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Diffusion in planar Liouville quantum gravity

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Abstract. We construct the natural diffusion in the random geometry of planar Liouville quantum gravity. Formally, this is the Brownian motion in a domain D of the complex plane for which the Riemannian metric tensor at a point $z \in D$ is given by $\exp(\gamma h(z))$, appropriately normalised. Here h is an instance of the Gaussian free field on D and $\gamma \in (0, 2)$ is a parameter. We show that the process is almost surely continuous and enjoys certain conformal invariance properties. We also estimate the Hausdorff dimension of times that the diffusion spends in the thick points of the Gaussian free field, and show that it spends Lebesgue-almost all its time in the set of γ-thick points, almost surely. Similar but deeper results have been independently and simultaneously proved by Garban, Rhodes and Vargas.

Résumé. Nous construisons une diffusion naturelle associée ê la géométrie aléatoire de la gravité quantique de Liouville. Formellement, il s'agît d'un mouvement Brownien dans un domaine D du plan complexe, muni d'un tenseur de Riemann donné par $\exp(\gamma h(z))$, correctement renomalisé. Ici h est une réalisation du champ libre Gaussien sur D, et $\gamma \in]0, 2[$ est un paramètre. Il est montré que ce processus est presque sûrement continu et possède certains propriétés d'invariance conforme. Une borne sur la dimension de Hausdorff des instants passés dans les points épais du champ libre Gaussien est obtenue, qui montre que cette diffusion passe Lebesue-presque tout son temps dans les points γ-épais, presque sûrement. Des résultats semblables mais plus profonds ont été indépendemment et simultanément obtenus par Garban, Rhodes et Vargas.

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Keywords: Brownian motion; Gaussian free field; Liouville quantum gravity; Thick points

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1. Introduction

This paper is motivated by a recent series of works on planar Liouville quantum gravity and the so-called KPZ relation (for Knizhnik, Polyakov and Zamolodchikov). The KPZ relation describes a way to relate geometric quantities associated with Euclidean models of statistical physics to their formulation in a setup governed by a certain *random geometry*, the so-called Liouville (critical) quantum gravity. This is a problem which has a long and distinguished history and for which we refer the interested reader to the recent breakthrough paper of Duplantier and Sheffield [4] and the excellent survey article by Garban [5].

A central problem in this area is the construction of a natural random metric in the plane, enjoying properties of conformal invariance, such that a KPZ relation holds. By this we mean that given a set A in the plane, the Hausdorff dimensions of A endowed either with the Euclidean metric or the random (quantum) metric are related by a deterministic transformation. Given these requirements, it is reasonably natural to look for or postulate that the local metric at a point z can be written in the form (up to normalising factor) $\exp(\gamma h(z))$, where h is a Gaussian free field and γ is a parameter. Unfortunately, h is not a function but a random distribution, and the exponential of a distribution is not in general well-defined.

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While making sense of this notion of random metric is still wide open, Duplantier and Sheffield, in the paper mentioned above, were able to construct a random measure, called the quantum gravity measure, which intuitively speaking corresponds to the volume measure of the metric. Remarkably, using this measure, they were able to define suitable notions of scaling dimensions for a set A and show that a KPZ relation holds, where the deterministic transformation involves a quadratic polynomial.

The purpose of this paper is to show that a natural notion of *diffusion* also makes sense in this context. Roughly speaking, one can summarise the main result by saying that, while we still do not know how to measure the distance between two points z and w, it is possible to say how long it would take a Brownian motion to go from z to w. The key idea is to note that, using conformal invariance of Brownian motion in two dimensions, it suffices to parametrise the Brownian motion correctly.

Note. *As I was preparing this paper, I learnt that Garban, Rhode and Vargas were working on a similar problem* [6]. *Their results are of a similar nature, though are more precise in some aspects.*

1.1. *Looking for the right object*

What follows is an informal discussion which is aimed to explain where the definition comes from. By *local metric* $\rho(z)$ at a point $z \in D$, we mean that small segments of Euclidean length ε are in the Riemannian metric considered to have distance $\rho(z)\varepsilon$ at the first order when $\varepsilon \to 0$.

Let U, D be two proper simply connected domains, and let $f : U \to D$ be a conformal isomorphism. We think of U as being a (wild) domain endowed with the random geometry, and D a nice domain such as the unit disc, in which we read this geometry. If $(W_t, t \geq 0)$ is a standard Brownian motion in U (i.e., stopped upon leaving U), then we simply wish to describe how $(W_t, t \geq 0)$ is parametrized by D. To do this, it suffices to consider $X_t = f(W_t)$. By Itô's formula,

$$Z_t = f(W_t) = B_{\int_0^t |f'(W_s)|^2\,\mathrm{d}s}; \tag{1}$$

and hence Z is a time-change of a Brownian motion $(B_t, t \geq 0)$ in D. This cannot directly be used as a definition as the time-change still involves W and we only want to define the process Z in terms of B and the metric $\rho(z)$ in D derived from mapping the metric in U via f. Clearly, $\rho(z)$ is simply equal to $1/|f'(w)|^2 = |g'(z)|^2$, where $g = f^{-1}$ (see Figure 1).

The reader can then easily check that setting

$$Z_t = B(\mu_t^{-1}); \quad \text{where } \mu_t = \int_0^t |g'(B_s)|^2\,\mathrm{d}s, \tag{2}$$

and $\mu_t^{-1} := \inf\{s > 0 \colon \mu_s > t\}$, gives the same process as (1). The advantage of this way of writing Z is that it involves only the standard Brownian motion $(B_t, t \geq 0)$ in the nice domain D and the local metric $\rho(z)$ at any point $z \in D$, which we assume to be given.

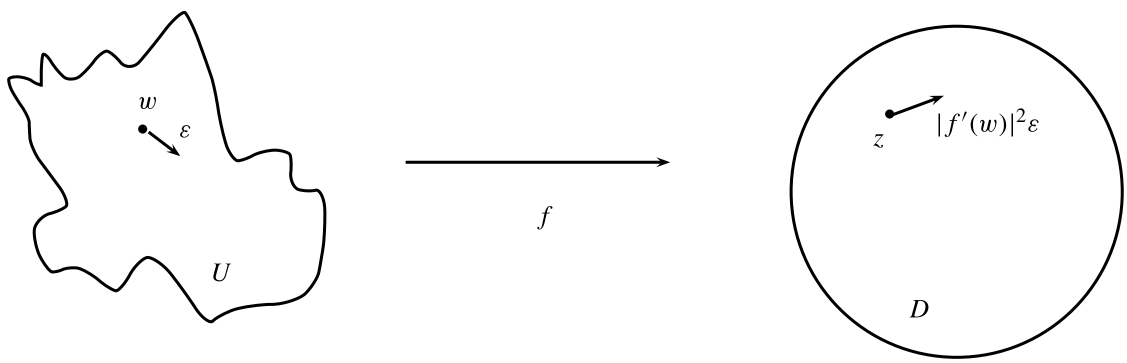

Fig. 1. Local metric and conformal map.

1.2. *Statements*

We will thus use (2) as our definition. Fix a proper connected domain $D \subset \mathbb{C}$, and let h be an instance of the Gaussian free field in D. (We use the Duplantier–Sheffield normalisation of the Green function.) Formally, h is a centered Gaussian process indexed by the Sobolev space $H_0^1(D)$, which is the completion of $C_K^\infty(D)$ with respect to the scalar product

$$(f,g)_\nabla = \frac{1}{2\pi}\int_D (\nabla f)\cdot(\nabla g).$$

Then h is a centered Gaussian process such that if $(h,f)_\nabla$ is the value of the field at the function $f \in H_0^1(D)$, then

$$\operatorname{Cov}\bigl[(h,f)_\nabla,(h,g)_\nabla\bigr] = (f,g)_\nabla$$

by definition.

For $z \in D$ and ε sufficiently small, we let $h_\varepsilon(z)$ be the well-defined average of h over a circle of radius ε about z. (We refer the reader to [13] for a proof that this is indeed well-defined and other general facts about the Gaussian free field.) Then we define a process $(Z_\varepsilon(t), t \ge 0)$ as in (2). That is, let $z \in U$ and let $(B_t, t \ge 0)$ be a planar Brownian motion such that $Z_0 = z$ almost surely. We put

$$Z_\varepsilon(t) = B\bigl(\mu_\varepsilon^{-1}(t)\bigr), \quad \text{where } \mu_\varepsilon(t) = \int_0^{t\wedge T} \mathrm{e}^{\gamma h_\varepsilon(B_s)}\varepsilon^{\gamma^2/2}\,\mathrm{d}s,$$

and $\mu_\varepsilon^{-1}(t) = \inf\{s \ge 0\colon\ \mu_\varepsilon(s) > t\}$, $T = \inf\{t \ge 0\colon\ B_s \notin D\}$.

Definition 1.1. *The Liouville diffusion, if it exists, is the limit as $\varepsilon \to 0$ of the process Z_ε.*

Obviously, since B does not depend on ε, the issue of convergence of the process Z_ε reduces to that of the *quantum clock process* $(\mu_\varepsilon(t), t \le T)$.

Theorem 1.2. *Assume $0 \le \gamma < 2$. Then $(Z_\varepsilon(t), t \ge 0)$ converges almost surely along the dyadic sequence $\varepsilon = 2^{-k}$ as $k \to \infty$ to a random process $(Z(t), t \ge 0)$ which is almost surely continuous up to the hitting time of ∂D.*

Remark 1.3. *It can be seen from the proof that Z does not stay stuck anywhere almost surely, in the sense that $\mu^{-1}(t)$ is strictly increasing.*

We now address conformal invariance properties. Let D, $\tilde{D}$ be two simply connected domains and let $\phi : D \to \tilde{D}$ be a conformal isomorphism (a bijective conformal map with conformal inverse).

