

# SAMPLE PATH BEHAVIOR OF A LÉVY INSURANCE RISK PROCESS APPROACHING RUIN, UNDER THE CRAMÉR–LUNDBERG AND CONVOLUTION EQUIVALENT CONDITIONS<sup>1</sup>

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Recent studies have demonstrated an interesting connection between the asymptotic behavior at ruin of a Lévy insurance risk process under the Cramér–Lundberg and convolution equivalent conditions. For example, the limiting distributions of the overshoot and the undershoot are strikingly similar in these two settings. This is somewhat surprising since the global sample path behavior of the process under these two conditions is quite different. Using tools from excursion theory and fluctuation theory, we provide a means of transferring results from one setting to the other which, among other things, explains this connection and leads to new asymptotic results. This is done by describing the evolution of the sample paths from the time of the last maximum prior to ruin until ruin occurs.

**1. Introduction.** It is becoming increasingly popular to model insurance risk processes with a general Lévy process. In addition to new and interesting mathematics, this approach allows for direct modeling of aggregate claims which can then be calibrated against real aggregate data, as opposed to the traditional approach of modeling individual claims. Whether this approach is superior remains to be seen, but it offers, at a minimum, an alternative, to the traditional approach. The focus of this paper will be on two such Lévy models, and their sample path behavior as ruin approaches.

Let  $X = \{X_t : t \geq 0\}$ ,  $X_0 = 0$ , be a Lévy process with characteristics  $(\gamma, \sigma^2, \Pi_X)$ . The characteristic function of  $X$  is given by the Lévy–Khintchine representation,  $E e^{i\theta X_t} = e^{t\Psi_X(\theta)}$ , where

$$\Psi_X(\theta) = i\theta\gamma - \sigma^2\theta^2/2 + \int_{\mathbf{R}} (e^{i\theta x} - 1 - i\theta x \mathbf{1}_{\{|x|<1\}}) \Pi_X(dx) \quad \text{for } \theta \in \mathbf{R}.$$

To avoid trivialities, we assume  $X$  is nonconstant. In the insurance risk model,  $X$  represents the excess in claims over premium. An insurance company starts with an initial positive reserve  $u$ , and ruin occurs if this level is exceeded by  $X$ .

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To reflect the insurance company’s desire to collect sufficient premia to prevent almost certain ruin, it is assumed that  $X_t \rightarrow -\infty$  a.s. This is the *general Lévy insurance risk model*, which we will investigate under two distinct conditions. The first is the well-known Cramér–Lundberg condition:

$$(1.1) \quad Ee^{\alpha X_1} = 1 \quad \text{and} \quad EX_1 e^{\alpha X_1} < \infty \quad \text{for some } \alpha > 0.$$

The second, introduced by Klüppelberg, Kyprianou and Maller [24], is the convolution equivalent condition:

$$(1.2) \quad Ee^{\alpha X_1} < 1 \quad \text{and} \quad X_1^+ \in \mathcal{S}^{(\alpha)} \quad \text{for some } \alpha > 0,$$

where  $\mathcal{S}^{(\alpha)}$  denotes the class of convolution equivalent distributions of index  $\alpha$ . The formal description of  $\mathcal{S}^{(\alpha)}$  will be given in Section 7. Typical examples of distributions in  $\mathcal{S}^{(\alpha)}$  are those with tails of the form

$$P(X_1 > x) \sim \frac{e^{-\alpha x}}{x^p} \quad \text{for } p > 1.$$

Under (1.2),  $Ee^{\theta X_1} = \infty$  for all  $\theta > \alpha$ , so (1.1) must fail. Hence, conditions (1.1) and (1.2) are mutually exclusive. For a further comparison, see the introduction to [20].

Historically, the first insurance risk model to be extensively studied was the *compound Poisson model*.<sup>2</sup> This arises when  $X$  is a spectrally positive compound Poisson process with negative drift. In recent years, attention has turned to the general Lévy insurance risk model (see Kyprianou [25] for a detailed discussion of the general model), where considerable progress has been made in calculating the limiting distribution of several variables related to ruin; see Doney, Klüppelberg and Maller [12], Doney and Kyprianou [13], Griffin and Maller [20], Klüppelberg, Kyprianou and Maller [24] and the references therein. To give some examples, particularly relevant to this paper, we first need a little notation. Set

$$\bar{X}_t = \sup_{0 \leq s \leq t} X_s,$$

$$\tau(u) = \inf\{t : X(t) > u\},$$

and let  $P^{(u)}$  denote the probability measure  $P^{(u)}(\cdot) = P(\cdot | \tau(u) < \infty)$ . Let  $H$  be the ascending ladder height process, and  $\Pi_H$ ,  $d_H$  and  $q$  its Lévy measure, drift and killing rate, respectively; see Section 2 for more details. Then under the Cramér–Lundberg condition (1.1), it was shown in [21] that the limiting distributions of the shortfall at, and the minimum surplus prior to, ruin are given by

$$(1.3) \quad \begin{aligned} P^{(u)}(X_{\tau(u)} - u \in dx) &\xrightarrow{w} q^{-1} \alpha \left[ d_H \delta_0(dx) + \int_{y \geq 0} e^{\alpha y} \Pi_H(y + dx) dy \right], \\ P^{(u)}(u - \bar{X}_{\tau(u)-} \in dy) &\xrightarrow{w} q^{-1} \alpha [d_H \delta_0(dy) + e^{\alpha y} \bar{\Pi}_H(y) dy], \end{aligned}$$

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<sup>2</sup>This is often called the Cramér–Lundberg model, as opposed to the Cramér–Lundberg condition (1.1).

where  $\xrightarrow{w}$  denotes weak convergence and  $\delta_0$  is a point mass at 0. Under the convolution equivalent condition (1.2), it follows from Theorem 4.2 in [24] and Theorem 10 in [13] (see also Section 7 of [20]) that the corresponding limits are

$$\begin{aligned}
 P^{(u)}(X_{\tau(u)} - u \in dx) &\xrightarrow{w} q^{-1}\alpha \left[ -\ln(Ee^{\alpha H_1})e^{-\alpha x} dx + d_H\delta_0(dx) \right. \\
 (1.4) \qquad \qquad \qquad &\qquad \qquad \qquad \qquad \qquad \qquad \left. + \int_{y \geq 0} e^{\alpha y} \Pi_H(y + dx) dy \right], \\
 P^{(u)}(u - \bar{X}_{\tau(u)-} \in dy) &\xrightarrow{v} q^{-1}\alpha [d_H\delta_0(dy) + e^{\alpha y} \bar{\Pi}_H(y) dy],
 \end{aligned}$$

where  $\xrightarrow{v}$  denotes vague convergence of measures on  $[0, \infty)$ .<sup>3</sup>

The resemblance between the results in (1.3) and (1.4) is striking and in many ways quite surprising, since the paths resulting in ruin behave very differently in the two cases as we now explain. Under the Cramér–Lundberg condition, with  $b = EX_1e^{\alpha X_1}$ , we have

$$\frac{\tau(u)}{u} \rightarrow b^{-1} \quad \text{in } P^{(u)} \text{ probability}$$

and

$$\sup_{t \in [0, 1]} \left| \frac{X(t\tau(u))}{\tau(u)} - bt \right| \rightarrow 0 \quad \text{in } P^{(u)} \text{ probability,}$$

indicating that ruin occurs due to the build up of small claims which cause  $X$  to behave as though it had positive drift; see Theorem 8.3.5 of [16].<sup>4</sup> By contrast in the convolution equivalent case, asymptotically, ruin occurs in finite time (in distribution), and for ruin to occur, the process must take a large jump from a neighborhood of the origin to a neighborhood of  $u$ . This jump may result in ruin, but if not, the resulting process  $X - u$  subsequently behaves like  $X$  conditioned to hit  $(0, \infty)$ . This representation of the limiting conditioned process leads to a straightforward proof of (1.4); see [20]. However, the description in the Cramér–Lundberg case is not sufficiently precise to yield (1.3). What is needed is a more refined characterization of the process as ruin approaches, specifically, a limiting description of the path from the time of the last strict maximum before time  $\tau(u)$  up until time  $\tau(u)$ .

In the discrete time setting, such a result was proved by Asmussen [1]. Let  $Z_k$  be i.i.d., nonlattice and set  $S_n = Z_1 + \dots + Z_n$ . Assume the Cramér–Lundberg condition,

$$Ee^{\alpha Z_1} = 1 \quad \text{and} \quad EZ_1e^{\alpha Z_1} < \infty \quad \text{for some } \alpha > 0.$$

<sup>3</sup>In (1.3) and (1.4), it is assumed that  $X_1$  has a nonlattice distribution. Similar results hold in the lattice case if the limit is taken through points in the lattice span, but to avoid repetition we will henceforth make the nonlattice assumption.

<sup>4</sup>The result cited in [16] follows from the work of Asmussen [1], which is for the compound Poisson model, but the result remains true for the general model.

As above, let  $\tau(u)$  be the first passage time of  $S_n$  over level  $u$  and  $\sigma(u)$  the time of the last strict ladder epoch prior to passage [thus  $\sigma(0) = 0$ ]. Set

$$Z(u) = (Z_{\sigma(u)+1}, \dots, Z_{\tau(u)}).$$

It follows from Section 8 of [1] that for  $G$  bounded and continuous

$$(1.5) \quad \begin{aligned} & E^{(u)} G(Z(u), S_{\tau(u)} - u) \\ & \rightarrow \frac{\int_0^\infty e^{\alpha y} E\{G(Z(0), S_{\tau(0)} - y); S_{\tau(0)} > y, \tau(0) < \infty\} dy}{C E(S_{\tau(0)}) e^{\alpha S_{\tau(0)}}; \tau(0) < \infty}, \end{aligned}$$

where  $C = \lim_{u \rightarrow \infty} e^{\alpha u} P(\tau(u) < \infty)$  and  $E^{(u)}$  denotes expectation with respect to the conditional probability  $P^{(u)}(\cdot) = P(\cdot | \tau(u) < \infty)$ . This result describes the limit of the conditioned process from the time of the last strict ladder epoch prior to first passage over a high level, up until the time of first passage. From it, the limiting distribution of several quantities related to first passage, such as those in (1.3), may be found in the random walk setting.

As it stands, the formulation in (1.5) makes no sense for a general Lévy process. To apply even to the compound Poisson model, the most popular risk model, some reformulation is needed. Furthermore, to prove (1.5), Asmussen derives a renewal equation by considering the two cases  $\tau(0) = \tau(u)$  and  $\tau(0) < \tau(u)$ . This is a standard renewal theoretic device which has no hope of success in the general Lévy insurance risk model since typically  $\tau(0) = 0$ . To circumvent these problems, we apply arguments from fluctuation theory and excursion theory. This allows us to describe, for any Lévy process, the final segment of the path from the time of the last maximum prior to ruin, up until the time of ruin. This description is in terms of the renewal measure  $V$  of the ascending ladder height process and the excursion measure of  $X$  below its running supremum  $\bar{X}$ . The key observation that ties together the two cases (1.1) and (1.2), and allows proof of convergence as  $u \rightarrow \infty$ , is that in either case,  $(\bar{V} \circ \ln)$  is regularly varying at infinity with index  $-\alpha$ , where  $\bar{V}(u) = V(\infty) - V(u)$ . This allows us to derive not only new results in the Cramér–Lundberg setting, but also to provide a tool for transferring results from one setting to the other, and in particular, to explain the striking similarity between results under (1.1) and (1.2).

A very different description of the sample paths which lead to ruin under (1.1), can be found in Barczy and Bertoin [4]. Building on results from Bertoin and Savov [7], they describe the sample paths in reverse time, from the time of ruin, in terms of the associated exponentially tilted process conditioned to stay positive and started with the limiting distribution of the undershoot  $u - X_{\tau(u)-}$ . These two approaches are quite distinct and the aim of [4] is somewhat different from here. An interesting example related to the post ruin process is discussed in [4], but the paper is not specifically directed at insurance risk. The limiting process here is described in forward time, and the convergence is stronger than in [4], in that it

also applies to certain discontinuous and unbounded functionals of the path. Additionally, the results of [4] do not apply to the convolution equivalent setting and so cannot explain the connection between results such as (1.3) and (1.4). The approach in this paper may also prove useful in establishing similar connections for related processes. For example, Mijatovic and Pistorius [26] recently showed that the joint limit law of the undershoot and overshoot for the reflected process under (1.1) is the same as for the processes itself. It now seems reasonable to conjecture that the analogous result holds under (1.2) and, furthermore, that this is a consequence of a more general result related to the sample path behavior of the process and the reflected processes under (1.1) and (1.2) as first passage approaches.

Although not directly related to the current work, a description of the sample paths leading to ruin has also been obtained for the heavy tailed subexponential class of general Lévy insurance risk processes. This class was studied in the compound Poisson model by Asmussen and Klüppelberg [3] and later for spectrally positive process by Klüppelberg and Kyprianou [23]. Results for the general Lévy insurance risk process were obtained recently by Doney, Klüppelberg and Maller [12]. The behavior of the paths is diametrically opposite to that in the Cramér–Lundberg case, with ruin being a consequence of one extremely large jump.

We conclude the [Introduction](#) with a brief outline of the paper. Section 2 contains the necessary fluctuation theory and excursion theory to give a precise statement of the results. The main results can then be found in Section 3 together with an outline of the general approach to their proof. Further results and proofs related to Section 2 are given in Section 4 and the proof of a preliminary result from Section 3 is in Section 5. Proof of the main convergence result under the Cramér–Lundberg condition is given in Section 6 and under the convolution equivalent condition in Section 7. The special case where  $(0, \infty)$  is irregular is then briefly discussed in Section 8. Specific calculations of limiting distributions as well as a Gerber–Shiu EDPF are given in Section 9. Finally, the [Appendix](#) contains a result in the case that  $X$  is compound Poisson which, as is often the case, needs to be treated separately. Throughout  $C, C_1, C_2, \dots$  will denote constants whose value is unimportant and may change from one usage to the next.

**2. Fluctuation variables and excursion measure.** Let  $L_t, t \geq 0$ , denote the local time at 0 of the process  $X - \bar{X}$ , normalized by

$$(2.1) \quad E \int_0^\infty e^{-t} dL_t = 1.$$

Here, we are following Chaumont [9] in our choice of normalization. When 0 is regular for  $[0, \infty)$ ,  $L$  is the unique increasing, continuous, additive functional satisfying (2.1) such that the support of the measure  $dL_t$  is the closure of the set  $\{t : \bar{X}_t = X_t\}$  and  $L_0 = 0$  a.s. If 0 is irregular for  $[0, \infty)$ , the set  $\{s : X_s > \bar{X}_{s-}\}$  of times of strict new maxima of  $X$  is discrete. Let  $R_t = |\{s \in (0, t] : X_s > \bar{X}_{s-}\}|$  and

define the local time of  $X - \bar{X}$  at 0 by

$$(2.2) \quad L_t = \sum_{k=0}^{R_t} \mathbf{e}_k,$$

where  $\mathbf{e}_k$ ,  $k = 0, 1, \dots$  is an independent sequence of i.i.d. exponentially distributed random variables with parameter

$$(2.3) \quad p = \frac{1}{1 - E(e^{-\tau(0)}; \tau(0) < \infty)}.$$

Note that in this latter case,  $dL_t$  has an atom of mass  $\mathbf{e}_0$  at  $t = 0$  and thus the choice of  $p$  ensures that (2.1) holds. Let  $L^{-1}$  be the right continuous inverse of  $L$  and  $H_s = \bar{X}_{L_s^{-1}}$ . Then  $(L_s^{-1}, H_s)_{s \geq 0}$  is the (weakly) ascending bivariate ladder process.

We will also need to consider the strictly ascending bivariate ladder process, which requires a slightly different definition for  $L$ . Specifically, when 0 is regular for  $(0, \infty)$ ,  $L$  is the unique increasing, continuous, additive functional as above. When 0 is irregular for  $(0, \infty)$ ,  $L$  is defined by (2.2). Thus, the only difference is for the compound Poisson process, where the  $L$  switches from being continuous to being given by (2.2). In this case, *that is*, when  $X$  is compound Poisson, the normalization (2.1) still holds, but now the support of the measure  $dL_t$  is the set of times of strict maxima of  $X$ , as opposed to the closure of the set  $\{t : \bar{X}_t = X_t\}$ .  $L^{-1}$  and  $H$  are then defined as before in terms of  $L$  and  $\bar{X}$ , and  $(L_s^{-1}, H_s)_{s \geq 0}$  is the strictly ascending bivariate ladder process. See [5, 11] and particularly Chapter 6 of [25].

