary matrix in $\mathcal{V}_{m,n}$ and R is upper triangular matrix with non-negative diagonal entries. Then

$$M_j = \sum_{i=1}^{j} Q_i R_{i,j},$$

where M_j and Q_j are jth columns of M and Q respectively. We would like to write Lebesgue measure on M in terms of Haar measure on Q and Lebesgue measure on R.

Note $M_1 = R_{1,1}Q_1$ where Q_1 is unit vector orthogonal to $e_2, e_3, \dots, e_m$, where $e_1, e_2, \dots, e_n$ are standard basis vectors in $\mathbb{C}^n$. So Lebesgue measure on M_1, $|\mathrm{D}M_1| = R_{1,1}^{2(n-m+1)-1}\,\mathrm{d}R_{1,1}\,\mathrm{d}\sigma_{\mathbb{T}^n\cap\{e_2,\dots,e_m\}^\perp}(Q_1)$ and $\mathrm{d}\sigma_{\mathbb{T}^n\cap\{e_2,\dots,e_m\}^\perp}$ denotes volume measure on manifold $\mathbb{T}^n \cap \{e_2, \dots, e_m\}^\perp$ in $\mathbb{C}^n$. Once Q_1 is fixed, second column M_2 can be written as

$$M_2 = Q_1 R_{1,2} + Q_2 R_{2,2},$$

where Q_2 is unit vector orthogonal to $Q_1, e_3, \dots, e_m$, and $R_{2,2} \ge 0$. By unitary invariance of Lebesgue measure, Lebesgue measure on M_2 can be written as

$$|\mathrm{D}M_2| = R_{2,2}^{2(n-m+1)-1}\,\mathrm{d}R_{2,2}|\mathrm{d}R_{1,2}|^2\,\mathrm{d}\sigma_{\mathbb{T}^n\cap\{Q_1,e_3,\dots,e_m\}^\perp}(Q_2).$$

Continuing this way, Lebesgue measure on M_i for $i < m$ is given by

$$|\mathrm{D}M_i| = R_{i,i}^{2(n-m+1)-1}\,\mathrm{d}R_{i,i}|\mathrm{d}R_{1,i}|^2|\mathrm{d}R_{2,i}|^2\cdots|\mathrm{d}R_{i-1,i}|^2\,\mathrm{d}\sigma_{\mathbb{T}^n\cap\{Q_1,\dots,Q_{i-1},e_{i+1},\dots,e_m\}^\perp}(Q_i)$$

and for $i \ge m$ is given by

$$|\mathrm{D}M_i| = R_{i,i}^{2(n-i+1)-1}\,\mathrm{d}R_{i,i}|\mathrm{d}R_{1,i}|^2|\mathrm{d}R_{2,i}|^2\cdots|\mathrm{d}R_{i-1,i}|^2\,\mathrm{d}\sigma_{\mathbb{T}^n\cap\{Q_1,\dots,Q_{i-1}\}^\perp}(Q_i).$$

Therefore

$$
\begin{aligned}
|\mathrm{D}M| &= \prod_{i=1}^{n} |\mathrm{D}M_i| \\
&= \left[\prod_{i=1}^{m-1} R_{i,i}^{2(n-m+1)-1}\,\mathrm{d}R_{i,i}\right]\left[\prod_{i=m}^{n} R_{i,i}^{2(n-i+1)-1}\,\mathrm{d}R_{i,i}\right]\left[\prod_{i<j} |\mathrm{d}R_{i,j}|^2\right] \\
&\quad \times \left[\prod_{i=1}^{m-1} \mathrm{d}\sigma_{\mathbb{T}^n \cap \{Q_1,\ldots,Q_{i-1},e_{i+1},\ldots,e_m\}^\perp}(Q_i)\right]\left[\prod_{i=m}^{n} \mathrm{d}\sigma_{\mathbb{T}^n \cap \{Q_1,Q_2,\ldots,Q_{i-1}\}^\perp}(Q_i)\right].
\end{aligned}
$$

We can see that measure given by

$$
\left[\prod_{i=1}^{m-1} \mathrm{d}\sigma_{\mathbb{T}^n \cap \{Q_1,\ldots,Q_{i-1},e_{i+1},\ldots,e_m\}^\perp}(Q_i)\right]\left[\prod_{i=m}^{n} \mathrm{d}\sigma_{\mathbb{T}^n \cap \{Q_1,Q_2,\ldots,Q_{i-1}\}^\perp}(Q_i)\right]
$$

is Haar measure on $\mathcal{V}_{m,n}$. So, finally we have

$$
\begin{aligned}
|\mathrm{D}M| &= \left[\prod_{i=1}^{m-1} R_{i,i}^{2(n-m+1)-1}\,\mathrm{d}R_{i,i}\right]\left[\prod_{i=m}^{n} R_{i,i}^{2(n-i+1)-1}\,\mathrm{d}R_{i,i}\right] \\
&\quad \times \left[\prod_{i<j} |\mathrm{d}R_{i,j}|^2\right]\big|\mathrm{d}H_{\mathcal{V}}(Q)\big|.
\end{aligned} \tag{54}
$$

