ERGODIC CONTROL OF MULTI-CLASS $M/M/N + M$ QUEUES IN THE HALFIN–WHITT REGIME

BY ARI ARAPOSTATHIS[*,1], ANUP BISWAS[*,2] AND GUODONG PANG[†,3]

*The University of Texas at Austin** and Pennsylvania State University[†]

We study a dynamic scheduling problem for a multi-class queueing network with a large pool of statistically identical servers. The arrival processes are Poisson, and service times and patience times are assumed to be exponentially distributed and class dependent. The optimization criterion is the expected long time average (ergodic) of a general (nonlinear) running cost function of the queue lengths. We consider this control problem in the Halfin–Whitt (QED) regime, that is, the number of servers n and the total offered load $\mathbf{r}$ scale like $n \approx \mathbf{r} + \hat{\rho}\sqrt{\mathbf{r}}$ for some constant $\hat{\rho}$. This problem was proposed in [*Ann. Appl. Probab.* **14** (2004) 1084–1134, Section 5.2].

The optimal solution of this control problem can be approximated by that of the corresponding ergodic diffusion control problem in the limit. We introduce a broad class of ergodic control problems for controlled diffusions, which includes a large class of queueing models in the diffusion approximation, and establish a complete characterization of optimality via the study of the associated HJB equation. We also prove the asymptotic convergence of the values for the multi-class queueing control problem to the value of the associated ergodic diffusion control problem. The proof relies on an approximation method by *spatial truncation* for the ergodic control of diffusion processes, where the Markov policies follow a fixed priority policy outside a fixed compact set.

.

CONTENTS

1. Introduction.   One of the classical problems in queueing theory is to schedule the customers/jobs in a network in an optimal way. These problems are known as the scheduling problems which arise in a wide variety of applications, in particular, whenever there are different customer classes present in the network and competing for the same resources. The optimal scheduling problem has a long history in the literature. One of the appealing scheduling rules is the well-known $c\mu$ rule. This is a static priority policy in which it is assumed that each class-i customer has a marginal delay cost c_i and an average service time $1/\mu_i$, and the classes are prioritized in the decreasing order of $c_i\mu_i$. This static priority rule has proven asymptotically optimal in many settings [4, 28, 32]. In [11], a single-server Markov modulated queueing network is considered and an *averaged $c\mu$*-rule is shown asymptotically optimal for the discounted control problem.

An important aspect of queueing networks is abandonment/reneging, that is, customers/jobs may choose to leave the system while being in the queue before their service. Therefore, it is important to include customer abandonment in modeling queueing systems. In [5, 6], Atar et al. considered a multi-class $M/M/N + M$ queueing network with customer abandonment and proved that a modified priority policy, referred to as $c\mu/\theta$ rule, is asymptotically optimal for the long run average cost in the fluid scale. Dai and Tezcan [13] showed the asymptotic optimality of a static priority policy on a finite time interval for a parallel server model under the assumed conditions on the ordering of the abandonment rates and running costs. Although static priority policies are easy to implement, it may not be optimal for control problems of many multi-server queueing systems. For the same multi-class $M/M/N + M$ queueing network, discounted cost control problems are studied in [3, 7, 22], and asymptotically optimal controls for these problems are

constructed from the minimizer of a Hamilton–Jacobi–Bellman (HJB) equation associated with the controlled diffusions in the Halfin–Whitt regime.

In this article, we are interested in an ergodic control problem for a multi-class $M/M/N + M$ queueing network in the Halfin–Whitt regime. The network consists of a single pool of n statistically identical servers and a buffer of infinite capacity. There are d customer classes and arrivals of jobs/customers are d independent Poisson processes with parameters λ_i^n, $i = 1, \ldots, d$. The service rate for class-i customers is μ_i^n, $i = 1, \ldots, d$. Customers may renege from the queue if they have not started to receive service before their patience times. Class-i customers renege from the queue at rates $\gamma_i^n > 0$, $i = 1, \ldots, d$. The scheduling policies are *work-conserving*, that is, no server stays idle if any of the queues is nonempty. We assume the system operates in the Halfin–Whitt regime, where the arrival rates and the number of servers are scaled appropriately in a manner that the traffic intensity of the system satisfies

$$\sqrt{n}\left(1 - \sum_{i=1}^{d} \frac{\lambda_i^n}{n\mu_i^n}\right) \xrightarrow[n\to\infty]{} \hat{\rho} \in \mathbb{R}.$$

In this regime, the system operations achieve both high quality (high server levels) and high efficiency (high servers' utilization), and hence it is also referred to as the Quality-and-Efficiency-Driven (QED) regime; see, for example, [7, 16, 17, 19, 21] on the many-server regimes. We consider an ergodic cost function given by

$$\limsup_{T\to\infty} \frac{1}{T} \mathbb{E}\left[\int_0^T r\big(\hat{Q}^n(s)\big) \, ds\right],$$

where the *running cost r* is a nonnegative, convex function with polynomial growth and $\hat{Q}^n = (\hat{Q}_1^n, \ldots, \hat{Q}_d^n)^\top$ is the diffusion-scaled queue length process. It is worth mentioning that in addition to the running cost above which is based on the queue-length, we can add an idle-server cost provided that it has at most polynomial growth. For such, a running cost structure the same analysis goes through. The control is the allocation of servers to different classes of customers at the service completion times. The value function is defined to be the infimum of the above cost over all *admissible* controls (among all work-conserving scheduling policies). In this article, we are interested in the existence and uniqueness of asymptotically optimal stable stationary Markov controls for the ergodic control problem, and the asymptotic behavior of the value functions as n tends to infinity. In [7], Section 5.2, it is stated that analysis of this type of problems is important for modeling call centers.

1.1. *Contributions and comparisons.* The usual methodology for studying these problems is to consider the associated continuum model, which is the controlled diffusion limit in a heavy-traffic regime, and to study the ergodic control

problem for the controlled diffusion. Ergodic control problems governed by controlled diffusions have been well studied in literature [1, 9] for models that fall in these two categories: (a) the running cost is *near-monotone*, which is defined by the requirement that its value outside a compact set exceeds the optimal average cost, thus penalizing unstable behavior (see Assumption 3.4.2 in [1] for details), or (b) the controlled diffusion is uniformly stable, that is, every stationary Markov control is stable and the collection of invariant probability measures corresponding to the stationary Markov controls is tight. However, the ergodic control problem at hand does not fall under any of these frameworks. First, the running cost we consider here is not near-monotone because the total queue length can be 0 when the total number of customers in the system are $\mathcal{O}(n)$. On the other hand, it is not at all clear that the controlled diffusion is uniformly stable (unless one imposes nontrivial hypotheses on the parameters), and this remains an open problem. One of our main contributions in this article is that we solve the ergodic control problem for a broad class of nondegenerate controlled diffusions, that in a certain way can be viewed as a mixture of the two categories mentioned above. As we show in Section 3, stability of the diffusion under any optimal stationary Markov control occurs due to certain interplay between the drift and the running cost. The model studied in Section 3 is far more general than the queueing problem described, and thus it is of separate interest for ergodic control. We present a comprehensive study of this broad class of ergodic control problems that includes existence of a solution to the ergodic HJB equation, its stochastic representation and verification of optimality (Theorem 3.4), uniqueness of the solution in a certain class (Theorem 3.5), and convergence of the vanishing discount method (Theorem 3.6). These results extend the well-known results for near-monotone running costs. The assumptions in these theorems are verified for the multi-class queueing model and the corresponding characterization of optimality is obtained (Corollary 3.1), which includes growth estimates for the solution of the HJB.

We also introduce a new approximation technique, *spatial truncation*, for the controlled diffusion processes; see Section 4. It is shown that if we freeze the Markov controls to a fixed stable Markov control outside a compact set, then we can still obtain nearly optimal controls in this class of Markov controls for large compact sets. We should keep in mind that this property is not true in general. This method can also be thought of as an approximation by a class of controlled diffusions that are uniformly stable.

We remark that for a fixed control, the controlled diffusions for the queueing model can be regarded as a special case of the piecewise linear diffusions considered in [14]. It is shown in [14] that these diffusions are stable under *constant* Markov controls. The proof is via a suitable Lyapunov function. We conjecture that uniform stability holds for the controlled diffusions associated with the queueing model. For the same multi-class Markovian model, Gamarnik and Stolyar show that the stationary distributions of the queue lengths are tight under any work-conserving policy [15], Theorem 2. We also wish to remark here that we allow $\hat{\rho}$

to be negative, assuming abandonment rates are strictly positive, while in [15], $\hat{\rho} > 0$ and abandonment rates can be zero.

Another important contribution of this work is the convergence of the value functions associated with the sequence of multi-class queueing models to the value of the ergodic control problem, say ϱ_*, corresponding to the controlled diffusion model. It is not obvious that one can have asymptotic optimality from the existence of optimal stable controls for the HJB equations of controlled diffusions. This fact is relatively straightforward when the cost under consideration is discounted. In that situation, the tightness of paths on a finite time horizon is sufficient to prove asymptotic optimality [7]. But we are in a situation where any finite time behavior of the stochastic process plays no role in the cost. In particular, we need to establish the convergence of the controlled steady states. Although uniform stability of stationary distributions for this multi-class queueing model in the case where $\hat{\rho} > 0$ and abandonment rates can be zero is established in [15], it is not obvious that the stochastic model considered here has the property of uniform stability. Therefore, we use a different method to establish the asymptotic optimality. First, we show that the value functions are asymptotically bounded below by ϱ_*. To study the upper bound, we construct a sequence of Markov scheduling policies that are uniformly stable (see Lemma 5.1). The key idea used in establishing such stability results is a spatial truncation technique, under which the Markov policies follow a fixed priority policy outside a given compact set. We believe these techniques can also be used to study ergodic control problems for other many-server queueing models.

The scheduling policies we consider in this paper allow preemption, that is, a customer in service can be interrupted for the server to serve a customer of a different class and her service will be resumed later. In fact, the asymptotic optimality is shown within the class of the work-conserving preemptive policies. In [7], both preemptive and nonpreemptive policies are studied, where a nonpreemptive scheduling control policy is constructed from the HJB equation associated with preemptive policies and thus is shown to be asymptotically optimal. However, as far as we know, the optimal nonpreemptive scheduling problem under the ergodic cost remains open.

For a similar line of work in uncontrolled settings, we refer the reader to [16, 19]. Admission control of the single class $M/M/N + M$ model with an ergodic cost criterion in the Halfin–Whitt regime is studied in [26]. For controlled problems and for finite server models, asymptotic optimality is obtained in [12] in the conventional heavy-traffic regime. The main advantage in [12] is the uniform exponential stability of the stochastic processes, which is obtained by using properties of the Skorohod reflection map. A recent work studying ergodic control of a multi-class single-server queueing network is [25].

To summarize our main contributions in this paper:

– We introduce a new class of ergodic control problems and a framework to solve them.

– We establish an approximation technique by spatial truncation.
– We provide, to the best of our knowledge, the first treatment of ergodic control problems at the diffusion scale for many server models.
– We establish asymptotic optimality results.

1.2. *Organization.* In Section 1.3, we summarize the notation used in the paper. In Section 2, we introduce the multi-class many server queueing model and describe the Halfin–Whitt regime. The ergodic control problem under the heavy-traffic setting is introduced in Section 2.2, and the main results on asymptotic convergence are stated as Theorems 2.1 and 2.2. Section 3 introduces a class of controlled diffusions and associated ergodic control problems, which contains the queueing models in the diffusion scale. The key structural assumptions are in Section 3.2 and these are verified for a generic class of queueing models in Section 3.3, which are characterized by piecewise linear controlled diffusions. Section 3.4 concerns the existence of optimal controls under the general hypotheses, while Section 3.5 contains a comprehensive study of the HJB equation. Section 3.6 is devoted to the proofs of the results in Section 3.5. The spatial truncation technique is introduced and studied in Section 4. Finally, in Section 5 we prove the results of asymptotic optimality.

1.3. *Notation.* The standard Euclidean norm in $\mathbb{R}^d$ is denoted by $|\cdot|$. The set of nonnegative real numbers is denoted by $\mathbb{R}_+$, $\mathbb{N}$ stands for the set of natural numbers, and $\mathbb{I}$ denotes the indicator function. By $\mathbb{Z}_+^d$ we denote the set of d-vectors of nonnegative integers. The closure, the boundary and the complement of a set $A \subset \mathbb{R}^d$ are denoted by $\overline{A}$, ∂A and A^c, respectively. The open ball of radius R around 0 is denoted by B_R. Given two real numbers a and b, the minimum (maximum) is denoted by $a \wedge b$ ($a \vee b$), respectively. Define $a^+ := a \vee 0$ and $a^- := -(a \wedge 0)$. The integer part of a real number a is denoted by $\lfloor a \rfloor$. We use the notation $e_i, i = 1, \ldots, d$, to denote the vector with ith entry equal to 1 and all other entries equal to 0. We also let $e := (1, \ldots, 1)^\mathsf{T}$. Given any two vectors $x, y \in \mathbb{R}^d$ the inner product is denoted by $x \cdot y$. By δ_x we denote the Dirac mass at x. For any function $f : \mathbb{R}^d \to \mathbb{R}$ and domain $D \subset \mathbb{R}$ we define the oscillation of f on D as follows:

$$\operatorname*{osc}_{D}(f) := \sup\{f(x) - f(y) : x, y \in D\}.$$

For a nonnegative function $g \in \mathcal{C}(\mathbb{R}^d)$, we let $\mathcal{O}(g)$ denote the space of functions $f \in \mathcal{C}(\mathbb{R}^d)$ satisfying $\sup_{x \in \mathbb{R}^d} \frac{|f(x)|}{1+g(x)} < \infty$. This is a Banach space under the norm

$$\|f\|_g := \sup_{x \in \mathbb{R}^d} \frac{|f(x)|}{1 + g(x)}.$$

We also let $\mathfrak{o}(g)$ denote the subspace of $\mathcal{O}(g)$ consisting of those functions f satisfying

$$\limsup_{|x|\to\infty} \frac{|f(x)|}{1+g(x)} = 0.$$

By a slight abuse of notation, we also denote by $\mathcal{O}(g)$ and $\mathfrak{o}(g)$ a generic member of these spaces. For two nonnegative functions f and g, we use the notation $f \sim g$ to indicate that $f \in \mathcal{O}(g)$ and $g \in \mathcal{O}(f)$.

We denote by $L^p_{\mathrm{loc}}(\mathbb{R}^d)$, $p \geq 1$, the set of real-valued functions that are locally p-integrable and by $\mathcal{W}^{k,p}_{\mathrm{loc}}(\mathbb{R}^d)$ the set of functions in $L^p_{\mathrm{loc}}(\mathbb{R}^d)$ whose ith weak derivatives, $i = 1, \ldots, k$, are in $L^p_{\mathrm{loc}}(\mathbb{R}^d)$. The set of all bounded continuous functions is denoted by $\mathcal{C}_b(\mathbb{R}^d)$. By $\mathcal{C}^{k,\alpha}_{\mathrm{loc}}(\mathbb{R}^d)$ we denote the set of functions that are k-times continuously differentiable and whose kth derivatives are locally Hölder continuous with exponent α. We define $\mathcal{C}^k_b(\mathbb{R}^d)$, $k \geq 0$, as the set of functions whose ith derivatives, $i = 1, \ldots, k$, are continuous and bounded in $\mathbb{R}^d$ and denote by $\mathcal{C}^k_c(\mathbb{R}^d)$ the subset of $\mathcal{C}^k_b(\mathbb{R}^d)$ with compact support. For any path $X(\cdot)$, we use the notation $\Delta X(t)$ to denote the jump at time t. Given any Polish space $\mathcal{X}$, we denote by $\mathcal{P}(\mathcal{X})$ the set of probability measures on $\mathcal{X}$ and we endow $\mathcal{P}(\mathcal{X})$ with the Prokhorov metric. For $\nu \in \mathcal{P}(\mathcal{X})$ and a Borel measurable map $f : \mathcal{X} \to \mathbb{R}$, we often use the abbreviated notation

$$\nu(f) := \int_{\mathcal{X}} f \, d\nu.$$

The quadratic variation of a square integrable martingale is denoted by $\langle \cdot, \cdot \rangle$ and the optional quadratic variation by $[\cdot, \cdot]$. For presentation purposes we use the time variable as the subscript for the diffusion processes. Also $\kappa_1, \kappa_2, \ldots$ and $C_1, C_2, \ldots$ are used as generic constants whose values might vary from place to place.

2. The controlled system in the Halfin–Whitt regime.

2.1. *The multi-class Markovian many-server model.* Let $(\Omega, \mathfrak{F}, \mathbb{P})$ be a given complete probability space and all the stochastic variables introduced below are defined on it. The expectation w.r.t. $\mathbb{P}$ is denoted by $\mathbb{E}$. We consider a multi-class Markovian many-server queueing system which consists of d customer classes and n parallel servers capable of serving all customers (see Figure 1).

The system buffer is assumed to have infinite capacity. Customers of class $i \in \{1, \ldots, d\}$ arrive according to a Poisson process with rate $\lambda^n_i > 0$. Customers enter the queue of their respective classes upon arrival if not being processed. Customers of each class are served in the first-come-first-serve (FCFS) service discipline. While waiting in queue, customers can abandon the system. The service times and patience times of customers are class-dependent and both are assumed to be exponentially distributed, that is, class i customers are served at rate μ^n_i and renege at rate γ^n_i. We assume that customer arrivals, service and abandonment of all classes are mutually independent.
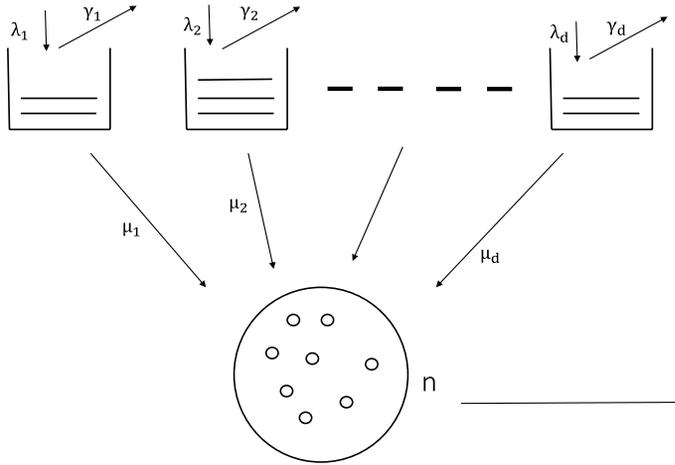
FIG. 1. *A schematic model of the system.*

The Halfin–Whitt regime. We study this queueing model in the Halfin–Whitt regime [or the Quality-and-Efficiency-Driven (QED) regime]. Consider a sequence of such systems indexed by n, in which the arrival rates λ_i^n and the number of servers n both increase appropriately. Let $\mathbf{r}_i^n := \lambda_i^n / \mu_i^n$ be the mean offered load of class i customers. The traffic intensity of the nth system is given by $\rho^n = n^{-1} \sum_{i=1}^d \mathbf{r}_i^n$. In the Halfin–Whitt regime, the parameters are assumed to satisfy the following: as $n \to \infty$,

$$\frac{\lambda_i^n}{n} \to \lambda_i > 0, \qquad \mu_i^n \to \mu_i > 0, \qquad \gamma_i^n \to \gamma_i > 0,$$

(2.1) $$\frac{\lambda_i^n - n\lambda_i}{\sqrt{n}} \to \hat{\lambda}_i, \qquad \sqrt{n}(\mu_i^n - \mu_i) \to \hat{\mu}_i,$$

$$\frac{\mathbf{r}_i^n}{n} \to \rho_i := \frac{\lambda_i}{\mu_i} < 1, \qquad \sum_{i=1}^d \rho_i = 1.$$

This implies that

$$\sqrt{n}(1 - \rho^n) \to \hat{\rho} := \sum_{i=1}^d \frac{\rho_i \hat{\mu}_i - \hat{\lambda}_i}{\mu_i} \in \mathbb{R}.$$

The above scaling is common in multi-class multi-server models [7, 22]. Note that we do not make any assumption on the sign of $\hat{\rho}$.

State descriptors. Let $X_i^n = \{X_i^n(t) : t \geq 0\}$ be the total number of class i customers in the system, $Q_i^n = \{Q_i^n(t) : t \geq 0\}$ the number of class i customers in the queue and $Z_i^n = \{Z_i^n(t) : t \geq 0\}$ the number of class i customers in service.

The following basic relationships hold for these processes: for each $t \geq 0$ and $i = 1, \ldots, d$,

(2.2)
$$X_i^n(t) = Q_i^n(t) + Z_i^n(t),$$

$$Q_i^n(t) \geq 0, \qquad Z_i^n(t) \geq 0 \quad \text{and} \quad e \cdot Z^n(t) \leq n.$$

We can describe these processes using a collection $\{A_i^n, S_i^n, R_i^n, i = 1, \ldots, d\}$ of independent rate-1 Poisson processes. Define

$$\tilde{A}_i^n(t) := A_i^n(\lambda_i^n t),$$

$$\tilde{S}_i^n(t) := S_i^n\left(\mu_i^n \int_0^t Z_i^n(s)\, ds\right),$$

$$\tilde{R}_i^n(t) := R_i^n\left(\gamma_i^n \int_0^t Q_i^n(s)\, ds\right).$$

Then the dynamics take the form

(2.3) $\quad X_i^n(t) = X_i^n(0) + \tilde{A}_i^n(t) - \tilde{S}_i^n(t) - \tilde{R}_i^n(t), \qquad t \geq 0, i = 1, \ldots, d.$

Scheduling control. Following [7, 22], we only consider work-conserving policies that are nonanticipative and allow preemption. When a server becomes free and there are no customers waiting in any queue, the server stays idle, but if there are customers of multiple classes waiting in the queue, the server has to make a decision on the customer class to serve. Service preemption is allowed, that is, service of a customer class can be interrupted at any time to serve some other class of customers and the original service is resumed at a later time. A scheduling control policy determines the processes Z^n, which must satisfy the constraints in (2.2) and the work-conserving constraint, that is,

$$e \cdot Z^n(t) = \left(e \cdot X^n(t)\right) \wedge n, \qquad t \geq 0.$$

Define the action set $\mathbb{A}^n(x)$ as

$$\mathbb{A}^n(x) := \left\{a \in \mathbb{Z}_+^d : a \leq x \text{ and } e \cdot a = (e \cdot x) \wedge n\right\}.$$

Thus, we can write $Z^n(t) \in \mathbb{A}^n(X^n(t))$ for each $t \geq 0$. We also assume that all controls are nonanticipative. Define the σ-fields

$$\mathcal{F}_t^n := \sigma\left\{X^n(0), \tilde{A}_i^n(t), \tilde{S}_i^n(t), \tilde{R}_i^n(t) : i = 1, \ldots, d, 0 \leq s \leq t\right\} \vee \mathcal{N}$$

and

$$\mathcal{G}_t^n := \sigma\left\{\delta\tilde{A}_i^n(t, r), \delta\tilde{S}_i^n(t, r), \delta\tilde{R}_i^n(t, r) : i = 1, \ldots, d, r \geq 0\right\},$$

where

$$\delta\tilde{A}_i^n(t, r) := \tilde{A}_i^n(t + r) - \tilde{A}_i^n(t),$$

$$\delta\tilde{S}_i^n(t, r) := S_i^n\left(\mu_i^n \int_0^t Z_i^n(s)\, ds + \mu_i^n r\right) - \tilde{S}_i^n(t),$$

$$\delta\tilde{R}_i^n(t, r) := R_i^n\left(\gamma_i^n \int_0^t Q_i^n(s)\, ds + \gamma_i^n r\right) - \tilde{R}_i^n(t),$$

and $\mathcal{N}$ is the collection of all $\mathbb{P}$-null sets. The filtration $\{\mathcal{F}_t^n, t \geq 0\}$ represents the information available up to time t while $\mathcal{G}_t^n$ contains the information about future increments of the processes.

We say that a working-conserving control policy is *admissible* if:

(i) $Z^n(t)$ is adapted to $\mathcal{F}_t^n$,

(ii) $\mathcal{F}_t^n$ is independent of $\mathcal{G}_t^n$ at each time $t \geq 0$,

(iii) for each $i = 1, \ldots, d$, and $t \geq 0$, the process $\delta \tilde{S}_i^n(t, \cdot)$ agrees in law with $S_i^n(\mu_i^n \cdot)$, and the process $\delta \tilde{R}_i^n(t, \cdot)$ agrees in law with $R_i^n(\gamma_i^n \cdot)$.

We denote the set of all admissible control policies $(Z^n, \mathcal{F}^n, \mathcal{G}^n)$ by $\mathfrak{U}^n$.

