WEAK REFLECTION PRINCIPLE FOR LÉVY PROCESSES

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In this paper, we develop a new mathematical technique which allows us to express the joint distribution of a Markov process and its running maximum (or minimum) through the marginal distribution of the process itself. This technique is an extension of the classical reflection principle for Brownian motion, and it is obtained by weakening the assumptions of symmetry required for the classical reflection principle to work. We call this method a weak reflection principle and show that it provides solutions to many problems for which the classical reflection principle is typically used. In addition, unlike the classical reflection principle, the new method works for a much larger class of stochastic processes which, in particular, do not possess any strong symmetries. Here, we review the existing results which establish the weak reflection principle for a large class of time-homogeneous diffusions on a real line and then proceed to extend this method to the Lévy processes with one-sided jumps (subject to some admissibility conditions). Finally, we demonstrate the applications of the weak reflection principle in financial mathematics, computational methods and inverse problems.

1. Introduction.

1.1. *Classical reflection principle and its applications.* We start with a brief review of the classical reflection principle for Brownian motion. Denote by B^x the Brownian motion on a real line, started form x. Given arbitrary levels $U > 0$ and $K < U$, we can compute the joint distribution of $B_t = B_t^0$ and its running maximum $M_t = \sup_{u \in [0,t]} B_u$ as follows:

$$
\begin{aligned}
\mathbb{P}(B_t &\leq K, M_t > U) \\
&= \mathbb{P}(B_{t-T_U+T_U} \leq K, T_U < t) = \mathbb{P}(B_{t-T_U}^U \leq K, T_U < t) \\
&= \mathbb{P}(2U - B_{t-T_U}^U \leq K, T_U < t) = \mathbb{P}(B_t \geq 2U - K, M_t > U) \\
&= \mathbb{P}(B_t \geq 2U - K),
\end{aligned}
\tag{1}
$$

where T_U is the first hitting time of U by B, and $B_s^U = B_{T_U+s}$. The above formula first appeared in the work of Bachelier [1], followed by a more rigorous treatment, for example, by Lévy [16]. Notice that the above derivations are based on the following well-known properties of Brownian motion:

- *Strong Markov property*: $B_{T_U+t} - B_{T_U}$ is a standard Brownian motion (started from zero), independent of $\mathcal{F}_{T_U}$ (where the filtration is generated by B).
- *Continuity*: since the paths of B are continuous, $B_{T_U} = U$, and in view of the above, B_{T_U+t} is a Brownian motion started from U and independent of $\mathcal{F}_{T_U}$.
- *Symmetry*: the distribution of B_t^U is symmetric with respect to the initial level U, that is, $\mathrm{Law}(B_t^U) = \mathrm{Law}(2U - B_t^U)$.

We will come back to the above observations in the next subsection, but first let us outline several applications of the classical reflection principle. One obvious application is the computation of the joint distribution. Since the marginal distribution of a Brownian motion is available in closed form, the above formula gives us a closed form expression for the joint distribution of the process and its running maximum, at any given time. A more subtle application, which requires the use of reflection principle itself (rather than the resulting formula for the joint distribution) comes from financial mathematics. Namely, the reflection principle turns out to be very useful in the problem of hedging barrier options. To simplify the notation, throughout the rest of the paper, we consider $U = 0$. Let us assume that the risk-neutral evolution of the underlying is described by a Brownian motion B^x started from $x \leq 0$ (we assume no discounting). Consider an up-and-out option, written on this underlying, with a terminal payoff function h, such that $\mathrm{supp}(h) \subset (-\infty, 0)$. The payoff of such barrier option, at the time of maturity T, is given by

$$h\big(B_T^x\big)\mathbf{1}_{\{\sup_{u\in[0,T]} B_u^x < 0\}}.$$

To find the price of the option, we need to compute the expectation of the above random variable. This problem can be solved by applying the formula for the joint distribution of $(B_T, \sup_{u\in[0,T]} B_u)$, given in (1). However, making use of the reflection principle itself, we can obtain more than just a price: in fact, we can find a static hedging strategy for a given barrier option via the European-type ones. Recall that a European-type option pays, at the time of maturity, a certain function of the terminal value of the underlying. Hence, we need to find a function G, such that up until hitting the barrier, the price of the target barrier option coincides with the price of a European-type option with maturity T and payoff $G(B_T^x)$,

$$\mathbb{E}\big(h\big(B_T^x\big)\mathbf{1}_{\{\sup_{u\in[0,T]} B_u^x < 0\}}|\mathcal{F}_{t\wedge T_0}\big) = \mathbb{E}\big(G\big(B_T^x\big)|\mathcal{F}_{t\wedge T_0}\big),$$

where the filtration is generated by B^x, and T_0 is the first hitting time of zero by B^x. In fact, due to the tower property, it is enough to ensure that the above identity holds for $t = T$. To do this, we consider two cases: $T_0 \geq T$ and $T_0 < T$. In the first case, the above equation reduces to

$$h\big(B_T^x\big)\mathbf{1}_{\{B_T^x < 0\}} = G\big(B_T^x\big)\mathbf{1}_{\{B_T^x < 0\}},$$

which yields that, below the barrier, the payoff function G has to coincide with h: $G(z)\mathbf{1}_{\{z<0\}} = h(z)$. Thus we can search for G in the form $G(z) = h(z) - g(z)$,

where $\text{supp}(g) \subset (0, \infty)$. Considering the second case, and making use of the *strong Markov property* and *continuity* of B, we obtain

$$(2) \qquad \left(\mathbb{E}h(B_\tau^0) - \mathbb{E}g(B_\tau^0)\right)_{\tau=T-T\wedge T_0} \mathbf{1}_{\{T_0 < T\}} = 0,$$

where $B_s^0 = B_{T_0+s}^x$ is a new Brownian motion started from zero, which is independent of $\mathcal{F}_{T_0}$. We emphasize that the main difficulty in finding g that satisfies (2) is the requirement that *h and g have supports on the opposite sides of the barrier $U = 0$*. In the case of a Brownian motion, we can make use of its *symmetry*, to conclude that the function $g(z) = h(-z)$ fulfills (2). In fact, such choice of g satisfies

$$(3) \qquad \mathbb{E}(h(B_t^0)) = \mathbb{E}(g(B_t^0)) \qquad \text{for all } t > 0,$$

which is sufficient for (2) to hold. Thus $G(z) = h(z) - h(-z)$ is the solution to the static hedging problem.

It is worth mentioning that the solution to the integral equation (3) has an interpretation through the partial differential equations (PDEs). Namely, a function g that satisfies (3) and has support in $[0, \infty)$ provides a solution to the following inverse problem: assume that the function h, with support in $(-\infty, 0]$, serves as the initial condition to the following heat equation:

$$(4) \qquad \begin{cases} \partial_t u - \Delta u = 0, & x \in \mathbb{R}, t > 0, \\ u(x, 0) = h(x). \end{cases}$$

The Feynman–Kac formula implies that $u(x, t) = \mathbb{E}h(B_t^x)$. Then it follows from (3) that replacing h with g, we obtain a solution to (4) which coincides with the original solution at $x = 0$, for all $t > 0$. Next, assume that the initial condition in (4) has an arbitrary support in $\mathbb{R}$. The above observation implies that we can modify this initial condition (only) in $[0, \infty)$, to ensure that the resulting solution is zero at $x = 0$, for all $t > 0$. One can think of the initial temperature distribution, which we only control on the positive half line, and only at the initial time. The goal is then to ensure that the temperature at $x = 0$ remains zero at all times. Of course, in the case of a Brownian motion, the solution to this problem is obvious due to the symmetry of the system, $g(x) = h(-x)$. Equivalently, this follows from the symmetry of the associated heat equation: its first order coefficient is zero, and the second order coefficient is constant, which corresponds to constant heat conductivity at all points of the real line. However, a similar problem can formulated for diffusion processes that do not posses any symmetries. In this case, the Laplace operator in (4) is replaced by a more general elliptic operator, whose coefficients may not possess the desired symmetries (e.g., this corresponds to nonconstant and asymmetric heat conductivity). Although the straightforward approach does not work in this case, the *weak symmetry mapping*, introduced in the next section, allows us to solve this problem.

1.2. *Weak reflection principle.* From the above description of the classical reflection principle, it is easy to see that this method continues to work if we substitute the assumptions of strong Markov property, continuity and symmetry with the following: for any $t > 0$, the conditional distribution of $B_{T_U + t}$, given $\mathcal{F}_{T_U}$, is symmetric with respect to the initial level U. Thus the key property is the *symmetry* of the underlying stochastic process. The strong Markov property and continuity are only needed to pass from the conditional distributions to the unconditional ones. This observation immediately yields several extensions of the classical reflection principle. First, it is well known that the reflection principle also works for an exponential of a Brownian motion. Indeed, for any monotone continuous function F, defined on a real line, the process $F(B^x)$ is still strongly Markov and continuous. Assuming the range of F contains $U = 0$, we conclude that the random variable $X_t = F(B_t^V)$, with $V = F^{-1}(0)$, has the same distribution as $F(2V - F^{-1}(X_t)) = F(2V - B_t^V)$. Moreover, the function $x \mapsto F(2V - F^{-1}(x))$ maps $(-\infty, 0]$ into $[0, \infty)$, and vice versa. Then, for any function h, with support in the negative half line, the function

$$g(x) = h\big(F\big(2V - F^{-1}(x)\big)\big)$$

has support in the positive half line and solves the desired integral equation

$$\mathbb{E}h(X_t) = \mathbb{E}h\big(F\big(B_t^V\big)\big) = \mathbb{E}h\big(F\big(2V - B_t^V\big)\big) = \mathbb{E}g(X_t) \qquad \text{for all } t > 0.$$

In general, we say that a real-valued stochastic process X, with $X_0 = 0$, possesses a *strong symmetry* (with respect to zero) if there exists a mapping $\mathbf{S} : \mathbb{R} \to \mathbb{R}$, such that $x\mathbf{S}(x) \le 0$, for all $x \in \mathbb{R}$, and $\mathrm{Law}(\mathbf{S}(X_t)) = \mathrm{Law}(X_t)$, for all $t > 0$. Then, for any given function h, with support on one side of zero, the target equation

$$\mathbb{E}h(X_t) = \mathbb{E}g(X_t) \qquad \text{for all } t > 0,$$

has the solution $g(x) = h(\mathbf{S}(x))$, and the function g has support on the opposite side of zero. Thus the reflection principle can be extended easily to any strong Markov process that possesses a strong symmetry and does not jump across the barrier $U = 0$. In particular, it holds for any diffusion martingale whose coefficient is an even function, since any such process possesses a strong symmetry, with $\mathbf{S} : x \mapsto -x$.

