Uniform asymptotic properties of a nonparametric regression estimator of conditional tails

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Abstract. We consider a nonparametric regression estimator of conditional tails introduced by Goegebeur, Y., Guillou, A., Schorgen, G. (2013). *Nonparametric regression estimation of conditional tails – the random covariate case*. It is shown that this estimator is uniformly strongly consistent on compact sets and its rate of convergence is given.

Résumé. Nous considérons l'estimateur à noyau de l'indice des valeurs extrêmes conditionnel présenté dans Goegebeur, Y., Guillou, A., Schorgen, G. (2013). *Nonparametric regression estimation of conditional tails – the random covariate case*. Nous montrons la consistance uniforme presque sûre de cet estimateur sur les compacts et nous calculons sa vitesse de convergence presque sûre.

1. Introduction

Extreme value analysis has attracted considerable attention in many fields of application, such as hydrology, biology and finance, for instance. The main result of extreme value theory asserts that the asymptotic distribution of the – properly rescaled – maximum of a sequence $(Y_1, \ldots, Y_n)$ of independent copies of a random variable Y with distribution function F is a distribution having the form

$$G_\gamma(x) = \exp\big(-(1 + \gamma x)_+^{-1/\gamma}\big), \quad \text{where } y_+ = \max(0, y)$$

for some $\gamma \in \mathbb{R}$, with $G_0(x) = \exp(-\mathrm{e}^{-x})$. The distribution function F is then said to belong to the maximum domain of attraction of G_γ and the parameter γ is called the extreme value index. Many applications in the areas of finance, insurance and geology, to name a few, can be found in the case when $\gamma > 0$, where F is a heavy-tailed distribution i.e. the associated survival function $\overline{F} := 1 - F$ satisfies $\overline{F}(x) = x^{-1/\gamma} L(x)$, where γ shall now be referred to as the tail-index and L is a slowly varying function at infinity: namely, L satisfies, for all $\lambda > 0$, $L(\lambda x)/L(x) \to 1$ as x goes to infinity. In this case, the parameter γ clearly drives the tail behavior of F; its estimation is in general a first step of extreme value analysis. For instance, if the idea is to estimate extreme quantiles – namely, quantiles with order $\alpha_n > 1 - 1/n$, where n is the sample size – then one has to extrapolate beyond the available data using an extreme value model which depends on the tail-index. For this reason, the problem of estimating γ has been extensively studied in the literature. Recent overviews on univariate tail-index estimation can be found in the monographs of [2] and [17].

In practice, it is often useful to link the variable of interest Y to a covariate X. In this situation, the tail-index depends on the observed value x of the covariate X and shall be referred to, in the following, as the conditional tail-index. Its estimation has been addressed in the recent extreme value literature, albeit mostly when the covariates are nonrandom. In [31] and [10] a parametric regression model was considered while [19] used a semi-parametric approach to estimate the conditional tail-index. Fully nonparametric methods have been considered using splines (see [4]), local polynomials (see [9]), a moving window approach (see [12]), or a nearest neighbor approach (see [13]), among others.

Less attention though has been paid to the random covariate case, despite its practical interest. One can recall the works of [33], based on a maximum likelihood approach in the Hall class of distribution functions (see [18]), [7] who use a fixed number of nonparametric conditional quantile estimators to estimate the conditional tail-index, later generalized in [6] to a regression context with response distributions belonging to the general max-domain of attraction, and [16] and [14] who both provide adaptations of Hill's estimator, [22], the latter also studying an average of Hill-type statistics to improve the finite sample performance of the method.

In this paper, we focus on a nonparametric regression estimator of conditional tails introduced by [16]. The particular structure of this estimator makes it possible to study its uniform properties. Note that uniform properties of estimators of the conditional tail-index are seldom considered in the literature. One can think of the work of [14], who study the uniform weak consistency of their estimator. Outside the field of conditional tail-index estimation, uniform convergence of the Parzen–Rosenblatt density estimator ([28] and [29]) was first considered by [27]. His results were then improved by [30] and [32], the latter proving a law of the iterated logarithm in this context. Analogous results on kernel regression estimators were obtained by, among others, [26], [20] and [11]. Uniform consistency of isotonized versions of order-α quantile estimators introduced in [1] was shown in [8]. The case of estimators of the left-truncated quantiles is considered in [25]. Finally, the uniform strong consistency of a frontier estimator using kernel regression on high order moments was shown in [15].

The paper is organised as follows. Our main results are stated in Section 2. The estimator is shown to be uniformly strongly consistent on compact sets in a semiparametric framework. The rate of convergence is provided when a further condition on the bias is satisfied. The rate of uniform convergence is closely linked to the rate of pointwise convergence in distribution established in [16]. The proofs of the main results are given in Section 3. Auxiliary results are postponed to the Appendix.

2. Main results

We assume that the covariate X takes its values in $\mathbb{R}^d$ for some $d \geq 1$. We shall work in the following semiparametric framework:

(*SP*) X has a probability density function f with support $S \subset \mathbb{R}^d$ having nonempty interior and the conditional survival function of Y given $X = x$ is such that

$$\forall x \in S, \forall y \geq 1, \quad \overline{F}(y|x) = y^{-1/\gamma(x)} L(y|x),$$

where $\gamma(x) > 0$ and $L(\cdot|x)$ is a slowly varying function at infinity.

The estimator of the conditional tail-index we shall study in this paper is defined as

$$\widehat{\gamma}_n(x) := \frac{\sum_{i=1}^n K_h(x - X_i)(\log Y_i - \log \omega_{n,x})_+ \mathbb{1}_{\{Y_i > \omega_{n,x}\}}}{\sum_{i=1}^n K_h(x - X_i) \mathbb{1}_{\{Y_i > \omega_{n,x}\}}}. \tag{1}$$

Here $K_h(u) := h^{-d} K(u/h)$ where K is a probability density function on $\mathbb{R}^d$ and $h := h_n$ is a positive sequence tending to 0 while for all x, $(\omega_{n,x})$ is a positive sequence tending to infinity. Note that $\widehat{\gamma}_n(x) = T_n^{(1)}(x)/T_n^{(0)}(x)$ where, for all $t \geq 0$,

$$T_n^{(t)}(x) := \frac{1}{n} \sum_{i=1}^n K_h(x - X_i)(\log Y_i - \log \omega_{n,x})_+^t \mathbb{1}_{\{Y_i > \omega_{n,x}\}}.$$

The estimator (1) is an element of the family of estimators introduced in [16], which can be seen as an adaptation of the classical Hill estimator of the tail-index for univariate distributions (see [22]). Note that the threshold $\omega_{n,x}$ is local, i.e. it depends on the point x where the estimation is to be made, while the bandwidth h is global.

We first wish to state the uniform strong consistency of our estimator on an arbitrary compact subset Ω of $\mathbb{R}^d$ contained in the interior of S. To this end, we first assume that for every $x \in S$ the slowly varying function $L(\cdot|x)$ appearing in $\overline{F}(\cdot|x)$ is normalised (see [3]):

(A_1) For all $x \in S$ and $y \geq 1$,

$$L(y|x) = c_L(x) \exp\left(\int_1^y \frac{\alpha(v|x)}{v} \, dv \right),$$

where $c_L(x) > 0$ and $\alpha(\cdot|x)$ is a function converging to 0 at infinity.

Let $\|\cdot\|$ be a norm on $\mathbb{R}^d$ and for $r > 0$, let Ω^r be the set of those points in $\mathbb{R}^d$ whose distance to Ω is not more than r:

$$\Omega^r = \left\{ x \in \mathbb{R}^d | \exists x' \in \Omega, \left\| x - x' \right\| \leq r \right\}.$$

Remark that since Ω is contained in the interior of the closed set S, the distance of Ω to the boundary of S must be positive. As a consequence, the set Ω^r is contained in S for all $r > 0$ small enough. We can therefore introduce some classical regularity assumptions:

(A_2) For some $r > 0$, on Ω^r, the functions f and γ are positive Hölder continuous functions, $\log c_L$ is a Hölder continuous function and $\alpha(y|\cdot)$ is a Hölder continuous function uniformly in $y \geq 1$: for all $x, x' \in \Omega^r$,

$$\left| f(x) - f\left(x'\right) \right| \leq M_f \left\| x - x' \right\|^{\eta_f},$$

$$\left| \gamma(x) - \gamma\left(x'\right) \right| \leq M_\gamma \left\| x - x' \right\|^{\eta_\gamma},$$

$$\left| \log c_L(x) - \log c_L\left(x'\right) \right| \leq M_{c_L} \left\| x - x' \right\|^{\eta_{c_L}},$$

$$\sup_{y \geq 1} \left| \alpha(y|x) - \alpha\left(y|x'\right) \right| \leq M_\alpha \left\| x - x' \right\|^{\eta_\alpha}.$$

Let moreover $\eta := \eta_\gamma \wedge \eta_{c_L} \wedge \eta_\alpha$. We introduce the oscillation of $x \mapsto \log \omega_{n,x}$ at a point $x \in \mathbb{R}^d$ over the ball $B(x, \varepsilon)$:

$$\forall \varepsilon > 0, \quad \Delta(\log \omega_{n,x})(\varepsilon) := \sup_{z \in B(x,\varepsilon)} |\log \omega_{n,x} - \log \omega_{n,z}|$$

and the quantity $\overline{\alpha}(y|x) := \sup_{t \geq y} |\alpha(t|x)|$ for all $y \geq 1$. Our results are established under the following classical regularity condition on the kernel:

(K) K is a probability density function which is Hölder continuous with Hölder exponent $\eta_K > 0$: for all $x, x' \in \mathbb{R}^d$,

$$\left| K(x) - K\left(x'\right) \right| \leq M_K \left\| x - x' \right\|^{\eta_K}$$

and its support is included in the unit ball B of $\mathbb{R}^d$.

Especially, if (K) holds then K is bounded with compact support. Let

$$v_n(x) := \sqrt{\frac{nh^d}{\log n} \overline{F}(\omega_{n,x}|x)}$$

and introduce the hypothesis

(C) For some $b > 0$, it holds that $\limsup_{n \to \infty} \sup_{x \in \Omega} v_n(x) \Delta(\log \omega_{n,x})(n^{-b}) < \infty$.

Our uniform strong consistency result may now be stated:

Theorem 1. *Assume that* (SP), (K), (A$_1$) *and* (A$_2$) *hold and that*:

- $\inf_{x \in \Omega} v_n(x) \to \infty$;
- $\inf_{x \in \Omega} \omega_{n,x} \to \infty$;
- $h^\eta \sup_{x \in \Omega} \log \omega_{n,x} \to 0$;
- $\sup_{x \in \Omega} \Delta(\log \omega_{n,x})(h) \to 0$;
- $\sup_{x \in \Omega} \overline{\alpha}(y|x) \to 0$ *as* $y \to \infty$.

Assume moreover that condition (C) *is satisfied. Then it holds that*

$$\sup_{x \in \Omega} |\widehat{\gamma}_n(x) - \gamma(x)| \to 0 \quad \text{almost surely as } n \to \infty.$$

Note that the hypotheses $\inf_{x \in \Omega} \omega_{n,x} \to \infty$ and $\sup_{x \in \Omega} \overline{\alpha}(y|x) \to 0$ as $y \to \infty$ imply the convergence

$$\sup_{x \in \Omega} \overline{\alpha}(\omega_{n,x}|x) \to 0$$

which shall frequently be used in the proofs of our results. Besides, using the mean value theorem, it holds that $|e^u - 1| \le 2|u|$ for $u \in \mathbb{R}$ such that $|u|$ is sufficiently small. As a consequence, using the condition $\sup_{x \in \Omega} \Delta(\log \omega_{n,x})(h) \to 0$, this inequality implies that for n large enough

$$\sup_{x \in \Omega} \sup_{z \in B(x,h)} \left| \frac{\omega_{n,x}}{\omega_{n,z}} - 1 \right| \le 2 \sup_{x \in \Omega} \Delta(\log \omega_{n,x})(h) \to 0. \tag{2}$$

Finally, the conditions

$$\sup_{x \in \Omega} \Delta(\log \omega_{n,x})(h) \to 0 \quad \text{and} \quad \limsup_{n \to \infty} \sup_{x \in \Omega} v_n(x) \Delta(\log \omega_{n,x})(n^{-b}) < \infty$$

are satisfied if for instance $\omega_{n,x} = n^{g(x)}$ where $g : S \to \mathbb{R}$ is a positive Hölder continuous function whose Hölder exponent is not less than η. In other words, Theorem 1 requires that a continuity property on $x \mapsto \log \omega_{n,x}$ be satisfied.