2.2. *The ergodic control problem in the Halfin–Whitt regime.* Define the diffusion-scaled processes

$$\hat{X}^n = (\hat{X}_1^n, \ldots, \hat{X}_d^n)^\mathsf{T}, \qquad \hat{Q}^n = (\hat{Q}_1^n, \ldots, \hat{Q}_d^n)^\mathsf{T} \quad \text{and} \quad \hat{Z}^n = (\hat{Z}_1^n, \ldots, \hat{Z}_d^n)^\mathsf{T},$$

by

$$
\begin{aligned}
\hat{X}_i^n(t) &:= \frac{1}{\sqrt{n}}(X_i^n(t) - \rho_i n t), \\
\hat{Q}_i^n(t) &:= \frac{1}{\sqrt{n}} Q_i^n(t), \\
\hat{Z}_i^n(t) &:= \frac{1}{\sqrt{n}}(Z_i^n(t) - \rho_i n t)
\end{aligned}
$$
(2.4)

for $t \geq 0$. By (2.3), we can express $\hat{X}_i^n$ as

$$
\begin{aligned}
\hat{X}_i^n(t) = \hat{X}_i^n(0) + \ell_i^n t &- \mu_i^n \int_0^t \hat{Z}_i^n(s)\, \mathrm{d}s - \gamma_i^n \int_0^t \hat{Q}_i^n(s)\, \mathrm{d}s \\
&+ \hat{M}_{A,i}^n(t) - \hat{M}_{S,i}^n(t) - \hat{M}_{R,i}^n(t),
\end{aligned}
$$
(2.5)

where $\ell^n = (\ell_1^n, \ldots, \ell_d^n)^\mathsf{T}$ is defined as

$$\ell_i^n := \frac{1}{\sqrt{n}}(\lambda_i^n - \mu_i^n \rho_i n),$$

and

$$
\begin{aligned}
\hat{M}_{A,i}^n(t) &:= \frac{1}{\sqrt{n}}\big(A_i^n(\lambda_i^n t) - \lambda_i^n t\big), \\
\hat{M}_{S,i}^n(t) &:= \frac{1}{\sqrt{n}}\bigg(S_i^n\bigg(\mu_i^n \int_0^t Z_i^n(s)\, \mathrm{d}s\bigg) - \mu_i^n \int_0^t Z_i^n(s)\, \mathrm{d}s\bigg), \\
\hat{M}_{R,i}^n(t) &:= \frac{1}{\sqrt{n}}\bigg(R_i^n\bigg(\gamma_i^n \int_0^t Q_i^n(s)\, \mathrm{d}s\bigg) - \gamma_i^n \int_0^t Q_i^n(s)\, \mathrm{d}s\bigg)
\end{aligned}
$$
(2.6)

are square integrable martingales w.r.t. the filtration $\{\mathcal{F}_t^n\}$.

Note that

$$\ell_i^n = \frac{1}{\sqrt{n}}(\lambda_i^n - \lambda_i n) - \rho_i \sqrt{n}(\mu_i^n - \mu_i) \xrightarrow[n\to\infty]{} \ell_i := \frac{(\hat{\lambda}_i - \rho_i \hat{\mu}_i)}{\mu_i}.$$

Define

$$\mathcal{S} := \{u \in \mathbb{R}_+^d : e \cdot u = 1\}.$$

For $Z^n \in \mathfrak{U}^n$ we define, for $t \geq 0$ and for adapted $\hat{U}^n(t) \in \mathcal{S}$,

(2.7)
$$\hat{Q}^n(t) := (e \cdot \hat{X}^n(t))^+ \hat{U}^n(t),$$
$$\hat{Z}^n(t) := \hat{X}^n(t) - (e \cdot \hat{X}^n(t))^+ \hat{U}^n(t).$$

If $\hat{Q}^n(t) = 0$, we define $\hat{U}^n(t) := e_d = (0, \ldots, 0, 1)^\mathsf{T}$. Thus, $\hat{U}_i^n$ represents the fraction of class-i customers in the queue when the total queue size is positive. As we show later, it is convenient to view $\hat{U}^n(t)$ as the control. Note that the controls are nonanticipative and preemption is allowed.

2.2.1. *The cost minimization problem.* We next introduce the running cost function for the control problem. Let $r : \mathbb{R}_+^d \to \mathbb{R}_+$ be a given function satisfying

(2.8) $$c_1 |x|^m \leq r(x) \leq c_2(1 + |x|^m) \qquad \text{for some } m \geq 1,$$

and some positive constants c_i, $i = 1, 2$. We also assume that r is locally Lipschitz. This assumption includes linear and convex running cost functions. For example, if we let h_i be the holding cost rate for class i customers, then some of the typical running cost functions are the following:

$$r(x) = \sum_{i=1}^d h_i x_i^m, \qquad m \geq 1.$$

These running cost functions evidently satisfy the condition in (2.8).

Given the initial state $X^n(0)$ and a work-conserving scheduling policy $Z^n \in \mathfrak{U}^n$, we define the diffusion-scaled cost function as

(2.9) $$J(\hat{X}^n(0), \hat{Z}^n) := \limsup_{T\to\infty} \frac{1}{T} \mathbb{E}\left[\int_0^T r(\hat{Q}^n(s)) \, ds \right],$$

where the running cost function r satisfies (2.8). Note that the running cost is defined using the scaled version of Z^n. Then the associated cost minimization problem becomes

(2.10) $$\hat{V}^n(\hat{X}^n(0)) := \inf_{Z^n \in \mathfrak{U}^n} J(\hat{X}^n(0), \hat{Z}^n).$$

We refer to $\hat{V}^n(\hat{X}^n(0))$ as the *diffusion-scaled value function* given the initial state $\hat{X}^n(0)$ in the nth system.

From (2.7), it is easy to see that by redefining r as $r(x, u) = r((e \cdot x)^+ u)$ we can rewrite the control problem as

$$\hat{V}^n(\hat{X}^n(0)) = \inf \tilde{J}(\hat{X}^n(0), \hat{U}^n),$$

where

(2.11) $$\tilde{J}(\hat{X}^n(0), \hat{U}^n) := \limsup_{T \to \infty} \frac{1}{T} \mathbb{E}\left[\int_0^T r(\hat{X}^n(s), \hat{U}^n(s)) \, \mathrm{d}s\right],$$

and the infimum is taken over all admissible pairs $(\hat{X}^n, \hat{U}^n)$ satisfying (2.7).

For simplicity, we assume that the initial condition $\hat{X}^n(0)$ is deterministic and $\hat{X}^n(0) \to x$ as $n \to \infty$ for some $x \in \mathbb{R}^d$.

2.2.2. *The limiting controlled diffusion process.* As in [7, 22], one formally deduces that, provided $\hat{X}^n(0) \to x$, there exists a limit X for $\hat{X}^n$ on every finite time interval, and the limit process X is a d-dimensional diffusion process with independent components, that is,

(2.12) $$\mathrm{d}X_t = b(X_t, U_t) \, \mathrm{d}t + \Sigma \, \mathrm{d}W_t,$$

with initial condition $X_0 = x$. In (2.12), the drift $b(x, u) : \mathbb{R}^d \times \mathcal{S} \to \mathbb{R}^d$ takes the form

(2.13) $$b(x, u) = \ell - R(x - (e \cdot x)^+ u) - (e \cdot x)^+ \Gamma u,$$

with

$$\ell := (\ell_1, \ldots, \ell_d)^{\mathsf{T}},$$
$$R := \mathrm{diag}(\mu_1, \ldots, \mu_d),$$
$$\Gamma := \mathrm{diag}(\gamma_1, \ldots, \gamma_d).$$

The control U_t lives in $\mathcal{S}$ and is nonanticipative, $W(t)$ is a d-dimensional standard Wiener process independent of the initial condition $X_0 = x$, and the covariance matrix is given by

$$\Sigma \Sigma^{\mathsf{T}} = \mathrm{diag}(2\lambda_1, \ldots, 2\lambda_d).$$

A formal derivation of the drift in (2.13) can be obtained from (2.5) and (2.7). A detailed description of equation (2.12) and related results are given in Section 3. Let $\mathfrak{U}$ be the set of all admissible controls for the diffusion model (for a definition see Section 3).

2.2.3. *The ergodic control problem in the diffusion scale.* Define $\tilde{r} : \mathbb{R}^d_+ \times \mathbb{R}^d_+ \to \mathbb{R}_+$ by

$$\tilde{r}(x, u) := r\big((e \cdot x)^+ u\big),$$

where r is the same function as in (2.9). In analogy with (2.11) we define the ergodic cost associated with the controlled diffusion process X and the running cost function $\tilde{r}(x, u)$ as

$$J(x, U) := \limsup_{T \to \infty} \frac{1}{T} \mathbb{E}^U_x \left[\int_0^T \tilde{r}(X_t, U_t) \, \mathrm{d}t \right], \qquad U \in \mathfrak{U}.$$

We consider the ergodic control problem

$$(2.14) \qquad \qquad \varrho_*(x) = \inf_{U \in \mathfrak{U}} J(x, U).$$

We call $\varrho_*(x)$ the optimal value at the initial state x for the controlled diffusion process X. It is shown later that $\varrho_*(x)$ is independent of x. A detailed treatment and related results corresponding to the ergodic control problem are given in Section 3.

We next state the main results of this section, the proof of which can be found in Section 5.

THEOREM 2.1. *Let $\hat{X}^n(0) \to x \in \mathbb{R}^d$ as $n \to \infty$. Also assume that (2.1) and (2.8) hold. Then*

$$\liminf_{n \to \infty} \hat{V}^n\big(\hat{X}^n(0)\big) \geq \varrho_*(x),$$

where $\varrho_*(x)$ is given by (2.14).

THEOREM 2.2. *Suppose the assumptions of Theorem 2.1 hold. In addition, assume that r in (2.9) is convex. Then*

$$\limsup_{n \to \infty} \hat{V}^n\big(\hat{X}^n(0)\big) \leq \varrho_*(x).$$

Thus, we conclude that for any convex running cost function r, Theorems 2.1 and 2.2 establish the asymptotic convergence of the ergodic control problem for the queueing model.

3. A broad class of ergodic control problems for diffusions.

3.1. *The controlled diffusion model.* The dynamics are modeled by a controlled diffusion process $X = \{X_t, t \geq 0\}$ taking values in the d-dimensional Euclidean space $\mathbb{R}^d$, and governed by the Itô stochastic differential equation

$$(3.1) \qquad \qquad \mathrm{d}X_t = b(X_t, U_t) \, \mathrm{d}t + \sigma(X_t) \, \mathrm{d}W_t.$$

All random processes in (3.1) live in a complete probability space $(\Omega, \mathfrak{F}, \mathbb{P})$. The process W is a d-dimensional standard Wiener process independent of the initial condition X_0. The control process U takes values in a compact, metrizable set $\mathbb{U}$, and $U_t(\omega)$ is jointly measurable in $(t, \omega) \in [0, \infty) \times \Omega$. Moreover, it is *nonanticipative*: for $s < t$, $W_t - W_s$ is independent of

$$\mathfrak{F}_s := \text{the completion of } \sigma\{X_0, U_r, W_r, r \le s\} \text{ relative to } (\mathfrak{F}, \mathbb{P}).$$

Such a process U is called an *admissible control*, and we let $\mathfrak{U}$ denote the set of all admissible controls.

We impose the following standard assumptions on the drift b and the diffusion matrix σ to guarantee existence and uniqueness of solutions to equation (3.1).

(A1) *Local Lipschitz continuity*: The functions

$$b = [b^1, \ldots, b^d]^{\mathsf{T}} : \mathbb{R}^d \times \mathbb{U} \to \mathbb{R}^d \quad \text{and} \quad \sigma = [\sigma^{ij}] : \mathbb{R}^d \to \mathbb{R}^{d \times d}$$

are locally Lipschitz in x with a Lipschitz constant $C_R > 0$ depending on $R > 0$. In other words, for all $x, y \in B_R$ and $u \in \mathbb{U}$,

$$\big|b(x, u) - b(y, u)\big| + \big\|\sigma(x) - \sigma(y)\big\| \le C_R |x - y|.$$

We also assume that b is continuous in (x, u).

(A2) *Affine growth condition*: b and σ satisfy a global growth condition of the form

$$\big|b(x, u)\big|^2 + \big\|\sigma(x)\big\|^2 \le C_1 \big(1 + |x|^2\big) \qquad \forall (x, u) \in \mathbb{R}^d \times \mathbb{U},$$

where $\|\sigma\|^2 := \text{trace}(\sigma \sigma^{\mathsf{T}})$.

(A3) *Local nondegeneracy*: For each $R > 0$, it holds that

$$\sum_{i, j=1}^{d} a^{ij}(x) \xi_i \xi_j \ge C_R^{-1} |\xi|^2 \qquad \forall x \in B_R,$$

for all $\xi = (\xi_1, \ldots, \xi_d)^{\mathsf{T}} \in \mathbb{R}^d$, where $a := \sigma \sigma^{\mathsf{T}}$.

In integral form, (3.1) is written as

$$(3.2) \qquad X_t = X_0 + \int_0^t b(X_s, U_s) \, ds + \int_0^t \sigma(X_s) \, dW_s.$$

The third term on the right-hand side of (3.2) is an Itô stochastic integral. We say that a process $X = \{X_t(\omega)\}$ is a solution of (3.1), if it is $\mathfrak{F}_t$-adapted, continuous in t, defined for all $\omega \in \Omega$ and $t \in [0, \infty)$, and satisfies (3.2) for all $t \in [0, \infty)$ a.s. It is well known that under (A1)–(A3), for any admissible control there exists a unique solution of (3.1) [1], Theorem 2.2.4.

We define the family of operators $L^u : \mathcal{C}^2(\mathbb{R}^d) \to \mathcal{C}(\mathbb{R}^d)$, where $u \in \mathbb{U}$ plays the role of a parameter, by

$$(3.3) \qquad L^u f(x) := \tfrac{1}{2} a^{ij}(x) \partial_{ij} f(x) + b^i(x, u) \partial_i f(x), \qquad u \in \mathbb{U}.$$

We refer to L^u as the *controlled extended generator* of the diffusion. In (3.3) and elsewhere in this paper, we have adopted the notation $\partial_i := \frac{\partial}{\partial x_i}$ and $\partial_{ij} := \frac{\partial^2}{\partial x_i \, \partial x_j}$. We also use the standard summation rule that repeated subscripts and superscripts are summed from 1 through d. In other words, the right-hand side of (3.3) stands for

$$\frac{1}{2} \sum_{i,j=1}^d a^{ij}(x) \frac{\partial^2 f}{\partial x_i \, \partial x_j}(x) + \sum_{i=1}^d b^i(x, u) \frac{\partial f}{\partial x_i}(x).$$

Of fundamental importance in the study of functionals of X is Itô's formula. For $f \in \mathcal{C}^2(\mathbb{R}^d)$ and with L^u as defined in (3.3), it holds that

$$(3.4) \qquad f(X_t) = f(X_0) + \int_0^t L^{U_s} f(X_s) \, \mathrm{d}s + M_t, \qquad \text{a.s.,}$$

where

$$M_t := \int_0^t \langle \nabla f(X_s), \sigma(X_s) \, \mathrm{d}W_s \rangle$$

is a local martingale. Krylov's extension of Itô's formula [27], page 122, extends (3.4) to functions f in the local Sobolev space $\mathcal{W}^{2,p}_{\text{loc}}(\mathbb{R}^d)$, $p \geq d$.

Recall that a control is called *Markov* if $U_t = v(t, X_t)$ for a measurable map $v : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{U}$, and it is called *stationary Markov* if v does not depend on t, that is, $v : \mathbb{R}^d \to \mathbb{U}$. Correspondingly, (3.1) is said to have a *strong solution* if given a Wiener process $(W_t, \mathfrak{F}_t)$ on a complete probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, there exists a process X on $(\Omega, \mathfrak{F}, \mathbb{P})$, with $X_0 = x_0 \in \mathbb{R}^d$, which is continuous, $\mathfrak{F}_t$-adapted, and satisfies (3.2) for all t a.s. A strong solution is called *unique*, if any two such solutions X and X' agree $\mathbb{P}$-a.s., when viewed as elements of $\mathcal{C}([0, \infty), \mathbb{R}^d)$. It is well known that under assumptions (A1)–(A3), for any Markov control v, (3.1) has a unique strong solution [20].

Let $\mathfrak{U}_{\text{SM}}$ denote the set of stationary Markov controls. Under $v \in \mathfrak{U}_{\text{SM}}$, the process X is strong Markov, and we denote its transition function by $P^t_v(x, \cdot)$. It also follows from the work of [8, 31] that under $v \in \mathfrak{U}_{\text{SM}}$, the transition probabilities of X have densities which are locally Hölder continuous. Thus, L^v defined by

$$L^v f(x) := \tfrac{1}{2} a^{ij}(x) \partial_{ij} f(x) + b^i(x, v(x)) \partial_i f(x), \qquad v \in \mathfrak{U}_{\text{SM}},$$

for $f \in \mathcal{C}^2(\mathbb{R}^d)$, is the generator of a strongly-continuous semi-group on $\mathcal{C}_b(\mathbb{R}^d)$, which is strong Feller. We let $\mathbb{P}^v_x$ denote the probability measure and $\mathbb{E}^v_x$ the expectation operator on the canonical space of the process under the control $v \in \mathfrak{U}_{\text{SM}}$, conditioned on the process X starting from $x \in \mathbb{R}^d$ at $t = 0$.

We need the following definition.

DEFINITION 3.1. A function $h : \mathbb{R}^d \times \mathbb{U} \to \mathbb{R}$ is called *inf-compact* on a set $A \subset \mathbb{R}^d$ if the set $\bar{A} \cap \{x : \min_{u \in \mathbb{U}} h(x, u) \leq \beta\}$ is compact (or empty) in $\mathbb{R}^d$ for all $\beta \in \mathbb{R}$. When this property holds for $A \equiv \mathbb{R}^d$, then we simply say that h is inf-compact.

Recall that control $v \in \mathfrak{U}_{\text{SM}}$ is called *stable* if the associated diffusion is positive recurrent. We denote the set of such controls by $\mathfrak{U}_{\text{SSM}}$, and let μ_v denote the unique invariant probability measure on $\mathbb{R}^d$ for the diffusion under the control $v \in \mathfrak{U}_{\text{SSM}}$. We also let $\mathcal{M} := \{\mu_v : v \in \mathfrak{U}_{\text{SSM}}\}$. Recall that $v \in \mathfrak{U}_{\text{SSM}}$ if and only if there exists an inf-compact function $\mathcal{V} \in \mathcal{C}^2(\mathbb{R}^d)$, a bounded domain $D \subset \mathbb{R}^d$, and a constant $\varepsilon > 0$ satisfying

$$L^v \mathcal{V}(x) \leq -\varepsilon \qquad \forall x \in D^c.$$

We denote by $\tau(A)$ the *first exit time* of a process $\{X_t, t \in \mathbb{R}_+\}$ from a set $A \subset \mathbb{R}^d$, defined by

$$\tau(A) := \inf\{t > 0 : X_t \notin A\}.$$

The open ball of radius R in $\mathbb{R}^d$, centered at the origin, is denoted by B_R, and we let $\tau_R := \tau(B_R)$, and $\breve{\tau}_R := \tau(B_R^c)$.

We assume that the running cost function $r(x, u)$ is nonnegative, continuous and locally Lipschitz in its first argument uniformly in $u \in \mathbb{U}$. Without loss of generality, we let κ_R be a Lipschitz constant of $r(\cdot, u)$ over B_R. In summary, we assume that

(A4) $r : \mathbb{R}^d \times \mathbb{U} \to \mathbb{R}_+$ is continuous and satisfies, for some constant $C_R > 0$

$$\left| r(x, u) - r(y, u) \right| \leq C_R |x - y| \qquad \forall x, y \in B_R, \forall u \in \mathbb{U},$$

and all $R > 0$.

In general, $\mathbb{U}$ may not be a convex set. It is therefore often useful to enlarge the control set to $\mathcal{P}(\mathbb{U})$. For any $v(\mathrm{d}u) \in \mathcal{P}(\mathbb{U})$ we can redefine the drift and the running cost as

$$(3.5) \qquad \bar{b}(x, v) := \int_{\mathbb{U}} b(x, u)v(\mathrm{d}u) \quad \text{and} \quad \bar{r}(x, v) := \int_{\mathbb{U}} r(x, u)v(\mathrm{d}u).$$

It is easy to see that the drift and running cost defined in (3.5) satisfy all the aforementioned conditions (A1)–(A4). In what follows, we assume that all the controls take values in $\mathcal{P}(\mathbb{U})$. These controls are generally referred to as *relaxed* controls. We endow the set of relaxed stationary Markov controls with the following topology: $v_n \to v$ in $\mathfrak{U}_{\text{SM}}$ if and only if

$$\int_{\mathbb{R}^d} f(x) \int_{\mathbb{U}} g(x, u)v_n(\mathrm{d}u|x) \, \mathrm{d}x \xrightarrow[n \to \infty]{} \int_{\mathbb{R}^d} f(x) \int_{\mathbb{U}} g(x, u)v(\mathrm{d}u|x) \, \mathrm{d}x$$

for all $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and $g \in \mathcal{C}_b(\mathbb{R}^d \times \mathbb{U})$. Then $\mathfrak{U}_{\text{SM}}$ is a compact metric space under this topology [1], Section 2.4. We refer to this topology as the *topology of Markov controls*. A control is said to be *precise* if it takes value in $\mathbb{U}$. It is easy to see that any precise control U_t can also be understood as a relaxed control by $U_t(\mathrm{d}u) = \delta_{U_t}$. Abusing the notation, we denote the drift and running cost by b and r, respectively, and the action of a relaxed control on them is understood as in (3.5).

3.2. *Structural assumptions.* Assumptions 3.1 and 3.2, described below, are in effect throughout the analysis, unless otherwise stated.

ASSUMPTION 3.1. For some open set $\mathcal{K} \subset \mathbb{R}^d$, the following hold:

(i) The running cost r is inf-compact on $\mathcal{K}$.
(ii) There exist inf-compact functions $\mathcal{V} \in \mathcal{C}^2(\mathbb{R}^d)$ and $h \in \mathcal{C}(\mathbb{R}^d \times \mathbb{U})$, such that

(3.6)
$$L^u \mathcal{V}(x) \leq 1 - h(x, u) \qquad \forall (x, u) \in \mathcal{K}^c \times \mathbb{U},$$
$$L^u \mathcal{V}(x) \leq 1 + r(x, u) \qquad \forall (x, u) \in \mathcal{K} \times \mathbb{U}.$$

Without loss of generality, we assume that $\mathcal{V}$ and h are nonnegative.

REMARK 3.1. In the statement of Assumption 3.1, we refrain from using any constants in the interest of notational economy. There is no loss of generality in doing so, since the functions $\mathcal{V}$ and h can always be scaled to eliminate unnecessary constants.

The next assumption is not a structural one, but rather the necessary requirement that the value of the ergodic control problem is finite. Otherwise, the problem is vacuous. For $U \in \mathfrak{U}$, define

(3.7)
$$\varrho_U(x) := \limsup_{T \to \infty} \frac{1}{T} \mathbb{E}_x^U \left[\int_0^T r(X_s, U_s) \, ds \right].$$

ASSUMPTION 3.2. There exists $U \in \mathfrak{U}$ such that $\varrho_U(x) < \infty$ for some $x \in \mathbb{R}^d$.

Assumption 3.2 alone does not imply that $\varrho_v < \infty$ for some $v \in \mathfrak{U}_{\text{SSM}}$. However, when combined with Assumption 3.1, this is the case as the following lemma asserts.

LEMMA 3.1. *Let Assumptions 3.1 and 3.2 hold. Then there exists $u_0 \in \mathfrak{U}_{\text{SSM}}$ such that $\varrho_{u_0} < \infty$. Moreover, there exists a nonnegative inf-compact function $\mathcal{V}_0 \in \mathcal{C}^2(\mathbb{R}^d)$, and a positive constant η such that*

(3.8)
$$L^{u_0} \mathcal{V}_0(x) \leq \eta - r(x, u_0(x)) \qquad \forall x \in \mathbb{R}^d.$$

Conversely, if (3.8) *holds, then Assumption 3.2 holds.*

PROOF. The first part of the result follows from Theorem 3.1(e) and (3.23) whereas the converse part follows from Lemma 3.2. These proofs are stated later in the paper. □

REMARK 3.2. There is no loss of generality in using only the constant η in Assumption 3.2, since $\mathcal{V}_0$ can always be scaled to achieve this.

We also observe that for $\mathcal{K} = \mathbb{R}^d$ the problem reduces to an ergodic control problem with near-monotone cost, and for $\mathcal{K} = \varnothing$ we obtain an ergodic control problem under a uniformly stable controlled diffusion.

3.3. *Piecewise linear controlled diffusions.* The controlled diffusion process in (2.12) belongs to a large class of controlled diffusion processes, called piecewise linear controlled diffusions [14]. We describe this class of controlled diffusions and show that it satisfies the assumptions in Section 3.2.

DEFINITION 3.2. A square matrix R is said to be an M-matrix if it can be written as $R = sI - N$ for some $s > 0$ and nonnegative matrix N with property that $\rho(N) \leq s$, where $\rho(N)$ denotes the spectral radius of N.

Let $\Gamma = [\gamma^{ij}]$ be a given matrix whose diagonal elements are positive, $\gamma^{id} = 0$ for $i = 1, \ldots, d - 1$, and the remaining elements are in $\mathbb{R}$. (Note that for the queueing model, Γ is a positive diagonal matrix. Our results below hold for the more general Γ.) Let $\ell \in \mathbb{R}^d$ and R be a nonsingular M-matrix. Define

$$(3.9) \qquad b(x, u) := \ell - R\big(x - (e \cdot x)^+ u\big) - (e \cdot x)^+ \Gamma u,$$

with $u \in \mathcal{S} := \{u \in \mathbb{R}_+^d : e \cdot u = 1\}$. Assume that

$$e^\mathsf{T} R \geq 0^\mathsf{T}.$$

We consider the following controlled diffusion in $\mathbb{R}^d$:

$$(3.10) \qquad \mathrm{d}X_t = b(X_t, U_t)\,\mathrm{d}t + \Sigma\,\mathrm{d}W_t,$$

where Σ is a constant matrix such that $\Sigma\Sigma^\mathsf{T}$ is invertible. It is easy to see that (3.10) satisfies conditions (A1)–(A3).

Analysis of these types of diffusion approximations is an established tradition in queueing systems. It is often easy to deal with the limiting object and it also helps to obtain information on the behavior of the actual queueing model.