Theorem 1.4. *Let $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$ and $\psi = \phi^{-1}$. Then we can write*

$$\phi(B_{\mu^{-1}(t)}) = \tilde{B}_{\tilde{\mu}_\psi^{-1}(t)},$$

where $\tilde{B}$ is a Brownian motion in $\tilde{D}$,

$$\tilde{\mu}_\psi(t) = \lim_{\varepsilon\to 0}\int_0^t \varepsilon^{\gamma^2/2}\mathrm{e}^{\gamma[\tilde{h}_\varepsilon + Q\log|\psi'|](\tilde{B}_s)}\,\mathrm{d}s$$

and $\tilde{h}$ is the Gaussian free field in $\tilde{D}$.

In other words, mapping the Liouville diffusion $Z(t)$ by the transformation ϕ, one obtains the corresponding Liouville diffusion in $\tilde{D}$, except that the Gaussian free field $\tilde{h}$ in $\tilde{D}$ has been replaced by $\tilde{h} + Q\log|\psi'|$. (This is similar to Proposition 2.1 in Duplantier and Sheffield [4].)

Finally, it is of interest to quantify how much time the Brownian motion spends at points for which the value of the field h is unusually big. Consider the *thick points* of the Gaussian free field: for $\alpha > 0$, let

$$\begin{cases} \mathcal{T}_\alpha^- := \{z \in D\colon \liminf_{\varepsilon\to 0} \frac{h_\varepsilon(z)}{\log(1/\varepsilon)} \le \alpha\}, \\ \mathcal{T}_\alpha^+ := \{z \in D\colon \limsup_{\varepsilon\to 0} \frac{h_\varepsilon(z)}{\log(1/\varepsilon)} \ge \alpha\}. \end{cases} \tag{3}$$

Hu, Miller and Peres [7, Theorem 1.2] proved that the Hausdorff dimension of $\mathcal{T}_\alpha$ is a.s. $(2-\alpha^2/2) \vee 0$. (Note that these authors were working with a slightly different normalisation of the Gaussian free field which differs by a factor of $\sqrt{2\pi}$, and our α is $\sqrt{2a}$ in the notations of that paper.)

Theorem 1.5. *Let $0 < \gamma < 2$ and let $\alpha > \gamma$. Then almost surely,*

$$\dim\{t\colon Z(t) \in \mathcal{T}_\alpha^+\} \le \frac{2-\alpha^2/2}{2-\alpha\gamma+\gamma^2/2}. \tag{4}$$

The same result holds when $\alpha < \gamma$ and $\mathcal{T}_\alpha^-$ replaced with $\mathcal{T}_\alpha^+$.

This upper bound is enough to deduce the following result:

Corollary 1.6. *With probability one,*

$$\mathrm{Leb}\{t\colon Z(t) \notin \mathcal{T}_\gamma^=\} = 0,$$

where

$$\mathcal{T}_\alpha^= = \left\{z \in D\colon \lim_{\varepsilon\to 0} \frac{h_\varepsilon(z)}{\log(1/\varepsilon)} = \alpha\right\}.$$

By contrast, using the methods of Benjamini and Schramm [2] (see also Rhode and Vargas [12]) it is possible to show the following analogue of the KPZ relation.

Proposition 1.7. *Fix $A \subset D$ a nonrandom Borel set and let $d_0 = \dim(A)$ where* dim *refers to the* (*Euclidean*) *Hausdroff dimension*. *Then almost surely,*

$$\dim\{t\colon Z_t \in A\} = d, \tag{5}$$

where d solves the equation $d_0 + d^2\gamma^2/2 - d(2+\gamma^2/2) = 0$.

Recall that in the case where $A = \mathcal{T}_\alpha^\pm$, as mentionned above, the Hausdorff dimension is $(2-\alpha^2/2) \vee 0$. Nevertheless, the formula in (5) does not match that from Theorem 1.5. This is of course because $\mathcal{T}_\alpha^\pm$ depends very strongly on the Gaussian free field. The difficulty in Theorem 1.5 is thus essentially to understand the effect on the clock process of coming near a thick point, and hence to disentangle the separate effects linked on the one hand to the trajectory of a standard Brownian motion and on the other hand to the frequency of those thick points.

2. Proof of Theorem 1.2: Convergence

For the rest of the paper, with a slight abuse of notation, we call $(B_t, t \ge 0)$ a Brownian motion stopped at time $T := T_r = \inf\{t > 0\colon \mathrm{dist}(B_t, \partial D) \le r\}$. Here $r > 0$ is a small arbitrary number. We will still call

$$Z_\varepsilon(t) = B\bigl(\mu_\varepsilon^{-1}(t)\bigr), \quad \text{where } \mu_\varepsilon(t) = \int_0^{t\wedge T} \mathrm{e}^{\gamma h_\varepsilon(B_s)} \varepsilon^{\gamma^2/2}\,\mathrm{d}s,$$

and $\mu_\varepsilon^{-1}(t) = \inf\{s \ge 0\colon \mu_\varepsilon(s) > t\}$, $T = T_r$.

In this section we prove that the clock process $\mu_\varepsilon(t)$ converges as $\varepsilon \to 0$ to a limit (which might still be degenerate). By Proposition 3.2 in [4],

$$\operatorname{Var}\bigl(h_\varepsilon(z)\bigr) = -\log\varepsilon + \log\bigl(R(z; D)\bigr)$$

for all ε such that $B(z,\varepsilon) \subset D$, where $R(z; D)$ is the conformal radius of z in D. That is, $R(z; D) = 1/|\phi'(z)|$, where $\phi : D \to \mathbb{D}$ is a conformal map such that $\phi(z) = 0$. Note that for $t \le T = T_r$, there are two nonrandom constants c_1, c_2 such that

$$c_1 \le R(B_t; D) \le c_2.$$

We will denote $\bar{h}_\varepsilon(z) = \gamma h_\varepsilon(z) + (\gamma^2/2)\log\varepsilon$.

Proof. We start by pointing out a potential source of confusion. Note that for each *fixed* $z \in D$, the sequence $\alpha_\varepsilon(z) = \mathrm{e}^{\gamma h_\varepsilon(z) - (\gamma^2/2)\operatorname{Var} h_\varepsilon(z)}$ (viewed as a function of ε) forms a nonnegative martingale, in its own filtration. Hence by Fubini's theorem,

$$\mathbb{E}\biggl(\int_0^t \alpha_\varepsilon(B_s)\,\mathrm{d}s\biggr) = t.$$

Nevertheless, the above does *not* imply that $\alpha_\varepsilon(t)$ is a martingale as a function of ε: this is because the martingale property of $\alpha_\varepsilon(z)$ ceases to hold when the filtration contains all the information about $(h_\varepsilon(w), w \in D)$.

Nevertheless, the random variables $\mu_\varepsilon(t)$ converge as $\varepsilon \to 0$ almost surely to a limit. We now prove this statement. In fact we only prove this along the subsequence $\varepsilon = 2^{-k}, k \ge 1$. With an abuse of notation we write μ_k for $\mu_{2^{-k}}$ and h_k for $h_{2^{-k}}$. Then it suffices to prove that $|\mu_k - \mu_{k+1}| \le Cr^k$ for some $r < 1$ and $C < \infty$, almost surely. Assume without loss of generality that $t = 1$ and let $s \in [0, 1]$. Let $S_k^s = [0, 1] \cap \{s + 2^{-2k}\mathbb{Z}\}$, and let

$$X_k(s) = \frac{1}{2^{2k}} \sum_{t \in S_k^s} \mathrm{e}^{\bar{h}_k(B_t)}$$

and

$$Y_k(s) = \frac{1}{2^{2k}} \sum_{t \in S_k^s} \mathrm{e}^{\bar{h}_{k+1}(B_t)}.$$

Note that $\mu_k = \int_0^1 X_k(s)\,\mathrm{d}s$ and $\mu_{k+1} = \int_0^1 Y_k(s)\,\mathrm{d}s$ so it suffices to prove that

$$\bigl|X_k(s) - Y_k(s)\bigr| \le Cr^k \tag{6}$$

for some $C, r < 1$ uniformly in $s \in [0, 1]$. As in [4], we start with the case $\gamma < \sqrt{2}$ where an easy second moment argument suffices. Let $\tilde{\mathbb{E}}(\cdot) = \mathbb{E}(\cdot|\sigma(B_s, s \le t))$. Then note that

$$\tilde{\mathbb{E}}\bigl(\bigl|X_k(s) - Y_k(s)\bigr|^2\bigr) = \frac{1}{2^{4k}} \sum_{t,t' \in S_k^s} \tilde{\mathbb{E}}\bigl[\bigl(\mathrm{e}^{\bar{h}_k(B_t)} - \mathrm{e}^{\bar{h}_{k+1}(B_t)}\bigr)\bigl(\mathrm{e}^{\bar{h}_k(B_{t'})} - \mathrm{e}^{\bar{h}_{k+1}(B_{t'})}\bigr)\bigr].$$

Let $t, t' \in S_k^s$ and assume that $|B_t - B_{t'}| > 2^{-k}$. Then conditionally on $h_k(B_t)$ and $h_k(B_{t'})$, the random variables $h_{k+1}(B_t)$ and $h_{k+1}(B_{t'})$ are independent Gaussian random variables with mean $h_k(B_t)$ (resp. $h_k(B_{t'})$) and variance $\log 2$. Thus, in that case,

$$\begin{aligned}\tilde{\mathbb{E}}\bigl[\bigl(\mathrm{e}^{\bar{h}_k(B_t)} - \mathrm{e}^{\bar{h}_{k+1}(B_t)}\bigr)\bigl(\mathrm{e}^{\bar{h}_k(B_{t'})} - \mathrm{e}^{\bar{h}_{k+1}(B_{t'})}\bigr)|h_k(B_t), h_k(B_{t'})\bigr] &= \tilde{\mathbb{E}}\bigl[\mathrm{e}^{\bar{h}_k(B_t)} - \mathrm{e}^{\bar{h}_{k+1}(B_t)}|h_k(B_t), h_k(B_{t'})\bigr] \\ &\quad\times \tilde{\mathbb{E}}\bigl[\mathrm{e}^{\bar{h}_k(B_{t'})} - \mathrm{e}^{\bar{h}_{k+1}(B_{t'})}|h_k(B_t), h_k(B_{t'})\bigr].\end{aligned}$$