However, the assumption of the existence of a strong symmetry excludes many processes important for applications (some of these examples are discussed in [8]). In the present paper, we develop a weak formulation of the reflection principle, which can be applied to a large class of stochastic processes that *do not posses any strong symmetries*. This new formulation, albeit weaker than the standard one, is sufficient to solve the problems outlined in Section 1.1. Herein, we restrict our analysis to the strong Markov processes which do not jump across the given upper (lower) barrier from below (above). Then, in view of the above discussion, it suffices to consider the unconditional, as opposed to conditional, distributions of the

process. Consider a stochastic process X, defined on a real line and started from zero. Consider two spaces, $\mathcal{B}^-$ and $\mathcal{B}^+$, consisting of all Lebesgue measurable functions h, such that $h(X_t)$ has finite expectation, for all $t \geq 0$, and if $h \in \mathcal{B}^-$, then h has support in $(-\infty, 0]$, while if $h \in \mathcal{B}^+$, then h has support in $[0, \infty)$. We say that X possesses an *upper weak symmetry* (with respect to zero)[2] if there exists a mapping $\mathbf{W}^+$ from a space of test functions $\mathcal{B}_0^- \subset \mathcal{B}^-$ into $\mathcal{B}^+$, such that

$$\mathbb{E}h(X_t) = \mathbb{E}\big(\mathbf{W}^+ h(X_t)\big) \qquad \text{for all } t > 0,$$

for any $h \in \mathcal{B}_0^-$. Analogously, one can define the *lower weak symmetry*, along with the mapping $\mathbf{W}^- : \mathcal{B}_0^+ \to \mathcal{B}^-$. We do not insist on a particular choice of the spaces of test functions $\mathcal{B}_0^\pm$, but, in what follows, we choose the spaces that include all smooth functions with compact support. We will refer to $\mathbf{W}^\pm$ as the *weak symmetry mapping*, although one should remember that in the present setting, the specific form of the symmetry may vary significantly depending on the underlying process X. For example, if the distribution of X_t is not symmetric with respect to zero, the image of a piece-wise linear function may be a curve with nonzero curvature at every point. Thus the *weak reflection principle* consists of the application of the strong Markov property, the (semi-) continuity and the weak symmetry mapping $\mathbf{W}^\pm$.

It is worth mentioning that there is a particular choice of weak symmetry that has received a lot of attention in the existing literature.[3] This symmetry is related to the geometric Brownian motion with drift. Notice that a Brownian motion with a constant (nonzero) drift does not possess any strong symmetries, and neither does its exponential—the geometric Brownian motion—which is the main component of the celebrated Black–Scholes–Merton model. Nevertheless, Carr and Chou [6] eliminate the drift by a Girsanov change of measure and derive the following relation:

$$(5) \qquad \mathbb{E}\left(f\left(\frac{S_T}{S_{T \wedge \tau}}\right) \Big| \mathcal{F}_{T \wedge \tau} \right) = \mathbb{E}\left(\left(\frac{S_T}{S_{T \wedge \tau}}\right)^\alpha f\left(\frac{S_{T \wedge \tau}}{S_T}\right) \Big| \mathcal{F}_{T \wedge \tau} \right),$$

which holds for any (admissible) function f, with S being a geometric Brownian motion and τ being the first hitting time of a (strictly positive) barrier. The value of α is given explicitly via the drift and volatility of S. Relation (5) is extended to an arbitrary stopping time τ in [7], where it is also shown that (5) still holds if S is given by a geometric Brownian motion run on an independent continuous stochastic clock. The equivalent formulations of (5), and the additional properties of stochastic processes S that satisfy this relation are established in [20]. Finally, the authors of [17] and [18] further develop the analysis of (5), which they call the (*quasi*) *self-duality*, in the case when S is a Lévy process and in the multivariate

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[2]Similarly, one can define the weak symmetry with respect to any level U.

[3]We thank the anonymous referee for pointing this out.

case. It is important to observe that, in fact, (5) specifies a weak symmetry mapping associated with $X = \log(S)$. To see this, assume that X is a strongly Markov process, which does not jump across the barrier from below (resp., above), and that τ is the first time when $S = \exp(X)$ hits a given upper (resp., lower) barrier. Without loss of generality, we can also assume that the barrier is equal to one and that f has support in $(0, 1)$ [resp., $(1, \infty)$]. Then, repeating the derivation of (2), we conclude that

$$\mathbb{E} f\big(\exp(X_t)\big) = \mathbb{E}\big(e^{\alpha X_t} f\big(\exp(-X_t)\big)\big) \qquad \text{for all } t > 0,$$

is a sufficient condition for (5) to hold, which also becomes necessary if the distribution of τ has full support in $(0, \infty)$. Thus, in a Markovian setting, (5) can be viewed as the following weak symmetry mapping:

$$(6) \qquad \mathbf{W}^{\pm} h(x) = e^{\alpha x} h(-x).$$

It is shown in [7, 17, 20] and [18] that many popular stochastic processes admit (6) as the weak symmetry mapping. However, it is also easy to see (cf. [8]) that there are many important Markov processes whose weak symmetry mapping is different from (6) and cannot be obtained as a composition of the right-hand side of (6) with a monotone function of x; this extension of (6) is studied in [7]. Therefore, in the present paper, we do *not* focus on the properties of stochastic processes X which admit the particular weak symmetry mapping (6). Instead, *we show in Theorem* 1 *that any Lévy process with one-sided jumps* (*subject to some regularity assumptions*) *possesses a weak symmetry*, which is given by (32) and does not have to coincide with (6).

REMARK 1. Having established the weak reflection principle for a Markov process X, with the weak symmetry mapping $\mathbf{W}^{\pm}$, one can easily extend it to the process that arises as an independent continuous stochastic time change of X, with the same $\mathbf{W}^{\pm}$. This observation is made, for example, in [7] for the particular choice of weak symmetry mapping (6), but, of course, it remains valid for an arbitrary $\mathbf{W}^{\pm}$.

Next, let us demonstrate that the weak reflection principle provides solutions to the problems outlined in Section 1.1, for a class of processes that may not posses any strong symmetries. Without loss of generality, we focus on the weak reflection principle with an upper barrier. Namely, assuming that the process X, with $X_0 = 0$, is strongly Markov, that it does not jump across the barrier $U = 0$ from below and that it possesses the upper weak symmetry $\mathbf{W}^+$, we propose the following applications of our method.

First of all, the weak reflection principle allows us to solve the static hedging problem in a model where $S = X^x$ is the underlying, with $X_0^x = x \leq 0$. Namely, for any admissible function h, with support in $(-\infty, 0)$, at any time up until and

including the time when S hits 0, the price of a European-type option with maturity T and the payoff $h(S_T) - \mathbf{W}^+ h(S_T)$ is given by

(7)
$$
\begin{aligned}
\mathbb{E}&\big(h(S_T) - \mathbf{W}^+ h(S_T)|\mathcal{F}_{t \wedge T_0}\big) \\
&= \mathbb{E}\big(\mathbb{E}\big((h(S_T) - \mathbf{W}^+ h(S_T))\mathbf{1}_{\{T_0 < T\}}|\mathcal{F}_{T \wedge T_0}\big)|\mathcal{F}_{t \wedge T_0}\big) \\
&\quad + \mathbb{E}\big(\mathbb{E}\big((h(S_T) - \mathbf{W}^+ h(S_T))\mathbf{1}_{\{T_0 \geq T\}}|\mathcal{F}_{T \wedge T_0}\big)|\mathcal{F}_{t \wedge T_0}\big) \\
&= \mathbb{E}\big((\mathbb{E}h(X_\tau^s) - \mathbb{E}\mathbf{W}^+ h(X_\tau^s))_{\tau = T - T \wedge T_0, s = S_{T \wedge T_0}} \mathbf{1}_{\{T_0 < T\}}|\mathcal{F}_{t \wedge T_0}\big) \\
&\quad + \mathbb{E}\big(h(S_T)\mathbf{1}_{\{T_0 \geq T\}}|\mathcal{F}_{t \wedge T_0}\big) \\
&= \mathbb{E}\big(h(S_T)\mathbf{1}_{\{\sup_{t \in [0,T]} S_t < 0\}}|\mathcal{F}_{t \wedge T_0}\big),
\end{aligned}
$$

which coincides with the price of an up-and-out barrier option with the terminal payoff function h. In the above, X^s is a copy of the original Markov process which is independent of $\mathcal{F}_{T \wedge T_0}$. Then, the second equality follows from the strong Markov property of S and the fact that $\mathbf{W}^+ h$ is supported in $[0, \infty)$. The last equality, in turn, follows from the continuity of the running maximum of S (which implies $S_{T_0} = 0$) and the definition of $\mathbf{W}^+ h$ as the image of h under the weak symmetry mapping. Thus, in order to offset the risks associated with holding an up-and-out barrier option with the terminal payoff function h (i.e., hedge the barrier option), one needs to sell the European-type option with the payoff function $h - \mathbf{W}^+ h$, and buy it back (at a zero price) when, and if, the underlying hits 0. The static hedge payoff corresponding to the up-and-out put option [i.e., with $h(x) = (K - x)^+$] is computed in Section 3 for a particular choice of Lévy process X.

The weak reflection principle can also be used to express the joint distribution of the process and its running maximum via the marginal distribution of the process. Assuming again that $S = X^x$ with $X_0^x = x \leq 0$, we make use of (7), to obtain, for any $K < 0$,

(8)
$$
\begin{aligned}
\mathbb{P}\Big(S_T \leq K, \sup_{t \in [0,T]} S_t \geq 0\Big) &= \mathbb{E}h_K(S_T) - \mathbb{E}\big(h_K(S_T)\mathbf{1}_{\{\sup_{t \in [0,T]} S_t < 0\}}\big) \\
&= \mathbb{E}\mathbf{W}^+ h_K(S_T),
\end{aligned}
$$

with $h_K = \mathbf{1}_{(-\infty, K]}$.[4] It is worth mentioning that, unlike the static hedging problem, the computation of the joint law of a process does not require the knowledge of the image of h under the weak symmetry mapping. Indeed, we only need to know the integral of $\mathbf{W}^+ h$ with respect to the distribution of S_T. As a result, even

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[4]There is a subtlety hidden in equation (8). Namely, due to the discontinuity of h_K, its weak symmetry image may be a *generalized*, rather than classical, function. Thus, the right-hand side of (8) should be understood as the action of $\mathbf{W}^+ h_K$ on the density of S_T. Section 3 shows how to make this interpretation rigorous.

though in certain cases it may be advantageous to use (8) for the computation of the joint law, this application does not fully utilize the power of the weak reflection principle. In Section 3, we present the numerical implementation of (8) for a particular choice of Lévy process X, and discuss the complexity of this method relative to the existing ones; cf. [13–15, 19] and references therein.