Proof of Lemma 16. Any $n \times n$ complex matrix X of rank n admits QR-decomposition

$$
X = US,
$$

where U is unitary matrix, S is upper triangular matrix with positive real diagonal entries. Then by (53),

$$
|\mathrm{D}X| = J(S)\big|\mathrm{d}H_{\mathcal{U}(n)}(U)\big||\mathrm{D}S|,
$$

where $J(S)$ is the Jacobian determinant of transformation due to QR-decomposition and is given by

$$
J(S) = \prod_{i=1}^{n} |S_{i,i}|^{2(n-i+1)-1}.
$$

So we get

$$
\frac{\int_{\|X^*X-I\|<\varepsilon} f(X)|\mathrm{D}X|}{\int_{\|X^*X-I\|<\varepsilon} |\mathrm{D}X|} = \frac{\int_{\|S^*S-I\|<\varepsilon} f(US)J(S)|\mathrm{d}H_{\mathcal{U}(n)}(U)||\mathrm{D}S|}{\int_{\|S^*S-I\|<\varepsilon} J(S)|\mathrm{d}H_{\mathcal{U}(n)}(U)||\mathrm{D}S|}.
$$

Since f is uniformly continuous on the region $\{X\colon \|X^*X - I\| < \varepsilon\}$, given any $r \in \mathbb{N}$, there exists $\varepsilon_r > 0$ such that

$$
\big|f(US) - f(U)\big| < \frac{1}{2^r} \quad \text{for all } \big\|S^*S - I\big\| < \varepsilon_r.
$$

Therefore

$$
\left|\frac{\int_{\|S^*S-I\|<\varepsilon_r} f(US)J(S)|\mathrm{d}H_{\mathcal{U}(n)}(U)||\mathrm{D}S|}{\int_{\|S^*S-I\|<\varepsilon_r} J(S)|\mathrm{d}H_{\mathcal{U}(n)}(U)||\mathrm{D}S|} - \int f(U)\big|\mathrm{d}H_{\mathcal{U}(n)}(U)\big|\right| < \frac{1}{2^r}
$$

and hence

$$\lim_{\varepsilon\to 0}\frac{\int_{\|X^*X-I\|<\varepsilon} f(X)|\mathrm{D}X|}{\int_{\|X^*X-I\|<\varepsilon}|\mathrm{D}X|}=\int f(U)\big|\mathrm{d}H_{\mathcal{U}(n)}(U)\big|.$$

This completes the proof. □

Proof of Lemma 17. Let $X\in\mathcal{N}_{m,n}$ be of full rank, then by QR-decomposition, we have

$$X=VS,\quad V\in\mathcal{V}:=\mathcal{N}_{m,n}\cap\mathcal{U}(n),$$

where V is $n\times n$ unitary matrix whose (i,j)th entry is zero for $1\le j<i\le m$ and S is upper triangular matrix with positive real diagonal entries. Then by (54)

$$\mathrm{d}X=J_m(S)\big|\mathrm{d}H_{\mathcal{V}}(V)\big||\mathrm{D}S|,$$

where

$$J_m(S)=\prod_{i=1}^{m}|S_{i,i}|^{2(n-m)+1}\prod_{i=m+1}^{n}|S_{i,i}|^{2(n-i)+1}.$$

Using this decomposition we get

$$\frac{\int_{\|X^*X-I\|<\varepsilon} f(X)|\mathrm{D}X|}{\int_{\|X^*X-I\|<\varepsilon}|\mathrm{D}X|}=\frac{\int_{\|S^*S-I\|<\varepsilon} f(VS)J_m(S)|\mathrm{d}H_{\mathcal{V}}(V)||\mathrm{D}S|}{\int_{\|S^*S-I\|<\varepsilon} J_m(S)|\mathrm{d}H_{\mathcal{V}}(V)||\mathrm{D}S|}.$$