In the following paragraph,  $(L_s^{-1}, H_s)_{s \geq 0}$  can be either the weakly ascending or strictly ascending bivariate ladder process. When  $X_t \rightarrow -\infty$  a.s.,  $L_\infty$  has an exponential distribution with some parameter  $q > 0$ , and the defective process  $(L^{-1}, H)$  may be obtained from a nondefective process  $(\mathcal{L}^{-1}, \mathcal{H})$  by independent exponential killing at rate  $q > 0$ . We denote the bivariate Lévy measure of  $(\mathcal{L}^{-1}, \mathcal{H})$  by  $\Pi_{\mathcal{L}^{-1}, \mathcal{H}}(\cdot, \cdot)$ . The Laplace exponent  $\kappa(a, b)$  of  $(L^{-1}, H)$ , defined by

$$e^{-\kappa(a,b)} = E(e^{-aL_1^{-1} - bH_1}; 1 < L_\infty) = e^{-q} Ee^{-a\mathcal{L}_1^{-1} - b\mathcal{H}_1}$$

for values of  $a, b \in \mathbf{R}$  for which the expectation is finite, may be written

$$\kappa(a, b) = q + d_{L^{-1}}a + d_Hb + \int_{t \geq 0} \int_{x \geq 0} (1 - e^{-at - bx}) \Pi_{L^{-1}, H}(dt, dx),$$

where  $d_{L^{-1}} \geq 0$  and  $d_H \geq 0$  are drift constants. Observe that the normalization (2.1) results in  $\kappa(1, 0) = 1$ . The bivariate renewal function of  $(L^{-1}, H)$ , given by

$$\begin{aligned} V(t, x) &= \int_0^\infty e^{-qs} P(\mathcal{L}_s^{-1} \leq t, \mathcal{H}_s \leq x) ds \\ &= \int_0^\infty P(L_s^{-1} \leq t, H_s \leq x; s < L_\infty) ds, \end{aligned}$$

has Laplace transform

$$(2.4) \quad \int_{t \geq 0} \int_{x \geq 0} e^{-at-bx} V(dt, dx) = \int_{s=0}^{\infty} e^{-qs} E(e^{-a\mathcal{L}_s^{-1}-b\mathcal{H}_s}) ds = \frac{1}{\kappa(a, b)},$$

provided  $\kappa(a, b) > 0$ . We will also frequently consider the renewal function of  $\widehat{H}$ , defined on  $\mathbf{R}$  by

$$V(x) = \int_0^{\infty} e^{-qs} P(\mathcal{H}_s \leq x) ds = \lim_{t \rightarrow \infty} V(t, x).$$

Observe that  $V(x) = 0$  for  $x < 0$ , while  $V(0) > 0$  iff  $\mathcal{H}$  is compound Poisson. Also

$$(2.5) \quad V(\infty) := \lim_{x \rightarrow \infty} V(x) = q^{-1}.$$

From this point on, we will take  $(L^{-1}, H)$  to be the strictly ascending bivariate ladder processes of  $X$ . Let  $\widehat{X}_t = -X_t, t \geq 0$  denote the dual process, and  $(\widehat{L}^{-1}, \widehat{H})$  the weakly ascending bivariate ladder processes of  $\widehat{X}$ . This is opposite to the usual convention, and means some care needs to be taken when citing the literature in the compound Poisson case. This choice is made because it leads to more natural results and a direct analogue of (1.5) when  $X$  is compound Poisson. All quantities relating to  $\widehat{X}$  will be denoted in the obvious way, for example,  $\widehat{\tau}(0), \widehat{p}, \Pi_{\widehat{L}^{-1}, \widehat{H}}, \widehat{\kappa}$  and  $\widehat{V}$ . With these choices of bivariate ladder processes, together with the normalization of the local times implying  $\kappa(1, 0) = \widehat{\kappa}(1, 0) = 1$ , the Wiener–Hopf factorization takes the form

$$(2.6) \quad \kappa(a, -ib)\widehat{\kappa}(a, ib) = a - \Psi_X(b), \quad a \geq 0, b \in \mathbf{R}.$$

If  $\alpha > 0$  and  $Ee^{\alpha X_1} < \infty$ , then by analytically extending  $\kappa, \widehat{\kappa}$  and  $\Psi_X$ , it follows from (2.6) that

$$\kappa(a, -z)\widehat{\kappa}(a, z) = a - \Psi_X(-iz) \quad \text{for } a \geq 0, 0 \leq \Re z \leq \alpha.$$

If further  $Ee^{\alpha X_1} < 1$ , for example, when (1.2) holds, then  $\Psi_X(-i\alpha) < 0$  and since trivially  $\widehat{\kappa}(a, \alpha) > 0$ , we have

$$(2.7) \quad \kappa(a, -\alpha) > 0 \quad \text{for } a \geq 0.$$

Let  $D$  be the Skorohod space of functions  $w : [0, \infty) \rightarrow R$  which are right continuous with left limits, equipped with the usual Skorohod topology. The lifetime of a path  $w \in D$  is defined to be  $\zeta(w) = \inf\{t \geq 0 : w(s) = w(t) \text{ for all } s \geq t\}$ , where we adopt the standard convention that  $\inf \emptyset = \infty$ . If  $\zeta(w) = \infty$  then  $w(\zeta)$  is taken to be some cemetery point. Thus, for example, if  $w(\zeta) > y$  for some  $y$  then necessarily  $\zeta < \infty$ . The jump in  $w$  at time  $t$  is given by  $\Delta w_t = w(t) - w(t-)$ . We assume that  $X$  is given as the coordinate process on  $D$ , and the usual right continuous completion of the filtration generated by the coordinate maps will be denoted  $\{\mathcal{F}_t\}_{t \geq 0}$ .  $P_z$  is the probability measure induced on  $\mathcal{F} = \bigvee_{t \geq 0} \mathcal{F}_t$  by the Lévy process starting at  $z \in \mathbf{R}$ , and we usually write  $P$  for  $P_0$ .

Let  $G = \{L_t^{-1} : \Delta L_t^{-1} > 0\}$  and  $D = \{L_t^{-1} : \Delta L_t^{-1} > 0\}$  denote the set of left and right endpoints of excursion intervals of  $X - \bar{X}$ . For  $g \in G$ , let  $d \in D$  be the corresponding right endpoint of the excursion interval ( $d = \infty$  if the excursion has infinite lifetime), and set

$$\epsilon_g(t) = X_{(g+t) \wedge d} - \bar{X}_g, \quad t \geq 0.$$

Note, these are  $X$ -excursions in the terminology of Greenwood and Pitman; see Remark 4.6 of [17], as opposed to  $X - \bar{X}$  excursions. Let

$$\mathcal{E} = \{w \in D : w(t) \leq 0 \text{ for all } 0 \leq t < \zeta(w)\}$$

and  $\mathcal{F}^E$  the restriction of  $\mathcal{F}$  to  $\mathcal{E}$ . Then  $\epsilon_g \in \mathcal{E}$  for each  $g \in G$ , and  $\zeta(\epsilon_g) = d - g$ . The characteristic measure on  $(\mathcal{E}, \mathcal{F}^E)$  of the  $X$ -excursions will be denoted  $n$ .

For fixed  $u > 0$ , let

$$G_{\tau(u)-} = \begin{cases} g, & \text{if } \tau(u) = d \text{ for some excursion interval } (g, d) \\ \tau(u), & \text{else.} \end{cases}$$

If  $X$  is compound Poisson, then  $G_{\tau(u)-}$  is the first time of the last maximum prior to  $\tau(u)$ . When  $X$  is not compound Poisson,  $G_{\tau(u)-}$  is the left limit at  $\tau(u)$  of  $G_t = \sup\{s \leq t : X_s = \bar{X}_s\}$ , explaining the reason behind this common notation.

Set

$$Y_u(t) = X_{(G_{\tau(u)-} + t) \wedge \tau(u)} - \bar{X}_{\tau(u)-}, \quad t \geq 0.$$

Clearly,  $\zeta(Y_u) = \tau(u) - G_{\tau(u)-}$ . If  $\zeta(Y_u) > 0$  then  $G_{\tau(u)-} \in G$ ,  $\bar{X}_{\tau(u)-} = \bar{X}_{G_{\tau(u)-}}$  and  $Y_u \in \mathcal{E}$ . If in addition  $\tau(u) < \infty$ , equivalently  $\zeta(Y_u) < \infty$ , then  $Y_u$  is the excursion which leads to first passage over level  $u$ . To cover the possibility that first passage does not occur at the end of an excursion interval, introduce

$$\bar{\mathcal{E}} = \mathcal{E} \cup \{\mathbf{x} : x \geq 0\},$$

where  $\mathbf{x} \in D$  is the path which is identically  $x$ . On the event  $\zeta(Y_u) = 0$ , that is  $G_{\tau(u)-} = \tau(u)$ , either  $X$  creeps over  $u$  in which case  $Y_u = \mathbf{0}$ , or  $X$  jumps over  $u$  from its current strict maximum in which case  $Y_u = \mathbf{x}$  where  $x = \Delta X_{\tau(u)} > 0$  is the size of the jump at time  $\tau(u)$ . In all cases,  $Y_u \in \bar{\mathcal{E}}$ .

Let  $\mathcal{F}^{\bar{\mathcal{E}}}$  be the restriction of  $\mathcal{F}$  to  $\bar{\mathcal{E}}$ . We extend  $n$  trivially to a measure on  $\mathcal{F}^{\bar{\mathcal{E}}}$  by setting  $n(\bar{\mathcal{E}} \setminus \mathcal{E}) = 0$ . Let  $\tilde{n}$  denote the measure on  $\mathcal{F}^{\bar{\mathcal{E}}}$  obtained by pushing forward the measure  $\Pi_X^+$  with the mapping  $x \rightarrow \mathbf{x}$ , where  $\Pi_X^+$  is the restriction of  $\Pi_X$  to  $[0, \infty)$ . Thus,  $\tilde{n}(\mathcal{E}) = 0$ , and for any Borel set  $B \subset [0, \infty)$ ,  $\tilde{n}(\{\mathbf{x} : x \in B\}) = \Pi_X^+(B)$ . Finally, let  $\bar{n} = n + d_{L^{-1}} \tilde{n}$ .

For  $u > 0, s \geq 0, y \geq 0, \epsilon \in \bar{\mathcal{E}}$  define

$$(2.8) \quad \begin{aligned} & Q_u(ds, dy, d\epsilon) \\ &= P(G_{\tau(u)-} \in ds, u - \bar{X}_{\tau(u)-} \in dy, Y_u \in d\epsilon, \tau(u) < \infty). \end{aligned}$$

The starting point for our investigation is the following result, to be proved in Section 4, which provides a description of the sample paths from the (first) time of the last maximum prior to  $\tau(u)$  until the time of first passage over  $u$ . It may be



viewed as an extension of the quintuple law of Doney and Kyprianou [13]; see the discussion following Proposition 4.3.

**THEOREM 2.1.** For  $u > 0, s \geq 0, y \geq 0, \epsilon \in \bar{\mathcal{E}}$ ,

$$(2.9) \quad \begin{aligned} Q_u(ds, dy, d\epsilon) &= I(y \leq u)V(ds, u - dy)\bar{n}(d\epsilon, \epsilon(\zeta) > y) \\ &\quad + d_H \frac{\partial_-}{\partial_- u} V(ds, u)\delta_0(dy)\delta_{\mathbf{0}}(d\epsilon), \end{aligned}$$

where  $\frac{\partial_-}{\partial_- u}$  denotes left derivative and  $\frac{\partial_-}{\partial_- u} V(ds, u)$  is the Lebesgue–Stieltjes measure associated with the function  $\frac{\partial_-}{\partial_- u} V(s, u)$  (which is increasing in  $s$  by (1.2) and (3.5) of [19]).

**3. Statement of results and a unified approach.** In this section, we state the main results and outline a unified approach to proving them under (1.1) and (1.2). We assume from now on that  $X_t \rightarrow -\infty$ . We will be interested in a marginalized version of (2.8) conditional on  $\tau(u) < \infty$ . Thus, for  $u > 0, y \geq 0$  and  $\epsilon \in \bar{\mathcal{E}}$  define

$$Q^{(u)}(dy, d\epsilon) = P^{(u)}(u - \bar{X}_{\tau(u)-} \in dy, Y_u \in d\epsilon),$$

where recall  $P^{(u)}(\cdot) = P(\cdot | \tau(u) < \infty)$ . Setting  $\bar{V}(u) = V(\infty) - V(u)$ , and using the Pollacek–Khintchine formula,

$$(3.1) \quad P(\tau(u) < \infty) = q\bar{V}(u),$$

see Proposition 2.5 of [24], it follows from (2.9) that

$$Q^{(u)}(dy, d\epsilon) = I(y \leq u) \frac{V(u - dy)}{q\bar{V}(u)} \bar{n}(d\epsilon, \epsilon(\zeta) > y) + d_H \frac{V'(u)}{q\bar{V}(u)} \delta_0(dy)\delta_{\mathbf{0}}(d\epsilon).$$

Here, we have used the fact that  $V$  is differentiable when  $d_H > 0$ , see Theorem VI.19 of [5]. Now under either the Cramér–Lundberg condition (1.1) or the convolution equivalent condition (1.2),

$$(3.2) \quad \frac{V(u - dy)}{q\bar{V}(u)} \xrightarrow{v} \frac{\alpha}{q} e^{\alpha y} dy \quad \text{and} \quad d_H \frac{V'(u)}{q\bar{V}(u)} \rightarrow d_H \frac{\alpha}{q}$$

as  $u \rightarrow \infty$ ;

see Sections 6 and 7 below. This suggests that under suitable conditions on  $F : [0, \infty) \times \bar{\mathcal{E}} \rightarrow \mathbf{R}$ ,

$$(3.3) \quad \int_{[0, \infty) \times \bar{\mathcal{E}}} F(y, \epsilon) Q^{(u)}(dy, d\epsilon) \rightarrow \int_{[0, \infty) \times \bar{\mathcal{E}}} F(y, \epsilon) Q^{(\infty)}(dy, d\epsilon),$$

where

$$(3.4) \quad Q^{(\infty)}(dy, d\epsilon) = \frac{\alpha}{q} e^{\alpha y} dy \bar{n}(d\epsilon, \epsilon(\zeta) > y) + d_H \frac{\alpha}{q} \delta_0(dy)\delta_{\mathbf{0}}(d\epsilon),$$

thus yielding a limiting description of the process as ruin approaches. Observe that (3.3) may be rewritten as

$$(3.5) \quad E^{(u)} F(u - \bar{X}_{\tau(u)-}, Y_u) \rightarrow \int_{[0, \infty)} \frac{\alpha}{q} e^{\alpha y} dy \int_{\bar{\mathcal{E}}} F(y, \epsilon) \bar{n}(d\epsilon, \epsilon(\zeta) > y) + d_H \frac{\alpha}{q} F(0, \mathbf{0}),$$

indicating how the limiting behavior of many functionals of the process related to ruin may be calculated.

To determine a broad class of functions  $F$  for which (3.3) holds, first introduce

$$(3.6) \quad h(y) = \int_{\bar{\mathcal{E}}} F(y, \epsilon) \bar{n}(d\epsilon, \epsilon(\zeta) > y).$$

We emphasize that throughout,  $h$  will always depend on  $F$ , but we will suppress this dependence to ease notation. Since, by (3.2), (3.3) is equivalent to

$$(3.7) \quad \int_0^u h(y) \frac{V(u - dy)}{q \bar{V}(u)} \rightarrow \int_0^\infty h(y) \frac{\alpha e^{\alpha y}}{q} dy,$$

it will be of interest to know when  $h$  is continuous a.e. with respect to Lebesgue measure  $m$ . The most obvious setting in which the condition on  $B_y$  below holds, is when  $F$  is continuous in  $y$  for each  $\epsilon$ . In particular, it holds when  $F$  is jointly continuous. The boundedness condition holds when  $F$  is bounded, but applies to certain unbounded functions.

**PROPOSITION 3.1.** *Assume  $Ee^{\alpha X_1} < \infty$ , and  $F : [0, \infty) \times \bar{\mathcal{E}} \rightarrow \mathbf{R}$  is product measurable, and  $F(y, \epsilon)e^{-\alpha(\epsilon(\zeta) - y)} I(\epsilon(\zeta) > y)$  is bounded in  $(y, \epsilon)$ . Further assume  $\bar{n}(B_y^c) = 0$  for a.e.  $y$  with respect to Lebesgue measure  $m$ , where*

$$B_y = \{\epsilon : F(\cdot, \epsilon) \text{ is continuous at } y\}.$$

*Then  $h$  is continuous a.e. w.r.t.  $m$ .*

We can now state the main results.