We next introduce the running cost function. Let $r : \mathbb{R}^d \times \mathcal{S} \to [0, \infty)$ be locally Lipschitz with polynomial growth and

$$(3.11) \qquad c_1\big[(e \cdot x)^+\big]^m \leq r(x, u) \leq c_2\big(1 + \big[(e \cdot x)^+\big]^m\big),$$

for some $m \geq 1$ and positive constants c_1 and c_2 that do not depend on u. Some typical examples of such running costs are

$$r(x, u) = \big[(e \cdot x)^+\big]^m \sum_{i=1}^d h_i u_i^m \qquad \text{with } m \geq 1,$$

for some positive vector $(h_1, \ldots, h_d)^\mathsf{T}$.

REMARK 3.3. The controlled dynamics in (3.9) and running cost in (3.11) are clearly more general than the model described in Section 2.2. In (3.10), X denotes the diffusion approximation for the number customers in the system in the Halfin–Whitt regime and its ith component X^i denotes the diffusion approximation of the number of class i customers. Therefore, $(e \cdot X)^+$ denotes the total number of customers in the queue. For R and Γ diagonal as in (2.13), the diagonal entries of R and Γ denote the service and abandonment rates, respectively, of the customer classes. The ith coordinate of U denotes the fraction of class-i customers waiting in the queue. Therefore, the vector-valued process $X_t - (e \cdot X_t)^+ U_t$ denotes the diffusion approximation of the numbers of customers in service from different customer classes.

PROPOSITION 3.1. *Let b and r be given by* (3.9) *and* (3.11), *respectively. Then* (3.10) *satisfies Assumptions* 3.1 *and* 3.2, *with $h(x) = c_0|x|^m$ and*

$$(3.12) \qquad \mathcal{K} := \big\{ x : \delta|x| < (e \cdot x)^+ \big\}$$

for appropriate positive constants c_0 and δ.

PROOF. We recall that if R is a nonsingular M-matrix, then there exists a positive definite matrix Q such that $QR + R^{\mathsf{T}}Q$ is strictly positive definite [14]. Therefore, for some positive constant κ_0 it holds that

$$\kappa_0|y|^2 \le y^{\mathsf{T}}[QR + R^{\mathsf{T}}Q]y \le \kappa_0^{-1}|y|^2 \qquad \forall y \in \mathbb{R}^d.$$

The set $\mathcal{K}$ in (3.12), where $\delta > 0$ is chosen later, is an open convex cone, and the running cost function r is inf-compact on $\mathcal{K}$. Let $\mathcal{V}$ be a nonnegative function in $\mathcal{C}^2(\mathbb{R}^d)$ such that $\mathcal{V}(x) = [x^{\mathsf{T}}Qx]^{m/2}$ for $|x| \ge 1$, where the constant m is as in (3.11).

Let $|x| \ge 1$ and $u \in \mathcal{S}$. Then

$$
\begin{aligned}
\nabla \mathcal{V}(x) \cdot b(x, u) = {} & \ell \cdot \nabla \mathcal{V}(x) - \frac{m[x^{\mathsf{T}}Qx]^{m/2-1}}{2} x^{\mathsf{T}}[QR + R^{\mathsf{T}}Q]x \\
& + m[x^{\mathsf{T}}Qx]^{m/2-1}Qx \cdot (R - \Gamma)(e \cdot x)^+ u \\
\le {} & \ell \cdot \nabla \mathcal{V}(x) - m[x^{\mathsf{T}}Qx]^{m/2-1}\left(\frac{\kappa_0}{2}|x|^2 - C|x|(e \cdot x)^+ \right)
\end{aligned}
$$

for some positive constant C. If we choose $\delta = \frac{\kappa_0}{4C}$, then on $\mathcal{K}^c \cap \{|x| \ge 1\}$ we have the estimate

$$(3.13) \qquad \nabla \mathcal{V}(x) \cdot b(x, u) \le \ell \cdot \nabla \mathcal{V}(x) - \frac{m\kappa_0}{4}[x^{\mathsf{T}}Qx]^{m/2-1}|x|^2.$$

Note that $\ell \cdot \mathcal{V}$ is globally bounded for $m = 1$. For any $m \in (1, \infty)$, it follows by (3.13) that

$$
\begin{aligned}
(3.14) \qquad \nabla \mathcal{V}(x) \cdot b(x, u) \le {} & m(\ell^{\mathsf{T}}Qx)[x^{\mathsf{T}}Qx]^{m/2-1} - \frac{m\kappa_0}{4}[x^{\mathsf{T}}Qx]^{m/2-1}|x|^2 \\
\le {} & \frac{m|\ell^{\mathsf{T}}Q|(\overline{\lambda}(Q))^{m/2}}{\underline{\lambda}(Q)}|x|^{m-1} - \frac{m\kappa_0(\underline{\lambda}(Q))^{m/2}}{4\overline{\lambda}(Q)}|x|^m
\end{aligned}
$$

for $x \in \mathcal{K}^c \cap \{|x| \geq 1\}$, where $\underline{\lambda}(Q)$ and $\overline{\lambda}(Q)$ are the smallest and largest eigenvalues of Q, respectively. We use Young's inequality

$$|ab| \leq \frac{|a|^m}{m} + \frac{m-1}{m}|b|^{m/(m-1)}, \qquad a, b \geq 0,$$

in (3.14) to obtain the bound

$$(3.15) \qquad \nabla \mathcal{V}(x) \cdot b(x, u) \leq \kappa_1 - \frac{m\kappa_0}{8\overline{\lambda}(Q)}(\underline{\lambda}(Q))^{m/2}|x|^m$$

for some constant $\kappa_1 > 0$. A similar calculation shows for some constant $\kappa_2 > 0$ it holds that

$$(3.16) \qquad \nabla \mathcal{V}(x) \cdot b(x, u) \leq \kappa_2\big(1 + \big[(e \cdot x)^+\big]^m\big) \qquad \forall x \in \mathcal{K} \cap \{|x| \geq 1\}.$$

Also note that we can select $\kappa_3 > 0$ large enough such that

$$(3.17) \qquad \frac{1}{2}\big|\text{trace}\big(\Sigma\Sigma^\mathsf{T}\nabla^2\mathcal{V}(x)\big)\big| \leq \kappa_3 + \frac{m\kappa_0}{16\overline{\lambda}(Q)}(\underline{\lambda}(Q))^{m/2}|x|^m.$$

Hence, by (3.13)–(3.17) there exists $\kappa_4 > 0$ such that

$$(3.18) \qquad L^u\mathcal{V}(x) \leq \kappa_4 - \frac{m\kappa_0}{16\overline{\lambda}(Q)}(\underline{\lambda}(Q))^{m/2}|x|^m \mathbb{I}_{\mathcal{K}^c}(x) + \kappa_2\big[(e \cdot x)^+\big]^m\mathbb{I}_{\mathcal{K}}(x)$$

for all $x \in \mathbb{R}^d$. It is evident that we can scale $\mathcal{V}$, by multiplying it with a constant, so that (3.18) takes the form

$$(3.19) \qquad L^u\mathcal{V}(x) \leq 1 - c_0|x|^m\mathbb{I}_{\mathcal{K}^c}(x) + c_1\big[(e \cdot x)^+\big]^m\mathbb{I}_{\mathcal{K}}(x) \qquad \forall x \in \mathbb{R}^d.$$

By (3.11), the running cost r is inf-compact on $\mathcal{K}$. It then follows from (3.11) and (3.19) that (3.6) is satisfied with $h(x) := c_0|x|^m$.

We next show that (3.10) satisfies Assumption 3.2. Let

$$u_0(\cdot) \equiv e_d = (0, \ldots, 0, 1)^\mathsf{T}.$$

Then we can write (3.10) as

$$(3.20) \qquad dX_t = \big(\ell - R\big(X_t - (e \cdot X_t)^+u_0\big) - (e \cdot x)^+\Gamma u_0\big)\,dt + \Sigma\,dW_t.$$

It is shown in [14] that the solution X_t in (3.20) is positive recurrent and, therefore, u_0 is a stable Markov control. This is done by finding a suitable Lyapunov function. In particular, in [14], Theorem 3, it is shown that there exists a positive definite matrix $\tilde{Q}$ such that if we define

$$(3.21) \qquad \psi(x) := (e \cdot x)^2 + \tilde{\kappa}\big[x - e_d\phi(e \cdot x)\big]^\mathsf{T}\tilde{Q}\big[x - e_d\phi(e \cdot x)\big],$$

for some suitably chosen constant $\tilde{\kappa}$ and a function $\phi \in \mathcal{C}^2(\mathbb{R})$, given by

$$\phi(y) = \begin{cases} y, & \text{if } y \geq 0, \\ -\frac{1}{2}\tilde{\delta}, & \text{if } y \leq -\tilde{\delta}, \\ \text{smooth}, & \text{if } -\tilde{\delta} < y < 0, \end{cases}$$

where $\tilde{\delta} > 0$ is a suitable constant and $0 \le \phi'(y) \le 1$, then it holds that

$$(3.22) \qquad L^{u_0}\psi(x) \le -\tilde{\kappa}_1|x|^2,$$

for $|x|$ large enough and positive constant $\tilde{\kappa}_1$. We define $\mathcal{V}_0 := e^{a\psi}$ where a is to be determined later. Note that $|\nabla\psi(x)| \le \tilde{\kappa}_2(1 + |x|)$ for some constant $\tilde{\kappa}_2 > 0$. Hence, a straightforward calculation shows that if we choose a small enough, then for some constant $\tilde{\kappa}_3 > 0$ it holds that

$$L^{u_0}\mathcal{V}_0(x) \le \left(-\tilde{\kappa}_1 a|x|^2 + a^2\|\Sigma\|^2\tilde{\kappa}_2(1 + |x|)^2\right)\mathcal{V}_0(x)$$
$$\le -\tilde{\kappa}_3|x|^2\mathcal{V}_0(x),$$

for all $|x|$ large enough. Since $\mathcal{V}_0(x) > [(e \cdot x)^+]^m$, $m \ge 1$, for all large enough $|x|$ we see that $\mathcal{V}_0$ satisfies (3.8) with control u_0. Hence, Assumption 3.2 holds by Lemma 3.1. $\square$

3.4. *Existence of optimal controls.*

DEFINITION 3.3. Recall the definition of ϱ_U in (3.7). For $\beta > 0$, we define

$$\mathfrak{U}^\beta := \{U \in \mathfrak{U} : \varrho_U(x) \le \beta \text{ for some } x \in \mathbb{R}^d\}.$$

We also let $\mathfrak{U}^\beta_{\mathrm{SM}} := \mathfrak{U}^\beta \cap \mathfrak{U}_{\mathrm{SM}}$, and

$$\hat{\varrho}_* := \inf\{\beta > 0 : \mathfrak{U}^\beta \neq \varnothing\},$$
$$\varrho_* := \inf\{\beta > 0 : \mathfrak{U}^\beta_{\mathrm{SM}} \neq \varnothing\},$$
$$\tilde{\varrho}_* := \inf\{\pi(r) : \pi \in \mathcal{G}\},$$

where

$$\mathcal{G} := \left\{\pi \in \mathcal{P}(\mathbb{R}^d \times \mathbb{U}) : \int_{\mathbb{R}^d \times \mathbb{U}} L^u f(x)\pi(dx, du) = 0 \; \forall f \in \mathcal{C}^\infty_c(\mathbb{R}^d)\right\},$$

and $L^u f(x)$ is given by (3.3). It is well known that $\mathcal{G}$ is the set of ergodic occupation measures of the controlled process in (3.1), and that $\mathcal{G}$ is a closed and convex subset of $\mathcal{P}(\mathbb{R}^d \times \mathbb{U})$ [1], Lemmas 3.2.2 and 3.2.3. We use the notation π_v when we want to indicate the ergodic occupation measure associated with the control $v \in \mathfrak{U}_{\mathrm{SSM}}$. In other words,

$$\pi_v(dx, du) := \mu_v(dx)v(du|x).$$

LEMMA 3.2. *If (3.8) holds for some $\mathcal{V}_0$ and u_0, then we have $\pi_{u_0}(r) \le \eta$. Therefore, $\hat{\varrho}_* < \infty$.*

PROOF.   Let $(X_t, u_0(X_t))$ be the solution of (3.1). Recall that τ_R is the first exit time from B_R for $R > 0$. Then by Itô's formula

$$\mathbb{E}_x^{u_0}\big[\mathcal{V}_0(X_{T \wedge \tau_R})\big] - \mathcal{V}_0(x) \le \eta T - \mathbb{E}_x^{u_0}\bigg[\int_0^{T \wedge \tau_R} r\big(X_s, u_0(X_s)\big)\,\mathrm{d}s\bigg].$$

Therefore, letting $R \to \infty$ and using Fatou's lemma, we obtain the bound

$$\mathbb{E}_x^{u_0}\bigg[\int_0^T r\big(X_s, u_0(X_s)\big)\,\mathrm{d}s\bigg] \le \eta T + \mathcal{V}_0(x) - \min_{\mathbb{R}^d} \mathcal{V}_0,$$

and thus

$$\limsup_{T \to \infty} \frac{1}{T}\mathbb{E}_x^{u_0}\bigg[\int_0^T r\big(X_s, u_0(X_s)\big)\,\mathrm{d}s\bigg] \le \eta. \qquad \square$$

In the analysis, we use a function $\tilde{h} \in \mathcal{C}(\mathbb{R}^d \times \mathbb{U})$ which, roughly speaking, is of the same order as r in $\mathcal{K} \times \mathbb{U}$ and lies between r and a multiple of $r + h$ on $\mathcal{K}^c \times \mathbb{U}$, with $\mathcal{K}$ as in Assumption 3.1. The existence of such a function is guaranteed by Assumption 3.1 as the following lemma shows.

LEMMA 3.3.   *Define*

$$\mathcal{H} := (\mathcal{K} \times \mathbb{U}) \cup \big\{(x, u) \in \mathbb{R}^d \times \mathbb{U} : r(x, u) > h(x, u)\big\},$$

where $\mathcal{K}$ is the open set in Assumption 3.1. Then there exists an inf-compact function $\tilde{h} \in \mathcal{C}(\mathbb{R}^d \times \mathbb{U})$ which is locally Lipschitz in its first argument uniformly w.r.t. its second argument, and satisfies

$$(3.23) \quad r(x, u) \le \tilde{h}(x, u) \le \frac{k_0}{2}\big(1 + h(x, u)\mathbb{I}_{\mathcal{H}^c}(x, u) + r(x, u)\mathbb{I}_{\mathcal{H}}(x, u)\big)$$

for all $(x, u) \in \mathbb{R}^d \times \mathbb{U}$, and for some positive constant $k_0 \ge 2$. Moreover,

$$(3.24) \qquad L^u \mathcal{V}(x) \le 1 - h(x, u)\mathbb{I}_{\mathcal{H}^c}(x, u) + r(x, u)\mathbb{I}_{\mathcal{H}}(x, u)$$

for all $(x, u) \in \mathbb{R}^d \times \mathbb{U}$, where $\mathcal{V}$ is the function in Assumption 3.1.

PROOF.   If $f(x, u)$ denotes the right-hand side of (3.23), with $k_0 = 4$, then

$$f(x, u) - r(x, u) > h(x, u)\mathbb{I}_{\mathcal{H}^c}(x, u) + r(x, u)\mathbb{I}_{\mathcal{H}}(x, u)$$

$$\ge h(x, u)\mathbb{I}_{\mathcal{K}^c}(x) + r(x, u)\mathbb{I}_{\mathcal{K}}(x),$$

since $r(x, u) > h(x, u)$ on $\mathcal{H} \setminus (\mathcal{K} \times \mathbb{U})$. Therefore, by Assumption 3.1, the set $\{(x, u) : f(x, u) - r(x, u) \le n\}$ is bounded in $\mathbb{R}^d \times \mathbb{U}$ for every $n \in \mathbb{N}$. Hence, there exists an increasing sequence of open balls D_n, $n = 1, 2, \dots$, centered at 0 in $\mathbb{R}^d$ such that $f(x, u) - r(x, u) \ge n$ for all $(x, u) \in D_n^c \times \mathbb{U}$. Let $g : \mathbb{R}^d \to \mathbb{R}$ be any nonnegative smooth function such that $n - 1 \le g(x) \le n$ for $x \in D_{n+1} \setminus D_n$, $n = 1, 2, \dots$, and $g(x) = 0$ on D_1. Clearly, $\tilde{h} := r + g$ is continuous, inf-compact, locally Lipschitz in its first argument, and satisfies (3.23). That (3.24) holds is clear from (3.6) and the fact that $\mathcal{H} \supseteq \mathcal{K} \times \mathbb{U}$. $\square$

REMARK 3.4. It is clear from the proof of Lemma 3.3 that we could fix the value of the constant k_0 in (3.23), say $k_0 = 4$. However, we keep the variable k_0 because this provides some flexibility in the choice of $\tilde{h}$, and also in order to be able to trace it along the different calculations.

REMARK 3.5. Note that if $h \geq r$ and r is inf-compact, then $\mathcal{H} = \mathcal{K} \times \mathbb{U}$ and $\tilde{h} := r$ satisfies (3.23). Note also, that in view of (3.11) and Proposition 3.1, for the multi-class queueing model we have

$$
\begin{aligned}
r(x, u) &\leq c_2 \big(1 + [(e \cdot x)^+]^m\big) \\
&\leq \frac{c_2 d^{m-1}}{1 \wedge c_0} \big(1 + (1 \wedge c_0)|x|^m\big) \\
&\leq \frac{c_2 d^{m-1}}{1 \wedge c_0} \left(1 + c_0|x|^m \mathbb{I}_{\mathcal{K}^c}(x) + \frac{1}{\delta^m}[(e \cdot x)^+]^m \mathbb{I}_{\mathcal{K}}(x)\right) \\
&\leq \frac{c_2 d^{m-1}}{1 \wedge c_0} \left(1 + h(x)\mathbb{I}_{\mathcal{K}^c}(x) + \frac{1}{c_1 \delta^m} r(x, u)\mathbb{I}_{\mathcal{K}}(x)\right) \\
&\leq \frac{c_2 d^{m-1}}{1 \wedge c_0 \wedge c_1 \delta^m} \big(1 + h(x)\mathbb{I}_{\mathcal{K}^c}(x) + r(x, u)\mathbb{I}_{\mathcal{K}}(x)\big) \\
&\leq \frac{c_2 d^{m-1}}{1 \wedge c_0 \wedge c_1 \delta^m} \big(1 + h(x)\mathbb{I}_{\mathcal{H}^c}(x, u) + r(x, u)\mathbb{I}_{\mathcal{H}}(x, u)\big).
\end{aligned}
$$

Therefore, $\tilde{h}(x, u) := c_2 + c_2 d^{m-1}|x|^m$ satisfies (3.23).

REMARK 3.6. We often use the fact that if $g \in \mathcal{C}(\mathbb{R}^d \times \mathbb{U})$ is bounded below, then the map $\mathcal{P}(\mathbb{R}^d \times \mathbb{U}) \ni \nu \mapsto \nu(g)$ is lower semi-continuous. This easily follows from two facts: (a) g can be expressed as an increasing limit of bounded continuous functions, and (b) if g is bounded and continuous, then $\pi \mapsto \pi(g)$ is continuous.

THEOREM 3.1. Let $\beta \in (\hat{\varrho}_*, \infty)$. Then:

(a) For all $U \in \mathfrak{U}^\beta$ and $x \in \mathbb{R}^d$ such that $\varrho_U(x) \leq \beta$, then

$$
(3.25) \qquad \limsup_{t \to \infty} \frac{1}{T} \mathbb{E}_x^U \left[\int_0^T \tilde{h}(X_s, U_s)\, ds\right] \leq k_0(1 + \beta).
$$

(b) $\hat{\varrho}_* = \varrho_* = \tilde{\varrho}_*$.

(c) For any $\beta \in (\varrho_*, \infty)$, we have $\mathfrak{U}_{\mathrm{SM}}^\beta \subset \mathfrak{U}_{\mathrm{SSM}}$.

(d) The set of invariant probability measures $\mathcal{M}^\beta$ corresponding to controls in $\mathfrak{U}_{\mathrm{SM}}^\beta$ satisfies

$$
\int_{\mathbb{R}^d} \tilde{h}(x, v(x))\mu_v(dx) \leq k_0(1 + \beta) \qquad \forall \mu_v \in \mathcal{M}^\beta.
$$

In particular, $\mathfrak{U}_{\mathrm{SM}}^\beta$ is tight in $\mathcal{P}(\mathbb{R}^d)$.

(e) *There exists $(\tilde{V}, \tilde{\varrho}) \in \mathcal{C}^2(\mathbb{R}^d) \times \mathbb{R}_+$, with $\tilde{V}$ inf-compact, such that*

$$(3.26) \qquad \min_{u \in \mathbb{U}} [L^u \tilde{V}(x) + \tilde{h}(x, u)] = \tilde{\varrho}.$$

PROOF. Using Itô's formula, it follows by (3.24) that

$$\frac{1}{T} \left(\mathbb{E}_x^U [\mathcal{V}(X_{T \wedge \tau_R})] - \mathcal{V}(x) \right)$$

$$(3.27) \qquad \leq 1 - \frac{1}{T} \mathbb{E}_x^U \left[\int_0^{T \wedge \tau_R} h(X_s, U_s) \mathbb{I}_{\mathcal{H}^c}(X_s, U_s) \, ds \right]$$

$$+ \frac{1}{T} \mathbb{E}_x^U \left[\int_0^{T \wedge \tau_R} r(X_s, U_s) \mathbb{I}_{\mathcal{H}}(X_s, U_s) \, ds \right].$$

Since $\mathcal{V}$ is inf-compact, (3.27) together with (3.23) implies that

$$\frac{2}{k_0} \limsup_{T \to \infty} \frac{1}{T} \mathbb{E}_x^U \left[\int_0^T \tilde{h}(X_s, U_s) \, ds \right]$$

$$(3.28) \qquad \leq 2 + 2 \limsup_{T \to \infty} \frac{1}{T} \mathbb{E}_x^U \left[\int_0^T r(X_s, U_s) \, ds \right].$$

Part (a) then follows from (3.28).

Now fix $U \in \mathfrak{U}^\beta$ and $x \in \mathbb{R}^d$ such that $\varrho_U(x) \leq \beta$. The inequality in (3.25) implies that the set of *mean empirical measures* $\{\zeta_{x,t}^U : t \geq 1\}$, defined by

$$\zeta_{x,t}^U(A \times B) := \frac{1}{t} \mathbb{E}_x^U \left[\int_0^t \mathbb{I}_{A \times B}(X_s, U_s) \, ds \right]$$

for any Borel sets $A \subset \mathbb{R}^d$ and $B \subset \mathbb{U}$, is tight. It is the case that any limit point of the mean empirical measures in $\mathcal{P}(\mathbb{R}^d \times \mathbb{U})$ is an ergodic occupation measure [1], Lemma 3.4.6. Then in view of Remark 3.6 we obtain

$$(3.29) \qquad \pi(r) \leq \limsup_{t \to \infty} \zeta_{x,t}^U(r) \leq \beta$$

for some ergodic occupation measure π. Therefore, $\tilde{\varrho}_* \leq \hat{\varrho}_*$. Disintegrating the measure π as $\pi(dx, du) = v(du|x)\mu_v(dx)$, we obtain the associated control $v \in \mathfrak{U}_{\mathrm{SSM}}$. From ergodic theory [33], we also know that

$$\limsup_{T \to \infty} \frac{1}{T} \mathbb{E}_x^v \left[\int_0^T r(X_s, v(X_s)) \, ds \right] = \pi_v(r) \qquad \text{for almost every } x.$$

It follows that $\varrho_* \leq \tilde{\varrho}_*$, and since it is clear that $\hat{\varrho}_* \leq \varrho_*$, equality must hold among the three quantities.

If $v \in \mathfrak{U}_{\mathrm{SM}}^\beta$, then (3.28) implies that (3.29) holds with $U \equiv v$ and $\pi \equiv \pi_v$. Therefore, parts (c) and (d) follow.

Existence of $(\tilde{V}, \tilde{\varrho})$, satisfying (3.26), follows from Assumption 3.2 and [1], Theorem 3.6.6. The inf-compactness of $\tilde{V}$ follows from the stochastic representation of $\tilde{V}$ in [1], Lemma 3.6.9. This proves (e). □

Existence of a stationary Markov control that is optimal is asserted by the following theorem.