Provided that the Markov process X has a partial (integro-) differential equation, or P(I)DE, associated with it (which is the case for diffusions and Lévy processes), the weak symmetry mapping $\mathbf{W}^+$ allows us to solve the inverse problem associated with this equation. This problem is described briefly at the end of Section 1.1, in the context of diffusion processes (Brownian motion, in particular). Note that the weak reflection principle allows us to solve this inverse problem even when the associated equation does not possess any symmetries. Section 3 provides a more detailed discussion of this application for a particular choice of Lévy process X.

1.3. *Prior results.* In view of the strong Markov assumption, it is natural to formulate the problem of weak symmetry for jump-diffusions. In [8], the weak symmetry mapping is constructed for a class of diffusion processes whose coefficients are only required to satisfy some regularity conditions and do not have to be symmetric. Namely, consider a diffusion process given by

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dB_t,$$

where B is a Brownian motion. We assume that $\inf_{x\in\mathbb{R}}\sigma(x) > 0$, the functions μ and σ belong to $C^3(\mathbb{R})$, the functions themselves and their first three derivatives have finite limits at $-\infty$, and, for any $k = 1, 2, 3$, the functions $e^{(3-k)x}|\mu^{(k)}(x)|$ and $e^{(3-k)x}\sigma^{(k)}(x)$ are bounded over all $x > 0$. Then Theorem 2.8 in [8] provides an explicit integral representation for the weak symmetry mapping associated with X. For the sake of completeness, we present a simplified corollary of this theorem here.

PROPOSITION 1 (Carr and Nadtochiy [8]). *Let X be as above, and let h be a once weakly differentiable function, with support in $(-\infty, 0)$, such that its derivative is locally integrable and has a modification with finite variation over $(-\infty, 0)$. Then there exists a continuous and exponentially bounded function g, with support in $(0, \infty)$, such that*

$$(9) \qquad \mathbb{E}h(X_t) = \mathbb{E}g(X_t) \qquad \text{for all } t > 0.$$

Moreover, for any large enough $\gamma > 0$, the function g can be computed as follows:

$$g(x) = \frac{2}{\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{w\psi_1(x, w)}{\partial_x\psi_1(0, w) - \partial_x\psi_2(0, w)}$$
$$\times \int_{-\infty}^{0} \frac{\psi_1(z, w)}{\sigma^2(z)} \exp\left(-2\int_0^z \frac{\mu(y)}{\sigma^2(y)}\,dy\right) h(z)\,dz\,dw,$$

where ψ_1 and ψ_2 are the fundamental solutions of the associated Sturm–Liouville equation

$$(10) \qquad \frac{1}{2}\sigma^2(x)\frac{\partial^2}{\partial x^2}\psi(x,w) + \mu(x)\frac{\partial}{\partial x}\psi(x,w) - w^2\psi(x,w) = 0,$$

determined uniquely, for all complex w with large enough $\mathrm{Re}(w) > 0$, by the following conditions: $\psi_1(\cdot, w)$ *is square integrable on* $(-\infty, 0)$, $\psi_2(\cdot, w)$ *is square integrable on* $(0, \infty)$ *and* $\psi_1(0, w) = \psi_2(0, w) = 1$.

The above result shows that any regular enough diffusion possesses a weak symmetry given by an explicit integral transform. Of course, in order to implement this transform numerically, one needs to know the fundamental solutions of the associated Sturm–Liouville equation. These functions can be approximated efficiently by expanding them into power series of w; cf. [21]. Alternatively, one can notice that if $\mu \equiv 0$ and σ is piecewise constant, then ψ_i's are piecewise linear-exponential (linear combinations of exponentials). Thus we can approximate any function σ with the piecewise constant ones, then compute ψ_i's in closed form, and finally, obtain g via numerical integration. Examples of functions g corresponding to a piecewise linear function h can be found in [8].

The remainder of this paper is organized as follows. Section 2.1 introduces the notation, states the main assumptions and formulates the weak symmetry problem for spectrally-negative Lévy processes. Section 2.2 contains the technical lemmas needed for the proof of the main results, which are given in Section 2.3. Theorem 1 provides the weak symmetry mapping for spectrally-negative Lévy processes, and Corollary 2 addresses the computational aspects. Finally, Section 3 illustrates the applications of the weak reflection principle, for a particular choice of the Lévy process, and Section 4 summarizes the results and outlines the future research directions.

2. Weak symmetry of spectrally negative Lévy processes.

2.1. *Problem formulation.* Consider a Lévy process $(X_t)_{t\geq 0}$, given by its initial condition $X_0 = 0$ and the Laplace exponent ψ,

$$(11) \qquad \psi(\lambda) = \mu\lambda + \frac{\sigma^2}{2}\lambda^2 + \int_{-\infty}^{0} (e^{\lambda x} - 1 - \lambda x)\Pi(dx),$$

where Π is the Lévy measure of X, and

$$\mathbb{E}e^{\lambda X_t} = e^{t\psi(\lambda)},$$

for all complex λ for which both sides of the above equation are well defined. To make sure that the above expressions are well defined, at least, for all λ with positive real part, as well as to simplify some of the derivations that follow, we make the following assumption on the Lévy triplet (μ, σ, Π).

ASSUMPTION 1.   We assume that $\mu \in \mathbb{R}$, $\sigma > 0$ and that Π is a σ-finite Borel measure on $(-\infty, 0)$, satisfying

$$\int_{-\infty}^{0} x^2 \Pi(dx) < \infty \quad \text{and} \quad \int_{-\infty}^{-1} |x| e^{-\zeta x} \Pi(dx) < \infty,$$

with some $\zeta \geq 0$.

The process X is called *spectrally negative* because it is only allowed to have negative (i.e., downward) jumps; cf. [15]. The reason for such a restriction is that the process must not jump across the upper barrier in order for the weak reflection principle to hold. Of course, in the case of a lower barrier, one needs to consider X with positive jumps only.

Our goal is to construct a weak symmetry mapping for the process X. Namely, for any given admissible function $h : \mathbb{R} \to \mathbb{R}$, with $\operatorname{supp}(h) \subset (-\infty, 0)$, we would like to find a measurable function $g : \mathbb{R} \to \mathbb{R}$, with $\operatorname{supp}(g) \subset [0, \infty)$, such that

$$(12) \qquad \mathbb{E}h(X_t) = \mathbb{E}g(X_t) \qquad \forall t > 0.$$

To ensure that the expectation of $h(X_t)$ is well defined, we need to make some additional assumptions on h and Π.

ASSUMPTION 2.   We assume that $\operatorname{supp}(h) \subset (-\infty, 0)$ and that there exists $\hat{h} \in \mathbb{L}^1(\mathbb{R})$, such that the function $x \mapsto e^{\zeta x} h(x)$, defined for all $x \in \mathbb{R}$, is a Fourier transform of $\hat{h}$ (with ζ given in Assumption 1).[5]

Taking the Laplace transform with respect to t on both sides of (12), we obtain

$$(13) \qquad \mathbb{E} \int_0^\infty e^{-\lambda t} h(X_t) \, dt = \mathbb{E} \int_0^\infty e^{-\lambda t} g(X_t) \, dt.$$

Assumption 2 implies that

$$(14) \qquad \begin{aligned} \mathbb{E} \int_0^\infty e^{-\lambda t} |h(X_t)| \, dt &\leq \int_0^\infty e^{-\lambda t} \mathbb{E} e^{-\zeta X_t} \, dt \\ &= \int_0^\infty e^{-(\lambda - \psi(-\zeta))t} \, dt < \infty \end{aligned}$$

holds for all real $\lambda > \psi(-\zeta) \vee 0$. Due to the uniqueness of the Laplace inverse, our problem is equivalent to the following: find a measurable function $g : \mathbb{R} \to \mathbb{R}$, with $\operatorname{supp}(g) \subset [0, \infty)$, such that (13) holds for all large enough $\lambda > 0$.

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[5]We use the convention that Assumptions 1 and 2 hold for all the subsequent derivations, except the statements of lemmas, theorems and corollaries, where all assumptions are stated explicitly.

Next, recall that the Laplace transform in time of the expectation of a function of a Markov process is given by the value of an integral operator, called the *resolvent operator*, applied to this function. In particular, as follows, for example, from Theorem 2.7 in [15], for the spectrally negative Lévy process X, we have

$$\mathbb{E}\int_0^\infty e^{-\lambda t} f(X_t)\,dt = \int_{\mathbb{R}} \big(\Phi'(\lambda)e^{-\Phi(\lambda)x} - W^\lambda(-x)\big)f(x)\,dx,$$

which holds for any measurable function f for which the integral on the left-hand side is absolutely convergent, with

$$\Phi(\lambda) := \sup\{q \geq 0 : \psi(q) = \lambda\}, \qquad \lambda \geq 0$$

and with W^λ being the λ-*scale function* of X. Recall that, for every real $\lambda \geq 0$, the λ-scale function $W^\lambda : \mathbb{R} \to [0, \infty)$ is uniquely defined as the right-continuous function that takes value zero in $(-\infty, 0)$ and satisfies

$$\int_{\mathbb{R}} e^{-wx} W^\lambda(x)\,dx = \frac{1}{\psi(w) - \lambda},$$

for all $w \geq \Phi(\lambda)$. For further details on the theory of scale functions, we refer the reader to [15] and the references therein. Thus (13) is equivalent to

$$\int_0^\infty \Phi'(\lambda)e^{-\Phi(\lambda)x} g(x)\,dx = \Upsilon(\lambda),$$

where

(15) $$\Upsilon(\lambda) := \mathbb{E}\int_0^\infty e^{-\lambda t} h(X_t)\,dt = \int_0^\infty \big(\Phi'(\lambda)e^{\Phi(\lambda)x} - W^\lambda(x)\big)h(-x)\,dx.$$

Notice that ψ is continuous and strictly increasing on $(|\mu|/\sigma^2, \infty)$, exploding at infinity. Hence, we can change the variables in the above equations to obtain an equivalent formulation of the problem. Namely, we search for a measurable function g, such that, for all large enough real $\lambda > 0$, the following holds:

(16) $$\int_0^\infty e^{-\lambda x} g(x)\,dx = \psi'(\lambda)\Upsilon\big(\psi(\lambda)\big),$$

with

(17) $$\psi'(\lambda)\Upsilon\big(\psi(\lambda)\big) = \int_0^\infty \big(e^{\lambda x} - \psi'(\lambda)W^{\psi(\lambda)}(x)\big)h(-x)\,dx.$$

Notice that the weak symmetry problem (16) now looks exactly like the Laplace transform inversion. However, there is a major difference. In the classical problem of inverting a Laplace transform, we typically know that the right-hand side is a Laplace transform of some function, and we need to find a mapping that would recover this function. In the present case, we need to prove that $\psi'(\lambda)\Upsilon(\psi(\lambda))$ is, indeed, a Laplace transform of some function g, which is not obvious a priori. In addition, we need to propose a method to recover g. Of course, there exist several

methods for inverting the Laplace transform; cf. [10, 22]. However, the conditions that are required for some of these methods to succeed are expressed through the original function (in our case, g), rather than the transformed one [in the present case, $\psi'(\lambda)\Upsilon(\psi(\lambda))$]. Since, a priori, we know very little about function g (e.g., we do not even know if it exists), we cannot apply any of the existing results on the Laplace transform inversion to solve the weak symmetry problem. Instead, we will show, by hand, that the desired function g exists and that it can be recovered via the classical Bromwich integral; cf. [10].