**THEOREM 3.1.** *If (1.1) holds, and  $F \geq 0$  satisfies the hypotheses for Proposition 3.1, then (3.3), equivalently (3.5), holds. In particular,*

$$Q^{(u)}(dy, d\epsilon) \xrightarrow{w} Q^{(\infty)}(dy, d\epsilon).$$

**THEOREM 3.2.** *If (1.2) holds,  $F \geq 0$  satisfies the hypotheses of Proposition 3.1, and*

$$(3.8) \quad F(y, \epsilon) I(\epsilon(\zeta) > y) \rightarrow 0 \quad \text{uniformly in } \epsilon \in \bar{\mathcal{E}} \text{ as } y \rightarrow \infty,$$

*then (3.3), equivalently (3.5), holds. Condition (3.8) holds if, for example,  $F$  has compact support. In particular,*

$$(3.9) \quad Q^{(u)}(dy, d\epsilon) \xrightarrow{v} Q^{(\infty)}(dy, d\epsilon).$$

While the outline of the proofs of these two results is the same, the details need to be handled differently. In the Cramér–Lundberg case, the key renewal theorem is used, whereas in the convolution equivalent case, special properties of convolution equivalent distributions are used. This results in different classes of functions for which (3.3) holds. The extra condition (3.8) in Theorem 3.2 cannot be dispensed with, else the convergence in (3.9) could be improved to weak convergence. However, as we show later in (7.4), the total mass of  $Q^{(\infty)}$  under (1.2) is less than one. The convergence in (3.5) may be expressed alternatively in terms of the overshoot  $X_{\tau(u)} - u$  rather than the undershoot  $u - \bar{X}_{\tau(u)-}$ , making it more analogous to (1.5); see Theorems 6.1 and 7.1.

Evaluation, or even simplification, of the limit in (3.5) for a specific functional of the path is, in general, difficult to achieve. Here, we give an example where it is possible and which arises naturally in risk theory. Note that the function  $f$  below can grow exponentially in the overshoot variable. This allows for the calculation of certain unbounded Gerber–Shiu expected discounted penalty functions; see Section 9.

**THEOREM 3.3.** *Assume that (1.1) holds, and  $f : [0, \infty)^4 \rightarrow [0, \infty)$  is a Borel function which is jointly continuous in the first three variables and  $e^{-\alpha x} f(y, x, v, t)$  is bounded. Then*

$$\begin{aligned}
 & E^{(u)} f(u - \bar{X}_{\tau(u)-}, X_{\tau(u)} - u, u - X_{\tau(u)-}, \tau(u) - G_{\tau(u)-}) \\
 & \rightarrow \int_{x \geq 0} \int_{y \geq 0} \int_{v \geq 0} \int_{t \geq 0} f(y, x, v, t) \\
 (3.10) \quad & \quad \times \frac{\alpha}{q} e^{\alpha y} dy I(v \geq y) \widehat{V}(dt, dv - y) \Pi_X(v + dx) \\
 & + d_H \frac{\alpha}{q} f(0, 0, 0, 0).
 \end{aligned}$$

*In particular, we have joint convergence; for  $y \geq 0, x \geq 0, v \geq 0, t \geq 0$*

$$\begin{aligned}
 & P^{(u)}(u - \bar{X}_{\tau(u)-} \in dy, X_{\tau(u)} - u \in dx, u - X_{\tau(u)-} \in dv, \\
 & \quad \tau(u) - G_{\tau(u)-} \in dt) \\
 (3.11) \quad & \xrightarrow{w} \frac{\alpha}{q} e^{\alpha y} dy I(v \geq y) \widehat{V}(dt, dv - y) \Pi_X(v + dx) \\
 & + d_H \frac{\alpha}{q} \delta_{(0,0,0,0)}(dx, dy, dv, dt).
 \end{aligned}$$

In the convolution equivalent setting, the following result extends Theorem 10 of [13].

**THEOREM 3.4.** *Assume (1.2) holds and that  $f : [0, \infty)^4 \rightarrow [0, \infty)$  satisfies (9.12), is jointly continuous in the first three variables,  $e^{-\alpha x} f(y, x, v, t)$  is bounded, and*

$$\sup_{x>0, t \geq 0, v \geq y} f(y, x, v, t) \rightarrow 0 \quad \text{as } y \rightarrow \infty.$$

Then

$$\begin{aligned} & E^{(u)} f(u - \bar{X}_{\tau(u)-}, X_{\tau(u)} - u, u - X_{\tau(u)-}, \tau(u) - G_{\tau(u)-}) \\ & \rightarrow \int_{x \geq 0} \int_{y \geq 0} \int_{v \geq 0} \int_{t \geq 0} f(x, y, v, t) \\ & \quad \times \frac{\alpha}{q} e^{\alpha y} dy I(v \geq y) \widehat{V}(dt, dv - y) \Pi_X(v + dx) \\ & \quad + d_H \frac{\alpha}{q} f(0, 0, 0, 0). \end{aligned} \tag{3.12}$$

In particular, we have joint convergence; for  $y \geq 0, x \geq 0, v \geq 0, t \geq 0$

$$\begin{aligned} & P^{(u)}(u - \bar{X}_{\tau(u)-} \in dy, X_{\tau(u)} - u \in dx, u - X_{\tau(u)-} \in dv, \\ & \quad \tau(u) - G_{\tau(u)-} \in dt) \\ & \xrightarrow{v} \frac{\alpha}{q} e^{\alpha y} dy I(v \geq y) \widehat{V}(dt, dv - y) \Pi_X(v + dx) \\ & \quad + d_H \frac{\alpha}{q} \delta_{(0,0,0,0)}(dx, dy, dv, dt). \end{aligned} \tag{3.13}$$

Theorems 3.1 and 3.2 describe, in a very general sense, how to transfer results from the Cramér–Lundberg setting to the convolution equivalent setting and vice versa. Theorems 3.3 and 3.4 provide a specific example of this. However, since the mode of convergence is  $\xrightarrow{w}$  under (1.1) and  $\xrightarrow{v}$  under (1.2), some subtleties may arise. For example, the marginal distributions of the limit in (3.11) can be readily calculated using (4.8) below, and consequently under (1.1) we obtain, in addition to (1.3),<sup>5</sup>

$$\begin{aligned} & P^{(u)}(u - X_{\tau(u)-} \in dx) \xrightarrow{w} q^{-1} \alpha d_H \delta_0(dx) \\ & \quad + q^{-1} \alpha e^{\alpha x} \bar{\Pi}_X(x) dx \int_{0 \leq v \leq x} e^{-\alpha v} \widehat{V}(dv), \\ & P^{(u)}(\tau(u) - G_{\tau(u)-} \in dt) \xrightarrow{w} q^{-1} (\alpha d_H \delta_0(dt) + K(dt)), \end{aligned}$$

---

<sup>5</sup>Strictly speaking, the proof of (1.3) in [21] assumes that  $(L^{-1}, H)$  is the weakly ascending ladder process, whereas the marginals of (3.11) yield the same formulae as (1.3) but with  $(L^{-1}, H)$  the strictly ascending ladder process. Thus, as can be easily checked directly, the limiting expressions must agree irrespective of the choice of ascending ladder process. This remark applies to (1.4) and several other limiting distributions discussed here.

where

$$(3.14) \quad K(dt) = \int_{z \geq 0} (e^{\alpha z} - 1) \Pi_{L^{-1}, H}(dt, dz).$$

Under (1.2), some care is needed. The marginals of the limit in (3.13) are the same as in (3.11), but they all have mass less than one. This does not mean that we can simply replace weak convergence of the marginals under (1.1) with vague convergence under (1.2). For the undershoots of  $X$  and  $\bar{X}$ , this is correct, but the overshoot and  $\tau(u) - G_{\tau(u)-}$  both converge weakly under (1.2); indeed they converge jointly, as will be shown in Proposition 9.2. Consequently, an extra term appears in the limit of the overshoot in (1.4) to account for the missing mass. Similarly, for  $\tau(u) - G_{\tau(u)-}$ , see (9.19).

Based on the outline of the proofs of Theorems 3.1 and 3.2 given at the beginning of this section, it is natural to ask if any other limits are possible in (3.2), thus leading to different forms of the limit in (3.5). However, this is not the case. More precisely, if  $V(u - dy)/\bar{V}(u)$  converges vaguely to a nonzero locally finite Borel measure, then  $\bar{V}(u - y)/\bar{V}(u)$  converges as  $u \rightarrow \infty$  on a dense set of  $y$ . Hence, by Theorem 1.4.3 of [8],  $\bar{V}(\ln u)$  is regularly varying at infinity with some index  $-\alpha$ . Consequently, the limit in (3.2) must be of the form given. The only general classes of processes that the author is aware of which satisfy (3.2) are those studied in this paper, namely the Cramér–Lundberg and convolution equivalent cases.

**4. Proof of Theorem 2.1 and related results.** The following result will be needed in the proof of Theorem 2.1.

PROPOSITION 4.1. *If  $X$  is not compound Poisson, then for  $s \geq 0, x \geq 0$ ,*

$$d_{L^{-1}} V(ds, dx) = P(\bar{X}_s = X_s \in dx) ds.$$

PROOF. For any  $s \geq 0, x \geq 0$

$$\begin{aligned} d_{L^{-1}} \int_0^{L^\infty} I(L_t^{-1} \leq s, H_t \leq x) dt &= d_{L^{-1}} \int_0^{L^\infty} I(L_t^{-1} \leq s, \bar{X}_{L_t^{-1}} \leq x) dt \\ &= d_{L^{-1}} \int_0^s I(\bar{X}_r \leq x) dL_r \\ &= \int_0^s I(\bar{X}_r \leq x) I(\bar{X}_r = X_r) dr, \end{aligned}$$

by Theorem 6.8 and Corollary 6.11 of Kyprianou [25], which apply since  $X$  is not compound Poisson, [Kyprianou’s  $(L^{-1}, H)$  is the weakly ascending ladder process in which case the result holds in the compound Poisson case also]. Taking expectations completes the proof.  $\square$

PROOF OF THEOREM 2.1. There are three possible ways in which  $X$  can first cross level  $u$ ; by a jump at the end of an excursion interval, by a jump from a current strict maximum or by creeping. We consider each in turn.

Let  $f, h$  and  $j$  be nonnegative bounded continuous functions. Since  $\bar{X}_{L_t^-}$  is left continuous, we may apply the master formula of excursion theory, Corollary IV.11 of [5], to obtain

$$\begin{aligned}
 & E\{f(G_{\tau(u)-})h(u - \bar{X}_{\tau(u)-})j(Y_u); X_{\tau(u)} > u, G_{\tau(u)-} < \tau(u) < \infty\} \\
 &= E \sum_{g \in G} f(g)h(u - \bar{X}_g)j(\epsilon_g)I(\bar{X}_g \leq u, \epsilon_g(\zeta) > u - \bar{X}_g) \\
 &= E \int_0^\infty dL_t \int_{\mathcal{E}} f(t)h(u - \bar{X}_t)j(\epsilon)I(\bar{X}_t \leq u, \epsilon(\zeta) > u - \bar{X}_t)n(d\epsilon) \\
 (4.1) \quad &= \int_{\mathcal{E}} j(\epsilon)n(d\epsilon)E \int_0^{L_\infty} f(L_r^{-1})h(u - H_r)I(H_r \leq u, \epsilon(\zeta) > u - H_r) dr \\
 &= \int_{\mathcal{E}} j(\epsilon)n(d\epsilon) \int_{s \geq 0} \int_{0 \leq y \leq u} f(s)h(u - y)I(\epsilon(\zeta) > u - y)V(ds, dy) \\
 &= \int_{\mathcal{E}} j(\epsilon)n(d\epsilon) \int_{s \geq 0} \int_{0 \leq y \leq u} f(s)h(y)I(\epsilon(\zeta) > y)V(ds, u - dy) \\
 &= \int_{s \geq 0} \int_{0 \leq y \leq u} \int_{\mathcal{E}} f(s)h(y)j(\epsilon)V(ds, u - dy)n(d\epsilon, \epsilon(\zeta) > y).
 \end{aligned}$$

Next, define  $\tilde{j} : [0, \infty) \rightarrow \mathbf{R}$  by  $\tilde{j}(x) = j(\mathbf{x})$ . Then, since  $Y_u(t) = \Delta X_{\tau(u)}$  for all  $t \geq 0$  on  $\{X_{\tau(u)} > u, G_{\tau(u)-} = \tau(u) < \infty\}$ , we have by the compensation formula,

$$\begin{aligned}
 & E\{f(G_{\tau(u)-})h(u - \bar{X}_{\tau(u)-})j(Y_u); X_{\tau(u)} > u, G_{\tau(u)-} = \tau(u) < \infty\} \\
 &= E\{f(G_{\tau(u)-})h(u - \bar{X}_{\tau(u)-})\tilde{j}(\Delta X_{\tau(u)}); \\
 &\quad X_{\tau(u)} > u, G_{\tau(u)-} = \tau(u) < \infty\} \\
 &= E \sum_s f(s)h(u - \bar{X}_{s-})\tilde{j}(\Delta X_s)I(X_{s-} = \bar{X}_{s-} \leq u, \Delta X_s > u - \bar{X}_{s-}) \\
 (4.2) \quad &= E \int_0^\infty f(s)h(u - \bar{X}_s)I(X_s = \bar{X}_s \leq u) ds \int_{\xi} \tilde{j}(\xi)I(\xi > u - \bar{X}_s)\Pi_X(d\xi) \\
 &= \int_0^\infty f(s) \int_{0 \leq y \leq u} h(y) \int_{\xi} \tilde{j}(\xi)I(\xi > y)\Pi_X(d\xi)P(\bar{X}_s = X_s \in u - dy) ds \\
 &= \int_{s \geq 0} f(s) \int_{0 \leq y \leq u} h(y) \int_{\mathcal{E}} j(\epsilon)\tilde{n}(d\epsilon, \epsilon(\zeta) > y)P(\bar{X}_s = X_s \in u - dy) ds \\
 &= \int_{s \geq 0} \int_{0 \leq y \leq u} \int_{\mathcal{E}} f(s)h(y)j(\epsilon)\tilde{n}(d\epsilon, \epsilon(\zeta) > y)d_{L^{-1}}V(ds, u - dy),
 \end{aligned}$$

where the final equality follows from Proposition 4.1 if  $X$  is not compound Poisson. If  $X$  is compound Poisson the first and last formulas of (4.2) are equal because  $P(G_{\tau(u)-} = \tau(u)) = 0$  and  $d_{L^{-1}} = 0$  [recall that  $(L^{-1}, H)$  is the strictly ascending ladder process].

Finally,

$$\begin{aligned}
 (4.3) \quad & E\{f(G_{\tau(u)-})h(u - \bar{X}_{\tau(u)-})j(Y_u); X_{\tau(u)} = u, \tau(u) < \infty\} \\
 & = h(0)j(\mathbf{0})E\{f(\tau(u)); X_{\tau(u)} = u, \tau(u) < \infty\} \\
 & = d_H h(0)j(\mathbf{0}) \int_s f(s) \frac{\partial_-}{\partial_- u} V(ds, u)
 \end{aligned}$$

if  $d_H > 0$ , by (3.5) of [19]. If  $d_H = 0$ , then  $X$  does not creep, and so  $P(X_{\tau(u)} = u) = 0$ . Thus, (4.3) holds in this case also. Combining the three terms (4.1), (4.2) and (4.3) gives the result.  $\square$

The next two results will be used to calculate limits such as those of the form (3.11) and (3.13).

PROPOSITION 4.2. For  $t \geq 0, z \geq 0$ ,

$$(4.4) \quad \widehat{V}(dt, dz) = n(\epsilon(t) \in -dz, \zeta > t) dt + d_{L^{-1}} \delta_{(0,0)}(dt, dz).$$

PROOF. If  $X$  is not compound Poisson nor  $|X|$  a subordinator, (4.4) follows from (5.9) of [9] applied to the dual process  $\widehat{X}$ .

If  $X$  is a subordinator, but not compound Poisson, then  $n$  is the zero measure and  $d_{L^{-1}} = 1$  by (2.1). On the other hand,  $(\widehat{L}_t^{-1}, \widehat{H}_t)$  remains at  $(0, 0)$  for an exponential amount of time with parameter  $\widehat{p} = 1$ , by (2.3), and is then killed. Hence, (4.4) holds.

If  $-X$  is a subordinator, then  $(\widehat{L}_t^{-1}, \widehat{H}_t) = (t, \widehat{X}_t)$  and so  $\widehat{V}(dt, dz) = P(X_t \in -dz) dt$ . On the other hand,  $n$  is proportional to the first, and only, excursion, so  $n(\epsilon(t) \in -dz, \zeta > t) = cP(X_t \in -dz)$  for some  $c > 0$ . Since  $d_{L^{-1}} = 0$ , we thus only need check that  $|n| = 1$ . But  $G = \{0\}$ , and so by the master formula

$$1 = E \sum_{g \in G} e^{-g} = E \int_0^\infty e^{-t} dL_t \int_{\mathcal{E}} n(d\epsilon) = |n|.$$

To complete the proof, it thus remains to prove (4.4) when  $X$  is compound Poisson. We defer this case to the Appendix.  $\square$

For notational convenience, we define  $\epsilon(0-) = 0$  for  $\epsilon \in \bar{\mathcal{E}}$ . Thus, in particular,  $\mathbf{x}(\zeta-) = 0$  since  $\zeta(\mathbf{x}) = 0$ . Note also that  $\mathbf{x}(\zeta) = x$ .

