Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2015, Vol. 51, No. 3, 965–992
DOI: 10.1214/14-AIHP606

ANNALES DE L'INSTITUT HENRI POINCARÉ
PROBABILITÉS ET STATISTIQUES
www.imstat.org/aihp

Propagation of chaos for a subcritical Keller–Segel model

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Received 20 June 2013; revised 13 December 2013; accepted 23 January 2014

Abstract. This paper deals with a subcritical Keller–Segel equation. Starting from the stochastic particle system associated with it, we show well-posedness results and the propagation of chaos property. More precisely, we show that the empirical measure of the system tends towards the unique solution of the limit equation as the number of particles goes to infinity.

Résumé. Cet article traite de l'équation de Keller–Segel dans un cadre sous-critique. À l'aide du système de particules en lien avec cette équation, nous montrons des résultats d'existence et d'unicité, puis la propagation du chaos pour ce dernier. Plus précisément, nous montrons que la mesure empirique du système tend vers l'unique solution de l'équation limite lorsque le nombre de particules tend vers l'infini.

MSC: 65C35

Keywords: Keller–Segel; Propagation of chaos; Well-posedness

1. Introduction and main results

The subject of this paper is the convergence of a stochastic particle system to a nonlinear and nonlocal equation which can be seen as a subcritical version of the classical Keller–Segel equation.

1.1. *The subcritical Keller–Segel equation*

Consider the equation:

$$\frac{\partial f_t(x)}{\partial t} = \chi \nabla_x \cdot \big((K * f_t)(x) f_t(x)\big) + \Delta_x f_t(x), \tag{1.1}$$

where $f : \mathbb{R}_+ \times \mathbb{R}^2 \to \mathbb{R}$ and $\chi > 0$. The force field kernel $K : \mathbb{R}^2 \to \mathbb{R}^2$ comes from an attractive potential $\Phi : \mathbb{R}^2 \to \mathbb{R}$ and is defined by

$$K(x) := \frac{x}{|x|^{\alpha+1}} = -\nabla \underbrace{\left(\frac{1}{\alpha - 1}|x|^{1-\alpha}\right)}_{\Phi(x)}, \quad \alpha \in (0, 1). \tag{1.2}$$

The standard Keller–Segel equation correspond to the critical case $K(x) = x/|x|^2$ (i.e., more singular at $x = 0$) and it describes a model of chemotaxis, i.e., the movement of cells (usually bacteria or amoebae) which are attracted by some chemical substance that they produce. This equation has been first introduced by Keller and Segel in [15,16]. Blanchet, Dolbeault and Perthame showed in [4] some nice results on existence of global weak solutions if the non-negative parameter χ (which is the sensitivity of the bacteria to the chemo-attractant) is smaller than $8\pi/M$ where M is the initial mass (here M will always be 1 since we will deal with probability measures). For more details on the subject, see [12,13].

1.2. *The particle system*

We consider the following system of particles

$$\forall i = 1, \dots, N, \quad X_t^{i,N} = X_0^{i,N} - \frac{\chi}{N} \sum_{j=1, j\neq i}^{N} \int_0^t K\big(X_s^{i,N} - X_s^{j,N}\big)\,\mathrm{d}s + \sqrt{2}B_t^i, \tag{1.3}$$

where $(B^i)_{i=1,\dots,N}$ is an independent family of 2D standard Brownian motions and K is defined in (1.2). We will show in the sequel that there is propagation of chaos to the solution of the following nonlinear S.D.E. linked with (1.1) (see the next paragraph)

$$X_t = X_0 - \chi \int_0^t \int_{\mathbb{R}^2} K(X_s - x) f_s(\mathrm{d}x)\,\mathrm{d}s + \sqrt{2}B_t, \tag{1.4}$$

where $f_t = \mathcal{L}(X_t)$ ($\mathcal{L}(X_t)$ denotes the law of X_t).

1.3. *Weak solution for the P.D.E.*

For any Polish space E, we denote by $\mathbf{P}(E)$ the set of all probability measures on E which we endow with the topology of weak convergence defined by duality against functions of $C_b(E)$. We give the notion of weak solution that we use in this paper.

Definition 1.1. *We say that $f = (f_t)_{t\geq 0} \in C([0,\infty), \mathbf{P}(\mathbb{R}^2))$ is a weak solution to (1.1) if*

$$\forall T > 0, \quad \int_0^T \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \big|K(x-y)\big| f_s(\mathrm{d}x) f_s(\mathrm{d}y)\,\mathrm{d}s < \infty, \tag{1.5}$$

and if for all $\varphi \in C_b^2(\mathbb{R}^2)$, all $t \geq 0$,

$$\begin{aligned} \int_{\mathbb{R}^2} \varphi(x) f_t(\mathrm{d}x) &= \int_{\mathbb{R}^2} \varphi(x) f_0(\mathrm{d}x) + \int_0^t \int_{\mathbb{R}^2} \triangle_x \varphi(x) f_s(\mathrm{d}x)\,\mathrm{d}s \\ &\quad - \chi \int_0^t \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} K(x-y) \cdot \nabla_x \varphi(x) f_s(\mathrm{d}y) f_s(\mathrm{d}x)\,\mathrm{d}s. \end{aligned} \tag{1.6}$$

Remark 1.2. *We can see easily that if $(X_t)_{t\geq 0}$ is a solution to (1.4), then setting $f_t = \mathcal{L}(X_t)$ for any $t \geq 0$, $(f_t)_{t\geq 0}$ is a weak solution of (1.1) in the sense of Definition 1.1 provided it satisfies (1.5). Indeed, by Itô's formula, we find that for $\varphi \in C_b^2(\mathbb{R}^2)$,*

$$\begin{aligned} \varphi(X_t) &= \varphi(X_0) - \chi \int_0^t \nabla_x \varphi(X_s) \cdot \int_{\mathbb{R}^2} K(X_s - y) f_s(\mathrm{d}y)\,\mathrm{d}s \\ &\quad + \int_0^t \sqrt{2} \nabla_x \varphi(X_s) \cdot \mathrm{d}B_s + \int_0^t \triangle_x \varphi(X_s)\,\mathrm{d}s. \end{aligned}$$

Taking expectations, we get (1.6).

1.4. *Notation and propagation of chaos*

For $N \geq 2$, we denote by $\mathbf{P}_{\text{sym}}(E^N)$ the set of symmetric probability measures on E^N, i.e., the set of probability measures which are laws of exchangeable E^N-valued random variables.

We consider for any $F \in \mathbf{P}_{\text{sym}}((\mathbb{R}^2)^N)$ with a density (a finite moment of positive order is also required in order to define the entropy) the Boltzmann entropy and the Fisher information which are defined by

$$H(F) := \frac{1}{N} \int_{(\mathbb{R}^2)^N} F(x) \log F(x) \, \mathrm{d}x \quad \text{and} \quad I(F) := \frac{1}{N} \int_{(\mathbb{R}^2)^N} \frac{|\nabla F(x)|^2}{F(x)} \, \mathrm{d}x.$$

We also define ($x_i \in \mathbb{R}^2$ stands for the ith coordinate of $x \in (\mathbb{R}^2)^N$), for $k \geq 0$,

$$M_k(F) := \frac{1}{N} \int_{(\mathbb{R}^2)^N} \sum_{i=1}^N |x_i|^k F(\mathrm{d}x).$$

Observe that we proceed to the normalization by $1/N$ in order to have, for any $f \in \mathbf{P}(\mathbb{R}^2)$,

$$H\big(f^{\otimes N}\big) = H(f), \qquad I\big(f^{\otimes N}\big) = I(f) \quad \text{and} \quad M_k\big(f^{\otimes N}\big) = M_k(f).$$

We introduce the space $\mathbf{P}_1(\mathbb{R}^2) := \{f \in \mathbf{P}(\mathbb{R}^2), M_1(f) < \infty\}$ and we recall the definition of the Wasserstein distance: if $f, g \in \mathbf{P}_1(\mathbb{R}^2)$,

$$\mathcal{W}_1(f, g) = \inf\left\{\int_{\mathbb{R}^2 \times \mathbb{R}^2} |x - y| R(\mathrm{d}x, \mathrm{d}y)\right\},$$

where the infimum is taken over all probability measures R on $\mathbb{R}^2 \times \mathbb{R}^2$ with f for first marginal and g for second marginal. It is known that the infimum is reached. See, e.g., Villani [21] for many details on the subject.

We now define the notion of propagation of chaos.

Definition 1.3. *Let X be some E-valued random variable. A sequence $(X_1^N, \ldots, X_N^N)$ of exchangeable E-valued random variables is said to be X-chaotic if one of the three following equivalent conditions is satisfied*:

(i) *(X_1^N, X_2^N) goes in law to 2 independent copies of X as $N \to +\infty$;*
(ii) *for all $j \geq 1$, $(X_1^N, \ldots, X_j^N)$ goes in law to j independent copies of X as $N \to +\infty$;*
(iii) *the empirical measure $\mu_{X^N}^N := \frac{1}{N} \sum_{i=1}^N \delta_{X_i^N} \in \mathbf{P}(E)$ goes in law to the constant $\mathcal{L}(X)$ as $N \to +\infty$.*

We refer to [19] for the equivalence of the three conditions or [11], Theorem 1.2, where the equivalence is established in a quantitative way.

Propagation of chaos in the sense of Sznitman holds for a system of N exchangeable particles evolving in time if when the initial conditions $(X_0^{1,N}, \ldots, X_0^{N,N})$ are X_0-chaotic, the trajectories $((X_t^{1,N})_{t\geq 0}, \ldots, (X_t^{N,N})_{t\geq 0})$ are $(X_t)_{t\geq 0}$-chaotic, where $(X_t)_{t\geq 0}$ is the (unique) solution of the expected (one-particle) limit model.

We finally recall a stronger (see [11]) sense of chaos introduced by Kac in [14] and formalized recently in [6]: the entropic chaos.

Definition 1.4. *Let f be some probability measure on E. A sequence (F^N) of symmetric probability measures on E^N is said to be entropically f-chaotic if*

$$F_1^N \to f \quad \textit{weakly in } \mathbf{P}(E) \quad \textit{and} \quad H\big(F^N\big) \to H(f) \quad \textit{as } N \to \infty,$$

where F_1^N stands for the first marginal of F^N.

We can observe that since the entropy is lower semi continuous (so that $H(f) \leq \liminf_N H(F^N)$) and is convex, the entropic chaos (which requires $\lim_N H(F^N) = H(f)$) is a stronger notion of convergence which implies that for all $j \geq 1$, the density of the law of $(X_1^N, \ldots, X_j^N)$ goes to $f^{\otimes j}$ strongly in L^1 as $N \to \infty$ (see [3]).

1.5. *Main results*

We first give a result of existence and uniqueness for (1.1).

Theorem 1.5. *Let $\alpha \in (0,1)$. Assume that $f_0 \in \mathbf{P}_1(\mathbb{R}^2)$ is such that $H(f_0) < \infty$.*

(i) *There exists a unique weak solution f to (1.1) such that*

$$f \in L^\infty_{\mathrm{loc}}\big([0,\infty), \mathbf{P}_1\big(\mathbb{R}^2\big)\big) \cap L^1_{\mathrm{loc}}\big([0,\infty); L^p\big(\mathbb{R}^2\big)\big) \quad \textit{for some } p > \frac{2}{1-\alpha}. \tag{1.7}$$

(ii) *This solution furthermore satisfies that for all $T > 0$,*

$$\int_0^T I(f_s)\,\mathrm{d}s < \infty, \tag{1.8}$$

for any $q \in [1,2)$ and for all $T > 0$,

$$\nabla_x f \in L^{2q/(3q-2)}\big(0,T; L^q\big(\mathbb{R}^2\big)\big), \tag{1.9}$$

for any $p \geq 1$,

$$f \in C\big([0,\infty); L^1\big(\mathbb{R}^2\big)\big) \cap C\big((0,\infty); L^p\big(\mathbb{R}^2\big)\big), \tag{1.10}$$

and that for any $\beta \in C^1(\mathbb{R}) \cap W^{2,\infty}_{\mathrm{loc}}(\mathbb{R})$ such that β'' is piecewise continuous and vanishes outside a compact set,

$$\begin{aligned} \partial_t \beta(f) &= \chi(K * f)\cdot \nabla_x\big(\beta(f)\big) + \triangle_x \beta(f) \\ &\quad - \beta''(f)|\nabla_x f|^2 + \chi\beta'(f_s) f_s(\nabla_x \cdot K * f_s), \end{aligned} \tag{1.11}$$

on $[0,\infty) \times \mathbb{R}^2$ in the distributional sense.

We denote by F_0^N the law of $(X_0^{i,N})_{i=1,\dots,N}$. We assume that for some $f_0 \in \mathbf{P}(\mathbb{R}^2)$,

$$\begin{cases} F_0^N \in \mathbf{P}_{\mathrm{sym}}((\mathbb{R}^2)^N) & \text{is } f_0\text{-chaotic;} \\ \sup_{N\geq 2} M_1(F_0^N) < \infty, & \sup_{N\geq 2} H(F_0^N) < \infty. \end{cases} \tag{1.12}$$

Observe that this condition is satisfied if the random variables $(X_0^{i,N})_{i=1,\dots,N}$ are i.i.d. with law $f_0 \in \mathbf{P}_1(\mathbb{R}^2)$ such that $H(f_0) < \infty$. The next result states the well-posedness for the particle system (1.3).

Theorem 1.6. *Let $\alpha \in (0,1)$.*

(i) *Let $N \geq 2$ be fixed and assume that $M_1(F_0^N) < \infty$ and $H(F_0^N) < \infty$. There exists a unique strong solution $(X_t^{i,N})_{t\geq 0, i=1,\dots,N}$ to (1.3). Furthermore, the particles a.s. never collapse, i.e., it holds that a.s., for any $t \geq 0$ and $i \neq j$, $X_t^{i,N} \neq X_t^{j,N}$.*

(ii) *Assume (1.12). If for all $t \geq 0$, we denote by $F_t^N \in \mathbf{P}_{\mathrm{sym}}((\mathbb{R}^2)^N)$ the law of $(X_t^{i,N})_{i=1,\dots,N}$, then there exists a constant C depending on χ, $\sup_{N\geq 2} H(F_0^N)$ and $\sup_{N\geq 2} M_1(F_0^N)$ such that for all $t \geq 0$ and $N \geq 2$*

$$H\big(F_t^N\big) \leq C(1+t), \qquad M_1\big(F_t^N\big) \leq C(1+t), \qquad \int_0^t I\big(F_s^N\big)\,\mathrm{d}s \leq C(1+t).$$

Furthermore for any $T > 0$,

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \big|X_t^{1,N}\big|\Big] \leq C(1+T). \tag{1.13}$$

We also have

$$H(F_t^N) + \int_0^t I(F_s^N)\,\mathrm{d}s \le H(F_0^N) + \frac{\chi}{N^2}\sum_{i\neq j}\int_0^t \mathbb{E}\big[\operatorname{div} K(X_s^{i,N} - X_s^{j,N})\big]\,\mathrm{d}s. \tag{1.14}$$

We next state a well-posedness result for the nonlinear S.D.E. (1.4).

Theorem 1.7. *Let $\alpha \in (0,1)$ and $f_0 \in \mathbf{P}_1(\mathbb{R}^2)$ such that $H(f_0) < \infty$. There exists a unique strong solution $(X_t)_{t\ge 0}$ to (1.4) such that for some $p > 2/(1-\alpha)$,*

$$(f_t)_{t\ge 0} \in L^\infty_{\mathrm{loc}}\big([0,\infty), \mathbf{P}_1(\mathbb{R}^2)\big) \cap L^1_{\mathrm{loc}}\big([0,\infty); L^p(\mathbb{R}^2)\big), \tag{1.15}$$

where f_t is the law of X_t. Furthermore, $(f_t)_{t\ge 0}$ is the unique solution to (1.1) given in Theorem 1.5.

We finally give the result about propagation of chaos.