Our second aim is to compute the rate of uniform strong consistency of the estimator (1):

Theorem 2. *Assume that the conditions of Theorem 1 are satisfied. If moreover*

$$\limsup_{n \to \infty} \left(\sup_{x \in \Omega} v_n(x) \left\{ \overline{\alpha}(\omega_{n,x}|x) \vee h^{\eta_f} \vee h^\eta \log \omega_{n,x} \vee \Delta(\log \omega_{n,x})(h) \right\} \right) < \infty \tag{3}$$

then it holds that

$$\sup_{x \in \Omega} v_n(x) |\widehat{\gamma}_n(x) - \gamma(x)| = O(1) \quad \text{almost surely as } n \to \infty.$$

Let us highlight that condition (3) controls the bias of the estimator $\widehat{\gamma}_n$. The terms h^{η_f} and $h^\eta \log \omega_{n,x}$ correspond to the bias which stems from the use of a kernel regression, while the presence of the other terms is due to the particular structure of the semiparametric model (SP). Besides, as pointed out in [16], the rate of pointwise convergence of $\widehat{\gamma}_n(x)$ to $\gamma(x)$ is $[nh^d \overline{F}(\omega_{n,x}|x)]^{1/2}$. Up to the term $[\log n]^{1/2}$, the rate of uniform convergence of $\widehat{\gamma}_n$ to γ is therefore the infimum (over Ω) of the rate of pointwise convergence of $\widehat{\gamma}_n(x)$ to $\gamma(x)$.

3. Proofs of the main results

Before starting the proof of Theorem 1, let us note that assuming that (SP), (A_1) and (A_2) hold then it is easy to show that there exists a positive constant $M_{\overline{F}}$ such that the function $(x, y) \mapsto \log \overline{F}(y|x)$ has the following property: for all $x, x' \in \Omega^r$ such that $\|x - x'\| \le 1$ and $y, y' \ge e$,

$$\left| \log \frac{\overline{F}(y|x)}{\overline{F}(y'|x')} \right| \le M_{\overline{F}} \|x - x'\|^{\eta} \log y + \left(\frac{1}{\gamma(x')} + \overline{\alpha}(y \wedge y'|x') \right) |\log y - \log y'|. \tag{4}$$

Moreover, if (A_2) holds then one may take a positive number r such that the four conditions of the hypothesis hold on Ω^{2r}. Since Ω^r is compact, $\overline{f} := \sup_{\Omega^r} f < \infty$ and $\underline{f} := \inf_{\Omega^r} f > 0$. As a consequence, the uniform relative oscillation of f over the ball $B(x, h)$ can be controlled as

$$\sup_{x \in \Omega^r} \sup_{z \in B(x,h)} \left| \frac{f(z)}{f(x)} - 1 \right| = O(h^{\eta_f}) \to 0. \tag{5}$$

Second, $\overline{\gamma} := \sup_{\Omega^r} \gamma < \infty$ and $\underline{\gamma} := \inf_{\Omega^r} \gamma > 0$ and we thus have

$$\sup_{x \in \Omega^r} \sup_{z \in B(x,h)} \left| \frac{\gamma(z)}{\gamma(x)} - 1 \right| = O(h^{\eta_\gamma}) \to 0. \tag{6}$$

Third, we can write for all $x, x' \in \Omega^r$ and $t \ge 1$

$$\alpha(t|x) \le \alpha(t|x') + |\alpha(t|x) - \alpha(t|x')|$$

and the roles of x and x' are symmetric in the above inequality, so that taking the supremum over $t \ge y$ on both sides yields

$$\forall y \ge 1, \quad |\overline{\alpha}(y|x) - \overline{\alpha}(y|x')| \le M_\alpha \|x - x'\|^{\eta_\alpha}. \tag{7}$$

We may now prove the key result for the proof of Theorem 1, which is a uniform law of large numbers for $T_n^{(0)}(x)$ and $T_n^{(1)}(x)$. In what follows, we let $\mu_n^{(t)}(x) := \mathbb{E}(T_n^{(t)}(x))$.

Proposition 1. *Assume that the conditions of Theorem 1 are satisfied. Then for every $t \in \{0, 1\}$ it holds that*

$$\sup_{x \in \Omega} v_n(x) \left| \frac{T_n^{(t)}(x)}{\mu_n^{(t)}(x)} - 1 \right| = O(1) \quad \text{almost surely as } n \to \infty.$$

Proof. The proof is based on that of Lemma 1 in [21]: we shall in fact show complete convergence in the sense of [24]. Since Ω is a compact subset of $\mathbb{R}^d$, we may, for every $n \in \mathbb{N} \setminus \{0\}$, find a finite subset Ω_n of Ω such that:

$$\forall x \in \Omega, \exists \chi(x) \in \Omega_n, \quad \|x - \chi(x)\| \le n^{-b} \quad \text{and} \quad \exists c > 0, \quad |\Omega_n| = O(n^c),$$

where b, which we may take to be not less than $1/d + 1/2\eta_K$, is given by condition (C) and $|\Omega_n|$ stands for the cardinality of Ω_n. Notice that, since $nh^d \to \infty$, one has $n^{-b}/h \to 0$, so that one can assume that eventually $\chi(x) \in B(x, h)$ for all $x \in \Omega$. Next, remark that $\|x - \chi(x)\| \le n^{-b} \le h \le 1$ and that since $n^{-b} \le h$ the convergences

$$n^{-b\eta} \sup_{x \in \Omega} \log \omega_{n,x} \le h^{\eta} \sup_{x \in \Omega} \log \omega_{n,x} \to 0 \quad \text{and} \quad \sup_{x \in \Omega} \Delta(\log \omega_{n,x})(n^{-b}) \le \sup_{x \in \Omega} \Delta(\log \omega_{n,x})(h) \to 0$$

hold. Consequently, Lemma 1 entails

$$\sup_{x \in \Omega} \left| \frac{v_n(x)}{v_n(\chi(x))} - 1 \right| = \sup_{x \in \Omega} \left| \sqrt{\frac{\overline{F}(\omega_{n,x}|x)}{\overline{F}(\omega_{n,\chi(x)}|\chi(x))}} - 1 \right| \to 0. \tag{8}$$

Pick $\varepsilon > 0$ and an arbitrary sequence of positive numbers (δ_n) converging to 0; using together (8) and the triangular inequality thus shows that for n large enough

$$\mathbb{P}\left(\delta_n \sup_{x \in \Omega} v_n(x) \left| \frac{T_n^{(t)}(x)}{\mu_n^{(t)}(x)} - 1 \right| > \varepsilon \right) \le R_{1,n} + R_{2,n},$$

where

$$R_{1,n} := \sum_{z \in \Omega_n} \mathbb{P}\left(\delta_n v_n(z) \left| \frac{T_n^{(t)}(z)}{\mu_n^{(t)}(z)} - 1 \right| > \frac{\varepsilon}{4} \right)$$

and

$$R_{2,n} := \mathbb{P}\left(\delta_n \sup_{x \in \Omega} v_n(x) \left| \frac{T_n^{(t)}(x)}{\mu_n^{(t)}(x)} - \frac{T_n^{(t)}(\chi(x))}{\mu_n^{(t)}(\chi(x))} \right| > \frac{\varepsilon}{2} \right).$$

The goal of the proof is now to show that the series $\sum_n R_{1,n}$ and $\sum_n R_{2,n}$ converge. The result of Proposition 1 shall then be an easy consequence of Borel–Cantelli's lemma and Lemma 6.

We start by controlling $R_{1,n}$. To this end, apply Lemma 3 to get that there exists a positive constant κ such that for n large enough,

$$\forall z \in \Omega_n, \quad \mathbb{P}\left(\delta_n v_n(z) \left| \frac{T_n^{(t)}(z)}{\mu_n^{(t)}(z)} - 1 \right| > \frac{\varepsilon}{4} \right) \le 2 \exp\left[-\frac{\kappa}{16} \varepsilon^2 \frac{nh^d \overline{F}(\omega_{n,z}|z)}{\delta_n^2 v_n^2(z)} \right].$$

Use now the definition of $v_n(z)$ to get

$$R_{1,n} = O\left(n^c \exp\left[-\frac{\kappa}{16} \varepsilon^2 \frac{\log n}{\delta_n^2} \right] \right).$$

Hence $\sum_n R_{1,n}$ converges.

We now turn to $R_{2,n}$. Using the triangular inequality gives

$$R_{2,n} \le \mathbb{P}\left(\delta_n \sup_{x \in \Omega} v_n(x) S_{1,n}(x) > \frac{\varepsilon}{4} \right) + \mathbb{P}\left(\delta_n \sup_{x \in \Omega} v_n(x) S_{2,n}(x) > \frac{\varepsilon}{4} \right) =: R_{3,n} + R_{4,n},$$

where

$$S_{1,n}(x) := \frac{1}{n} \sum_{i=1}^{n} \left| \frac{K_h(x - X_i)}{\mu_n^{(t)}(x)} - \frac{K_h(\chi(x) - X_i)}{\mu_n^{(t)}(\chi(x))} \right| (\log Y_i - \log \omega_{n,\chi(x)})_+^t \mathbb{1}_{\{Y_i > \omega_{n,\chi(x)}\}},$$

$$S_{2,n}(x) := \frac{1}{n} \sum_{i=1}^{n} \frac{K_h(x - X_i)}{\mu_n^{(t)}(x)} \left| (\log Y_i - \log \omega_{n,x})_+^t \mathbb{1}_{\{Y_i > \omega_{n,x}\}} - (\log Y_i - \log \omega_{n,\chi(x)})_+^t \mathbb{1}_{\{Y_i > \omega_{n,\chi(x)}\}} \right|,$$

and it is enough to show that the series $\sum_n R_{3,n}$ and $\sum_n R_{4,n}$ converge.

To deal with $\sum_n R_{3,n}$ use once again the triangular inequality to obtain

$$\mu_n^{(t)}(\chi(x)) \left| \frac{K_h(x - X_i)}{\mu_n^{(t)}(x)} - \frac{K_h(\chi(x) - X_i)}{\mu_n^{(t)}(\chi(x))} \right| \le \left| K_h(x - X_i) - K_h(\chi(x) - X_i) \right|$$

$$+ \left| \frac{\mu_n^{(t)}(\chi(x))}{\mu_n^{(t)}(x)} - 1 \right| K_h(x - X_i).$$

Using hypothesis (K) and Lemma 4, there exists a positive constant M such that for n large enough:

$$\forall x \in \Omega, \quad \mu_n^{(t)}\big(\chi(x)\big)\left|\frac{K_h(x - X_i)}{\mu_n^{(t)}(x)} - \frac{K_h(\chi(x) - X_i)}{\mu_n^{(t)}(\chi(x))}\right| \leq \frac{M}{h^d}\left\{\left[\frac{n^{-b}}{h}\right]^{\eta_K} \vee \Delta(\log \omega_{n,x})\big(n^{-b}\big)\right\}.$$

Besides

$$\widetilde{\mathfrak{m}}_n^{(t)}(z) := \frac{1}{n}\sum_{i=1}^n \mathcal{K}_{2h}(z - X_i)(\log Y_i - \log \omega_{n,z})_+^t \mathbb{1}_{\{Y_i > \omega_{n,z}\}}$$

is the empirical analogue of $\mathfrak{m}_n^{(t)}(z)$ defined before Lemma 4; since the support of the random variable $K_h(x - X_i)$ is included in $B(\chi(x), 2h)$, one has for n large enough

$$\forall x \in \Omega, \quad v_n(x)S_{1,n}(x) \leq 2^d \mathcal{V} M v_n(x)\left\{\left[\frac{n^{-b}}{h}\right]^{\eta_K} \vee \Delta(\log \omega_{n,x})\big(n^{-b}\big)\right\}\frac{\widetilde{\mathfrak{m}}_n^{(t)}(\chi(x))}{\mu_n^{(t)}(\chi(x))}.$$

Moreover, since $\widetilde{\mathfrak{m}}_n^{(t)}(z)$ is a kernel estimator of $m_n^{(t)}(z, z)$ for which the conditions of Lemma 2 are satisfied, we get for n large enough:

$$\forall z \in \Omega_n, \quad \delta_n v_n(z)\frac{\widetilde{\mathfrak{m}}_n^{(t)}(z)}{\mu_n^{(t)}(z)} \leq 2\delta_n v_n(z)\left[1 + \left|\frac{\widetilde{\mathfrak{m}}_n^{(t)}(z)}{\mathfrak{m}_n^{(t)}(z)} - 1\right|\right].$$

The fact that $b \geq 1/d + 1/2\eta_K$ gives

$$\sup_{z \in \Omega_n} v_n(z)\left[\frac{n^{-b}}{h}\right]^{\eta_K} \leq \sqrt{n}\left[\frac{n^{-b}}{h}\right]^{\eta_K} \leq \left[\frac{1}{nh^d}\right]^{\eta_K/d} \to 0.$$

Using first this convergence together with hypothesis (C) and (8) and then Lemma 3 entails for n large enough:

$$R_{3,n} \leq \sum_{z \in \Omega_n} \mathbb{P}\left(\delta_n v_n(z)\left|\frac{\widetilde{\mathfrak{m}}_n^{(t)}(z)}{\mathfrak{m}_n^{(t)}(z)} - 1\right| > \varepsilon\right) = O\left(n^c \exp\left[-\kappa' \varepsilon^2 \frac{\log n}{\delta_n^2}\right]\right),$$

where κ' is a positive constant. Hence $\sum_n R_{3,n}$ converges.