PROPOSITION 4.3. For  $t \geq 0, z \geq 0$  and  $x > 0$ ,

$$(4.5) \quad \bar{n}(\zeta \in dt, \epsilon(\zeta-) \in -dz, \epsilon(\zeta) \in dx) = \widehat{V}(dt, dz) \Pi_X(z + dx).$$

PROOF. First consider the case  $t > 0, z \geq 0$  and  $x > 0$ . For any  $0 < s < t$ , using the Markov property of the excursion measure  $n$ , we have

$$\begin{aligned} &\bar{n}(\zeta \in dt, \epsilon(\zeta-) \in -dz, \epsilon(\zeta) \in dx) \\ &= \int_{y \geq 0} n(\epsilon(s) \in -dy, \zeta > s) \\ &\quad \times P_{P-y}(\tau(0) \in dt - s, X_{\tau(0)-} \in -dz, X_{\tau(0)} \in dx) \\ &= \int_{y \geq 0} n(\epsilon(s) \in -dy, \zeta > s) \\ &\quad \times P(\tau(y) \in dt - s, X_{\tau(y)-} \in y - dz, X_{\tau(y)} \in y + dx). \end{aligned}$$

By the compensation formula, for any positive bounded Borel function  $f$ ,

$$\begin{aligned} &E\{f(\tau(y)); X_{\tau(y)-} \in y - dz, X_{\tau(y)} \in y + dx\} \\ &= E\left\{\sum_r f(r); \bar{X}_{r-} \leq y, X_{r-} \in y - dz, X_r \in y + dx\right\} \\ &= \int_{r=0}^{\infty} f(r)P(\bar{X}_{r-} \leq y, X_{r-} \in y - dz) dr \Pi_X(z + dx). \end{aligned}$$

Thus,

$$\begin{aligned} &P(\tau(y) \in dt - s, X_{\tau(y)-} \in y - dz, X_{\tau(y)} \in y + dx) \\ &= P(\bar{X}_{t-s} \leq y, X_{t-s} \in y - dz) dt \Pi_X(z + dx). \end{aligned}$$

Hence,

$$\begin{aligned} &\bar{n}(\zeta \in dt, \epsilon(\zeta-) \in -dz, \epsilon(\zeta) \in dx) \\ &= \int_{y \geq 0} n(\epsilon(s) \in -dy, \zeta > s)P(\bar{X}_{t-s} \leq y, X_{t-s} \in y - dz) dt \Pi_X(z + dx) \\ &= n(\epsilon(t) \in -dz, \zeta > t) dt \Pi_X(z + dx) \\ &= \widehat{V}(dt, dz)\Pi_X(z + dx) \end{aligned}$$

by (4.4).

Finally, if  $t = 0$ , then for any positive bounded Borel function,

$$\begin{aligned} &\int_{\{(t,z,x): t=0, z \geq 0, x > 0\}} f(t, z, x)\bar{n}(\zeta \in dt, \epsilon(\zeta-) \in -dz, \epsilon(\zeta) \in dx) \\ &= d_{L^{-1}} \int_{x > 0} f(0, 0, x)\tilde{n}(\epsilon(\zeta) \in dx) \\ &= d_{L^{-1}} \int_{x > 0} f(0, 0, x)\Pi_X^+(dx) \\ &= \int_{\{(t,z,x): t=0, z \geq 0, x > 0\}} f(t, z, x)\widehat{V}(dt, dz)\Pi_X(z + dx) \end{aligned}$$



by Proposition 4.2.  $\square$

As mentioned earlier, Theorem 2.1 may be viewed as an extension of the quintuple law of Doney and Kyprianou [13]. To see this, observe that from Theorem 2.1 and Proposition 4.3, for  $u > 0, s \geq 0, t \geq 0, 0 \leq y \leq u \wedge z, x \geq 0$ ,

$$\begin{aligned}
 &P(G_{\tau(u)-} \in ds, u - \bar{X}_{\tau(u)-} \in dy, \tau(u) - G_{\tau(u)-} \in dt, \\
 &\quad u - X_{\tau(u)-} \in dz, X_{\tau(u)} - u \in dx) \\
 (4.6) \quad &= P(G_{\tau(u)-} \in ds, u - \bar{X}_{\tau(u)-} \in dy, \zeta(Y_u) \in dt, \\
 &\quad Y_u(\zeta-) \in y - dz, Y_u(\zeta) \in y + dx) \\
 &= I(x > 0)V(ds, u - dy)\bar{n}(\zeta \in dt, \epsilon(\zeta-) \in y - dz, \epsilon(\zeta) \in y + dx) \\
 &\quad + d_H \frac{\partial_-}{\partial_- u} V(ds, u)\delta_{(0,0,0,0)}(dt, dx, dz, dy) \\
 &= I(x > 0)V(ds, u - dy)\widehat{V}(dt, dz - y)\Pi_X(z + dx) \\
 &\quad + d_H \frac{\partial_-}{\partial_- u} V(ds, u)\delta_{(0,0,0,0)}(dt, dx, dz, dy).
 \end{aligned}$$

When  $X$  is not compound Poisson, this is the statement of Theorem 3 of [13] with the addition of the term due to creeping; see Theorem 3.2 of [19]. When  $X$  is compound Poisson the quintuple law, though not explicitly stated in [13], remains true and can be found in [14]. In that case, the result is slightly different from (4.6) since the definitions of  $G_{\tau(u)-}$ ,  $V$  and  $\widehat{V}$  then differ due to the choice of  $(L^{-1}, H)$  as the weakly ascending ladder process in [13] and [14]. Thus, we point out that Vigon’s *équation amicale inversée*, [30],

$$(4.7) \quad \Pi_H(dx) = \int_{z \geq 0} \widehat{V}(dz)\Pi_X(z + dx), \quad x > 0,$$

and Doney and Kyprianou’s extension,

$$(4.8) \quad \Pi_{L^{-1}, H}(dt, dx) = \int_{v \geq 0} \widehat{V}(dt, dv)\Pi_X(v + dx), \quad x > 0, t \geq 0,$$

continue to hold with our choice of  $(L^{-1}, H)$  as the strongly ascending ladder process. The proof of (4.8) is analogous to the argument in Corollary 6 of [13], using (4.6) instead of Doney and Kyprianou’s quintuple law, and (4.7) follows immediately from (4.8).

COROLLARY 4.1. For  $x > 0$ ,

$$(4.9) \quad \bar{n}(\epsilon(\zeta) \in dx) = \Pi_H(dx).$$

PROOF. Integrating out in (4.5),

$$\bar{n}(\epsilon(\zeta) \in dx) = \int_{z \geq 0} \widehat{V}(dz) \Pi_X(z + dx) = \Pi_H(dx),$$

by (4.7).  $\square$

**5. Proof of Proposition 3.1.** From (3.6),

$$h(z) = \int_{\bar{\mathcal{E}}} f_z(\epsilon) \bar{n}(d\epsilon),$$

where

$$f_z(\epsilon) = F(z, \epsilon) I(\epsilon(\zeta) > z).$$

Fix  $y > 0$  and assume  $|z - y| < y/2$ . Then for some constant  $C$ , independent of  $z$  and  $\epsilon$ ,

$$(5.1) \quad |f_z(\epsilon)| \leq C e^{\alpha(\epsilon(\zeta) - z)} I(\epsilon(\zeta) > z) \leq C e^{\alpha(\epsilon(\zeta) - y/2)} I(\epsilon(\zeta) > y/2).$$

By (4.9),

$$(5.2) \quad \int_{\bar{\mathcal{E}}} e^{\alpha(\epsilon(\zeta) - y/2)} I(\epsilon(\zeta) > y/2) \bar{n}(d\epsilon) = \int_{x > y/2} e^{\alpha(x - y/2)} \Pi_H(dx),$$

and since  $E e^{\alpha X_1} < \infty$ , this last integral is finite by Proposition 7.1 of [18].

Now let  $A = \{y : \bar{n}(B_y^c) = 0\}$  and  $C_H = \{y : \Pi_H(\{y\}) = 0\}$ . Then  $C_H^c$  is countable and

$$\bar{n}(\epsilon(\zeta) = y) = \Pi_H(\{y\}) = 0 \quad \text{if } y \in C_H.$$

Thus, if  $y > 0$ ,  $y \in A \cap C_H$  and  $z \rightarrow y$ , then  $f_z(\epsilon) \rightarrow f_y(\epsilon)$  a.e.  $\bar{n}$ . Hence, by (5.1) and (5.2), we can apply dominated convergence to obtain continuity of  $h$  at such  $y$ . Since  $m(A^c) = m(C_H^c) = 0$ , this completes the proof.

**6. Proofs under the Cramér–Lundberg condition.** In studying the process  $X$  under the Cramér–Lundberg condition (1.1), it is useful to introduce the Esscher transform. Thus, let  $P^*$  be the measure on  $\mathcal{F}$  defined by

$$dP^* = e^{\alpha X_t} dP \quad \text{on } \mathcal{F}_t,$$

for all  $t \geq 0$ . Then  $X$  under  $P^*$  is the Esscher transform of  $X$ . It is itself a Lévy process with  $E^* X > 0$ ; see Section 3.3 of Kyprianou [25].

When (1.1) holds, Bertoin and Doney [6] extended the classical Cramér–Lundberg estimate for ruin to a general Lévy process; assume  $X$  is nonlattice in the case that  $X$  is compound Poisson, then

$$(6.1) \quad \lim_{u \rightarrow \infty} e^{\alpha u} P(\tau(u) < \infty) = \frac{q}{\alpha m^*},$$

where  $m^* = E^*H_1$ . Under  $P^*$ ,  $H$  is a nondefective subordinator with drift  $d_H^*$  and Lévy measure  $\Pi_H^*$  given by

$$d_H^* = d_H \quad \text{and} \quad \Pi_H^*(dx) = e^{\alpha x} \Pi_H(dx),$$

and so

$$(6.2) \quad E^*H_1 = d_H^* + \int_0^\infty x \Pi_H^*(dx) = d_H + \int_0^\infty x e^{\alpha x} \Pi_H(dx).$$

Combining (6.1) with the Pollacek–Khintchine formula (3.1), we obtain

$$(6.3) \quad \lim_{u \rightarrow \infty} \frac{\bar{V}(u - y)}{\bar{V}(u)} = e^{\alpha y},$$

and hence the first result in (3.2) holds as claimed. The second result in (3.2) is a consequence of (4.15) in [21], for example. Since  $\bar{V}(\ln x)$  is regularly varying, note that the convergence in (6.3) is uniform on compact subsets of  $\mathbf{R}$ ; see Theorem 1.2.1 of [8].

Let

$$(6.4) \quad V^*(x) := \int_0^\infty P^*(H_s \leq x) ds = \int_{y \leq x} e^{\alpha y} V(dy),$$

see [6] or Section 7.2 of [25]. Then  $V^*$  is a renewal function, and so by the key renewal theorem

$$(6.5) \quad \int_0^u g(y) V^*(u - dy) \rightarrow \frac{1}{m^*} \int_0^\infty g(y) dy,$$

if  $g \geq 0$  is directly Riemann integrable on  $[0, \infty)$ . We will make frequent use of the following criterion for direct Riemann integrability. *If  $g \geq 0$  is continuous a.e. and dominated by a bounded, nonincreasing integrable function on  $[0, \infty)$ , then  $g$  is directly Riemann integrable on  $[0, \infty)$ .* See Chapter V.4 of [2] for information about the key renewal theorem and direct Riemann integrability.

The function to which we would like to apply (6.5), namely  $e^{\alpha y} h(y)$  where  $h$  is given by (3.6), is typically unbounded at 0. To overcome this difficulty, we use the following result.

**PROPOSITION 6.1.** *If (1.1) holds,  $h \geq 0$ ,  $e^{\alpha y} h(y) 1_{[\varepsilon, \infty)}(y)$  is directly Riemann integrable for every  $\varepsilon > 0$ , and*

$$(6.6) \quad \limsup_{\varepsilon \rightarrow 0} \limsup_{u \rightarrow \infty} \int_{[0, \varepsilon)} h(y) \frac{V(u - dy)}{\bar{V}(u)} = 0,$$

then (3.3) holds.

**PROOF.** By (3.1) and (6.4),

$$\frac{V(u - dy)}{q \bar{V}(u)} = \frac{e^{\alpha y} V^*(u - dy)}{e^{\alpha u} P(\tau(u) < \infty)}.$$

Hence, by (6.1), and (6.5) applied to  $e^{\alpha y}h(y)1_{[\varepsilon, \infty)}(y)$ ,

$$\int_{\varepsilon}^u h(y) \frac{V(u - dy)}{q\bar{V}(u)} = \int_{\varepsilon}^u e^{\alpha y}h(y) \frac{V^*(u - dy)}{e^{\alpha u}P(\tau(u) < \infty)} \rightarrow \int_{\varepsilon}^{\infty} e^{\alpha y}h(y) \frac{\alpha}{q} dy$$

as  $u \rightarrow \infty$ . Combined with (6.6) and monotone convergence, this proves (3.7), which in turn is equivalent to (3.3).  $\square$

The next result gives conditions on  $F$  which ensure that  $h$ , defined by (3.6), satisfies the hypotheses of Proposition 6.1.

PROPOSITION 6.2. *Assume  $F \geq 0$  satisfies the hypotheses of Proposition 3.1, and further that  $EX_1e^{\alpha X_1} < \infty$ . Then  $h$  satisfies the hypotheses of Proposition 6.1.*

PROOF. For any  $y \geq 0$ ,

$$\begin{aligned} e^{\alpha y}h(y) &= \int_{\bar{\varepsilon}} e^{\alpha y}F(y, \epsilon)\bar{n}(d\epsilon, \epsilon(\zeta) > y) \\ (6.7) \quad &= \int_{\bar{\varepsilon}} I(\epsilon(\zeta) > y)e^{-\alpha(\epsilon(\zeta)-y)}F(y, \epsilon)e^{\alpha\epsilon(\zeta)}\bar{n}(d\epsilon) \\ &\leq C \int_{(y, \infty)} e^{\alpha x}\Pi_H(dx), \end{aligned}$$

by (4.9). Further,

$$\int_{y \geq 0} \int_{x > y} e^{\alpha x}\Pi_H(dx) dy = \int_{x \geq 0} xe^{\alpha x}\Pi_H(dx) < \infty,$$

by an argument analogous to Proposition 7.1 of [18]. Thus,  $e^{\alpha y}h(y)$  is dominated by a nonincreasing integrable function on  $[0, \infty)$ , and hence, for each  $\varepsilon > 0$ ,  $e^{\alpha y}h(y)1_{[\varepsilon, \infty)}(y)$  is dominated by a bounded nonincreasing integrable function on  $[0, \infty)$ . Additionally, by Proposition 3.1,  $h$  is continuous a.e. with respect to Lebesgue measure. Consequently,  $e^{\alpha y}h(y)1_{[\varepsilon, \infty)}(y)$  is directly Riemann integrable for every  $\varepsilon > 0$ .

Next, since the convergence is uniform on compact in (6.3), for any  $x \geq 0$ ,

$$(6.8) \quad \int_{[0, x)} \frac{V(u - dy)}{\bar{V}(u)} \rightarrow e^{\alpha x} - 1$$

as  $u \rightarrow \infty$ . Thus, by (6.7) and (6.8), if  $\varepsilon < 1$  and  $u$  is sufficiently large

$$\begin{aligned} &\int_{[0, \varepsilon)} h(y) \frac{V(u - dy)}{\bar{V}(u)} \\ &\leq C \int_{[0, \varepsilon)} \frac{V(u - dy)}{\bar{V}(u)} \int_{(y, \infty)} e^{\alpha x}\Pi_H(dx) \\ &= C \int_{[0, \varepsilon)} e^{\alpha x}\Pi_H(dx) \int_{[0, x)} \frac{V(u - dy)}{\bar{V}(u)} \end{aligned}$$

$$\begin{aligned}
 &+ C \int_{[\varepsilon, \infty)} e^{\alpha x} \Pi_H(dx) \int_{[0, \varepsilon)} \frac{V(u - dy)}{\bar{V}(u)} \\
 &\leq C_1 \int_{[0, \varepsilon)} x e^{\alpha x} \Pi_H(dx) + C_1 \varepsilon \int_{[\varepsilon, \infty)} e^{\alpha x} \Pi_H(dx) \\
 &\leq C_1 e^\alpha \int_{[0, \varepsilon)} x \Pi_H(dx) + C_1 e^\alpha \varepsilon \bar{\Pi}_H(\varepsilon) + C_1 \varepsilon \int_{[1, \infty)} e^{\alpha x} \Pi_H(dx).
 \end{aligned}$$

Now  $\int_{[0, 1)} x \Pi_H(dx) < \infty$ , since  $H$  is a subordinator, thus  $\int_{[0, \varepsilon)} x \Pi_H(dx) \rightarrow 0$  and  $\varepsilon \bar{\Pi}_H(\varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . Combined with  $\int_{x \geq 1} e^{\alpha x} \Pi_H(dx) < \infty$ , this shows that the final expression approaches 0 as  $\varepsilon \rightarrow 0$ .  $\square$

As a consequence, we have the following.