2.2. *A priori estimates.* In this subsection, we establish the analytic continuation of $\psi'(\lambda)\Upsilon(\psi(\lambda))$ to a complex half plane of the form $H_R = \{w : \operatorname{Re}(w) > R\}$ and provide some useful estimates of its absolute value.

REMARK 2. Notice that $\Upsilon(\lambda)$ can be easily extended to a half plane H_R, via its probabilistic representation given by the first identity in (15). However, under the change of variables $\lambda \mapsto \psi(\lambda)$, the half plane transforms into a smaller domain which is not sufficient for our purposes. To obtain an analytic extension of $\Upsilon(\psi(\lambda))$, one would need the probabilistic representation in (15) to hold in a domain where the real part of λ is unbounded from below, which is usually impossible. More precisely, the conditions that function h has to satisfy, in order for the probabilistic representation in (15) to hold in the desired domain, are extremely restrictive and rather implicit. For example, if X is a martingale, these conditions exclude all convex functions h, except zero. In addition, even for those functions h for which the probabilistic representation in (15) holds in the desired domain (although we do not know how to characterize this set explicitly), the standard estimates do not provide sufficient information about the asymptotic behavior of $\Upsilon(\psi(\lambda))$, as $|\lambda| \to \infty$, which is needed to solve the weak symmetry problem (16).

Alternatively, one might be tempted to use the analytic continuation of $W^{\psi(\lambda)}$ to extend $\Upsilon(\psi(\lambda))$ to a complex half plane via (17). Indeed, it is well known (cf. [15]) that W^λ can be extended analytically to the entire complex plane. However, to the best of our knowledge, there exist no estimates of this extension [more precisely, we need an estimate of $e^{\lambda x} - \psi'(\lambda)W^{\psi(\lambda)}(x)$], which, in particular, would guarantee that the integral on the right-hand side of (17) is well defined for all λ in a half plane H_R. In fact, it is easy to construct an example of a Lévy process whose infinitesimal generator has a nontrivial spectrum, and hence the integral on the right-hand side of (17) is not well defined for some λ.

The difficulties described above explain why we are forced to construct the analytic continuation of $\Upsilon(\psi(\lambda))$, and investigate its asymptotic behavior, by hand.

It turns out that Fourier transform offers a natural way to obtain the desired analytic continuation. Recall that, due to Assumption 2,

$$h(x) = e^{-\zeta x} \int_{\mathbb{R}} e^{-ixz} \hat{h}(z) \, dz.$$

Making use of (14), we apply Fubini's theorem to obtain

$$\psi'(\lambda)\Upsilon(\psi(\lambda))$$

(18)
$$= \int_0^\infty (e^{\lambda x} - \psi'(\lambda)W^{\psi(\lambda)}(x))h(-x)\,dx$$

$$= \int_{\mathbb{R}} \int_0^\infty (e^{\lambda x} - \psi'(\lambda)W^{\psi(\lambda)}(x))e^{(\zeta+iz)x}\,dx\,\hat{h}(z)\,dz,$$

for all large enough real $\lambda > 0$. Our next goal is to extend the above representation to a complex half plane H_R and estimate its absolute value from above. Let us analyze the inner integral on the right-hand side of (18). Notice that it can be viewed as

$$\int_0^\infty (e^{\lambda x} - \psi'(\lambda)W^{\psi(\lambda)}(x))e^{-wx}\,dx,$$

evaluated at $w = -\zeta - iz$. For $\lambda > |\mu|/\sigma^2$ and $w \in (\lambda, \infty)$, the above integral can be computed explicitly. Using the definition of the scale function $W^{\psi(\lambda)}$ (cf. [15]), we obtain

$$\int_0^\infty (e^{\lambda x} - \psi'(\lambda)W^{\psi(\lambda)}(x))e^{-wx}\,dx$$

(19)
$$= \int_0^\infty e^{\lambda x - wx}\,dx - \psi'(\lambda)\int_0^\infty e^{-wx}W^{\psi(\lambda)}(x)\,dx$$

$$= \frac{1}{w-\lambda} - \frac{\psi'(\lambda)}{\psi(w) - \psi(\lambda)}$$

$$= \frac{\psi(w) - \psi(\lambda) - \psi'(\lambda)(w-\lambda)}{(w-\lambda)^2}\frac{w-\lambda}{\psi(w) - \psi(\lambda)}.$$

The right-hand side of the above is analytic everywhere in $w \in H_{-\zeta} = \{w : \operatorname{Re}(w) > -\zeta\}$, except, possibly, the zeros of $\psi(w) - \psi(\lambda)$, where it has poles (on every compact, there is at most a finite number of such points). The left-hand side of the above is analytic in $H_{-\zeta}$ (we can differentiate with respect to w inside the integral, and the resulting expression is bounded, uniformly over w, by an absolutely integrable function of x). Therefore, the right-hand side, in fact, does not have any poles in $H_{-\zeta}$, and the above equality holds for all $w \in H_{-\zeta}$. By continuity, the equality holds at $w = -\zeta - iz$

(20)
$$\int_0^\infty (e^{\lambda x} - \psi'(\lambda)W^{\psi(\lambda)}(x))e^{(\zeta+iz)x}\,dx$$

$$= \frac{\psi'(\lambda)}{\psi(\lambda) - \psi(-\zeta - iz)} - \frac{1}{\lambda + \zeta + iz},$$

for all large enough real $\lambda > 0$. Next, we need to estimate the absolute value of the right-hand side of the above. To do this, we will use the following lemmas,

which are also crucial for the proofs of the main result. These lemmas describe the asymptotic behavior of $1/(\psi(\lambda) - \psi(-\zeta - iz))$, and they can be viewed as the core technical result of the paper. Their proofs are given in Appendix A.

LEMMA 1. *Let Assumption 1 hold. For all $u, z \in \mathbb{R}$ and $v > 0$, we have*

$$\text{Im}(\psi(v + iu) - \psi(-\zeta - iz)) = \mu(u + z) + (vu - z\zeta)\sigma^2 + \bar{\bar{o}}(uv)$$
$$+ \text{sign}(u)|\bar{\bar{o}}(u^2)| + \text{sign}(z)|\bar{\bar{o}}(z^2)| + \bar{\bar{o}}(z),$$

where we denote by $\bar{\bar{o}}(f(u, v, z))$ any function of (u, v, z), such that $\bar{\bar{o}}(f(u, v, z))/f(u, v, z)$ is absolutely bounded over all $u, z \in \mathbb{R}$ and $v > 0$, and it converges to zero, as $|f(u, v, z)| \to \infty$.

LEMMA 2. *Let Assumption 1 hold. For all $u, z \in \mathbb{R}$ and $v > 0$, we have*

$$\text{Im}(\psi(v + iu) - \psi(-\zeta - iz)) = \mu(u + z) + (vu - z\zeta)\sigma^2 + \bar{\bar{o}}(uv) + \bar{\bar{o}}(z^2 - u^2)$$
$$+ \text{sign}(z + u)|\bar{\bar{o}}((z + u)^2)| + \bar{\bar{o}}(z),$$

where $\bar{\bar{o}}(f(u, v, z))$ has the same meaning as in Lemma 1.

LEMMA 3. *Let Assumption 1 hold. For all $u, z \in \mathbb{R}$ and $v > 0$, we have*

$$\text{Re}(\psi(v + iu) - \psi(-\zeta - iz)) = \mu(v + \zeta) + (v^2 + z^2 - u^2)\sigma^2/2 + \bar{\bar{o}}(z^2 - u^2)$$
$$+ |\bar{\bar{o}}(uv)| + |\bar{\bar{o}}(v^2)| + \bar{\bar{o}}(z) + \bar{\bar{o}}(1),$$

where $\bar{\bar{o}}(f(u, v, z))$ has the same meaning as in Lemma 1.

LEMMA 4. *Let Assumption 1 hold. For any $\varepsilon \in (0, 1)$, there exist $R_1 > 0$, $R_2 > 0$, $c_1 > 0$ and $c_2 > 0$, such that the inequality*

$$|\psi(v + iu) - \psi(-\zeta - iz)| \geq c_1 v(|z| + v)\mathbf{1}_{\{||u| - \sqrt{z^2 + v^2}| \leq \varepsilon v\}}$$
$$+ c_2|u^2 - (z^2 + v^2)|\mathbf{1}_{\{||u| - \sqrt{z^2 + v^2}| > \varepsilon v\}}$$

holds for all $u, v, z \in \mathbb{R}$, with $v > R_1$ and $|z| > R_2$.

LEMMA 5. *Let Assumption 1 hold. There exists a constant $R_1 > 0$, such that, for any $v > R_1$, there exist $R_2 = R_2(v) > 0$, $R_3 = R_3(v) > 0$ and $c = c(v) > 0$, such that the inequality*

$$|\psi(v + iu) - \psi(-\zeta - iz)|^2 \geq c((z^2 - u^2)^2 + z^2),$$

holds for all $u, z \in \mathbb{R}$, with $|z| > R_2$ and $|u| > R_3$.

The following lemma makes use of the above results to show that the right-hand side of (20) is absolutely integrable with respect to $\hat{h}(z) \, dz$, even for complex λ.