To control $\sum_n R_{4,n}$ first use Lemmas 2(iv) and 4 to get, for n large enough

$$\sup_{x \in \Omega}\frac{\mathfrak{m}_n^{(t)}(\chi(x))}{\mu_n^{(t)}(x)} = \sup_{x \in \Omega}\left\{\frac{\mathfrak{m}_n^{(t)}(\chi(x))}{\mu_n^{(t)}(\chi(x))}\frac{\mu_n^{(t)}(\chi(x))}{\mu_n^{(t)}(x)}\right\} \leq 2.$$

Therefore, since the support of the random variable $K_h(x - X_i)$ is included in $B(\chi(x), 2h)$, one has for n large enough and all $x \in \Omega$

$$S_{2,n}(x) \leq 2^{d+1} \mathcal{V}\|K\|_\infty S_{3,n}(x),$$

where $\|K\|_\infty := \sup_B K$ and

$$S_{3,n}(x) := \frac{1}{n}\sum_{i=1}^n \frac{\mathcal{K}_{2h}(\chi(x) - X_i)}{\mathfrak{m}_n^{(t)}(\chi(x))}\left|(\log Y_i - \log \omega_{n,x})_+^t \mathbb{1}_{\{Y_i > \omega_{n,x}\}} - (\log Y_i - \log \omega_{n,\chi(x)})_+^t \mathbb{1}_{\{Y_i > \omega_{n,\chi(x)}\}}\right|.$$

We then get

$$R_{4,n} \leq \mathbb{P}\left(\delta_n \sup_{x \in \Omega} v_n(x)S_{3,n}(x) > \frac{\varepsilon}{2^{d+3}\mathcal{V}\|K\|_\infty}\right) =: R_{5,n}$$

and it is enough to control $\sum_n R_{5,n}$. We start by considering the case $t = 0$. In this case, $S_{3,n}(x)$ reduces to

$$S_{3,n}(x) = \frac{1}{n} \sum_{i=1}^{n} \frac{\mathcal{K}_{2h}(\chi(x) - X_i)}{\mathfrak{m}_n^{(0)}(\chi(x))} \mathbb{1}_{\{\omega_{n,x} \wedge \omega_{n,\chi(x)} < Y_i \leq \omega_{n,x} \vee \omega_{n,\chi(x)}\}}.$$

Letting $\rho_{n,x} := 2\Delta(\log \omega_{n,x})(n^{-b})$ and using (2), we have $\sup_{x \in \Omega} \rho_{n,x} \to 0$ and for n large enough

$$\forall x \in \Omega, \quad (1 - \rho_{n,\chi(x)})\omega_{n,\chi(x)} \leq \omega_{n,x} \leq (1 + \rho_{n,\chi(x)})\omega_{n,\chi(x)}.$$

As a consequence, for n large enough it holds that

$$\forall x \in \Omega, \quad S_{3,n}(x) \leq \frac{1}{n} \sum_{i=1}^{n} \frac{\mathcal{K}_{2h}(\chi(x) - X_i)}{\mathfrak{m}_n^{(0)}(\chi(x))} \mathbb{1}_{\{(1-\rho_{n,\chi(x)})\omega_{n,\chi(x)} < Y_i \leq (1+\rho_{n,\chi(x)})\omega_{n,\chi(x)}\}}.$$

Similarly to Lemma 5, let

$$M_n(x) := \mathbb{E}\big(\mathcal{K}_{2h}(x - X)\mathbb{1}_{\{(1-\rho_{n,x})\omega_{n,x} < Y < (1+\rho_{n,x})\omega_{n,x}\}}\big)$$

and

$$U_n(x) := \frac{1}{n} \sum_{i=1}^{n} \mathcal{K}_{2h}(x - X_i)\mathbb{1}_{\{(1-\rho_{n,x})\omega_{n,x} < Y_i < (1+\rho_{n,x})\omega_{n,x}\}}.$$

Write

$$\forall x \in \Omega, \quad \delta_n v_n(x) S_{3,n}(x) \leq \delta_n v_n(x) \frac{M_n(\chi(x))}{\mathfrak{m}_n^{(0)}(\chi(x))}\left[1 + \left|\frac{U_n(\chi(x))}{M_n(\chi(x))} - 1\right|\right].$$

Use together Lemmas 2(iv) and 5 along with (8) to get for n large enough

$$\forall x \in \Omega, \quad \delta_n v_n(x) S_{3,n}(x) \leq \frac{4}{\gamma(\chi(x))} \delta_n v_n\big(\chi(x)\big)\rho_{n,\chi(x)}\left[1 + \left|\frac{U_n(\chi(x))}{M_n(\chi(x))} - 1\right|\right].$$

Recall that $\rho_{n,x} = 2\Delta(\log \omega_{n,x})(n^{-b})$ and that condition (C) is satisfied to obtain

$$\delta_n \sup_{z \in \Omega_n} v_n(z)\rho_{n,z} \to 0.$$

Therefore, since $0 < \underline{\gamma} \leq \gamma(\chi(x))$, the triangular inequality implies that

$$R_{5,n} \leq \sum_{z \in \Omega_n} \mathbb{P}\left(\delta_n v_n(z)\rho_{n,z}\left|\frac{U_n(z)}{M_n(z)} - 1\right| > \frac{\varepsilon \underline{\gamma}}{2^{d+6}\mathcal{V}\|K\|_\infty}\right)$$

for n large enough. Lemma 5 now makes it clear that

$$R_{5,n} = \mathrm{O}\left(n^c \sup_{z \in \Omega_n} \exp\left(-\kappa'' \frac{\varepsilon \underline{\gamma}}{2^{d+6}\mathcal{V}\|K\|_\infty} v_n(z)\frac{\log n}{\delta_n}\right)\right) = \mathrm{o}\left(n^c \exp\left(-\kappa'' \varepsilon \frac{\log n}{\delta_n}\right)\right)$$

which proves that $\sum_n R_{5,n}$ converges in this case.

If now $t = 1$, we recall (45) in the proof of Lemma 4 to get for n large enough and for all $x \in \Omega$

$$S_{3,n}(x) = \left|\log \frac{\omega_{n,x}}{\omega_{n,\chi(x)}}\right| \frac{\mathfrak{m}_n^{(0)}(\chi(x))}{\mathfrak{m}_n^{(1)}(\chi(x))} \frac{1}{n} \sum_{i=1}^{n} \frac{\mathcal{K}_{2h}(\chi(x) - X_i)}{\mathfrak{m}_n^{(0)}(\chi(x))} \mathbb{1}_{\{Y_i > \omega_{n,x} \wedge \omega_{n,\chi(x)}\}}.$$

Use (2) and Lemma 2(iv) to get for n large enough

$$\forall x \in \Omega, \quad S_{3,n}(x) \le \frac{2}{\underline{\gamma}} \Delta(\log \omega_{n,x})(n^{-b}) \frac{1}{n} \sum_{i=1}^{n} \frac{\mathcal{K}_{2h}(\chi(x) - X_i)}{\mathfrak{m}_n^{(0)}(\chi(x))} \mathbb{1}_{\{Y_i > \omega_{n,\chi(x)}/2\}}$$

$$\le \frac{2}{\underline{\gamma}} \Delta(\log \omega_{n,x})(n^{-b}) \frac{\nu_n(\chi(x))}{\mathfrak{m}_n^{(0)}(\chi(x))} \left[1 + \left|\frac{V_n(\chi(x))}{\nu_n(\chi(x))} - 1\right|\right], \tag{9}$$

where

$$\nu_n(x) := \mathbb{E}\big(\mathcal{K}_{2h}(x - X)\mathbb{1}_{\{Y > \omega_{n,x}/2\}}\big) \quad \text{and} \quad V_n(x) := \frac{1}{n} \sum_{i=1}^{n} \mathcal{K}_{2h}(x - X_i)\mathbb{1}_{\{Y_i > \omega_{n,x}/2\}}.$$

The family of sequences $(\omega_{n,x}/2)$ clearly satisfies the hypotheses of Lemmas 2 and 3: in particular

$$\sup_{x \in \Omega} \left|\frac{\nu_n(x)}{\mathfrak{m}_n^{(0)}(x)} \frac{\overline{F}(\omega_{n,x}|x)}{\overline{F}(\omega_{n,x}/2|x)} - 1\right| \to 0 \tag{10}$$

and there exists a positive constant κ''' such that for n large enough

$$\forall x \in \Omega, \quad \mathbb{P}\left(\left|\frac{V_n(x)}{\nu_n(x)} - 1\right| > \varepsilon\right) \le 2\exp\big(-\kappa''' \varepsilon^2 n h^d \overline{F}(\omega_{n,x}|x)\big), \tag{11}$$

where the inequality $\overline{F}(\omega_{n,x}/2|x) \ge \overline{F}(\omega_{n,x}|x)$ was used. We conclude by noting that according to (4),

$$\limsup_{n \to \infty} \sup_{x \in \Omega} \left|\log \frac{\overline{F}(\omega_{n,x}|x)}{\overline{F}(\omega_{n,x}/2|x)}\right| \le \frac{\log 2}{\underline{\gamma}} < \infty \quad \Rightarrow \quad 0 < \limsup_{n \to \infty} \sup_{x \in \Omega} \frac{\overline{F}(\omega_{n,x}|x)}{\overline{F}(\omega_{n,x}/2|x)} < \infty.$$

This property together with (10) entails the convergences

$$\delta_n \sup_{x \in \Omega} \frac{\nu_n(x)}{\mathfrak{m}_n^{(0)}(x)} \to 0 \quad \text{and} \quad \sup_{x \in \Omega} \Delta(\log \omega_{n,x})(n^{-b}) \frac{\nu_n(x)}{\mathfrak{m}_n^{(0)}(x)} \to 0. \tag{12}$$

Reporting (10) along with (12) into (9), recalling condition (C) and using the triangular inequality together with (8) shows that for n large enough,

$$R_{5,n} \le \sum_{z \in \Omega_n} \mathbb{P}\left(\delta_n \nu_n(z)\left|\frac{V_n(x)}{\nu_n(x)} - 1\right| > \varepsilon\right) = \mathrm{O}\left(n^c \exp\left(-\kappa''' \varepsilon^2 \frac{\log n}{\delta_n^2}\right)\right),$$

where (11) was used in the last step. As a consequence, $\sum_n R_{5,n}$ converges in this case as well. This completes the proof of Proposition 1. □

With Proposition 1 at hand, we can now prove Theorems 1 and 2.

Proof of Theorem 1. Notice that

$$\widehat{\gamma}_n(x) = \frac{\mu_n^{(1)}(x)}{\mu_n^{(0)}(x)} \frac{T_n^{(1)}(x)}{\mu_n^{(1)}(x)} \frac{\mu_n^{(0)}(x)}{T_n^{(0)}(x)}. \tag{13}$$

Applying Proposition 1 twice yields

$$\sup_{x \in \Omega} \left|\frac{T_n^{(1)}(x)}{\mu_n^{(1)}(x)} \frac{\mu_n^{(0)}(x)}{T_n^{(0)}(x)} - 1\right| \to 0 \quad \text{almost surely as } n \to \infty. \tag{14}$$

Moreover, recalling that γ is continuous and therefore bounded on the compact set Ω, using Lemma 2(i) and (iv) twice entails

$$\sup_{x \in \Omega} \left| \frac{\mu_n^{(1)}(x)}{\mu_n^{(0)}(x)} - \gamma(x) \right| \to 0 \quad \text{as } n \to \infty. \tag{15}$$

The result follows by reporting (14) and (15) into (13). □

Proof of Theorem 2. Note that because $nh^d \to \infty$, the hypothesis

$$\limsup_{n \to \infty} \sup_{x \in \Omega} v_n(x) \Delta(\log \omega_{n,x})(h) < \infty$$

entails condition (C). Besides, Proposition 1 yields

$$\sup_{x \in \Omega} v_n(x) \left| \frac{\mu_n^{(0)}(x)}{T_n^{(0)}(x)} - 1 \right| = \mathrm{O}(1) \quad \text{and} \quad \sup_{x \in \Omega} v_n(x) \left| \frac{T_n^{(1)}(x)}{\mu_n^{(1)}(x)} - 1 \right| = \mathrm{O}(1) \tag{16}$$

almost surely as $n \to \infty$. Moreover, Lemma 2(iv) gives

$$\sup_{x \in \Omega} \frac{1}{\overline{\alpha}(\omega_{n,x}|x) \vee h^{\eta_f} \vee h^\eta \log \omega_{n,x}} \left| \frac{\mu_n^{(t)}(x)}{f(x)\overline{F}(\omega_{n,x}|x)} - \gamma^t(x) \right| = \mathrm{O}(1)$$

for $t \in \{0, 1\}$, so that using condition (3),

$$\sup_{x \in \Omega} v_n(x) \left| \frac{\mu_n^{(1)}(x)}{\mu_n^{(0)}(x)} - \gamma(x) \right| = \mathrm{O}(1). \tag{17}$$

The result follows by reporting (16) and (17) into (13). □

Appendix: Auxiliary results and proofs

The first lemma of this section is a technical result that gives an upper bound for the oscillation of the log-conditional survival function.