**PROOF OF THEOREM 3.1.** This follows immediately from Propositions 6.1 and 6.2.  $\square$

The convergence in Theorem 3.1 may alternatively be expressed in terms of the overshoot  $X_{\tau(u)} - u$  rather than the undershoot of the maximum  $u - \bar{X}_{\tau(u)-}$ .

**THEOREM 6.1.** Assume  $G: \bar{\mathcal{E}} \times [0, \infty) \rightarrow [0, \infty)$  is product measurable,  $e^{-\alpha x} G(\varepsilon, x)$  is bounded in  $(\varepsilon, x)$  and  $G(\varepsilon, \cdot)$  is continuous for a.e.  $\varepsilon$  w.r.t.  $\bar{\nu}$ . Then under (1.1),

$$\begin{aligned}
 &E^{(u)} G(Y_u, X_{\tau(u)} - u) \\
 (6.9) \quad &\rightarrow \int_{[0, \infty)} \frac{\alpha}{q} e^{\alpha y} dy \int_{\bar{\mathcal{E}} \times (0, \infty)} G(\varepsilon, x) \bar{\nu}(d\varepsilon, \varepsilon(\zeta) \in y + dx) \\
 &\quad + d_H \frac{\alpha}{q} G(\mathbf{0}, 0).
 \end{aligned}$$

**PROOF.** Let  $F(y, \varepsilon) = G(\varepsilon, \varepsilon(\zeta) - y) I(\varepsilon(\zeta) \geq y)$ . Then  $F$  satisfies the conditions of Theorem 3.1, and

$$G(Y_u, X_{\tau(u)} - u) = F(u - \bar{X}_{\tau(u)-}, Y_u)$$

on  $\{\tau(u) < \infty\}$ . Consequently, (3.5) yields

$$\begin{aligned}
 &E^{(u)} G(Y_u, X_{\tau(u)} - u) \\
 &\rightarrow \int_{[0, \infty)} \frac{\alpha}{q} e^{\alpha y} dy \int_{\bar{\mathcal{E}}} G(\varepsilon, \varepsilon(\zeta) - y) \bar{\nu}(d\varepsilon, \varepsilon(\zeta) > y) + d_H \frac{\alpha}{q} G(\mathbf{0}, 0) \\
 &= \int_{[0, \infty)} \frac{\alpha}{q} e^{\alpha y} dy \int_{\bar{\mathcal{E}} \times (0, \infty)} G(\varepsilon, x) \bar{\nu}(d\varepsilon, \varepsilon(\zeta) \in y + dx) + d_H \frac{\alpha}{q} G(\mathbf{0}, 0),
 \end{aligned}$$

completing the proof.  $\square$

**7. Proofs under the convolution equivalent condition.** We begin with the definition of the class  $\mathcal{S}^{(\alpha)}$ . As mentioned previously, we will restrict ourselves to the nonlattice case, with the understanding that the alternative can be handled by obvious modifications. A distribution  $F$  on  $[0, \infty)$  with tail  $\bar{F} = 1 - F$  belongs to the class  $\mathcal{S}^{(\alpha)}$ ,  $\alpha > 0$ , if  $\bar{F}(u) > 0$  for all  $u > 0$ ,

$$(7.1) \quad \lim_{u \rightarrow \infty} \frac{\bar{F}(u+x)}{\bar{F}(u)} = e^{-\alpha x} \quad \text{for } x \in (-\infty, \infty),$$

and

$$(7.2) \quad \lim_{u \rightarrow \infty} \frac{\overline{F^{2*}}(u)}{\bar{F}(u)} \quad \text{exists and is finite,}$$

where  $F^{2*} = F * F$ . Distributions in  $\mathcal{S}^{(\alpha)}$  are called *convolution equivalent* with index  $\alpha$ . When  $F \in \mathcal{S}^{(\alpha)}$ , the limit in (7.2) must be of the form  $2\delta_\alpha^F$ , where  $\delta_\alpha^F := \int_{[0, \infty)} e^{\alpha x} F(dx)$  is finite. Much is known about the properties of such distributions; see, for example, [10, 15, 22, 27, 28] and [31]. In particular, the class is closed under tail equivalence, that is, if  $F \in \mathcal{S}^{(\alpha)}$  and  $G$  is a distribution function for which

$$\lim_{u \rightarrow \infty} \frac{\bar{G}(u)}{\bar{F}(u)} = c \quad \text{for some } c \in (0, \infty),$$

then  $G \in \mathcal{S}^{(\alpha)}$ .

The convolution equivalent model (1.2) was introduced by Klüppelberg, Kyprianou and Maller [24].<sup>6</sup> As noted earlier, when (1.2) holds,  $Ee^{\theta X_1} = \infty$  for all  $\theta > \alpha$ , so (1.1) must fail. Nevertheless, (3.2) continues to hold under (1.2). This is because by (2.5),  $F(u) = qV(u)$  is a distribution function, and combining several results in [24] (see (4) of [13]), together with closure of  $\mathcal{S}^{(\alpha)}$  under tail equivalence, it follows that  $F \in \mathcal{S}^{(\alpha)}$ . Hence, the first condition in (3.2) follows from (7.1). The second condition, which corresponds to asymptotic creeping, again follows from results in [24] and can also be found in [13].

We begin with a general result about convolution equivalent distributions.

**LEMMA 7.1.** *If  $F \in \mathcal{S}^{(\alpha)}$ , and  $g \geq 0$  is continuous a.e. (Lebesgue) with  $g(y)/\bar{F}(y) \rightarrow L$  as  $y \rightarrow \infty$ , then*

$$\int_{0 \leq y \leq u} g(y) \frac{F(u-dy)}{\bar{F}(u)} \rightarrow \int_0^\infty g(y) \alpha e^{\alpha y} dy + L \int_0^\infty e^{\alpha y} F(dy) \quad \text{as } u \rightarrow \infty.$$

---

<sup>6</sup>In [24], (1.2) is stated in terms of  $\Pi_X^+(\cdot \cap [1, \infty))/\Pi_X^+([1, \infty)) \in \mathcal{S}^{(\alpha)}$ . This is equivalent to  $X_1^+ \in \mathcal{S}^{(\alpha)}$  by Watanabe [31].

PROOF. Fix  $K \in (0, \infty)$  and write

$$\begin{aligned}
 (7.3) \quad & \int_{0 \leq y \leq u} g(y) \frac{F(u - dy)}{\bar{F}(u)} \\
 &= \left( \int_{0 \leq y \leq K} + \int_{K < y < u - K} + \int_{u - K \leq y \leq u} \right) g(y) \frac{F(u - dy)}{\bar{F}(u)} \\
 &= \text{I} + \text{II} + \text{III}.
 \end{aligned}$$

By vague convergence,

$$\text{I} \rightarrow \int_0^K g(y) \alpha e^{\alpha y} dy.$$

Next,

$$\text{III} = \int_{0 \leq y \leq K} g(u - y) \frac{F(dy)}{\bar{F}(u)} = \int_{0 \leq y \leq K} \frac{g(u - y)}{\bar{F}(u - y)} \frac{\bar{F}(u - y)}{\bar{F}(u)} F(dy).$$

For large  $u$ , the integrand is bounded by  $2Le^{\alpha K}$  and converges to  $Le^{\alpha y}$ , thus by bounded convergence,

$$\text{III} \rightarrow L \int_0^K e^{\alpha y} F(dy).$$

Finally,

$$\limsup_{K \rightarrow \infty} \limsup_{u \rightarrow \infty} \text{II} \leq \limsup_{K \rightarrow \infty} \limsup_{u \rightarrow \infty} \sup_{y \geq K} \frac{g(y)}{\bar{F}(y)} \int_{K < y < u - K} \bar{F}(y) \frac{F(u - dy)}{\bar{F}(u)} = 0,$$

by Lemma 7.1 of [24]. Thus, the result follows by letting  $u \rightarrow \infty$  and then  $K \rightarrow \infty$  in (7.3).  $\square$

We now turn to conditions under which (3.3) holds in terms of  $h$  given by (3.6).

PROPOSITION 7.1. *If (1.2) holds, and  $h \geq 0$  is continuous a.e. (Lebesgue) with  $h(y)/\bar{V}(y) \rightarrow 0$  as  $y \rightarrow \infty$ , then (3.3) holds. More generally, assume  $h(y)/\bar{V}(y) \rightarrow L$ , then an extra term needs to be added to the RHS of (3.3), namely  $L/q\kappa(0, -\alpha)$ .*

PROOF. As noted above,  $qV(u)$  is a distribution function in  $\mathcal{S}^{(\alpha)}$ . Thus, by Lemma 7.1,

$$\int_{0 \leq y \leq u} h(y) \frac{V(u - dy)}{\bar{V}(u)} \rightarrow \int_0^\infty h(y) \alpha e^{\alpha y} dy + L \int_0^\infty e^{\alpha y} V(dy).$$

Dividing through by  $q$  and using (2.4) and (2.7) gives

$$\int_{0 \leq y \leq u} h(y) \frac{V(u - dy)}{q\bar{V}(u)} \rightarrow \int_0^\infty h(y) \frac{\alpha e^{\alpha y}}{q} dy + \frac{L}{q\kappa(0, -\alpha)}.$$

With  $L = 0$ , this is (3.7) which is equivalent to (3.3).  $\square$

The next result gives conditions on  $F$  in (3.6) which ensures convergence of  $h(y)/\bar{V}(y)$  as  $y \rightarrow \infty$ .

PROPOSITION 7.2. *If (1.2) holds and*

$$|F(y, \epsilon) - L|I(\epsilon(\zeta) > y) \rightarrow 0 \quad \text{uniformly in } \epsilon \in \bar{\mathcal{E}} \text{ as } y \rightarrow \infty,$$

then  $h(y)/\bar{V}(y) \rightarrow L\kappa^2(0, -\alpha)$ .

PROOF. By (3.6),

$$h(y) \sim L\bar{n}(\epsilon(\zeta) > y) = L\bar{\Pi}_H(y) \sim L\kappa^2(0, -\alpha)\bar{V}(y)$$

by (3.1) together with (4.4) of [24].  $\square$

As a consequence, we have the following.

PROOF OF THEOREM 3.2. This follows immediately from Propositions 7.1 and 7.2.  $\square$

REMARK 7.1. Another condition under which (3.8) from Theorem 3.2 holds, other than when  $F$  has compact support, is if  $F(y, \epsilon) = \tilde{F}(y, \epsilon)I(\epsilon(\zeta) \leq K)$  for some function  $\tilde{F}$  and some  $K \geq 0$ . In particular if  $\tilde{F} \geq 0$  satisfies the hypotheses of Proposition 3.1, then  $F$  satisfies all the hypotheses of Theorem 3.2.

The convergence in (3.9) cannot be improved to  $\xrightarrow{w}$  since from (3.4) the total mass of  $Q^{(\infty)}$  is given by

$$\begin{aligned} |Q^{(\infty)}| &= \frac{\alpha}{q} \int_{[0, \infty) \times \bar{\mathcal{E}}} e^{\alpha y} \bar{n}(d\epsilon, \epsilon(\zeta) > y) dy + d_H \frac{\alpha}{q} \\ &= \frac{\alpha}{q} \int_{[0, \infty)} e^{\alpha y} \bar{\Pi}_H(y) dy + d_H \frac{\alpha}{q} \\ (7.4) \quad &= \frac{1}{q} \int_{[0, \infty)} (e^{\alpha y} - 1) \Pi_H(dy) + d_H \frac{\alpha}{q} \\ &= 1 - \frac{\kappa(0, -\alpha)}{q}. \end{aligned}$$

Under (1.1),  $\kappa(0, -\alpha) = 0$  so  $|Q^{(\infty)}| = 1$ , but under (1.2),  $\kappa(0, -\alpha) > 0$  and so  $|Q^{(\infty)}| < 1$ .

As with Theorem 3.1, the convergence in Theorem 3.2 may alternatively be expressed in terms of the overshoot  $X_{\tau(u)} - u$  rather than the undershoot  $u - \bar{X}_{\tau(u)-}$ .



**THEOREM 7.1.** *Assume  $G : \bar{\mathcal{E}} \times [0, \infty) \rightarrow [0, \infty)$  is product measurable,  $e^{-\alpha x} G(\epsilon, x)$  is bounded in  $(\epsilon, x)$  and  $G(\epsilon, \cdot)$  is continuous for a.e.  $\epsilon$  w.r.t.  $\bar{\pi}$ . Further assume that*

$$G(\epsilon, \epsilon(\zeta) - y)I(\epsilon(\zeta) > y) \rightarrow 0 \quad \text{uniformly in } \epsilon \in \bar{\mathcal{E}} \text{ as } y \rightarrow \infty.$$

Then under (1.2),

$$\begin{aligned} & E^{(u)} G(Y_u, X_{\tau(u)} - u) \\ (7.5) \quad & \rightarrow \int_{[0, \infty)} \frac{\alpha}{q} e^{\alpha y} dy \int_{\bar{\mathcal{E}} \times (0, \infty)} G(\epsilon, x) \bar{\pi}(d\epsilon, \epsilon(\zeta) \in y + dx) \\ & + d_H \frac{\alpha}{q} G(\mathbf{0}, 0). \end{aligned}$$

**8. The irregular case.** We briefly consider the special case of Theorems 6.1 and 7.1 where 0 is irregular for  $(0, \infty)$  for  $X$ . In addition to covering the natural Lévy process version of Asmussen’s random walk result (1.5), that is when  $X$  is compound Poisson, it also includes the widely studied *compound Poisson model*, which recall includes a negative drift. We begin by identifying  $\bar{\pi}$  in terms of the stopped process  $X_{[0, \tau(0)]}$  where

$$(8.1) \quad X_{[0, \tau(0)]}(t) := X_{t \wedge \tau(0)}, \quad t \geq 0.$$

**PROPOSITION 8.1.** *Assume 0 is irregular for  $(0, \infty)$  for  $X$ , then*

$$(8.2) \quad P(\tau(0) < \infty) \bar{\pi}(d\epsilon) = |\Pi_H| P(X_{[0, \tau(0)]} \in d\epsilon).$$

**PROOF.** By construction, or using the compensation formula as in Theorem 2.1, for some constant  $c \in (0, \infty)$ ,

$$(8.3) \quad \bar{\pi}(d\epsilon) = n(d\epsilon) = cP(X_{[0, \tau(0)]} \in d\epsilon).$$

Since  $P(X_{\tau(0)} = 0, \tau(0) < \infty) = 0$ , this implies

$$(8.4) \quad cP(\tau(0) < \infty) = \bar{\pi}(\epsilon(\zeta) > 0, \zeta < \infty) = |\Pi_H|$$

by (4.9). Combining (8.3) and (8.4) proves (8.2).  $\square$

**PROPOSITION 8.2.** *Assume 0 is irregular for  $(0, \infty)$  for  $X$  and either,  $G$  is as in Theorem 6.1 and (1.1) holds, or  $G$  is as in Theorem 7.1 and (1.2) holds, then*

$$(8.5) \quad \begin{aligned} & E^{(u)} G(Y_u, X_{\tau(u)} - u) \\ & \rightarrow \frac{\alpha |\Pi_H| \int_0^\infty e^{\alpha y} E\{G(X_{[0, \tau(0)]}, X_{\tau(0)} - y); X_{\tau(0)} > y, \tau(0) < \infty\} dy}{q P(\tau(0) < \infty)}. \end{aligned}$$

PROOF. Since (1.1) or (1.2) holds, we have  $P(\tau(0) < \infty) > 0$ . Thus, by (8.2), if  $y \geq 0, x \geq 0$ , then

$$\bar{n}(d\epsilon, \epsilon(\zeta) \in y + dx) = \frac{|\Pi_H|P(X_{[0,\tau(0)]} \in d\epsilon, X_{\tau(0)} \in y + dx, \tau(0) < \infty)}{P(\tau(0) < \infty)}.$$

Since  $H$  is compound Poisson when 0 is irregular for  $(0, \infty)$ , we have  $d_H = 0$ . Consequently (6.9) or (7.5) yields

$$E^{(u)}G(Y_u, X_{\tau(u)} - u) \rightarrow \int_{[0,\infty)} \frac{\alpha}{q} e^{\alpha y} dy \int_{\bar{\mathcal{E}} \times (0,\infty)} G(\epsilon, x) \bar{n}(d\epsilon, \epsilon(\zeta) \in y + dx),$$

which, by (8), is equivalent to (8.5).  $\square$

Proposition 8.2 thus provides a natural Lévy process version of (1.5) under (1.2) as well as under (1.1). We conclude this section by confirming that the constants outside the integrals in (1.5) and (8.5) are in agreement when (1.1) holds. To be precise, by (6.1), the natural Lévy process form of the constant in (1.5), when (1.1) holds, is

$$\frac{\alpha m^*}{q E(X_{\tau(0)} e^{\alpha X_{\tau(0)}}; \tau(0) < \infty)}.$$

To see that this agrees with the constant in (8.5), it suffices to prove the following.