LEMMA 6. *Let Assumption 1 hold. Then there exist constants c and R_1, such that the following holds, for all $u \in \mathbb{R}$, all $v \geq R_1$ and all $\hat{h} \in \mathbb{L}^1 \cup \mathbb{L}^2$: $\psi(v + iu) \neq \psi(-\zeta - iz)$, for all $z \in \mathbb{R}$ and*

$$(21) \qquad \begin{aligned} \int_{\mathbb{R}} \left| \frac{\psi'(v + iu)}{\psi(v + iu) - \psi(-\zeta - iz)} - \frac{1}{v + iu + \zeta + iz} \right| \left| \hat{h}(z) \right| dz \\ \leq c |v + iu| \left(\|\hat{h}\|_{\mathbb{L}^1(\mathbb{R})} \wedge \|\hat{h}\|_{\mathbb{L}^2(\mathbb{R})} \right). \end{aligned}$$

PROOF. Fix arbitrary $\varepsilon \in (0, 1)$. Lemma 4 yields that there exist $R_1, R_2 > 0$ and c_i's, such that the following estimates hold, for all $u \in \mathbb{R}$, all $v \geq R_1$ and all $|z| \geq R_2$:

$$(22) \qquad \begin{aligned} \left| \frac{1}{\psi(v + iu) - \psi(-\zeta - iz)} \right|^2 \\ \leq \frac{c_1}{v^2 (|z| + v)^2} \mathbf{1}_{\{||u| - \sqrt{z^2 + v^2}| \leq \varepsilon v\}} + \frac{c_2}{\varepsilon^2 v^2 (v^2 + z^2)} \mathbf{1}_{\{||u| - \sqrt{z^2 + v^2}| > \varepsilon v\}} \\ \leq \frac{c_3}{(1 + |z|)^2}. \end{aligned}$$

It is also easy to see [by a direct examination of (11)] that, for all large enough $|\lambda|$, such that $\text{Re}(\lambda) \geq -\zeta$, we have

$$(23) \qquad \left| \psi''(\lambda) \right| \leq c_4, \qquad \left| \psi'(\lambda) \right| \leq c_5 |\lambda|, \qquad c_6 |\lambda|^2 \leq \left| \psi(\lambda) \right| \leq c_7 |\lambda|^2,$$

with some strictly positive constants c_i. Collecting the above and possibly increasing R_1, we obtain

$$\begin{aligned} \int_{\mathbb{R}} &\left| \frac{\psi'(v + iu)}{\psi(v + iu) - \psi(-\zeta - iz)} - \frac{1}{v + iu + \zeta + iz} \right| \left| \hat{h}(z) \right| dz \\ &\leq \frac{c_8 |v + iu|}{|v + iu|^2 - R_2^2} \int_{|z| \leq R_2} \left| \hat{h}(z) \right| dz \\ &\quad + \int_{|z| > R_2} \left| \frac{\psi'(v + iu)}{\psi(v + iu) - \psi(-\zeta - iz)} \right| \left| \hat{h}(z) \right| dz \\ &\quad + \int_{\mathbb{R}} \left| \frac{1}{v + iu + \zeta + iz} \right| \left| \hat{h}(z) \right| dz \\ &\leq \frac{c_8 |v + iu|}{|v + iu|^2 - R_2^2} \int_{|z| \leq R_2} \left| \hat{h}(z) \right| dz + c_9 |v + iu| \int_{|z| > R_2} \frac{1}{1 + |z|} \left| \hat{h}(z) \right| dz \\ &\quad + c_{10} \int_{\mathbb{R}} \frac{1}{v + \zeta + |z + u|} \left| \hat{h}(z) \right| dz, \end{aligned}$$

which yields (21), after an application of the Cauchy inequality. □

Recall that, due to equations (18) and (20), the representation

$$(24) \quad \psi'(\lambda)\Upsilon(\psi(\lambda)) = \int_{\mathbb{R}} \left(\frac{\psi'(\lambda)}{\psi(\lambda) - \psi(-\zeta - iz)} - \frac{1}{\lambda + \zeta + iz} \right) \hat{h}(z)\, dz$$

is well defined and holds for all large enough real $\lambda > 0$. Lemma 6 shows that the right-hand side of the above is well defined for all $\lambda \in H_{R_1} = \{\lambda : \operatorname{Re}(\lambda) > R_1\}$, with $R_1 > 0$ given in Lemma 6. In fact, it is easy to deduce that the right-hand side of (24) is analytic in H_{R_1}. To see this, first notice that the integrand in (24) is analytic in $\lambda \in H_{R_1}$ (as the denominators cannot vanish). Then differentiate, formally, inside the integral, and apply the same estimates as in the proof of Lemma 6 to show that the integral of the derivative is absolutely convergent, for any $\lambda \in H_{R_1}$. Thus we have proved the following corollary.

COROLLARY 1. *Let Assumptions* 1 *and* 2 *hold, and let* $R_1 > 0$ *be the constant appearing in Lemma* 6. *Then the function* $\lambda \mapsto \psi'(\lambda)\Upsilon(\psi(\lambda))$ [*defined in* (17), *for all large enough real* $\lambda > 0$] *can be extended analytically to* $H_{R_1} = \{\lambda : \operatorname{Re}(\lambda) > R_1\}$ *via* (24).

2.3. *Main results.* Now, we have everything we need to solve the weak symmetry problem (16). First, for any $r > 0$, we introduce the following function of $x \in \mathbb{R}$:

$$(25) \quad g_r(x) = \frac{1}{2\pi} \int_{-r}^{r} e^{(\gamma + iu)x} \int_{\mathbb{R}} \left(\frac{\psi'(\gamma + iu)}{\psi(\gamma + iu) - \psi(-\zeta - iz)} \right. \\ \left. - \frac{1}{\gamma + iu + \zeta + iz} \right) \hat{h}(z)\, dz\, du,$$

with a large (but fixed) constant $\gamma > 0$. To ensure that g_r is well defined, we assume that $\gamma > R_1$, where R_1 is the constant appearing in Lemma 6. In this section, we show that g_r has a limit g, as $r \to \infty$, and that g is the weak symmetry image of h (as discussed in Section 1.2). However, before we present the main result of the paper, we need to relax our assumptions on function h. Notice that Assumption 2 excludes some functions h that are important for applications, such as the indicator functions.[6] Indeed, if $h = \mathbf{1}_{(-\infty, K]}$, then

$$\hat{h}(z) = \frac{\exp(K(\zeta + iz))}{2\pi(\zeta + iz)},$$

which is not absolutely integrable over $\mathbb{R}$. Nevertheless, the above function belongs to $\mathbb{L}^2(\mathbb{R})$, and the right-hand side of (25) is well defined for all $\hat{h} \in \mathbb{L}^1(\mathbb{R}) \cup \mathbb{L}^2(\mathbb{R})$, due to Lemma 6. Thus we extend the scope of our analysis to include all functions h that satisfy the following assumption (which is a strictly weaker version of Assumption 2).

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[6]We thank the anonymous referee for pointing this out.

ASSUMPTION 3.   We assume that supp(h) $\subset (-\infty, 0)$ and that there exists $\hat{h} \in \mathbb{L}^1(\mathbb{R}) \cup \mathbb{L}^2(\mathbb{R})$, such that the function $x \mapsto e^{\zeta x} h(x)$, defined for all $x \in \mathbb{R}$, coincides almost everywhere with the Fourier transform of $\hat{h}$ (with ζ given in Assumption 1).

Recall that, in order to construct the weak symmetry image of h, we need to consider the expectation of $h(X_t)$. However, if $\hat{h} \in \mathbb{L}^2(\mathbb{R})$, we cannot guarantee that the expectation of $h(X_t)$ is well defined: in this case, h may not be locally bounded. Nevertheless, if X_t has a well-behaved density p_t [such that $e^{-\zeta x} p_t(x)$ is square integrable over $x \in \mathbb{R}$], the expectation of $h(X_t)$ is well defined since the associated integral

$$\mathbb{E}h(X_t) = \int_{\mathbb{R}} \left(e^{\zeta x} h(x)\right)\left(e^{-\zeta x} p_t(x)\right) dx$$

is absolutely convergent, which follows from the Cauchy inequality. The following discussion shows that p_t does possess the desired properties and in addition, provides some auxiliary constructions needed to formulate the main result.

Denote by $\mathcal{D}$ the space of all functions $f : \mathbb{R} \to \mathbb{R}$, which are 3 times continuously differentiable and satisfy

$$\sup_{x \in \mathbb{R}} \left| e^{kx} f^{(n)}(x) \right| < \infty, \qquad n = 0, 1, 2, 3, k = 0, 1, 2 \ldots.$$

We equip $\mathcal{D}$ with the topology generated by the above family of semi-norms and consider $\mathcal{D}^*$—the dual of $\mathcal{D}$, consisting of all continuous linear functionals on $\mathcal{D}$. Using Fubini's theorem, it is easy to see that $g_r \in \mathcal{D}^*$, for any $r > 0$. In addition, Assumption 1 implies that the marginal density p_t of X_t is well defined, for all $t > 0$. For any fixed $t > 0$ and $k = 0, 1, 2, \ldots$, the Fourier transform of $x \mapsto e^{kx} p_t(x)$ is given by

$$(26) \qquad \int_{\mathbb{R}} e^{-iux} e^{kx} p_t(x) dx = \mathbb{E}e^{(k-iu)X_t} = \exp\left(t\psi(k - iu)\right),$$

which can be computed for all $u \in \mathbb{R}$ as an analytic continuation of its values in the domain $u \in \{-iy | y \in (k, \infty)\}$, where the integral on the left-hand side is a priori known to be absolutely convergent. Applying standard estimates to the integral term in $\psi(k + iu)$ [cf. (11)], we can easily deduce that

$$(27) \qquad \psi(k + iu) \sim -cu^2, \qquad |u| \to \infty,$$

where c is a positive constant. This, in particular, shows that the Fourier transform of $x \mapsto e^{kx} p_t(x)$ belongs to the Schwartz space (i.e., the space of infinitely smooth functions, decaying at $\pm\infty$ faster than any power). Using the standard properties of Fourier transform, we conclude that $x \mapsto e^{kx} p_t(x)$ belongs to the Schwartz space as well. In addition, we notice that relations (26)–(27) hold for all $k \geq -\zeta$, due to Assumption 1. Thus, we have proved the following lemma.

LEMMA 7. *Let Assumption 1 hold. For any $t > 0$, we have $p_t \in \mathcal{D}$, and the mapping $x \mapsto e^{-\zeta x} p_t(x)$ belongs to $\mathbb{L}^1(\mathbb{R}) \cap \mathbb{L}^2(\mathbb{R})$.*

Finally, we present the main result of this paper.