Now, observe that $\bar{h}_{k+1}(B_t) = \bar{h}_k(B_t) + \gamma X - (\gamma^2/2)\log 2$ where X is a centred Gaussian random variable with variance $\log 2$, which is independent from $\tilde{B}, h_k(B_t), h_k(B_{t'})$. Hence

$$\tilde{\mathbb{E}}\big[\mathrm{e}^{\bar{h}_k(B_t)} - \mathrm{e}^{\bar{h}_{k+1}(B_t)} | h_k(B_t), h_k(B_{t'})\big] = \mathrm{e}^{\bar{h}_k(B_t)}\big[1 - \mathrm{e}^{-(\gamma^2/2)\log 2}\tilde{\mathbb{E}}\big(\mathrm{e}^{\gamma X}\big)\big] = 0.$$

Of course the same also holds once we uncondition on $h_k(B_t), h_k(B_{t'})$. It follows by Cauchy–Schwarz's inequality that

$$\begin{aligned}
\tilde{\mathbb{E}}\big(\big|X_k(s) - Y_k(s)\big|^2\big) &= \frac{1}{2^{4k}} \sum_{t,t' \in S_k^s} 1_{|B_t - B_{t'}| \le 2^{-k}} \tilde{\mathbb{E}}\big[\big(\mathrm{e}^{\bar{h}_k(B_t)} - \mathrm{e}^{\bar{h}_{k+1}(B_t)}\big)\big(\mathrm{e}^{\bar{h}_k(B_{t'})} - \mathrm{e}^{\bar{h}_{k+1}(B_{t'})}\big)\big] \\
&\le \frac{1}{2^{4k}} \sum_{t,t' \in S_k^s} 1_{|B_t - B_{t'}| \le 2^{-k}} \sqrt{\tilde{\mathbb{E}}\big[\big(\mathrm{e}^{\bar{h}_k(B_t)} - \mathrm{e}^{\bar{h}_{k+1}(B_t)}\big)^2\big]\tilde{\mathbb{E}}\big[\big(\mathrm{e}^{\bar{h}_k(B_{t'})} - \mathrm{e}^{\bar{h}_{k+1}(B_{t'})}\big)^2\big]} \\
&= \frac{C}{2^{4k}} \sum_{t,t' \in S_k^s} 1_{|B_t - B_{t'}| \le 2^{-k}} \mathbb{E}\big[\big(\mathrm{e}^{\bar{h}_k(z)} - \mathrm{e}^{\bar{h}_{k+1}(z)}\big)^2\big],
\end{aligned} \tag{7}$$

where z is any point in D such that $\mathrm{dist}(z, \partial D) \ge r$. To compute the expectation in the sum, we condition on $h_k(z)$ and get, letting $\varepsilon = 2^{-k}$,

$$\begin{aligned}
\mathbb{E}\big[\big(\mathrm{e}^{\bar{h}_k(z)} - \mathrm{e}^{\bar{h}_{k+1}(z)}\big)^2\big] &\le \mathbb{E}\big(\mathrm{e}^{2\bar{h}_k(z)}\big) = \varepsilon^{\gamma^2}\mathbb{E}\big(\mathrm{e}^{2\gamma h_k(z)}\big) \\
&\le C\varepsilon^{\gamma^2 - 4\gamma^2/2} = C\varepsilon^{-\gamma^2}.
\end{aligned}$$

Therefore,

$$\tilde{\mathbb{E}}\big(\big|X_k(s) - Y_k(s)\big|^2\big) \le C\varepsilon^4\varepsilon^{-\gamma^2} \sum_{t,t' \in S_k^s} 1_{|B_t - B_{t'}| \le 2^{-k}}. \tag{8}$$

We will need the following lemma on two-dimensional Brownian motion:

Lemma 2.1. *There exists $C = C(\omega)$ depending only on the realisation of B, such that, uniformly in $s \in [0, 1]$:*

$$\sum_{t,t' \in S_k^s} 1_{|B_t - B_{t'}| \le 2^{-k}} \le C2^{2k}k^4,$$

almost surely.

Proof. Key to the proof will be a result of Dembo, Peres, Rosen and Zeitouni [3]. Let μ denote the occupation measure of Brownian motion at time 1. Then Theorem 1.2 of [3] states that

$$\lim_{\delta \to 0} \sup_{x \in \mathbb{R}^2} \frac{\mu(D(x, \delta))}{\delta^2(\log(1/\delta))^2} = 2 \tag{9}$$

almost surely. In particular, there exists $M(\omega)$ such that $\mu(D(x, \delta)) \le M(\omega)\delta^2(\log(1/\delta))^2$.

Let $\mathcal{A}$ denote the event where there is some $t \in [0, 1]$ such that

$$\sum_{t' \in S_k^t} 1_{|B_t - B_{t'}| \le 2^{-k}} > Ck^4,$$

where $C = C(\omega)$ is chosen suitably. By Lévy's modulus of continuity theorem, we know that

$$\sup_{|s-t| \le 2^{-2k}} |B_s - B_t| \le \delta := c2^{-k}\sqrt{k}$$

almost surely for some universal $c > 0$ and for all k sufficiently large. On the event $\mathcal{A}$, we can thus find $x = B_t$ where $\mu(D(x,\delta)) \geq Ck^4 2^{-2k}$ for k sufficiently large. But note that $\delta^2(\log(1/\delta))^2 \leq c2^{-2k}k^3$ for some nonrandom $c > 0$. Hence choosing $C(\omega) = cM(\omega)$ we see from (9) that $\mathbb{P}(\mathcal{A}) = 0$, as desired. $\square$

Plugging the estimate of Lemma 2.1 into (8), we get

$$\tilde{\mathbb{E}}\big(\big|X_k(s) - Y_k(s)\big|^2\big) \leq C\varepsilon^{4-\gamma^2-2}(\log 1/\varepsilon)^4,$$

which proves (6) at least if $\gamma < \sqrt{2}$.

To prove (6) in the general case ($\gamma < 2$), we introduce the set

$$\tilde{S}_k^s = \big\{t \in S_k^s \colon h_\varepsilon(B_t) < -\alpha \log\big(\varepsilon/R(B_t; D)\big)\big\},$$

where $\alpha > \gamma$ is a fixed parameter which will be chosen close enough to γ later on, and $R(z; D)$ denotes the conformal radius at the point $z \in D$. We let $T_k^s = S_k^s \setminus \tilde{S}_k^s$. Then we have

$$X_k(s) = \frac{1}{2^{2k}} \sum_{t \in T_k^s} \mathrm{e}^{\bar{h}_k(B_t)} + \frac{1}{2^{2k}} \sum_{t \in \tilde{S}_k^s} \mathrm{e}^{\bar{h}_k(B_t)}.$$

It is easy to show that the first is negligible. If $\tilde{\mathbb{Q}}$ denotes the law of the exponential tilting of $\tilde{\mathbb{P}}$ by $\mathrm{e}^{\gamma h_k(B_t)}$, i.e.,

$$\frac{d\tilde{\mathbb{Q}}}{d\tilde{\mathbb{P}}}(\omega) = \frac{\mathrm{e}^{\gamma h_k(B_t)}}{\tilde{\mathbb{E}}(\mathrm{e}^{\gamma h_k(B_t)})},$$

then letting $\sigma^2 = -\log(\varepsilon/R(B_t; D))$,

$$\tilde{\mathbb{Q}}\big(h_k(B_t) \in dx\big) = \frac{\mathrm{e}^{-x^2/(2\sigma^2)}\mathrm{e}^{\gamma x}\,\mathrm{d}x/\sqrt{2\pi\sigma^2}}{\int \mathrm{e}^{-x^2/(2\sigma^2)}\mathrm{e}^{\gamma x}\,\mathrm{d}x/\sqrt{2\pi\sigma^2}} = \mathrm{e}^{-(x-m)^2/(2\sigma^2)}\frac{\mathrm{d}x}{\sqrt{2\pi\sigma^2}}, \tag{10}$$

where $m = \gamma\sigma^2$, and hence the law of $h_K(B_t)$ under $\tilde{\mathbb{Q}}$ is $\mathcal{N}(m, \sigma^2)$. Thus

$$\begin{aligned}\tilde{\mathbb{E}}\big(1_{t \in T_k^s}\mathrm{e}^{\bar{h}_k(B_t)}\big) &= \varepsilon^{\gamma^2/2}\tilde{\mathbb{Q}}\big(h_k(B_t) > \alpha\sigma^2\big) \times \tilde{\mathbb{E}}\big(\mathrm{e}^{\gamma h_k(B_t)}\big) \\ &\leq \varepsilon^{\gamma^2/2}\exp\bigg(-\frac{1}{2}(\alpha-\gamma)^2\sigma^2\bigg)R(B_t; D)^{\gamma^2/2},\end{aligned}$$

where the bound above is obtained by using standard bounds on the normal tail distribution. This decays exponentially fast with k uniformly in $t \leq T$.

Likewise, by conditioning on $\bar{h}_k(B_t)$, we get that

$$\tilde{\mathbb{E}}\frac{1}{2^{2k}}\sum_{t \in T_k^s}\mathrm{e}^{\bar{h}_{k+1}(B_t)} \leq C\tilde{\mathbb{E}}\frac{1}{2^{2k}}\sum_{t \in T_k^s}\mathrm{e}^{\bar{h}_k(B_t)},$$

where $C < \infty$ depends only on γ, and thus this tends to 0 exponentially fast.