LEMMA 8.1. *If 0 is irregular for  $(0, \infty)$  for  $X$  and (1.1) holds, then*

$$|\Pi_H|E(X_{\tau(0)} e^{\alpha X_{\tau(0)}}; \tau(0) < \infty) = P(\tau(0) < \infty)E^*H_1.$$

PROOF. By (4.9) and (8.2),

$$|\Pi_H|P(X_{\tau(0)} \in dx, \tau(0) < \infty) = P(\tau(0) < \infty)\Pi_H(dx),$$

and so

$$|\Pi_H|E(X_{\tau(0)} e^{\alpha X_{\tau(0)}}; \tau(0) < \infty) = P(\tau(0) < \infty) \int_0^\infty x e^{\alpha x} \Pi_H(dx).$$

Since  $d_H = 0$  when 0 is irregular for  $(0, \infty)$ , the result now follows from (6.2).  $\square$

**9. Proofs of Theorems 3.3, 3.4 and related results.** To calculate the limits in Theorems 3.3 and 3.4, we consider a particular form for  $F$  in Theorems 3.1 and 3.2. Let  $f : [0, \infty)^4 \rightarrow [0, \infty)$  be a Borel function, and set

$$(9.1) \quad F(y, \epsilon) = f(y, \epsilon(\zeta) - y, y - \epsilon(\zeta -), \zeta)I(\epsilon(\zeta) \geq y).$$

Then

$$F(u - \bar{X}_{\tau(u)-}, Y_u) = f(u - \bar{X}_{\tau(u)-}, X_{\tau(u)} - u, u - X_{\tau(u)-}, \tau(u) - G_{\tau(u)-})$$

on  $\{\tau(u) < \infty\}$ . To calculate the limit in this case, we need the following.

LEMMA 9.1. *If  $F$  is of the form (9.1) then for every  $y \geq 0$ ,*

$$(9.2) \quad \int_{\mathcal{E}} F(y, \epsilon) \bar{n}(d\epsilon, \epsilon(\zeta) > y) \\ = \int_{x>0} \int_{v \geq 0} \int_{t \geq 0} f(y, x, v, t) I(v \geq y) \widehat{V}(dt, dv - y) \Pi_X(v + dx).$$

*If in addition,  $Ee^{\alpha X_1} < \infty$ ,  $f$  is jointly continuous in the first three variables and  $e^{-\alpha x} f(y, x, v, t)$  is bounded, then  $F$  satisfies the hypotheses of Proposition 3.1.*

PROOF. Using Proposition 4.3 in the third equality, we have

$$\int_{\mathcal{E}} F(y, \epsilon) \bar{n}(d\epsilon, \epsilon(\zeta) > y) \\ = \int_{\mathcal{E}} f(y, \epsilon(\zeta) - y, y - \epsilon(\zeta -), \zeta) \bar{n}(d\epsilon, \epsilon(\zeta) > y) \\ = \int_{x>0} \int_{z \geq 0} \int_{t \geq 0} f(y, x, y + z, t) \bar{n}(\epsilon(\zeta) \in y + dx, \epsilon(\zeta -) \in -dz, \zeta \in dt) \\ = \int_{x>0} \int_{z \geq 0} \int_{t \geq 0} f(y, x, y + z, t) \widehat{V}(dt, dz) \Pi_X(z + y + dx) \\ = \int_{x>0} \int_{v \geq 0} \int_{t \geq 0} f(y, x, v, t) I(v \geq y) \widehat{V}(dt, dv - y) \Pi_X(v + dx),$$

which proves (9.2).

For the second statement, we only need to check  $\bar{n}(B_y^c) = 0$  for a.e.  $y$ . But  $B_y^c \subset \{\epsilon : \epsilon(\zeta) = y\}$ , and so  $\bar{n}(B_y^c) \leq \Pi_H(\{y\}) = 0$  except for at most countably many  $y$ .  $\square$

REMARK 9.1. Lemma 9.1 remains true if  $f$  is replaced by  $\phi(y) f(y, x, v, t)$  where  $\phi$  is bounded and continuous a.e., since in that case  $\bar{n}(B_y^c) = 0$  except when  $y$  is a point of discontinuity of  $\phi$  or  $\Pi_H(\{y\}) = 0$ .

For reference below we note that if  $e^{-\alpha x} f(y, x, v, t)$  is bounded then

$$(9.3) \quad \int_{x=0} \int_{y \geq 0} \int_{v \geq 0} \int_{t \geq 0} f(y, x, v, t) e^{\alpha y} dy I(v \geq y) \\ \times \widehat{V}(dt, dv - y) \Pi_X(v + dx) = 0,$$

since

$$\int_{x=0} \int_{y \geq 0} \int_{v \geq 0} \int_{t \geq 0} e^{\alpha y} dy I(v \geq y) \widehat{V}(dt, dv - y) \Pi_X(v + dx) \\ = \int_{y \geq 0} \int_{v \geq 0} e^{\alpha y} dy I(v \geq y) \widehat{V}(dv - y) \Pi_X(\{v\}) \\ = \int_{v \geq 0} \widehat{V}(dv) \int_{y \geq 0} e^{\alpha y} \Pi_X(\{v + y\}) dy = 0.$$

We first consider limiting results for  $F$  of the form (9.1) in the Cramér–Lundberg setting, beginning with Theorem 3.3.

PROOF OF THEOREM 3.3. Define  $F$  by (9.1). Then by Lemma 9.1,  $F$  satisfies the hypotheses of Theorem 3.1, and hence the result follows from (3.5), (9.2) and (9.3).  $\square$

Marginal convergence in each of the first three variables in (3.11) was shown in [21]. Equation (3.11) exhibits the stronger joint convergence and includes the additional time variable  $\tau(u) - G_{\tau(u)-}$ . Note also that in the time variable, there is no restriction on  $f$  beyond bounded, and hence the convergence is stronger than weak convergence in this variable.

As an illustration of (3.10) we obtain, for any  $\lambda \leq 0, \eta \leq \alpha, \rho \leq 0$  and  $\delta \geq 0$ ,

$$\begin{aligned}
 & E^{(u)} e^{\lambda(u - \bar{X}_{\tau(u)-}) + \eta(X_{\tau(u)} - u) + \rho(u - X_{\tau(u)-}) - \delta(\tau(u) - G_{\tau(u)-})} \\
 (9.4) \quad & \rightarrow \int_{x \geq 0} \int_{y \geq 0} \int_{v \geq 0} \int_{t \geq 0} e^{\lambda y + \eta x + \rho v - \delta t} \frac{\alpha}{q} e^{\alpha y} dy I(v \geq y) \\
 & \quad \times \widehat{V}(dt, dv - y) \Pi_X(v + dx) + d_H \frac{\alpha}{q}.
 \end{aligned}$$

This gives the future value, at time  $G_{\tau(u)-}$ , of a Gerber–Shiu expected discounted penalty function (EDPF) as  $u \rightarrow \infty$ . The present value is zero since  $\tau(u) \rightarrow \infty$  in  $P^{(u)}$  probability as  $u \rightarrow \infty$ . The limit can be simplified if  $\rho = 0$ . From (4.8) and (9.3), we obtain

$$\begin{aligned}
 & E^{(u)} e^{\lambda(u - \bar{X}_{\tau(u)-}) + \eta(X_{\tau(u)} - u) - \delta(\tau(u) - G_{\tau(u)-})} \\
 & \rightarrow \int_{x > 0} \int_{y \geq 0} \int_{t \geq 0} e^{\lambda y + \eta x - \delta t} \frac{\alpha}{q} e^{\alpha y} dy \int_{v \geq y} \widehat{V}(dt, dv - y) \Pi_X(v + dx) \\
 & \quad + d_H \frac{\alpha}{q} \\
 (9.5) \quad & = \int_{x > 0} \int_{y \geq 0} \int_{t \geq 0} e^{\lambda y + \eta x - \delta t} \frac{\alpha}{q} e^{\alpha y} dy \Pi_{L^{-1}, H}(dt, y + dx) + d_H \frac{\alpha}{q} \\
 & = \int_{t \geq 0} \int_{y \geq 0} \int_{x > y} e^{\lambda y + \eta(x - y) - \delta t} \frac{\alpha}{q} e^{\alpha y} dy \Pi_{L^{-1}, H}(dt, dx) + d_H \frac{\alpha}{q} \\
 & = \frac{\alpha}{q(\eta - \lambda - \alpha)} \int_{t \geq 0} \int_{x > 0} (e^{\eta x} - e^{(\lambda + \alpha)x}) e^{-\delta t} \Pi_{L^{-1}, H}(dt, dx) + d_H \frac{\alpha}{q} \\
 & = \frac{\alpha(\kappa(\delta, -(\lambda + \alpha)) - \kappa(\delta, -\eta))}{q(\eta - \lambda - \alpha)}.
 \end{aligned}$$

Under (1.1), it is possible that  $E e^{\theta X_1} = \infty$  for all  $\theta > \alpha$ , but it is often the case that  $E e^{\theta X_1} < \infty$  for some  $\theta > \alpha$ . The next result extends Theorem 3.3 to include

this possibility, and also provides more information when restricted to the former setting. This is done by taking advantage of the special form of  $F$  in (9.1), whereas Theorem 3.3 was derived from the general convergence result in Theorem 3.1. It is interesting to note how the exponential moments may be spread out over the undershoot variables. The EDPF results in (9.4) and (9.5) also have obvious extensions to this setting.

**THEOREM 9.1.** *Assume (1.1) holds and  $f : [0, \infty)^4 \rightarrow [0, \infty)$  is a Borel function which is jointly continuous in the first three variables. Assume  $\theta \geq \alpha$  and one of the following three conditions holds:*

- (i)  $Ee^{\theta X_1} < \infty$ ,  $\rho < \theta$  and  $\lambda + \rho < \theta - \alpha$ ;
- (ii)  $EX_1 e^{\theta X_1} < \infty$ ,  $\rho \leq \theta$  and  $\lambda + \rho \leq \theta - \alpha$ , with at least one of these inequalities being strict;
- (iii)  $EX_1^2 e^{\theta X_1} < \infty$ ,  $\rho \leq \theta$  and  $\lambda + \rho \leq \theta - \alpha$ .

If  $e^{-\lambda y - \theta x - \rho v} f(y, x, v, t)$  is bounded, then

$$\begin{aligned}
 & E^{(u)} f(u - \bar{X}_{\tau(u)-}, X_{\tau(u)} - u, u - X_{\tau(u)-}, \tau(u) - G_{\tau(u)-}) \\
 & \rightarrow \int_{x \geq 0} \int_{y \geq 0} \int_{v \geq 0} \int_{t \geq 0} f(y, x, v, t) \frac{\alpha}{q} e^{\alpha y} dy I(v \geq y) \\
 (9.6) \quad & \times \widehat{V}(dt, dv - y) \Pi_X(v + dx) \\
 & + d_H \frac{\alpha}{q} f(0, 0, 0, 0).
 \end{aligned}$$

**PROOF.** Define  $F$  by (9.1) and then  $h$  by (3.6). We will show that  $h$  satisfies the hypotheses of Proposition 6.1. Let  $\tilde{f}(y, x, v, t) = e^{-\lambda y - \theta x - \rho v} f(y, x, v, t)$ . Then  $\tilde{f}$  is bounded, jointly continuous in the first three variables, and by (9.2), for every  $y \geq 0$ ,

$$\begin{aligned}
 h(y) &= \int_{\bar{\mathcal{E}}} F(y, e) \bar{n}(de, e(\zeta) > y) \\
 &= \int_{x > 0} \int_{v \geq 0} \int_{t \geq 0} \tilde{f}(y, x, v, t) I(v \geq y) e^{\lambda y + \theta x + \rho v} \\
 (9.7) \quad & \times \widehat{V}(dt, dv - y) \Pi_X(v + dx) \\
 &= e^{(\lambda + \rho - \theta)y} \int_{x > 0} \int_{v \geq 0} \int_{t \geq 0} \tilde{f}(y, x - v - y, v + y, t) I(x > v + y) \\
 & \times e^{\theta x + (\rho - \theta)v} \widehat{V}(dt, dv) \Pi_X(dx)
 \end{aligned}$$

after a change of variables. Let  $g_y(x, v, t) = \tilde{f}(y, x - v - y, v + y, t) I(x > v + y) e^{\theta x + (\rho - \theta)v}$ . Then clearly  $g_z(x, v, t) \rightarrow g_y(x, v, t)$  as  $z \downarrow y$ , for every  $y \geq 0$ . Now

fix  $y > 0$  and let  $|z - y| < y/2$ . Then for some constant  $C$  independent of  $z, x, v$  and  $t$ ,

$$|g_z(x, v, t)| \leq CI(x > v + y/2)e^{\theta x + (\rho - \theta)v}$$

and

$$(9.8) \quad \int_{x>0} \int_{v \geq 0} \int_{t \geq 0} I(x > v + y/2)e^{\theta x + (\rho - \theta)v} \widehat{V}(dt, dv) \Pi_X(dx) \leq \int_{x>y/2} e^{\theta x} \Pi_X(dx) \int_{0 \leq v < x-y/2} e^{(\rho - \theta)v} \widehat{V}(dv).$$

If we show this last integral is finite, then by dominated convergence,  $h(z) \rightarrow h(y)$  as  $z \downarrow y$  for every  $y > 0$ , showing that  $h$  is right continuous on  $(0, \infty)$ , and consequently continuous a.e. The final expression in (9.8) is decreasing in  $y$ , hence to prove finiteness it suffices to prove the following stronger result, which will be needed below; for every  $\varepsilon > 0$ ,

$$(9.9) \quad I_\varepsilon := \int_\varepsilon^\infty e^{(\alpha + \lambda + \rho - \theta)y} dy \int_{x>y} e^{\theta x} \Pi_X(dx) \int_{0 \leq v < x-y} e^{(\rho - \theta)v} \widehat{V}(dv) < \infty.$$

We will need the following consequence of Proposition 3.1 of Bertoin [5]; for every  $y > 0$  there is a constant  $c = c(y)$  such that

$$(9.10) \quad \widehat{V}(v) \leq cv \quad \text{for } v \geq y.$$

First assume  $\rho < \theta$ . Integrating by parts and using (9.10) shows that the last of the three integrals in (9.9) is bounded independently of  $x$  and  $y$ , hence

$$I_\varepsilon \leq C \int_{x>\varepsilon} e^{\theta x} \Pi_X(dx) \int_{\varepsilon < y < x} e^{(\alpha + \lambda + \rho - \theta)y} dy \leq C \begin{cases} \int_{x>\varepsilon} e^{\theta x} \Pi_X(dx), & \text{if } \alpha + \lambda + \rho - \theta < 0, \\ \int_{x>\varepsilon} x e^{\theta x} \Pi_X(dx), & \text{if } \alpha + \lambda + \rho - \theta = 0. \end{cases}$$

Thus,  $I_\varepsilon < \infty$  under each of the assumptions (i), (ii) and (iii) by Theorem 25.3 of Sato [29]. Now assume  $\rho = \theta$ . Then

$$I_\varepsilon = \int_{x>\varepsilon} e^{\theta x} \Pi_X(dx) \int_{\varepsilon < y < x} e^{(\alpha + \lambda + \rho - \theta)y} \widehat{V}(x - y) dy.$$

If  $\alpha + \lambda + \rho - \theta = 0$ , then we are in case (iii) and

$$I_\varepsilon \leq \int_{x>\varepsilon} x \widehat{V}(x) e^{\theta x} \Pi_X(dx) \leq C \int_{x>\varepsilon} x^2 e^{\theta x} \Pi_X(dx),$$

which is finite under (iii). Finally, if  $\alpha + \lambda + \rho - \theta < 0$  then we are in case (ii) or (iii). We break  $I_\varepsilon$  into two parts  $I_\varepsilon(1) + I_\varepsilon(2)$  where

$$\begin{aligned} I_\varepsilon(1) &= \int_{x>\varepsilon} e^{\theta x} \Pi_X(dx) \int_{\varepsilon \vee (x-1) < y < x} e^{(\alpha+\lambda+\rho-\theta)y} \widehat{V}(x-y) dy \\ &\leq \widehat{V}(1) \int_{x>\varepsilon} e^{\theta x} \Pi_X(dx) \end{aligned}$$

and

$$\begin{aligned} I_\varepsilon(2) &= \int_{x>\varepsilon} e^{\theta x} \Pi_X(dx) \int_{\varepsilon < y \leq \varepsilon \vee (x-1)} e^{(\alpha+\lambda+\rho-\theta)y} \widehat{V}(x-y) dy \\ &\leq c(1) \int_{x>\varepsilon} e^{\theta x} \Pi_X(dx) \int_{\varepsilon < y \leq \varepsilon \vee (x-1)} e^{(\alpha+\lambda+\rho-\theta)y} (x-y) dy \\ &\leq C \int_{x>\varepsilon} x e^{\theta x} \Pi_X(dx). \end{aligned}$$

Thus,  $I_\varepsilon$  is finite in this case also, completing the proof of (9.9).