THEOREM 1. *Let Assumptions 1 and 3 hold, and let g_r be given by (25), with an arbitrary $\gamma > R_1$, where R_1 is the constant appearing in Lemma 6. Then, the following holds.*

- *As $r \to \infty$, g_r converges weakly to a generalized function $g \in \mathcal{D}^*$, which has support in $[0, \infty)$ and satisfies*

$$\langle g, p_t \rangle = \mathbb{E}h(X_t) \qquad \text{for all } t > 0.$$

Moreover, there exist constants $c_1, c_2 > 0$, independent of $\hat{h}$, r and t, such that, for all $r > 0$ and all $t > 0$, we have

$$(28) \qquad |\langle g - g_r, p_t \rangle| \le c_1 (\|\hat{h}\|_{\mathbb{L}^1(\mathbb{R})} \wedge \|\hat{h}\|_{\mathbb{L}^2(\mathbb{R})}) \int_r^\infty u \exp(-c_2 t u^2) \, du.$$

- *If, in addition, Assumption 2 holds, then the restriction of g to the interval $(0, \infty)$ coincides with a continuous function, which has at most exponential growth at infinity [i.e., $|g(x)|$ is bounded by a constant times an exponential, for all large enough values of x]. Moreover, there exists a constant $c > 0$, independent of $\hat{h}$, r and x, such that, for all large enough $r > 0$ and all $x > 0$, we have*

$$(29) \qquad |g(x) - g_r(x)| \le c \frac{e^{\gamma x}}{x} \left(\frac{1}{r} \|\hat{h}\|_{\mathbb{L}^1(\mathbb{R})} + \int_{|z| > r/2} |\hat{h}(z)| \, dz \right).$$

- *If, in addition, Assumption 2 holds and*

$$(30) \qquad \int_0^\infty \left(\int_{|z| > |u|} |\hat{h}(z)| \, dz \right)^2 du < \infty,$$

then g is locally integrable in $\mathbb{R}$ and continuous in $\mathbb{R} \setminus \{0\}$, with at most exponential growth at infinity, and

$$(31) \qquad \mathbb{E}g(X_t) = \mathbb{E}h(X_t) \qquad \text{for all } t > 0.$$

REMARK 3. Notice that the uniqueness of function g, having support in $[0, \infty)$ and satisfying (31), follows from the uniqueness of the Laplace inverse and from equation (16), derived in Section 2.1.

The proof of the above theorem is given in Appendix B. Theorem 1 shows that any spectrally negative Lévy process X, satisfying Assumption 1, possesses an upper weak symmetry, with the space of test functions $\mathcal{B}_0^-$ consisting of all functions

that satisfy Assumption 2. The associated weak symmetry image transformation is given by

$$
\begin{aligned}
\mathbf{W}^+ h(x) &= g(x) \\
&= \frac{1}{2\pi} \int_{\mathbb{R}} e^{(\gamma+iu)x} \int_{\mathbb{R}} \left(\frac{\psi'(\gamma+iu)}{\psi(\gamma+iu) - \psi(-\zeta-iz)} \right. \\
&\qquad\qquad \left. - \frac{1}{\gamma+iu+\zeta+iz} \right) \hat{h}(z)\, dz\, du.
\end{aligned}
\tag{32}
$$

To implement this transformation, one needs to find the Fourier inverse of $x \mapsto e^{\zeta x} h(x)$ and evaluate the above integral numerically, by truncating the domain of the integration

$$
\begin{aligned}
g_{r,R}(x) &= \frac{1}{2\pi} \int_{-r}^{r} e^{(\gamma+iu)x} \int_{-R}^{R} \left(\frac{\psi'(\gamma+iu)}{\psi(\gamma+iu) - \psi(-\zeta-iz)} \right. \\
&\qquad\qquad \left. - \frac{1}{\gamma+iu+\zeta+iz} \right) \hat{h}(z)\, dz\, du,
\end{aligned}
\tag{33}
$$

and, then, consider $r, R \to \infty$. The next corollary provides the rate of convergence of $g_{r,R}$ to g. Its proof is given in Appendix B.

COROLLARY 2. *Let Assumptions 1 and 2 hold, and let $g_{r,R}$ be given by (33), with an arbitrary $\gamma > R_1$, where R_1 is the constant appearing in Lemma 6. Then, there exists a constant $c > 0$, independent of $\hat{h}$, r, R and x, such that, for all large enough $r, R > 0$ and all $x > 0$, we have*

$$
\left| g(x) - g_{r,R}(x) \right| \le c \frac{e^{\gamma x}}{x} \left(\frac{1}{r} \|\hat{h}\|_{\mathbb{L}^1(\mathbb{R})} + \int_{|z| > (r/2) \wedge R} \left| \hat{h}(z) \right| dz \right).
\tag{34}
$$

3. Examples and implementation. Consider a Lévy process X given by the sum of a scaled Brownian motion and a negative Gamma process. In other words,

$$
X_t = \sigma B_t - \Gamma_t,
\tag{35}
$$

where B is a standard Brownian motion, and Γ is a Gamma process with parameters $\alpha > 0$ and $\beta > 0$; cf. [4, 19]. In this case, the characteristic triplet of X, as defined in (11), is given by

$$
\left(\mu = -\frac{\beta}{\alpha}, \sigma = \sigma, \Pi(dx) = \beta \frac{e^{-\alpha|x|}}{|x|} \mathbf{1}_{(-\infty,0)}(x)\, dx \right),
$$

and in particular,

$$
\psi(\lambda) = \frac{1}{2} \sigma^2 \lambda^2 - \beta \log\left(1 + \frac{\lambda}{\alpha} \right).
$$

Consider an up-and-out put option, with maturity T, strike $K < 0$ and barrier 0, written on the underlying process S, which has the following payoff:

$$(K - S_T)^+ \mathbf{1}_{\{\sup_{t \in [0,T]} S_t < 0\}}.$$

Here, for simplicity, we assume that the underlying process S can take negative values. Assume that the risk neutral evolution of the underlying is given by $S_t = x + X_t$, with some $x \in (-\infty, 0)$ and with X given by (35). Assume that we need to find a static hedging strategy for this barrier option using the European-type options. Then, following the algorithm presented in Sections 1.1 and 1.2, we need to construct the weak symmetry mapping $\mathbf{W}^+$ and apply it to the hockey-stick function $h^1(x) = (K - x)^+$. Note that Assumption 1 is always satisfied for the process X, Assumption 2 holds with any $\zeta \in (0, \alpha)$ and (30) is satisfied. Indeed, it is easy to see that the inverse Fourier transform of the function $x \mapsto e^{\zeta x} h(x)$ is given by

$$\hat{h}^1(z) = \frac{\exp(K(\zeta + iz))}{2\pi(\zeta + iz)^2}.$$

Therefore, we apply the last assertion of Theorem 1 and make use of (7) to conclude that, at all times up until and including the first time when S hits 0, the price of up-and-out put option coincides with the price of a European-type option, which has maturity T and the following payoff function:

$$(K - S_T)^+ - g_1(S_T),$$

with

$$
\begin{aligned}
g^1(x) &= \mathbf{W}^+ h^1(x) \\
&= \frac{1}{4\pi^2 i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda x} \int_{\mathbb{R}} \Bigg((\sigma^2 \lambda - \beta/(\lambda + \alpha)) \\
&\qquad\qquad \Bigg/ \left(\frac{\sigma^2 \lambda^2}{2} - \beta \log\left(1 + \frac{\lambda}{\alpha}\right) \right. \\
&\qquad\qquad\qquad - \frac{\sigma^2(\zeta + iz)^2}{2} + \beta \log\left(1 - \frac{\zeta + iz}{\alpha}\right) \Bigg) \\
&\qquad\qquad\qquad\qquad\qquad \left. - \frac{1}{\lambda + \zeta + iz} \right) \\
&\qquad\qquad \times \frac{e^{K(\zeta + iz)}}{(\zeta + iz)^2} \, dz \, d\lambda,
\end{aligned}
$$

(36)

for $x > 0$. Thus, in order to offset the risks associated with holding an up-and-out put option (i.e., hedge the barrier option), one needs to sell the European-
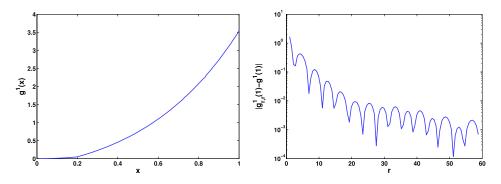
FIG. 1. *On the left*: *the values of* $g^1(x)$, *defined in* (36), *for various* $x > 0$. *On the right*: *the convergence rate of the truncated integrals* $g^1_{r,r}(1)$, *defined in* (33), *to the value of* $g^1(1)$ (*on a logarithmic scale*). *The parameter values are*: $K = -0.2$, $\alpha = \beta = \sigma = 1$, $\zeta = 0.9$, $\gamma = 4$.

type option with the above payoff, and buy it back (at a zero price) if and when the underlying hits 0. The results of numerical integration are presented in Figure 1. Notice that, in the present case, Corollary 2 and the asymptotic relation $|\hat{h}^1(z)| \sim \text{const} \cdot z^{-2}$ imply that it is optimal to approximate the double integral in (36) by integrating over squares in the Im(λ) $\times z$ domain. The convergence rate of $g^1_{r,r}(1)$ [cf. (33)], as $r \to \infty$, is shown on the right-hand side of Figure 1 (on a logarithmic scale). Notice that the convergence rate seems to be polynomial (rather than exponential), as predicted by Corollary 2. The numerical integration over a finite domain, required to compute $g^1_{r,r}$, is performed via the MatLab function `quad2d`. For the set of parameters used to generate Figure 1, the CPU time required to compute $g^1(1)_{r,r}$, with $r = 60$, is 1.27 seconds.[7]

Next, we illustrate how the weak reflection principle can be used for numerical computation of the joint marginal distribution of a stochastic process and its running maximum. Assume that X is given by (35), and we need to approximate numerically the value of

$$(37) \qquad \mathbb{P}\Big(X_T \leq K + x, \sup_{t \in [0,T]} X_t \geq x\Big),$$

with some $T > 0$, $K < 0$ and $x \geq 0$. The first assertion of Theorem 1 implies that

$$
\begin{aligned}
&\mathbb{P}\Big(X_T \leq K + x, \sup_{t \in [0,T]} X_t \geq x\Big) \\
(38) \qquad &= \mathbb{E}h^2(X_T - x) - \mathbb{E}\big(h^2(X_T - x)\mathbf{1}_{\{\sup_{t \in [0,T]}(X_t - x) < 0\}}\big) \\
&= \langle g^2, p_T(\cdot - x)\rangle,
\end{aligned}
$$