Define now $\tilde{X}_k(s) = \frac{1}{2^{2k}}\sum_{t \in \tilde{S}_k^s}\mathrm{e}^{\bar{h}_k(B_t)}$ and $\tilde{Y}_k(s) = \frac{1}{2^{2k}}\sum_{t \in \tilde{S}_k^s}\mathrm{e}^{\bar{h}_{k+1}(B_t)}$. We wish to bound $\mathbb{E}((\tilde{X}_k(s) - \tilde{Y}_k(s))^2)$. Applying the same reasoning as in (7) shows that

$$\mathbb{E}\big(\big(\tilde{X}_k(s) - \tilde{Y}_k(s)\big)^2\big) \leq \frac{C}{2^{4k}}\sum_{t,t' \in \tilde{S}_k^s} 1_{|B_t - B_{t'}| \leq 2^{-k}}\sqrt{\mathbb{E}\big(\mathrm{e}^{2\bar{h}_k(B_t)}\big)\mathbb{E}\big(\mathrm{e}^{2\bar{h}_k(B_{t'})}\big)}.$$

Now when $t \in \tilde{S}_k^s$, using the same reasoning as in (10) but with tilting proportional to $\mathrm{e}^{2\gamma h_k(B_t)}$ instead

$$\mathbb{E}\big(1_{t \in \tilde{S}_k^s}\mathrm{e}^{2\bar{h}_k(B_t)}\big) = \varepsilon^{\gamma^2}\mathbb{Q}\big(X < \alpha\sigma^2\big)\mathbb{E}\big(\mathrm{e}^{2\gamma h_k(B_t)}\big),$$

where $X \sim \mathcal{N}(2\gamma\sigma^2, \sigma^2)$. We may if we wish assume that $\alpha < 2\gamma$, so

$$\mathbb{Q}\big(X < \alpha\sigma^2\big) \le \exp\left(-\frac{1}{2}(2\gamma - \alpha)^2\sigma^2\right) \le C\exp\left(\frac{1}{2}(2\gamma-\alpha)^2 \log\varepsilon\right).$$

Thus using Lemma 2.1 again,

$$\begin{aligned}\mathbb{E}\big(\big(\tilde{X}_k(s) - \tilde{Y}_k(s)\big)^2\big) &\le C\varepsilon^4 \times \big(\varepsilon^{-2}\log(1/\varepsilon)^4\big) \times \varepsilon^{\gamma^2+(2\gamma-\alpha)^2/2}\varepsilon^{-4\gamma^2/2} \\ &\le C(\log 1/\varepsilon)^4 \varepsilon^{2+(2\gamma-\alpha)^2/2-\gamma^2}.\end{aligned}$$

Choosing α arbitrarily close to γ we find that the exponent of ε is arbitrarily close to $2-\gamma^2/2$ which is positive since $\gamma < 2$. Thus we can find α close enough to γ such that the exponent is positive, in which case (6) follows.

As discussed at the beginning of the section, this implies almost sure convergence of $\mu_\varepsilon(t)$ to a limit $\mu(t)$ (which might still be identically zero at this stage). □

3. Proof of Theorem 1.2: Nondegeneracy

Let $r > 0$ and let $T_r = \inf\{t \ge 0\colon\ \mathrm{dist}(B_t, \partial U) \le r\}$.

We will first show that

$$\mathbb{P}_z\left\{\lim_{\varepsilon\to 0}\int_0^{T_r} \mathrm{e}^{\gamma h_\varepsilon(B_s)}\varepsilon^{\gamma^2/2}\,\mathrm{d}s > 0\right\} > 0.$$

It suffices to show that the integral is bounded in L^q for some $q > 1$. Our strategy is inspired by work of Bacry and Muzy [1] on multifractal random measures. Since the proof can appear a bit convoluted, we start by explaining what lies behind it. Essentially, the qth moment of the integral can be understood as the sum of two terms: one diagonal term which gives the sum of the local contribution of the field at each point, and a cross-diagonal term which evaluates how these various bits interact with one another. Consider a square S in the domain and such that $z \in S$. The strategy will be to slice the square into many squares of sidelength 2^{-m}, where m will be a large but finite number. The key part of the estimate is to show that the sum of the contributions inside each smaller square is small. To achieve this, we use a scaling argument, as the Gaussian free field in a small square can be thought of as a general 'background' height plus an independent Gaussian free field in the square.

Without loss of generality, we will assume that $z \in S = (0,1)^2$ the unit square, and D contains the square S', where S' is the square centered on S whose sidelength is 3 (i.e., $S' = (-1,2)^2$). Then it will suffice to check that

$$\int_0^\tau \mathrm{e}^{\gamma h_\varepsilon(B_s)}\varepsilon^{\gamma^2/2}\,\mathrm{d}s \text{ is bounded in } L^q,$$

where $\tau = \inf\{t \ge 0\colon\ B_t \notin S\}$. We will in fact show the slightly stronger statement that

$$\int_0^T \mathrm{e}^{\gamma h_\varepsilon(B_s)}\varepsilon^{\gamma^2/2}\mathbf{1}_{\{B_s\in S\}}\,\mathrm{d}s \text{ is bounded in } L^q, \tag{11}$$

where $T = \inf\{t \ge 0\colon\ B_t \notin S'\}$.

3.1. *Auxiliary field*

Fix a bounded continuous function $\phi : [0,\infty) \to [0,\infty)$ which is a bounded positive definite function and such that $\phi(x) = 0$ if $x \ge 1$. For instance, we choose $\phi(x) = \sqrt{(1-|x|)_+}$, see [10] and the discussion in Example 2.3 in [11]. Define an auxiliary centered Gaussian random field $(X_\varepsilon(x))_{x\in\mathbb{R}^d}$ by specifying its covariance

$$c_\varepsilon(x,y) := \mathbb{E}\big(X_\varepsilon(x)X_\varepsilon(y)\big) = \log_+\left(\frac{8}{|x-y| \vee \varepsilon}\right) + \phi\left(\frac{|y-x|}{\varepsilon}\right).$$

Define also the normalized field to be $\bar{X}_\varepsilon(x) = \gamma X_\varepsilon(x) - (\gamma^2/2)\sigma_\varepsilon$, with $\sigma_\varepsilon = c_\varepsilon(0,0) = \log(8/\varepsilon) + 1$, so that $\mathbb{E}(\mathrm{e}^{\bar{X}_\varepsilon(x)}) = 1$. Because we have assumed that $S' \subset D$, it is easy to check that the covariance structure of $\bar{X}_\varepsilon$ and γh_ε are very close: more precisely, there are constants a and b, independent of ε, such that

$$c_\varepsilon(x,y) - a \le \mathbb{E}\big(\gamma h_\varepsilon(x)\gamma h_\varepsilon(y)\big) \le c_\varepsilon(x,y) + b \tag{12}$$

for all $x, y \in S'$. Condition for a moment on the trajectory of the Brownian path $(B_s, s \le T)$, and let $\tilde{\mathbb{E}}$ denote the corresponding conditional expectation. Define a measure μ_ε to be the (random) Borel measure on $[0,T]$ whose density with respect to Lebesgue measure is $\mathrm{e}^{\gamma h_\varepsilon(B_s)}\varepsilon^{\gamma^2/2}\mathbf{1}_{\{B_s\in S\}}$, $s \in [0,T]$. Let $I = \{s \in [0,T]\colon\ B_s \in S\}$. Note that conditionally given B, the process $(h_\varepsilon(B_s))_{s\in I}$ is a centered Gaussian process indexed by I with covariance function $\eta_\varepsilon(B_s, B_t)$, where $\eta_\varepsilon(x,y)$ is the covariance function of the (unconditional) Gaussian field $(h_\varepsilon(x))_{x\in D}$. By Theorem 2 of Kahane [8], we deduce from the right-hand side of (12) that

$$\tilde{\mathbb{E}}\big[\big(\mu_\varepsilon(0,T)\big)^q\big] \le \tilde{\mathbb{E}}\bigg[\bigg(\int_0^T \mathrm{e}^{Y_\varepsilon(s)-(1/2)\tilde{\mathbb{E}}(Y_\varepsilon(s))^2}\mathbf{1}_{\{B_s\in S\}}\,\mathrm{d}s\bigg)^q\bigg], \tag{13}$$

where for $s \in I$, $Y_\varepsilon(s)$ is a Gaussian centered field with covariance $c_\varepsilon(B_s, B_t) + b$. Thus $Y_\varepsilon(s)$ may be realized as $Y_\varepsilon(s) = \bar{X}_\varepsilon(B_s) + W$, where W is a fixed independent centered normal random variable of variance b. We deduce that

$$\begin{aligned}\tilde{\mathbb{E}}\big[\big(\mu_\varepsilon(0,T)\big)^q\big] &\le \tilde{\mathbb{E}}\big(\mathrm{e}^{qW-qb/2}\big)\tilde{\mathbb{E}}\bigg[\bigg(\int_0^T \mathrm{e}^{\bar{X}_\varepsilon(B_s)}\mathbf{1}_{\{B_s\in S\}}\,\mathrm{d}s\bigg)^q\bigg] \\ &= \mathrm{e}^{q(q-1)b/2}\tilde{\mathbb{E}}\bigg[\bigg(\int_0^T \mathrm{e}^{\bar{X}_\varepsilon(B_s)}\mathbf{1}_{\{B_s\in S\}}\,\mathrm{d}s\bigg)^q\bigg].\end{aligned}$$

Taking expectations,

$$\mathbb{E}\big[\big(\mu_\varepsilon(0,T)\big)^q\big] \le \mathrm{e}^{q(q-1)b/2}\mathbb{E}\bigg[\bigg(\int_0^T \mathrm{e}^{\bar{X}_\varepsilon(B_s)}\mathbf{1}_{\{B_s\in S\}}\,\mathrm{d}s\bigg)^q\bigg]. \tag{14}$$

Reasoning similarly with the left-hand side of (12) gives us

$$\mathbb{E}\big[\big(\mu_\varepsilon(0,T)\big)^q\big] \ge \mathrm{e}^{-q(q-1)a/2}\mathbb{E}\bigg[\bigg(\int_0^T \mathrm{e}^{\bar{X}_\varepsilon(B_s)}\mathbf{1}_{\{B_s\in S\}}\,\mathrm{d}s\bigg)^q\bigg]. \tag{15}$$

The crucial observation about X_ε (and the reason why we introduce it) is that it enjoys an exact scaling relation, as follows:

Lemma 3.1. *For* $\lambda < 1$,

$$\big(X_{\lambda\varepsilon}(\lambda x)\big)_{x\in B(0,4)} =_d \big(\Omega_\lambda + X_\varepsilon(x)\big)_{x\in B(0,4)},$$

where Ω_λ *is an independent centered Gaussian random variable with variance* $\log(1/\lambda)$.