THEOREM 3.2. *Let $\mathcal{G}$ denote the set of ergodic occupation measures corresponding to controls in $\mathfrak{U}_{\mathrm{SSM}}$, and $\mathcal{G}^\beta$ those corresponding to controls in $\mathfrak{U}_{\mathrm{SM}}^\beta$, for $\beta > \varrho_*$. Then:*

(a) *The set $\mathcal{G}^\beta$ is compact in $\mathcal{P}(\mathbb{R}^d)$ for any $\beta > \varrho_*$.*
(b) *There exists $v \in \mathfrak{U}_{\mathrm{SM}}$ such that $\varrho_v = \varrho_*$.*

PROOF. By Theorem 3.1(d), the set $\mathcal{G}^\beta$ is tight for any $\beta > \varrho_*$. Let $\{\pi_n\} \subset \mathcal{G}^\beta$, for some $\beta > \varrho_*$, be any convergent sequence in $\mathcal{P}(\mathbb{R}^d)$ such that $\pi_n(r) \to \varrho_*$ as $n \to \infty$ and denote its limit by π_*. Since $\mathcal{G}$ is closed, $\pi_* \in \mathcal{G}$, and since the map $\pi \to \pi(r)$ is lower semi-continuous, it follows that $\pi_*(r) \le \varrho_*$. Therefore, $\mathcal{G}^\beta$ is closed, and hence compact. Since $\pi(r) \ge \varrho_*$ for all $\pi \in \mathcal{G}$, the equality $\pi_*(r) = \varrho_*$ follows. Also v is obtained by disintegrating π_*. □

REMARK 3.7. The reader might have noticed at this point that Assumption 3.1 may be weakened significantly. What is really required is the existence of an open set $\hat{\mathcal{H}} \subset \mathbb{R}^d \times \mathbb{U}$ and inf-compact functions $\mathcal{V} \in \mathcal{C}^2(\mathbb{R}^d)$ and $h \in \mathcal{C}(\mathbb{R}^d \times \mathbb{U})$, satisfying

(H1) $\inf_{\{u \,:\, (x,u) \in \hat{\mathcal{H}}\}} r(x, u) \xrightarrow[|x| \to \infty]{} \infty.$

(H2) $L^u \mathcal{V}(x) \le 1 - h(x, u) \mathbb{I}_{\hat{\mathcal{H}}^c}(x, u) + r(x, u) \mathbb{I}_{\hat{\mathcal{H}}}(x, u) \ \forall (x, u) \in \mathbb{R}^d \times \mathbb{U}.$

In (H1), we use the convention that the 'inf' of the empty set is $+\infty$. Also note that (H1) is equivalent to the statement that $\{(x, u) : r(x, u) \le c\} \cap \hat{\mathcal{H}}$ is bounded in $\mathbb{R}^d \times \mathbb{U}$ for all $c \in \mathbb{R}_+$. If (H1)–(H2) are met, we define $\mathcal{H} := \hat{\mathcal{H}} \cup \{(x, u) \in \mathbb{R}^d \times \mathbb{U} : r(x, u) > h(x, u)\}$, and following the proof of Lemma 3.3, we assert the existence of an inf-compact $\tilde{h} \in \mathcal{C}(\mathbb{R}^d \times \mathbb{U})$ satisfying (3.23). In fact, throughout the rest of the paper, Assumption 3.1 is not really invoked. We only use (3.24), the inf-compact function $\tilde{h}$ satisfying (3.23), and, naturally, Assumption 3.2.

3.5. *The HJB equation.* For $\varepsilon > 0$, let

$$r_\varepsilon(x, u) := r(x, u) + \varepsilon \tilde{h}(x, u).$$

By Theorem 3.1(d), for any $\pi \in \mathcal{G}^\beta$, $\beta > \varrho_*$, we have the bound

$$(3.30) \qquad\qquad \pi(r_\varepsilon) \le \beta + \varepsilon k_0 (1 + \beta).$$

Therefore, since r_ε is near-monotone, that is,

$$\liminf_{|x| \to \infty} \min_{u \in \mathbb{U}} r_\varepsilon(x, u) > \inf_{\pi \in \mathcal{G}} \pi(r_\varepsilon),$$

there exists $\pi_\varepsilon \in \arg\min_{\pi \in \mathcal{G}} \pi(r_\varepsilon)$. Let $\pi_* \in \mathcal{G}$ be as in the proof of Theorem 3.2. The sub-optimality of π_* relative to the running cost r_ε and (3.30) imply that

$$
\begin{aligned}
\pi_\varepsilon(r) &\leq \pi_\varepsilon(r_\varepsilon) \\
&\leq \pi_*(r_\varepsilon) \\
&\leq \varrho_* + \varepsilon k_0(1 + \varrho_*) \qquad \forall \varepsilon > 0.
\end{aligned}
$$
(3.31)

It follows from (3.31) and Theorem 3.1(d) that $\{\pi_\varepsilon : \varepsilon \in (0, 1)\}$ is tight. Since $\pi_\varepsilon \mapsto \pi_\varepsilon(r)$ is lower semi-continuous, if $\bar{\pi}$ is any limit point of π_ε as $\varepsilon \searrow 0$, then taking limits in (3.31), we obtain

$$\bar{\pi}(r) \leq \limsup_{\varepsilon \searrow 0} \pi_\varepsilon(r) \leq \varrho_*.$$
(3.32)

Since $\mathcal{G}$ is closed, $\bar{\pi} \in \mathcal{G}$, which implies that $\bar{\pi}(r) \geq \varrho_*$. Therefore, equality must hold in (3.32), or in other words, $\bar{\pi}$ is an optimal ergodic occupation measure.

THEOREM 3.3. *There exists a unique function $V^\varepsilon \in \mathcal{C}^2(\mathbb{R}^d)$ with $V^\varepsilon(0) = 0$, which is bounded below in $\mathbb{R}^d$, and solves the HJB*

$$\min_{u \in \mathbb{U}} \left[L^u V^\varepsilon(x) + r_\varepsilon(x, u) \right] = \varrho_\varepsilon,$$
(3.33)

where $\varrho_\varepsilon := \inf_{\pi \in \mathcal{G}} \pi(r_\varepsilon)$, or in other words, ϱ_ε is the optimal value of the ergodic control problem with running cost r_ε. Also a stationary Markov control v_ε is optimal for the ergodic control problem relative to r_ε if and only if it satisfies

$$H_\varepsilon\big(x, \nabla V^\varepsilon(x)\big) = b\big(x, v_\varepsilon(x)\big) \cdot \nabla V^\varepsilon(x) + r_\varepsilon\big(x, v_\varepsilon(x)\big) \qquad a.e.\ in\ \mathbb{R}^d,$$
(3.34)

where

$$H_\varepsilon(x, p) := \min_{u \in \mathbb{U}} \big[b(x, u) \cdot p + r_\varepsilon(x, u) \big].$$
(3.35)

Moreover:

(a) *for every $R > 0$, there exists k_R such that*

$$\operatorname*{osc}_{B_R} V^\varepsilon \leq k_R;$$
(3.36)

(b) *if v_ε is a measurable a.e. selector from the minimizer of the Hamiltonian in (3.35), that is, if it satisfies (3.33), then for any $\delta > 0$,*

$$V^\varepsilon(x) \geq \mathbb{E}_x^{v_\varepsilon}\left[\int_0^{\breve{\tau}_\delta} \big(r_\varepsilon(X_s, v_\varepsilon(X_s)) - \varrho_\varepsilon \big) \, \mathrm{d}s \right] + \inf_{B_\delta} V^\varepsilon;$$

(c) *for any stationary control $v \in \mathfrak{U}_{\mathrm{SSM}}$ and for any $\delta > 0$,*

$$V^\varepsilon(x) \leq \mathbb{E}_x^v\left[\int_0^{\check{\tau}_\delta}\left(r_\varepsilon(X_s, v(X_s)) - \varrho_\varepsilon\right)\mathrm{d}s + V^\varepsilon(X_{\check{\tau}_\delta})\right],$$

where $\check{\tau}_\delta$ is hitting time to the ball B_δ.

THEOREM 3.4. *Let V^ε, ϱ_ε, and v_ε, for $\varepsilon > 0$, be as in Theorem 3.3. The following hold:*

(a) *The function V^ε converges to some $V_* \in \mathcal{C}^2(\mathbb{R}^d)$, uniformly on compact sets, and $\varrho_\varepsilon \to \varrho_*$, as $\varepsilon \searrow 0$, and V_* satisfies*

(3.37) $$\min_{u \in \mathbb{U}}\left[L^u V_*(x) + r(x, u)\right] = \varrho_*.$$

Also, any limit point v_* (in the topology of Markov controls) as $\varepsilon \searrow 0$ of the set $\{v_\varepsilon\}$ satisfies

$$L^{v_*}V_*(x) + r(x, v_*(x)) = \varrho_* \qquad a.e.\ in\ \mathbb{R}^d.$$

(b) *A stationary Markov control v is optimal for the ergodic control problem relative to r if and only if it satisfies*

(3.38) $$H(x, \nabla V_*(x)) = b(x, v(x)) \cdot \nabla V_*(x) + r(x, v(x)) \qquad a.e.\ in\ \mathbb{R}^d,$$

where

$$H(x, p) := \min_{u \in \mathbb{U}}\left[b(x, u) \cdot p + r(x, u)\right].$$

Moreover, for an optimal $v \in \mathfrak{U}_{\mathrm{SM}}$, we have

$$\lim_{T \to \infty}\frac{1}{T}\mathbb{E}_x^v\left[\int_0^T r(X_s, v(X_s))\,\mathrm{d}s\right] = \varrho_* \qquad \forall x \in \mathbb{R}^d.$$

(c) *The function V_* has the stochastic representation*

$$V_*(x) = \lim_{\delta \searrow 0}\inf_{v \in \bigcup_{\beta > 0}\mathfrak{U}_{\mathrm{SM}}^\beta}\mathbb{E}_x^v\left[\int_0^{\check{\tau}_\delta}\left(r(X_s, v(X_s)) - \varrho_*\right)\mathrm{d}s\right]$$

(3.39)

$$= \lim_{\delta \searrow 0}\mathbb{E}_x^{\bar{v}}\left[\int_0^{\check{\tau}_\delta}\left(r(X_s, v_*(X_s)) - \varrho_*\right)\mathrm{d}s\right]$$

for any $\bar{v} \in \mathfrak{U}_{\mathrm{SM}}$ that satisfies (3.38).

(d) *If $\mathbb{U}$ is a convex set, $u \mapsto \{b(x, u) \cdot p + r(x, u)\}$ is strictly convex whenever it is not constant, and $u \mapsto \tilde{h}(x, u)$ is strictly convex for all x, then any measurable minimizer of (3.33) converges pointwise, and thus in $\mathfrak{U}_{\mathrm{SM}}$, to the minimizer of (3.37).*

Theorem 3.4 guarantees the existence of an optimal stable control, which is made precise by (3.38), for the ergodic diffusion control problem with the running cost function r. Moreover, under the convexity property in part (d), the optimal stable control can be obtained as a pointwise limit from the minimizing selector of (3.33). For instance, if we let

$$r(x, u) = (e \cdot x)^+ \sum_{i=1}^{d} h_i u_i^m, \qquad m > 1,$$

then by choosing h and $\tilde{h} + |u|^2$ as in Proposition 3.1, we see that the approximate value function V^ε and approximate control v_ε converge to the desired optimal value function V_* and optimal control v_*, respectively.

Concerning the uniqueness of the solution to the HJB equation in (3.37), recall that in the near-monotone case the existing uniqueness results are as follows: there exists a unique solution pair (V, ϱ) of (3.37) with V in the class of functions $\mathcal{C}^2(\mathbb{R}^d)$ which are bounded below in $\mathbb{R}^d$. Moreover, it satisfies $V(0) = 0$ and $\varrho \le \varrho_*$. If the restriction $\varrho \le \varrho_*$ is removed, then in general, there are multiple solutions. Since in our model r is not near-monotone in $\mathbb{R}^d$, the function V_* is not, in general, bounded below. However, as we show later in Lemma 3.10 the negative part of V_* grows slower than $\mathcal{V}$, that is, it holds that $V_*^- \in \mathfrak{o}(\mathcal{V})$, with $\mathfrak{o}(\cdot)$ as defined in Section 1.3. Therefore, the second part of the theorem that follows may be viewed as an extension of the well-known uniqueness results that apply to ergodic control problems with near-monotone running cost. The third part of the theorem resembles the hypotheses of uniqueness that apply to problems under a blanket stability hypothesis.

THEOREM 3.5.   *Let $(\hat{V}, \hat{\varrho})$ be a solution of*

(3.40) $$\min_{u \in \mathbb{U}} \big[L^u \hat{V}(x) + r(x, u) \big] = \hat{\varrho},$$

such that $\hat{V}^- \in \mathfrak{o}(\mathcal{V})$ and $\hat{V}(0) = 0$. Then the following hold:

(a) *Any measurable selector $\hat{v}$ from the minimizer of the associated Hamiltonian in (3.38) is in $\mathfrak{U}_{\mathrm{SSM}}$ and $\varrho_{\hat{v}} < \infty$.*
(b) *If $\hat{\varrho} \le \varrho_*$ then necessarily $\hat{\varrho} = \varrho_*$ and $\hat{V} = V_*$.*
(c) *If $\hat{V} \in \mathcal{O}(\min_{u \in \mathbb{U}} \tilde{h}(\cdot, u))$, then $\hat{\varrho} = \varrho_*$ and $\hat{V} = V_*$.*

Applying these results to the multi-class queueing diffusion model, we have the following corollary.

COROLLARY 3.1.   *For the queueing diffusion model with controlled dynamics given by (3.10), drift given by (3.9), and running cost as in (3.11), there exists a unique solution V, satisfying $V(0) = 0$, to the associated HJB in the class of functions $\mathcal{C}^2(\mathbb{R}^d) \cap \mathcal{O}(|x|^m)$, whose negative part is in $\mathfrak{o}(|x|^m)$. This solution agrees with V_* in Theorem 3.4.*

PROOF. Existence of a solution V follows by Theorem 3.4. Select $\mathcal{V} \sim |x|^m$ as in the proof of Proposition 3.1. That the solution V is in the stated class then follows by Lemma 3.10 and Corollary 4.1 that appear later in Sections 3.6 and 4, respectively. With $h \sim |x|^m$ as in the proof of Proposition 3.1, it follows that $\min_{u \in \mathbb{U}} \tilde{h}(x, u) \in \mathcal{O}(|x|^m)$. Therefore, uniqueness follows by Theorem 3.5. $\square$

We can also obtain the HJB equation in (3.37) via the traditional vanishing discount approach as the following theorem asserts. Similar results are shown for a one-dimensional degenerate ergodic diffusion control problem in [29] and certain multi-dimensional ergodic diffusion control problems (allowing degeneracy and spatial periodicity) in [2].

THEOREM 3.6. *Let V_* and ϱ_* be as in Theorem 3.4. For $\alpha > 0$, we define*

$$V_\alpha(x) := \inf_{U \in \mathfrak{U}} \mathbb{E}_x^U \left[\int_0^\infty e^{-\alpha t} r(X_t, U_t) \, dt \right].$$

The function $V_\alpha - V_\alpha(0)$ converges, as $\alpha \searrow 0$, to V_*, uniformly on compact subsets of $\mathbb{R}^d$. Moreover, $\alpha V_\alpha(0) \to \varrho_*$, as $\alpha \searrow 0$.

The proofs of the Theorems 3.3–3.6 are given in Section 3.6. The following result, which follows directly from (3.31), provides a way to find ε-optimal controls.

PROPOSITION 3.2. *Let $\{v_\varepsilon\}$ be the minimizing selector from Theorem 3.3 and $\{\mu_{v_\varepsilon}\}$ be the corresponding invariant probability measures. Then almost surely for all $x \in \mathbb{R}^d$,*

$$\lim_{T \to \infty} \frac{1}{T} \mathbb{E}_x^{v_\varepsilon} \left[\int_0^T r(X_s, v_\varepsilon(X_s)) \, ds \right] = \int_{\mathbb{R}^d} r(x, v_\varepsilon(x)) \mu_{v_\varepsilon}(dx)$$

$$\leq \varrho_* + \varepsilon k_0 (1 + \varrho_*).$$

3.6. *Technical proofs.* Recall that $r_\varepsilon(x, u) = r(x, u) + \varepsilon \tilde{h}(x, u)$, with $\tilde{h}$ as in Lemma 3.3. We need the following lemma.

For $\alpha > 0$ and $\varepsilon \geq 0$, we define

(3.41) $$V_\alpha^\varepsilon(x) := \inf_{U \in \mathfrak{U}} \mathbb{E}_x^U \left[\int_0^\infty e^{-\alpha t} r_\varepsilon(X_t, U_t) \, dt \right],$$

where we set $r_0 \equiv r$. Clearly, when $\varepsilon = 0$, we have $V_\alpha^0 \equiv V_\alpha$.

We quote the following result from [1], Theorem 3.5.6, Remark 3.5.8.

LEMMA 3.4. *Provided $\varepsilon > 0$, then V_α^ε defined above is in $\mathcal{C}^2(\mathbb{R}^d)$ and is the minimal nonnegative solution of*

$$\min_{u \in \mathbb{U}} \left[L^u V_\alpha^\varepsilon(x) + r_\varepsilon(x, u) \right] = \alpha V_\alpha^\varepsilon(x).$$

The HJB in Lemma 3.4 is similar to the equation in [7], Theorem 3, which concerns the characterization of the discounted control problem.

LEMMA 3.5.   *Let u be any precise Markov control and L^u be the corresponding generator. Let $\varphi \in C^2(\mathbb{R}^d)$ be a nonnegative solution of*

$$L^u \varphi - \alpha \varphi = g,$$

where $g \in L^\infty_{loc}(\mathbb{R}^d)$. Let $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$ be any nondecreasing function such that $\|g\|_{L^\infty(B_R)} \leq \kappa(R)$ for all $R > 0$. Then for any $R > 0$ there exists a constant $D(R)$ which depends on $\kappa(4R)$, but not on u, or φ, such that

$$\underset{B_R}{\operatorname{osc}}\, \varphi \leq D(R)\Big(1 + \alpha \inf_{B_{4R}} \varphi\Big).$$

PROOF.   Define $\tilde{g} := \alpha(g - 2\kappa(4R))$ and $\tilde{\varphi} := 2\kappa(4R) + \alpha\varphi$. Then $\tilde{g} \leq 0$ in B_{4R} and $\tilde{\varphi}$ solves

$$L^u \tilde{\varphi} - \alpha\tilde{\varphi} = \tilde{g} \qquad \text{in } B_{4R}.$$

Also

$$\begin{aligned}
\|\tilde{g}\|_{L^\infty(B_{4R})} &\leq \alpha\big(2\kappa(4R) + \|g\|_{L^\infty(B_{4R})}\big) \\
&\leq 3\alpha\big(2\kappa(4R) - \|g\|_{L^\infty(B_{4R})}\big) \\
&= 3 \inf_{B_{4R}} |\tilde{g}| \\
&\leq 3|B_{4R}|^{-1} \|\tilde{g}\|_{L^1(B_{4R})}.
\end{aligned}$$

Hence by [1], Theorem A.2.13, there exists a positive constant $\tilde{C}_H$ such that

$$\sup_{x \in B_{3R}} \tilde{\varphi}(x) \leq \tilde{C}_H \inf_{x \in B_{3R}} \tilde{\varphi}(x),$$

implying that

(3.42) $$\alpha \sup_{x \in B_{3R}} \varphi(x) \leq \tilde{C}_H \Big(2\kappa(4R) + \inf_{x \in B_{3R}} \alpha\varphi(x)\Big).$$

We next consider the solution of

$$L^u \psi = 0 \qquad \text{in } B_{3R}, \qquad \psi = \varphi \qquad \text{on } \partial B_{3R}.$$

Then

$$L^u(\varphi - \psi) = \alpha\varphi + g \qquad \text{in } B_{3R}.$$

If $\varphi(\hat{x}) = \inf_{x \in B_{3R}} \varphi(x)$, then applying the maximum principle ([1], Theorem A.2.1, [18]) it follows from (3.42) that

(3.43) $$\sup_{x \in B_{3R}} |\varphi - \psi| \leq \hat{C}\big(1 + \alpha\varphi(\hat{x})\big).$$

Again ψ attains its minimum at the boundary ([1], Theorem A.2.3, [18]). Therefore, $\psi - \varphi(\hat{x})$ is a nonnegative function, and hence by the Harnack inequality, there exists a constant $C_H > 0$ such that

$$\psi(x) - \varphi(\hat{x}) \le C_H(\psi(\hat{x}) - \varphi(\hat{x})) \le C_H \hat{C}(1 + \alpha\varphi(\hat{x})) \qquad \forall x \in B_{2R}.$$

Thus, combining the above display with (3.43) we obtain

$$\operatorname*{osc}_{B_{2R}} \varphi \le \sup_{B_{2R}}(\varphi - \psi) + \sup_{B_{2R}} \psi - \varphi(\hat{x}) \le \hat{C}(1 + C_H)(1 + \alpha\varphi(\hat{x})).$$

This completes the proof. □

LEMMA 3.6. *Let V_α^ε be as in Lemma 3.4. Then for any $R > 0$, there exists a constant $k_R > 0$ such that*

$$\operatorname*{osc}_{B_R} V_\alpha^\varepsilon \le k_R \qquad \text{for all } \alpha \in (0, 1] \text{ and } \varepsilon \in [0, 1].$$

PROOF. Recall that μ_{u_0} is the stationary probability distribution for the process under the control $u_0 \in \mathfrak{U}_{\mathrm{SSM}}$ in Lemma 3.1. Since u_0 is sub-optimal for the α-discounted criterion in (3.41), and V_α^ε is nonnegative, then for any ball B_R, using Fubini's theorem, we obtain

$$
\begin{aligned}
\mu_{u_0}(B_R) \inf_{B_R} V_\alpha^\varepsilon &\le \int_{\mathbb{R}^d} V_\alpha^\varepsilon(x)\mu_{u_0}(\mathrm{d}x) \\
&\le \int_{\mathbb{R}^d} \mathbb{E}_x^{u_0}\left[\int_0^\infty e^{-\alpha t} r_\varepsilon(X_t, u_0(X_t))\, \mathrm{d}t\right]\mu_{u_0}(\mathrm{d}x) \\
&= \frac{1}{\alpha}\mu_{u_0}(r_\varepsilon) \\
&\le \frac{1}{\alpha}(\eta + \varepsilon k_0(1 + \eta)),
\end{aligned}
$$

where for the last inequality we used Lemma 3.2 and Theorem 3.1(a).

Therefore, we have the estimate

$$\alpha \inf_{B_R} V_\alpha^\varepsilon \le \frac{\eta + \varepsilon k_0(1 + \eta)}{\mu_{u_0}(B_R)}.$$

The result then follows by Lemma 3.5. □

We continue with the proof of Theorem 3.3.

PROOF OF THEOREM 3.3. Consider the function $\bar{V}_\alpha^\varepsilon := V_\alpha^\varepsilon - V_\alpha^\varepsilon(0)$. In view of Lemma 3.5 and Lemma 3.6, we see that $\bar{V}_\alpha^\varepsilon$ is locally bounded uniformly in $\alpha \in (0, 1]$ and $\varepsilon \in (0, 1]$. Therefore, by standard elliptic theory, $\bar{V}_\alpha^\varepsilon$ and its first- and second-order partial derivatives are uniformly bounded in $L^p(B)$, for any $p > 1$,

in any bounded ball $B \subset \mathbb{R}^d$, that is, for some constant C_B depending on B and p, $\|\bar{V}_\alpha^\varepsilon\|_{\mathcal{W}^{2,p}(B)} \leq C_B$ ([18], Theorem 9.11, page 117). Therefore, we can extract a subsequence along which $\bar{V}_\alpha^\varepsilon$ converges. Then the result follows from Theorems 3.6.6, Lemma 3.6.9 and Theorem 3.6.10 in [1]. The proof of (3.36) follows from Lemma 3.5 and Lemma 3.6. □

REMARK 3.8. In the proof of the following lemma, and elsewhere in the paper, we use the fact that if $\mathcal{U} \subset \mathfrak{U}_{\mathrm{SSM}}$ is a any set of controls such that the corresponding set $\{\mu_v : v \in \mathcal{U}\} \subset \mathcal{M}$ of invariant probability measures is tight then the map $v \mapsto \pi_v$ from the closure of $\mathcal{U}$ to $\mathcal{P}(\mathbb{R}^d \times \mathbb{U})$ is continuous, and so is the map $v \mapsto \mu_v$. In fact, the latter is continuous under the total variation norm topology [1], Lemma 3.2.6. We also recall that $\mathcal{G}$ and $\mathcal{M}$ are closed and convex subsets of $\mathcal{P}(\mathbb{R}^d \times \mathbb{U})$ and $\mathcal{P}(\mathbb{R}^d)$. Therefore, $\{\pi_v : v \in \bar{\mathcal{U}}\}$ is compact in $\mathcal{G}$. Note also that since $\mathbb{U}$ is compact, tightness of a set of invariant probability measures is equivalent to tightness of the corresponding set of ergodic occupation measures.

LEMMA 3.7. *If $\{v_\varepsilon : \varepsilon \in (0, 1]\}$ is a collection of measurable selectors from the minimizer of* (3.33), *then the corresponding invariant probability measures $\{\mu_\varepsilon : \varepsilon \in (0, 1]\}$ are tight. Moreover, if $v_{\varepsilon_n} \to v_*$ along some subsequence $\varepsilon_n \searrow 0$, then the following hold*:

(a) $\mu_{\varepsilon_n} \to \mu_{v_*}$ *as $\varepsilon_n \searrow 0$*,
(b) v_* *is a stable Markov control*,
(c) $\int_{\mathbb{R}^d} r(x, v_*(x))\mu_{v_*}(\mathrm{d}x) = \lim_{\varepsilon \searrow 0} \varrho_\varepsilon = \varrho_*$.

PROOF. By (3.25) and (3.31), the set of ergodic occupation measures corresponding to $\{v_\varepsilon : \varepsilon \in (0, 1]\}$ is tight. By Remark 3.8, the same applies to the set $\{\mu_\varepsilon : \varepsilon \in (0, 1]\}$, and also part (a) holds. Part (b) follows from the equivalence of the existence of an invariant probability measure for a controlled diffusion and the stability of the associated stationary Markov control (see [1], Theorem 2.6.10). Part (c) then follows since equality holds in (3.32). □

We continue with the following lemma that asserts the continuity of the mean hitting time of a ball with respect to the stable Markov controls.