---

[7]All computations are performed on a standard laptop, 1.8 GHz Intel Core i5, 4 GB RAM.

where $h^2 = \mathbf{1}_{(-\infty, K]}$ is the indicator function, p_T is the density of X_T and $g^2 = \mathbf{W}^+ h^2$ is the weak symmetry image of h^2,

$$
\begin{aligned}
\langle g^2, & p_T(\cdot - x) \rangle \\
&= \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \int_{\mathbb{R}} e^{\lambda y} p_T(y - x) \, dy \int_{\mathbb{R}} \left(\frac{\psi'(\lambda)}{\psi(\lambda) - \psi(-\zeta - iz)} \right. \\
&\hspace{8cm} \left. - \frac{1}{\lambda + \zeta + iz} \right) \hat{h}^2(z) \, dz \, d\lambda \\
&= \frac{1}{4\pi^2 i} \int_{\gamma - i\infty}^{\gamma + i\infty} \exp\left(\lambda x + T\left(\frac{1}{2} \sigma^2 \lambda^2 - \beta \log(1 + \lambda/\alpha) \right) \right) \\
&\hspace{2cm} \times \int_{\mathbb{R}} \left((\sigma^2 \lambda - \beta/(\lambda + \alpha)) \right. \\
&\hspace{3cm} \left/ \left(\frac{\sigma^2 \lambda^2}{2} - \beta \log\left(1 + \frac{\lambda}{\alpha} \right) \right. \right. \\
&\hspace{4cm} \left. - \frac{\sigma^2(\zeta + iz)^2}{2} + \beta \log\left(1 - \frac{\zeta + iz}{\alpha} \right) \right) \\
&\hspace{6cm} \left. - \frac{1}{\lambda + \zeta + iz} \right) \\
&\hspace{1cm} \times \frac{e^{K(\zeta + iz)}}{(\zeta + iz)} \, dz \, d\lambda,
\end{aligned}
\tag{39}
$$

and the associated $\hat{h}^2$ is given by

$$
\hat{h}^2(z) = -\frac{\exp(K(\zeta + iz))}{2\pi(\zeta + iz)}.
$$

Notice that, strictly speaking, Theorem 1 yields (38) only for $x = 0$. To show that (38) holds for all $x \geq 0$, we use g_r^2 in place of g^2 and repeat (7), to estimate the absolute value of the difference between the left and the right-hand sides of (38) by

$$
\sup_{t \in [0, T]} |\langle g^2, p_t \rangle - \mathbb{E} g_r^2(X_t)| = \sup_{t \in [0, T]} |\langle g^2 - g_r^2, p_t \rangle|.
$$

Then we make use of (28), to pass to the limit as $r \to \infty$ and obtain (38). Figure 2 shows the convergence rate of the numerical approximation of the right-hand side of (39). As before, we used the MatLab function `quad2d` to evaluate the integral in (39) numerically, with the parameters' values specified in Figure 2. The CPU time required to compute $\langle g_{r,r}^2, p_T(\cdot - x) \rangle$, with $r = 60$, $x = 0.1$ and $T = 1$, is 1.69 seconds. It is worth mentioning that, as discussed in Section 1.2, the computation of the joint marginal distribution of a process and its running maximum is *not* the main application of the weak reflection principle. In fact, our method
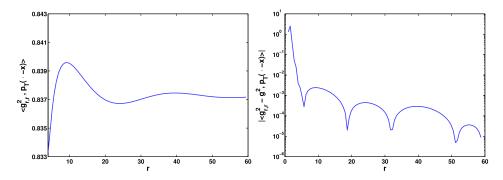
FIG. 2. *On the left*: *the values of* $\langle g_{r,r}^2, p_T(\cdot - x)\rangle$, *with* $g_{r,r}^2$ *given by* (36), *for various* $r > 0$. *On the right*: *the convergence rate of* $\langle g_{r,r}^2, p_T(\cdot - x)\rangle$ *to* $\langle g^2, p_T(\cdot - x)\rangle$, *as* $r \to \infty$ (*on a logarithmic scale*). *The parameter values are* $x = 0.1$, $T = 1$, $K = -0.2$, $\alpha = \beta = \sigma = 1$, $\zeta = 0.9$, $\gamma = 4$.

produces more than just the value of the expectation of a function of X_T and $\sup_{t \in [0,T]} X_t$: it allows to express this value via the expectation of a function of X_T alone. The latter amounts to solving an inverse problem, as opposed to the direct problem of computing the expectation. Thus the use of the weak reflection principle for computing the joint probabilities (as opposed to static hedging) does not fully utilize the power of the method. As a result, in general, our method may not outperform the existing algorithms for the computation of the joint law, which are based on the Wiener–Hopf factorization; cf. [13–15, 19] and references therein. Recall that the Wiener–Hopf factorization allows one to compute the joint probability (37) by evaluating an integral over a vertical line on a complex plane, very much like the integral with respect to λ in (39). The value of the integrand at each point is, in turn, computed by a Fourier inversion applied to the Wiener–Hopf factors; see, for example, equation (18) in [14].[8] Thus, in general, the computation of the integral in (39) seems to have the same order of complexity as the classical method [in addition, (38)–(39) only apply to spectrally-negative Lévy processes, with nontrivial Brownian component]. However, in some cases, the algorithm described by (38)–(39) may be more efficient. Namely, it is advantageous to use the above method if the joint probability (37) needs to be computed for multiple x and T (which corresponds to varying the initial condition and the time horizon). Notice that the inner integral on the right-hand side of (39) is independent of x and T. Then the computational complexity can be reduced by reusing, for different x and T, the same values of the inner integral (as a function of λ, computed on a given grid or via basis expansion), in the integration with respect to λ.[9] Figure 3 shows the numerical approximation of $\langle g^2, p_T(\cdot - x)\rangle$, given by the right-hand

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[8]As described in [13], for certain families of Lévy processes, the required Fourier inversion can be reduced to a series expansion, which is more computationally efficient.

[9]Of course, this only works if the x and T very over a reasonably small range of values, so that there is no need to change the precision with which the inner integral in (39) is computed. Also, by
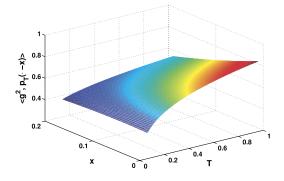
FIG. 3. *The values of $\langle g^2_{r,r}, p_T(\cdot - x) \rangle$, for $x \in [0, 0.2]$ and $T \in [0.1, 1]$. The parameter values are*: $r = 60$, $K = -0.2$, $\alpha = \beta = \sigma = 1$, $\zeta = 0.9$, $\gamma = 4$.

side of (39), for 10,000 different pairs (x, T). In this computation, we approximate the integral with respect to λ, in (39), by a simple Riemann sum (with the uniform partition of diameter 0.6) and reuse the same values of the inner integral in (39), for different x and T (the inner integral is computed using the MatLab function `quad`). As a result, the total CPU time required to finish all the computations is only 20.46 seconds [compared to 1.69 seconds required to evaluate (39) for a single pair (x, T)]. Another advantage of (38)–(39) is that these formulas enable a straightforward computation of the derivatives of the joint probability (37) with respect to x and T (which provide sensitivities with respect to the initial condition and the time horizon). Indeed, it follows from (27) and (21) that the absolute value of the integrand on the right-hand side of (39) decays faster than any exponential, as a function of $\text{Im}(\lambda)$. Therefore, the same conclusion holds for any derivative of the integrand with respect to x and T. Thus we can differentiate with respect to x and T inside the integral in (39) an arbitrary number of times.

Finally, we illustrate the PIDE interpretation of the weak symmetry. Recall that every Lévy process has an infinitesimal generator $\mathcal{L}$ associated with it; cf. [19]. This generator can be viewed as a pseudo-differential operator, which acts on all infinitely smooth functions ϕ with compact support as follows:

$$\mathcal{L}\phi(x) = \frac{\sigma^2}{2} \partial_x^2 \phi(x) + \mu \, \partial_x \phi(x) + \int_{\mathbb{R}} (\phi(x + z) - \phi(x) - z \, \partial_x \phi(x)) \Pi(dz).$$

In the present case, we obtain

$$\mathcal{L}\phi(x) = \frac{\sigma^2}{2} \partial_x^2 \phi(x) - \frac{\beta}{\alpha} \, \partial_x \phi(x)$$
$$- \beta \int_{-\infty}^{0} (\phi(x + z) - \phi(x) - z \, \partial_x \phi(x)) \frac{e^{\alpha z}}{z} \, dz.$$

---

changing the order of integration, one can use the same idea to compute the right-hand side of (39) for multiple K (with x and T fixed) in a more efficient way.

Let us introduce

$$u(x, t) = \mathbb{E}h(x + X_t).$$

Then the Feynman–Kac formula for the Lévy process X (see, e.g., [3, 9, 12] and [11]) implies that u is a solution of the following initial value problem:

(40)
$$\begin{cases} \partial_t u - \mathcal{L}u = 0, & x \in \mathbb{R}, t > 0, \\ u(x, 0) = h(x). \end{cases}$$

The exact definition of a solution to the above problem depends on the regularity assumptions on function h and is discussed in the aforementioned references. The existence of the weak symmetry mapping $\mathbf{W}^+$ for the process X implies that, for any admissible h, with support in $(-\infty, 0)$, we can find a function $g = \mathbf{W}^+ h$, with support in $[0, \infty)$, such that the solution to (40), with h replaced by g, coincides with the original solution at $x = 0$, for all $t > 0$. In particular, this means that we can modify any initial condition for $x \in [0, \infty)$ to ensure that $u(0, t) = 0$, for all $t > 0$. This, in turn, implies that we can reduce an initial-boundary value problem to an initial-value problem. Namely, any solution to the PIDE in (40), defined for $(x, t) \in (-\infty, 0) \times (0, \infty)$, with zero boundary condition at $x = 0$ and with initial condition h, can be represented as a solution to the initial value problem (40), with h replaced by $h - g$. Notice that equations of the form (40) may be interesting on their own. For example, it is discussed in [5] and [2] that the relativistic Schrödinger equation can be reduced to a PIDE associated with the Lévy process known as the normal inverse Gaussian (NIG) process. Unfortunately, Assumption 1 excludes NIG from the scope of the present work, since NIG is a pure jump process with both positive and negative jumps.[10] However, we believe that future research will extend the results presented herein, to construct weak symmetry mappings for NIG and other important Lévy processes with two-sided jumps and/or missing Brownian component.