Proof. One easily checks that for all $x, y \in \mathbb{R}^2$ such that $\|x-y\| \le 8$, $c_{\lambda\varepsilon}(\lambda x, \lambda y) = \log(1/\lambda) + c_\varepsilon(x,y)$. □

3.2. *Moments of order* $q > 1$

Therefore, fix $1 < q < 2$ and consider for $z \in S = [0,1]^2$ the unit square,

$$f_\varepsilon(z) = \mathbb{E}_z\bigg[\bigg(\int_0^T \mathrm{e}^{\bar{X}_\varepsilon(B_s)}\mathbf{1}_{\{B_s\in S\}}\,\mathrm{d}s\bigg)^q\bigg], \tag{16}$$

where as before, $T = \inf\{t \geq 0\colon\ B_t \notin S'\}$, and $S' = [-1,2]^2 \subset B(0,4)$. We let $M_\varepsilon = \sup_{z\in S} f_\varepsilon(z)$. Our goal will be to show that

$$M_\varepsilon \text{ is uniformly bounded in } \varepsilon \text{ for some choice of } q > 1. \tag{17}$$

The strategy for the proof of (17) will be the following. We fix $m \geq 1$, which we will choose suitably large (but fixed) at some point. We split the square S into a checkerboard pattern of squares S_i, each of which has sidelength 2^{-m}. By Minkowski's inequality, it suffices to show that

$$\sum_{i\in I} \int_0^T \mathrm{e}^{\bar{X}_\varepsilon(B_s)} \mathbf{1}_{\{B_s\in S_i\}}\,\mathrm{d}s$$

is uniformly bounded in L^q, where $(S_i)_{i\in I}$ is a maximal subset of squares such that $|z-w| > 2^{-m}$ for $z \in S_i$, $w \in S_j$ and $i \neq j \in I$. In words, we have retained "every other subsquare" in I. Note that there are at most $|I| \leq 4^m$ such subsquares.

Since $q < 2$, the function $x \mapsto x^{q/2}$ is concave. Being equal to 0 at $x = 0$, it is therefore subadditive. Hence letting $d_i = \int_0^T \mathrm{e}^{\bar{X}_\varepsilon(B_s)} \mathbf{1}_{\{B_s\in S_i\}}\,\mathrm{d}s$,

$$\mathbb{E}_z\biggl[\biggl(\sum_{i\in I}\int_0^T \mathrm{e}^{\bar{X}_\varepsilon(B_s)}\mathbf{1}_{\{B_s\in S_i\}}\,\mathrm{d}s\biggr)^q\biggr] \leq \mathbb{E}_z\biggl[\biggl(\sum_{i\in I} d_i^{q/2}\biggr)^2\biggr] = \sum_{i\in I}\mathbb{E}_z\bigl(d_i^q\bigr) + \sum_{i\neq j}\mathbb{E}_z\bigl(d_i^{q/2}d_j^{q/2}\bigr).$$

We treat separately the diagonal terms and the nondiagonal ones. We start by the diagonal terms.

Lemma 3.2. *There exists C independent of m, q and ε such that*

$$\sum_{i\in I}\mathbb{E}_z\bigl(d_i^q\bigr) \leq Cm2^{m(\zeta(q)+2)} M_{\varepsilon 2^m},$$

where

$$\zeta(q) := q^2\frac{\gamma^2}{2} - q\biggl(2 + \frac{\gamma^2}{2}\biggr). \tag{18}$$

Proof. Let $U_i = \inf\{t \geq 0\colon\ B_t \in S_i\}$ and let $T_i = \inf\{t \geq U_i\colon\ B_s \notin S_i'\}$, where S_i' is the square centered on S_i containing the 8 adjacent dyadic squares of same size as S_i. Let N_i denote the number of times that the path of the Brownian motion returns to S_i after having touched the boundary of S_i', before T.

Then applying the Markov property at each such return, we get

$$\max_{z\in S}\mathbb{E}_z\bigl(d_i^q\bigr) \leq C\max_{w\in S_i}\mathbb{E}_w\bigl(\beta_i^q\bigr)\mathbb{E}_z(N_i), \tag{19}$$

where

$$\beta_i = \int_0^{T_i} \mathrm{e}^{\bar{X}_\varepsilon(B_s)}\mathbf{1}_{\{B_s\in S_i\}}\,\mathrm{d}s.$$

(Note that by the strong Markov property, (19) holds even when $z \notin S_i$, as before the first hitting time of S_i the contribution to the integral in d_i is zero.) Now, a simple martingale argument shows that for some constant $C > 0$,

$$\mathbb{E}_z(N_i) \leq C\log\bigl(2^m\bigr),$$

uniformly in $z \in S$ and $i \in I$.

We now use the scaling properties of both B and X_ε (i.e., Lemma 3.1) to estimate the diagonal terms. Let $\lambda = 1/2^m$, and write $2^m B_s = \tilde{B}_{s2^{2m}}$, where $\tilde{B}$ is a planar Brownian motion starting from $\tilde{w} = 2^m w$. Let $\tilde{S}_i = 2^m S_i$, $\tilde{S}_i' = 2^m S_i'$, and let $\tilde{T}_i = 2^{2m} T_i = \inf\{t \geq 0\colon\ \tilde{B}_t \notin \tilde{S}_i'\}$. Note that we have $S' \subset B(0,4)$. Hence for $w \in S_i$, performing a change of variables $u = 2^{2m} s$,

$$\begin{aligned}
\mathbb{E}_w\big(\beta_i^q\big) &= \mathbb{E}_w\left[\left(\int_0^{T_i} \mathbf{1}_{\{2^m B_s \in 2^m S_i\}} \mathrm{e}^{\gamma \Omega_\lambda + \gamma X_{\varepsilon 2^m}(2^m B_s)} \varepsilon^{\gamma^2/2}\, \mathrm{d}s\right)^q\right] \\
&= \mathbb{E}\big(\mathrm{e}^{q\gamma\Omega_\lambda}\big)\mathbb{E}_{\tilde{w}}\left[\left(\int_0^{T_i} \mathbf{1}_{\{\tilde{B}_{2^{2m}s} \in \tilde{S}_i\}} \mathrm{e}^{\gamma X_{\varepsilon 2^m}(\tilde{B}_{2^{2m}s})} \varepsilon^{\gamma^2/2}\, \mathrm{d}s\right)^q\right] \\
&\leq C \mathrm{e}^{q^2\gamma^2 \log(2^m)/2} 2^{-2qm - qm\gamma^2/2} \mathbb{E}_{\tilde{w}}\left[\left(\int_0^{\tilde{T}_i} \mathbf{1}_{\{\tilde{B}_u \in \tilde{S}_i\}} \mathrm{e}^{\gamma X_{\varepsilon 2^m}(\tilde{B}_u)} \big(2^m \varepsilon\big)^{\gamma^2/2}\, \mathrm{d}u\right)^q\right] \\
&\leq C 2^{m(q^2\gamma^2/2 - q(2+\gamma^2/2))} M_{2^m \varepsilon} = C 2^{m\zeta(q)} M_{2^m\varepsilon}
\end{aligned} \tag{20}$$

for a constant C that does not depend on m, q or ε.

Since there are at most $|I| = 4^m$ terms, we deduce that the contribution of the diagonal terms is at most

$$\sum_{i \in I} \mathbb{E}_z\big(d_i^q\big) \leq C m 2^{m(\zeta(q)+2)} M_{\varepsilon 2^m},$$

where $\zeta(q)$ is defined in (18). This finishes the proof of Lemma 3.2. □

3.3. *Interaction term*

We now look at the cross-diagonal terms.

Lemma 3.3. *There exists $C_{m,q}$ which may depend on m and q but not ε, such that*

$$\sum_{i \neq j} \mathbb{E}_z\big(d_i^{q/2} d_j^{q/2}\big) \leq C_{m,q}.$$

Proof. Since $q/2 \leq 1$, by Hölder's inequality,

$$\mathbb{E}_z\big(d_i^{q/2} d_j^{q/2}\big) \leq \mathbb{E}_z(d_i d_j)^{q/2}.$$

Now, let $\tilde{\mathbb{E}}$ denote the conditional expectation given $(B_s, s \leq T)$. Then by Fubini's theorem

$$\begin{aligned}
\tilde{\mathbb{E}}(d_i d_j) &= \int_0^T \int_0^T \tilde{\mathbb{E}}\big(\exp\big(\bar{X}_\varepsilon(B_s) + \bar{X}_\varepsilon(B_t)\big)\big) \mathbf{1}_{\{B_s \in S_i, B_t \in S_j\}}\, \mathrm{d}s\, \mathrm{d}t \\
&= \int_0^T \int_0^T (\varepsilon/8)^{\gamma^2} \exp\left(\frac{-2\gamma^2 \log(\varepsilon/8) + 2\gamma^2 \log_+ 8/(\varepsilon \vee |B_s - B_t|)}{2}\right) \mathbf{1}_{\{B_s \in S_i, B_t \in S_j\}}\, \mathrm{d}s\, \mathrm{d}t \\
&\leq \int_0^T \int_0^T \left(\frac{8}{|B_s - B_t|}\right)^{\gamma^2} \mathbf{1}_{\{B_s \in S_i, B_t \in S_j\}}\, \mathrm{d}s\, \mathrm{d}t \\
&\leq \big(8 \cdot 2^m\big)^{\gamma^2} T^2.
\end{aligned}$$

Hence, taking expectations, $\mathbb{E}_z(d_i d_j) \leq C_{m,q} \mathbb{E}_z(T^2) \leq C_{m,q}$. Taking the $(q/2)$th power, and summing over $i \neq j$, we get Lemma 3.3. □

Putting together these two lemmas, we immediately obtain

$$M_\varepsilon \leq C m 2^{m(\zeta(q)+2)} M_{2^m \varepsilon} + C_{m,q}. \tag{21}$$

The key fact is that the exponent $2+\zeta(q)$ may be chosen to be negative for some $q>1$. Indeed, note that $\zeta(1)=\gamma^2/2-2-\gamma^2/2=-2$, and $\zeta'(1)=(\gamma^2/2)-2<0$ if and only if $\gamma^2<4$ i.e. $\gamma<2$. Since this is the assumption of the theorem, we deduce $\zeta'(1)<0$ and hence it is possible to choose $q>1$ such that $2+\zeta(q)<0$.