Theorem 1.8. *Let $\alpha \in (0,1)$. Assume (1.12). For each $N \ge 2$, consider the unique solution $(X_t^{i,N})_{i=1,\dots,N,t\ge 0}$ to (1.3). Let $(X_t)_{t\ge 0}$ be the unique solution to (1.4).*

(i) *The sequence $(X_t^{i,N})_{i=1,\dots,N,t\ge 0}$ is $(X_t)_{t\ge 0}$-chaotic. In particular, the empirical measure $Q^N := \frac{1}{N}\sum_{i=1}^N \delta_{(X_t^{i,N})_{t\ge 0}}$ goes in law to $\mathcal{L}((X_t)_{t\ge 0})$ in $\mathbf{P}(C((0,\infty),\mathbb{R}^2))$.*
(ii) *Assume furthermore that $\lim_N H(F_0^N) = H(f_0)$. For all $t \ge 0$, the sequence $(X_t^{i,N})_{i=1,\dots,N}$ is then X_t-entropically chaotic. In particular, for any $j \ge 1$ and any $t \ge 0$, denoting by F_{tj}^N the density of the law of $(X_t^{1,N},\dots,X_t^{j,N})$, it holds that*

$$\lim_{N\to\infty} \big\| F_{tj}^N - f_t^{\otimes j}\big\|_{L^1((\mathbb{R}^2)^j)} = 0.$$

We can observe that the condition $\lim_N H(F_0^N) = H(f_0)$ is satisfied if the random variables $(X_0^{i,N})_{i=1,\dots,N}$ are i.i.d. with law f_0 such that $H(f_0) < \infty$.

1.6. *Comments*

This paper is some kind of adaptation of the work of Fournier, Hauray and Mischler in [8] where they show the propagation of chaos of some particle system for the 2D viscous vortex model. We use the same methods for a subcritical Keller–Segel equation. The proofs are thus sometimes very similar to those in [8] but there are some differences due to the facts that (i) there are no circulation parameter ($\mathcal{M}_i^N$ in [8]): this simplify the situation since we thus deal with solutions which are probabilities, (ii) $\alpha \neq 1$ so when we use Hardy–Littlewood–Sobolev's inequality an extra change of variables for the time variable is needed (see Step 1 in the proof of Theorem 1.5 in Section 6) and (iii) the kernel is not the same: it is not divergence-free and we thus have to deal with some additional terms in our computations (see the comments before Proposition 3.1 and in the proof of Theorem 1.5). We can also notice that due to this fact, we have no already known result for the existence and uniqueness of the particle system that we consider. The methods used to prove uniqueness for the Keller–Segel equation (1.1) and its associated S.D.E. (1.4), and to prove the entropic chaos are also different.

The proof of Theorem 1.5 follows the ideas of renormalisation solutions to a P.D.E. introduced by DiPerna and Lions in [7] and developed since then. The key point is to be able to find good *a priori* estimates which allow us to approximate the weak solutions by regular functions, i.e., to use C^k functions instead of L^1. Then, using these estimates, one can pass to the limit and go back to the initial problem. One can further see that the uniqueness result is proven based on coupling methods and the Wasserstein distance. This will allow us to use more general initial conditions than we could use in a strictly deterministic framework.

The proof of existence and uniqueness for the particle system (1.3) (Theorem 1.6) use some nice arguments. Like for S.D.E.s with locally Lipschitz coefficients, we show existence and uniqueness up to an explosion time and the interesting part of the proof is to show that this explosion time is infinite a.s.

To our knowledge, there is no other work that give a convergence result of some particle system for a chemotaxis model with a singular kernel K and without cutoff parameter. In [17] and [18], Stevens studies a particle system with two kinds of particles corresponding to bacteria and chemical substance. She shows convergence of the system for smooth initial data (lying in $C_b^3(\mathbb{R}^d)$) and for regular kernels (continuously differentiable and bounded together with their derivatives). In [9] and [10], Haskovec and Schmeiser consider a kernel with a cutoff parameter $K_\varepsilon(x) = \frac{x}{|x|(|x|+\varepsilon)}$. They get some well-posedness result for the particle system and they show the weak convergence of subsequences due to a tightness result (observe that here we have propagation of chaos and also entropic chaos). In a recent work [5], Calvez and Corrias work on some one-dimensional Keller–Segel model. They study a dynamical particle system for which they give a global existence result under some assumptions on the initial distribution of the particles that prevents collisions. They also give two blow-up criteria for the particle system they do not state a convergence result for this system.

Finally, it is important to notice that the present method can not be directly adapted for the standard case $\alpha = 1$ because in this last situation the entropy and the Fisher information are not controlled.

1.7. *Plan of the paper*

In the next section, we give some preliminary results. In Section 3, we establish the well-posedness of the particle system (1.3). In Section 4, we prove the tightness of the particle system and we show that any limit point belongs to the set of solutions to the nonlinear S.D.E. (1.4). In Section 5, we show that the P.D.E. (1.1) and the nonlinear S.D.E. (1.4) are well-posed and we show the propagation of chaos. Finally, in the last section, we improve the regularity of the solution, give some renormalization results for the solution to (1.1) and we conclude with the entropic chaos.

2. Preliminaries

In this section, we recall some lemmas stated in [8] and [11] and we state a result on the regularity of the kernel K defined in (1.2). The first result tells us that pairs of particles which law have finite Fisher information cannot be too close.

Lemma 2.1 ([8], Lemma 3.3). *Consider $F \in \mathbf{P}(\mathbb{R}^2 \times \mathbb{R}^2)$ with finite Fisher information and (X_1, X_2) a random variable with law F. Then for any $\gamma \in (0,2)$ and any $\beta > \gamma/2$ there exists $C_{\gamma,\beta}$ so that*

$$\mathbb{E}\big[|X_1 - X_2|^{-\gamma}\big] = \int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{F(x_1,x_2)}{|x_1-x_2|^\gamma}\,\mathrm{d}x_1\,\mathrm{d}x_2 \le C_{\gamma,\beta}\big(I(F)^\beta + 1\big).$$

In the next lemma, we see that the Fisher information of the marginals of some $F \in \mathbf{P}_{\mathrm{sym}}((\mathbb{R}^2)^N)$ is smaller than the Fisher information of F.

Lemma 2.2 ([11], Lemma 3.7). *For any $F \in \mathbf{P}_{\mathrm{sym}}((\mathbb{R}^2)^N)$ and $1 \le l \le N$, $I(F_l) \le I(F)$, where $F_l \in \mathbf{P}_{\mathrm{sym}}((\mathbb{R}^2)^l)$ denotes the marginal probability of F on the lth block of variables.*

The following lemma allows us to control from below the entropy of some $F \in \mathbf{P}_k((\mathbb{R}^2)^N)$ by its moment of order k for any $k > 0$.

Lemma 2.3 ([8], Lemma 3.1). *For any $k, \lambda \in (0,\infty)$, there is a constant $C_{k,\lambda} \in \mathbb{R}$ such that for any $N \ge 1$, any $F \in \mathbf{P}_k((\mathbb{R}^2)^N)$,*

$$H(F) \ge -C_{k,\lambda} - \lambda M_k(F).$$

The next result tells us that a probability measure on $\mathbb{R}^2$ with finite Fisher information belongs to L^p for any $p \ge 1$ and its derivatives, to L^q for any $q \in [1,2)$.

Lemma 2.4 ([8], Lemma 3.2). *For any $f \in \mathbf{P}(\mathbb{R}^2)$ with finite Fisher information, there holds*

$$\forall p \in [1,\infty), \quad \|f\|_{L^p(\mathbb{R}^2)} \le C_p I(f)^{1-1/p},$$
$$\forall q \in [1,2), \quad \|\nabla_x f\|_{L^q(\mathbb{R}^2)} \le C_q I(f)^{3/2-1/q}.$$

We end this section with the following result on K.

Lemma 2.5. *Let $\alpha \in (0,1)$. There exists a constant C_α such that for all $x, y \in \mathbb{R}^2$*

$$\bigl|K(x)-K(y)\bigr| \le C_\alpha |x-y|\biggl(\frac{1}{|x|^{\alpha+1}}+\frac{1}{|y|^{\alpha+1}}\biggr).$$

Proof. We have

$$\begin{aligned}\bigl|K(x)-K(y)\bigr| &= \biggl|x\biggl(\frac{1}{|x|^{\alpha+1}}-\frac{1}{|y|^{\alpha+1}}\biggr)+\frac{x-y}{|y|^{\alpha+1}}\biggr| \\ &\le |x||x-y|(\alpha+1)\max\biggl(\frac{1}{|x|^{\alpha+2}},\frac{1}{|y|^{\alpha+2}}\biggr)+\frac{|x-y|}{|y|^{\alpha+1}}.\end{aligned}$$

By symmetry, we also have

$$\bigl|K(x)-K(y)\bigr| \le |y||x-y|(\alpha+1)\max\biggl(\frac{1}{|x|^{\alpha+2}},\frac{1}{|y|^{\alpha+2}}\biggr)+\frac{|x-y|}{|x|^{\alpha+1}}.$$

So we deduce that

$$\begin{aligned}\bigl|K(x)-K(y)\bigr| &\le |x-y|\biggl[(\alpha+1)\min\bigl(|x|,|y|\bigr)\max\biggl(\frac{1}{|x|^{\alpha+2}},\frac{1}{|y|^{\alpha+2}}\biggr) \\ &\qquad +\frac{1}{|x|^{\alpha+1}}+\frac{1}{|y|^{\alpha+1}}\biggr] \\ &\le |x-y|\biggl[(\alpha+1)\frac{1}{\min(|x|,|y|)^{\alpha+1}}+\frac{1}{|x|^{\alpha+1}}+\frac{1}{|y|^{\alpha+1}}\biggr] \\ &\le (\alpha+2)|x-y|\biggl(\frac{1}{|x|^{\alpha+1}}+\frac{1}{|y|^{\alpha+1}}\biggr),\end{aligned}$$

which concludes the proof. □

3. Well-posedness for the system of particles

Let's now introduce another particle system with a regularized kernel. We set, for $\varepsilon \in (0,1)$,

$$K_\varepsilon(x) = \frac{x}{\max(|x|,\varepsilon)^{\alpha+1}}, \tag{3.1}$$

which obviously satisfies $|K_\varepsilon(x)-K_\varepsilon(y)| \le C_{\alpha,\varepsilon}|x-y|$ and we consider the following system of S.D.E.s

$$\forall i=1,\dots,N, \quad X_t^{i,N,\varepsilon} = X_0^{i,N} - \frac{\chi}{N}\sum_{j=1, j\neq i}^{N}\int_0^t K_\varepsilon\bigl(X_s^{i,N,\varepsilon}-X_s^{j,N,\varepsilon}\bigr)\,\mathrm{d}s + \sqrt{2}B_t^i, \tag{3.2}$$

for which strong existence and uniqueness thus holds.

The following result will be useful for the proof of Theorem 1.6. Its proof is very similar to the proof of [8], Proposition 5.1. Nevertheless, due to the fact that the kernel is not divergence-free, there is an additional term in the dissipation of entropy's formula (3.3) which will lead to additional computations to control it.

Proposition 3.1. *Let $\alpha \in (0,1)$.*

(i) *Let $N \geq 2$ be fixed. Assume that $M_1(F_0^N) < \infty$ and $H(F_0^N) < \infty$. For all $t \geq 0$, we denote by $F_t^{N,\varepsilon} \in \mathbf{P}_{\mathrm{sym}}((\mathbb{R}^2)^N)$ the law of $(X_t^{i,N,\varepsilon})_{i=1,\dots,N}$. Then*

$$H\bigl(F_t^{N,\varepsilon}\bigr) = H\bigl(F_0^N\bigr) + \frac{\chi}{N^2}\sum_{i\neq j}\int_0^t \int_{(\mathbb{R}^2)^N} \operatorname{div} K_\varepsilon(x_i - x_j) F_s^{N,\varepsilon}(x)\,\mathrm{d}s\,\mathrm{d}x - \int_0^t I\bigl(F_s^{N,\varepsilon}\bigr)\,\mathrm{d}s. \tag{3.3}$$

(ii) *There exists a constant C which depends on χ, $H(F_0^N)$ and $M_1(F_0^N)$ (but not on ε) such that for all $t \geq 0$ and $N \geq 2$,*

$$H\bigl(F_t^{N,\varepsilon}\bigr) \leq C(1+t), \qquad M_1\bigl(F_t^{N,\varepsilon}\bigr) \leq C(1+t), \qquad \int_0^t I\bigl(F_s^{N,\varepsilon}\bigr)\,\mathrm{d}s \leq C(1+t). \tag{3.4}$$

Furthermore,

$$\mathbb{E}\Bigl[\sup_{[0,T]}\bigl|X_t^{1,N,\varepsilon}\bigr|\Bigr] \leq C(1+T). \tag{3.5}$$

Proof. Let $\varphi \in C_b^2((\mathbb{R}^2)^N)$, and $t \geq 0$ be fixed. Using Itô's formula, we compute the expectation of $\varphi(X_t^{1,N,\varepsilon},\dots, X_t^{N,N,\varepsilon})$ and get (recall that $x_i \in \mathbb{R}^2$ stands for the ith coordinate of $x \in (\mathbb{R}^2)^N$)

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{(\mathbb{R}^2)^N} \varphi(x) F_t^{N,\varepsilon}(\mathrm{d}x) = -\frac{\chi}{N}\int_{(\mathbb{R}^2)^N} \sum_{i\neq j} K_\varepsilon(x_i - x_j)\cdot \nabla_{x_i}\varphi(x) F_t^{N,\varepsilon}(\mathrm{d}x) + \int_{(\mathbb{R}^2)^N} \triangle_x \varphi(x) F_t^{N,\varepsilon}(\mathrm{d}x). \tag{3.6}$$

We deduce that $F^{N,\varepsilon}$ is a weak solution to

$$\partial_t F_t^{N,\varepsilon}(x) = \frac{\chi}{N}\sum_{i\neq j} \operatorname{div}_{x_i}\bigl(F_t^{N,\varepsilon}(x) K_\varepsilon(x_i - x_j)\bigr) + \triangle_x F_t^{N,\varepsilon}(x). \tag{3.7}$$

We are now able to compute the evolution of the entropy.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} H\bigl(F_t^{N,\varepsilon}\bigr) &= \frac{1}{N}\int_{(\mathbb{R}^2)^N} \partial_t F_t^{N,\varepsilon}(x)\bigl(1 + \log F_t^{N,\varepsilon}(x)\bigr)\,\mathrm{d}x \\
&= \frac{\chi}{N^2}\sum_{i\neq j}\int_{(\mathbb{R}^2)^N} \operatorname{div}_{x_i}\bigl(F_t^{N,\varepsilon}(x) K_\varepsilon(x_i - x_j)\bigr)\bigl(1 + \log F_t^{N,\varepsilon}(x)\bigr)\,\mathrm{d}x \\
&\quad + \frac{1}{N}\int_{(\mathbb{R}^2)^N} \triangle_x F_t^{N,\varepsilon}(x)\bigl(1 + \log F_t^{N,\varepsilon}(x)\bigr)\,\mathrm{d}x.
\end{aligned}$$

Performing some integrations by parts, we get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} H\bigl(F_t^{N,\varepsilon}\bigr) &= -\frac{\chi}{N^2}\sum_{i\neq j}\int_{(\mathbb{R}^2)^N} K_\varepsilon(x_i - x_j)\cdot \nabla_{x_i} F_t^{N,\varepsilon}(x)\,\mathrm{d}x - I\bigl(F_t^{N,\varepsilon}\bigr) \\
&= \frac{\chi}{N^2}\sum_{i\neq j}\int_{(\mathbb{R}^2)^N} \operatorname{div} K_\varepsilon(x_i - x_j) F_t^{N,\varepsilon}(x)\,\mathrm{d}x - I\bigl(F_t^{N,\varepsilon}\bigr),
\end{aligned}$$

and (3.3) follows. Using that div $K_\varepsilon(x) = \frac{1-\alpha}{|x|^{\alpha+1}}\mathbb{1}_{\{|x|\geq\varepsilon\}} + \frac{2}{\varepsilon^{\alpha+1}}\mathbb{1}_{\{|x|<\varepsilon\}} \leq \frac{2}{|x|^{\alpha+1}}$ and the exchangeability of the particles, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}H\big(F_t^{N,\varepsilon}\big) &\leq \frac{2\chi}{N^2}\sum_{i\neq j}\int_{(\mathbb{R}^2)^N}\frac{F_t^{N,\varepsilon}(x)}{|x_i-x_j|^{\alpha+1}}\,\mathrm{d}x - I\big(F_t^{N,\varepsilon}\big)\\ &\leq 2\chi\int_{(\mathbb{R}^2)^N}\frac{F_t^{N,\varepsilon}(x)}{|x_1-x_2|^{\alpha+1}}\,\mathrm{d}x - I\big(F_t^{N,\varepsilon}\big).\end{aligned}$$