Lemma 1. *Assume that* (SP), (A_1) *and* (A_2) *hold. Let moreover* $\varepsilon := \varepsilon_n$, $\varepsilon' := \varepsilon'_n$ *and* $\varepsilon'' := \varepsilon''_n$ *be three positive sequences tending to* 0 *and assume that*:

- $\inf_{x \in \Omega} \omega_{n,x} \to \infty$;
- $\varepsilon''^\eta \sup_{x \in \Omega} \log \omega_{n,x} \to 0$;
- $\sup_{x \in \Omega} \Delta(\log \omega_{n,x})(\varepsilon') \to 0$;
- $\sup_{x \in \Omega} \overline{\alpha}(y|x) \to 0$ *as* $y \to \infty$.

Then it holds that, for n large enough,

$$\forall (x, x') \in \Omega \times \Omega^\varepsilon, \forall (z, z') \in B(x, \varepsilon') \times B(x', \varepsilon''),$$

$$\left| \log \frac{\overline{F}(\omega_{n,z}|z')}{\overline{F}(\omega_{n,x}|x')} \right| \le M_{\overline{F}} \varepsilon''^\eta \log \omega_{n,z} + \frac{2}{\underline{\gamma}} \Delta(\log \omega_{n,x})(\varepsilon').$$

In particular,

$$\sup_{x \in \Omega} \sup_{x' \in \Omega^\varepsilon} \sup_{z \in B(x, \varepsilon')} \sup_{z' \in B(x', \varepsilon'')} \frac{1}{\varepsilon''^\eta \log \omega_{n,x} \vee \Delta(\log \omega_{n,x})(\varepsilon')} \left| \frac{\overline{F}(\omega_{n,z}|z')}{\overline{F}(\omega_{n,x}|x')} - 1 \right| = \mathrm{O}(1).$$

Proof. Pick $(x, x') \in \Omega \times \Omega^\varepsilon$ and $(z, z') \in B(x, \varepsilon') \times B(x', \varepsilon'')$. Use (4) to get for n large enough

$$\left| \log \frac{\overline{F}(\omega_{n,z}|z')}{\overline{F}(\omega_{n,x}|x')} \right| \le M_{\overline{F}} \|x' - z'\|^\eta \log \omega_{n,z} + \left(\frac{1}{\gamma(x')} + \overline{\alpha}(\omega_{n,z} \wedge \omega_{n,x}|x') \right) |\log \omega_{n,x} - \log \omega_{n,z}|.$$

For every $y \ge 1$, inequality (7) entails

$$\sup_{x \in \Omega^\varepsilon} \overline{\alpha}(y|x) \le \sup_{x \in \Omega} \overline{\alpha}(y|x) + M_\alpha \varepsilon^{\eta_\alpha}. \tag{18}$$

Using then (2) with ε' instead of h, we get $\inf_{x \in \Omega} \inf_{z \in B(x,\varepsilon')} \omega_{n,z} \wedge \omega_{n,x} = \inf_{x \in \Omega} \omega_{n,x}(1 + o(1)) \to \infty$, so that

$$\sup_{x \in \Omega} \sup_{x' \in \Omega^\varepsilon} \sup_{z \in B(x,\varepsilon')} \overline{\alpha}(\omega_{n,z} \wedge \omega_{n,x}|x') \to 0.$$

Especially, since $0 < \underline{\gamma} \le \gamma(x')$, we obtain for n large enough:

$$\forall (x, x') \in \Omega \times \Omega^\varepsilon, \forall (z, z') \in B(x, \varepsilon') \times B(x', \varepsilon''),$$

$$\left| \log \frac{\overline{F}(\omega_{n,z}|z')}{\overline{F}(\omega_{n,x}|x')} \right| \le M_{\overline{F}} \varepsilon''^\eta \log \omega_{n,z} + \frac{2}{\underline{\gamma}} \Delta(\log \omega_{n,x})(\varepsilon')$$

which is the first part of the result. To prove the second part, note that because $\sup_{x \in \Omega} \Delta(\log \omega_{n,x})(\varepsilon') \to 0$ it holds that for n large enough

$$\forall (x, x') \in \Omega \times \Omega^\varepsilon, \forall (z, z') \in B(x, \varepsilon') \times B(x', \varepsilon''),$$

$$\left| \log \frac{\overline{F}(\omega_{n,z}|z')}{\overline{F}(\omega_{n,x}|x')} \right| \le 2 M_{\overline{F}} \varepsilon''^\eta \log \omega_{n,x} + \frac{2}{\underline{\gamma}} \Delta(\log \omega_{n,x})(\varepsilon').$$

Consequently

$$\sup_{x \in \Omega} \sup_{x' \in \Omega^\varepsilon} \sup_{z \in B(x,\varepsilon')} \sup_{z' \in B(x',\varepsilon'')} \frac{1}{\varepsilon''^\eta \log \omega_{n,x} \vee \Delta(\log \omega_{n,x})(\varepsilon')} \left| \log \frac{\overline{F}(\omega_{n,z}|z')}{\overline{F}(\omega_{n,x}|x')} \right| = O(1).$$

Using the equivalent $e^u - 1 = u(1 + o(1))$ therefore completes the proof of Lemma 1. $\square$

The second lemma examines the behavior of the conditional moment

$$m_n^{(t)}(x, z) := \mathbb{E}\left((\log Y - \log \omega_{n,x})_+^t \mathbb{1}_{\{Y > \omega_{n,x}\}} | X = z \right)$$

and that of its smoothed version $\mu_n^{(t)}(x) = \mathbb{E}(K_h(x - X) m_n^{(t)}(x, X))$. Let Γ be Euler's Gamma function:

$$\forall t > 0, \quad \Gamma(t) := \int_0^{+\infty} v^{t-1} e^{-v} \, dv.$$

Lemma 2. *Assume that (SP), (A_1) and (A_2) hold. Pick $t \ge 0$ and assume that K is a bounded probability density function on $\mathbb{R}^d$ with support included in B. If moreover*:

- $\inf_{x \in \Omega} \omega_{n,x} \to \infty$;
- $h^\eta \sup_{x \in \Omega} \log \omega_{n,x} \to 0$;
- $\sup_{x \in \Omega} \overline{\alpha}(y|x) \to 0$ *as* $y \to \infty$

then, as $n \to \infty$, the following estimations hold:

(i) $\sup_{x \in \Omega} \sup_{z \in B(x,h)} \frac{1}{\overline{\alpha}(\omega_{n,x}|x) \vee h^{\eta_\alpha}} \left| \frac{m_n^{(t)}(x,z)}{\gamma^t(z)\Gamma(t+1)\overline{F}(\omega_{n,x}|z)} - 1 \right| = O(1),$

(ii) $\sup_{x \in \Omega} \sup_{z \in B(x,h)} \frac{1}{\overline{\alpha}(\omega_{n,x}|x) \vee h^\eta \log \omega_{n,x}} |\frac{m_n^{(t)}(x,z)}{m_n^{(t)}(x,x)} - 1| = O(1)$,

(iii) $\sup_{x \in \Omega} \frac{1}{\overline{\alpha}(\omega_{n,x}|x) \vee h^{\eta_f} \vee h^\eta \log \omega_{n,x}} |\frac{\mu_n^{(t)}(x)}{f(x)m_n^{(t)}(x,x)} - 1| = O(1)$,

(iv) $\sup_{x \in \Omega} \frac{1}{\overline{\alpha}(\omega_{n,x}|x) \vee h^{\eta_f} \vee h^\eta \log \omega_{n,x}} |\frac{\mu_n^{(t)}(x)}{f(x)\gamma^t(x)\Gamma(t+1)\overline{F}(\omega_{n,x}|x)} - 1| = O(1)$.

Proof. (i) When $t = 0$, there is nothing to prove, since $m_n^{(0)}(x,z) = \overline{F}(\omega_{n,x}|z)$ and $\Gamma(1) = 1$. In the case $t > 0$, an integration by parts yields

$$m_n^{(t)}(x,z) = \int_{\omega_{n,x}}^{+\infty} t \frac{(\log y - \log \omega_{n,x})^{t-1}}{y} \overline{F}(y|z) \, dy = t\overline{F}(\omega_{n,x}|z) \int_1^{+\infty} (\log r)^{t-1} \frac{\overline{F}(r\omega_{n,x}|z)}{r\overline{F}(\omega_{n,x}|z)} \, dr.$$

From (SP) and (A_1), one has

$$\left| \frac{\overline{F}(r\omega_{n,x}|z)}{r\overline{F}(\omega_{n,x}|z)} - r^{-1/\gamma(z)-1} \right| = r^{-1/\gamma(z)-1} \left| \exp\left(\int_{\omega_{n,x}}^{r\omega_{n,x}} \frac{\alpha(v|z)}{v} \, dv \right) - 1 \right|. \tag{19}$$

For all $y \in \mathbb{R}$, the mean value theorem yields $|e^y - 1| \le |y|e^{|y|}$. Meanwhile,

$$\left| \int_{\omega_{n,x}}^{r\omega_{n,x}} \frac{\alpha(v|z)}{v} \, dv \right| \le \overline{\alpha}(\omega_{n,x}|z) \log r. \tag{20}$$

Choosing n so large that $\sup_{x \in \Omega} \sup_{z \in B(x,h)} \overline{\alpha}(\omega_{n,x}|z) < 1/2\overline{\gamma}$, (18), (19) and (20) together imply that, for all $x \in \Omega$ and $z \in B(x,h)$,

$$\left| \int_1^{+\infty} (\log r)^{t-1} \left[\frac{\overline{F}(r\omega_{n,x}|z)}{r\overline{F}(\omega_{n,x}|z)} - r^{-1/\gamma(z)-1} \right] dr \right| \le \left(\overline{\alpha}(\omega_{n,x}|x) + M_\alpha h^{\eta_\alpha} \right) \int_1^{+\infty} (\log r)^t r^{-1/2\overline{\gamma}-1} \, dr$$

which, since the integral on the right-hand side of this inequality converges, gives

$$\sup_{x \in \Omega} \sup_{z \in B(x,h)} \frac{1}{\overline{\alpha}(\omega_{n,x}|x) \vee h^{\eta_\alpha}} \left| \int_1^{+\infty} (\log r)^{t-1} \left[\frac{\overline{F}(r\omega_{n,x}|z)}{r\overline{F}(\omega_{n,x}|z)} - r^{-1/\gamma(z)-1} \right] dr \right| = O(1)$$

as $n \to \infty$. An elementary change of variables and the well-known equality $t\Gamma(t) = \Gamma(t+1)$ thus entail

$$\sup_{x \in \Omega} \sup_{z \in B(x,h)} \frac{1}{\overline{\alpha}(\omega_{n,x}|x) \vee h^{\eta_\alpha}} \left| \frac{m_n^{(t)}(x,z)}{\overline{F}(\omega_{n,x}|z)} - \gamma^t(z)\Gamma(t+1) \right| = O(1)$$

as $n \to \infty$ and (i) is proven.

(ii) Since for all $x \in \Omega$, $0 < \underline{\gamma} \le \gamma(x) \le \overline{\gamma} < \infty$, applying (i) entails

$$\sup_{x \in \Omega} \sup_{z \in B(x,h)} \frac{1}{\overline{\alpha}(\omega_{n,x}|x) \vee h^{\eta_\alpha}} \left| \frac{m_n^{(t)}(x,z)}{\gamma^t(z)\Gamma(t+1)\overline{F}(\omega_{n,x}|z)} \frac{\gamma^t(x)\Gamma(t+1)\overline{F}(\omega_{n,x}|x)}{m_n^{(t)}(x,x)} - 1 \right| = O(1). \tag{21}$$

Moreover, hypothesis (A_2) and the mean value theorem yield

$$\left| \frac{\gamma^t(x)}{\gamma^t(z)} - 1 \right| \le \left[\frac{1}{\underline{\gamma}^t} \sup_{\underline{\gamma} \le r \le \overline{\gamma}} |tr^{t-1}| \right] \sup_{x \in \Omega} \sup_{z \in B(x,h)} |\gamma(x) - \gamma(z)| = O(h^{\eta_\gamma}). \tag{22}$$

Besides, using Lemma 1 gives

$$\sup_{x \in \Omega} \sup_{z \in B(x,h)} \frac{1}{h^\eta \log \omega_{n,x}} \left| \frac{\overline{F}(\omega_{n,x}|x)}{\overline{F}(\omega_{n,x}|z)} - 1 \right| = O(1). \tag{23}$$

Note finally that since $\eta \le \eta_\gamma \wedge \eta_\alpha$ and $\inf_{x \in \Omega} \omega_{n,x} \to \infty$ one has

$$\sup_{x \in \Omega} \frac{h^{\eta_\gamma} \vee h^{\eta_\alpha}}{h^\eta \log \omega_{n,x}} \to 0.$$

Using then (22) and (23) together with (21) yields (ii).