By (9.7), for every  $y \geq 0$ ,

$$\begin{aligned} e^{\alpha y} h(y) &\leq C e^{(\alpha+\lambda+\rho-\theta)y} \int_{v \geq 0} \int_{x > 0} I(x > v + y) e^{\theta x + (\rho-\theta)v} \widehat{V}(dv) \Pi_X(dx) \\ (9.11) \quad &=: k(y) \end{aligned}$$

say. Clearly,  $k$  is nonincreasing on  $[0, \infty)$ , and for every  $\varepsilon > 0$ ,

$$\begin{aligned} \int_\varepsilon^\infty k(y) dy &= C \int_\varepsilon^\infty e^{(\alpha+\lambda+\rho-\theta)y} dy \int_{x>y} e^{\theta x} \Pi_X(dx) \int_{0 \leq v < x-y} e^{(\rho-\theta)v} \widehat{V}(dv) \\ &< \infty, \end{aligned}$$

by (9.9) under (i), (ii) or (iii). Hence, in each case  $e^{\alpha y} h(y) 1_{[\varepsilon, \infty)}(y)$  is directly Riemann integrable for every  $\varepsilon > 0$ .

Finally, from (9.11), for  $\varepsilon \in (0, 1)$ ,

$$\begin{aligned} \int_{[0, \varepsilon)} h(y) \frac{V(u-dy)}{\overline{V}(u)} &\leq C \int_{v \geq 0} \int_{x > v} e^{\theta x + (\rho-\theta)v} \widehat{V}(dv) \Pi_X(dx) \int_{[0, \varepsilon)} I(y < x-v) \frac{V(u-dy)}{\overline{V}(u)}. \end{aligned}$$

By the uniform convergence on compact sets in (6.3), it follows that for large  $u$ ,

$$\begin{aligned} \int_{[0, \varepsilon)} h(y) \frac{V(u-dy)}{\overline{V}(u)} &\leq C_1 \left( \int_{v \leq 1} + \int_{v > 1} \right) \int_{x > v} e^{\theta x + (\rho-\theta)v} [(x-v) \wedge \varepsilon] \widehat{V}(dv) \Pi_X(dx) \\ &= \text{I} + \text{II}. \end{aligned}$$

Now, using (4.7),

$$\begin{aligned} I &\leq C_1 \int_{v \leq 1} \int_{x > 0} (x \wedge \varepsilon) e^{\theta x} \widehat{V}(dv) \Pi_X(v + dx) \\ &\leq C_1 \int_{x > 0} (x \wedge \varepsilon) e^{\theta x} \Pi_H(dx) \rightarrow 0 \end{aligned}$$

as  $\varepsilon \rightarrow 0$  by dominated convergence, since  $\int_{x > 1} e^{\theta x} \Pi_H(dx) < \infty$  by Proposition 7.1 of [18] and  $\int_{x \leq 1} x e^{\theta x} \Pi_H(dx) < \infty$  because  $H$  is a subordinator. For the second term,

$$II \leq C_1 \varepsilon \int_{x > 1} e^{\theta x} \Pi_X(dx) \int_{0 \leq v < x} e^{(\rho - \theta)v} \widehat{V}(dv) \rightarrow 0$$

as  $\varepsilon \rightarrow 0$ , since the integral is easily seen to be finite from (9.9) and the renewal theorem. Thus we may apply Proposition 6.1 to  $h$ , and (9.6) follows after observing that the integral over  $x = 0$  in (9.6) is zero by (9.3).  $\square$

REMARK 9.2. As the proof shows, conditions under which (9.6) holds can be stated in terms of the integral condition (9.9) on the renewal function  $\widehat{V}$ , rather than in terms of conditions (i)–(iii). Specifically, assume (1.1) holds,  $Ee^{\theta X_1} < \infty$  for some  $\theta \geq \alpha$ , and  $f : [0, \infty)^4 \rightarrow [0, \infty)$  is a Borel function which is jointly continuous in the first three variables and  $e^{-\lambda y - \theta x - \rho v} f(y, x, v, t)$  is bounded where  $\lambda + \rho \leq \theta - \alpha$  and  $\rho \leq \theta$ . If (9.9) holds for every  $\epsilon > 0$ , then (9.6) holds.

We now turn to the convolution equivalent setting. In this case, we need to impose an extra condition on  $f$  in (9.1).

PROPOSITION 9.1. Assume  $F$  is given by (9.1) where

$$(9.12) \quad \sup_{x > 0, t \geq 0, v \geq y} f(y, x, v, t) \rightarrow 0 \quad \text{as } y \rightarrow \infty.$$

Then (3.8) holds.

PROOF. From (9.1),

$$\begin{aligned} \sup_{\epsilon \in \overline{\mathcal{E}}} F(y, \epsilon) I(\epsilon(\zeta) > y) &= \sup_{\epsilon \in \overline{\mathcal{E}}} f(y, \epsilon(\zeta) - y, y - \epsilon(\zeta -), \zeta) I(\epsilon(\zeta) > y) \\ &\leq \sup_{x > 0, t \geq 0, v \geq y} f(y, x, v, t) \rightarrow 0 \end{aligned}$$

as  $y \rightarrow \infty$  by (9.12).  $\square$

As an immediate consequence, we have the following.

PROOF OF THEOREM 3.4. Define  $F$  by (9.1). By Lemma 9.1 and Proposition 9.1,  $F$  satisfies the hypotheses of Theorem 3.2. Thus, (3.3) holds which is equivalent to (3.12) by (9.2) and (9.3).  $\square$



Theorem 3.4 imposes the extra condition (9.12) on  $f$  compared with Theorem 3.3. As a typical example, any function  $f$  satisfying the conditions of Theorem 3.3, when multiplied by a bounded continuous function with compact support  $\phi(y)$ , trivially satisfies the conditions of Theorem 3.4. This manifests itself in the convergence of (3.13) only being vague convergence rather than weak convergence. It cannot be improved to the weak convergence of (3.11) since, as noted earlier in (7.4), the total mass of the limit in (3.13) is  $1 - \kappa(0, -\alpha)q^{-1}$ .

Another example of the effect of condition (9.12) is in the calculation of the EDPF analogous (9.4). Using Remark 9.1, the continuity assumption on  $\phi$  above can be weakened to continuous a.e. Hence, we may take  $\phi(y) = I(y \leq K)$  for some  $K \geq 0$ . Thus, applying (3.12) to the function  $f(y, x, v, t) = e^{\lambda y + \eta x + \rho v - \delta t} I(y \leq K)$  where  $K > 0, \lambda \leq 0, \eta \leq \alpha, \rho \leq 0$  and  $\delta \geq 0$ , we obtain

$$\begin{aligned} & E^{(u)} \left\{ e^{\lambda(u - \bar{X}_{\tau(u)-}) + \eta(X_{\tau(u)} - u) + \rho(u - X_{\tau(u)-}) - \delta(\tau(u) - G_{\tau(u)-})}; u - \bar{X}_{\tau(u)-} \leq K \right\} \\ & \rightarrow \int_{0 \leq y \leq K} \int_{x \geq 0} \int_{v \geq 0} \int_{t \geq 0} e^{\lambda y + \eta x + \rho v - \delta t} \frac{\alpha}{q} e^{\alpha y} dy I(v \geq y) \\ & \quad \times \widehat{V}(dt, dv - y) \Pi_X(v + dx) + d_H \frac{\alpha}{q}. \end{aligned}$$

The restriction imposed by  $K$  cannot be removed as will be apparent from Proposition 9.2 below. Finally, we point out that there is no extension of Theorem 3.4 to the setting of Theorem 9.1 since  $Ee^{\theta X_1} = \infty$  for all  $\theta > \alpha$ .

We next address convergence of the marginals in (3.11) and (3.13). As indicated in Section 3, some care is needed under (1.2) since, in (3.13), the limit of the marginals is not the marginal of the limits for the case of the overshoot and  $\tau(u) - G_{\tau(u)-}$ . If  $F$  is given by (9.1) where  $f$  depends only on  $x$  and  $t$ , then, using (4.8), (9.2) reduces to

$$\begin{aligned} (9.13) \quad & \int_{\bar{\mathcal{E}}} F(y, \epsilon) \bar{n}(d\epsilon, \epsilon(\zeta) > y) \\ & = \int_{x > 0} \int_{t \geq 0} f(x, t) \Pi_{L^{-1}, H}(dt, y + dx), \quad y \geq 0. \end{aligned}$$

In particular, under (1.1), by Theorem 3.3, for  $x \geq 0, t \geq 0$ ,

$$\begin{aligned} (9.14) \quad & P^{(u)}(X_{\tau(u)} - u \in dx, \tau(u) - G_{\tau(u)-} \in dt) \\ & \xrightarrow{w} \frac{\alpha}{q} \int_{y \geq 0} e^{\alpha y} \Pi_{L^{-1}, H}(dt, y + dx) dy + d_H \frac{\alpha}{q} \delta_{(0,0)}(dx, dt). \end{aligned}$$

Under (1.2), the mass of the limit in (9.14) is less than one. In this case, an extra term appears in the limit. The distribution of this additional mass and proof of joint weak convergence under (1.2) is given in the following result.

PROPOSITION 9.2. Assume (1.2) holds and that  $f : [0, \infty)^2 \rightarrow [0, \infty)$  is a Borel function which is continuous in the first variable, and  $e^{-\beta x} f(x, t)$  is bounded for some  $\beta < \alpha$ . Then

$$\begin{aligned}
 & E^{(u)} f(X_{\tau(u)} - u, \tau(u) - G_{\tau(u)-}) \\
 & \rightarrow \frac{-\Psi_X(-i\alpha)}{q} \int_{x \geq 0} \int_{t \geq 0} f(x, t) \alpha e^{-\alpha x} dx \int_{v \geq 0} e^{-\alpha v} \widehat{V}(dt, dv) \\
 (9.15) \quad & + \frac{\alpha}{q} \int_{x \geq 0} \int_{t \geq 0} f(x, t) \int_{y \geq 0} e^{\alpha y} \Pi_{L^{-1}, H}(dt, y + dx) dy \\
 & + d_H \frac{\alpha}{q} f(0, 0).
 \end{aligned}$$

In particular, we have joint convergence; for  $x \geq 0, t \geq 0$ ,

$$\begin{aligned}
 & P^{(u)}(X_{\tau(u)} - u \in dx, \tau(u) - G_{\tau(u)-} \in dt) \\
 (9.16) \quad & \xrightarrow{w} \frac{-\Psi_X(-i\alpha)}{q} \alpha e^{-\alpha x} dx \int_{v \geq 0} e^{-\alpha v} \widehat{V}(dt, dv) \\
 & + \frac{\alpha}{q} \int_{y \geq 0} e^{\alpha y} \Pi_{L^{-1}, H}(dt, y + dx) dy + d_H \frac{\alpha}{q} \delta_{(0,0)}(dx, dt).
 \end{aligned}$$

PROOF. We will use Proposition 7.1. Let

$$F(y, \epsilon) = f(\epsilon(\zeta) - y, \zeta) I(\epsilon(\zeta) \geq y).$$

By Lemma 9.1,  $F$  satisfies the hypothesis of Proposition 3.1, hence  $h$  is continuous a.e. Next we evaluate the limit of  $h(y)/\overline{\Pi}_X(y)$  as  $y \rightarrow \infty$ . By (9.2), for  $y \geq 0$ ,

$$\begin{aligned}
 h(y) &= \int_{\overline{\mathcal{E}}} F(y, \epsilon) \overline{n}(d\epsilon, \epsilon(\zeta) > y) \\
 &= \int_{x > 0} \int_{v \geq 0} \int_{t \geq 0} f(x, t) I(v \geq y) \widehat{V}(dt, dv - y) \Pi_X(v + dx) \\
 &= \int_{t \geq 0} \int_{v \geq 0} \widehat{V}(dt, dv) \int_{x > 0} f(x, t) \Pi_X(y + v + dx).
 \end{aligned}$$

Observe that for  $v \geq 0$ , from footnote 6 (page 381) and (7.1),

$$\frac{\Pi_X(y + v + dx)}{\overline{\Pi}_X(y)} \xrightarrow{w} \alpha e^{-\alpha(v+x)} dx \quad \text{on } [0, \infty) \text{ as } y \rightarrow \infty.$$

Further, by Potter’s bounds (see, e.g., (4.10) of [20]), if  $\gamma \in (\beta, \alpha)$  then

$$\frac{\overline{\Pi}_X(y + v)}{\overline{\Pi}_X(y)} \leq C e^{-\gamma v} \quad \text{if } v \geq 0, y \geq 1,$$

where  $C$  depends only on  $\gamma$ . Thus, for any  $y \geq 1, v \geq 0$  and  $K \geq 0$ ,

$$\begin{aligned}
 \int_{x>K} e^{\beta x} \frac{\Pi_X(y+v+dx)}{\overline{\Pi}_X(y)} &= \int_{x>K} \beta e^{\beta x} \frac{\overline{\Pi}_X(y+v+x)}{\overline{\Pi}_X(y)} dx \\
 &\quad + \frac{e^{\beta K} \overline{\Pi}_X(y+v+K)}{\overline{\Pi}_X(y)} \\
 (9.17) \qquad \qquad \qquad &\leq C \int_{x>K} \beta e^{\beta x} e^{-\gamma(v+x)} dx + C e^{\beta K} e^{-\gamma(v+K)} \\
 &\leq C e^{-\gamma v - (\gamma - \beta)K}.
 \end{aligned}$$

Now, for any  $v \geq 0$  and  $K \geq 0$  write

$$\begin{aligned}
 \int_{x>0} f(x, t) \frac{\Pi_X(y+v+dx)}{\overline{\Pi}_X(y)} \\
 (9.18) \qquad \qquad \qquad &= \left( \int_{0<x \leq K} + \int_{x>K} \right) f(x, t) \frac{\Pi_X(y+v+dx)}{\overline{\Pi}_X(y)} \\
 &= I + II.
 \end{aligned}$$

By weak convergence,

$$I \rightarrow \int_{0<x \leq K} f(x, t) \alpha e^{-\alpha(v+x)} dx \quad \text{as } y \rightarrow \infty,$$

and by monotone convergence,

$$\int_{0<x \leq K} f(x, t) \alpha e^{-\alpha(v+x)} dx \rightarrow \int_{x \geq 0} f(x, t) \alpha e^{-\alpha(v+x)} dx \quad \text{as } K \rightarrow \infty.$$

On the other hand, by (9.17),

$$II \leq C e^{-\gamma v - (\gamma - \beta)K} \quad \text{for } y \geq 1.$$

Thus, letting  $y \rightarrow \infty$  then  $K \rightarrow \infty$  in (9.18) gives

$$\int_{x>0} f(x, t) \frac{\Pi_X(y+v+dx)}{\overline{\Pi}_X(y)} \rightarrow \int_{x \geq 0} f(x, t) \alpha e^{-\alpha(v+x)} dx.$$

Further, by (9.17) with  $K = 0$ , for every  $v \geq 0$

$$\int_{x>0} f(x, t) \frac{\Pi_X(y+v+dx)}{\overline{\Pi}_X(y)} \leq C e^{-\gamma v}.$$

Hence, by dominated convergence,

$$\frac{h(y)}{\overline{\Pi}_X(y)} \rightarrow \int_{x \geq 0} \int_{t \geq 0} f(x, t) \alpha e^{-\alpha x} dx \int_{v \geq 0} e^{-\alpha v} \widehat{V}(dt, dv).$$

Since

$$\lim_{y \rightarrow \infty} \frac{\bar{\Pi}_X(y)}{\bar{V}(y)} = \kappa^2(0, -\alpha)\widehat{\kappa}(0, \alpha)$$

by (3.1), together with (4.4) and Proposition 5.3 of [24], we thus have

$$\frac{h(y)}{\bar{V}(y)} \rightarrow \kappa^2(0, -\alpha)\widehat{\kappa}(0, \alpha) \int_{x \geq 0} \int_{t \geq 0} f(x, t)\alpha e^{-\alpha x} dx \int_{v \geq 0} e^{-\alpha v} \widehat{V}(dt, dv).$$