Since $\alpha\in(0,1)$, we can use Lemma 2.1 with $\gamma=\alpha+1$ and β such that $\frac{\alpha+1}{2}<\beta<1$, which gives

$$\int_{(\mathbb{R}^2)^N}\frac{F_t^{N,\varepsilon}(x)\,\mathrm{d}x}{|x_1-x_2|^{\alpha+1}} \leq C\big(I\big(F_{t2}^{N,\varepsilon}\big)^\beta+1\big),$$

where $F_{t2}^{N,\varepsilon}$ is the two-marginal of $F_t^{N,\varepsilon}$. By Lemma 2.2, we have $I(F_{t2}^{N,\varepsilon})\leq I(F_t^{N,\varepsilon})$. Using that $Cx^\beta\leq C'+\frac{x}{6\chi}$ for a constant C' sufficiently large, we thus get

$$\frac{\mathrm{d}}{\mathrm{d}t}H\big(F_t^{N,\varepsilon}\big)\leq C-\frac{2}{3}I\big(F_t^{N,\varepsilon}\big),$$

and thus

$$H\big(F_t^{N,\varepsilon}\big)+\frac{2}{3}\int_0^t I\big(F_s^{N,\varepsilon}\big)\,\mathrm{d}s\leq H\big(F_0^N\big)+Ct. \tag{3.8}$$

We now compute $M_1(F_t^{N,\varepsilon})$. We first observe that

$$M_1\big(F_t^{N,\varepsilon}\big)=\frac{1}{N}\int_{(\mathbb{R}^2)^N}\sum_{i=1}^N|x_i|F_t^{N,\varepsilon}(\mathrm{d}x)=\mathbb{E}\big[\big|X_t^{1,N,\varepsilon}\big|\big],$$

since the particles are exchangeable. We will need to control $\mathbb{E}[\sup_{[0,T]}|X_t^{1,N,\varepsilon}|]$ in the sequel. We have

$$\begin{aligned}\mathbb{E}\Big[\sup_{[0,T]}\big|X_t^{1,N,\varepsilon}\big|\Big] &\leq C\bigg(\mathbb{E}\big[\big|X_0^1\big|\big]+\mathbb{E}\Big[\sup_{[0,T]}\big|B_t^1\big|\Big]\\ &\qquad+\mathbb{E}\bigg[\sup_{t\in[0,T]}\bigg|\frac{1}{N}\sum_{j\neq 1}\int_0^t K_\varepsilon\big(X_s^{1,N,\varepsilon}-X_s^{j,N,\varepsilon}\big)\,\mathrm{d}s\bigg|\bigg]\bigg)\\ &\leq C\bigg(\mathbb{E}\big[\big|X_0^1\big|\big]+T+\frac{1}{N}\sum_{j\neq 1}\int_0^T\mathbb{E}\big[\big|K_\varepsilon\big(X_s^{1,N,\varepsilon}-X_s^{j,N,\varepsilon}\big)\big|\big]\,\mathrm{d}s\bigg)\\ &\leq C\bigg(\mathbb{E}\big[\big|X_0^1\big|\big]+T+\int_0^T\mathbb{E}\bigg[\frac{1}{|X_s^{1,N,\varepsilon}-X_s^{2,N,\varepsilon}|^\alpha}\bigg]\,\mathrm{d}s\bigg).\end{aligned} \tag{3.9}$$

Using Lemma 2.1 with $\gamma=\alpha$ and β such that $\frac{\alpha}{2}<\beta<1$ and recalling that $I(F_{t2}^{N,\varepsilon})\leq I(F_t^{N,\varepsilon})$, we get

$$\begin{aligned}M_1\big(F_t^{N,\varepsilon}\big)&\leq C\bigg(M_1\big(F_0^N\big)+T+\int_0^t I\big(F_t^{N,\varepsilon}\big)^\beta\,\mathrm{d}s\bigg)\\ &\leq C\big(M_1\big(F_0^N\big)+T\big)+\frac{1}{3}\int_0^t I\big(F_t^{N,\varepsilon}\big)\,\mathrm{d}s,\end{aligned} \tag{3.10}$$

where we used that $Cx^\beta \le C' + \frac{x}{3}$ for a constant C' sufficiently large. Summing (3.8) and (3.10), we thus find

$$H\big(F_t^{N,\varepsilon}\big) + M_1\big(F_t^{N,\varepsilon}\big) + \frac{1}{3}\int_0^t I\big(F_s^{N,\varepsilon}\big)\,\mathrm{d}s \le H\big(F_0^N\big) + Ct + C\big(1 + M_1\big(F_0^N\big)\big).$$

Since the quantities M_1 and I are positive, we immediately get $H(F_t^{N,\varepsilon}) \le C(1+t)$. Using Lemma 2.3, we have $H(F_t^{N,\varepsilon}) \ge -C - M_1(F_t^{N,\varepsilon})/2$, so that

$$M_1\big(F_t^{N,\varepsilon}\big) + \frac{1}{3}\int_0^t I\big(F_s^{N,\varepsilon}\big)\,\mathrm{d}s \le C(1+t) + M_1\big(F_t^{N,\varepsilon}\big)/2.$$

Using again the positivity of M_1 and I, we easily get (3.4). Coming back to (3.9), we finally observe that

$$\mathbb{E}\Big[\sup_{[0,T]}\big|X_t^{1,N,\varepsilon}\big|\Big] \le C\bigg(\mathbb{E}\big[\big|X_0^1\big|\big] + T + \int_0^T I\big(F_s^{N,\varepsilon}\big)\,\mathrm{d}s\bigg) \le C\big(1 + \mathbb{E}\big[\big|X_0^1\big|\big] + T\big),$$

which gives (3.5) and concludes the proof. □

We can now give the proof of existence and uniqueness for the particle system (1.3).

Proof of Theorem 1.6. Like in [20], the key point of the proof is to show that particles of the system (1.3) a.s. never collide. We divide the proof in three steps. The first step consists in showing that a.s. there are no collisions between particles for the system (3.2). In the second step, we deduce that the particles of the system (1.3) also never collide, which ensures global existence and uniqueness for (1.3). In the last step, we establish the estimates about the entropy, Fisher information and the first moment. We fix $N \ge 2$ and for all $\varepsilon \in (0,1)$, we consider $(X_t^{i,N,\varepsilon})_{i=1,\dots,N,t\ge 0}$ the unique solution to (3.2).

Step 1. Let $\tau_\varepsilon := \inf\{t \ge 0, \exists i \ne j, |X_t^{i,N,\varepsilon} - X_t^{j,N,\varepsilon}| \le \varepsilon\}$. The aim of this step is to prove that $\lim_{\varepsilon\to 0}\mathbb{P}[\tau_\varepsilon < T] = 0$ for all $T > 0$. We fix $T > 0$ and introduce

$$S_t^\varepsilon := \frac{1}{N^2}\sum_{i\ne j}\log\big|X_t^{i,N,\varepsilon} - X_t^{j,N,\varepsilon}\big|. \tag{3.11}$$

For any $A > 1$, we have

$$\begin{aligned}
\mathbb{P}[\tau_\varepsilon < T] &\le \mathbb{P}\Big[\inf_{[0,T]} S^\varepsilon_{t\wedge\tau_\varepsilon} \le S^\varepsilon_{\tau_\varepsilon}\Big] \\
&\le \mathbb{P}\big[\exists i, \exists t \in [0,T], \big|X_t^{i,N,\varepsilon}\big| > A\big] \\
&\quad + \mathbb{P}\Big[\forall i, \forall t \in [0,T], \big|X_t^{i,N,\varepsilon}\big| \le A, \inf_{[0,T]} S^\varepsilon_{t\wedge\tau_\varepsilon} \le S^\varepsilon_{\tau_\varepsilon}\Big] \\
&\le \frac{N\mathbb{E}[\sup_{[0,T]}|X_t^{1,N,\varepsilon}|]}{A} + \mathbb{P}\Big[\inf_{[0,T]} S^\varepsilon_{t\wedge\tau_\varepsilon} \le \frac{\log\varepsilon}{N^2} + \log 2A\Big] \\
&\le \frac{C(1+T)N}{A} + \mathbb{P}\Big[\inf_{[0,T]} S^\varepsilon_{t\wedge\tau_\varepsilon} \le \frac{\log\varepsilon}{N^2} + \log 2A\Big],
\end{aligned} \tag{3.12}$$

where we used (3.5). We thus want to compute $\mathbb{P}[\inf_{[0,T]} S^\varepsilon_{t\wedge\tau_\varepsilon} \le -M]$ for all (large) $M > 0$. Using Itô's formula, that $K_\varepsilon(x) = K(x)$ for any $|x| \ge \varepsilon$ (see (3.1)) and that $\Delta(\log|x|) = 0$ on $\{x \in \mathbb{R}^2, |x| > \varepsilon\}$, we have

$$\begin{aligned}
\log\big|X^{i,N,\varepsilon}_{t\wedge\tau_\varepsilon} - X^{j,N,\varepsilon}_{t\wedge\tau_\varepsilon}\big| &= \log\big|X_0^{i,N} - X_0^{j,N}\big| + M^{i,j,\varepsilon}_{t\wedge\tau_\varepsilon} \\
&\quad - \frac{\chi}{N}\int_0^{t\wedge\tau_\varepsilon}\bigg[\sum_{k\ne i,j}\big(K\big(X_s^{i,N,\varepsilon} - X_s^{k,N,\varepsilon}\big) - K\big(X_s^{j,N,\varepsilon} - X_s^{k,N,\varepsilon}\big)\big)
\end{aligned}$$

$$
\begin{aligned}
&\quad + 2K\big(X_s^{i,N,\varepsilon} - X_s^{j,N,\varepsilon}\big)\Big] \cdot \frac{X_s^{i,N,\varepsilon} - X_s^{j,N,\varepsilon}}{|X_s^{i,N,\varepsilon} - X_s^{j,N,\varepsilon}|^2}\,\mathrm{d}s \\
&=: \log\big|X_0^{i,N} - X_0^{j,N}\big| + M_{t\wedge\tau_\varepsilon}^{i,j,\varepsilon} + R_{t\wedge\tau_\varepsilon}^{i,j,\varepsilon},
\end{aligned}
$$

where $M_t^{i,j,\varepsilon}$ is a martingale. Setting $S_0 := \frac{1}{N^2}\sum_{i\neq j}\log|X_0^{i,N} - X_0^{j,N}|$, $M_t^\varepsilon := \frac{1}{N^2}\sum_{i\neq j} M_{t\wedge\tau_\varepsilon}^{i,j,\varepsilon}$ and $R_t^\varepsilon := \frac{1}{N^2}\sum_{i\neq j} R_{t\wedge\tau_\varepsilon}^{i,j,\varepsilon}$, we thus have

$$
S_{t\wedge\tau_\varepsilon}^\varepsilon = S_0 + M_t^\varepsilon + R_t^\varepsilon,
$$

so that

$$
\begin{aligned}
\mathbb{P}\Big(\inf_{[0,T]} S_{t\wedge\tau_\varepsilon}^\varepsilon \le -M\Big) &\le \mathbb{P}(S_0 \le -M/3) + \mathbb{P}\Big(\inf_{[0,T]} M_t^\varepsilon \le -M/3\Big) \\
&\quad + \mathbb{P}\Big(\inf_{[0,T]} R_t^\varepsilon \le -M/3\Big).
\end{aligned}
\tag{3.13}
$$

Using first Lemma 2.5 and that $|K(x)| = |x|^{-\alpha}$, and then exchangeability, we clearly have for some constant C independent of N and ε,

$$
\begin{aligned}
\mathbb{E}\Big[\sup_{[0,T]}\big|R_t^\varepsilon\big|\Big] &\le \frac{C}{\chi N^3}\sum_{i\neq j}\sum_{k\neq i,j}\bigg(\mathbb{E}\bigg[\frac{1}{|X_s^{i,N,\varepsilon} - X_s^{k,N,\varepsilon}|^{\alpha+1}}\bigg] + \mathbb{E}\bigg[\frac{1}{|X_s^{j,N,\varepsilon} - X_s^{k,N,\varepsilon}|^{\alpha+1}}\bigg] \\
&\qquad + \mathbb{E}\bigg[\frac{1}{|X_s^{i,N,\varepsilon} - X_s^{j,N,\varepsilon}|^{\alpha+1}}\bigg]\bigg)\,\mathrm{d}s \\
&\le C\chi\int_0^T \mathbb{E}\bigg[\frac{1}{|X_s^{1,N,\varepsilon} - X_s^{2,N,\varepsilon}|^{\alpha+1}}\bigg]\,\mathrm{d}s \\
&\le C\chi\int_0^T \big(1 + I\big(F_{s2}^{N,\varepsilon}\big)\big)\,\mathrm{d}s \\
&\le C(1+T),
\end{aligned}
\tag{3.14}
$$

where we used Lemma 2.1, the fact that $I(F_{t2}^{N,\varepsilon}) \le I(F_t^{N,\varepsilon})$ by Lemma 2.2, and finally Proposition 3.1. We thus get

$$
\mathbb{P}\Big(\inf_{[0,T]} R_t^\varepsilon \le -M/3\Big) \le \mathbb{P}\Big(\sup_{[0,T]}\big|R_t^\varepsilon\big| \ge M/3\Big) \le \frac{C(1+T)}{M}.
\tag{3.15}
$$

We now want to compute $\mathbb{P}(\inf_{[0,T]} M_t^\varepsilon \le -M/3)$. Using that $\log|x| \le |x|$, we have

$$
S_t^\varepsilon \le \frac{1}{N^2}\sum_{i\neq j}\big(\big|X_t^{i,N,\varepsilon}\big| + \big|X_t^{j,N,\varepsilon}\big|\big) \le \frac{2}{N}\sum_i\big|X_t^{i,N,\varepsilon}\big|.
$$

Consequently,

$$
\begin{aligned}
M_t^\varepsilon &\le S_{t\wedge\tau_\varepsilon}^\varepsilon + \sup_{s\in[0,T]}\big|R_s^\varepsilon\big| - S_0 \\
&\le \frac{2}{N}\sum_i \sup_{s\in[0,T]}\big|X_s^{i,N,\varepsilon}\big| + \sup_{s\in[0,T]}\big|R_s^\varepsilon\big| - S_0 =: K^\varepsilon - S_0 =: Z^\varepsilon.
\end{aligned}
$$

We have

$$
\mathbb{P}\Big(\inf_{[0,T]} M_t^\varepsilon \le -M/3\Big) \le \mathbb{P}\big(Z^\varepsilon \ge \sqrt{M/3}\big) + \mathbb{P}\Big(\inf_{[0,T]} M_t^\varepsilon \le -M/3, Z^\varepsilon < \sqrt{M/3}\Big).
\tag{3.16}
$$