LEMMA 3.8. *Let $\{v_n : n \in \mathbb{N}\} \subset \mathfrak{U}_{\mathrm{SM}}^\beta$, for some $\beta > 0$, be a collection of Markov controls such that $v_n \to \hat{v}$ in the topology of Markov controls as $n \to \infty$. Let μ_n, $\hat{\mu}$ be the invariant probability measures corresponding to the controls v_n, $\hat{v}$, respectively. Then for any $\delta > 0$, it holds that*

$$\mathbb{E}_x^{v_n}[\breve{\tau}_\delta] \xrightarrow[n \to \infty]{} \mathbb{E}_x^{\hat{v}}[\breve{\tau}_\delta] \qquad \forall x \in B_\delta^c.$$

PROOF. Define $H(x) := \min_{u \in \mathbb{U}} \tilde{h}(x, u)$. It is easy to see that H is inf-compact and locally Lipschitz. Therefore, by Theorem 3.1(d) we have

$$\sup_{n \in \mathbb{N}} \mu_n(H) \leq k_0(1 + \beta),$$

and since $\mu_n \to \hat{\mu}$, we also have $\hat{\mu}(H) \leq k_0(1 + \beta)$. Then by [1], Lemma 3.3.4, we obtain

$$(3.44) \qquad \sup_{n \in \mathbb{N}} \mathbb{E}_x^{v_n}\left[\int_0^{\breve{\tau}_\delta} H(X_s) \, ds\right] + \mathbb{E}_x^{\hat{v}}\left[\int_0^{\breve{\tau}_\delta} H(X_s) \, ds\right] < \infty.$$

Let R be a positive number greater than $|x|$. Then by (3.44), there exists a positive k such that

$$\mathbb{E}_x^v\left[\int_0^{\breve{\tau}_\delta} \mathbb{I}_{\{H > R\}}(X_s) \, ds\right] \leq \frac{1}{R}\mathbb{E}_x^v\left[\int_0^{\breve{\tau}_\delta} H(X_s)\mathbb{I}_{\{H > R\}}(X_s) \, ds\right] \leq \frac{k}{R}$$

for $v \in \{\{v_n\}, \hat{v}\}$. From this assertion and (3.44), we see that

$$\sup_{v \in \{\{v_n\}, \hat{v}\}} \mathbb{E}_x^v\left[\int_0^{\breve{\tau}_\delta} \mathbb{I}_{\{H > R\}}(X_s) \, ds\right] \underset{R \to \infty}{\longrightarrow} 0.$$

Therefore, in order to prove the lemma it is enough to show that, for any $R > 0$, we have

$$\mathbb{E}_x^{v_n}\left[\int_0^{\breve{\tau}_\delta} \mathbb{I}_{\{H \leq R\}}(X_s) \, ds\right] \underset{n \to \infty}{\longrightarrow} \mathbb{E}_x^{\hat{v}}\left[\int_0^{\breve{\tau}_\delta} \mathbb{I}_{\{H \leq R\}}(X_s) \, ds\right].$$

But this follows from [1], Lemma 2.6.13(iii). □

LEMMA 3.9. *Let $(V^\varepsilon, \varrho_\varepsilon)$ be as in Theorem 3.3, and v_ε satisfy (3.35). There exists a subsequence $\varepsilon_n \searrow 0$, such that V^{ε_n} converges to some $V_* \in \mathcal{C}^2(\mathbb{R}^d)$, uniformly on compact sets, and V_* satisfies*

$$(3.45) \qquad \min_{u \in \mathbb{U}}[L^u V_*(x) + r(x, u)] = \varrho_*.$$

Also, any limit point v_* (in the topology of Markov controls) of the set $\{v_\varepsilon\}$, as $\varepsilon \searrow 0$, satisfies

$$(3.46) \qquad L^{v_*} V_*(x) + r(x, v_*(x)) = \varrho_* \qquad \text{a.e. in } \mathbb{R}^d.$$

Moreover, V_* admits the stochastic representation

$$
\begin{aligned}
V_*(x) &= \inf_{v \in \bigcup_{\beta > 0} \mathfrak{U}_{\mathrm{SM}}^\beta} \mathbb{E}_x^v\left[\int_0^{\breve{\tau}_\delta} \left(r(X_s, v(X_s)) - \varrho_*\right) ds + V_*(X_{\breve{\tau}_\delta})\right] \\
(3.47) \qquad & = \mathbb{E}_x^{v_*}\left[\int_0^{\breve{\tau}_\delta} \left(r(X_s, v_*(X_s)) - \varrho_*\right) ds + V_*(X_{\breve{\tau}_\delta})\right].
\end{aligned}
$$

It follows that V_* is the unique limit point of V^ε as $\varepsilon \searrow 0$.

PROOF. From (3.36), we see that the family $\{V^\varepsilon : \varepsilon \in (0, 1]\}$ is uniformly locally bounded. Hence, applying the theory of elliptic PDE, it follows that $\{V^\varepsilon : \varepsilon \in (0, 1]\}$ is uniformly bounded in $\mathcal{W}^{2,p}_{\mathrm{loc}}(\mathbb{R}^d)$ for $p > d$. Consequently, $\{V^\varepsilon : \varepsilon \in (0, 1]\}$ is uniformly bounded in $\mathcal{C}^{1,\gamma}_{\mathrm{loc}}$ for some $\gamma > 0$. Therefore, along some subsequence $\varepsilon_n \searrow 0$, $V^{\varepsilon_n} \to V_* \in \mathcal{W}^{2,p} \cap \mathcal{C}^{1,\gamma}$, as $n \to \infty$, uniformly on compact sets. Also, $\lim_{\varepsilon \searrow 0} \varrho_\varepsilon = \varrho_*$ by Lemma 3.6(c). Therefore, passing to the limit we obtain the HJB equation in (3.45). It is straightforward to verify that (3.46) holds [1], Lemma 2.4.3.

By Theorem 3.3(c), taking limits as $\varepsilon \searrow 0$, we obtain

$$(3.48) \quad V_*(x) \le \inf_{v \in \bigcup_{\beta > 0} \mathfrak{U}^\beta_{\mathrm{SM}}} \mathbb{E}^v_x \left[\int_0^{\breve{\tau}_\delta} \left(r(X_s, v(X_s)) - \varrho_* \right) \mathrm{d}s + V_*(X_{\breve{\tau}_\delta}) \right].$$

Also by Theorem 3.3(b) we have the bound

$$V^\varepsilon(x) \ge -\varrho_\varepsilon \mathbb{E}^{v_\varepsilon}_x [\breve{\tau}_\delta] + \inf_{B_\delta} V^\varepsilon.$$

Using Lemma 3.8 and taking limits as $\varepsilon_n \searrow 0$, we obtain the lower bound

$$(3.49) \quad V_*(x) \ge -\varrho_* \mathbb{E}^{v_*}_x [\breve{\tau}_\delta] + \inf_{B_\delta} V_*.$$

By Lemma 3.7(c) and Theorem 3.1(d), $v_* \in \mathfrak{U}_{\mathrm{SSM}}$, and $\pi_{v_*}(\tilde{h}) \le k_0(1 + \varrho_*)$. Define

$$(3.50) \quad \varphi(x) := \mathbb{E}^{v_*}_x \left[\int_0^{\breve{\tau}_\delta} \tilde{h}(X_s, v_*(X_s)) \, \mathrm{d}s \right].$$

For $|x| > \delta$, we have

$$\mathbb{E}^{v_*}_x \left[\mathbb{I}_{\{\tau_R < \breve{\tau}_\delta\}} \varphi(X_{\tau_R}) \right] = \mathbb{E}^{v_*}_x \left[\mathbb{I}_{\{\tau_R < \breve{\tau}_\delta\}} \int_{\tau_R \wedge \breve{\tau}_\delta}^{\breve{\tau}_\delta} \tilde{h}(X_s, v_*(X_s)) \, \mathrm{d}s \right].$$

Therefore, by the dominated convergence theorem and the fact that $\varphi(x) < \infty$ we obtain

$$\mathbb{E}^{v_*}_x \left[\varphi(X_{\tau_R}) \mathbb{I}_{\{\tau_R < \breve{\tau}_\delta\}} \right] \underset{R \nearrow \infty}{\longrightarrow} 0.$$

By (3.48) and (3.49), we have $|V_*| \in \mathcal{O}(\varphi)$. Thus (3.49) and (3.50) imply that

$$\liminf_{R \nearrow \infty} \mathbb{E}^{v_*}_x \left[V_*(X_{\tau_R}) \mathbb{I}_{\{\tau_R < \breve{\tau}_\delta\}} \right] = 0,$$

and thus

$$(3.51) \quad \liminf_{R \nearrow \infty} \mathbb{E}^{v_*}_x \left[V_*(X_{\tau_R \wedge \breve{\tau}_\delta}) \right] = \mathbb{E}^{v_*}_x \left[V_*(X_{\breve{\tau}_\delta}) \right].$$

Applying Itô's formula to (3.46), we obtain

$$(3.52) \quad V_*(x) = \mathbb{E}^{v_*}_x \left[\int_0^{\breve{\tau}_\delta \wedge \tau_R} \left(r(X_s, v_*(X_s)) - \varrho_* \right) \mathrm{d}s + V_*(X_{\breve{\tau}_\delta \wedge \tau_R}) \right].$$

Taking limits as $R \to \infty$, and using the dominated convergence theorem, we obtain (3.47) from (3.48). $\square$

Recall the definition of $\mathfrak{o}(\cdot)$ from Section 1.3. We need the following lemma.

LEMMA 3.10. *Let V_* be as in Lemma 3.9. It holds that $V_*^- \in \mathfrak{o}(\mathcal{V})$.*

PROOF. Let v_* be as in Lemma 3.9. Applying Itô's formula to (3.24) with $u \equiv v_*$ we obtain

$$
(3.53) \quad
\begin{aligned}
&\mathbb{E}_x^{v_*}\left[\int_0^{\breve{\tau}_\delta} h\big(X_s, v_*(X_s)\big)\mathbb{I}_{\mathcal{H}^c}\big(X_s, v_*(X_s)\big)\,\mathrm{d}s\right] \\
&\qquad \leq \mathbb{E}_x^{v_*}\left[\int_0^{\breve{\tau}_\delta} r\big(X_s, v_*(X_s)\big)\mathbb{I}_{\mathcal{H}}\big(X_s, v_*(X_s)\big)\,\mathrm{d}s\right] + \mathbb{E}_x^{v_*}[\breve{\tau}_\delta] + \mathcal{V}(x).
\end{aligned}
$$

Therefore, adding the term

$$
\mathbb{E}_x^{v_*}\left[\int_0^{\breve{\tau}_\delta} r\big(X_s, v_*(X_s)\big)\mathbb{I}_{\mathcal{H}}\big(X_s, v_*(X_s)\big)\,\mathrm{d}s\right] - (1 + 2\varrho_*)\mathbb{E}_x^{v_*}[\breve{\tau}_\delta]
$$

to both sides of (3.53) and using the stochastic representation of V_* we obtain

$$
(3.54) \quad
\begin{aligned}
F(x) &:= 2k_0^{-1}\mathbb{E}_x^{v_*}\left[\int_0^{\breve{\tau}_\delta} \tilde{h}\big(X_s, v_*(X_s)\big)\,\mathrm{d}s\right] - 2(1 + \varrho_*)\mathbb{E}_x^{v_*}[\breve{\tau}_\delta] \\
&\leq 2V_*(x) + \mathcal{V}(x) - 2\inf_{B_\delta} V_*.
\end{aligned}
$$

From the stochastic representation of V_* we have $V_*^-(x) \leq \varrho_*\mathbb{E}_x^{v_*}[\breve{\tau}_\delta] - \inf_{B_\delta} V_*$. For any $R > \delta$, we have

$$
(3.55) \quad \mathbb{E}_x^{v_*}\left[\int_0^{\breve{\tau}_\delta} \tilde{h}\big(X_s, v_*(X_s)\big)\,\mathrm{d}s\right] \geq \Big(\inf_{B_R^c \times \mathbb{U}} \tilde{h}\Big)\mathbb{E}_x[\breve{\tau}_R] \qquad \forall x \in B_R^c.
$$

It is also straightforward to show that $\lim_{|x| \to \infty} \frac{\mathbb{E}_x[\breve{\tau}_R]}{\mathbb{E}_x[\breve{\tau}_\delta]} = 1$. Therefore, since $\tilde{h}$ is inf-compact, it follows by (3.54) and (3.55) that the map $x \mapsto \mathbb{E}_x^{v_*}[\breve{\tau}_\delta]$ is in $\mathfrak{o}(F)$, which implies that $V_*^- \in \mathfrak{o}(F)$. On the other hand, by (3.54) we obtain $F(x) \leq \mathcal{V}(x) - 2\sup_{B_\delta} V_*$ for all x such that $V_*(x) \leq 0$, which implies that the restriction of F to the support of V_*^- is in $\mathcal{O}(\mathcal{V})$. It follows that $V_*^- \in \mathfrak{o}(\mathcal{V})$. $\square$

We next prove Theorem 3.4.

PROOF OF THEOREM 3.4. Part (a) is contained in Lemma 3.9.
To prove part (b), let $\bar{v}$ be any control satisfying (3.38). By Lemma 3.10 the map $\mathcal{V} + 2V_*$ is inf-compact and by Theorem 3.4 and (3.24) it satisfies

$$
\begin{aligned}
L^{\bar{v}}(\mathcal{V} + 2V_*)(x) &\leq 1 + 2\varrho_* - r\big(x, \bar{v}(x)\big) - h\big(x, \bar{v}(x)\big)\mathbb{I}_{\mathcal{H}^c}\big(x, \bar{v}(x)\big) \\
&\leq 2 + 2\varrho_* - 2k_0^{-1}\tilde{h}\big(x, \bar{v}(x)\big) \qquad \forall x \in \mathbb{R}^d.
\end{aligned}
$$

This implies that $\bar{v} \in \mathfrak{U}_{\mathrm{SSM}}$. Applying Itô's formula, we obtain

$$(3.56) \qquad \limsup_{T \to \infty} \frac{1}{T} \mathbb{E}_x^{\bar{v}} \left[\int_0^T \tilde{h}(X_s, \bar{v}(X_s)) \, \mathrm{d}s \right] \leq k_0 (1 + \varrho_*).$$

Therefore, $\pi_{\bar{v}}(\tilde{h}) < \infty$. By (3.24), we have

$$\mathbb{E}_x^{\bar{v}} [\mathcal{V}(X_t)] \leq \mathcal{V}(x) + t + \mathbb{E}_x^{\bar{v}} \left[\int_0^t r(X_s, \bar{v}(X_s)) \, \mathrm{d}s \right],$$

and since $r \leq \tilde{h}$, this implies by (3.56) that

$$(3.57) \qquad \limsup_{T \to \infty} \frac{1}{T} \mathbb{E}_x^{\bar{v}} [\mathcal{V}(X_T)] \leq 1 + k_0 (1 + \varrho_*).$$

Since $V_*^- \in \mathfrak{o}(\mathcal{V})$, it follows by (3.57) that

$$\limsup_{T \to \infty} \frac{1}{T} \mathbb{E}_x^{\bar{v}} [V_*^-(X_T)] = 0.$$

Therefore, by Itô's formula, we deduce from (3.37) that

$$(3.58) \qquad \limsup_{T \to \infty} \frac{1}{T} \mathbb{E}_x^{\bar{v}} \left[\int_0^T r(X_s, \bar{v}(X_s)) \, \mathrm{d}s \right] \leq \varrho_*.$$

On the other hand, since the only limit point of the mean empirical measures $\zeta_{x,t}^{\bar{v}}$, as $t \to \infty$, is $\pi_{\bar{v}}$, and $\pi_{\bar{v}}(r) = \varrho_*$, then in view of Remark 3.6, we obtain $\liminf_{t \to \infty} \zeta_{x,t}^{\bar{v}}(r) \geq \varrho_*$. This proves that equality holds in (3.58) and that the "lim sup" may be replaced with "lim."

Conversely, suppose $v \in \mathfrak{U}_{\mathrm{SM}}$ is optimal but does not satisfy (3.38). Then there exists $R > 0$ and a nontrivial nonnegative $f \in L^\infty(B_R)$ such that

$$f_\varepsilon(x) := \mathbb{I}_{B_R}(x) \left(L^v V^\varepsilon(x) + r_\varepsilon(x, v(x)) - \varrho_\varepsilon \right)$$

converges to f, weakly in $L^1(B_R)$, along some subsequence $\varepsilon \searrow 0$. By applying Itô's formula to (3.33), we obtain

$$(3.59) \qquad \begin{aligned} &\frac{1}{T} \left(\mathbb{E}_x^v [V^\varepsilon(X_{T \wedge \tau_R})] - V^\varepsilon(x) \right) + \frac{1}{T} \mathbb{E}_x^v \left[\int_0^{T \wedge \tau_R} r_\varepsilon(X_s, v(X_s)) \, \mathrm{d}s \right] \\ &\qquad \geq \varrho_\varepsilon + \frac{1}{T} \mathbb{E}_x^v \left[\int_0^{T \wedge \tau_R} f_\varepsilon(X_s, v(X_s)) \, \mathrm{d}s \right]. \end{aligned}$$

Define, for some $\delta > 0$,

$$G(x) := \mathbb{E}_x^v \left[\int_0^{\breve{\tau}_\delta} r_\varepsilon(X_s, v(X_s)) \, \mathrm{d}s \right].$$

Since V^ε is bounded from below, by Theorem 3.3(c) we have $V^\varepsilon \in \mathcal{O}(G)$. Invoking [1], Corollary 3.7.3, we obtain

$$\lim_{T \to \infty} \frac{1}{T} \mathbb{E}_x^v [V^\varepsilon(X_T)] = 0,$$

and

$$\lim_{R \to \infty} \mathbb{E}_x^v \big[V^\varepsilon (X_{T \wedge \tau_R}) \big] = \mathbb{E}_x^v \big[V^\varepsilon (X_T) \big].$$

Therefore, taking limits in (3.59), first as $R \nearrow \infty$, and then as $T \to \infty$, we obtain

(3.60) $$\pi_v(r_\varepsilon) \geq \varrho_\varepsilon + \pi_v(f_\varepsilon).$$

Taking limits as $\varepsilon \searrow 0$ in (3.60), since μ_v has a strictly positive density in B_R, we obtain

$$\pi_v(r) \geq \varrho_* + \pi_v(f) > \varrho_*,$$

which is a contradiction. This completes the proof of part (b).

The first equality (3.39) follows by Lemma 3.9, taking limits as $\delta \searrow 0$. To show that the second equality holds for any optimal control, suppose $\bar{v}$ satisfies (3.38). By (3.24) we have, for $\delta > 0$ and $|x| > \delta$,

$$\mathbb{E}_x^{\bar{v}} \big[\mathcal{V}(X_{\tau_R}) \mathbb{I}_{\{\tau_R < \check{\tau}_\delta\}} \big] \leq \mathcal{V}(x) + \sup_{B_\delta} \mathcal{V}^- + \mathbb{E}_x^{\bar{v}} \bigg[\int_0^{\tau_R \wedge \check{\tau}_\delta} \big(1 + r(X_s, \bar{v}(X_s)) \big) \, \mathrm{d}s \bigg].$$

It follows that (see [1], Lemma 3.3.4)

$$\limsup_{R \to \infty} \mathbb{E}_x^{\bar{v}} \big[\mathcal{V}(X_{\tau_R}) \mathbb{I}_{\{\tau_R < \check{\tau}_\delta\}} \big] < \infty,$$

and since $V_*^- \in \mathfrak{o}(\mathcal{V})$ we must have

$$\limsup_{R \to \infty} \mathbb{E}_x^{\bar{v}} \big[V_*^- (X_{\tau_R}) \mathbb{I}_{\{\tau_R < \check{\tau}_\delta\}} \big] = 0.$$

By the first equality in (3.47), we obtain $V_*^+ \in \mathcal{O}(\varphi)$, with φ as defined in (3.50) with v_* replaced by $\bar{v}$. Thus, in analogy to (3.51), we obtain

$$\liminf_{R \nearrow \infty} \mathbb{E}_x^{\bar{v}} \big[V_*(X_{\tau_R \wedge \check{\tau}_\delta}) \big] = \mathbb{E}_x^{\bar{v}} \big[V_*(X_{\check{\tau}_\delta}) \big].$$

The rest follows as in the proof of Lemma 3.9 via (3.52).

We next prove part (d). We assume that $\mathbb{U}$ is a convex set and that

$$c(x, u, p) := \big\{ b(x, u) \cdot p + r(x, u) \big\}$$

is strictly convex in u if it is not identically a constant for fixed x and p. We fix some point $\bar{u} \in \mathbb{U}$. Define

$$\mathcal{B} := \big\{ x \in \mathbb{R}^d : c(x, \cdot, p) = c(x, \bar{u}, p) \text{ for all } p \big\}.$$

It is easy to see that on $\mathcal{B}$ both b and r do not depend on u. It is also easy to check that $\mathcal{B}$ is a closed set. Let (V_*, v_*) be the limit of $(V^\varepsilon, v_\varepsilon)$, where V_* is the solution to (3.37) and v_* is the corresponding limit of v_ε. We have already shown that v_* is a stable Markov control. We next show that it is, in fact, a precise Markov control. By our assumption, v_ε is the unique minimizing selector in (3.34)

and, moreover, v_ε is continuous in x. By the definition of r_ε it is clear that the restriction of v_ε to $\mathcal{B}$ does not depend on ε. Let $v_\varepsilon(x) = v'(x)$ on $\mathcal{B}$. Using the strict convexity property of $c(x, \cdot, \nabla V_*)$ it is easy to verify that v_ε converges to the unique minimizer of (3.37) on $\mathcal{B}^c$. In fact, since $\mathcal{B}^c$ is open, then for any sequence $x^\varepsilon \to x \in \mathcal{B}^c$ it holds that $v_\varepsilon(x^\varepsilon) \to v_*(x)$. This follows from the definition of the minimizer and the uniform convergence of ∇V^ε to ∇V_*. Therefore, we see that v_* is a precise Markov control, $v_* = v'$ on $\mathcal{B}$, and $v_\varepsilon \to v_*$ pointwise as $\varepsilon \to 0$. It is also easy to check that pointwise convergence implies convergence in the topology of Markov controls. $\quad\square$

We now embark on the proof of Theorem 3.5.

PROOF OF THEOREM 3.5. The hypothesis that $\hat{V}^- \in \mathfrak{o}(\mathcal{V})$ implies that the map $\mathcal{V} + 2\hat{V}$ is inf-compact. Also by (3.24) and (3.40), it satisfies

$$L^{\hat{v}}(\mathcal{V} + 2\hat{V})(x) \le 1 + 2\hat\varrho - r(x, \hat{v}(x)) - h(x, \hat{v}(x))\mathbb{I}_{\mathcal{H}^c}(x, \hat{v}(x))$$
$$\le 2 + 2\hat\varrho - 2k_0^{-1}\tilde{h}(x, \hat{v}(x)) \qquad \forall x \in \mathbb{R}^d.$$

Therefore, $\int \tilde{h}(x, \hat{v}(x))\,d\pi_{\hat{v}} < \infty$ from which it follows that $\varrho_{\hat{v}} < \infty$. This proves part (a).

By (3.24), we have

$$\mathbb{E}_x^{\hat{v}}[\mathcal{V}(X_t)] \le \mathcal{V}(x) + t + \mathbb{E}_x^{\hat{v}}\left[\int_0^t r(X_s, \hat{v}(X_s))\,ds\right],$$

and since $\varrho_{\hat{v}} < \infty$, this implies that

(3.61) $$\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}_x^{\hat{v}}[\mathcal{V}(X_T)] \le 1 + \varrho_{\hat{v}}.$$

Since $\hat{V}^- \in \mathfrak{o}(\mathcal{V})$, it follows by (3.61) that

$$\limsup_{T\to\infty} \frac{1}{T}\mathbb{E}_x^{\hat{v}}[\hat{V}^-(X_T)] = 0.$$

Therefore, by Itô's formula, we deduce from (3.40) that

$$\limsup_{T\to\infty}\left(\frac{1}{T}\mathbb{E}_x^{\hat{v}}[\hat{V}^+(X_T)] + \frac{1}{T}\mathbb{E}_x^{\hat{v}}\left[\int_0^T r(X_s, \hat{v}(X_s))\,ds\right]\right) = \hat\varrho.$$

This implies that $\varrho_{\hat{v}} \le \hat\varrho$ and since by hypothesis $\hat\varrho \le \varrho_*$ we must have $\hat\varrho = \varrho_*$.

Again by (3.24), we have

$$\mathbb{E}_x^{\hat{v}}[\mathcal{V}(X_{\tau_R})\mathbb{I}_{\{\tau_R < \check{\tau}_\delta\}}] \le \mathcal{V}(x) + \sup_{B_\delta}\mathcal{V}^- + \mathbb{E}_x^{\hat{v}}\left[\int_0^{\tau_R \wedge \check{\tau}_\delta}(1 + r(X_s, \hat{v}(X_s)))\,ds\right].$$

It follows by [1], Lemma 3.3.4, that

$$\limsup_{R\to\infty}\mathbb{E}_x^{\hat{v}}[\mathcal{V}(X_{\tau_R})\mathbb{I}_{\{\tau_R < \check{\tau}_\delta\}}] < \infty,$$

and since $\hat{V}^- \in \mathfrak{o}(\mathcal{V})$ we must have

(3.62)
$$\limsup_{R \to \infty} \mathbb{E}_x^{\hat{v}}\big[\hat{V}^-(X_{\tau_R})\mathbb{I}_{\{\tau_R < \check{\tau}_\delta\}}\big] = 0.$$

Using (3.62) and following the steps in the proof of the second equality in (3.47), we obtain

$$\hat{V}(x) \geq \mathbb{E}_x^{\hat{v}}\left[\int_0^{\check{\tau}_\delta}\big(r(X_s, \hat{v}(X_s)) - \varrho_*\big)\,\mathrm{d}s\right] + \inf_{B_\delta} \hat{V}$$

$$\geq V_*(x) - \sup_{B_\delta} V_* + \inf_{B_\delta} \hat{V}.$$

Taking limits as $\delta \searrow 0$, we have $V_* \leq \hat{V}$. Since $L^{\hat{v}}(V_* - \hat{V}) \geq 0$ and $V_*(0) = \hat{V}(0)$, we must have $\hat{V} = V_*$ on $\mathbb{R}^d$, and the proof of part (b) is complete.