Hence, by (2.6) and Proposition 7.1,

$$\begin{aligned} & E^{(u)} f(X_{\tau(u)} - u, \tau(u) - G_{\tau(u)-}) \\ & \rightarrow \frac{-\Psi_X(-i\alpha)}{q} \int_{x \geq 0} \int_{t \geq 0} f(x, t)\alpha e^{-\alpha x} dx \int_{v \geq 0} e^{-\alpha v} \widehat{V}(dt, dv) \\ & \quad + \int_{y \geq 0} \frac{\alpha}{q} e^{\alpha y} dy \int_{\mathcal{E}} F(y, \epsilon) \bar{n}(d\epsilon, \epsilon(\zeta) > y) + d_H \frac{\alpha}{q} f(0, 0) \\ & = \frac{-\Psi_X(-i\alpha)}{q} \int_{x \geq 0} \int_{t \geq 0} f(x, t)\alpha e^{-\alpha x} dx \int_{v \geq 0} e^{-\alpha v} \widehat{V}(dt, dv) \\ & \quad + \frac{\alpha}{q} \int_{x > 0} \int_{t \geq 0} f(x, t) \int_{y \geq 0} e^{\alpha y} \Pi_{L^{-1}, H}(dt, y + dx) dy + d_H \frac{\alpha}{q} f(0, 0) \end{aligned}$$

by (9.13). This proves (9.15) since the integral over  $\{x = 0\}$  in the final expression vanishes.  $\square$

From (9.16), a simple calculation shows that the limiting distribution of the overshoot is as given in (1.4), and

$$(9.19) \quad \begin{aligned} & P^{(u)}(\tau(u) - G_{\tau(u)-} \in dt) \\ & \xrightarrow{w} q^{-1}(-\Psi_X(-i\alpha)\delta_{-\alpha}^{\widehat{V}}(dt) + \alpha d_H \delta_0(dt) + K(dt)), \end{aligned}$$

where  $K(dt)$  is given by (3.14) and

$$\delta_{-\alpha}^{\widehat{V}}(dt) = \int_{v \geq 0} e^{-\alpha v} \widehat{V}(dt, dv).$$

Using (9.15), we can calculate the limiting value of an EDPF similar to (9.5); for any  $\beta < \alpha$  and  $\delta \geq 0$ ,

$$\begin{aligned} & E^{(u)} e^{\beta(X_{\tau(u)} - u) - \delta(\tau(u) - G_{\tau(u)-})} \\ & \rightarrow \frac{-\Psi_X(-i\alpha)}{q} \int_{x \geq 0} \alpha e^{-(\alpha - \beta)x} dx \int_{t \geq 0} \int_{v \geq 0} e^{-\delta t - \alpha v} \widehat{V}(dt, dv) \\ & \quad + \frac{\alpha}{q} \int_{x \geq 0} \int_{t \geq 0} e^{\beta x - \delta t} \int_{y \geq 0} e^{\alpha y} \Pi_{L^{-1}, H}(dt, y + dx) dy + d_H \frac{\alpha}{q} f(0, 0) \\ & = \frac{-\alpha \Psi_X(-i\alpha)}{q(\alpha - \beta)\widehat{\kappa}(\delta, \alpha)} + \frac{\alpha(\kappa(\delta, -\beta) - \kappa(\delta, -\alpha))}{q(\alpha - \beta)} \end{aligned}$$

by the same calculation as (9.5).

The results of this section, in the convolution equivalent case, can be derived from a path decomposition for the limiting process given in [20]. The main result in [20], Theorem 3.1, makes precise the idea that under  $P^{(u)}$  for large  $u$ ,  $X$  behaves like an Esscher transform of  $X$  up to an independent exponential time  $\tau$ . At this time, the process makes a large jump into a neighborhood of  $u$ , and if  $W_t = X_{\tau+t} - u$  then

$$P(W \in dw) = \kappa(0, -\alpha) \int_{z \in \mathbf{R}} \alpha e^{-\alpha z} \bar{V}(-z) dz P_z(X \in dw | \tau(0) < \infty),$$

$w \in D,$

where we set  $\bar{V}(y) = q^{-1}$  for  $y < 0$ . Thus,  $W$  has the law of  $X$  conditioned on  $\tau(0) < \infty$  and started with initial distribution

$$P(W_0 \in dz) = \kappa(0, -\alpha) \alpha e^{-\alpha z} \bar{V}(-z) dz, \quad z \in \mathbf{R}.$$

In the Cramér–Lundberg case, there is no comparable decomposition for the entire path since there is no “large jump” at which to do the decomposition. One of the aims of this paper is to offer an alternative approach by describing the path from the time of the last maximum prior to first passage until the time of first passage. This allows the limiting distribution of many variables associated with ruin to be readily calculated.

**APPENDIX: COMPLETION OF THE PROOF OF PROPOSITION 4.2  
WHEN  $X$  IS COMPOUND POISSON**

For  $\varepsilon > 0$ , let

$$X_t^\varepsilon = X_t - \varepsilon t.$$

If  $X$  is compound Poisson, then Proposition 4.2 holds for  $X^\varepsilon$ . The aim is then to take limits as  $\varepsilon \rightarrow 0$  and check that (4.4) continues to hold in the limit. We begin with an alternative characterization of the constants in (8.2). Recall the notation of (8.1).

LEMMA A.1. *Assume 0 is irregular for  $(0, \infty)$ , then*

$$d_{\hat{\tau}^{-1}} n(d\varepsilon) = P(X_{[0, \tau(0)]} \in d\varepsilon).$$

PROOF. If  $s \notin G$  set  $\varepsilon_s = \Delta$  where  $\Delta$  is a cemetery state. Then  $\{(t, \varepsilon_{L_{t-}^{-1}}) : t \geq 0, \varepsilon_{L_{t-}^{-1}} \neq \Delta\}$  is a Poisson point process with characteristic measure  $dt \otimes n(d\varepsilon)$ . By construction,  $n$  is proportional to the law of the first excursion, thus

$$(A.1) \quad n(d\varepsilon) = |n| P(X_{[0, \tau(0)]} \in d\varepsilon).$$

Now let  $\sigma = \inf\{t : \varepsilon_{L_{t-}^{-1}} \neq \Delta\}$ . Then  $\sigma$  is exponentially distributed with parameter  $|n|$ . On the other hand  $\sigma$  is the time of the first jump of  $L^{-1}$  and hence is exponential with parameter  $p$  given by (2.3). A short calculation using duality (see,

e.g., the paragraph following (2.7) in [9]) shows that if 0 is irregular for  $(0, \infty)$ , then

$$(A.2) \quad p d_{\widehat{\mathcal{L}}^{-1}} = 1.$$

Hence,  $|n|^{-1} = d_{\widehat{\mathcal{L}}^{-1}}$  and the result follows from (A.1).  $\square$

Let  $n^\varepsilon$  denote the excursion measure of  $X^\varepsilon$ , with similar notation for all other quantities related to  $X^\varepsilon$  or  $\widehat{X}^\varepsilon$ . To ease the notational complexity, we will write

$$\widehat{d}_\varepsilon = d_{(\widehat{\mathcal{L}}^\varepsilon)^{-1}} \quad \text{and} \quad \widehat{d} = d_{\widehat{\mathcal{L}}^{-1}}.$$

LEMMA A.2. *Assume 0 is irregular for  $(0, \infty)$ , then  $\widehat{d}_\varepsilon$  is nondecreasing, and for any  $\delta \geq 0$*

$$\widehat{d}_\varepsilon \downarrow \widehat{d}_\delta \quad \text{as } \varepsilon \downarrow \delta.$$

PROOF. Clearly, for  $0 \leq \delta < \varepsilon$ , we have  $\tau^\delta(0) \leq \tau^\varepsilon(0)$  and  $\tau^\varepsilon(0) \downarrow \tau^\delta(0)$  as  $\varepsilon \downarrow \delta$ . Thus,

$$E(e^{-\tau^\varepsilon(0)}; \tau^\varepsilon(0) < \infty) \uparrow E(e^{-\tau^\delta(0)}; \tau^\delta(0) < \infty),$$

and so from (2.3),  $p^\varepsilon \uparrow p^\delta$ . Hence, by (A.2),  $\widehat{d}_\varepsilon \downarrow \widehat{d}_\delta$ .  $\square$

PROPOSITION A.1. *Assume  $X$  is compound Poisson and  $f : [0, \infty)^2 \rightarrow [0, \infty)$  is continuous with compact support. Then*

$$\begin{aligned} & \int_{t \geq 0} \int_{z \geq 0} f(t, z) n^\varepsilon(\epsilon(t) \in -dz, \zeta > t) dt \\ & \rightarrow \int_{t \geq 0} \int_{z \geq 0} f(t, z) n(\epsilon(t) \in -dz, \zeta > t) dt \quad \text{as } \varepsilon \rightarrow 0. \end{aligned}$$

PROOF. Assume  $f$  vanishes for  $t \geq r$ . Then

$$(A.3) \quad f(t, -X_t^\varepsilon) I(\tau^\varepsilon(0) > t) \leq \|f\|_\infty I(t \leq r).$$

Thus, using Lemma A.1,

$$\begin{aligned} & \int_{t \geq 0} \int_{z \geq 0} f(t, z) n^\varepsilon(\epsilon(t) \in -dz, \zeta > t) dt \\ & = \widehat{d}_\varepsilon^{-1} \int_{t \geq 0} \int_{z \geq 0} f(t, z) P(X_t^\varepsilon \in -dz, \tau^\varepsilon(0) > t) dt \\ & = \widehat{d}_\varepsilon^{-1} \int_{t=0}^\infty E(f(t, -X_t^\varepsilon); \tau^\varepsilon(0) > t) dt \\ & \rightarrow \widehat{d}^{-1} \int_{t=0}^\infty E(f(t, -X_t); \tau(0) > t) dt \\ & = \int_{t \geq 0} \int_{z \geq 0} f(t, z) n(\epsilon(t) \in -dz, \zeta > t) dt \end{aligned}$$

by (A.3) and dominated convergence, since  $X_t^\varepsilon \rightarrow X_t$ ,  $\tau^\varepsilon(0) \rightarrow \tau(0)$  and  $P(\tau(0) = t) = 0$ .  $\square$

PROPOSITION A.2. *Assume  $X$  is compound Poisson and  $f : [0, \infty)^2 \rightarrow [0, \infty)$  is continuous with compact support. Then*

$$\int_{t \geq 0} \int_{z \geq 0} f(t, z) \widehat{V}^\varepsilon(dt, dz) \rightarrow \int_{t \geq 0} \int_{z \geq 0} f(t, z) \widehat{V}(dt, dz) \quad \text{as } \varepsilon \rightarrow 0.$$

PROOF. We will show

$$(A.4) \quad (\widehat{L}^\varepsilon)_s^{-1} \rightarrow \widehat{L}_s^{-1}, \quad \widehat{H}_s^\varepsilon \rightarrow \widehat{H}_s \quad \text{for all } s \geq 0 \text{ as } \varepsilon \rightarrow 0,$$

and that the family

$$(A.5) \quad f((\widehat{L}^\varepsilon)_s^{-1}, \widehat{H}_s^\varepsilon), \quad 0 < \varepsilon \leq 1,$$

is dominated by an integrable function with respect to  $P \times ds$ . Then

$$\begin{aligned} \int_{t \geq 0} \int_{z \geq 0} f(t, z) \widehat{V}^\varepsilon(dt, dz) &= \int_{t=0}^\infty E f((\widehat{L}^\varepsilon)_s^{-1}, \widehat{H}_s^\varepsilon) ds \\ &\rightarrow \int_{t=0}^\infty E f(\widehat{L}_s^{-1}, \widehat{H}_s) ds \\ &= \int_{t \geq 0} \int_{z \geq 0} f(t, z) V(dt, dz). \end{aligned}$$

For  $\varepsilon \geq 0$ , let  $A_\varepsilon = \{s : \widehat{X}_s^\varepsilon = \widehat{X}_s\}$ . Then for  $0 \leq \delta < \varepsilon$ ,  $A_\delta \subset A_\varepsilon$ . Further, for any  $T$ , if  $\varepsilon$  is sufficiently close to 0, then  $A_0 \cap [0, T] = A_\varepsilon \cap [0, T]$ . Thus, by Theorem 6.8 and Corollary 6.11 of [25],

$$\widehat{d}_\delta \widehat{L}_t^\delta = \int_0^t I_{A_\delta}(s) ds \leq \int_0^t I_{A_\varepsilon}(s) ds = \widehat{d}_\varepsilon \widehat{L}_t^\varepsilon, \quad \text{all } 0 \leq \delta < \varepsilon,$$

and

$$\widehat{d} \widehat{L}_t = \widehat{d}_\varepsilon \widehat{L}_t^\varepsilon, \quad 0 \leq t \leq T,$$

if  $\varepsilon$  is sufficiently close to 0. Hence, for all  $0 \leq \delta < \varepsilon$ ,

$$(A.6) \quad (\widehat{L}^\delta)_s^{-1} = \inf\{t : \widehat{L}_t^\delta > s\} \geq \inf\{t : (\widehat{d}_\varepsilon / \widehat{d}_\delta) \widehat{L}_t^\varepsilon > s\} = (\widehat{L}^\varepsilon)_{(\widehat{d}_\delta / \widehat{d}_\varepsilon)s}^{-1},$$

with equality if  $\delta = 0$  and  $\varepsilon$  is sufficiently close to 0.

Fix  $s \geq 0$  and assume  $\varepsilon$  is sufficiently close to 0 that equality holds in (A.6) with  $\delta = 0$ . Thus,

$$(A.7) \quad (\widehat{L}^\varepsilon)_s^{-1} = \widehat{L}_{(\widehat{d}_\varepsilon / \widehat{d})s}^{-1}.$$

Since  $(\widehat{X}^\varepsilon)_t = \widehat{X}_t + J_{\varepsilon,t}$  where

$$(A.8) \quad 0 \leq J_{\varepsilon,t} \leq \varepsilon t,$$

it then follows that

$$(A.9) \quad \widehat{H}_s^\varepsilon = (\widehat{X^\varepsilon})_{(\widehat{L}^\varepsilon)_s^{-1}} = \widehat{X}_{\widehat{L}_{(\widehat{d}_\varepsilon/\widehat{d})_s}^{-1}} + J_{\varepsilon, \widehat{L}_{(\widehat{d}_\varepsilon/\widehat{d})_s}^{-1}} = \widehat{H}_{(\widehat{d}_\varepsilon/\widehat{d})_s} + J_{\varepsilon, \widehat{L}_{(\widehat{d}_\varepsilon/\widehat{d})_s}^{-1}}.$$

Hence, using Lemma A.2, (A.4) follows from (A.7), (A.8) and (A.9).

Now let  $0 \leq \varepsilon \leq 1$ . Then by (A.6),

$$(\widehat{L}^\varepsilon)_s^{-1} \geq (\widehat{L}^1)_{(\widehat{d}_\varepsilon/\widehat{d}_1)_s}^{-1}.$$

Thus, by monotonicity of  $\widehat{d}_\varepsilon$ ,

$$I((\widehat{L}^\varepsilon)_s^{-1} \leq r) \leq I((\widehat{L}^1)_{(\widehat{d}_\varepsilon/\widehat{d}_1)_s}^{-1} \leq r) \leq I((\widehat{L}^1)_{(\widehat{d}_1/\widehat{d}_1)_s}^{-1} \leq r).$$

Hence, if  $f$  vanishes for  $t \geq r$ , then

$$f((\widehat{L}^\varepsilon)_s^{-1}, \widehat{H}_s^\varepsilon) \leq \|f\|_\infty I((\widehat{L}^1)_{(\widehat{d}_1/\widehat{d}_1)_s}^{-1} \leq r),$$

where

$$E \int_0^\infty I((\widehat{L}^1)_{(\widehat{d}_1/\widehat{d}_1)_s}^{-1} \leq r) ds = (\widehat{d}_1/\widehat{d}) E \int_0^\infty I((\widehat{L}^1)_s^{-1} \leq r) ds < \infty,$$

which proves (A.5).  $\square$

**PROOF OF PROPOSITION 4.2 WHEN  $X$  IS COMPOUND POISSON.** Assume  $X$  is compound Poisson. Since  $d_{L^{-1}} = 0$  whenever 0 is irregular for  $(0, \infty)$ , it follows that  $d_{(L^\varepsilon)^{-1}} = d_{L^{-1}} = 0$ . Further, (4.4) holds for  $X^\varepsilon$ . Hence, (4.4) for  $X$  follows from Propositions A.1 and A.2.  $\square$

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