Since $(M_t^\varepsilon)_{t\geq 0}$ is a continuous local martingale, there exists a Brownian motion β such that $M_t^\varepsilon = \beta_{\langle M^\varepsilon\rangle_t}$. For $x\in\mathbb{R}$, we set $\sigma_x := \inf\{t\geq 0, \beta_t = x\}$. Using that $\sup_{[0,T]} M_t^\varepsilon \leq Z^\varepsilon$ a.s.,

$$\begin{aligned}\mathbb{P}\Big(\inf_{[0,T]} M_t^\varepsilon \leq -M/3, Z^\varepsilon < \sqrt{M/3}\Big) &\leq \mathbb{P}\Big(\inf_{[0,T]} M_t^\varepsilon \leq -M/3, \sup_{[0,T]} M_t^\varepsilon < \sqrt{M/3}\Big)\\ &\leq \mathbb{P}(\sigma_{-M/3} \leq \sigma_{\sqrt{M/3}})\\ &= \frac{\sqrt{M/3}}{M/3+\sqrt{M/3}} \leq \sqrt{\frac{3}{M}},\end{aligned} \tag{3.17}$$

by classical results on the Brownian motion. Using (3.5) and (3.14), we get that $\mathbb{E}[K^\varepsilon]\leq C(1+T)$ where C does not depend on ε. So using the Markov inequality,

$$\begin{aligned}\mathbb{P}\big(Z^\varepsilon \geq \sqrt{M/3}\big) &= \mathbb{P}\big(K^\varepsilon - S_0 \geq \sqrt{M/3}\big)\\ &\leq \mathbb{P}\big(K^\varepsilon \geq \sqrt{M/12}\big) + \mathbb{P}(-S_0 \geq \sqrt{M/12})\\ &\leq \frac{C(1+T)}{\sqrt{M}} + \mathbb{P}(-S_0\geq \sqrt{M/12}).\end{aligned} \tag{3.18}$$

Gathering (3.16), (3.17) and (3.18), we find that

$$\mathbb{P}\Big(\inf_{[0,T]} M_t^\varepsilon \leq -M/3\Big) \leq \frac{C(1+T)}{\sqrt{M}} + \mathbb{P}(-S_0 \geq \sqrt{M/12}). \tag{3.19}$$

Coming back to (3.12) and (3.13), using (3.15) and (3.19) with $M = -\frac{\log\varepsilon}{N^2} - \log 2A$, we finally get that for any $\varepsilon\in(0,1)$, any $A>1$ such that $\frac{\log\varepsilon}{N^2} + \log 2A < 0$,

$$\begin{aligned}\mathbb{P}(\tau_\varepsilon < T) \leq{}& \frac{C(1+T)N}{A} + \mathbb{P}\bigg(S_0 \leq \Big(\frac{\log\varepsilon}{N^2} + \log 2A\Big)\Big/3\bigg)\\ &+ \frac{C(1+T)}{-(\log\varepsilon/N^2) - \log 2A} + \frac{C(1+T)}{\sqrt{(-\log\varepsilon/N^2)-\log 2A}}\\ &+ \mathbb{P}\bigg(S_0 \leq -\sqrt{\Big(-\frac{\log\varepsilon}{N^2} - \log 2A\Big)\Big/12}\bigg).\end{aligned}$$

Observe finally that $S_0 > -\infty$ a.s. (because F_0^N has a density since $H(F_0^N)<\infty$) so that $\lim_{M\to+\infty}\mathbb{P}(S_0 < -M) = 0$. Letting $\varepsilon\to 0$ in the above formula, we get that for all $A>1$,

$$\limsup_{\varepsilon} \mathbb{P}(\tau_\varepsilon < T) \leq \frac{C(1+T)N}{A}.$$

It only remains to make A go to ∞ to conclude this step.

Step 2. Since K is Lipschitz-continuous outside 0, classical arguments give existence and uniqueness of a solution to (1.3) until the explosion time $\tau = \inf\{t\geq 0, \exists i\neq j, X_t^{i,N} = X_t^{j,N}\}$. We can observe that since $K_\varepsilon(x) = K(x)$ for any $|x|\geq\varepsilon$, $(X^{i,N,\varepsilon})_{i=1,\dots,N}$ is solution to (1.3) on $[0,\tau_\varepsilon]$ so that for any $i=1,\dots,N$, $X_t^{i,N} = X_t^{i,N,\varepsilon}$ on $[0,\tau_\varepsilon]$. We thus have $\tau_\varepsilon < \tau$ for any $\varepsilon\in(0,1)$ a.s. so that, using Step 1, we have for any $T>0$

$$\mathbb{P}(\tau < T) \leq \mathbb{P}(\tau_\varepsilon < T) \overset{\varepsilon\to 0}{\longrightarrow} 0.$$

Thus $\tau=\infty$ a.s. which proves global existence and uniqueness for (1.3).

Step 3. Using that the functionals H, I and M_1 are lower semi-continuous and Proposition 3.1, we have

$$
\begin{aligned}
H\big(F_t^N\big) &\le \liminf_\varepsilon H\big(F_t^{N,\varepsilon}\big) \le C(1+t),\\
\int_0^t I\big(F_s^N\big)\,\mathrm{d}s &\le \liminf_\varepsilon \int_0^t I\big(F_s^{N,\varepsilon}\big)\,\mathrm{d}s \le C(1+t)
\end{aligned}
\tag{3.20}
$$

and

$$
M_1\big(F_t^N\big) \le \liminf_\varepsilon M_1\big(F_t^{N,\varepsilon}\big) \le C(1+t).
$$

Using Fatou's lemma and (3.5), we get

$$
\mathbb{E}\Big[\sup_{[0,T]}\big|X_t^{1,N}\big|\Big] \le \liminf_\varepsilon \mathbb{E}\Big[\sup_{[0,T]}\big|X_t^{1,N,\varepsilon}\big|\Big] \le C(1+T),
$$

and (1.13) is proven. It remains to prove (1.14). Using again that the functionals H and I are lower semi-continuous and using (3.3), we get

$$
\begin{aligned}
H\big(F_t^N\big) + \int_0^t I\big(F_s^N\big)\,\mathrm{d}s &\le \liminf_\varepsilon \Big[H\big(F_t^{N,\varepsilon}\big) + \int_0^t I\big(F_s^{N,\varepsilon}\big)\,\mathrm{d}s\Big]\\
&\le H\big(F_0^N\big) + \liminf_\varepsilon \frac{\chi}{N^2}\int_0^t \sum_{i\neq j}\mathbb{E}\big[\operatorname{div} K_\varepsilon\big(X_s^{i,N,\varepsilon} - X_s^{j,N,\varepsilon}\big)\big]\,\mathrm{d}s.
\end{aligned}
$$

By exchangeability, it suffices to prove that, as $\varepsilon \to 0$,

$$
D_\varepsilon := \int_0^t \mathbb{E}\big[\operatorname{div} K_\varepsilon\big(X_s^{1,N,\varepsilon} - X_s^{2,N,\varepsilon}\big)\big]\,\mathrm{d}s \to \int_0^t \mathbb{E}\big[\operatorname{div} K\big(X_s^{1,N} - X_s^{2,N}\big)\big]\,\mathrm{d}s =: D.
$$

By Step 2, we have $X_s^{i,N} = X_s^{i,N,\varepsilon}$ for any i and $s \le \tau_\varepsilon$ and thus recalling that $K_\varepsilon(x) = K(x)$ for any $|x| \ge \varepsilon$, we get that a.s. for any $s < \tau_\varepsilon$

$$
\operatorname{div} K_\varepsilon\big(X_s^{1,N,\varepsilon} - X_s^{2,N,\varepsilon}\big) = \operatorname{div} K\big(X_s^{1,N,\varepsilon} - X_s^{2,N,\varepsilon}\big) = \operatorname{div} K\big(X_s^{1,N} - X_s^{2,N}\big).
$$

So using that $\operatorname{div} K(x) \le 2|x|^{-\alpha-1}$ and $\operatorname{div} K_\varepsilon(x) \le 2|x|^{-\alpha-1}$, we get

$$
|D - D_\varepsilon| \le C\int_0^t \mathbb{E}\Big[\mathbb{1}_{\{\tau_\varepsilon < s\}}\Big(\frac{1}{|X_s^{1,N,\varepsilon} - X_s^{2,N,\varepsilon}|^{\alpha+1}} + \frac{1}{|X_s^{1,N} - X_s^{2,N}|^{\alpha+1}}\Big)\Big]\,\mathrm{d}s.
$$

Let $a \in (0, \frac{1-\alpha}{1+\alpha})$ (in order to have $(1+a)(\alpha+1) < 2$). Using first the Hölder inequality with $p = 1+a$ and q such that $1/p + 1/q = 1$, and then Lemma 2.1 with $\beta = 1$, we get

$$
\begin{aligned}
|D - D_\varepsilon| &\le C\int_0^t \mathbb{P}(\tau_\varepsilon < s)^{1/q}\mathbb{E}\Big[\Big(\frac{1}{|X_s^{1,N,\varepsilon} - X_s^{2,N,\varepsilon}|^{(\alpha+1)(1+a)}}\\
&\qquad + \frac{1}{|X_s^{1,N} - X_s^{2,N}|^{(\alpha+1)(1+a)}}\Big)\Big]^{1/p}\,\mathrm{d}s\\
&\le C\mathbb{P}(\tau_\varepsilon < t)^{1/q}\int_0^t \big[1 + I\big(F_s^{N,\varepsilon}\big) + I\big(F_s^N\big)\big]\,\mathrm{d}s\\
&\le C(1+t)\mathbb{P}(\tau_\varepsilon < t)^{1/q},
\end{aligned}
$$

by (3.4) and (3.20). This tends to 0 as $\varepsilon \to 0$ by Step 1 and concludes the proof. □

4. Convergence of the particle system

We start this section with a tightness result for the particle system (1.3).

Lemma 4.1. *Let $\alpha \in (0,1)$. Assume (1.12). For each $N \geq 2$, let $(X_t^{i,N})_{i=1,\dots,N}$ be the unique solution to (1.3) and $Q^N := \frac{1}{N}\sum_{i=1}^N \delta_{(X_t^{i,N})_{t\geq 0}}$.*

(i) *The family $\{\mathcal{L}((X_t^{1,N})_{t\geq 0}), N \geq 2\}$ is tight in $\mathbf{P}(C([0,\infty),\mathbb{R}^2))$.*
(ii) *The family $\{\mathcal{L}(Q^N), N \geq 2\}$ is tight in $\mathbf{P}(\mathbf{P}(C([0,\infty),\mathbb{R}^2)))$.*

Proof. Since the system is exchangeable, we deduce (ii) from (i) by [19], Proposition 2.2. Let's prove (i). Let thus $\eta > 0$ and $T > 0$ be fixed. To prove the tightness of $\{\mathcal{L}((X_t^{1,N})_{t\geq 0}), N \geq 2\}$ in $\mathbf{P}(C([0,\infty),\mathbb{R}^2))$, we have to find a compact subset $\mathcal{K}_{\eta,T}$ of $C([0,T],\mathbb{R}^2)$ such that $\sup_N \mathbb{P}[(X_t^{1,N})_{t\in[0,T]} \notin \mathcal{K}_{\eta,T}] \leq \eta$. We first set $Z_T := \sup_{0<s<t<T} \sqrt{2}|B_t^1 - B_s^1|/|t-s|^{1/3}$. This random variable is a.s. finite since the paths of a Brownian motion are a.s. Hölder continuous with index $1/3$. We can also notice that the law of Z_T does not depend on N. Using the Hölder inequality with $p = 3$ and $q = 3/2$, we get that for all $0 < s < t < T$,

$$\begin{aligned}
\left|\frac{\chi}{N}\sum_{j=2}^N \int_s^t K\big(X_u^{1,N} - X_u^{j,N}\big)\,\mathrm{d}u\right| &\leq \frac{\chi}{N}\sum_{j=2}^N \int_s^t \frac{\mathrm{d}u}{|X_u^{1,N} - X_u^{j,N}|^\alpha} \\
&\leq \frac{\chi}{N}(t-s)^{1/3}\sum_{j=2}^N \left(\int_0^T \frac{\mathrm{d}u}{|X_u^{1,N} - X_u^{j,N}|^{3\alpha/2}}\right)^{2/3} \\
&\leq (t-s)^{1/3}\left(\chi + \frac{\chi}{N}\sum_{j=2}^N \int_0^T \frac{\mathrm{d}u}{|X_u^{1,N} - X_u^{j,N}|^{3\alpha/2}}\right) \\
&=: (t-s)^{1/3} U_T^N.
\end{aligned}$$

Using Lemma 2.1 with $\gamma = 3\alpha/2$ and $\beta = 1$, the exchangeability of the system of particles, and denoting by F_{u2}^N the two-marginal of F_u^N, we have

$$\begin{aligned}
\mathbb{E}\big(U_T^N\big) &= \chi + \chi\frac{N-1}{N}\int_0^T \mathbb{E}\left(\frac{1}{|X_u^{1,N} - X_u^{2,N}|^{3\alpha/2}}\right)\mathrm{d}u \leq \chi + C\int_0^T \big(1 + I\big(F_{u2}^N\big)\big)\,\mathrm{d}u \\
&\leq \chi + C\int_0^T \big(1 + I\big(F_u^N\big)\big)\,\mathrm{d}u \\
&\leq C(1+T),
\end{aligned}$$

where we used that $I(F_{t2}^N) \leq I(F_t^N)$ by Lemma 2.2 and Theorem 1.6. We thus have $\sup_{N\geq 2}\mathbb{E}(U_T^N) < \infty$. Furthermore, Z_T is also a.s. finite so that we can find $R > 0$ such that $\mathbb{P}(Z_T + U_T^N > R) \leq \eta/2$ for all $N \geq 2$. Recalling (1.12), we can also find $a > 0$ such that $\sup_{N\geq 2}\mathbb{P}(X_0^{1,N} > a) \leq \eta/2$. We now consider

$$\mathcal{K}_{\eta,T} := \big\{f \in C\big([0,T],\mathbb{R}^2\big), \big|f(0)\big| \leq a, \big|f(t) - f(s)\big| \leq R(t-s)^{1/3}\ \forall 0 < s < t < T\big\},$$

which is a compact subset of $C([0,T],\mathbb{R}^2)$ by Ascoli's theorem. Observing that for all $0 < s < t < T$, $|X_t^{1,N} - X_s^{1,N}| \leq (Z_T + U_t^N)(t-s)^{1/3}$, we get

$$\mathbb{P}\big[\big(X_t^{1,N}\big)_{t\in[0,T]} \notin \mathcal{K}_{\eta,T}\big] \leq \mathbb{P}\big(\big|X_0^{1,N}\big| > a\big) + \mathbb{P}\big(Z_T + U_T^N > R\big) \leq \eta,$$

which concludes the proof. □

We define $\mathcal{S}$ as the set of all probability measures $f \in \mathbf{P}(C([0,\infty),\mathbb{R}^2))$ such that f is the law of $(X_t)_{t\geq 0}$ solution to (1.4) satisfying (setting $f_t = \mathcal{L}(X_t)$)

$$\forall T > 0, \quad \int_0^T I(f_s)\,\mathrm{d}s < \infty \quad \text{and} \quad \sup_{[0,T]} M_1(f_s) < \infty. \tag{4.1}$$

Observe that by Lemma 2.4, (4.1) implies (1.7). The condition $p > \frac{2}{1-\alpha}$ in (1.7) is asked in order to use (5.1) with $\gamma = -(\alpha+1)$ (see the beginning of Section 5).