To prove part (c) note that by part (a) we have $\varrho_{\hat{v}} < \infty$. Therefore, $\int \tilde{h}\,\mathrm{d}\pi_{\hat{v}} \leq \infty$ by Theorem 3.1(a), which implies that $\int |\hat{V}|\,\mathrm{d}\mu_{\hat{v}} \leq \infty$ by the hypothesis. Therefore, $\mathbb{E}_x^{\hat{v}}(|\hat{V}(X_t)|)$ converges as $t \to \infty$ by [23], Proposition 2.6, which of course implies that $\frac{1}{t}\mathbb{E}_x^{\hat{v}}(|\hat{V}(X_t)|)$ tends to 0 as $t \to \infty$. Similarly, we deduce that $\frac{1}{t}\mathbb{E}_x^{v_*}(|\hat{V}(X_t)|)$ as $t \to \infty$. Applying Itô's formula to (3.40), with $u \equiv v_*$, we obtain $\hat{\varrho} \leq \varrho_*$. Another application with $u \equiv \hat{v}$ results in $\hat{\varrho} = \varrho_{\hat{v}}$. Therefore, $\hat{\varrho} = \varrho_*$. The result then follows by part (b). $\quad\square$

We finish this section with the proof of Theorem 3.6.

PROOF OF THEOREM 3.6.   We first show that $\lim_{\alpha \searrow 0}\alpha V_\alpha(0) = \varrho_*$. Let $\tilde{\mathcal{V}}(t, x) := \mathrm{e}^{-\alpha t}\mathcal{V}(x)$, and $\tau_n(t) := \tau_n \wedge t$. Applying Itô's formula to (3.24), we obtain

$$\mathbb{E}_x^U\big[\tilde{\mathcal{V}}\big(\tau_n(t), X_{\tau_n(t)}\big)\big] \leq \mathcal{V}(x) - \mathbb{E}_x^U\left[\int_0^{\tau_n(t)} \alpha\tilde{\mathcal{V}}(s, X_s)\,\mathrm{d}s\right]$$

$$+ \mathbb{E}_x^U\left[\int_0^{\tau_n(t)} \mathrm{e}^{-\alpha s}\big(1 - h(X_s, U_s)\big)\mathbb{I}_{\mathcal{H}^c}(X_s, U_s)\,\mathrm{d}s\right]$$

$$+ \mathbb{E}_x^U\left[\int_0^{\tau_n(t)} \mathrm{e}^{-\alpha s}\big(1 + r(X_s, U_s)\big)\mathbb{I}_{\mathcal{H}}(X_s, U_s)\,\mathrm{d}s\right].$$

It follows that

(3.63)
$$\mathbb{E}_x^U\left[\int_0^{\tau_n(t)} \mathrm{e}^{-\alpha s}h(X_s, U_s)\mathbb{I}_{\mathcal{H}^c}(X_s, U_s)\,\mathrm{d}s\right]$$
$$\leq \frac{1}{\alpha} + \mathcal{V}(x) + \mathbb{E}_x^U\left[\int_0^{\tau_n(t)} \mathrm{e}^{-\alpha s}r(X_s, U_s)\mathbb{I}_{\mathcal{H}}(X_s, U_s)\,\mathrm{d}s\right].$$

Taking limits first as $n \nearrow \infty$ and then as $t \nearrow \infty$ in (3.63), and evaluating U at an optimal α-discounted control v_α^*, relative to r we obtain the estimate, using

also (3.23),

$$(3.64) \quad 2k_0^{-1} \mathbb{E}_x^{v_\alpha^*} \left[\int_0^\infty e^{-\alpha s} \tilde{h}(X_s, v_\alpha^*(X_s)) \, ds \right] \le \frac{2}{\alpha} + \mathcal{V}(x) + 2V_\alpha(x).$$

By (3.23) and (3.64), it follows that

$$V_\alpha(x) \le V_\alpha^\varepsilon(x) \le \mathbb{E}_x^{v_\alpha^*} \left[\int_0^\infty e^{-\alpha s} r_\varepsilon(X_s, v_\alpha^*(X_s)) \, ds \right]$$

$$\le V_\alpha(x) + \varepsilon k_0 (\alpha^{-1} + \mathcal{V}(x) + V_\alpha(x)).$$

Multiplying by α and taking limits as $\alpha \searrow 0$ we obtain

$$\limsup_{\alpha \searrow 0} \alpha V_\alpha(0) \le \varrho_\varepsilon \le (1 + \varepsilon k_0) \limsup_{\alpha \searrow 0} \alpha V_\alpha(0) + \varepsilon k_0.$$

The same inequalities hold for the "lim inf." Therefore, $\lim_{\alpha \searrow 0} \alpha V_\alpha(0) = \varrho_*$.

Let

$$\tilde{V} := \lim_{\alpha \searrow 0} (V_\alpha - V_\alpha(0)).$$

(Note that a similar result as Lemma 3.5 holds.) Then $\tilde{V}$ satisfies

$$\tilde{V}(x) \le \lim_{\delta \searrow 0} \mathbb{E}_x^v \left[\int_0^{\breve{\tau}_\delta} (r(X_s, v(X_s)) - \varrho_*) \, ds \right] \qquad \forall v \in \bigcup_{\beta > 0} \mathfrak{U}_{\mathrm{SM}}^\beta.$$

This can be obtained without the near-monotone assumption on the running cost; see, for example, [1], Lemma 3.6.9 or Lemma 3.7.8. It follows from (3.39) that $\tilde{V} \le V_*$. On the other hand, since $L^{v_*}(\tilde{V} - V_*) \ge 0$, and $\tilde{V}(0) = V_*(0)$, we must have $\tilde{V} = V_*$ by the strong maximum principle. $\square$

4. Approximation via spatial truncations. We introduce an approximation technique which is in turn used to prove the asymptotic convergence results in Section 5.

Let $v_0 \in \mathfrak{U}_{\mathrm{SSM}}$ be any control such that $\pi_{v_0}(r) < \infty$. We fix the control v_0 on the complement of the ball $\bar{B}_l$ and leave the parameter u free inside. In other words, for each $l \in \mathbb{N}$ we define

$$b_l(x, u) := \begin{cases} b(x, u), & \text{if } (x, u) \in \bar{B}_l \times \mathbb{U}, \\ b(x, v_0(x)), & \text{otherwise}, \end{cases}$$

$$r_l(x, u) := \begin{cases} r(x, u), & \text{if } (x, u) \in \bar{B}_l \times \mathbb{U}, \\ r(x, v_0(x)), & \text{otherwise}. \end{cases}$$

We consider the family of controlled diffusions, parameterized by $l \in \mathbb{N}$, given by

$$(4.1) \qquad dX_t = b_l(X_t, U_t) \, dt + \sigma(X_t) \, dW_t,$$

with associated running costs $r_l(x, u)$. We denote by $\mathfrak{U}_{SM}(l, v_0)$ the subset of $\mathfrak{U}_{SM}$ consisting of those controls v which agree with v_0 on $\bar{B}_l^c$. Let $\eta_0 := \pi_{v_0}(r)$. It is well known that there exists a nonnegative solution $\varphi_0 \in \mathcal{W}_{loc}^{2,p}(\mathbb{R}^d)$, for any $p > d$, to the Poisson equation (see [1], Lemma 3.7.8(ii))

$$L^{v_0}\varphi_0(x) = \eta_0 - \tilde{h}(x, v_0(x)) \qquad x \in \mathbb{R}^d,$$

which is inf-compact, and satisfies, for all $\delta > 0$,

$$\varphi_0(x) = \mathbb{E}_x^{v_0}\left[\int_0^{\breve{\tau}_\delta} (\tilde{h}(X_s, v_0(X_s)) - \eta_0)\, ds + \varphi_0(X_{\breve{\tau}_\delta})\right] \qquad \forall x \in \mathbb{R}^d.$$

We recall the Lyapunov function $\mathcal{V}$ from Assumption 3.1. We have the following theorem.

THEOREM 4.1. *Let Assumptions 3.1 and 3.2 hold. Then for each $l \in \mathbb{N}$ there exists a solution V^l in $\mathcal{W}_{loc}^{2,p}(\mathbb{R}^d)$, for any $p > d$, with $V^l(0) = 0$, of the HJB equation*

$$(4.2) \qquad \min_{u \in \mathbb{U}}[L_l^u V^l(x) + r_l(x, u)] = \varrho_l,$$

where L_l^u is the elliptic differential operator corresponding to the diffusion in (4.1). Moreover, the following hold:

  (i) *ϱ_l is nonincreasing in l;*
  (ii) *there exists a constant C_0, independent of l, such that $V^l(x) \leq C_0 + 2\varphi_0(x)$ for all $l \in \mathbb{N}$;*
  (iii) *$(V^l)^- \in \mathfrak{o}(\mathcal{V} + \varphi_0)$ uniformly over $l \in \mathbb{N}$;*
  (iv) *the restriction of V^l on B_l is in $\mathcal{C}^2$.*

PROOF. As earlier, we can show that

$$V_\alpha^l(x) := \inf_{U \in \mathfrak{U}} \mathbb{E}_x^U\left[\int_0^\infty e^{-\alpha s} r_l(X_s, U_s)\, ds\right]$$

is the minimal nonnegative solution to

$$(4.3) \qquad \min_{u \in \mathbb{U}}[L_l^u V_\alpha^l(x) + r_l(x, u)] = \alpha V_\alpha^l(x),$$

and $V_\alpha^l \in \mathcal{W}_{loc}^{2,p}(\mathbb{R}^d)$, $p > d$. Moreover, any measurable selector from the minimizer in (4.3) is an optimal control. A similar estimate as in Lemma 3.5 holds and, therefore, there exists a subsequence $\{\alpha_n\}$, along which $V_{\alpha_n}^l(x) - V_{\alpha_n}^l(0)$ converges to V^l in $\mathcal{W}_{loc}^{2,p}(\mathbb{R}^d)$, $p > d$, and $\alpha_n V_{\alpha_n}^l(0) \to \varrho_l$ as $\alpha_n \searrow 0$, and (V^l, ϱ_l) satisfies (4.2) (see also [1], Lemma 3.7.8).

To show that $\pi_{v^l}(r) = \varrho_l$, v^l is a minimizing selector in (4.2), we use the following argument. Since $\pi_{v_0}(r) < \infty$, we claim that there exists a nonnegative, inf-compact function $g \in \mathcal{C}(\mathbb{R}^d)$ such that $\pi_{v_0}(g \cdot (1 + r)) < \infty$. Indeed, this is true

since integrability and uniform integrability of a function under any given measure are equivalent (see also the proof of [1], Lemma 3.7.2). Since every control in $\mathfrak{U}_{\mathrm{SM}}(l, v_0)$ agrees with v_0 on B_l^c, then for any $x_0 \in \bar{B}_l^c$ the map

$$v \mapsto \mathbb{E}_{x_0}^v \left[\int_0^{\check{\tau}_l} g(X_s)\big(1 + r(X_s, v(X_s))\big) \, \mathrm{d}s \right]$$

is constant on $\mathfrak{U}_{\mathrm{SM}}(l, v_0)$. By the equivalence of (i) and (iii) in Lemma 3.3.4 of [1], this implies that

$$\sup_{v \in \mathfrak{U}_{\mathrm{SM}}(l, v_0)} \pi_v\big(g \cdot (1 + r)\big) < \infty \qquad \forall l \in \mathbb{N},$$

and thus r is uniformly integrable with respect to the family $\{\pi_v : v \in \mathfrak{U}_{\mathrm{SM}}(l, v_0)\}$ for any $l \in \mathbb{N}$. It then follows by [1], Theorem 3.7.11, that

$$(4.4) \qquad \varrho_l = \inf_{v \in \mathfrak{U}_{\mathrm{SM}}(l, v_0)} \pi_v(r), \qquad l \in \mathbb{N}.$$

This yields part (i). Moreover, in view of Lemmas 3.5 and 3.6, we deduce that for any $\delta > 0$ it holds that $\sup_{B_\delta} |V^l| \leq \kappa_\delta$, where κ_δ is a constant independent of $l \in \mathbb{N}$. It is also evident by (4.4) that ϱ_l is decreasing in l and $\varrho_l \leq \eta_0$ for all $l \in \mathbb{N}$. Fix δ such that $\min_{u \in \mathbb{U}} \tilde{h}(x, u) \geq 2\eta_0$ on B_δ^c. Since φ_0 is nonnegative, we obtain

$$(4.5) \qquad \mathbb{E}_x^{v_0} \left[\int_0^{\check{\tau}_\delta} \big(\tilde{h}(X_s, v_0(X_s)) - \eta_0\big) \, \mathrm{d}s \right] \leq \varphi_0(x) \qquad \forall x \in \mathbb{R}^d.$$

Using an analogous argument as the one used in the proof of [1], Lemma 3.7.8, we have

$$(4.6) \quad V^l(x) \leq \mathbb{E}_x^v \left[\int_0^{\check{\tau}_\delta} \big(r_l(X_s, v(X_s)) - \varrho_l\big) \, \mathrm{d}s \right] + \kappa_\delta \qquad \forall v \in \mathfrak{U}_{\mathrm{SM}}(l, v_0).$$

Thus, by (4.5) and (4.6), and since by the choice of $\delta > 0$, it holds that $r \leq \tilde{h} \leq 2(\tilde{h} - \eta_0)$ on B_δ^c, we obtain

$$(4.7) \qquad \begin{aligned} V^l(x) &\leq \mathbb{E}_x^{v_0} \left[\int_0^{\check{\tau}_\delta} 2\big(\tilde{h}(X_s, v_0(X_s)) - \eta_0\big) \, \mathrm{d}s \right] + \kappa_\delta \\ &\leq \kappa_\delta + 2\varphi_0(x) \qquad \forall x \in \mathbb{R}^d. \end{aligned}$$

This proves part (ii).

Now fix $l \in \mathbb{N}$. Let v_α^l be a minimizing selector of (4.3). Note then that $v_\alpha^l \in \mathfrak{U}_{\mathrm{SM}}(l, v_0)$. Therefore, v_α^l is a stable Markov control. Let $v_{\alpha_n}^l \to v^l$ in the topology of Markov controls along the same subsequence as above. Then it is evident that $v^l \in \mathfrak{U}_{\mathrm{SM}}(l, v_0)$. Also from Lemma 3.8, we have

$$\mathbb{E}_x^{v_{\alpha_n}^l}[\check{\tau}_\delta] \xrightarrow[\alpha_n \searrow 0]{} \mathbb{E}_x^{v^l}[\check{\tau}_\delta] \qquad \forall x \in B_\delta^c, \forall \delta > 0.$$

Using [1], Lemma 3.7.8, we obtain the lower bound

$$V^l(x) \geq -\varrho_l \mathbb{E}_x^{v^l}[\check{\tau}_\delta] - \kappa_\delta. \tag{4.8}$$

By [1], Theorem 3.7.12(i) (see also (3.7.50) in [1]), it holds that

$$
\begin{aligned}
V^l(x) &= \mathbb{E}_x^{v^l}\left[\int_0^{\check{\tau}_\delta} \left(r_l(X_s, v^l(X_s)) - \varrho_l\right) ds + V^l(X_{\check{\tau}_\delta})\right] \\
&\geq \mathbb{E}_x^{v^l}\left[\int_0^{\check{\tau}_\delta} r_l(X_s, v^l(X_s)) ds\right] - \varrho_l \mathbb{E}_x^{v^l}[\check{\tau}_\delta] - \kappa_\delta \qquad \forall x \in B_l^c.
\end{aligned}
\tag{4.9}
$$

By (3.23), we have

$$2k_0^{-1}\tilde{h}(x, u)\mathbb{I}_{\mathcal{H}}(x, u) \leq 1 + r(x, u)\mathbb{I}_{\mathcal{H}}(x, u).$$

Therefore, using the preceding inequality and (4.9), we obtain

$$
\begin{aligned}
&V^l(x) + (1 + \varrho_l)\mathbb{E}_x^{v^l}[\check{\tau}_\delta] + \kappa_\delta \\
&\qquad \geq \frac{2}{k_0}\mathbb{E}_x^{v^l}\left[\int_0^{\check{\tau}_\delta} \tilde{h}(X_s, v^l(X_s))\mathbb{I}_{\mathcal{H}}(X_s, v^l(X_s)) ds\right].
\end{aligned}
\tag{4.10}
$$

By (3.24), (4.9) and the fact that $\mathcal{V}$ is nonnegative, we have

$$
\begin{aligned}
&\frac{2}{k_0}\mathbb{E}_x^{v^l}\left[\int_0^{\check{\tau}_\delta} \tilde{h}(X_s, v^l(X_s))\mathbb{I}_{\mathcal{H}^c}(X_s, v^l(X_s)) ds\right] - \mathcal{V}(x) - \mathbb{E}_x^{v^l}[\check{\tau}_\delta] \\
&\qquad \leq \mathbb{E}_x^{v^l}\left[\int_0^{\check{\tau}_\delta} r(X_s, v^l(X_s))\mathbb{I}_{\mathcal{H}}(X_s, v^l(X_s)) ds\right] \\
&\qquad \leq V^l(x) + \varrho_l \mathbb{E}_x^{v^l}[\check{\tau}_\delta] + \kappa_\delta.
\end{aligned}
\tag{4.11}
$$

Combining (4.7), (4.10) and (4.11), we obtain

$$
\begin{aligned}
\mathbb{E}_x^{v^l}\left[\int_0^{\check{\tau}_\delta} \tilde{h}(X_s, v^l(X_s)) ds\right] &\leq k_0(1 + \varrho_l)\mathbb{E}_x^{v^l}[\check{\tau}_\delta] \\
&\quad + \frac{k_0}{2}\mathcal{V}(x) + 2k_0\left(\varphi_0(x) + \kappa_\delta\right)
\end{aligned}
$$

for all $l \in \mathbb{N}$. As earlier, using the inf-compact property of $\tilde{h}$ and the fact that $\varrho_l \leq \eta_0$ is bounded, we can choose δ large enough such that

$$\eta_0 \mathbb{E}_x^{v^l}[\check{\tau}_\delta] \leq \mathbb{E}_x^{v^l}\left[\int_0^{\check{\tau}_\delta} \tilde{h}(X_s, v^l(X_s)) ds\right] \leq k_0\mathcal{V}(x) + 4k_0\left(\varphi_0(x) + \kappa_\delta\right) \tag{4.12}$$

for all $l \in \mathbb{N}$. Since $\tilde{h}$ is inf-compact, part (iii) follows by (4.8) and (4.12).

Part (iv) is clear from regularity theory of elliptic PDE [18], Theorem 9.19, page 243. □

Similar to Theorem 3.3, we can show that oscillations of $\{V^l\}$ are uniformly bounded on compacts. Therefore, if we let $l \to \infty$ we obtain a HJB equation

$$\min_{u \in \mathbb{U}}\left[L^u \hat{V}(x) + r(x, u)\right] = \hat{\varrho}, \tag{4.13}$$

with $\hat{V} \in \mathcal{C}^2(\mathbb{R}^d)$ and $\lim_{l \to \infty} \varrho_l = \hat{\varrho}$. By Theorem 4.1, we have the bound

$$\hat{V}(x) \le C_0 + 2\varphi_0(x), \tag{4.14}$$

for some positive constant C_0. This of course, implies that $\hat{V}^+(x) \le C_0 + 2\varphi_0(x)$. Moreover, it is straightforward to show that for any $v \in \mathfrak{U}_{\mathrm{SSM}}$ with $\varrho_v < \infty$, we have

$$\limsup_{t \to \infty} \frac{1}{t}\mathbb{E}_x^v\left[\mathcal{V}(X_t)\right] < \infty.$$

Therefore, if in addition, we have

$$\limsup_{t \to \infty} \frac{1}{t}\mathbb{E}_x^v\left[\varphi_0(X_t)\right] < \infty,$$

then it follows by Theorem 4.1(iii) that

$$\limsup_{t \to \infty} \frac{1}{t}\hat{V}^-(X_t) \xrightarrow[t \to \infty]{} 0. \tag{4.15}$$

THEOREM 4.2. *Suppose that $\varphi_0 \in \mathcal{O}(\min_{u \in \mathbb{U}} \tilde{h}(\cdot, u))$. Then, under the assumptions of Theorem 4.1, we have $\lim_{l \to \infty} \varrho_l = \hat{\varrho} = \varrho_*$, and $\hat{V} = V_*$. Moreover, $V_* \in \mathcal{O}(\varphi_0)$.*

PROOF. Let $\{\hat{v}_l\}$ be any sequence of measurable selectors from the minimizer of (4.2) and $\{\pi_l\}$ the corresponding sequence of ergodic occupation measures. Since by Theorem 3.1 $\{\pi_l\}$ is tight, then by Remark 3.8 if $\hat{v}$ is a limit point of a subsequence $\{\hat{v}_l\}$, which we also denote by $\{\hat{v}_l\}$, then $\hat{\pi} = \pi_{\hat{v}}$ is the corresponding limit point of $\{\pi_l\}$. Therefore, by the lower semi-continuity of $\pi \to \pi(r)$ we have

$$\hat{\varrho} = \lim_{l \to \infty} \pi_l(r) \ge \hat{\pi}(r) = \varrho_{\hat{v}}.$$

It also holds that

$$L^{\hat{v}}\hat{V}(x) + r(x, \hat{v}(x)) = \hat{\varrho}, \qquad \text{a.s.} \tag{4.16}$$

By (4.15), we have

$$\liminf_{T \to \infty} \frac{1}{T}\mathbb{E}_x^{\hat{v}}\left[\hat{V}(X_T)\right] = 0,$$

and hence applying Itô's rule on (4.16) we obtain $\varrho_{\hat{v}} \le \hat{\varrho}$. On the other hand, if v_* is an optimal stationary Markov control, then by the hypothesis $\varphi_0 \in \mathcal{O}(\tilde{h})$,

the fact that $\pi_{v_*}(\tilde{h}) < \infty$, (4.14) and [23], Proposition 2.6, we deduce that $\mathbb{E}_x^{v_*}[\hat{V}^+(X_t)]$ converges as $t \to \infty$, which of course together with (4.15) implies that $\frac{1}{t}\mathbb{E}_x^{\hat{v}}[\hat{V}(X_t)]$ tends to 0 as $t \to \infty$. Therefore, evaluating (4.13) at v_* and applying Itô's rule we obtain $\varrho_{v_*} \geq \hat{\varrho}$. Combining the two estimates, we have $\varrho_{\hat{v}} \leq \hat{\varrho} \leq \varrho_*$, and thus equality must hold. Here, we have used the fact that there exists an optimal Markov control for r by Theorem 3.4.

Next, we use the stochastic representation in (4.9)

$$(4.17) \quad V^l(x) = \mathbb{E}_x^{\hat{v}_l}\left[\int_0^{\check{\tau}_\delta} \left(r(X_s, \hat{v}_l(X_s)) - \varrho_l\right) ds + V^l(X_{\check{\tau}_\delta})\right], \qquad x \in B_\delta^c.$$

Fix any $x \in B_\delta^c$. Since $\mathfrak{U}_{\mathrm{SM}}^{\varrho_{v_0}}$ is compact, it follows that for each δ and R with $0 < \delta < R$, the map $F_{\delta,R}(v): \mathfrak{U}_{\mathrm{SM}}^{\varrho_{v_0}} \to \mathbb{R}_+$ defined by

$$F_{\delta,R}(v) := \mathbb{E}_x^v\left[\int_0^{\check{\tau}_\delta \wedge \tau_R} r(X_s, v(X_s)) ds\right]$$

is continuous. Therefore, the map $\bar{F}_\delta := \lim_{R \nearrow \infty} F_{\delta,R}$ is lower semi-continuous. It follows that

$$(4.18) \qquad \mathbb{E}_x^{\hat{v}}\left[\int_0^{\check{\tau}_\delta} r(X_s, \hat{v}(X_s)) ds\right] \leq \lim_{l \to \infty} \mathbb{E}_x^{\hat{v}_l}\left[\int_0^{\check{\tau}_\delta} r(X_s, \hat{v}_l(X_s)) ds\right].$$

On the other hand, since $\tilde{h}$ is inf-compact, it follows by (4.12) that $\check{\tau}_\delta$ is uniformly integrable with respect to the measures $\{\mathbb{P}_x^{\hat{v}_l}\}$. Therefore, as also shown in Lemma 3.8, we have

$$(4.19) \qquad \lim_{l \to \infty} \mathbb{E}_x^{\hat{v}_l}[\check{\tau}_\delta] = \mathbb{E}_x^{\hat{v}}[\check{\tau}_\delta].$$

Since $V^l \to \hat{V}$, uniformly on compact sets, and $\varrho_l \to \varrho_*$, as $l \to \infty$, it follows by (4.17)–(4.19) that

$$\hat{V}(x) \geq \mathbb{E}_x^{\hat{v}}\left[\int_0^{\check{\tau}_\delta} \left(r(X_s, \hat{v}(X_s)) - \varrho_*\right) ds + \hat{V}(X_{\check{\tau}_\delta})\right], \qquad x \in B_\delta^c.$$

Therefore, by Theorem 3.4(b), for any $\delta > 0$ and $x \in B_\delta^c$ we obtain

$$V_*(x) \leq \mathbb{E}_x^{\hat{v}}\left[\int_0^{\check{\tau}_\delta} \left(r(X_s, \hat{v}(X_s)) - \varrho_*\right) ds + V_*(X_{\check{\tau}_\delta})\right]$$

$$\leq \hat{V}(x) + \mathbb{E}_x^{\hat{v}}[V^*(X_{\check{\tau}_\delta})] - \mathbb{E}_x^{\hat{v}}[\hat{V}(X_{\check{\tau}_\delta})],$$

and taking limits as $\delta \searrow 0$, using the fact that $\hat{V}(0) = V_*(0) = 0$, we obtain $V_* \leq \hat{V}$ on $\mathbb{R}^d$. Since $L^{\hat{v}}(V_* - \hat{V}) \geq 0$, we must have $V_* = \hat{V}$. By Theorem 4.1(ii), we have $V_* \in \mathcal{O}(\varphi_0)$. $\square$

REMARK 4.1. It can be seen from the proof of Theorem 4.2 that the assumption $\varphi_0 \in \mathcal{O}(\tilde{h})$ can be replaced by the weaker hypothesis that $\frac{1}{T}\mathbb{E}_x^{v_*}[\varphi_0(X_T)] \to 0$ as $T \to \infty$.