Since f is uniformly continuous on region $\{X\colon\ \|X^*X-I\|<\varepsilon\}$, given any $r\in\mathbb{N}$, there exists $\varepsilon_r>0$ such that

$$\big|f(VS)-f(V)\big|<\frac{1}{2^r}\quad\text{for all }\big\|S^*S-I\big\|<\varepsilon_r.$$

Therefore

$$\left|\frac{\int_{\|S^*S-I\|<\varepsilon_r} f(VS)J_m(S)|\mathrm{d}H_{\mathcal{V}}(V)||\mathrm{D}S|}{\int_{\|S^*S-I\|<\varepsilon_r} J_m(S)|\mathrm{d}H_{\mathcal{V}}(V)||\mathrm{D}S|}-\int f(V)\big|\mathrm{d}H_{\mathcal{V}}(V)\big|\right|<\frac{1}{2^r}$$

and hence

$$\lim_{\varepsilon\to 0}\frac{\int_{\|X^*X-I\|<\varepsilon} f(X)|\mathrm{D}X|}{\int_{\|X^*X-I\|<\varepsilon}|\mathrm{D}X|}=\int f(V)\big|\mathrm{d}H_{\mathcal{V}}(V)\big|.$$

This completes the proof of the lemma. □

Now we shall prove Lemma 18. The key ingredient of the proof is Co-area formula on manifold. Here we state the Co-area formula without proof. Before stating the Co-area formula we need to introduce some notation. Fix a smooth map $f: M\to N$ from an manifold of dimension n to a manifold of dimension k. We denote derivative of f at $p\in M$ by

$$D_p(f):\mathbb{T}_p(M)\to\mathbb{T}_{f(p)}N.$$

We denote

$$M_{\mathrm{reg}}:=\text{set of regular points of } f,$$

$$J\big(D_p(f)\big):=\text{generalized determinant of } D_p(f),$$

$$\rho_M:=\text{volume measure on } M.$$

Lemma 23 (Co-area formula). *With notation and setting as above, let ϕ be any non-negative Borel-measurable function on M. Then*

(1) *The function $p \mapsto J(D_p(f))$ on M is Borel-measurable.*
(2) *The function $q \mapsto \int \phi(p)\,\mathrm{d}\rho_{M_{\mathrm{reg}} \cap f^{-1}(q)}(p)$ on N is Borel-measurable.*
(3) *The integral formula:*

$$\int_M \phi(p) J\bigl(D_p(f)\bigr)\,\mathrm{d}\rho_M(p) = \int_N \left(\int \phi(p)\,\mathrm{d}\rho_{M_{\mathrm{reg}} \cap f^{-1}(q)}(p)\right)\mathrm{d}\rho_N(q) \tag{55}$$

holds.

For the proof of Co-area formula see [3] (p. 442).

Proof of Lemma 18. Recall from (26), that $\mathcal{V} := \mathcal{V}_{m,n} = \mathcal{N}_{m,n} \cap \mathcal{U}(n)$. Define

$$\mathcal{V}_0 = \bigl\{V_{n\times m}\colon\ V^*V = I, V_{ij} = 0\ \forall 1 \le j < i \le m\bigr\}$$

and $g : \mathcal{V} \to \mathcal{V}_0$ be projection map such that $g(V)$ is a matrix of dimension $n \times m$ by removing last $n - m$ columns from V. Now by Co-area formula (55), we have

$$\int f(\underline{z})\bigl|\mathrm{d}H_{\mathcal{V}}(V)\bigr| = \int \left(\int f(\underline{z})\bigl|\mathrm{d}H_{g^{-1}(V_0)}(V)\bigr|\right)\bigl|\mathrm{d}H_{\mathcal{V}_0}(V_0)\bigr|. \tag{56}$$

For a fixed $V_0 \in \mathcal{V}_0$ (so $z_1, z_2, \dots, z_m$ are also fixed), $g^{-1}(V_0)$ is a sub-manifold of $\mathcal{V}$. It is isometric to the set of $(n - m)$ tuples of orthonormal unit vectors in $\mathbb{C}^n$ which are orthogonal to m columns of V_0. So $g^{-1}(V_0)$ is isometric to the manifold $\mathcal{U}(n - m)$. Jacobian in the Co-area formula for projection maps is equal to one. So from (56), we get

$$\mathbf{E}\bigl[f(\underline{Z})\bigr] = C \int f(\underline{z})\bigl|\mathrm{d}H_{\mathcal{V}_0}(V_0)\bigr|,$$

where $z_i = V_0(i, i)$. Note that $\mathcal{V}_0$ is a manifold of dimension $2nm - 2m^2 + m$ in $\mathbb{R}^{2nm-m^2+m}$ and its normalized volume measure is denoted by $H_{\mathcal{V}_0}$. Similarly we define

$$\mathcal{V}_i = \bigl\{V_{n\times m-i}\colon\ V^*V = I, V_{s,t} = 0\ \forall 1 \le s < t \le m\bigr\}$$

and denote its normalized volume measure by $H_{\mathcal{V}_i}$. Here also we denote $V_i(\ell, \ell)$ by z_ℓ, where $V_i \in \mathcal{V}_i$.