Proposition 4.2. *Let $\alpha \in (0,1)$ and assume (1.12). For each $N \geq 2$, let $(X_0^{i,N})_{i=1,\dots,N}$ be F_0^N-distributed and consider the solution $(X_t^{i,N})_{i=1,\dots,N,t\geq 0}$ to (1.3). Assume that there is a subsequence of $\mathcal{Q}^N := \frac{1}{N}\sum_{i=1}^N \delta_{(X_t^{i,N})_{t\geq 0}}$ going in law to some $\mathbf{P}(C([0,\infty),\mathbb{R}^2))$-valued random variable $\mathcal{Q}$. Then $\mathcal{Q}$ a.s. belongs to $\mathcal{S}$.*

Proof. We consider a (not relabelled) subsequence of $\mathcal{Q}^N$ going in law to some $\mathcal{Q}$ and we introduce the identity map $\psi : C([0,\infty);\mathbb{R}^2) \to C([0,\infty);\mathbb{R}^2)$. Using the arguments of [8], Proposition 6.1, we have to prove that $\mathcal{Q}$ a.s. satisfies

(a) $\mathcal{Q} \circ (\psi(0))^{-1} = f_0$;
(b) setting $\mathcal{Q}_t = \mathcal{Q} \circ (\psi(t))^{-1}$, $(\mathcal{Q}_t)_{t\geq 0}$ satisfies (4.1);
(c) for all $0 < t_0 < \cdots < t_k < s < t$, $\varphi_1,\dots,\varphi_k \in C_b(\mathbb{R}^2)$, $\varphi \in C_b^2(\mathbb{R}^2)$, $\mathcal{F}(\mathcal{Q}) = 0$ where, for $f \in \mathbf{P}(C([0,\infty),\mathbb{R}^2))$,

$$\begin{aligned}\mathcal{F}(f) := \iint & f(\mathrm{d}\gamma) f(\mathrm{d}\tilde{\gamma}) \varphi_1(\gamma_{t_1}) \cdots \varphi_k(\gamma_{t_k}) \\ & \times \left[\varphi(\gamma_t) - \varphi(\gamma_s) + \chi \int_s^t \nabla_x \varphi(\gamma_u) \cdot K(\gamma_u - \tilde{\gamma}_u)\,\mathrm{d}u - \int_s^t \triangle_x \varphi(\gamma_u)\,\mathrm{d}u\right].\end{aligned}$$

For simplicity, we split the proof in many steps.

Step 1. By assumption (1.12), we have that F_0^N is f_0-chaotic which implies that $\mathcal{Q}_0^N = \mathcal{Q}^N \circ \psi(0)^{-1}$ goes weakly to f_0 in law, and, since f_0 is deterministic, also in probability. Hence $\mathcal{Q}_0 = f_0$ a.s. and thus $f \circ \psi(0)^{-1} = f_0$. Thus $\mathcal{Q}$ a.s. satisfies (a).

Step 2. Since $\frac{1}{N}\sum_{i=1}^N \delta_{X_t^{i,N}}$ goes weakly to $\mathcal{Q}_t$, for all $j \geq 1$, F_{tj}^N goes weakly to π_{tj}, where $\pi_t := \mathcal{L}(\mathcal{Q}_t)$ and $\pi_{tj} := \int_{\mathbf{P}(\mathbb{R}^2)} f^{\otimes j} \pi_t(\mathrm{d}f)$. We can thus apply [11], Theorem 5.7 (and then Fatou's lemma) to get

$$\begin{aligned}\mathbb{E}\left[\int_0^T I(\mathcal{Q}_s)\,\mathrm{d}s\right] &= \int_0^T \mathbb{E}\big[I(\mathcal{Q}_s)\big]\,\mathrm{d}s \leq \int_0^T \liminf_N I\big(F_s^N\big)\,\mathrm{d}s \\ &\leq \liminf_N \int_0^T I\big(F_s^N\big)\,\mathrm{d}s,\end{aligned}$$

which is finite by Theorem 1.6. We conclude that $\int_0^T I(\mathcal{Q}_s)\,\mathrm{d}s < \infty$ a.s. We also have, using Fatou's lemma and the exchangeability of the particles,

$$\begin{aligned}\mathbb{E}\Big[\sup_{[0,T]} M_1(\mathcal{Q}_t)\Big] &\leq \mathbb{E}\Big[\liminf_N \sup_{[0,T]} M_1\big(\mathcal{Q}_t^N\big)\Big] \\ &\leq \liminf_N \mathbb{E}\left[\sup_{[0,T]} \frac{1}{N}\sum_{i=1}^N \big|X_t^{i,N}\big|\right] \\ &\leq \liminf_N \mathbb{E}\Big[\sup_{[0,T]} \big|X_t^{1,N}\big|\Big] \leq C(1+T),\end{aligned}$$

by (1.13), so that $\sup_{[0,T]} M_1(\mathcal{Q}_t) < \infty$ a.s. Consequently, $\mathcal{Q}$ a.s. satisfies (b).

Step 3.1. Using Itô's formula

$$O_t^i := \varphi\big(X_s^{i,N}\big) + \frac{\chi}{N}\sum_{j\neq i}\int_0^t \nabla_x\varphi\big(X_s^{i,N}\big)\cdot K\big(X_s^{i,N}-X_s^{j,N}\big)\,\mathrm{d}s - \int_0^t \triangle_x\varphi\big(X_s^{i,N}\big)\,\mathrm{d}s$$

$$= \varphi\big(X_0^{i,N}\big) + \sqrt{2}\int_0^t \nabla_x\varphi\big(X_s^{i,N}\big)\cdot \mathrm{d}B_s^i.$$

But, using the last equality, we see that

$$\mathcal{F}\big(\mathcal{Q}^N\big) = \frac{1}{N}\sum_{i=1}^N \varphi_1\big(X_{t_1}^{i,N}\big)\cdots\varphi_k\big(X_{t_k}^{i,N}\big)\big[O_t^i - O_s^i\big]$$

$$= \frac{\sqrt{2}}{N}\sum_{i=1}^N \varphi_1\big(X_{t_1}^{i,N}\big)\cdots\varphi_k\big(X_{t_k}^{i,N}\big)\int_s^t \nabla_x\varphi\big(X_u^{i,N}\big)\cdot \mathrm{d}B_u^i.$$

From there, and thanks to the independence of the Brownian motions we conclude that (recall that the functions $\varphi_1,\dots,\varphi_k,\nabla_x\varphi$ are bounded)

$$\mathbb{E}\big[\big(\mathcal{F}\big(\mathcal{Q}^N\big)\big)^2\big] \le \frac{C}{N}.$$

Step 3.2. We also introduce the regularized version of $\mathcal{F}$. For $\varepsilon\in(0,1)$, we define $\mathcal{F}_\varepsilon$ replacing K by K_ε defined by (3.1). Since $f\mapsto\mathcal{F}_\varepsilon(f)$ is continuous and bounded from $\mathbf{P}(C([0,\infty);\mathbb{R}^2))$ to $\mathbb{R}$ and since $\mathcal{Q}^N$ goes in law to $\mathcal{Q}$, we deduce that for any $\varepsilon\in(0,1)$,

$$\mathbb{E}\big[\big|\mathcal{F}_\varepsilon(\mathcal{Q})\big|\big] = \lim_N \mathbb{E}\big[\big|\mathcal{F}_\varepsilon\big(\mathcal{Q}^N\big)\big|\big].$$

Step 3.3. Using that all the functions and their derivatives involved in $\mathcal{F}$ are bounded and that $|K_\varepsilon(x)-K(x)|\le |x|^{-\alpha}\mathbb{1}_{0\le|x|\le\varepsilon}$, we get

$$\big|\mathcal{F}(f)-\mathcal{F}_\varepsilon(f)\big| \le \chi C\iiint_0^t \big|\gamma(u)-\tilde{\gamma}(u)\big|^{-\alpha}\mathbb{1}_{0<|\gamma(u)-\tilde{\gamma}(u)|<\varepsilon}\,\mathrm{d}u f(\mathrm{d}\gamma)f(\mathrm{d}\tilde{\gamma})$$

$$\le C\varepsilon^{3/2-\alpha}\iiint_0^t \big|\gamma(u)-\tilde{\gamma}(u)\big|^{-3/2}\mathbb{1}_{\gamma(u)\neq\tilde{\gamma}(u)}\,\mathrm{d}u f(\mathrm{d}\gamma)f(\mathrm{d}\tilde{\gamma}).$$

Thus,

$$\big|\mathcal{F}\big(\mathcal{Q}^N\big)-\mathcal{F}_\varepsilon\big(\mathcal{Q}^N\big)\big| \le \frac{C\varepsilon^{3/2-\alpha}}{N^2}\sum_{i\neq j}\int_0^t \big|X_u^{i,N}-X_u^{j,N}\big|^{-3/2}\,\mathrm{d}u,$$

and by exchangeability

$$\mathbb{E}\big[\big|\mathcal{F}\big(\mathcal{Q}^N\big)-\mathcal{F}_\varepsilon\big(\mathcal{Q}^N\big)\big|\big] \le C\varepsilon^{3/2-\alpha}\int_0^t \mathbb{E}\big[\big|X_u^{1,N}-X_u^{2,N}\big|^{-3/2}\big]\,\mathrm{d}u.$$

Using Lemma 2.1 with $\gamma=3/2$ and $\beta=1$ and denoting by F_{u2}^N the two-marginal of F_u^N, we have

$$\mathbb{E}\big[\big|\mathcal{F}\big(\mathcal{Q}^N\big)-\mathcal{F}_\varepsilon\big(\mathcal{Q}^N\big)\big|\big] \le C\varepsilon^{3/2-\alpha}\int_0^t I\big(F_{u2}^N\big)\,\mathrm{d}u.$$

Using that $I(F_{t2}^N)\le I(F_t^N)$ by Lemma 2.2 and Theorem 1.6 we conclude that

$$\mathbb{E}\big[\big|\mathcal{F}\big(\mathcal{Q}^N\big)-\mathcal{F}_\varepsilon\big(\mathcal{Q}^N\big)\big|\big] \le C\varepsilon^{3/2-\alpha}.$$

Step 3.4. Now we see that

$$\left|\mathcal{F}(\mathcal{Q})-\mathcal{F}_\varepsilon(\mathcal{Q})\right|\le C\varepsilon^{3/2-\alpha}\int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}|x-y|^{-3/2}\mathcal{Q}_s(\mathrm{d}x)\mathcal{Q}_s(\mathrm{d}y)\,\mathrm{d}s.$$

Step 2 says that (4.1) holds true for $\mathcal{Q}_s$, then thanks to Lemma 2.4 we get that a.s., $\nabla_x\mathcal{Q}_s\in L^{2q/(3q-2)}(0,T;L^q(\mathbb{R}^2))$ for all $q\in[1,2)$. Then using [8], Lemma 3.5, for $\gamma=3/2$ we deduce that a.s.

$$\lim_{\varepsilon\to 0}\left|\mathcal{F}(\mathcal{Q})-\mathcal{F}_\varepsilon(\mathcal{Q})\right|=0.$$

Step 3.5. Using Steps 3.1, 3.2 and 3.3, we finally observe, using the same arguments as in [8], Proposition 6.1, Step 4.5, that

$$\mathbb{E}\left[\left|\mathcal{F}(\mathcal{Q})\right|\wedge 1\right]\le C\varepsilon^{3/2-\alpha}+\mathbb{E}\left[\left|\mathcal{F}(\mathcal{Q})-\mathcal{F}_\varepsilon(\mathcal{Q})\right|\wedge 1\right],$$

so that $\mathcal{F}(\mathcal{Q})=0$ a.s. by Step 3.4 thanks to dominated convergence and $\mathcal{Q}$ a.s. satisfies (c) which concludes the proof. □

5. Well-posedness and propagation of chaos

We start this section with the proof of existence and uniqueness for the nonlinear S.D.E. (1.4). We will use that for $\gamma\in(-2,0)$, for $p\in(2/(2+\gamma),\infty]$ and for any $h\in\mathbf{P}(\mathbb{R}^2)\cap L^p(\mathbb{R}^2)$,

$$\begin{aligned}\sup_{v\in\mathbb{R}^2}\int_{\mathbb{R}^2}h(v_*)|v-v_*|^\gamma\,\mathrm{d}v_* &\le \sup_{v\in\mathbb{R}^2}\int_{|v_*-v|<1}h(v_*)|v-v_*|^\gamma\,\mathrm{d}v_*\\ &\quad+\sup_{v\in\mathbb{R}^2}\int_{|v_*-v|\ge 1}h(v_*)\,\mathrm{d}v_*\\ &\le C_{\gamma,p}\|h\|_{L^p(\mathbb{R}^2)}+1,\end{aligned}\tag{5.1}$$

where

$$C_{\gamma,p}=\left[\int_{|v_*|\le 1}|v_*|^{\gamma p/(p-1)}\,\mathrm{d}v_*\right]^{(p-1)/p}<\infty,$$

since by assumption $\gamma p/(p-1)>-2$.

Proof of Theorem 1.7. The existence in law follows from Proposition 4.2 and Lemma 4.1 (see the comment after (4.1)). We now prove pathwise uniqueness which will also imply the strong existence. To this aim, we consider $(X_t)_{t\ge 0}$ and $(Y_t)_{t\ge 0}$ two solutions of (1.4) driven by the same Brownian motion and with same initial condition such that, setting $f_s:=\mathcal{L}(X_s)$ and $g_s:=\mathcal{L}(Y_s)$, $(f_t)_{t\ge 0}$ and $(g_t)_{t\ge 0}$ are in $L^\infty_{\mathrm{loc}}([0,\infty),\mathbf{P}_1(\mathbb{R}^2))\cap L^1_{\mathrm{loc}}([0,\infty);L^p(\mathbb{R}^2))$ for some $p>\frac{2}{1-\alpha}$. For any $s>0$, we consider the probability measure R_s on $\mathbb{R}^2\times\mathbb{R}^2$ with first (resp. second) marginal equal to f_s (resp. g_s) such that

$$\mathcal{W}_1(f_s,g_s)=\int_{\mathbb{R}^2\times\mathbb{R}^2}|x-y|R_s(\mathrm{d}x,\mathrm{d}y).$$

We have

$$\begin{aligned}X_t-Y_t&=-\chi\left(\int_0^t\int_{\mathbb{R}^2}K(X_s-x)f_s(\mathrm{d}x)\,\mathrm{d}s-\int_0^t\int_{\mathbb{R}^2}K(Y_s-y)g_s(\mathrm{d}y)\,\mathrm{d}s\right)\\ &=-\chi\int_0^t\int_{\mathbb{R}^2\times\mathbb{R}^2}\left[K(X_s-x)-K(Y_s-y)\right]R_s(\mathrm{d}x,\mathrm{d}y).\end{aligned}$$

Using Lemma 2.5 and recalling that $\mathcal{L}(X_t) = f_t$, $\mathcal{L}(Y_t) = g_t$, and that R_t has marginals f_t and g_t, this gives

$$\begin{aligned}\mathbb{E}\Big[\sup_{[0,T]}|X_t - Y_t|\Big] &\le C_\alpha\chi\int_0^T\int_{\mathbb{R}^2\times\mathbb{R}^2}\mathbb{E}\Big[\big(|X_s-Y_s|+|x-y|\big)\Big(\frac{1}{|X_s-x|^{\alpha+1}}+\frac{1}{|Y_s-y|^{\alpha+1}}\Big)\Big]R_s(\mathrm{d}x,\mathrm{d}y)\,\mathrm{d}s\\ &\le C_\alpha\chi\int_0^T\mathbb{E}\Big[|X_s-Y_s|\Big(\int_{\mathbb{R}^2}\frac{1}{|X_s-x|^{\alpha+1}}f_s(\mathrm{d}x)\\ &\quad+\int_{\mathbb{R}^2}\frac{1}{|Y_s-y|^{\alpha+1}}g_s(\mathrm{d}y)\Big)\Big]\,\mathrm{d}s\\ &\quad+C_\alpha\chi\int_0^T\int_{\mathbb{R}^2\times\mathbb{R}^2}|x-y|\mathbb{E}\Big[\frac{1}{|X_s-x|^{\alpha+1}}\\ &\quad+\frac{1}{|Y_s-y|^{\alpha+1}}\Big]R_s(\mathrm{d}x,\mathrm{d}y)\,\mathrm{d}s.\end{aligned}$$

Using (5.1), we thus have, since $\int_{\mathbb{R}^2\times\mathbb{R}^2}|x-y|R_s(\mathrm{d}x,\mathrm{d}y) = \mathcal{W}_1(f_s,g_s) \le \mathbb{E}[|X_s-Y_s|]$ by definition of $\mathcal{W}_1$,

$$\begin{aligned}\mathbb{E}\Big[\sup_{[0,T]}|X_t-Y_t|\Big] &\le C\int_0^T\mathbb{E}\big[|X_s-Y_s|\big]\big(1+\|f_s\|_{L^p}+\|g_s\|_{L^p}\big)\,\mathrm{d}s\\ &\quad+C\int_0^T\int_{\mathbb{R}^2\times\mathbb{R}^2}|x-y|\big(1+\|f_s\|_{L^p}+\|g_s\|_{L^p}\big)R_s(\mathrm{d}x,\mathrm{d}y)\,\mathrm{d}s\\ &\le C\int_0^T\mathbb{E}\big[|X_s-Y_s|\big]\big(1+\|f_s\|_{L^p}+\|g_s\|_{L^p}\big)\,\mathrm{d}s.\end{aligned}$$

By Grönwall's lemma, we thus get $\mathbb{E}(\sup_{[0,T]}|X_t-Y_t|) = 0$ and pathwise uniqueness is proven. □

The following lemma is useful for the uniqueness of (1.1).