REMARK 4.2. It is easy to see that if one replaces r_l by

$$r_l(x, u) = \begin{cases} r(x, u) + \dfrac{1}{l} f(u), & \text{for } x \in \bar{B}_l, \\ r\big(x, v_0(x)\big) + \dfrac{1}{l} f\big(v_0(x)\big), & \text{otherwise,} \end{cases}$$

for some positive valued continuous function f, the same conclusion of Theorem 4.2 holds.

If we consider the controlled dynamics given by (3.20), with running cost as in (3.11), then there exists a function $\mathcal{V} \sim |x|^m$ satisfying (3.6). This fact is proved in Proposition 3.1. There also exists a Lyapunov function $\mathcal{V}_0 \in \mathcal{O}(|x|^m)$, satisfying the assumption in Theorem 4.2, relative to any control v_0 with $\pi_{v_0}(\tilde{h}) < \infty$, where $\tilde{h}$ is selected as in Remark 3.5. Indeed, in order to construct $\mathcal{V}_0$ we recall the function ψ in (3.21). Let $\mathcal{V}_0 \in \mathcal{C}^2(\mathbb{R}^d)$ be any function such that $\mathcal{V}_0 = \psi^{m/2}$ on the complement of the unit ball centered at the origin. Observe that for some positive constants κ_1 and κ_2 it holds that

$$\kappa_1 |x|^2 \le \psi(x) \le \kappa_2 |x|^2.$$

Then a straightforward calculation from (3.22) shows that (3.8) holds with the above choice of $\mathcal{V}_0$. By the stochastic representation of φ_0, it follows that $\varphi_0 \in \mathcal{O}(\mathcal{V}_0)$. We have proved the following corollary.

COROLLARY 4.1. *For the queueing diffusion model with controlled dynamics given by* (3.20), *and running cost given by* (3.11), *there exists a solution* (*up to an additive constant*) *to the associated HJB in the class of functions in* $\mathcal{C}^2(\mathbb{R}^d)$ *whose positive part grows no faster than* $|x|^m$ *and whose negative part is in* $\mathfrak{o}(|x|^m)$.

We conclude this section with the following remark.

REMARK 4.3. Comparing the approximation technique introduced in this section with that in Section 3, we see that the spatial truncation technique relies on more restrictive assumption on the Lyapunov function $\mathcal{V}_0$ and the running cost function (Theorem 4.2). In fact, the growth of $\tilde{h}$ also restricts the growth of r by (3.23). Therefore, the class of ergodic diffusion control problems considered in this section is more restrictive. For example, if the running cost r satisfies (3.11) and $\tilde{h} \sim |x|^m$, then it is not obvious that one can obtain a Lyapunov function $\mathcal{V}_0$ with growth at most of order $|x|^m$. For instance, if the drift has strictly sub-linear growth, then it is expected that the Lyapunov function should have growth larger than $|x|^m$. Therefore, the class of problems considered in Section 3 is larger than those considered in this section.

5. Asymptotic convergence. In this section, we prove that the value of the ergodic control problem corresponding to the multi-class $M/M/N + M$ queueing network asymptotically converges to ϱ_*, the value of the ergodic control for the controlled diffusion.

Recall the diffusion-scaled processes $\hat{X}^n$, $\hat{Q}^n$ and $\hat{Z}^n$ defined in (2.4), and from (2.5) and (2.6) that

$$
\begin{aligned}
\hat{X}_i^n(t) = {} & \hat{X}_i^n(0) + \ell_i^n t - \mu_i^n \int_0^t \hat{Z}_i^n(s)\,\mathrm{d}s - \gamma_i^n \int_0^t \hat{Q}_i^n(s)\,\mathrm{d}s \\
& + \hat{M}_{A,i}^n(t) - \hat{M}_{S,i}^n(t) - \hat{M}_{R,i}^n(t),
\end{aligned}
$$
(5.1)

where $\hat{M}_{A,i}^n(t)$, $\hat{M}_{S,i}^n(t)$ and $\hat{M}_{R,i}^n(t)$, $i = 1, \ldots, d$, as defined in (2.6), are square integrable martingales w.r.t. the filtration $\{\mathcal{F}_t^n\}$ with quadratic variations

$$
\langle \hat{M}_{A,i}^n \rangle(t) = \frac{\lambda_i^n}{n} t,
$$

$$
\langle \hat{M}_{S,i}^n \rangle(t) = \frac{\mu_i^n}{n} \int_0^t Z_i^n(s)\,\mathrm{d}s,
$$

$$
\langle \hat{M}_{R,i}^n \rangle(t) = \frac{\gamma_i^n}{n} \int_0^t Q_i^n(s)\,\mathrm{d}s.
$$

5.1. *The lower bound.* In this section, we prove Theorem 2.1.

PROOF OF THEOREM 2.1. Recall the definition of $\hat{V}^n$ in (2.10), and consider a sequence such that $\sup_n \hat{V}^n(\hat{X}^n(0)) < \infty$. Let $\varphi \in \mathcal{C}^2(\mathbb{R}^d)$ be any function satisfying $\varphi(x) := |x|^m$ for $|x| \geq 1$. As defined in Section 1.3, $\Delta X(t)$ denotes the jump of the process X at time t. Applying Itô's formula on φ (see, e.g., [24], Theorem 26.7), we obtain from (5.1) that

$$
\begin{aligned}
\mathbb{E}\big[\varphi\big(\hat{X}_1^n(t)\big)\big] = {} & \mathbb{E}\big[\varphi\big(\hat{X}_1^n(0)\big)\big] + \mathbb{E}\bigg[\int_0^t \Theta_1^n\big(\hat{X}_1^n(s), \hat{Z}_1^n(s)\big)\varphi'\big(\hat{X}_1^n(s)\big)\,\mathrm{d}s\bigg] \\
& + \mathbb{E}\bigg[\int_0^t \Theta_2^n\big(\hat{X}_1^n(s), \hat{Z}_1^n(s)\big)\varphi''\big(\hat{X}_1^n(s)\big)\,\mathrm{d}s\bigg] \\
& + \mathbb{E}\sum_{s \leq t} \bigg(\Delta\varphi\big(\hat{X}_1^n(s)\big) - \varphi'\big(\hat{X}_1^n(s-)\big)\cdot\Delta\hat{X}_1^n(s) \\
& \qquad\qquad - \frac{1}{2}\varphi''\big(\hat{X}^n(s-)\big)\Delta\hat{X}_1^n(s)\Delta\hat{X}_1^n(s)\bigg),
\end{aligned}
$$

where

$$
\Theta_1^n(x, z) := \ell_1^n - \mu_1^n z - \gamma_1^n(x - z),
$$

$$
\Theta_2^n(x, z) := \frac{1}{2}\bigg(\mu_1^n \rho_1 + \frac{\lambda_1^n}{n} + \frac{\mu_1^n z + \gamma_1^n(x - z)}{\sqrt{n}}\bigg).
$$

Since $\{\ell_1^n\}$ is a bounded sequence, it is easy to show that for all n there exist positive constants κ_i, $i = 1, 2$, independent of n, such that

$$\Theta_1^n(x, z)\varphi'(x) \leq \kappa_1(1 + |(e \cdot x)^+|^m) - \kappa_2|x|^m,$$

$$\Theta_2^n(x, z)\varphi''(x) \leq \kappa_1(1 + |(e \cdot x)^+|^m) + \frac{\kappa_2}{4}|x|^m,$$

provided that $x - z \leq (e \cdot x)^+$ and $\frac{z}{\sqrt{n}} \leq 1$. We next compute the terms corresponding to the jumps. For that, first we see that the jump size is of order $\frac{1}{\sqrt{n}}$. We can also find a positive constant κ_3 such that

$$\sup_{|y-x|\leq 1} |\varphi''(y)| \leq \kappa_3(1 + |x|^{m-2}) \qquad \forall x \in \mathbb{R}^d.$$

Using Taylor's approximation, we obtain the inequality

$$\Delta\varphi(\hat{X}_1^n(s)) - \varphi'(\hat{X}_1^n(s-)) \cdot \Delta\hat{X}_1^n(s) \leq \frac{1}{2} \sup_{|y-\hat{X}_1^n(s-)|\leq 1} |\varphi''(y)|[\Delta(\hat{X}_1^n(s))]^2.$$

Hence, combining the above facts we obtain

$$\mathbb{E}\sum_{s\leq t}\left(\Delta\varphi(\hat{X}_1^n(s)) - \varphi'(\hat{X}_1^n(s-)) \cdot \Delta\hat{X}_1^n(s) - \frac{1}{2}\varphi''(\hat{X}_1^n(s-))\Delta\hat{X}_1^n(s)\Delta\hat{X}_1^n(s)\right)$$

$$\leq \mathbb{E}\sum_{s\leq t}\kappa_3(1 + |\hat{X}_1^n(s-)|^{m-2})(\Delta(\hat{X}_1^n(s)))^2$$

(5.2)

$$= \kappa_3\mathbb{E}\left[\int_0^t (1 + |\hat{X}_1^n(s)|^{m-2})\left(\frac{\lambda_1^n}{n} + \frac{\mu_1^n Z_1^n(s)}{n} + \frac{\gamma_1^n Q_1^n(s)}{n}\right)ds\right]$$

$$\leq \mathbb{E}\left[\int_0^t \left(\kappa_4 + \frac{\kappa_2}{4}|\hat{X}_1^n(s)|^m + \kappa_5((e \cdot \hat{X}^n(s))^+)^m\right)ds\right],$$

for some suitable positive constants κ_4 and κ_5, independent of n, where in the second inequality we use the fact that the optional martingale $[\hat{X}_1^n]$ is the sum of the squares of the jumps, and that $[\hat{X}_1^n] - \langle\hat{X}_1^n\rangle$ is a martingale. Therefore, for some positive constants C_1 and C_2 it holds that

$$0 \leq \mathbb{E}[\varphi(\hat{X}_1^n(t))]$$

(5.3)

$$\leq \mathbb{E}[\varphi(\hat{X}_1^n(0))] + C_1 t - \frac{\kappa_2}{2}\mathbb{E}\left[\int_0^t |\hat{X}_1^n(s)|^m ds\right]$$

$$+ C_2\mathbb{E}\left[\int_0^t ((e \cdot \hat{X}^n(s))^+)^m ds\right].$$

By (2.8), we have

$$r(\hat{Q}^n(s)) \geq \frac{c_1}{d^m}((e \cdot \hat{X}^n(s))^+)^m,$$

which, combined with the assumption that $\sup_n \hat{V}^n(\hat{X}^n(0)) < \infty$, implies that

$$\sup_n \limsup_{T \to \infty} \frac{1}{T} \mathbb{E}\left[\int_0^T ((e \cdot \hat{X}^n(s))^+)^m \, ds \right] < \infty.$$

In turn, from (5.3) we obtain

$$\sup_n \limsup_{T \to \infty} \frac{1}{T} \mathbb{E}\left[\int_0^T |\hat{X}_1^n(s)|^m \, ds \right] < \infty.$$

Repeating the same argument for coordinates $i = 2, \ldots, d$, we obtain

$$(5.4) \qquad \sup_n \limsup_{T \to \infty} \frac{1}{T} \mathbb{E}\left[\int_0^T |\hat{X}^n(s)|^m \, ds \right] < \infty.$$

We introduce the process

$$U_i^n(t) := \begin{cases} \dfrac{\hat{X}_i^n(t) - \hat{Z}_i^n(t)}{(e \cdot \hat{X}^n(t))^+}, & i = 1, \ldots, d, & \text{if } (e \cdot \hat{X}^n(t))^+ > 0, \\ e_d, & & \text{otherwise.} \end{cases}$$

Since Z^n is work-conserving, it follows that U^n takes values in $\mathcal{S}$, and $U_i^n(t)$ represents the fraction of class i customers in queue. Define the mean empirical measures

$$\Phi_T^n(A \times B) := \frac{1}{T} \mathbb{E}\left[\int_0^T \mathbb{I}_{A \times B}(\hat{X}^n(s), U^n(s)) \, ds \right]$$

for Borel sets $A \subset \mathbb{R}^d$ and $B \subset \mathcal{S}$.

From (5.4), we see that the family $\{\Phi_T^n : T > 0, n \geq 1\}$ is tight. Hence, for any sequence $T_k \to \infty$, there exists a subsequence, also denoted by T_k, such that $\Phi_{T_k}^n \to \pi^n$, as $k \to \infty$. It is evident that $\{\pi^n : n \geq 1\}$ is tight. Let $\pi^n \to \pi$ along some subsequence, with $\pi \in \mathcal{P}(\mathbb{R}^d \times \mathcal{S})$. Therefore, it is not hard to show that

$$\lim_{n \to \infty} \hat{V}^n(\hat{X}^n(0)) \geq \int_{\mathbb{R}^d \times \mathbb{U}} \tilde{r}(x, u) \pi(dx, du),$$

where, as defined earlier, $\tilde{r}(x, u) = r((e \cdot x)^+ u)$. To complete the proof of the theorem, we only need to show that π is an ergodic occupation measure for the diffusion. For that, consider $f \in \mathcal{C}_c^\infty(\mathbb{R}^d)$. Recall that $[\hat{X}_i^n, \hat{X}_j^n] = 0$ for $i \neq j$ [30], Lemmas 9.2 and 9.3. Therefore, using Itô's formula and the definition of Φ_T^n, we obtain

$$\frac{1}{T} \mathbb{E}[f(\hat{X}^n(T))]$$

$$= \frac{1}{T} \mathbb{E}[f(\hat{X}^n(0))]$$

$$(5.5) \qquad + \int_{\mathbb{R}^d \times \mathbb{U}} \left(\sum_{i=1}^d \mathcal{A}_i^n(x, u) \cdot f_{x_i}(x) + \mathcal{B}_i^n(x, u) f_{x_i x_i}(x) \right) \Phi_T^n(dx, du)$$

$$+ \frac{1}{T} \mathbb{E} \sum_{s \leq T} \left[\Delta f(\hat{X}^n(s)) - \sum_{i=1}^{d} f_{x_i}(\hat{X}^n(s-)) \cdot \Delta \hat{X}_i^n(s) \right.$$

$$\left. - \frac{1}{2} \sum_{i,j=1}^{d} f_{x_i x_j}(\hat{X}^n(s-)) \Delta \hat{X}_i^n(s) \Delta \hat{X}_j^n(s) \right],$$

where

$$\mathcal{A}_i^n(x, u) := \ell_i^n - \mu_i^n(x_i - (e \cdot x)^+ u_i) - \gamma_i^n (e \cdot x)^+ u_i,$$

$$\mathcal{B}_i^n(x, u) := \frac{1}{2} \left(\mu_i^n \rho_i + \frac{\lambda_i^n}{n} + \frac{\mu_i^n x_i + (\gamma_i^n - \mu_i^n)(e \cdot x)^+ u_i}{\sqrt{n}} \right).$$

We first bound the last term in (5.5). Using Taylor's formula, we see that

$$\Delta f(\hat{X}^n(s)) - \sum_{i=1}^{d} \nabla f(\hat{X}^n(s-)) \cdot \Delta \hat{X}^n(s)$$

$$- \frac{1}{2} \sum_{i,j=1}^{d} f_{x_i x_j}(\hat{X}^n(s-)) \Delta \hat{X}_i^n(s) \Delta \hat{X}_j^n(s)$$

$$= \frac{k \|f\|_{\mathcal{C}^3}}{\sqrt{n}} \sum_{i,j=1}^{d} |\Delta \hat{X}_i^n(s) \Delta \hat{X}_j^n(s)|$$

for some positive constant k, where we use the fact that the jump size is $\frac{1}{\sqrt{n}}$. Hence, using the fact that independent Poisson processes do not have simultaneous jumps w.p.1, using the identity $\hat{Q}_i^n = \hat{X}_i^n - \hat{Z}_i^n$, we obtain

$$\frac{1}{T} \mathbb{E} \sum_{s \leq T} \left[\Delta f(\hat{X}^n(s)) - \sum_{i=1}^{d} \nabla f(\hat{X}^n(s-)) \cdot \Delta \hat{X}^n(s) \right.$$

(5.6)
$$\left. - \frac{1}{2} \sum_{i,j=1}^{d} f_{x_i x_j}(\hat{X}^n(s-)) \Delta \hat{X}_i^n(s) \Delta \hat{X}_j^n(s) \right]$$

$$\leq \frac{k \|f\|_{\mathcal{C}^3}}{T \sqrt{n}} \mathbb{E} \left[\int_0^T \sum_{i=1}^{d} \left(\frac{\lambda_i^n}{n} + \frac{\mu_i^n Z_i^n(s)}{n} + \frac{\gamma_i^n Q_i^n(s)}{n} \right) ds \right].$$

Therefore, first letting $T \to \infty$ and using (5.2) and (5.4) we see that the expectation on the right-hand side of (5.6) is bounded above. Therefore, as $n \to \infty$, the left-hand side of (5.6) tends to 0. Thus, by (5.5) and the fact that f is compactly supported, we obtain

$$\int_{\mathbb{R}^d \times \mathbb{U}} L^u f(x) \pi(dx, du) = 0,$$

where

$$L^u f(x) = \lambda_i \partial_{ii} f(x) + \big(\ell_i - \mu_i\big(x_i - (e \cdot x)^+ u_i\big) - \gamma_i (e \cdot x)^+ u_i\big) \partial_i f(x).$$

Therefore, $\pi \in \mathcal{G}$. $\quad\square$

5.2. *The upper bound.* The proof of the upper bound in Theorem 2.2 is a little more involved than that of the lower bound. Generally, it is very helpful if one has uniform stability across $n \in \mathbb{N}$ (see, e.g., [12]). In [12], uniform stability is obtained from the reflected dynamics with the Skorohod mapping. However, here we establish the asymptotic upper bound by using the technique of spatial truncation that we have introduced in Section 4. Let v_δ be any precise continuous control in $\mathfrak{U}_{\mathrm{SSM}}$ satisfying $v_\delta(x) = u_0 = (0, \ldots, 0, 1)$ for $|x| > K > 1$.

First, we construct a work-conserving admissible policy for each $n \in \mathbb{N}$ (see [7]). Define a measurable map $\varpi : \{z \in \mathbb{R}^d_+ : e \cdot z \in \mathbb{Z}\} \to \mathbb{Z}^d_+$ as follows: for $z = (z_1, \ldots, z_d) \in \mathbb{R}^d$, let

$$\varpi(z) := \left(\lfloor z_1 \rfloor, \ldots, \lfloor z_{d-1} \rfloor, \lfloor z_d \rfloor + \sum_{i=1}^d (z_i - \lfloor z_i \rfloor) \right).$$

Note that $|\varpi(z) - z| \le 2d$. Define

$$u_h(x) := \varpi\big((e \cdot x - n)^+ v_\delta(\hat{x}^n)\big), \qquad x \in \mathbb{R}^d,$$

$$\hat{x}^n := \left(\frac{x_1 - \rho_1 n}{\sqrt{n}}, \ldots, \frac{x_d - \rho_d n}{\sqrt{n}} \right),$$

$$A_n := \left\{ x \in \mathbb{R}^d_+ : \sup_i |x_i - \rho_i n| \le K \sqrt{n} \right\}.$$

We define a state-dependent, work-conserving policy as follows:

$$(5.7) \qquad Z^n_i[X^n] := \begin{cases} X^n_i - u_h(X^n), & \text{if } X^n \in A_n, \\[2mm] X^n_i \wedge \left(n - \sum_{j=1}^{i-1} X^n_j \right)^+, & \text{otherwise.} \end{cases}$$

Therefore, whenever the state of the system is in A^c_n, the system works under the fixed priority policy with the least priority given to class-d jobs. First, we show that this is a well-defined policy for all large n. It is enough to show that $X^n_i - u_h(X^n) \ge 0$ for all i when $X^n \in A_n$. If not, then for some i, $1 \le i \le d$, we must have $X^n_i - u_h(X^n) < 0$ and so $X^n_i < (e \cdot X^n - n)^+ + d$. Since $X^n \in A_n$, we obtain

$$-K\sqrt{n} + \rho_i n \le X^n_i$$
$$< (e \cdot X^n - n)^+ + d$$

$$= \left(\sum_{i=1}^{d} (X_i^n - \rho_i n) \right)^{+} + d$$

$$\leq dK \sqrt{n} + d.$$

But this cannot hold for large n. Hence, this policy is well defined for all large n. Under the policy defined in (5.7), X^n is a Markov process and its generator given by

$$\mathcal{L}_n f(x) = \sum_{i=1}^{d} \lambda_i^n \big(f(x + e_i) - f(x) \big) + \sum_{i=1}^{d} \mu_i^n Z_i^n[x] \big(f(x - e_i) - f(x) \big)$$

$$+ \sum_{i=1}^{d} \gamma_i^n Q_i^n[x] \big(f(x - e_i) - f(x) \big), \qquad x \in \mathbb{Z}_+^d,$$

where $Z^n[x]$ is as above and $Q^n[x] := x - Z^n[x]$. It is easy to see that, for $x \notin A_n$,

$$Q_i^n[x] = \left[x_i - \left(n - \sum_{j=1}^{i-1} x_j \right)^{+} \right]^{+}.$$

LEMMA 5.1. *Let X^n be the Markov process corresponding to the above control. Let q be an even positive integer. Then there exists $n_0 \in \mathbb{N}$ such that*

$$\sup_{n \geq n_0} \limsup_{T \to \infty} \frac{1}{T} \mathbb{E} \left[\int_{0}^{T} |\hat{X}^n(s)|^q \, ds \right] < \infty,$$

where $\hat{X}^n = (\hat{X}_1^n, \ldots, \hat{X}_d^n)^\mathsf{T}$ is the diffusion-scaled process corresponding to the process X^n, as defined in (2.4).