Since $2+\zeta(q)<0$, we can choose m sufficiently large that $Cm2^{m(2+\zeta(q))}<1/2$, and so obtain that

$$M_\varepsilon \le \frac{1}{2}M_{2^m\varepsilon}+C_{m,q}.$$

Taking the supremum over $\varepsilon>\varepsilon_0$, we get

$$\begin{aligned}\sup_{\varepsilon>\varepsilon_0} M_\varepsilon &\le \frac{1}{2}\sup_{\varepsilon>2^m\varepsilon_0} M_\varepsilon + C_{m,q}\\ &\le \frac{1}{2}\sup_{\varepsilon>\varepsilon_0} M_\varepsilon + C_{m,q}\end{aligned}$$

and hence

$$\sup_{\varepsilon>\varepsilon_0} M_\varepsilon \le 2C_{m,q}.$$

This proves (17), and therefore (11). Thus $\alpha(T_r)>0$ with positive probability.

3.4. *Continuity*

Let μ_ε denote the random Borel measure on $\mathbb{R}$ obtained by

$$\mu_\varepsilon\big((s,t]\big)=\int_s^t \mathrm{e}^{\gamma h_\varepsilon(B_u)}\mathbf{1}_{\{u<T\}}\varepsilon^{\gamma^2/2}\,\mathrm{d}u.$$

Then by the first part of the argument, μ_ε converges to a measure μ and we have just shown that $\mu_\varepsilon(0,T)$ is bounded in L^q for some $q>1$, hence is uniformly integrable. Thus $\mathbb{E}(\mu(0,T))=\mathbb{E}(\int_0^T R(B_t;D)^{\gamma^2/2}\,\mathrm{d}t)>0$ and thus μ is positive at least with positive probability. We now check that this probability must in fact be equal to one.

Let D_i be the disc of radius 2^{-m-1} having the same centre as the square S_i. Let D_i' be the (open) disc of radius $2\cdot 2^{-m}$ having the same centre as D_i. Let $\sigma_i=\inf\{t\ge 0\colon\ B_t\in D_i\}$ and $\tau_i=\inf\{t\ge 0\colon\ B_t\notin D_i'\}$. Recall the subset I introduced earlier, which is a maximal subset of $\{1,\dots,2^{2m}\}$ such that the D_i' are pairwise disjoint. Note however each D_i' is tangent to four other discs D_j' (except near the boundary of S). Define the event

$$G_i=\left\{\int_{\sigma_i}^{\tau_i}\mathrm{d}\mu(t)>0\right\}.$$

Note that if just one of the G_i hold, then $\mu(0,T)>0$. We will in fact show that many G_i occur with high probability. Let $Z_m=\sum_{i\in I}\mathbf{1}_{G_i}$.

Using the Markov property of the Gaussian free field (see e.g. the statement of Proposition 2.3 in [7]), we see that conditionally given the values of $h|_U$ where $U=S\setminus\bigcup_{i\in I}D_i'$, we can write on each of the D_i, $i\in I$,

$$h=h_U+h^i, \tag{22}$$

where h_U is a.s. a harmonic function on $\bigcup_{i\in I}D_i'$ and h^i are independent Gaussian free field with zero boundary condition on $\partial D_i'$. Then note that the event G_i is unchanged if we replace h by h^i in the definition of $\mu(t)$. In other words, G_i is a function of h^i and of the Brownian path $(B_t,\sigma_i\le t\le\tau_i)$ only. By rotational invariance of h^i, and independence of the paths $(B_t,\sigma_i\le t\le\tau_i)$ up to rotations around the centre of D_i', we deduce that for each $i_1,\dots,i_k\in I$,

$$\mathbb{P}_z(G_{i_1}\cap\cdots\cap G_{i_k}|\sigma_{i_1}<\infty,\dots,\sigma_{i_k}<\infty)=\prod_{j=1}^k\mathbb{P}_z(G_{i_j}|\sigma_{i_j}<\infty).$$

In other words, the G_i are conditionally independent given $\{i\colon \sigma_i < \infty\}$. Moreover, $\mathbb{P}_z(G_i|\sigma_i > 0) = p$, where $p > 0$ does not depend on i, nor on m by an easy scale-invariance argument, and $p > 0$ because $\mathbb{E}_z(\mu[0, T]) > 0$.

Putting these pieces together, we find that Z_m has the same distribution as

$$Z_m = \sum_{i\in I} \xi_i \mathbf{1}_{\{\sigma_i<\infty\}}, \tag{23}$$

where ξ_i are i.i.d. Bernoulli random variable with parameter $p > 0$.

Lemma 3.4. *Assume that z is the centre of S and let $N_m = \sum_i \mathbf{1}_{\{\sigma_i<\infty\}}$. Then $N_m \to \infty$ almost surely as $m \to \infty$.*

Proof. Let $\sigma_i' = \inf\{t \geq 0\colon\ B_t \in D_i'\}$ then $N' = \sum_{i\in I} \mathbf{1}_{\{\sigma_i'<\infty\}}$. Note that we necessarily have $N' \geq 2^{m-1}$ almost surely. This is because the diameter of the $\{B_t, 0 \leq t \leq T\}$ is bounded by above $2^{-m+1}N'$ but is also at least equal to 1. Now, it is easy to see that for each new disc D_i' which is visited, there is a probability at least $1/2$ to visit D_i, independently for all i such that $\sigma_i' < \infty$. Hence $N_m \to \infty$ almost surely. □

Putting together (23) and Lemma 3.4 we deduce that

$$\mathbb{P}_z(Z_m > 0) \to 1$$

as $m \to \infty$. In particular, $\mathbb{P}(\mu(0, T) > 0) = 1$. Now, recall that $T = \inf\{t \geq 0\colon\ B_t \notin S\}$, and $S = [0, 1]^2$ and we have assumed without loss of generality $S \subset U$. The same holds with S replaced by any square containing z and which is a subset of U. Hence if $T^m = \inf\{t \geq 0\colon\ B_t \notin [-2^{-m}, 2^{-m}]^2\}$, we also deduce that $\mu[0, T^m] > 0$ almost surely. Since $T^m \to 0$ almost surely, this implies $\mu(0, t) > 0$ almost surely for all $t \leq T_0 = \inf\{t \geq 0\colon\ B_t \notin U\}$. Applying the Markov property, it follows from this that, almost surely, for all rationals $s < t < T_0$,

$$\mu\big((s, t]\big) > 0.$$

Hence, since μ is nondecreasing, this is also true for all times $s < t < T_0$ simultaneously.

Therefore $t \mapsto \mu^{-1}(t)$ is continuous with probability one, and so $t \mapsto Z(t) = B_{\mu^{-1}(t)}$ is also continuous with probability one.

Remark 3.5. *It is also easy to see from the proof that $\mu^{-1}(t)$ is a.s. increasing, and hence Z does not "stay stuck" anywhere. This argument is identical to that used in* [6]. *Indeed applying Kahane's convexity inequality as in* (13) *and scaling exactly as in* (20), *we see that*

$$\mathbb{E}_z\big[\mu(0, t)^q\big] \leq Ct^{-\zeta(q)/2}$$

and hence if $\mathcal{A}_x(s, u)$ is the event that $t \mapsto \mu(0, t)$ has a jump greater than x in the interval $[s, u)$

$$\begin{aligned}\mathbb{P}_z\big(\mathcal{A}_x(0, 1)\big) &\leq \sum_{j=1}^{n} \mathbb{P}_z\big(\mathcal{A}_x\big(j/n, (j + 1)/n\big)\big)\\ &\leq cnx^{-q}n^{\zeta(q)/2}\end{aligned}$$

by Chebyshev's inequality. But recall that $\zeta(q) + 2 < 0$ for some $q > 1$. Hence $n^{1+\zeta(q)/2} \to 0$ and $\mathbb{P}_z(\mathcal{A}_x(0, 1)) = 0$ and the result follows, since x is arbitrary.

4. Conformal invariance

Naturally, the Gaussian free field is conformally invariant as a random distribution. However, its regularisation h_ε is not, and so it is better to consider a different approximation of the Gaussian free field. Fix $f_1, \ldots$ an orthonormal basis of $H_0^1(D)$, say by considering normalised eigenvectors of $-\Delta$ with Dirichlet boundary conditions on ∂D.

Let $h^n(z) = \sum_{i=1}^n X_i f_i$, where X_i are i.i.d. standard normal random variables, and note that we can think of h^n as the orthogonal projection of h, which is formally the infinite sum $\sum_{i=1}^\infty X_i f_i$, onto $\mathrm{Span}(f_1, \dots, f_n)$.

Define

$$\mu^n\big([0,t]\big) = \int_0^t \mathbf{1}_{\{t<T\}} e^{\gamma h^n(B_t) - (\gamma^2/2) \operatorname{Var} h^n(B_t) + (\gamma^2/2) \log R(B_t; D)}\, dt. \tag{24}$$

The following proposition shows that approximating h by h_n does not change the limiting diffusion.

Proposition 4.1. *Almost surely for all $t \geq 0$,*

$$\mu^n\big([0,t]\big) \to \mu\big([0,t]\big)$$

as $n \to \infty$.