Lemma 5.1. *Let $p > 2/(1-\alpha)$ and consider a weak solution $(f_t)_{t\ge 0}$ to (1.1) lying in $L^\infty_{\mathrm{loc}}([0,\infty),\mathbf{P}_1(\mathbb{R}^2))\cap L^1_{\mathrm{loc}}([0,\infty);L^p(\mathbb{R}^2))$. Assume that for some $h=(h_t)_{t\ge0}$ lying in $L^\infty_{\mathrm{loc}}([0,\infty),\mathbf{P}_1(\mathbb{R}^2))\cap L^1_{\mathrm{loc}}([0,\infty);L^p(\mathbb{R}^2))$, for all $\varphi\in C^2_c(\mathbb{R}^2)$, all $t\ge 0$,*

$$\begin{aligned}\int_{\mathbb{R}^2}\varphi(x)h_t(\mathrm{d}x) &= \int_{\mathbb{R}^2}\varphi(x)f_0(\mathrm{d}x)+\int_0^t\int_{\mathbb{R}^2}\triangle_x\varphi(x)h_s(\mathrm{d}x)\,\mathrm{d}s\\ &\quad-\chi\int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}K(x-y)\cdot\nabla_x\varphi(x)f_s(\mathrm{d}y)h_s(\mathrm{d}x)\,\mathrm{d}s.\end{aligned}\tag{5.2}$$

Then $h = f$.

Proof. For any $\varphi\in C^2_c(\mathbb{R}^2)$ and any $t\ge 0$, we set

$$\mathcal{A}_t\varphi(x) = \triangle_x\varphi(x)-\chi\int_{\mathbb{R}^2}K(x-y)\cdot\nabla_x\varphi(x)f_t(\mathrm{d}y).$$

We will prove that for any $\mu\in\mathbf{P}_1(\mathbb{R}^2)$, there exists at most one h lying in $L^\infty_{\mathrm{loc}}([0,\infty),\mathbf{P}_1(\mathbb{R}^2))\cap L^1_{\mathrm{loc}}([0,\infty);L^p(\mathbb{R}^2))$ such that for all $t\ge0$, $\varphi\in C^2_c(\mathbb{R}^2)$,

$$\int_{\mathbb{R}^2}\varphi(x)h_t(\mathrm{d}x) = \int_{\mathbb{R}^2}\varphi(x)\mu(\mathrm{d}x)+\int_0^t\int_{\mathbb{R}^2}\mathcal{A}_s\varphi(x)h_s(\mathrm{d}x)\,\mathrm{d}s.\tag{5.3}$$

This will conclude the proof since f and h solve this equation with $\mu = f_0$ by assumption.

Step 1. Let $\mu \in \mathbf{P}_1(\mathbb{R}^2)$. A continuous adapted $\mathbb{R}^2$-valued process $(X_t)_{t\ge 0}$ on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, P)$ is said to solve the martingale problem $\mathrm{MP}((\mathcal{A}_t)_{\ge 0}, \mu)$ if $P \circ X_0^{-1} = \mu$ and if for all $\varphi \in C_c^2(\mathbb{R}^2)$, $(M_t^\varphi)_{t\ge 0}$ is a $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, P)$-martingale, where

$$M_t^\varphi = \varphi(X_t) - \int_0^t \mathcal{A}_s\varphi(X_s)\,\mathrm{d}s.$$

Using Bhatt and Karandikar [1], Theorem 5.2 (see also Remark 3.1 in [1]), uniqueness for (5.3) holds if

(i) there exists a countable subset $(\varphi_k)_{k\ge 1} \subset C_c^2$ such that for all $t \ge 0$, the closure (for the bounded pointwise convergence) of $\{(\varphi_k, \mathcal{A}_t\varphi_k), k \ge 1\}$ contains $\{(\varphi, \mathcal{A}_t\varphi), \varphi \in C_c^2\}$,
(ii) for each $x_0 \in \mathbb{R}^2$, there exists a solution to $\mathrm{MP}((\mathcal{A}_t)_{\ge 0}, \delta_{x_0})$,
(iii) for each $x_0 \in \mathbb{R}^2$, uniqueness (in law) holds for $\mathrm{MP}((\mathcal{A}_t)_{\ge 0}, \delta_{x_0})$.

Step 2. We first prove (i). Consider thus some countable $(\varphi_k)_{k\ge 1} \subset C_c^2$ dense in C_c^2, in the sense that for $\psi \in C_c^2$, there exists a subsequence φ_{k_n} such that $\lim_{n\to\infty}(\|\psi - \varphi_{k_n}\|_\infty + \|\psi' - \varphi'_{k_n}\|_\infty + \|\psi'' - \varphi''_{k_n}\|_\infty) = 0$. We then have to prove that, for $t \ge 0$,

(a) $\mathcal{A}_t\varphi_{k_n}(x)$ tends to $\mathcal{A}_t\psi(x)$ for all $x \in \mathbb{R}^2$,
(b) $\sup_n \|\mathcal{A}_t\varphi_{k_n}\|_\infty < \infty$.

Let $x \in \mathbb{R}^2$. Using that $|K(x)| = \frac{1}{|x|^\alpha}$, we have

$$\left|\mathcal{A}_t\varphi_{k_n}(x) - \mathcal{A}_t\psi(x)\right| \le \left\|\psi'' - \varphi''_{k_n}\right\|_\infty + \chi\left\|\psi' - \varphi'_{k_n}\right\|_\infty \int_{\mathbb{R}^2} \frac{1}{|x-y|^\alpha} f_t(\mathrm{d}y) \to 0,$$

since $\int_{\mathbb{R}^2} \frac{1}{|x-y|^\alpha} f_t(\mathrm{d}y) \le C(1 + \|f_t\|_{L^p})$ by (5.1). For (b), we can observe that setting $A := \sup_n(\|\varphi_{k_n}\|_\infty + \|\varphi_{k'_n}\|_\infty + \|\varphi''_{k_n}\|_\infty)$

$$|\mathcal{A}_t\varphi_{k_n}| \le A + \chi A \int_{\mathbb{R}^2} \frac{1}{|x-y|^\alpha} f_t(\mathrm{d}y) \le A + CA\big(1 + \|f_t\|_{L^p}\big),$$

which concludes this step.

Step 3. Using classical arguments, we observe that a process $(X_t)_{t\ge 0}$ is a solution to $\mathrm{MP}((\mathcal{A}_t)_{\ge 0}, \delta_{x_0})$ if and only if there exists, on a possibly enlarged probability space, a $(\mathcal{F}_t)_{t\ge 0}$-Brownian motion $(B_t)_{t\ge 0}$ such that

$$X_t = x_0 - \chi\int_0^t \int_{\mathbb{R}^2} K(X_s - x) f_s(\mathrm{d}x)\,\mathrm{d}s + \sqrt{2}B_t. \tag{5.4}$$

It thus suffices to prove existence and uniqueness in law for solutions to (5.4) to get (ii) and (iii).

Step 4. The proof of (pathwise) uniqueness for (5.4) is very similar with the proof of uniqueness for (1.4) which has already been done and we leave it to the reader.

Step 5. It remains to check (ii) to conclude. We thus have to prove the existence of a solution to (5.4). To this aim, we use a Picard iteration. We thus consider the constant process $X_t^0 = x_0$ and define recursively

$$X_t^{n+1} = x_0 - \chi\int_0^t \int_{\mathbb{R}^2} K\big(X_s^n - x\big) f_s(\mathrm{d}x)\,\mathrm{d}s + \sqrt{2}B_t.$$

Using the same kind of arguments as in the proof of Theorem 1.7, we get

$$\mathbb{E}\Big(\sup_{[0,T]}\big|X_t^{n+1} - X_t^n\big|\Big) \le C\int_0^T \mathbb{E}\big[\big|X_s^n - X_s^{n-1}\big|\big]\big(1 + \|f_s\|_{L^p}\big)\,\mathrm{d}s.$$

Since $\int_0^T (1 + \|f_s\|_{L^p})\,\mathrm{d}s < \infty$, we classically deduce that $\sum_n \mathbb{E}(\sup_{[0,T]}|X_t^{n+1} - X_t^n|) < \infty$, so that there is a continuous adapted process $(X_t)_{t\ge 0}$ such that for all $T > 0$, $\lim_n \mathbb{E}[\sup_{[0,T]}|X_t - X_t^n|] = 0$. This L^1 convergence implies that $(X_t)_{t\ge 0}$ is solution to (5.4), which concludes the proof. □

The following result ensures that uniqueness holds for (1.1).

Theorem 5.2. *Let f_0 and g_0 be two probability measures with finite first moment. Let $(f_t)_{t\geq 0}$ and $(g_t)_{t\geq 0}$ be two solutions to (1.1) lying in $L^\infty_{\rm loc}([0,\infty),\mathbf{P}_1(\mathbb{R}^2))\cap L^1_{\rm loc}([0,\infty);L^p(\mathbb{R}^2))$ for some $p>2/(1-\alpha)$ starting from f_0 and g_0, respectively. Then*

$$\mathcal{W}_1(f_t,g_t)\leq \mathcal{W}_1(f_0,g_0)\exp\biggl(C\int_0^t\bigl(1+\|f_s\|_{L^p}+\|g_s\|_{L^p}\bigr)\,\mathrm{d}s\biggr).$$

Proof. Let thus $p>2/(1-\alpha)$, $(f_t)_{t\geq 0}$ and $(g_t)_{t\geq 0}$ be two solutions to (1.1) lying in $L^\infty_{\rm loc}([0,\infty),\mathbf{P}_1(\mathbb{R}^2))\cap L^1_{\rm loc}([0,\infty);L^p(\mathbb{R}^2))$. For any $s\geq 0$, we consider the probability measure R_s on $\mathbb{R}^2\times\mathbb{R}^2$ with first (resp. second) marginal equal to f_s (resp. g_s) such that

$$\mathcal{W}_1(f_s,g_s)=\int_{\mathbb{R}^2\times\mathbb{R}^2}|x-y|R_s(\mathrm{d}x,\mathrm{d}y),$$

and we consider (X_0,Y_0) with law R_0. We finally set

$$X_t=X_0-\chi\int_0^t\int_{\mathbb{R}^2}K(X_s-x)f_s(\mathrm{d}x)\,\mathrm{d}s+\sqrt{2}B_t,$$
$$Y_t=Y_0-\chi\int_0^t\int_{\mathbb{R}^2}K(Y_s-x)g_s(\mathrm{d}x)\,\mathrm{d}s+\sqrt{2}B_t.$$

Using Itô's formula, we see that h defined by $h_t:=\mathcal{L}(X_t)$ satisfies (5.2) and Lemma 5.1 ensures us that $\mathcal{L}(X_t)=f_t$. Similarly, we also have $\mathcal{L}(Y_t)=g_t$. Using the same arguments as in the proof of Theorem 1.7, we easily get

$$\mathbb{E}\bigl(|X_t-Y_t|\bigr)\leq \mathbb{E}\bigl[|X_0-Y_0|\bigr]+C\int_0^t\mathbb{E}\bigl[|X_s-Y_s|\bigr]\bigl(1+\|f_s\|_{L^p}+\|g_s\|_{L^p}\bigr)\,\mathrm{d}s.$$

Using the Grönwall's lemma and recalling that $\mathbb{E}[|X_0-Y_0|]=\mathcal{W}_1(f_0,g_0)$, we get

$$\mathbb{E}\bigl(|X_t-Y_t|\bigr)\leq \mathcal{W}_1(f_0,g_0)\exp\biggl(C\int_0^t\bigl(1+\|f_s\|_{L^p}+\|g_s\|_{L^p}\bigr)\,\mathrm{d}s\biggr),$$

which concludes the proof since $\mathcal{W}_1(f_t,g_t)\leq\mathbb{E}(|X_t-Y_t|)$. □

We can now give the proof of our well-posedness result for (1.1).

Proof of Theorem 1.5(i). The existence follows by Theorem 1.7. Indeed consider $(X_t)_{t\geq 0}$ the unique solution of (1.4) with initial law f_0 and set for $t\geq 0$ $f_t:=\mathcal{L}(X_t)$. Thanks to the Remark 1.2, f_t is a weak solution to (1.1) in the sense given by Definition 1.1 and (1.15) is exactly (1.7).

For uniqueness, consider two weak solutions $(f_t)_{t\geq 0}$ and $(g_t)_{t\geq 0}$ of (1.1) satisfying (1.7) with the same initial condition $f_0\in\mathbf{P}_1(\mathbb{R}^2)$. Then Theorem 5.2 ensures that $\mathcal{W}_1(f_t,g_t)=0$ for any $t\geq 0$ which concludes the proof. □

We end this section with the proof of our propagation of chaos result.

Proof of Theorem 1.8(i). We consider $Q^N:=\frac{1}{N}\sum_{i=1}^N\delta_{(X_t^{i,N})_{t\geq 0}}$. By Lemma 4.1, the family $\{\mathcal{L}(Q^N),N\geq 2\}$ is tight in $\mathbf{P}(\mathbf{P}(C([0,\infty),\mathbb{R}^2)))$. Furthermore, by Proposition 4.2, any limit point of $\mathcal{Q}^N$ belongs a.s. to the set of all probability measures $f\in\mathbf{P}(C([0,\infty),\mathbb{R}^2))$ such that f is the law of a solution to (1.4) satisfying (1.9). But by Theorem 1.7, this set is reduced to $\mathcal{L}((X_t)_{t\geq 0})=:f$. We thus deduce that Q^N goes in law to f as $N\to\infty$ which concludes the proof of (i). □

6. Renormalization and entropic chaos

In this section, we first deal with the renormalization which will give us the dissipation of entropy for the solution to (1.1). From this, we will be able to show the entropic chaos for the system (1.3), which will conclude this paper.

Proof of Theorem 1.5(ii). We adapt the ideas used in [8] for the 2D vortex model to our case, which in particular has a nondivergence free kernel. We split the proof in four steps plus a Step 0 which is nothing but direct results of what we have already done. We consider the unique weak solution $f = (f_t)_{t\geq 0}$ of (1.1). In Step 1 we deal with the necessary estimates on $K * f$ and $\nabla \cdot (K * f)$ to regularize f. In Step 2 we show the convergence of a regular version of f towards f. In Step 3, we improve the regularity of the solution using a well-known bootstrap argument. Finally, in Step 4 we prove the renormalization property.

We first observe that by construction, f satisfies (1.8). Indeed, for any $t \geq 0$, we considered f_t as the law of X_t, where $(X_t)_{t\geq 0}$ is the unique solution to (1.4), obtained by Proposition 4.2 and Lemma 4.1, so that (4.1) (which englobes (1.8)) is satisfied.