PROOF. The proof technique is inspired by [6], Lemma 3.1. Define

$$f_n(x) := \sum_{i=1}^{d} \beta_i (x_i - \rho_i n)^q,$$

where β_i, $i = 1, \ldots, d$, are positive constants to be determined later. We first show that for a suitable choice of β_i, $i = 1, \ldots, d$, there exist constants C_i, $i = 1, 2$, independent of $n \geq n_0$, such that

(5.8) $$\mathcal{L}_n f_n(x) \leq C_1 n^{q/2} - C_2 f_n(x), \qquad x \in \mathbb{Z}_+^d.$$

Choose n large enough so that the policy is well defined. We define $Y_i^n := x_i - \rho_i n$. Note that

$$(a \pm 1)^q - a^q = \pm q a \cdot a^{q-2} + \mathcal{O}(a^{q-2}), \qquad a \in \mathbb{R}.$$

Also, $\mu_i^n Z_i^n[x] = \mu_i^n x_i - \mu_i^n Q_i^n[x]$. Then

$$
\mathcal{L}_n f_n(x) = \sum_{i=1}^d \beta_i \lambda_i^n [q Y_i^n |Y_i^n|^{q-2} + \mathcal{O}(|Y_i^n|^{q-2})]
$$

$$
- \sum_{i=1}^d \beta_i \mu_i^n x_i [q Y_i^n |Y_i^n|^{q-2} + \mathcal{O}(|Y_i^n|^{q-2})]
$$

$$
- \sum_{i=1}^d \beta_i (\gamma_i^n - \mu_i^n) Q_i^n[x][q Y_i^n |Y_i^n|^{q-2} + \mathcal{O}(|Y_i^n|^{q-2})]
$$

(5.9)
$$
\leq \sum_{i=1}^d \beta_i (\lambda_i^n + \mu_i^n x_i + |\gamma_i^n - \mu_i^n| Q_n^i[x]) \mathcal{O}(|Y_i^n|^{q-2})
$$

$$
+ \sum_{i=1}^d \beta_i q Y_i^n |Y_i^n|^{q-2} (\lambda_i^n - \mu_i^n x_i - (\gamma_i^n - \mu_i^n) Q_i^n[x])
$$

$$
\leq \sum_{i=1}^d \beta_i (\lambda_i^n + (\mu_i^n + |\gamma_i^n - \mu_i^n|)(Y_i^n + \rho_i n)) \mathcal{O}(|Y_i^n|^{q-2})
$$

$$
+ \sum_{i=1}^d \beta_i q Y_i^n |Y_i^n|^{q-2} (\lambda_i^n - \mu_i^n x_i - (\gamma_i^n - \mu_i^n) Q_i^n[x]),
$$

where in the last inequality we use the fact that $Q_i^n[x] \leq x_i$ for $x \in \mathbb{Z}_+^d$. Let

$$
\delta_i^n := \lambda_i^n - \mu_i^n \rho_i n = \mathcal{O}(\sqrt{n}).
$$

The last estimate is due to the assumptions in (2.1) concerning the parameters in the Halfin–Whitt regime. Then

(5.10)
$$
\sum_{i=1}^d \beta_i q Y_i^n |Y_i^n|^{q-2} (\lambda_i^n - \mu_i^n x_i - (\gamma_i^n - \mu_i^n) Q_i^n[x])
$$

$$
= -q \sum_{i=1}^d \beta_i \mu_i^n |Y_i^n|^q + \sum_{i=1}^d \beta_i q Y_i^n |Y_i^n|^{q-2} (\delta_i^n - (\gamma_i^n - \mu_i^n) Q_i^n[x]).
$$

If $x \in A_n$ and n is large, then

$$
Q_i^n[x] = u_h(x) = \varpi ((e \cdot x - n)^+ v_\delta(\hat{x}_n))
$$

$$
\leq (e \cdot x - n)^+ + d \leq 2dK\sqrt{n}.
$$

Let $x \in A_n^c$. We use the fact that for any $a, b \in \mathbb{R}$ it holds that $a^+ - b^+ = \xi[a - b]$ for some $\xi \in [0, 1]$. Also,

$$\left[n\rho_i - \left(n - \sum_{j=1}^{i-1} n\rho_j\right)^+\right]^+ = 0, \qquad i = 1, \dots, d.$$

Thus, we obtain maps $\xi, \tilde{\xi} : \mathbb{R}^d \to [0, 1]^d$ such that

$$-Q_i^n[x] = \left[n\rho_i - \left(n - \sum_{j=1}^{i-1} n\rho_j\right)^+\right]^+ - Q_i^n[x]$$

$$= \xi_i(x)(n\rho_i - x_i) - \tilde{\xi}_i(x) \sum_{j=1}^{i-1}(x_j - n\rho_j), \qquad x \in A_n^c.$$

Hence, from (5.10) we obtain

$$\sum_{i=1}^{d} \beta_i q Y_i^n |Y_i^n|^{q-2} (\lambda_i^n - \mu_i^n x_i - (\gamma_i^n - \mu_i^n) Q_i^n[x])$$

$$\leq \mathcal{O}(\sqrt{n}) q \sum_{i=1}^{d} \beta_i |Y_i^n|^{q-1} - q \sum_{i=1}^{d} \beta_i ((1 - \xi_i(x))\mu_i^n + \xi_i(x)\gamma_i^n) |Y_i^n|^q$$

$$+ q \sum_{i=1}^{d} \beta_i Y_i^n |Y_i^n|^{q-2} \left(\delta_i^n - (\gamma_i^n - \mu_i^n)\tilde{\xi}_i(x) \sum_{j=1}^{i-1} Y_j^n\right),$$

where we used the fact that on A_n we have

$$\left[x_i - \left(n - \sum_{j=1}^{i-1} x_j\right)^+\right]^+ = \mathcal{O}(\sqrt{n}) \qquad \forall i.$$

Observe that there exists $\vartheta > 0$, independent of n due to (2.1), such that

$$(1 - \xi_i(x))\mu_i^n + \xi_i(x)\gamma_i^n \geq \min(\mu_i^n, \gamma_i^n) \geq \vartheta$$

for all $n \in \mathbb{N}$, all $x \in \mathbb{R}^d$, and all $i = 1, \dots, d$. As a result, we obtain

$$\sum_{i=1}^{d} \beta_i q Y_i^n |Y_i^n|^{q-2} (\lambda_i^n - \mu_i^n x_i - (\gamma_i^n - \mu_i^n) Q_i^n[x])$$

(5.11) $$\leq \mathcal{O}(\sqrt{n}) q \sum_{i=1}^{d} \beta_i |Y_i^n|^{q-1} - q\vartheta \sum_{i=1}^{d} \beta_i |Y_i^n|^q$$

$$+ q \sum_{i=1}^{d} \beta_i Y_i^n |Y_i^n|^{q-2} \left(\delta_i^n - (\gamma_i^n - \mu_i^n)\tilde{\xi}_i(x) \sum_{j=1}^{i-1} Y_n^j\right).$$

We next estimate the last term on the right-hand side of (5.11). Let $\kappa :=$ $\sup_{n,i} |\gamma_i^n - \mu_i^n|$, and $\varepsilon_1 := \frac{\vartheta}{8\kappa}$. Using Young's inequality, we obtain the estimate

$$|Y_i^n|^{q-1}\left|\sum_{j=1}^{i-1} Y_n^j\right| \le \varepsilon_1 |Y_i^n|^q + \frac{1}{\varepsilon_1^{q-1}}\left|\sum_{j=1}^{i-1} Y_j^n\right|^q.$$

Therefore,

$$q \sum_{i=1}^{d} \beta_i Y_i^n |Y_i^n|^{q-2}\left(-(\gamma_i^n - \mu_i^n)\tilde{\xi}_i(x)\sum_{j=1}^{i-1} Y_j^n\right)$$

$$\le q\kappa \sum_{i=1}^{d}\left(\varepsilon_1\beta_i|Y_i^n|^q + \frac{\beta_i}{\varepsilon_1^{q-1}}\left|\sum_{j=1}^{i-1} Y_j^n\right|^q\right)$$

$$\le q\kappa \sum_{i=1}^{d}\left(\varepsilon_1\beta_i|Y_i^n|^q + \frac{\beta_i}{\varepsilon_1^{q-1}}d^{q-1}\sum_{j=1}^{i-1}|Y_j^n|^q\right)$$

$$= \frac{q\vartheta}{8}\sum_{i=1}^{d}\left(\beta_i|Y_i^n|^q + \frac{\beta_i}{\varepsilon_1^q}d^{q-1}\sum_{j=1}^{i-1}|Y_j^n|^q\right).$$

We choose $\beta_1 = 1$ and for $i \ge 2$, we define β_i by

$$\beta_i := \frac{\varepsilon_1^q}{d^q}\min_{j \le i-1} \beta_j.$$

With this choice of β_i it follows from above that

$$q \sum_{i=1}^{d} \beta_i Y_i^n |Y_i^n|^{q-2}\left(-(\gamma_i^n - \mu_i^n)\tilde{\xi}_i(x)\sum_{j=1}^{i-1} Y_n^j\right) \le \frac{q\vartheta}{4}\sum_{i=1}^{d} \beta_i|Y_i^n|^q.$$

Using the preceding inequality in (5.11), we obtain

(5.12)
$$\sum_{i=1}^{d} \beta_i q Y_i^n |Y_i^n|^{q-2}(\lambda_i^n - \mu_i^n x_i - (\gamma_i^n - \mu_i^n)Q_i^n[x])$$

$$\le \mathcal{O}(\sqrt{n})q\sum_{i=1}^{d}\beta_i|Y_i^n|^{q-1} - \frac{3}{4}q\vartheta\sum_{i=1}^{d}\beta_i|Y_i^n|^q.$$

Combining (5.9) and (5.12), we obtain

(5.13)
$$\mathcal{L}_n f_n(x) \le \sum_{i=1}^{d}\mathcal{O}(\sqrt{n})\mathcal{O}(|Y_i^n|^{q-1}) + \sum_{i=1}^{d}\mathcal{O}(n)\mathcal{O}(|Y_i^n|^{q-2})$$

$$- \frac{3}{4}q\vartheta\sum_{i=1}^{d}\beta_i|Y_i^n|^q.$$

By Young's inequality, for any $\varepsilon > 0$, we have the bounds

$$\mathcal{O}(\sqrt{n})\mathcal{O}\big(|Y_i^n|^{q-1}\big) \leq \varepsilon\big[\mathcal{O}\big(|Y_i^n|^{q-1}\big)\big]^{q/(q-1)} + \varepsilon^{(1-q)}\big[\mathcal{O}(\sqrt{n})\big]^q,$$

$$\mathcal{O}(n)\mathcal{O}\big(|Y_i^n|^{q-2}\big) \leq \varepsilon\big[\mathcal{O}\big(|Y_i^n|^{q-2}\big)\big]^{q/(q-2)} + \varepsilon^{(1-q/2)}\big[\mathcal{O}(n)\big]^{q/2}.$$

Thus, choosing ε properly in (5.13) we obtain (5.8).

We proceed to complete the proof of the lemma by applying (5.8). First, we observe that $\mathbb{E}[\sup_{s\in[0,T]}|X^n(s)|^p]$ is finite for any $p \geq 1$ as this quantity is dominated by the Poisson arrival process. Therefore, from (5.8) we see that

$$\mathbb{E}\big[f_n\big(X^n(T)\big)\big] - f_n\big(X^n(0)\big) = \mathbb{E}\bigg[\int_0^T \mathcal{L}_n f_n\big(X^n(s)\big)\,\mathrm{d}s\bigg]$$

$$\leq C_1 n^{q/2} T - C_2 \mathbb{E}\bigg[\int_0^T f_n\big(X^n(s)\big)\,\mathrm{d}s\bigg],$$

which implies that

$$C_2 \mathbb{E}\bigg[\int_0^T \sum_{i=1}^d \beta_i\big(\hat{X}_i^n(s)\big)^q\,\mathrm{d}s\bigg] \leq C_1 T + \sum_{i=1}^d \beta_i\big(\hat{X}_i^n(0)\big)^q.$$

Hence, the proof follows by dividing both sides by T and letting $T \to \infty$. $\quad\square$

PROOF OF THEOREM 2.2. Let r be the given running cost with polynomial growth with exponent m in (2.8). Let $q = 2(m + 1)$. Recall that $\tilde{r}(x, u) = r((e \cdot x)^+ u)$ for $(x, u) \in \mathbb{R}^d \times \mathcal{S}$. Then $\tilde{r}$ is convex in u and satisfies (3.11) with the same exponent m. For any $\delta > 0$, we choose $v_\delta \in \mathfrak{U}_{\mathrm{SSM}}$ such that v_δ is a continuous precise control with invariant probability measure μ_δ and

(5.14) $$\int_{\mathbb{R}^d} \tilde{r}\big(x, v_\delta(x)\big)\mu_\delta(\mathrm{d}x) \leq \varrho_* + \delta.$$

We also want the control v_δ to have the property that $v_\delta(x) = (0, \ldots, 0, 1)$ outside a large ball. To obtain such v_δ, we see that by Theorems 4.1, 4.2 and Remark 4.2 we can find v'_δ and a ball B_l for l large, such that $v'_\delta \in \mathfrak{U}_{\mathrm{SSM}}$, $v'_\delta(x) = e_d$ for $|x| > l$, v'_δ is continuous in B_l, and

$$\bigg|\int_{\mathbb{R}^d} \tilde{r}\big(x, v'_\delta(x)\big)\mu'_\delta(\mathrm{d}x) - \varrho_*\bigg| < \frac{\delta}{2},$$

where μ'_δ is the invariant probability measure corresponding to v'_δ. We note that v'_δ might not be continuous on ∂B_l. Let $\{\chi^n : n \in \mathbb{N}\}$ be a sequence of cut-off functions such that $\chi^n \in [0, 1]$, it vanishes on $B^c_{l-(1/n)}$, and it takes the value 1 on $B_{l-(2/n)}$. Define the sequence $v_\delta^n(x) := \chi^n(x)v'_\delta(x) + (1 - \chi^n(x))e_d$. Then $v_\delta^n \to v'_\delta$, as $n \to \infty$, and the convergence is uniform on the complement of any neighborhood

of ∂B_l. Also by Proposition 3.1 the corresponding invariant probability measures μ_δ^n are exponentially tight. Thus,

$$\left| \int_{\mathbb{R}^d} \tilde{r}(x, v_\delta'(x)) \mu_\delta'(\mathrm{d}x) - \int_{\mathbb{R}^d} \tilde{r}(x, v_\delta^n(x)) \mu_\delta^n(\mathrm{d}x) \right| \xrightarrow[n \to \infty]{} 0.$$

Combining the above two expressions, we can easily find v_δ which satisfies (5.14). We construct a scheduling policy as in Lemma 5.1. By Lemma 5.1, we see that for some constant K_1 it holds that

$$(5.15) \qquad \sup_{n \geq n_0} \limsup_{T \to \infty} \frac{1}{T} \mathbb{E}\left[\int_0^T |\hat{X}^n(s)|^q \, \mathrm{d}s \right] < K_1, \qquad q = 2(m+1).$$

Define

$$v_h(x) := \varpi\big((e \cdot x - n)^+ v_\delta(\hat{x}_n)\big),$$

$$\hat{v}_h(\hat{x}^n) := \varpi\big(\sqrt{n}(e \cdot \hat{x}^n)^+ v_\delta(\hat{x}^n)\big).$$

Since $v_\delta(\hat{x}^n) = (0, \ldots, 0, 1)$ when $|\hat{x}^n| \geq K$, it follows that

$$Q^n[X^n] = X^n - Z^n[X^n] = v_h(X^n)$$

for large n, provided that $\sum_{i=1}^{d-1} X_i^n \leq n$. Define

$$D_n := \left\{ x : \sum_{i=1}^{d-1} \hat{x}_i^n > \rho_d \sqrt{n} \right\}.$$

Then

$$r(\hat{Q}^n(t)) = r\left(\frac{1}{\sqrt{n}} \hat{v}_h(\hat{X}^n(t)) \right) + r(\hat{X}^n(t) - \hat{Z}^n(t)) \mathbb{I}_{\{\hat{X}^n(t) \in D_n\}}$$

$$- r\left(\frac{1}{\sqrt{n}} \hat{v}_h(\hat{X}^n(t)) \right) \mathbb{I}_{\{\hat{X}^n(t) \in D_n\}}.$$

Define, for each n, the mean empirical measure Ψ_T^n by

$$\Psi_T^n(A) := \frac{1}{T} \mathbb{E}\left[\int_0^T \mathbb{I}_A(\hat{X}^n(t)) \, \mathrm{d}t \right].$$

By (5.15), the family $\{\Psi_T^n : T > 0, n \geq 1\}$ is tight. We next show that

$$(5.16) \qquad \lim_{n \to \infty} \limsup_{T \to \infty} \frac{1}{T} \mathbb{E}\left[\int_0^T r(\hat{Q}^n(t)) \, \mathrm{d}t \right] = \int_{\mathbb{R}^d} r((e \cdot x)^+ v_\delta(x)) \mu_\delta(\mathrm{d}x).$$

For each n, select a sequence $\{T_k^n : k \in \mathbb{N}\}$ along which the "lim sup" in (5.16) is attained. By tightness, there exists a limit point Ψ^n of $\Psi_{T_k^n}^n$. Since Ψ^n has support on a discrete lattice, we have

$$\int_{\mathbb{R}^d} r\left(\frac{1}{\sqrt{n}} \hat{v}_h(x) \right) \Psi_{T_k^n}^n(\mathrm{d}x) \xrightarrow[k \to \infty]{} \int_{\mathbb{R}^d} r\left(\frac{1}{\sqrt{n}} \hat{v}_h(x) \right) \Psi^n(\mathrm{d}x).$$

Therefore,

$$\limsup_{T \to \infty} \frac{1}{T} \mathbb{E}\left[\int_0^T r(\hat{Q}^n(t))\,dt\right] \lessgtr \int_{\mathbb{R}^d} r\left(\frac{1}{\sqrt{n}}\hat{v}_h(x)\right)\Psi^n(dx) \pm \mathcal{E}^n,$$

where

$$\mathcal{E}^n = \limsup_{T \to \infty} \frac{1}{T}\mathbb{E}\left[\int_0^T \left(r(\hat{Q}^n(t)) + r\left(\frac{1}{\sqrt{n}}\hat{v}_h(\hat{X}^n(t))\right)\right)\mathbb{I}_{\{\hat{X}^n(t) \in D_n\}}\,dt\right].$$

By (5.15), the family $\{\Psi^n : n \geq 1\}$ is tight. Hence, it has a limit Ψ. By definition, we have

$$\left|\frac{1}{\sqrt{n}}\hat{v}_h(x) - (e \cdot x)^+ v_\delta(x)\right| \leq \frac{2d}{\sqrt{n}}.$$

Thus, using the continuity property of r and (2.8) it follows that

$$\int_{\mathbb{R}^d} r\left(\frac{1}{\sqrt{n}}\hat{v}_h(x)\right)\Psi^n(dx) \xrightarrow[n \to \infty]{} \int_{\mathbb{R}^d} r((e \cdot x)^+ v_\delta(x))\Psi(dx),$$

along some subsequence. Therefore, in order to complete the proof of (5.16) we need to show that

$$\limsup_{n \to \infty} \mathcal{E}^n = 0.$$

Since the policies are work-conserving, we observe that $0 \leq \hat{X}^n - \hat{Z}^n \leq (e \cdot \hat{X}^n)^+$, and therefore for some positive constants κ_1 and κ_2, we have

$$r\left(\frac{1}{\sqrt{n}}\hat{v}_h(\hat{X}^n(t))\right) \vee r(\hat{X}^n(t) - \hat{Z}^n(t)) \leq \kappa_1 + \kappa_2[(e \cdot \hat{X}^n)^+]^m.$$

Given $\varepsilon > 0$ we can choose n_1 so that for all $n \geq n_1$,

$$\limsup_{T \to \infty} \frac{1}{T}\mathbb{E}\left[\int_0^T [(e \cdot \hat{X}^n(s))^+]^m \mathbb{I}_{\{|\hat{X}^n(s)| > (\rho_d/\sqrt{d})\sqrt{n}\}}\,ds\right] \leq \varepsilon,$$

where we use (5.15). We observe that $D_n \subset \{|\hat{x}^n| > \rho_d\sqrt{d/n}\}$. Thus, (5.16) holds. In order to complete the proof, we only need to show that Ψ is the invariant probability measure corresponding to v_δ. This can be shown using the convergence of generators as in the proof of Theorem 2.1. $\square$

6. Conclusion. We have answered some of the most interesting questions for the ergodic control problem of the Markovian multi-class many-server queueing model. This current study has raised some more questions for future research. One of the interesting questions is to consider nonpreemptive policies and try to establish asymptotic optimality in the class of nonpreemptive admissible polices [7]. It will also be interesting to study a similar control problem when the system has multiple heterogeneous agent pools with skill-based routing.

It has been observed that customers' service requirements and patience times are nonexponential [10] in some situations. It is therefore important and interesting to address similar control problems under general assumptions on the service and patience time distributions.

REFERENCES

[1] ARAPOSTATHIS, A., BORKAR, V. S. and GHOSH, M. K. (2012). *Ergodic Control of Diffusion Processes. Encyclopedia of Mathematics and Its Applications* **143**. Cambridge Univ. Press, Cambridge. MR2884272

[2] ARISAWA, M. and LIONS, P.-L. (1998). On ergodic stochastic control. *Comm. Partial Differential Equations* **23** 2187–2217. MR1662180

[3] ATAR, R. (2005). Scheduling control for queueing systems with many servers: Asymptotic optimality in heavy traffic. *Ann. Appl. Probab.* **15** 2606–2650. MR2187306

[4] ATAR, R. and BISWAS, A. (2014). Control of the multiclass $G/G/1$ queue in the moderate deviation regime. *Ann. Appl. Probab.* **24** 2033–2069. MR3226171

[5] ATAR, R., GIAT, C. and SHIMKIN, N. (2010). The $c\mu/\theta$ rule for many-server queues with abandonment. *Oper. Res.* **58** 1427–1439. MR2560545

[6] ATAR, R., GIAT, C. and SHIMKIN, N. (2011). On the asymptotic optimality of the $c\mu/\theta$ rule under ergodic cost. *Queueing Syst.* **67** 127–144. MR2771197

[7] ATAR, R., MANDELBAUM, A. and REIMAN, M. I. (2004). Scheduling a multi class queue with many exponential servers: Asymptotic optimality in heavy traffic. *Ann. Appl. Probab.* **14** 1084–1134. MR2071417

[8] BOGACHEV, V. I., KRYLOV, N. V. and RÖCKNER, M. (2001). On regularity of transition probabilities and invariant measures of singular diffusions under minimal conditions. *Comm. Partial Differential Equations* **26** 2037–2080. MR1876411

[9] BORKAR, V. S. (1989). *Optimal Control of Diffusion Processes. Pitman Research Notes in Mathematics Series* **203**. Longman, Harlow. MR1005532

[10] BROWN, L., GANS, N., MANDELBAUM, A., SAKOV, A., SHEN, H., ZELTYN, S. and ZHAO, L. (2005). Statistical analysis of a telephone call center: A queueing-science perspective. *J. Amer. Statist. Assoc.* **100** 36–50. MR2166068

[11] BUDHIRAJA, A., GHOSH, A. and LIU, X. (2014). Scheduling control for Markov-modulated single-server multiclass queueing systems in heavy traffic. *Queueing Syst.* **78** 57–97. MR3238008

[12] BUDHIRAJA, A., GHOSH, A. P. and LEE, C. (2011). Ergodic rate control problem for single class queueing networks. *SIAM J. Control Optim.* **49** 1570–1606. MR2817491

[13] DAI, J. G. and TEZCAN, T. (2008). Optimal control of parallel server systems with many servers in heavy traffic. *Queueing Syst.* **59** 95–134. MR2430811

[14] DIEKER, A. B. and GAO, X. (2013). Positive recurrence of piecewise Ornstein–Uhlenbeck processes and common quadratic Lyapunov functions. *Ann. Appl. Probab.* **23** 1291–1317. MR3098433

[15] GAMARNIK, D. and STOLYAR, A. L. (2012). Multiclass multiserver queueing system in the Halfin-Whitt heavy traffic regime: Asymptotics of the stationary distribution. *Queueing Syst.* **71** 25–51. MR2925789

[16] GAMARNIK, D. and ZEEVI, A. (2006). Validity of heavy traffic steady-state approximation in generalized Jackson networks. *Ann. Appl. Probab.* **16** 56–90. MR2209336

[17] GARNETT, O., MANDELBAUM, A. and REIMAN, M. I. (2002). Designing a call center with impatient customers. *Manuf. Serv. Oper. Manag.* **4** 208–227.

[18] GILBARG, D. and TRUDINGER, N. S. (1983). *Elliptic Partial Differential Equations of Second Order*, 2nd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **224**. Springer, Berlin. MR0737190

[19] GURVICH, I. (2014). Diffusion models and steady-state approximations for exponentially ergodic Markovian queues. *Ann. Appl. Probab.* **24** 2527–2559. MR3262510

[20] GYÖNGY, I. and KRYLOV, N. (1996). Existence of strong solutions for Itô's stochastic equations via approximations. *Probab. Theory Related Fields* **105** 143–158. MR1392450

[21] HALFIN, S. and WHITT, W. (1981). Heavy-traffic limits for queues with many exponential servers. *Oper. Res.* **29** 567–588. MR0629195

[22] HARRISON, J. M. and ZEEVI, A. (2004). Dynamic scheduling of a multiclass queue in the Halfin–Whitt heavy traffic regime. *Oper. Res.* **52** 243–257. MR2066399

[23] ICHIHARA, N. and SHEU, S.-J. (2013). Large time behavior of solutions of Hamilton–Jacobi–Bellman equations with quadratic nonlinearity in gradients. *SIAM J. Math. Anal.* **45** 279–306. MR3032978

[24] KALLENBERG, O. (2002). *Foundations of Modern Probability*, 2nd ed. Springer, New York. MR1876169

[25] KIM, J. and WARD, A. R. (2013). Dynamic scheduling of a $GI/GI/1 + GI$ queue with multiple customer classes. *Queueing Syst.* **75** 339–384. MR3110643

[26] KOÇAĞA, Y. L. and WARD, A. R. (2010). Admission control for a multi-server queue with abandonment. *Queueing Syst.* **65** 275–323. MR2652045

[27] KRYLOV, N. V. (1980). *Controlled Diffusion Processes. Applications of Mathematics* **14**. Springer, New York. Translated from the Russian by A. B. Aries. MR0601776

[28] MANDELBAUM, A. and STOLYAR, A. L. (2004). Scheduling flexible servers with convex delay costs: Heavy-traffic optimality of the generalized $c\mu$-rule. *Oper. Res.* **52** 836–855. MR2104141

[29] OCONE, D. and WEERASINGHE, A. (2003). Degenerate variance control in the one-dimensional stationary case. *Electron. J. Probab.* **8** no. 24, 27 pp. (electronic). MR2041825

[30] PANG, G., TALREJA, R. and WHITT, W. (2007). Martingale proofs of many-server heavy-traffic limits for Markovian queues. *Probab. Surv.* **4** 193–267. MR2368951

[31] STANNAT, W. (1999). (Nonsymmetric) Dirichlet operators on L^1: Existence, uniqueness and associated Markov processes. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **28** 99–140. MR1679079

[32] VAN MIEGHEM, J. A. (1995). Dynamic scheduling with convex delay costs: The generalized $c\mu$ rule. *Ann. Appl. Probab.* **5** 809–833. MR1359830

[33] YOSIDA, K. (1980). *Functional Analysis*, 6th ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **123**. Springer, Berlin. MR0617913

A. ARAPOSTATHIS
A. BISWAS
DEPARTMENT OF ELECTRICAL
  AND COMPUTER ENGINEERING
THE UNIVERSITY OF TEXAS AT AUSTIN
1616 GUADALUPE ST., UTA 7.508
AUSTIN, TEXAS 78701
USA
E-MAIL: ari@ece.utexas.edu
        anupbiswas@utexas.edu

G. PANG
THE HAROLD AND INGE MARCUS DEPARTMENT
  OF INDUSTRIAL AND MANUFACTURING ENGINEERING
COLLEGE OF ENGINEERING
PENNSYLVANIA STATE UNIVERSITY
UNIVERSITY PARK, PENNSYLVANIA 16802
USA
E-MAIL: gup3@psu.edu