Proof. Define $h_n^\varepsilon(z)$ to be the average of $h^n(w)$ on a circle of radius ε about z. Then for each fixed $\varepsilon > 0$, the sequence $e^{\gamma h_\varepsilon^n(z) - (\gamma^2/2) \operatorname{Var} h_\varepsilon^n(z)}$ forms a nonnegative martingale with respect to n, and the filtration $\mathcal{H}_n = \sigma(h_i, 1 \leq i \leq n)$. The limit as $n \to \infty$ is naturally $e^{\gamma h_\varepsilon(z) - (\gamma^2/2) \operatorname{Var} h_\varepsilon(z)}$, which also has expectation equal to 1. Thus the martingale is uniformly integrable and we have

$$\mathbb{E}\big(\mu_\varepsilon\big([0,t]\big) | \mathcal{H}_n\big) = \int_0^t \mathbf{1}_{\{t<T\}} e^{\gamma h_\varepsilon^n(B_t) - (\gamma^2/2) \operatorname{Var} h_\varepsilon^n(B_t) + (\gamma^2/2) \log R(B_t; D)}\, dt.$$

Thus letting $\varepsilon \to 0$, since h_n and $\mathrm{Var}(h_n)$ are continuous,

$$\lim_{\varepsilon \to 0} \mathbb{E}\big(\mu_\varepsilon\big([0,t]\big) | \mathcal{H}_n\big) = \mu^n[0,t].$$

But by Fatou's lemma,

$$\mathbb{E}\big(\mu\big([0,t]\big) | \mathcal{H}_n\big) = \mathbb{E}\big(\liminf \mu_\varepsilon\big([0,t]\big) | \mathcal{H}_n\big) \leq \liminf_{\varepsilon \to 0} \mathbb{E}\big(\mu_\varepsilon\big([0,t]\big) | h^n\big) = \mu^n[0,t].$$

Hence, for all t, and all n, almost surely,

$$\mathbb{E}\big(\mu\big([0,t]\big) | \mathcal{H}_n\big) \leq \mu_n\big([0,t]\big).$$

But taking expectations, the left-hand side is equal to $\mathbb{E} \int_0^t \mathbf{1}_{\{t<T\}} (\log R(B_t; D))^{(\gamma^2/2)}\, dt$ as μ_ε is uniformly integrable, and the right-hand side is also equal to the same value. Since these two random variables are almost surely ordered and have the same expectation, they are almost surely equal.

We deduce

$$\mathbb{E}\big(\mu\big([0,t]\big) | \mathcal{H}_n\big) = \mu^n\big([0,t]\big).$$

By the martingale convergence, we deduce that $\mu^n([0,t]) \to \mu([0,t])$ as $n \to \infty$, almost surely. $\square$

Now, let $\phi : D \to \tilde{D}$ be a conformal transformation and let $\psi = \phi^{-1}$. Then writing $\tilde{f}_n = f_n \circ \psi$, we see that $\tilde{f}_n$ forms an orthonormal basis of $H_0^1(\tilde{D})$ (this is because $(\cdot,\cdot)_\nabla$ is conformally invariant). Thus let $\tilde{h}^n = h^n \circ \psi$, which is the projection of the Gaussian free field $h \circ \psi$ onto $\mathrm{Span}(\tilde{f}_1, \dots, \tilde{f}_n)$. Let $\mu_n = \mu^n$ and μ_n^{-1} be the inverse function of μ_n. Now by conformal invariance of ordinary Brownian motion,

$$\phi(B_t) = \tilde{B}_{\int_0^t |\phi'(B_s)|^2\, ds},$$

where $\tilde{B}$ is a killed Brownian motion in $\tilde{D}$. Thus

$$\phi(B_{\mu_n^{-1}(t)}) = \tilde{B}_{\sigma_n(t)} =: \tilde{Z}_n(t),$$

where, by definition, $\sigma_n(t) = \int_0^{\mu_n^{-1}(t)} |\phi'(B_s)|^2 \,\mathrm{d}s$. By the chain rule, if $\tilde{z} = \tilde{Z}_n(t)$ and $z = B_{\mu_n^{-1}(t)}$,

$$\sigma_n'(t) = \frac{\mathrm{d}}{\mathrm{d}t}\mu_n^{-1}(t)\left|\phi'(B_{\mu_n^{-1}(t)})\right|^2$$
$$= \frac{1}{\mathrm{e}^{\gamma h^n(z) - (\gamma^2/2)\operatorname{Var} h^n(z) + (\gamma^2/2)\log R(z;D)}|\psi'(\tilde{z})|^2}.$$

Observe that $\log R(\tilde{z}; \tilde{D}) = \log R(z, D) + \log|\phi'(z)|$ and hence

$$\sigma_n'(t) = \frac{1}{\mathrm{e}^{\gamma(\tilde{h}^n(\tilde{z}) + Q\log|\psi'(\tilde{z})|) - (\gamma^2/2)\operatorname{Var}\tilde{h}^n(\tilde{z}) + (\gamma^2/2)\log R(\tilde{z};\tilde{D})}}, \tag{25}$$

where $Q = \gamma/2 + 2/\gamma$. Thus define a field in $\tilde{D}$ by $\tilde{h}_\psi(w) = h \circ \psi + Q\log|\psi'|$ and let

$$\tilde{\mu}_\psi^n(t) = \int_0^t \mathbf{1}_{\{s<\tilde{T}\}} \mathrm{e}^{\gamma\tilde{h}_\psi^n(\tilde{B}_s) - (\gamma^2/2)\operatorname{Var}\tilde{h}_\psi^n(\tilde{B}_s) + (\gamma^2/2)\log R(\tilde{B}_s;\tilde{D})}\,\mathrm{d}s,$$

where $\tilde{T} = \inf\{t \ge 0\colon\ \tilde{B}_t \notin \tilde{D}\}$. Then a computation similar to (25) shows that $\mu(T) = \lim_{n\to\infty}\mu_n(T) = \tilde{\mu}_\psi(\tilde{T})$ and the right-hand side in (25) is simply the derivative of $\tilde{\mu}_\psi^n(t)^{-1}$. Hence we have proved, after taking limits as $n \to \infty$,

$$\bigl(\phi(Z_t), t < \mu(T)\bigr) = \bigl(\tilde{B}_{\tilde{\mu}_\psi^{-1}(t)}, t < \tilde{\mu}_\psi(\tilde{T})\bigr).$$

5. Proof of Theorem 1.5

We focus on the case $2 > \alpha > \gamma$ and consider the set $\{t\colon\ Z_t \in \mathcal{T}_\alpha^+\}$ (the other case is identical). To ease the proof we will drop the superscript $+$ from this notation and hence call $\mathcal{T}_\alpha := \mathcal{T}_\alpha^+$ in this proof. Let $d = (2 - \alpha^2/2)/(2 - \alpha\gamma + \gamma^2/2)$ so we wish to prove that $\dim\{t\colon\ Z_t \in \mathcal{T}_\alpha\} = d$ almost surely. Note that since $\alpha > \gamma$ we have $d < 1$; so we may choose $\eta > 0$ such that $q = d(1+\eta) < 1$. Since $\alpha < 2$, we have that $2 - \alpha^2/2 > 0$ and we may choose δ and $\varepsilon > 0$ small enough that $\eta(2 - \alpha^2/2) - \alpha\delta - 2\varepsilon > 0$. Then set

$$K = \frac{2}{\eta(2 - \alpha^2/2) - \alpha\delta - 2\varepsilon}$$

and choose $r_n = n^{-K}$ a sequence of scales. Let $t_{nj} = j r_n^{2+2\varepsilon}$, $1 \le j \le r_n^{-2-2\varepsilon}$ form a partition of $[0, 1]$ into intervals of size $r_n^{2+2\varepsilon}$.

If t_{nj} is the closest element of the net to t, then

$$\bigl|B(t) - B(t_{nj})\bigr| \le C\sqrt{r_n^{2+2\varepsilon}\log\bigl(1/r_n^{2+2\varepsilon}\bigr)}$$

by Lévy's result on the uniform modulus of continuity of Brownian motion [9]. Hence applying Proposition 2.1 in [7], for all $\zeta > 0$ and $\xi \in (0, 1/2)$,

$$\begin{aligned}\bigl|h_{r_n}\bigl(B(t)\bigr) - h_{r_n}\bigl(B(t_{nj})\bigr)\bigr| &\le MC^\xi(\log 1/r_n)^\zeta \frac{[\sqrt{r_n^{2+2\varepsilon}\log(1/r_n^{2+2\varepsilon})}]^\xi}{r_n^{\xi+\xi\varepsilon}} \\ &\le MC^\xi(2+2\varepsilon)^{\xi/2}(\log 1/r_n)^{\zeta+\xi/2} \\ &\le \delta\log(1/r_n)\end{aligned} \tag{26}$$

for n sufficiently large, provided that $\zeta + \xi/2 < 1$ (which we may assume if we wish). This motivates the following definition

$$\mathcal{I}_n = \bigl\{j\colon\ h_{r_n}\bigl(B(t_{nj})\bigr) \ge (\alpha - \delta)\log(1/r_n)\bigr\}.$$

The interest of introducing this set then comes from the fact that, by (26), if $B_t \in \mathcal{T}_\alpha$ then we can find arbitrarily large n such that $t \in [t_{nj} - r_n^2, t_{nj} + r_n^2]$ for some $j \in \mathcal{I}_n$. More precisely, recall the inverse clock function $t \mapsto \mu^{-1}(t)$ where with a slight abuse of notation we identify $\mu(t)$ and $\mu([0,t])$. Consider the image of $[t_{nj} - r_n^2, t_{nj} + r_n^2]$ by $t \mapsto \mu^{-1}(t)$. These are also intervals, which we denote by $[a_{nj}, b_{nj}]$. Let

$$J_N = \bigcup_{n \geq N} \{[a_{nj}, b_{nj}], j \in \mathcal{I}_n\}.$$

Hence a consequence of (26) is that for all $N \geq 1$, J_N covers $\{t\colon Z_t \in \mathcal{T}_\alpha\}$.