Step 0. *Direct estimates*. We start by noticing that Lemma 2.4 and (1.8) implies directly (1.9) and also that for any $p \in [1, \infty)$ and all $T > 0$,

$$f \in L^{p/(p-1)}\big(0, T; L^p\big(\mathbb{R}^2\big)\big). \tag{6.1}$$

Step 1. *First estimates*. The aim of this step is to prove that for any $q > 2/\alpha$ and all $T > 0$:

$$(K * f) \in L^{2q/(\alpha q-2)}\big(0, T; L^q\big(\mathbb{R}^2\big)\big) \tag{6.2}$$

and

$$\nabla_x \cdot (K * f) = K * (\nabla_x \cdot f) \in L^{2q/(q(1+\alpha)-2)}\big(0, T; L^q\big(\mathbb{R}^2\big)\big). \tag{6.3}$$

Let us remember the Hardy–Littlewood–Sobolev inequality in 2D: for $1 \leq p < 2/(2-\alpha)$,

$$\left\| \int_{\mathbb{R}^2} \frac{f(y)}{|\cdot - y|^{2-(2-\alpha)}}\,\mathrm{d}y \right\|_{2p/(2-(2-\alpha)p)} \leq C_{\alpha,p}\|f\|_p.$$

Using (6.1) we get that for any $p \in (1, 2/(2-\alpha))$ and all $T > 0$,

$$(K * f) \in L^{p/(p-1)}\big(0, T; L^{2p/(2-(2-\alpha)p)}\big(\mathbb{R}^2\big)\big),$$

and under the change of variables $q = 2p/(2-(2-\alpha)p)$ we easily deduce (6.2).

Similarly, but using (1.9) instead of (6.1), we get that for any $p \in (1, 2/(2-\alpha))$ and all $T > 0$,

$$\nabla_x \cdot (K * f) \in L^{2p/(3p-2)}\big(0, T; L^{2p/(2-(2-\alpha)p)}\big(\mathbb{R}^2\big)\big),$$

applying the same change of variables $q = 2p/(2-(2-\alpha)p)$ we get (6.3).

Step 2. *Continuity*. Consider $T > 0$ fixed. For $q > 2/\alpha$ we have that $2q/(q(1+\alpha)-2) > q/(q-1)$, then using (6.1) with $q_* = q/(q-1) > 1$, and (6.3), we get that $f\nabla_x \cdot (K * f)$ belongs to $L^1(0, T; L^1(\mathbb{R}^2))$. The following lemma follows directly:

Lemma 6.1. *Consider a mollifier sequence (ρ_n) on $\mathbb{R}^2$ and introduce the mollified function $f_t^n := f_t * \rho_n$. Clearly, $f_t^n \in C([0,\infty), L^1(\mathbb{R}^2))$. For all $T > 0$, there exists $r^n \in L^1(0, T; L^1_{\mathrm{loc}}(\mathbb{R}^2))$ that goes to 0 when $n \to \infty$, and such that*

$$\partial_t f^n - \chi\nabla_x \cdot \big((K * f) f^n\big) - \triangle_x f^n = r^n. \tag{6.4}$$

Remark 6.2. *The proof of the previous lemma is a modification of* [7], *Lemma* II.1(ii) *and Remark* 4. *In fact, for all $T > 0$, $f \in L^\infty(0, T; L^1(\mathbb{R}^2))$ and for any $q > 2/\alpha$, $(K * f) \in L^1(0, T; L^q(\mathbb{R}^2))$. That suffices for the existence of r^n*

given by

$$r^n := \chi\big[\big(\nabla\cdot\big((K*f)f\big)\big)*\rho^n - \nabla\cdot\big((K*f)f^n\big)\big],$$

which goes to 0 *if* $n\to\infty$ *in* $L^1(0,T;L^1_{\mathrm{loc}}(\mathbb{R}^2))$.

As a consequence of Lemma 6.1, the chain rule applied to the smooth f^n reads

$$\partial_t\beta\big(f^n\big) = \chi\big[(K*f)\cdot\nabla_x\beta\big(f^n\big)+\beta'\big(f^n\big)f^n\nabla_x\cdot(K*f)\big] + \triangle_x\beta\big(f^n\big)-\beta''\big(f^n\big)\big|\nabla_x f^n\big|^2+\beta'\big(f^n\big)r^n, \tag{6.5}$$

for any $\beta\in C^1(\mathbb{R})\cap W^{2,\infty}_{\mathrm{loc}}(\mathbb{R})$ such that β'' is piecewise continuous and vanishes outside of a compact set. Since equation (6.4) with $(K*f)$ fixed is linear in f^n, the difference $f^{n,k}:=f^n-f^k$ satisfies (6.4) with r^n replaced by $r^{n,k}:=r^n-r^k\to 0$ in $L^1(0,T;L^1_{\mathrm{loc}}(\mathbb{R}^2))$ and then also (6.5) (with again f^n and r^n changed in $f^{n,k}$ and $r^{n,k}$). Observe that the term $\beta'(f^n)f^n\nabla_x\cdot(K*f)$ is equal to 0 in [8].

Now, choosing $\beta(s)=\beta_1(s)$ where $\beta_1(s)=s^2/2$ for $|s|\le 1$ and $\beta_1(s)=|s|-1/2$ for $|s|\ge 1$. It is clear that $\beta\in C^1(\mathbb{R})$, that $\beta',\beta''\in L^\infty(\mathbb{R})$ and that the second derivative has compact support. For any nonnegative $\psi\in C^2_c(\mathbb{R}^2)$, we obtain

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^2}\beta_1\big(f^{n,k}(t,x)\big)\psi(x)\,\mathrm{d}x\\
&\quad=\int_{\mathbb{R}^2}\chi\big[(K*f)\cdot\nabla_x\beta_1\big(f^{n,k}\big)+\beta_1'\big(f^{n,k}\big)f^{n,k}\nabla_x\cdot(K*f)\big]\psi(x)\,\mathrm{d}x\\
&\qquad+\int_{\mathbb{R}^2}\big[\triangle_x\beta_1\big(f^{n,k}\big)-\beta_1''\big(f^{n,k}\big)\big|\nabla_x f^{n,k}\big|^2+\beta_1'\big(f^{n,k}\big)r^{n,k}\big]\psi(x)\,\mathrm{d}x\\
&\quad\le\int_{\mathbb{R}^2}\big|r^{n,k}(t,x)\big|\psi(x)\,\mathrm{d}x+\int_{\mathbb{R}^2}\beta_1\big(f^{n,k}\big)\triangle_x\psi\,\mathrm{d}x\\
&\qquad+\chi\int_{\mathbb{R}^2}\big|f^{n,k}\nabla_x\cdot(K*f)\big|\psi(x)\,\mathrm{d}x-\chi\int_{\mathbb{R}^2}\beta_1\big(f^{n,k}\big)\nabla_x\cdot\big((K*f)\psi(x)\big)\,\mathrm{d}x,
\end{aligned}$$

where we have used that $|\beta_1'|\le 1$ and that $\beta_1''\ge 0$. We know that $f_0\in L^1(\mathbb{R}^2)$ then $f^{n,k}(0)\to 0$ in $L^1(\mathbb{R}^2)$, also that $r^{n,k}\to 0$ in $L^1(0,T;L^1_{\mathrm{loc}}(\mathbb{R}^2))$. It is not difficult to see that $\beta_1(f^{n,k})(K*f)\to 0$ in $L^1(0,T;L^1_{\mathrm{loc}}(\mathbb{R}^2))$ (because β_1 is sub-linear, and for all $0<\alpha<1$ there is $q:=p/(p-1)>2/\alpha$, then using (6.1) and (6.2): $f^{n,k}\to 0$ in $L^{p/(p-1)}(0,T;L^p(\mathbb{R}^2))$, and $(K*f)\in L^{q/(q-1)}(0,T;L^q(\mathbb{R}^2))$).

The same arguments apply to $\beta_1(f^{n,k})\nabla_x\cdot(K*f)$ and $|f^{n,k}\nabla_x\cdot(K*f)|$, and then both goes to 0 as $n,k\to\infty$ in $L^1(0,T;L^1_{\mathrm{loc}}(\mathbb{R}^2))$. Finally, we get

$$\sup_{t\in[0,T]}\int_{\mathbb{R}^2}\beta_1\big(f^{n,k}(t,x)\big)\psi(x)\,\mathrm{d}x\underset{n,k\to\infty}{\longrightarrow}0.$$

Since ψ is arbitrary, we deduce that there exists $\bar f\in C([0,\infty);L^1_{\mathrm{loc}}(\mathbb{R}^2))$ so that $f^n\to\bar f$ in $C([0,\infty);L^1_{\mathrm{loc}}(\mathbb{R}^2))$ with the topology of the uniform convergence on any compact subset in time. Together with the convergence $f^n\to f$ in $C([0,\infty);\mathbf{P}(\mathbb{R}^2))$ we get that $f=\bar f$. We end this step by concluding that, with the same convention for the notion of convergence on $[0,\infty)$: $f^n\to f$ in $C([0,\infty);L^1(\mathbb{R}^2))$.

Step 3. *Additional estimates*. From (6.5), we know that for all $0<t_0<t_1$, all $\psi\in C^2_c(\mathbb{R}^2)$,

$$\begin{aligned}
&\int_{\mathbb{R}^2}\beta\big(f^n_{t_1}\big)\psi(x)\,\mathrm{d}x+\int_{t_0}^{t_1}\int_{\mathbb{R}^2}\beta''\big(f^n_s\big)\big|\nabla_x f^n_s\big|^2\psi(x)\,\mathrm{d}x\,\mathrm{d}s\\
&\quad=\int_{\mathbb{R}^2}\beta\big(f^n_{t_0}\big)\psi(x)\,\mathrm{d}x+\int_{t_0}^{t_1}\int_{\mathbb{R}^2}\beta'\big(f^n_s\big)r^n\psi(x)\,\mathrm{d}x\,\mathrm{d}s
\end{aligned}$$

$$+\int_{t_0}^{t_1}\int_{\mathbb{R}^2}\beta\big(f_s^n\big)\big[\triangle_x\psi(x)-\chi(K*f)\nabla_x\psi(x)\big]\,\mathrm{d}x\,\mathrm{d}s$$

$$+\chi\int_{t_0}^{t_1}\int_{\mathbb{R}^2}\big[\beta'\big(f_s^n\big)f_s^n-\beta\big(f_s^n\big)\big]\psi(x)\nabla_x\cdot(K*f)\,\mathrm{d}x\,\mathrm{d}s. \tag{6.6}$$

Let us choose $0\le\psi\in C_c^2(\mathbb{R}^2)$ and $\beta\in C^1(\mathbb{R})\cap W^{2,\infty}_{\mathrm{loc}}(\mathbb{R})$ convex such that β'' is nonnegative and vanishes outside of a compact set. Let us remark that there is a constant $C>0$ such that $s\beta'(s)\le C\beta(s)$, this will be very useful to deal with the last term which appears because the kernel is not divergence-free. We can pass to the limit as $n\to\infty$ (for details see Step 2) to get

$$\int_{\mathbb{R}^2}\beta(f_{t_1})\psi(x)\,\mathrm{d}x\le\int_{\mathbb{R}^2}\beta(f_{t_0})\psi(x)\,\mathrm{d}x+\int_{t_0}^{t_1}\int_{\mathbb{R}^2}\beta(f_s)\big[\triangle_x\psi(x)-\chi(K*f)\nabla_x\psi(x)\big]\,\mathrm{d}x\,\mathrm{d}s$$
$$+\chi\int_{t_0}^{t_1}\int_{\mathbb{R}^2}\big[-\beta(f_s)+\beta'(f_s)f_s\big]\psi(x)\nabla_x\cdot(K*f)\,\mathrm{d}x\,\mathrm{d}s.$$

It is not hard to deduce, by approximating $\psi\equiv1$ by a well-chosen sequence ψ_R that

$$\int_{\mathbb{R}^2}\beta(f_{t_1})\,\mathrm{d}x\le\int_{\mathbb{R}^2}\beta(f_{t_0})\,\mathrm{d}x+\chi\int_{t_0}^{t_1}\int_{\mathbb{R}^2}\big[-\beta(f_s)+\beta'(f_s)f_s\big]\nabla_x\cdot(K*f)\,\mathrm{d}x\,\mathrm{d}s,$$

whenever β is admissible.

Now we deal with the regularity in space of (1.10). Let us start by noticing that

$$\nabla_x\cdot(K*f)(x)=\int_{\mathbb{R}^2}\frac{(1-\alpha)f(y)}{|x-y|^{1+\alpha}}\,\mathrm{d}y, \tag{6.7}$$

so that taking $p>2/(1-\alpha)$ and using (5.1),

$$\int_0^T\big\|\nabla_x\cdot(K*f_s)\big\|_{L^\infty(\mathbb{R}^2)}\le C(\alpha,p)\int_0^T\big(\|f_s\|_{L^p(\mathbb{R}^2)}+1\big)<\infty,$$

and due to the fact that $s\beta'(s)\le C\beta(s)$, we get

$$\int_{\mathbb{R}^2}\beta(f_{t_1})\,\mathrm{d}x\le\int_{\mathbb{R}^2}\beta(f_{t_0})\,\mathrm{d}x$$
$$+(C+1)\chi\int_{t_0}^{t_1}\big\|\nabla_x\cdot(K*f)(x)\big\|_{L^\infty(\mathbb{R}^2)}\int_{\mathbb{R}^2}\beta(f_s)\,\mathrm{d}x\,\mathrm{d}s.$$

Then Grönwall's lemma implies that for all $0<t_0<t_1<T$,

$$\int_{\mathbb{R}^2}\beta(f_{t_1})\,\mathrm{d}x\le C(\alpha,T)\int_{\mathbb{R}^2}\beta(f_{t_0})\,\mathrm{d}x.$$

Finally letting $\beta(s)\to|s|^q/q$, we get that for all $q\ge1$ and all $0<t_0<t_1<T$,

$$\big\|f(t_1,\cdot)\big\|_{L^q(\mathbb{R}^2)}\le C(q,\alpha,T)\big\|f(t_0,\cdot)\big\|_{L^q(\mathbb{R}^2)}. \tag{6.8}$$

Coming back to (6.6) and using $\beta_M(s)=s^2/2$ for $|s|\le M$ and $\beta_M(s)=M|s|-M^2/2$ for $|s|\ge M$, we have

$$\int_{\mathbb{R}^2}\beta_M\big(f_{t_1}^n\big)\psi\,\mathrm{d}x+\int_{t_0}^{t_1}\int_{\mathbb{R}^2}\mathbb{1}_{|f_s|\le M}\big|\nabla_x f_s^n\big|^2\psi\,\mathrm{d}x\,\mathrm{d}s$$
$$=\int_{\mathbb{R}^2}\beta_M\big(f_{t_0}^n\big)\psi\,\mathrm{d}x+\int_{t_0}^{t_1}\int_{\mathbb{R}^2}\beta_M'\big(f_s^n\big)r^n\psi(x)\,\mathrm{d}x\,\mathrm{d}s$$

$$+\int_{t_0}^{t_1}\int_{\mathbb{R}^2}\beta_M\big(f_s^n\big)\big[\triangle\psi(x)-\chi(K*w)\nabla_x\psi(x)\big]\,\mathrm{d}x\,\mathrm{d}s$$
$$+\chi\int_{t_0}^{t_1}\int_{\mathbb{R}^2}\big[\beta_M'\big(f_s^n\big)f_s^n-\beta_M\big(f_s^n\big)\big]\psi(x)\nabla_x\cdot(K*f)\,\mathrm{d}x\,\mathrm{d}s,$$

similarly as above we first make $n\to\infty$, then we approximate $\psi\equiv 1$ by a well-chosen sequence ψ_R and make $R\to\infty$, and finally make the limit $M\to\infty$ to find that for every $T\geq t_1\geq t_0\geq 0$:

$$\int_{\mathbb{R}^2}|f_{t_1}|^2\,\mathrm{d}x+\int_{t_0}^{t_1}\int_{\mathbb{R}^2}|\nabla_x f_s|^2\,\mathrm{d}x\,\mathrm{d}s$$
$$\leq\int_{\mathbb{R}^2}|f_{t_0}|^2\,\mathrm{d}x+\chi\int_{t_0}^{t_1}\big\|\nabla_x(K*f)(x)\big\|_{L^\infty(\mathbb{R}^2)}\int_{\mathbb{R}^2}|f_s|^2\,\mathrm{d}x\,\mathrm{d}s.$$

We conclude, using (6.8), that for all $0<t_0<T$ and any $q\in[1,\infty)$:

$$f\in L^\infty\big(t_0,T;L^q\big(\mathbb{R}^2\big)\big)\quad\text{and}\quad\nabla_x f\in L^2\big((t_0,T)\times\mathbb{R}^2\big).\tag{6.9}$$