Let $g_0 : \mathcal{V}_0 \to \mathcal{V}_1$ be projection map such that $g_0(V_0)$ is a matrix of dimension $n \times (m - 1)$ by removing last column from V_0. Again by Co-area formula

$$\int f(\underline{z})\bigl|\mathrm{d}H_{\mathcal{V}_0}(V_0)\bigr| = \int \left(\int f(\underline{z})\bigl|\mathrm{d}H_{g_0^{-1}(V_1)}(V)\bigr|\right)\bigl|\mathrm{d}H_{\mathcal{V}_1}(V_1)\bigr|.$$

For a fixed $V_1 \in \mathcal{V}_1$ (so $z_1, z_2, \dots, z_{m-1}$ are also fixed), $g_0^{-1}(V_1)$ is a sub-manifold of $\mathcal{V}_0$. It is isometric to the set of unit vectors in $\mathbb{C}^n$ which are orthogonal to $m - 1$ columns of V_1 whose mth coordinates are zero. So $g_0^{-1}(V_1)$ is isometric to the manifold

$$\mathcal{T}_1 = \left\{(z_m, a_1, a_2, \dots, a_{n-m}) \in \mathbb{C}^{n-m+1}\colon\ |z_m|^2 + \sum_{i=1}^{n-m} |a_i|^2 = 1\right\}.$$

When integrating $f(\underline{z})$ with respect to $H_{g_0^{-1}(V_1)}$, because of z_m being the only $\underline{Z}$ variable involved, we get

$$\int f(\underline{z})\bigl|\mathrm{d}H_{\mathcal{V}_0}(V_0)\bigr| = \int f(\underline{z})|\mathrm{d}H_{\mathcal{T}_1}|\bigl|\mathrm{d}H_{\mathcal{V}_1}(V_1)\bigr|.$$

Now, by integrating out $a_1, a_2, \dots, a_{n-m}$, we get

$$\int f(\underline{z})\left|\mathrm{d}H_{\mathcal{V}_0}(V_0)\right| = C\int f(\underline{z})\left(1-|z_m|^2\right)^{n-m-1}\mathbf{1}_{\{|z_m|\le 1\}}(z_m)\left|\mathrm{d}H_{\mathcal{V}_1}(V_1)\right||\mathrm{d}z_m|^2. \tag{57}$$

Again by applying Co-area formula on the right hand side of (57) and using similar argument as above, we get that

$$\mathbf{E}\big[f(\underline{Z})\big] = C\int f(\underline{z})\prod_{\ell=m-1}^{m}\left(1-|z_\ell|^2\right)^{n-m-1}\mathbf{1}_{\{|z_\ell|\le 1\}}(z_\ell)\left|\mathrm{d}H_{\mathcal{V}_2}(V_2)\right|\prod_{\ell=m-1}^{m}|\mathrm{d}z_\ell|^2.$$

Thus by consecutive application of Co-area formula i times, we get

$$\mathbf{E}\big[f(\underline{Z})\big] = C\int f(\underline{z})\prod_{\ell=m-i+1}^{m}\left(1-|z_\ell|^2\right)^{n-m-1}\mathbf{1}_{\{|z_\ell|\le 1\}}(z_\ell)\left|\mathrm{d}H_{\mathcal{V}_i}(V_i)\right|\prod_{\ell=m-i+1}^{m}|\mathrm{d}z_\ell|^2.$$

Proceeding this way, finally we get

$$\mathbf{E}f(\underline{Z}) = C\int f(\underline{z})\prod_{\ell=1}^{m}\left(1-|z_\ell|^2\right)^{n-m-1}\mathbf{1}_{\{|z_\ell|\le 1\}}(z_\ell)\prod_{\ell=1}^{m}|\mathrm{d}z_\ell|^2$$

and this completes the proof. □

Acknowledgement

We thank Manjunath Krishnapur for many valuable suggestions and discussions. We thank the referee for her/his careful reading of the manuscript and helpful suggestions which have led to a significant improvement in presentation. Koushik Saha would like to thank the Department of Mathematics, University of California, Davis for its kind hospitality while part of this work was done.

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