(iii) Let us remark that for all $x \in \Omega$:

$$\frac{\mu_n^{(t)}(x)}{f(x) m_n^{(t)}(x, x)} = \int_B K(u) \frac{f(x - hu)}{f(x)} \frac{m_n^{(t)}(x, x - hu)}{m_n^{(t)}(x, x)} \, du.$$

From (5) and (ii) it follows that

$$\sup_{x \in \Omega} \sup_{z \in B(x, h)} \frac{1}{\overline{\alpha}(\omega_{n,x}|x) \vee h^{\eta_f} \vee h^\eta \log \omega_{n,x}} \left| \frac{f(z)}{f(x)} \frac{m_n^{(t)}(x, z)}{m_n^{(t)}(x, x)} - 1 \right| \to 0$$

as $n \to \infty$, which yields (iii).

(iv) This is a straightforward consequence of (i) and (iii). $\qquad \square$

The third lemma is essential to prove Proposition 1. It gives a uniform exponential bound for large deviations of $T_n^{(0)}$ and $T_n^{(1)}$.

Lemma 3. *Assume that* (SP), (A_1) *and* (A_2) *hold. Assume that K is a bounded probability density function on $\mathbb{R}^d$ with support included in B. If moreover*:

- $\inf_{x \in \Omega} \omega_{n,x} \to \infty$;
- $h^\eta \sup_{x \in \Omega} \log \omega_{n,x} \to 0$;
- $\sup_{x \in \Omega} \overline{\alpha}(y|x) \to 0$ *as* $y \to \infty$

then there exists a positive constant κ such that for all n large enough, one has for $t \in \{0, 1\}$ and every $\varepsilon > 0$ small enough:

$$\forall x \in \Omega, \quad \mathbb{P}\left(\left| \frac{T_n^{(t)}(x)}{\mu_n^{(t)}(x)} - 1 \right| > \varepsilon \right) \le 2 \exp\left(-\kappa \varepsilon^2 n h^d \overline{F}(\omega_{n,x}|x) \right).$$

Proof. For every $x \in \Omega$:

$$\mathbb{P}\left(\left| \frac{T_n^{(0)}(x)}{\mu_n^{(0)}(x)} - 1 \right| > \varepsilon \right) = \mathbb{P}\left(\left| h^d T_n^{(0)}(x) - h^d \mu_n^{(0)}(x) \right| > \varepsilon h^d \mu_n^{(0)}(x) \right).$$

Notice now that if $W_{n,i}(x) := h^d K_h(x - X_i) \mathbb{1}_{\{Y_i > \omega_{n,x}\}}$ then

$$h^d T_n^{(0)}(x) - h^d \mu_n^{(0)}(x) = \frac{1}{n} \sum_{i=1}^n \left[W_{n,i}(x) - \mathbb{E}\left(W_{n,i}(x) \right) \right]$$

is a mean of bounded, centered, independent and identically distributed random variables. Define

$$\tau_n(x) := \frac{\varepsilon}{\|K\|_\infty} n h^d \mu_n^{(0)}(x) \quad \text{and} \quad \lambda_n(x) := \frac{\varepsilon \|K\|_\infty h^d \mu_n^{(0)}(x)}{\text{Var}(W_{n,1}(x))}.$$

Bernstein's inequality (see [23]) yields, for all $\varepsilon > 0$:

$$\mathbb{P}\left(\left| \frac{T_n^{(0)}(x)}{\mu_n^{(0)}(x)} - 1 \right| > \varepsilon \right) \le 2 \exp\left(-\frac{\tau_n(x) \lambda_n(x)}{2(1 + \lambda_n(x)/3)} \right).$$

Applying Lemma 2(iii) yields for n large enough:

$$\inf_{x \in \Omega} \frac{\tau_n(x)}{nh^d \overline{F}(\omega_{n,x}|x)} \geq \frac{\varepsilon \underline{f}}{2\|K\|_\infty}. \tag{24}$$

Moreover, since $W_{n,1}(x)$ is bounded by $\|K\|_\infty$, it follows from the inequality $W_{n,1}^2(x) \leq \|K\|_\infty W_{n,1}(x)$ that

$$\sup_{x \in \Omega} \frac{1}{\lambda_n(x)} \leq \sup_{x \in \Omega} \frac{\mathbb{E}(W_{n,1}^2(x))}{\varepsilon \|K\|_\infty h^d \mu_n^{(0)}(x)} \leq \frac{1}{\varepsilon}. \tag{25}$$

Finally, it holds that

$$\frac{\tau_n(x)\lambda_n(x)}{2(1+\lambda_n(x)/3)} \geq \left\{ \inf_{x \in \Omega} \frac{\tau_n(x)}{nh^d \overline{F}(\omega_{n,x}|x)} \right\} \left\{ \inf_{x \in \Omega} \frac{1}{2(1/\lambda_n(x)+1/3)} \right\} nh^d \overline{F}(\omega_{n,x}|x).$$

Using (24), (25) and the fact that the function $t \mapsto 1/[2(t+1/3)]$ is decreasing on $\mathbb{R}_+$, it is then clear that for all n large enough, if $\varepsilon > 0$ is small enough, there exists a positive constant κ_1 that is independent of ε such that

$$\forall x \in \Omega, \quad \mathbb{P}\left(\left| \frac{T_n^{(0)}(x)}{\mu_n^{(0)}(x)} - 1 \right| > \varepsilon \right) \leq 2\exp\left[-\kappa_1 \varepsilon^2 nh^d \overline{F}(\omega_{n,x}|x)\right].$$

We now turn to $T_n^{(1)}(x)$. For every $x \in \Omega$, it holds that

$$\mathbb{P}\left(\left| \frac{T_n^{(1)}(x)}{\mu_n^{(1)}(x)} - 1 \right| > \varepsilon \right) = \mathbb{P}\left(\frac{T_n^{(1)}(x)}{\mu_n^{(1)}(x)} - 1 > \varepsilon \right) + \mathbb{P}\left(\frac{T_n^{(1)}(x)}{\mu_n^{(1)}(x)} - 1 < -\varepsilon \right) =: u_{1,n}(x) + u_{2,n}(x).$$

We shall then give a uniform Chernoff-type exponential bound (see [5]) for both terms on the right-hand side of the above inequality. We start by considering $u_{1,n}(x)$. Let

$$\varphi_n(s, x) := \mathbb{E}\left(\exp\left(s K_h(x - X)(\log Y - \log \omega_{n,x})_+ \mathbb{1}_{\{Y > \omega_{n,x}\}} \right) \right)$$

be the moment generating function of the random variable $K_h(x - X)(\log Y - \log \omega_{n,x})_+ \mathbb{1}_{\{Y > \omega_{n,x}\}}$. Markov's inequality entails, for every $q > 0$,

$$u_{1,n}(x) = \mathbb{P}\left(\exp\left(q \frac{T_n^{(1)}(x)}{\mu_n^{(1)}(x)} \right) > \exp\left(q[\varepsilon + 1] \right) \right) \leq \exp\left(-q[\varepsilon + 1] + n \log \varphi_n\left(\frac{q}{n\mu_n^{(1)}(x)}, x \right) \right). \tag{26}$$

Our goal is now to use inequality (26) with a suitable value $q^*(\varepsilon, x)$ for q. To this end, notice that

$$\varphi_n(s, x) = \int_{\mathbb{R}^d \setminus B(x,h)} f(z)\,dz + \int_{B(x,h)} \psi_n\left(s K_h(x - z)|x, z \right) f(z)\,dz,$$

where

$$\psi_n(s|x, z) := \mathbb{E}\left(\exp\left(s(\log Y - \log \omega_{n,x})_+ \mathbb{1}_{\{Y > \omega_{n,x}\}} \right) \mid X = z \right)$$

is the conditional moment generating function of the random variable $(\log Y - \log \omega_{n,x})_+ \mathbb{1}_{\{Y > \omega_{n,x}\}}$ given $X = z$. In particular, since f is a probability density function on $\mathbb{R}^d$,

$$\varphi_n(s, x) = 1 + \int_{B(x,h)} \left[\psi_n\left(s K_h(x - z)|x, z \right) - 1 \right] f(z)\,dz. \tag{27}$$

This equality makes it clear that it is enough to study the behavior of $\psi_n(\cdot|x, z)$. One has

$$\psi_n(s|x, z) = 1 - \overline{F}(\omega_{n,x}|z) + \mathbb{E}\left(\left[\frac{Y}{\omega_{n,x}} \right]^s \mathbb{1}_{\{Y > \omega_{n,x}\}} \mid X = z \right).$$

From this we deduce that

$$\psi_n(s|x,z) = 1 + \overline{F}(\omega_{n,x}|z) \int_1^{+\infty} st^s \frac{\overline{F}(t\omega_{n,x}|z)}{t\overline{F}(\omega_{n,x}|z)} \, dt.$$

A use of (19) and (20) therefore entails, for all $s < 1/\overline{\gamma}$,

$$\psi_n(s|x,z) = 1 + s\overline{F}(\omega_{n,x}|z)\left(\left[\frac{1}{\gamma(z)} - s\right]^{-1} + R_n(s|x,z)\right), \tag{28}$$

where $R_n(s|x,z)$ satisfies, for all $\delta > 0$, if n is large enough,

$$\sup_{x\in\Omega} \sup_{z\in B(x,h)} \left|R_n(s|x,z)\right| \leq \sup_{x\in\Omega} \sup_{z\in B(x,h)} \overline{\alpha}(\omega_{n,x}|z) \int_1^{+\infty} v^{s-1/\overline{\gamma}-1+\delta} \log v \, dv.$$

Since by (18) it holds that $\sup_{x\in\Omega} \sup_{z\in B(x,h)} \overline{\alpha}(\omega_{n,x}|z) \to 0$ we get, for all $\delta > 0$:

$$\sup_{s<1/\overline{\gamma}-\delta} \sup_{x\in\Omega} \sup_{z\in B(x,h)} \left|R_n(s|x,z)\right| \to 0 \tag{29}$$

as $n \to \infty$. We shall now derive a suitable value for the parameter q. Given $X = x$, if the remainder term R_n were identically 0, then one would have $m_n^{(1)}(x,x) = \gamma(x)\overline{F}(\omega_{n,x}|x)$ and thus an optimal value of q would be obtained by minimizing the function

$$q \mapsto -q[1+\varepsilon] + n\log\left[1 + \frac{q}{n}\left[1 - \frac{q}{n\overline{F}(\omega_{n,x}|x)}\right]^{-1}\right].$$

Straightforward but cumbersome computations lead to the optimal value

$$q_{c,+}^\star(\varepsilon) := n\overline{F}(\omega_{n,x}|x)\frac{[2 - \overline{F}(\omega_{n,x}|x)] - \sqrt{[2 - \overline{F}(\omega_{n,x}|x)]^2 - (4\varepsilon/(\varepsilon+1))[1 - \overline{F}(\omega_{n,x}|x)]}}{2[1 - \overline{F}(\omega_{n,x}|x)]}. \tag{30}$$

Since we are mostly interested in what happens in the limit $n \to \infty$ and $\varepsilon \to 0$, we may examine the behavior of $q_{c,+}^\star(\varepsilon)$ in this case. Using (30), we get the following asymptotic equivalent

$$q_{c,+}^*(\varepsilon) = n\overline{F}(\omega_{n,x}|x)\frac{\varepsilon}{2(\varepsilon+1)}.$$

Note that since $q_{c,+}^*(\varepsilon)/[nm_n^{(1)}(x,x)] = \varepsilon/[2\gamma(x)(\varepsilon+1)]$ is positive and converges to 0 as $\varepsilon \to 0$, the moment generating function $\psi_n(\cdot|x,x)$ at $q_{c,+}^*(\varepsilon)/[nm_n^{(1)}(x,x)]$ is well-defined and finite for ε small enough and therefore this choice of q is valid. Back to our original context, taking into account the presence of the covariate X motivates the following value for q:

$$q_{n,+}^*(\varepsilon,x) := \frac{M\varepsilon}{\varepsilon+1}nh^d f(x)\overline{F}(\omega_{n,x}|x),$$

where M is a positive constant to be chosen later. For ε small enough and for n so large that the quantity $\varphi_n(q_{n,+}^*(\varepsilon,x)/(n\mu_n^{(1)}(x)),x)$ is well-defined and finite for all $x \in \Omega$, replacing q by $q_{n,+}^*(\varepsilon,x)$ in the right-hand side of (26) gives

$$\forall x \in \Omega, \quad u_{1,n}(x) \leq \exp\left(-M\varepsilon nh^d f(x)\overline{F}(\omega_{n,x}|x) + n\log\varphi_n\left(\frac{q_{n,+}^*(\varepsilon,x)}{n\mu_n^{(1)}(x)}, x\right)\right). \tag{31}$$