To get the continuity in time of (1.10), we need to improve even more the estimates on f which will be achieved using a bootstrap argument. First, fixing $p>2/(2-\alpha)$ we notice that for all $t_0>0$

$$\|K*f_t\|_{L^\infty}\leq C(p)\big(1+\|f_t\|_{L^p}\big)\quad\Rightarrow\quad K*f_t\in L^\infty\big(t_0,T;L^\infty\big(\mathbb{R}^2\big)\big),$$

and thanks to (6.7) and (6.9):

$$\big\|\nabla_x(K*f_t)\big\|_{L^\infty}\leq C(p)\big(1+\|f_t\|_{L^p}\big)\quad\Rightarrow\quad\nabla_x(K*f_t)\in L^\infty\big(t_0,T;L^\infty\big(\mathbb{R}^2\big)\big),$$

we thus have

$$\partial_t f-\triangle_x f=\big[\chi f\nabla_x\cdot(K*f)+(K*f)\cdot\nabla_x f\big]\in L^2\big((t_0,T)\times\mathbb{R}^2\big),$$

and [2], Theorem X.11, provides the maximal regularity in L^2 spaces for the heat equation, in other words: for all $t_0>0$

$$f\in L^\infty\big(t_0,T;H^1\big(\mathbb{R}^2\big)\big)\cap L^2\big(t_0,T;H^2\big(\mathbb{R}^2\big)\big).$$

Remark 6.3. *We emphasize that the previous bound is true for all t_0. In fact, when $f_{t_0}\in H^1(\mathbb{R}^2)$, the maximal regularity implies the above bound in the time interval $[t_0,\infty)$. But thanks to (6.9), we can find t_0 arbitrary close to 0 such that $f_{t_0/2}\in H^1(\mathbb{R}^2)$, then we get the conclusion.*

Using now the interpolation inequality, there exists a constant $C>0$ such that

$$\|\nabla_x f\|_{L^3(\mathbb{R}^2)}\leq C\big\|D^2 f\big\|_{L^2(\mathbb{R}^2)}^{2/3}\|f\|_{L^2(\mathbb{R}^2)}^{1/3},$$

which implies

$$\int_{t_0}^{T}\|\nabla_x f\|_{L^3(\mathbb{R}^2)}^3\,\mathrm{d}s\leq C\int_{t_0}^{T}\big\|D^2 f\big\|_{L^2(\mathbb{R}^2)}^2\|f\|_{L^2(\mathbb{R}^2)}<\infty.$$

Thanks to the previous calculus and again [2], Theorem X.12, we conclude that $\partial_t f,\nabla_x f\in L^3((t_0,T)\times\mathbb{R}^2)$ and then Morrey's inequality implies that for all $t_0>0$

$$f\in C^0\big((t_0,T)\times\mathbb{R}^2\big),$$

all together allow us to deduce that

$$f \in C\big([0,T); L^1(\mathbb{R}^2)\big) \cap C\big((0,T); L^2(\mathbb{R}^2)\big).$$

We can go even further iterating this argument, using the interpolation inequality and the Sobolev inequality, to deduce that $\nabla_x f \in L^p((t_0,T)\times\mathbb{R}^2)$ for any $1<p<\infty$, $[\chi f \nabla_x \cdot (K*f) + (K*f)\cdot\nabla_x f] \in L^p((t_0,T)\times\mathbb{R}^2)$ for all $t_0>0$. Then the maximal regularity of the heat equation in L^p spaces (see [2], Theorem X.12) implies that for all $t_0>0$

$$\partial_t f, \nabla_x f \in L^p\big((t_0,T)\times\mathbb{R}^2\big),$$

and then using again Morrey's inequality: $f \in C^{0,\alpha}((t_0,T)\times\mathbb{R}^2)$ for any $0<\alpha<1$, and any $t_0>0$. All together allow us to prove (1.10).

Step 4. *Renormalization*. To end the proof we show (1.11). Let thus $\beta \in C^1(\mathbb{R}) \cap W^{2,\infty}_{\mathrm{loc}}(\mathbb{R})$ sub-linear, such that β'' is piecewise continuous and vanishes outside of a compact set. Thanks to (6.9), we can pass to the limit in the similar identity as (6.6) obtained for time dependent test functions $\psi \in C^2_c([0,\infty)\times\mathbb{R}^2)$ to get

$$\begin{aligned}\int_{t_0}^\infty \int_{\mathbb{R}^2} \beta''(f_s)|\nabla_x f_s|^2 \psi_s \,\mathrm{d}x\,\mathrm{d}s &= \int_{\mathbb{R}^2} \beta(f_{t_0})\psi_{t_0}\,\mathrm{d}x - \chi\int_{t_0}^\infty\int_{\mathbb{R}^2} \psi_s(x)\nabla_x\cdot(K*f)\big(f_s\beta'(f_s)-\beta(f_s)\big)\,\mathrm{d}x\,\mathrm{d}s \\ &\quad + \int_{t_0}^\infty\int_{\mathbb{R}^2} \beta(f_s)\big(\triangle_x\psi_s(x) - (K*f)\nabla_x\psi_s(x) + \partial_t\psi_s(x)\big)\,\mathrm{d}x\,\mathrm{d}s. \qquad (6.10)\end{aligned}$$

In the case $\psi \ge 0$ and $\beta'' \ge 0$ we can pass to the limit $t_0 \to 0$ thanks to monotonous convergence in the first term, the continuity property obtained in Step 2 in the second term, and the monotonous convergence in the other terms (recall that $s\beta'(s) \le \beta(s)$, β is sub-linear and $|f|(1+|K*f|+|\nabla\cdot(K*f)|)$ belongs to $L^1(0,T;L^1(\mathbb{R}^2))$ thanks to (6.2) and (6.3)). We get

$$\begin{aligned}\int_0^\infty\int_{\mathbb{R}^2}\beta''(f_s)|\nabla_x f_s|^2\psi_s\,\mathrm{d}x\,\mathrm{d}s &= \int_{\mathbb{R}^2}\beta(f_0)\psi_0\,\mathrm{d}x \\ &\quad + \int_0^\infty\int_{\mathbb{R}^2}\beta(f_s)\big[\triangle_x\psi_s - \chi\nabla_x\big((K*f)\cdot\psi_s\big) + \partial_t\psi_s\big]\,\mathrm{d}x\,\mathrm{d}s \\ &\quad + \chi\int_0^\infty\int_{\mathbb{R}^2}\beta'(f_s)f_s\psi_s(x)\nabla_x\cdot(K*f)\,\mathrm{d}x\,\mathrm{d}s, \qquad (6.11)\end{aligned}$$

and the bound given by (6.11) implies directly that we can pass to the limit $t_0\to 0$ in the general case for ψ in (6.10) which is nothing but (1.11) in the distributional sense. □

We now give a useful lemma for the entropic chaos.

Lemma 6.4. *Let $\alpha \in (0,1)$ and $f_0 \in \mathbf{P}_1(\mathbb{R}^2)$ such that $H(f_0)<\infty$. Let $(f_t)_{t\ge0}$ be the unique solution of (1.1) satisfying (1.7). Then*

$$H(f_t) + \int_0^t I(f_s)\,\mathrm{d}s = H(f_0) + \chi(1-\alpha)\int_0^t\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\frac{f_s(\mathrm{d}x)f_s(\mathrm{d}y)}{|x-y|^{\alpha+1}}\,\mathrm{d}s. \qquad (6.12)$$

Proof. For $m>1$, let us take $\beta_m \in C^1(\mathbb{R})\cap W^{2,\infty}_{\mathrm{loc}}(\mathbb{R})$ given by

$$\beta_m(s) = \begin{cases} s\log(s) + (1-s)/m & \text{for } m^{-1} \le s \le m, \\ \beta_m(m_-) + \beta_m'(m_-)(s-m) & \text{for } s>m, \\ \beta_m(m_+^{-1}) + \beta_m'(m_+^{-1})(s-m^{-1}) & \text{for } s<m^{-1}, \end{cases}$$

so that $\beta_m(s) \le Cs$ and $\beta_m \to s\log(s)$ for any $s>0$.

Since β_m is admissible (in the sense of Theorem 1.5), then using (1.11) we get that for any $\psi \in C_c^\infty(\mathbb{R}^2)$,

$$\begin{aligned}\int \beta_m(f_t)\psi \,\mathrm{d}x - \int \beta_m(f_0)\psi \,\mathrm{d}x &= \chi \int_0^t \int \nabla_x \cdot (K * f)\big(f\beta_m'(f) - \beta_m(f)\big)\psi \,\mathrm{d}x \,\mathrm{d}s \\ &\quad + \int_0^t \int \beta_m(f)\big(\triangle_x \psi - \chi(K * f)\cdot \nabla_x \psi\big)\,\mathrm{d}x\,\mathrm{d}s \\ &\quad - \int_0^t \int \beta_m''(f)|\nabla_x f|^2 \psi \,\mathrm{d}x\,\mathrm{d}s,\end{aligned}$$

using that $\beta_m''(s)$ is nonnegative, that β_m growths linearly at $+\infty$ and that $(f_s)_{s\geq 0}$ is nonnegative we can make $\psi \to 1$ to get

$$\begin{aligned}\int \beta_m(f_t)\,\mathrm{d}x - \int \beta_m(f_0)\,\mathrm{d}x &= \chi \int_0^t \int \nabla_x \cdot (K * f)\big(f\beta_m'(f) - \beta_m(f)\big)\,\mathrm{d}x\,\mathrm{d}s \\ &\quad - \int_0^t \int \beta_m''(f)|\nabla_x f|^2 \,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

In fact, the first and the second terms converge thanks to monotonous convergence and that $|\beta_m(s)| \leq C|s|$. The third term is a consequence of the monotonous convergence, that $\beta_m'(s)$ is bounded, and that $f\nabla\cdot(K * f)$ (resp. $|f(K * f)|$ for the fourth term) is integrable by (6.3) (resp. (6.2)). The last term is a consequence of (4.1).

Finally, we notice that in the interval $(0,1]$ the function $-\beta_m$ increases to $-s\log(s)$ while in the interval $[1,\infty)$, $\beta_m(s)$ increases to $s\log(s)$. Thanks to the monotonous convergence we can make $m \to \infty$ and using the integrability of all the limits we get (6.12). □

It remains to conclude with the proof of the entropic chaos.

Proof of Theorem 1.8(ii). We only have to prove that for each $t \geq 0$, $H(F_t^N)$ tends to $H(f_t)$. To this aim, we first show that for any $t \geq 0$

$$L := \limsup_N \left[H\big(F_t^N\big) + \int_0^t I\big(F_s^N\big)\,\mathrm{d}s\right] \leq H(f_t) + \int_0^t I(f_s)\,\mathrm{d}s. \tag{6.13}$$

Let $t \geq 0$ be fixed. Using (1.14) and recalling that $H(F_0^N) \to H(f_0)$ by assumption, we have

$$L \leq H(f_0) + \limsup_N \frac{\chi(1-\alpha)}{N^2} \sum_{i\neq j} \int_0^t \mathbb{E}\left[\frac{1}{|X_s^{i,N} - X_s^{j,N}|^{\alpha+1}}\right]\mathrm{d}s,$$

so that using that $H(f_t) + \int_0^t I(f_s)\,\mathrm{d}s = H(f_0) + \chi(1-\alpha)\int_0^t \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{f_s(\mathrm{d}x) f_s(\mathrm{d}y)}{|x-y|^{\alpha+1}}\,\mathrm{d}s$ by Lemma 6.4, we only have to prove that

$$\lim_{N\to\infty} \frac{1}{N^2} \int_0^t \mathbb{E}\left[\sum_{i\neq j} \frac{1}{|X_s^{i,N} - X_s^{j,N}|^{\alpha+1}}\right]\mathrm{d}s = \int_0^t \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{f_s(\mathrm{d}x) f_s(\mathrm{d}y)}{|x-y|^{\alpha+1}}\,\mathrm{d}s.$$

By exchangeability, it suffices to prove that, as $N \to \infty$,

$$D_N := \int_0^t \mathbb{E}\left[\frac{1}{|X_s^{1,N} - X_s^{2,N}|^{\alpha+1}}\right]\mathrm{d}s \to \int_0^t \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{f_s(\mathrm{d}x) f_s(\mathrm{d}y)}{|x-y|^{\alpha+1}}\,\mathrm{d}s =: D.$$

For any $\varepsilon > 0$, we have

$$|D - D_N| \leq |D - D_\varepsilon| + |D_\varepsilon - D_{N,\varepsilon}| + |D_{N,\varepsilon} - D_N|,$$

where $D_{N,\varepsilon} = \int_0^t \mathbb{E}[\frac{1}{(|X_s^{1,N}-X_s^{2,N}|\vee\varepsilon)^{\alpha+1}}]\,\mathrm{d}s$ and $D_\varepsilon = \int_0^t \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{f_s(\mathrm{d}x)f_s(\mathrm{d}y)}{(|x-y|\vee\varepsilon)^{\alpha+1}}\,\mathrm{d}s$. Using that for any $\varepsilon > 0$ fixed, the function $(x,y) \mapsto (|x-y|\vee\varepsilon)^{-\alpha-1}$ is bounded continuous and that $\mathcal{L}(X_s^{1,N}, X_s^{2,N})$ goes weakly to $f_s \otimes f_s$ for any $s \ge 0$, we have $\lim_N \mathbb{E}[\frac{1}{(|X_s^{1,N}-X_s^{2,N}|\vee\varepsilon)^{\alpha+1}}] = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{f_s(\mathrm{d}x)f_s(\mathrm{d}y)}{(|x-y|\vee\varepsilon)^{\alpha+1}}$. By dominated convergence, we thus get that $\lim_N |D_\varepsilon - D_{N,\varepsilon}| = 0$. We thus have

$$\limsup_N |D - D_N| \le |D - D_\varepsilon| + \limsup_N |D_{N,\varepsilon} - D_N| \quad \forall \varepsilon > 0.$$

Let $\tilde{\alpha}$ be such that $\alpha + 1 < \tilde{\alpha} < 2$. We have

$$\begin{aligned} |D - D_\varepsilon| &\le 2\int_0^t \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{f_s(\mathrm{d}x)f_s(\mathrm{d}y)}{|x-y|^{\alpha+1}} \mathbb{1}_{\{|x-y|<\varepsilon\}}\,\mathrm{d}s \\ &\le 2\varepsilon^{\tilde{\alpha}-\alpha-1} \int_0^t \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{f_s(\mathrm{d}x)f_s(\mathrm{d}y)}{|x-y|^{\tilde{\alpha}}}\,\mathrm{d}s \\ &\le C\varepsilon^{\tilde{\alpha}-\alpha-1} \int_0^t \big(1 + I(f_s)\big)\,\mathrm{d}s \le C(1+t)\varepsilon^{\tilde{\alpha}-\alpha-1}, \end{aligned}$$

by Lemma 2.1 (applied with $F = f_s \otimes f_s$, for which $I(F_s) = I(f_s)$) and (1.8). Using the same arguments, we also have for any $N \ge 2$,

$$|D_{N,\varepsilon} - D_N| \le C\varepsilon^{\tilde{\alpha}-\alpha-1} \int_0^t \big(1 + I\big(F_s^N\big)\big)\,\mathrm{d}s \le C(1+t)\varepsilon^{\tilde{\alpha}-\alpha-1}.$$

We thus get that $\limsup_N |D - D_N| = 0$ and (6.13) is proven.

Using [11], Theorem 3.4 and Theorem 5.7, we have

$$\liminf_N H\big(F_t^N\big) \ge H(f_t) \quad \text{and} \quad \liminf_N \int_0^t I\big(F_s^N\big)\,\mathrm{d}s \ge \int_0^t I(f_s)\,\mathrm{d}s. \tag{6.14}$$

Using (6.13) and (6.14), we easily conclude that

$$\lim_N H\big(F_t^N\big) = H(f_t) \quad \text{and} \quad \lim_N \int_0^t I\big(F_s^N\big)\,\mathrm{d}s = \int_0^t I(f_s)\,\mathrm{d}s,$$

which concludes the proof. □

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