Now, it is plain that

$$\mathbb{E}(|\mathcal{I}_n|) = r_n^{-2-2\varepsilon}\mathbb{P}(h_{r_n}(z) \geq (\alpha-\delta)\log(1/r_n)) \sim r_n^{-2+(\alpha-\delta)^2/2-2\varepsilon+\mathrm{o}(1)},$$

and thus

$$\mathbb{E}(|\mathcal{I}_n|) \leq r_n^{-2+\alpha^2/2-\delta\alpha-2\varepsilon+\mathrm{o}(1)}. \tag{27}$$

We now estimate the diameter of these intervals. We first need the following lemma:

Lemma 5.1. *Let $\varepsilon < r$ and $z, w \in D$ such that $B(z,r) \cup B(w,r) \subset D$.Then*

$$\mathrm{Cov}(h_\varepsilon(w), h_r(z)) \leq \log(1/r) + C + \mathrm{o}_\varepsilon(1), \tag{28}$$

where C depends only on $\max(\mathrm{dist}(z,\partial D), \mathrm{dist}(w,\partial D))$ *and* $\mathrm{o}_\varepsilon(1) \to 0$ *as* $\varepsilon \to 0$ *and r is fixed.*

Proof. It is obvious that the result is true for $w = z$; and intuitively $h_r(z)$ and $h_w(\varepsilon)$ are most highly correlated when $r = z$. By definition (see e.g. Proposition 3.2 in [4]), the left-hand side in (28) is equal to $(\xi_\varepsilon^z, \xi_r^w)_\nabla$, where

$$\xi_r^z(\cdot) = -\log(|z - \cdot| \vee r) + \bar{\phi}^z(\cdot),$$

where $\bar{\phi}^z$ is harmonic in D and is equal to $-\log|z - \cdot|$ on ∂D. Let $\phi_r^z(\cdot) = -\log|z - \cdot| \vee r$. Note that $\nabla\xi_r^z = 0$ in $B(z,r)$ and $\nabla\xi_\varepsilon^w = 0$ in $B(w,\varepsilon)$. By Green's identity, and since ξ_r^z is harmonic outside of $B(z,r) \cup B(z,\varepsilon)$,

$$\frac{1}{2\pi}\int_D \nabla\xi_r^z \cdot \nabla\xi_\varepsilon^w = \frac{1}{2\pi}\int_{\partial(B(0,r)\cup B(w,\varepsilon))} \xi_\varepsilon^w \nabla\xi_r^z \cdot n \,\mathrm{d}\sigma + \frac{1}{2\pi}\int_{\partial D} \xi_\varepsilon^w \nabla\xi_r^z \cdot n \,\mathrm{d}\sigma,$$

where n is the nomal unit vector. Now, the second integral is clearly $\mathrm{O}(1)$, and the contribution coming from $\bar{\phi}_r^z$ is also easily shown to be $\mathrm{O}(1)$ so it suffices to show that

$$\int_{\partial B(z,r)} \xi_\varepsilon^w \nabla\phi_r^z \cdot n \,\mathrm{d}\sigma + \int_{\partial B(w,\varepsilon)} \xi_\varepsilon^w \nabla\phi_r^z \cdot n \,\mathrm{d}\sigma \leq 2\pi\log(1/r) + \mathrm{O}(1). \tag{29}$$

Obviously, on $\partial B(z,r)$, $\nabla\phi_r^z \cdot n = 1/r$. Assume for ease of computations (without loss of generality) that $z = 0$. Hence the first integral on the left-hand side of (29) is equal to

$$\int_0^{2\pi} -\log|r\mathrm{e}^{\mathrm{i}\theta} - w| \,\mathrm{d}\theta = 2\pi\log(1/r) - \int_0^{2\pi} \log\left|\mathrm{e}^{\mathrm{i}\theta} - \frac{w}{r}\right| \mathrm{d}\theta.$$

The second term in the right-hand side is uniformly bounded when $|w| \leq 2r$ and negative for $|w| \geq 2r$.

On the other hand, the second integral in (29) is bounded by $(1/r)2\pi\varepsilon\log(1/\varepsilon) = \mathrm{o}_\varepsilon(1)$, so the result follows. $\square$

Lemma 5.2. *Let $1 \leq j \leq r_n^{-2-2\varepsilon}$. Given that $j \in \mathcal{I}_n$, for all $q \leq 1$,*

$$\mathbb{E}(\mathrm{Diam}([a_{nj}, b_{nj}])^q \mid j \in \mathcal{I}_n) \leq C r_n^{q(2-\gamma\alpha+\gamma^2/2)+\mathrm{o}_n(1)},$$

where $\mathrm{o}_n(1) \to 0$ as $n \to \infty$.

Proof. By Jensen's inequality it suffices to prove the result for $q=1$. Note that $\mathrm{Diam}([a_{nj},b_{nj}])=\mu([t_{nj}-r_n^2,t_{nj}+r_n^2])$. Hence, by uniform integrability of μ_ε,

$$\mathbb{E}\big(\mathrm{Diam}\big([a_{nj},b_{nj}]\big)|j\in\mathcal{I}_n\big)=\lim_{\varepsilon\to 0}\mathbb{E}\big(\mu_\varepsilon\big[t_{nj}-r_n^2,t_{nj}+r_n^2\big]|j\in\mathcal{I}_n\big).$$

Denote $r=r_n$. Note that by conditioning on B and letting $B_{t_{nj}}=z$ and $j\in\mathcal{I}_n$, we get

$$\begin{aligned}\tilde{\mathbb{E}}\big(\mu_\varepsilon\big[t_{nj}-r^2,t_{nj}+r^2\big]|j\in\mathcal{I}_n\big)&=\tilde{\mathbb{E}}\bigg(\int_{t_{nj}-r^2}^{t_{nj}+r^2}\mathrm{e}^{\gamma h_\varepsilon(B_s)+(\gamma^2/2)\log\varepsilon}|j\in\mathcal{I}_n\bigg)\,\mathrm{d}s\\&=\int_{t_{nj}-r^2}^{t_{nj}+r^2}\mathrm{d}s\,\tilde{\mathbb{E}}\big[\mathrm{e}^{\gamma h_\varepsilon(B_s)+(\gamma^2/2)\log\varepsilon}|h_r(z)>\alpha\log(1/r)\big].\end{aligned}$$

Now note that

$$\mathbb{E}\big[\mathrm{e}^{\gamma h_\varepsilon(B_s)+(\gamma^2/2)\log\varepsilon}|h_r(z)>\alpha\log(1/r)\big]=\tilde{\mathbb{P}}\big(h_r(z)>\alpha\log(1/r)\big)^{-1}\mathbb{Q}\big(h_r(z)>\alpha\log(1/r)\big),\tag{30}$$

where $d\mathbb{Q}/d\tilde{\mathbb{P}}=\mathrm{e}^{\gamma h_\varepsilon(z)+(\gamma^2/2)\log\varepsilon}$. Under $\mathbb{Q}$, it is easy to check that $h_r(z)$ is normal with mean γc where $c=\mathrm{Cov}(h_r(z),h_\varepsilon(B_s))$, and variance $\sigma^2=\log(1/r)+\mathrm{O}(1)$. Therefore, using Lemma 5.1,

$$\begin{aligned}\mathbb{Q}\big(h_r(z)>\alpha\log(1/r)\big)&=\mathbb{Q}\big(N\big(0,\sigma^2\big)>\alpha\log(1/r)-\gamma c\big)\\&\le\exp\bigg(-\frac{1}{2}(\alpha-\gamma)^2\log(1/r)+C+\mathrm{o}_\varepsilon(1)\bigg)\\&=Cr^{-(\alpha-\gamma)^2/2}\big(1+\mathrm{o}_\varepsilon(1)\big).\end{aligned}$$

Observing that $\tilde{\mathbb{P}}(h_r(z)>\alpha\log(1/r))=r^{\alpha^2/2+\mathrm{o}_r(1)}$ where $\mathrm{o}_r(1)\to 0$ as $r\to 0$ and does not depend on ε, and plugging in (30) this gives

$$\begin{aligned}\mathbb{E}\big[\mathrm{e}^{\gamma h_\varepsilon(B_s)+(\gamma^2/2)\log\varepsilon}|h_r(z)>\alpha\log(1/r)\big]&\le Cr^{\alpha^2/2+\mathrm{o}_r(1)}r^{-(\alpha-\gamma)^2/2}\big(1+\mathrm{o}_\varepsilon(1)\big)\\&=Cr^{-\alpha\gamma+\gamma^2/2+\mathrm{o}_r(1)}\big(1+\mathrm{o}_\varepsilon(1)\big).\end{aligned}$$

Integrating over $s\in[t_{nj}-r^2,t_{nj}+r^2]$ and taking expectations, we obtain

$$\mathbb{E}\big(\mu_\varepsilon\big(\big[t_{nj}-r_n^2,t_{nj}+r_n^2\big]\big)|j\in\mathcal{I}_n\big)\le Cr_n^{2-\gamma\alpha+\gamma^2/2+\mathrm{o}_n(1)}\big(1+\mathrm{o}_\varepsilon(1)\big),$$

from which the result follows after letting $\varepsilon\to 0$. $\square$

Now, recall that we have chosen $q=d(1+\eta)<1$, by assumption on η. By Lemma 5.2, and due to our choice of K,

$$\begin{aligned}\mathbb{E}\bigg(\sum_{n\ge N}\sum_{j\in\mathcal{I}_n}\mathrm{Diam}\big([a_{nj},b_{nj}]\big)^q\bigg)&=\mathrm{O}\bigg(\sum_{n\ge N}r_n^{\eta(2-\alpha^2/2)-\alpha\delta-2\varepsilon+\mathrm{o}(1)}\bigg)\\&=\mathrm{O}\bigg(\sum_{n\ge N}n^{-2+\mathrm{o}(1)}\bigg).\end{aligned}$$

This proves that the Hausdorff q-dimension of $\{t\colon Z_t\in\mathcal{T}_\alpha\}$ is 0, almost surely. Since $\eta>0$ is arbitrary, this proves the result.

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