Using the classical inequality $\log(1 + r) \leq r$ for all $r > 0$ together with (27) and (28), we obtain

$$\log \varphi_n(s, x) \leq \int_{B(x,h)} \left[\psi_n\big(s K_h(x - z)|x, z\big) - 1 \right] f(z) \, \mathrm{d}z$$

$$\leq \int_{B(x,h)} s K_h(x - z) \overline{F}(\omega_{n,x}|z) \left(\left[\frac{1}{\gamma(z)} - s K_h(x - z) \right]^{-1} + R_n\big(s K_h(x - z)|x, z\big) \right) f(z) \, \mathrm{d}z.$$

According to Lemma 2(iv),

$$\frac{q_{n,+}^*(\varepsilon, x)}{n \mu_n^{(1)}(x)} = \frac{M\varepsilon}{\varepsilon + 1} h^d f(x) \frac{\overline{F}(\omega_{n,x}|x)}{\mu_n^{(1)}(x)} = M \frac{\varepsilon h^d}{\gamma(x)(\varepsilon + 1)} \big[1 + r_{1,n}(x) \big], \tag{32}$$

where $r_{1,n}(x) \to 0$ as n goes to infinity, uniformly in $x \in \Omega$. As a consequence, using an elementary Taylor expansion, we get, for all $z \in B(x, h)$,

$$\left[\frac{1}{\gamma(z)} - \frac{q_{n,+}^*(\varepsilon, x)}{n \mu_n^{(1)}(x)} K_h(x - z) \right]^{-1} = \gamma(z) \left[1 + \frac{\gamma(z)}{\gamma(x)} \frac{M\varepsilon}{\varepsilon + 1} h^d \big[1 + r_{1,n}(x) \big] K_h(x - z) \right.$$

$$\left. + k \left(\frac{\gamma(z)}{\gamma(x)} \frac{M\varepsilon}{\varepsilon + 1} h^d \big[1 + r_{1,n}(x) \big] K_h(x - z) \right) \right],$$

where $k(r)/r \to 0$ as r goes to 0. Letting

$$p_n(x, z) := \frac{\gamma(z)}{\gamma(x)} \big[1 + r_{1,n}(x) \big] h^d K_h(x - z)$$

and using (6), the uniform convergence of $r_{1,n}$ to 0 and the fact that K is bounded yields

$$p_n(x, z) = h^d K_h(x - z) + r_{2,n}(x, z), \quad \text{where} \ \sup_{x \in \Omega} \sup_{z \in B(x,h)} \big| r_{2,n}(x, z) \big| \to 0$$

as n goes to infinity. Especially,

$$\left[\frac{1}{\gamma(z)} - \frac{q_{n,+}^*(\varepsilon, x)}{n \mu_n^{(1)}(x)} K_h(x - z) \right]^{-1} = \gamma(z) \left[1 + \frac{M\varepsilon}{\varepsilon + 1} h^d K_h(x - z) + \varepsilon r_{3,n}(\varepsilon, x, z) \right], \tag{33}$$

where $r_{3,n}(\varepsilon, x, z) \to 0$ as ε goes to 0 and n goes to infinity, uniformly in $x \in \Omega$ and $z \in B(x, h)$. Besides, since for every $\varepsilon_0 > 0$

$$\sup_{\varepsilon < \varepsilon_0} \sup_{x \in \Omega} \sup_{z \in B(x,h)} \left| \frac{q_{n,+}^*(\varepsilon, x)}{n \mu_n^{(1)}(x)} K_h(x - z) - \frac{M\varepsilon}{\varepsilon + 1} \frac{h^d K_h(x - z)}{\gamma(x)} \right| \to 0$$

as n goes to infinity and

$$\sup_{n \in \mathbb{N}} \sup_{x \in \Omega} \sup_{z \in B(x,h)} \left| \frac{M\varepsilon}{\varepsilon + 1} \frac{h^d K_h(x - z)}{\gamma(x)} \right| \to 0$$

as ε goes to 0, (29) yields for ε small enough

$$\sup_{x \in \Omega} \sup_{z \in B(x,h)} \left| R_n \left(\frac{q_{n,+}^*(\varepsilon, x)}{n \mu_n^{(1)}(x)} K_h(x - z) \Big| x, z \right) \right| \to 0 \tag{34}$$

as n goes to infinity. Using together (6), (32), (33) and (34) entails that there exist functions $r_{4,n} = r_{4,n}(x, z)$ and $r_{5,n} = r_{5,n}(\varepsilon, x, z)$ satisfying

$$\sup_{x \in \Omega} \sup_{z \in B(x,h)} \big| r_{4,n}(x, z) \big| \to 0 \quad \text{as } n \to \infty$$

and

$$\sup_{x \in \Omega} \sup_{z \in B(x,h)} \left| r_{5,n}(\varepsilon, x, z) \right| \to 0 \quad \text{as } \varepsilon \to 0 \text{ and } n \to \infty$$

such that

$$\log \varphi_n\left(\frac{q_{n,+}^*(\varepsilon, x)}{n\mu_n^{(1)}(x)}, x \right) \le \int_{B(x,h)} \frac{M\varepsilon}{\varepsilon+1} h^d \left[1 + \frac{M\varepsilon}{\varepsilon+1} h^d K_h(x-z) \right] \overline{F}(\omega_{n,x}|z) K_h(x-z) f(z) \, \mathrm{d}z$$
$$+ \frac{M\varepsilon}{\varepsilon+1} h^d \int_{B(x,h)} \overline{F}(\omega_{n,x}|z) \left[r_{4,n}(x,z) + \varepsilon r_{5,n}(\varepsilon, x, z) \right] K_h(x-z) f(z) \, \mathrm{d}z.$$

Recalling (5) and (23), we get, for n large enough and ε small enough, the inequality

$$\forall x \in \Omega, \quad \log \varphi_n\left(\frac{q_{n,+}^*(\varepsilon, x)}{n\mu_n^{(1)}(x)}, x \right) \le \frac{M\varepsilon}{\varepsilon+1} \left[1 + 2\frac{M\varepsilon}{\varepsilon+1} \|K\|_2^2 \right] h^d f(x) \overline{F}(\omega_{n,x}|x).$$

Using this result together with (31) and recalling that $0 < \underline{f} \le f(x)$ entails, for n large enough and ε small enough,

$$\forall x \in \Omega, \quad u_{1,n}(x) \le \exp\left(\underline{f} \left[-M\varepsilon + \frac{M\varepsilon}{\varepsilon+1} \left[1 + 2\frac{M\varepsilon}{\varepsilon+1} \|K\|_2^2 \right] \right] nh^d \overline{F}(\omega_{n,x}|x) \right).$$

A straightforward computation shows that $M_+^* := (\varepsilon+1)/(4\|K\|_2^2)$ is the optimal value for M in the above inequality; this value yields

$$\forall x \in \Omega, \quad u_{1,n}(x) \le \exp\left(-\frac{\varepsilon^2}{8\|K\|_2^2} \underline{f} nh^d \overline{F}(\omega_{n,x}|x) \right) = \exp\left(-\kappa_2 \varepsilon^2 nh^d \overline{F}(\omega_{n,x}|x) \right),$$

where κ_2 is a positive constant independent of ε.

Providing a uniform exponential bound for $u_{2,n}(x)$ starts by noticing that, for all $q > 0$,

$$u_{2,n}(x) \le \exp\left(-q[\varepsilon - 1] + n \log \varphi_n\left(-\frac{q}{n\mu_n^{(1)}(x)}, x \right) \right).$$

Recall (27) and use the inequality $\log(1-r) \le -r$ for all $r \in (0,1)$ to get

$$\log \varphi_n(-s, x) \le \int_{B(x,h)} \left[\psi_n\left(-sK_h(x-z)|x, z \right) - 1 \right] f(z) \, \mathrm{d}z.$$

We choose q as

$$q_{n,-}^*(\varepsilon, x) := \frac{\varepsilon}{4\|K\|_2^2} nh^d f(x) \overline{F}(\omega_{n,x}|x)$$

which, using the ideas developed to control $u_{1,n}(x)$, yields

$$\forall x \in \Omega, \quad u_{2,n}(x) \le \exp\left(-\kappa_2 \varepsilon^2 nh^d \overline{F}(\omega_{n,x}|x) \right)$$

for some constant $\kappa_2 > 0$. Setting $\kappa = \kappa_1 \wedge \kappa_2$ completes the proof of Lemma 3.   □

The fourth lemma of this section establishes a uniform control of the relative oscillation of $x \mapsto \mu_n^{(t)}(x)$. Before stating this result, we let

$$\mathfrak{m}_n^{(t)}(x) := \mathbb{E}\left(\mathcal{K}_{2h}(x - X) m_n^{(t)}(x, X) \right),$$

where $\mathcal{K} := \mathbb{1}_B / \mathcal{V}$ is the uniform kernel on $\mathbb{R}^d$, with $\mathcal{V}$ being the volume of the unit ball of $\mathbb{R}^d$; let further $\mathcal{K}_h(u) := h^{-d} \mathcal{K}(u/h)$.

Lemma 4. *Assume that (SP), (K), (A_1) and (A_2) hold. Pick $t \in \{0, 1\}$ and let $\varepsilon := \varepsilon_n$ be a sequence of positive real numbers such that $\varepsilon \le h$. If moreover*:

- $\inf_{x \in \Omega} \omega_{n,x} \to \infty$;
- $h^\eta \sup_{x \in \Omega} \log \omega_{n,x} \to 0$;
- $\sup_{x \in \Omega} \Delta(\log \omega_{n,x})(\varepsilon) \to 0$;
- $\sup_{x \in \Omega} \overline{\alpha}(y|x) \to 0$ *as $y \to \infty$*

then

$$\sup_{x \in \Omega} \sup_{z \in B(x,\varepsilon)} \frac{1}{[\varepsilon/h]^{\eta_K} \vee \Delta(\log \omega_{n,x})(\varepsilon)} \left| \frac{\mu_n^{(t)}(z)}{\mu_n^{(t)}(x)} - 1 \right| = \mathrm{O}(1).$$

Proof. For all $x \in \Omega$ and $z \in B(x, \varepsilon)$, we have

$$\left| \mu_n^{(0)}(x) - \mu_n^{(0)}(z) \right| \le \mathbb{E}\big(\left| K_h(x - X) - K_h(z - X) \right| \mathbb{1}_{\{Y > \omega_{n,x}\}} \big) + \mathbb{E}\big(K_h(z - X) | \mathbb{1}_{\{Y > \omega_{n,x}\}} - \mathbb{1}_{\{Y > \omega_{n,z}\}} | \big)$$

$$=: R_{1,n}^{(0)}(x, z) + R_{2,n}^{(0)}(x, z) \tag{35}$$

and we shall handle both terms in the right-hand side separately. Hypothesis (K) and the inclusion $B(z, h) \subset B(x, 2h)$ entail that

$$\left| K_h(x - X) - K_h(z - X) \right| \le \frac{M_K}{h^d} \left[\frac{\varepsilon}{h} \right]^{\eta_K} \mathbb{1}_{\{X \in B(x, 2h)\}}. \tag{36}$$

From (36), we get

$$\sup_{z \in B(x,\varepsilon)} R_{1,n}^{(0)}(x, z) \le 2^d M_K \mathcal{V} \mathfrak{m}_n^{(0)}(x) \left[\frac{\varepsilon}{h} \right]^{\eta_K}. \tag{37}$$

Because $\mathcal{K}$ is a probability density function on $\mathbb{R}^d$ with support included in B, applying Lemma 2(iii) implies that

$$\sup_{x \in \Omega} \left| \frac{\mathfrak{m}_n^{(0)}(x)}{\mu_n^{(0)}(x)} - 1 \right| \to 0 \quad \text{as } n \to \infty \tag{38}$$

which, together with (37), yields

$$\sup_{x \in \Omega} \sup_{z \in B(x,\varepsilon)} \left[\frac{\varepsilon}{h} \right]^{-\eta_K} \frac{R_{1,n}^{(0)}(x, z)}{\mu_n^{(0)}(x)} = \mathrm{O}(1). \tag{39}$$

We now turn to the second term. One has

$$R_{2,n}^{(0)}(x, z) = \mathbb{E}\big(K_h(z - X) \left| \overline{F}(\omega_{n,x}|X) - \overline{F}(\omega_{n,z}|X) \right| \big). \tag{40}$$

Furthermore, using Lemma 1 with $\varepsilon'' = 0$ entails

$$\sup_{x \in \Omega} \sup_{x' \in B(x,2h)} \sup_{z \in B(x,\varepsilon)} \frac{1}{\Delta(\log \omega_{n,x})(\varepsilon)} \left| \frac{\overline{F}(\omega_{n,z}|x')}{\overline{F}(\omega_{n,x}|x')} - 1 \right| = \mathrm{O}(1). \tag{41}$$

Besides, hypothesis (K) and the inclusion $B(z, h) \subset B(x, 2h)$ imply that

$$\mathbb{E}\big(K_h(z - X) m_n^{(0)}(x, X) \big) \le 2^d M_K \mathcal{V} \mathfrak{m}_n^{(0)}(x). \tag{42}$$

Using the obvious identity

$$\left| \overline{F}(\omega_{n,x}|X) - \overline{F}(\omega_{n,z}|X) \right| = m_n^{(0)}(x, X) \left| \frac{\overline{F}(\omega_{n,z}|X)}{\overline{F}(\omega_{n,x}|X)} - 1 \right| \tag{43}$$

and recalling that the support of the random variable $K_h(z - X)$ is contained in $B(z, h) \subset B(x, 2h)$, (40) and (41) yield:

$$\sup_{x \in \Omega} \sup_{z \in B(x, \varepsilon)} \frac{1}{\Delta(\log \omega_{n,x})(\varepsilon)} \frac{R_{2,n}^{(0)}(x, z)}{\mathfrak{m}_n^{(0)}(x)} = \mathrm{O}(1),$$

and (38) entails

$$\sup_{x \in \Omega} \sup_{z \in B(x, \varepsilon)} \frac{1}{\Delta(\log \omega_{n,x})(\varepsilon)} \frac{R_{2,n}^{(0)}(x, z)}{\mu_n^{(0)}(x)} = \mathrm{O}(1). \tag{44}$$

Applying (35) together with (39) and (44) gives

$$\sup_{x \in \Omega} \sup_{z \in B(x, \varepsilon)} \frac{1}{[\varepsilon/h]^{\eta_K} \vee \Delta(\log \omega_{n,x})(\varepsilon)} \left| \frac{\mu_n^{(0)}(z)}{\mu_n^{(0)}(x)} - 1 \right| = \mathrm{O}(1)$$

which shows Lemma 4 in this case.

We now turn to the case $t = 1$. Note that for all real numbers $a, b \geq 1$ such that $a \neq b$ one has

$$\forall y \geq 1, \quad \left| (\log y - \log a)_+ \mathbb{1}_{\{y > a\}} - (\log y - \log b)_+ \mathbb{1}_{\{y > b\}} \right| \leq |\log b - \log a| \mathbb{1}_{\{y > a \wedge b\}}. \tag{45}$$

Inequality (45) then implies, for all $x \in \Omega$ and $z \in B(x, \varepsilon)$:

$$
\begin{aligned}
\left| \mu_n^{(1)}(x) - \mu_n^{(1)}(z) \right| &\leq \mathbb{E}\left(\left| K_h(x - X) - K_h(z - X) \right| (\log Y - \log \omega_{n,x})_+ \mathbb{1}_{\{Y > \omega_{n,x}\}} \right) \\
&\quad + \left| \log \frac{\omega_{n,x}}{\omega_{n,z}} \right| \mathbb{E}\left(K_h(z - X) \mathbb{1}_{\{Y > \omega_{n,x} \wedge \omega_{n,z}\}} \right) \\
&=: R_{1,n}^{(1)}(x, z) + R_{2,n}^{(1)}(x, z)
\end{aligned}
\tag{46}
$$

and we shall once again take care of both terms in the right-hand side of this inequality. Start by using (36) to get

$$\sup_{z \in B(x, \varepsilon)} R_{1,n}^{(1)}(x, z) \leq 2^d M_K \mathcal{V} \mathfrak{m}_n^{(1)}(x) \left[\frac{\varepsilon}{h} \right]^{\eta_K}. \tag{47}$$

We now use the same idea developed to control $R_{1,n}^{(0)}(x, z)$: applying Lemma 2(iii) entails

$$\sup_{x \in \Omega} \left| \frac{\mathfrak{m}_n^{(1)}(x)}{\mu_n^{(1)}(x)} - 1 \right| \to 0 \quad \text{as } n \to \infty$$

which, together with (47), yields

$$\sup_{x \in \Omega} \sup_{z \in B(x, \varepsilon)} \left[\frac{\varepsilon}{h} \right]^{-\eta_K} \frac{R_{1,n}^{(1)}(x, z)}{\mu_n^{(1)}(x)} = \mathrm{O}(1). \tag{48}$$

To control the second term, write

$$\sup_{z \in B(x, \varepsilon)} R_{2,n}^{(1)}(x, z) \leq \Delta(\log \omega_{n,x})(\varepsilon) \sup_{z \in B(x, \varepsilon)} \mathbb{E}\left(K_h(z - X) \mathbb{1}_{\{Y > \omega_{n,x} \wedge \omega_{n,z}\}} \right).$$

Note that since $\omega_{n,x} \wedge \omega_{n,z}$ is either equal to $\omega_{n,x}$ or $\omega_{n,z}$, we can write, for all $z \in B(x, \varepsilon)$

$$\mathbb{E}\left(K_h(z - X) \mathbb{1}_{\{Y > \omega_{n,x} \wedge \omega_{n,z}\}} \right) \leq \mathbb{E}\left(K_h(z - X) m_n^{(0)}(x, X) \right) \vee \mathbb{E}\left(K_h(z - X) m_n^{(0)}(z, X) \right).$$

Recall now (41) and (43) to obtain, for n large enough, uniformly in $x \in \Omega$ and $z \in B(x, \varepsilon)$,

$$\mathbb{E}\big(K_h(z - X)\mathbb{1}_{\{Y > \omega_{n,x} \wedge \omega_{n,z}\}}\big) \leq 2\mathbb{E}\big(K_h(z - X)m_n^{(0)}(x, X)\big). \tag{49}$$

Finally, using (42) and (49) yields:

$$\sup_{x \in \Omega} \sup_{z \in B(x, \varepsilon)} \frac{1}{\Delta(\log \omega_{n,x})(\varepsilon)} \frac{R_{2,n}^{(1)}(x, z)}{\mathfrak{m}_n^{(0)}(x)} = \mathrm{O}(1),$$

and (38) entails

$$\sup_{x \in \Omega} \sup_{z \in B(x, \varepsilon)} \frac{1}{\Delta(\log \omega_{n,x})(\varepsilon)} \frac{R_{2,n}^{(1)}(x, z)}{\mu_n^{(0)}(x)} = \mathrm{O}(1)$$

so that Lemma 2(iv) gives

$$\sup_{x \in \Omega} \sup_{z \in B(x, \varepsilon)} \frac{1}{\Delta(\log \omega_{n,x})(\varepsilon)} \frac{R_{2,n}^{(1)}(x, z)}{\mu_n^{(1)}(x)} = \mathrm{O}(1). \tag{50}$$

Applying (46) together with (48) and (50) implies that

$$\sup_{x \in \Omega} \sup_{z \in B(x, \varepsilon)} \frac{1}{[\varepsilon/h]^{\eta_K} \vee \Delta(\log \omega_{n,x})(\varepsilon)} \left| \frac{\mu_n^{(1)}(z)}{\mu_n^{(1)}(x)} - 1 \right| = \mathrm{O}(1)$$

which completes the proof of Lemma 4. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The fifth lemma of this section provides a uniform control of both the difference of two versions of $\mu_n^{(0)}(x)$ for two families of thresholds that are uniformly asymptotically equivalent and the empirical analogue of this quantity.

Lemma 5. *Assume that* (SP), (A_1) *and* (A_2) *hold. Assume that K is a bounded probability density function on $\mathbb{R}^d$ with support included in B and that*:

- $\inf_{x \in \Omega} \omega_{n,x} \to \infty$;
- $h^\eta \sup_{x \in \Omega} \log \omega_{n,x} \to 0$;
- $\sup_{x \in \Omega} \overline{\alpha}(y|x) \to 0$ *as* $y \to \infty$.

For an arbitrary family of positive sequences $(\rho_{n,x})$ such that $\sup_{x \in \Omega} \rho_{n,x} \to 0$ as $n \to \infty$, let

$$M_n(x) := \mathbb{E}\big(K_h(x - X)\mathbb{1}_{\{(1-\rho_{n,x})\omega_{n,x} < Y \leq (1+\rho_{n,x})\omega_{n,x}\}}\big)$$

and

$$U_n(x) := \frac{1}{n} \sum_{i=1}^n K_h(x - X_i)\mathbb{1}_{\{(1-\rho_{n,x})\omega_{n,x} < Y_i \leq (1+\rho_{n,x})\omega_{n,x}\}}.$$

Then

$$\sup_{x \in \Omega} \left| \frac{\gamma(x)M_n(x)}{2f(x)\rho_{n,x}\overline{F}(\omega_{n,x}|x)} - 1 \right| \to 0$$

and there exists a positive constant κ such that for all n large enough, one has for every $\varepsilon > 0$ small enough:

$$\forall x \in \Omega, \quad \mathbb{P}\left(\rho_{n,x} \left| \frac{U_n(x)}{M_n(x)} - 1 \right| > \varepsilon \right) \leq 2\exp\big(-\kappa\varepsilon nh^d\overline{F}(\omega_{n,x}|x)\big).$$

Proof. We start by noting that

$$M_n(x) = \mathbb{E}\left(K_h(x - X)\rho_{n,x}\overline{F}(\omega_{n,x}|X)\left[\frac{\overline{F}((1 - \rho_{n,x})\omega_{n,x}|X)}{\rho_{n,x}\overline{F}(\omega_{n,x}|X)} - \frac{\overline{F}((1 + \rho_{n,x})\omega_{n,x}|X)}{\rho_{n,x}\overline{F}(\omega_{n,x}|X)} \right] \right).$$

Use then (*SP*) and (A_1) to get, for an arbitrary $z \in B(x, h)$,

$$\frac{\overline{F}((1 \pm \rho_{n,x})\omega_{n,x}|z)}{\rho_{n,x}\overline{F}(\omega_{n,x}|z)} = \frac{(1 \pm \rho_{n,x})^{-1/\gamma(z)}}{\rho_{n,x}} \exp\left(\int_{\omega_{n,x}}^{(1\pm\rho_{n,x})\omega_{n,x}} \frac{\alpha(v|z)}{v}\,\mathrm{d}v \right). \tag{51}$$

Since $0 < \underline{\gamma} \le \gamma(z)$ and $\sup_{x\in\Omega}\rho_{n,x} \to 0$, a Taylor expansion of the exponential function in a neighborhood of 0 yields

$$\frac{(1 \pm \rho_{n,x})^{-1/\gamma(z)}}{\rho_{n,x}} = \frac{1}{\rho_{n,x}} \mp \frac{1}{\gamma(z)}\big(1 + r_{1,\pm,n}(x, z)\big), \tag{52}$$

where $r_{1,+,n}(x, z)$ and $r_{1,-,n}(x, z)$ converge to 0 as $n \to \infty$, uniformly in $x \in \Omega$ and $z \in B(x, h)$. Besides, for all $u \in (-1, 1)$,

$$\left| \int_{\omega_{n,x}}^{(1+u)\omega_{n,x}} \frac{\alpha(v|z)}{v}\,\mathrm{d}v \right| \le \begin{cases} \overline{\alpha}(\omega_{n,x}|z)|\log(1+u)| & \text{if } u > 0, \\ \overline{\alpha}((1+u)\omega_{n,x}|z)|\log(1+u)| & \text{if } u < 0 \end{cases} \tag{53}$$

so that, because $\inf_{x\in\Omega}\omega_{n,x} \to \infty$, $\sup_{x\in\Omega}\rho_{n,x} \to 0$ and $\sup_{x\in\Omega}\overline{\alpha}(y|x) \to 0$ as $y \to \infty$:

$$\exp\left(\int_{\omega_{n,x}}^{(1\pm\rho_{n,x})\omega_{n,x}} \frac{\alpha(v|z)}{v}\,\mathrm{d}v \right) = 1 + \int_{\omega_{n,x}}^{(1\pm\rho_{n,x})\omega_{n,x}} \frac{\alpha(v|z)}{v}\,\mathrm{d}v\big(1 + r_{2,\pm,n}(x, z)\big),$$

where $r_{2,+,n}(x, z)$ and $r_{2,-,n}(x, z)$ converge to 0 as $n \to \infty$, uniformly in $x \in \Omega$ and $z \in B(x, h)$. Moreover, (53) yields

$$\sup_{x\in\Omega}\sup_{z\in B(x,h)} \frac{1}{\rho_{n,x}}\left| \int_{\omega_{n,x}}^{(1\pm\rho_{n,x})\omega_{n,x}} \frac{\alpha(v|z)}{v}\,\mathrm{d}v \right| \to 0 \quad \text{as } n \to \infty.$$

Plugging this together with (52) into (51) and recalling that $0 < \underline{\gamma} \le \gamma(z)$ entails

$$\sup_{x\in\Omega}\sup_{z\in B(x,h)}\left| \frac{\gamma(z)}{2}\left[\frac{\overline{F}((1 - \rho_{n,x})\omega_{n,x}|z)}{\rho_{n,x}\overline{F}(\omega_{n,x}|z)} - \frac{\overline{F}((1 + \rho_{n,x})\omega_{n,x}|z)}{\rho_{n,x}\overline{F}(\omega_{n,x}|z)} \right] - 1 \right| \to 0 \quad \text{as } n \to \infty.$$

Consequently,

$$\sup_{x\in\Omega}\left| \frac{M_n(x)}{2\mathbb{E}(K_h(x - X)\rho_{n,x}\overline{F}(\omega_{n,x}|X)/\gamma(X))} - 1 \right| \to 0 \quad \text{as } n \to \infty. \tag{54}$$

Recalling (5) and (6), we get

$$\sup_{x\in\Omega}\left| \frac{\mathbb{E}(K_h(x - X)\overline{F}(\omega_{n,x}|X)/\gamma(X))}{\mu_n^{(0)}(x)/\gamma(x)} - 1 \right| \to 0.$$

It only remains to recall (54) and to apply Lemma 2(iv) to obtain

$$\sup_{x\in\Omega}\left| \frac{\gamma(x)M_n(x)}{2f(x)\rho_{n,x}\overline{F}(\omega_{n,x}|x)} - 1 \right| \to 0. \tag{55}$$

We proceed by controlling $U_n(x)$. For every $x \in \Omega$,

$$\mathbb{P}\left(\rho_{n,x}\left|\frac{U_n(x)}{M_n(x)} - 1\right| > \varepsilon\right) = \mathbb{P}\left(\left|h^d U_n(x) - h^d M_n(x)\right| > \varepsilon\frac{h^d M_n(x)}{\rho_{n,x}}\right).$$

Notice now that if $Z_{n,i}(x) := h^d K_h(x - X_i)\mathbb{1}_{\{(1-\rho_{n,x})\omega_{n,x} < Y_i \leq (1+\rho_{n,x})\omega_{n,x}\}}$, then

$$h^d U_n(x) - h^d M_n(x) = \frac{1}{n}\sum_{i=1}^{n}\left[Z_{n,i}(x) - \mathbb{E}\left(Z_{n,i}(x)\right)\right]$$

is a mean of bounded, centered, independent and identically distributed random variables. Define

$$\tau_n(x) := \frac{\varepsilon}{\|K\|_\infty}\frac{nh^d M_n(x)}{\rho_{n,x}} \quad \text{and} \quad \lambda_n(x) := \varepsilon\|K\|_\infty\frac{h^d M_n(x)}{\rho_{n,x}}\frac{1}{\text{Var}(Z_{n,1}(x))}.$$

Bernstein's inequality (see [23]) yields, for all $\varepsilon > 0$,

$$\mathbb{P}\left(\rho_{n,x}\left|\frac{U_n(x)}{M_n(x)} - 1\right| > \varepsilon\right) \leq 2\exp\left(-\frac{\tau_n(x)\lambda_n(x)}{2(1 + \lambda_n(x)/3)}\right).$$

Applying (55) yields, for n large enough,

$$\inf_{x \in \Omega}\frac{\tau_n(x)}{nh^d\overline{F}(\omega_{n,x}|x)} \geq \frac{\varepsilon\underline{f}}{\overline{\gamma}\|K\|_\infty}. \tag{56}$$

Moreover, since $Z_{n,1}^2(x) \leq \|K\|_\infty Z_{n,1}(x)$, it follows that

$$\sup_{x \in \Omega}\frac{1}{\lambda_n(x)} \leq \sup_{x \in \Omega}\rho_{n,x}\frac{\mathbb{E}(Z_{n,1}^2(x))}{\varepsilon\|K\|_\infty h^d M_n(x)} \leq \frac{1}{\varepsilon}\sup_{x \in \Omega}\rho_{n,x} \to 0 \tag{57}$$

as $n \to \infty$. Finally, it holds that

$$\frac{\tau_n(x)\lambda_n(x)}{2(1 + \lambda_n(x)/3)} \geq \left\{\inf_{x \in \Omega}\frac{\tau_n(x)}{nh^d\overline{F}(\omega_{n,x}|x)}\right\}\left\{\inf_{x \in \Omega}\frac{1}{2(1/\lambda_n(x) + 1/3)}\right\}nh^d\overline{F}(\omega_{n,x}|x).$$

Using (56) and (57) it is then clear that, for all n large enough, if $\varepsilon > 0$ is small enough, there exists a positive constant κ that is independent of ε such that

$$\forall x \in \Omega, \quad \mathbb{P}\left(\rho_{n,x}\left|\frac{U_n(x)}{M_n(x)} - 1\right| > \varepsilon\right) \leq 2\exp\left[-\kappa\varepsilon nh^d\overline{F}(\omega_{n,x}|x)\right].$$

This completes the proof of Lemma 5. $\qquad\qquad\square$

The final lemma is the last step in the proof of Theorem 2.

Lemma 6. *Let (X_n) be a sequence of positive real-valued random variables such that for every positive nonrandom sequence (δ_n) converging to 0, the random sequence $(\delta_n X_n)$ converges to 0 almost surely. Then*

$$\mathbb{P}\left(\limsup_{n \to \infty}X_n = +\infty\right) = 0 \quad i.e. \quad X_n = \mathrm{O}(1) \quad almost\ surely.$$

Proof. Assume that there exists $\varepsilon > 0$ such that $\mathbb{P}(\limsup_{n \to \infty}X_n = +\infty) \geq \varepsilon$. Since by definition $\limsup_{n \to \infty}X_n = \lim_{n \to \infty}\sup_{p \geq n}X_p$ is the limit of a nonincreasing sequence, one has

$$\forall k \in \mathbb{N}, \forall n \in \mathbb{N}, \quad \mathbb{P}\left(\bigcup_{p \geq n}\{X_p \geq k\}\right) \geq \varepsilon \quad \Rightarrow \quad \forall k \in \mathbb{N}, \forall n \in \mathbb{N}, \exists n' \geq n, \quad \mathbb{P}\left(\bigcup_{p=n}^{n'}\{X_p \geq k\}\right) \geq \varepsilon/2.$$

It is thus easy to build an increasing sequence of integers (N_k) such that

$$\forall k \geq 1, \quad \mathbb{P}\left(\bigcup_{p=N_k}^{N_{k+1}-1} \{X_p \geq k\} \right) \geq \varepsilon/2.$$

Let $\delta_n = 1/k$ if $N_k \leq n < N_{k+1}$. It is clear that (δ_n) is a positive sequence which converges to 0. Besides, for all $k \in \mathbb{N} \setminus \{0\}$ it holds that

$$\mathbb{P}\left(\sup_{p \geq N_k} \delta_p X_p \geq 1 \right) = \mathbb{P}\left(\bigcup_{p \geq N_k} \{\delta_p X_p \geq 1\} \right) \geq \mathbb{P}\left(\bigcup_{p=N_k}^{N_{k+1}-1} \{\delta_p X_p \geq 1\} \right) = \mathbb{P}\left(\bigcup_{p=N_k}^{N_{k+1}-1} \{X_p \geq k\} \right) \geq \varepsilon/2.$$

Hence $(\delta_n X_n)$ does not converge almost surely to 0, from which the result follows. $\qquad\square$

Acknowledgements

References

[1] Y. Aragon, A. Daouia and C. Thomas-Agnan. Nonparametric frontier estimation: A conditional quantile-based approach. *Econometric Theory* **21** (2) (2005) 358–389. MR2179542

[2] J. Beirlant, Y. Goegebeur, J. Segers and J. Teugels. *Statistics of Extremes – Theory and Applications. Wiley Series in Probability and Statistics*. Wiley, Chichester, 2004. With contributions from Daniel de Waal and Chris Ferro. MR2108013

[3] N. H. Bingham, C. M. Goldie and J. L. Teugels. *Regular Variation*. Cambridge Univ. Press, Cambridge, 1987. MR0898871

[4] V. Chavez-Demoulin and A. C. Davison. Generalized additive modelling of sample extremes. *J. R. Stat. Soc. Ser. C. Appl. Stat.* **54** (2005) 207–222. MR2134607

[5] H. Chernoff. A measure of asymptotic efficiency for tests of a hypothesis based on the sum of observations. *Ann. Math. Statistics* **23** (4) (1952) 493–507. MR0057518

[6] A. Daouia, L. Gardes and S. Girard. On kernel smoothing for extremal quantile regression. *Bernoulli* **19** (5B) (2013) 2557–2589. MR3160564

[7] A. Daouia, L. Gardes, S. Girard and A. Lekina. Kernel estimators of extreme level curves. *Test* **20** (2) (2011) 311–333. MR2834049

[8] A. Daouia and L. Simar. Robust nonparametric estimators of monotone boundaries. *J. Multivariate Anal.* **96** (2005) 311–331. MR2204981

[9] A. C. Davison and N. I. Ramesh. Local likelihood smoothing of sample extremes. *J. R. Stat. Soc. Ser. B Stat. Methodol.* **62** (2000) 191–208. MR1747404

[10] A. C. Davison and R. L. Smith. Models for exceedances over high thresholds. *J. R. Stat. Soc. Ser. B Stat. Methodol.* **52** (1990) 393–442. MR1086795

[11] U. Einmahl and D. M. Mason. An empirical process approach to the uniform consistency of kernel-type function estimators. *J. Theoret. Probab.* **13** (1) (2000) 1–37. MR1744994

[12] L. Gardes and S. Girard. A moving window approach for nonparametric estimation of the conditional tail index. *J. Multivariate Anal.* **99** (2008) 2368–2388. MR2463396

[13] L. Gardes and S. Girard. Conditional extremes from heavy-tailed distributions: An application to the estimation of extreme rainfall return levels. *Extremes* **13** (2010) 177–204. MR2643556

[14] L. Gardes and G. Stupfler. Estimation of the conditional tail index using a smoothed local Hill estimator. *Extremes* **17** (2014) 45–75. MR3179970

[15] S. Girard, A. Guillou and G. Stupfler. Uniform strong consistency of a frontier estimator using kernel regression on high order moments. *ESAIM*. To appear, 2015. DOI:10.1051/ps/2013050.

[16] Y. Goegebeur, A. Guillou and A. Schorgen. Nonparametric regression estimation of conditional tails – the random covariate case. *Statistics* **48** (2014) 732–755.

[17] L. de Haan and A. Ferreira. *Extreme Value Theory: An Introduction*. Springer, New York, 2006. MR2234156

[18] P. Hall. On some simple estimates of an exponent of regular variation. *J. R. Stat. Soc. Ser. B Stat. Methodol.* **44** (1982) 37–42. MR0655370

[19] P. Hall and N. Tajvidi. Nonparametric analysis of temporal trend when fitting parametric models to extreme-value data. *Statist. Sci.* **15** (2000) 153–167. MR1788730

[20] W. Härdle, P. Janssen and R. Serfling. Strong uniform consistency rates for estimators of conditional functionals. *Ann. Statist.* **16** (1988) 1428–1449. MR0964932

[21] W. Härdle and J. S. Marron. Optimal bandwidth selection in nonparametric regression function estimation. *Ann. Statist.* **13** (4) (1985) 1465–1481. MR0811503

[22] B. M. Hill. A simple general approach to inference about the tail of a distribution. *Ann. Statist.* **3** (1975) 1163–1174. MR0378204

[23] W. Hoeffding. Probability inequalities for sums of bounded random variables. *J. Amer. Statist. Assoc.* **58** (1963) 13–30. MR0144363

[24] P. L. Hsu and H. Robbins. Complete convergence and the law of large numbers. *Proc. Nat. Acad. Sci. U.S.A.* **33** (1947) 25–31. MR0019852

[25] M. Lemdani, E. Ould-Saïd and N. Poulin. Asymptotic properties of a conditional quantile estimator with randomly truncated data. *J. Multivariate Anal.* **100** (2009) 546–559. MR2483437

[26] Y. P. Mack and B. W. Silverman. Weak and strong uniform consistency of kernel regression estimates. *Z. Wahrsch. Verw. Gebiete* **61** (1982) 405–415. MR0679685

[27] E. A. Nadaraya. On non-parametric estimates of density functions and regression curves. *Theory Probab. Appl.* **10** (1965) 186–190. MR0172400

[28] E. Parzen. On estimation of a probability density function and mode. *Ann. Math. Statist.* **33** (3) (1962) 1065–1076. MR0143282

[29] M. Rosenblatt. Remarks on some nonparametric estimates of a density function. *Ann. Math. Statist.* **27** (3) (1956) 832–837. MR0079873

[30] B. W. Silverman. Weak and strong uniform consistency of the kernel estimate of a density and its derivatives. *Ann. Statist.* **6** (1) (1978) 177–184. MR0471166

[31] R. L. Smith. Extreme value analysis of environmental time series: An application to trend detection in ground-level ozone (with discussion). *Statist. Sci.* **4** (1989) 367–393. MR1041763

[32] W. Stute. A law of the logarithm for kernel density estimators. *Ann. Probab.* **10** (1982) 414–422. MR0647513

[33] H. Wang and C. L. Tsai. Tail index regression. *J. Amer. Statist. Assoc.* **104** (487) (2009) 1233–1240. MR2750246