4. Summary and extensions. We have presented a new mathematical technique, which we christened the *weak reflection principle* and which is an extension of the well-known reflection principle for Brownian motion. This new form of reflection principle is obtained by weakening the notion of symmetry that is required for the classical reflection principle to hold. More precisely, our method is based on the notion of *weak symmetry*. We started by reviewing the existing results

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[10]Note that we focused on the Lévy processes with only negative jumps to ensure that the process does not jump across the barrier. Indeed, this is necessary for the weak reflection principle to hold, as we need to stop the process precisely at the barrier. However, in the PIDE interpretation, we do not need to use all the steps of the weak reflection principle. Namely, we do not need to stop the process at the barrier, but rather, we only need to construct the weak symmetry mapping, assuming the process starts from the barrier. Hence, in the context of PIDEs, it makes sense to consider processes with both positive and negative jumps.

which provide an explicit integral representation of the weak symmetry mapping for any time-homogeneous diffusion process on a real line (subject to some regularity conditions). Finally, for the most of the paper we focused on constructing the weak symmetry mapping for spectrally negative Lévy processes, thus extending the weak reflection principle to this new class of stochastic processes.

The weak reflection principle provides solutions to various problems for which the classical reflection principle can be used, even when the underlying process is not a Brownian motion and does not possess any strong symmetries. In particular, the weak reflection principle is a perfect tool for constructing the exact static hedging strategies of barrier options (in fact, this problem motivated the development of the method in the first place). Another application of this method is the computation of the joint distribution of a process and its running maximum (minimum). Of course, while this problem is quite relevant for diffusions, in the case of Lévy processes, there exist several alternative computational methods, based on the Wiener–Hopf factorization; cf. [13–15, 19]. Nevertheless, as shown in Section 3, there are cases when it is advantageous to use this method. Finally, the weak symmetry mapping allows us to solve an inverse problem for the parabolic partial PIDE associated with a Lévy process. Namely, using the weak symmetry, we can modify the initial condition of the PIDE on one half line only, so that its solution remains constant at $x = 0$, for all times. This, in particular, allows us to represent the solution of a PIDE with initial and boundary conditions via the solution of the same PIDE with initial condition only.

It is also worth mentioning that the technical Lemmas 4–6 and the resulting Corollary 1 describe a domain on which the resolvent function of a Lévy process is well defined. This domain is rather large, and in particular, the real parts of its elements are unbounded from below. Thus our results provide a nontrivial estimate of the spectrum of the integro-differential operator associated with any admissible spectrally-negative Lévy process. Recall that this operator is nonlocal and nonsymmetric, which makes it very hard to describe its spectrum using the general theory; see also Remark 2 for a description of the associated difficulties.

To date, the weak symmetry has only been established for diffusion processes and Lévy processes with one-sided jumps. However, we conjecture that these results can be extended to a larger class of time-homogeneous Markov processes—possibly all jump-diffusions satisfying some regularity conditions. Such an extension would allow us to solve the aforementioned problems for a larger class of stochastic processes. In particular, as discussed in Section 3, including the NIG process in the scope of our analysis would establish a connection with the relativistic Schrödinger equation which is important in Physics. Another possible extension is related to the domain with respect to which the weak symmetry is defined. Notice that, in the present case, we split the real line into two half lines and study the weak symmetry of the process with respect to the (unique) boundary point. It is interesting to extend these results to the case of a compact interval, whose boundary consists of two points (assuming the underlying Markov process

does not jump across the boundary points). In financial mathematics, this problem would correspond to the static hedging of double barrier options. More generally, one can investigate domains in higher dimension and try to establish the weak symmetry with respect to their boundaries.

APPENDIX A

PROOF OF LEMMA 1.

$$\text{Im}\big(\psi(v+iu) - \psi(-\zeta - iz)\big)$$

$$= \mu(u+z) + (vu - z\zeta)\sigma^2 + \int_{-\infty}^{0} \big(e^{vx}\sin(ux) - ux\big)\Pi(dx)$$

$$+ \int_{-\infty}^{0} \big(e^{-\zeta x}\sin(zx) - zx\big)\Pi(dx).$$

Notice that

$$\int_{-\infty}^{0} \big(e^{vx}\sin(ux) - ux\big)\Pi(dx)$$

$$= vu \int_{-\infty}^{0} \frac{e^{vx}-1}{vx}\frac{\sin(ux)}{ux}x^2\Pi(dx) + u^2 \int_{-\infty}^{0} \frac{\sin(ux)-ux}{u^2x^2}x^2\Pi(dx),$$

$$\int_{-\infty}^{0} \big(e^{-\zeta x}\sin(zx) - zx\big)\Pi(dx)$$

$$= z\zeta \int_{-\infty}^{0} \frac{1-e^{\zeta x}}{\zeta x}\frac{\sin(zx)}{zx}x^2 e^{-\zeta x}\Pi(dx) + z^2 \int_{-\infty}^{0} \frac{\sin(zx)-zx}{z^2x^2}x^2\Pi(dx).$$

Since x^2 and $x^2 e^{-\zeta x}$ are integrable with respect to $\Pi(dx)$, due to the dominated convergence theorem, the above integrals are absolutely bounded and vanish as the corresponding functions of (u, v, z) go to infinity. □

PROOF OF LEMMA 2.   We follow the proof of Lemma 1, except that at the end, we apply the following additional estimate:

$$u^2 \int_{-\infty}^{0} \frac{\sin(ux)-ux}{u^2x^2}x^2\Pi(dx) + z^2 \int_{-\infty}^{0} \frac{\sin(zx)-zx}{z^2x^2}x^2\Pi(dx)$$

$$= \int_{-\infty}^{0} \big(2\sin((z+u)x/2)\cos((z-u)x/2) - (z+u)x\big)\Pi(dx)$$

$$= (z^2 - u^2) \int_{-\infty}^{0} 2\frac{\sin((z+u)x/2)}{(z+u)x}\frac{\cos((z-u)x/2)-1}{(z-u)x}x^2\Pi(dx)$$

$$+ (z+u)^2 \int_{-\infty}^{0} \frac{2\sin((z+u)x/2) - (z+u)x}{(z+u)^2x^2}x^2\Pi(dx).$$

Applying the dominated convergence theorem, we complete the proof of the lemma. □

PROOF OF LEMMA 3.

$$\mathrm{Re}\big(\psi(v+iu)-\psi(-\zeta-iz)\big)$$
$$= \mu(v+\zeta)+(v^2-\zeta^2+z^2-u^2)\sigma^2/2$$
$$+\int_{-\infty}^{0}\big(e^{vx}\cos(ux)-1-vx\big)\Pi(dx)$$
$$-\int_{-\infty}^{0}\big(e^{-\zeta x}\cos(zx)-1+\zeta x\big)\Pi(dx).$$

Notice that

$$\int_{-\infty}^{0}\big(e^{vx}\cos(ux)-1-vx\big)\Pi(dx)-\int_{-\infty}^{0}\big(e^{-\zeta x}\cos(zx)-1+\zeta x\big)\Pi(dx)$$

$$=\int_{-\infty}^{0}e^{vx}\big(\cos(ux)-1\big)\Pi(dx)+\int_{-\infty}^{0}\big(e^{vx}-1-vx\big)\Pi(dx)$$

$$-\int_{-\infty}^{0}e^{-\zeta x}\big(\cos(zx)-1\big)\Pi(dx)-\int_{-\infty}^{0}\big(e^{-\zeta x}-1+\zeta x\big)\Pi(dx)$$

$$=\int_{-\infty}^{0}\big(\cos(ux)-1\big)\Pi(dx)+uv\int_{-\infty}^{0}\frac{e^{vx}-1}{vx}\frac{\cos(ux)-1}{ux}x^2\Pi(dx)$$

$$+v^2\int_{-\infty}^{0}\frac{e^{vx}-1-vx}{v^2x^2}x^2\Pi(dx)-\int_{-\infty}^{0}\big(\cos(zx)-1\big)\Pi(dx)$$

$$-z\zeta\int_{-\infty}^{0}\frac{1-e^{\zeta x}}{\zeta x}\frac{\cos(zx)-1}{zx}x^2e^{-\zeta x}\Pi(dx)$$

$$-\zeta^2\int_{-\infty}^{0}\frac{1-e^{\zeta x}+\zeta xe^{\zeta x}}{\zeta^2x^2}x^2e^{-\zeta x}\Pi(dx)$$

$$=\int_{-\infty}^{0}\big(\cos(ux)-\cos(zx)\big)\Pi(dx)+|\bar{\bar{o}}(uv)|+|\bar{\bar{o}}(v^2)|+\bar{\bar{o}}(z)+\bar{\bar{o}}(1)$$

$$=2(z^2-u^2)\int_{-\infty}^{0}\frac{\sin((u+z)x/2)}{(z+u)x}\frac{\sin((z-u)x/2)}{(z-u)x}x^2\Pi(dx)$$

$$+|\bar{\bar{o}}(uv)|+|\bar{\bar{o}}(v^2)|+\bar{\bar{o}}(z)+\bar{\bar{o}}(1)$$

$$=\bar{\bar{o}}(z^2-u^2)+|\bar{\bar{o}}(uv)|+|\bar{\bar{o}}(v^2)|+\bar{\bar{o}}(z)+\bar{\bar{o}}(1),$$

where we applied the dominated convergence theorem. □

PROOF OF LEMMA 4. Using Lemmas 2 and 3, we obtain

$$\big|\psi(v+iu) - \psi(-\zeta - iz)\big|$$

$$\geq \big|\mathrm{Im}\big(\psi(v+iu) - \psi(-\zeta - iz)\big)\big|\mathbf{1}_{\{||u|-\sqrt{z^2+v^2}|\leq \varepsilon v\}}$$

$$+ \big|\mathrm{Re}\big(\psi(v+iu) - \psi(-\zeta - iz)\big)\big|\mathbf{1}_{\{||u|-\sqrt{z^2+v^2}|> \varepsilon v\}}$$

$$(41)\quad = \big|\mu(u+z) + (vu - z\zeta)\sigma^2 + \overline{\overline{o}}(vu) + \overline{\overline{o}}(z)$$

$$+ \overline{\overline{o}}(z^2 - u^2) + \mathrm{sign}(z+u)|\overline{\overline{o}}((z+u)^2)|\big|\mathbf{1}_{\{||u|-\sqrt{z^2+v^2}|\leq \varepsilon v\}}$$

$$+ \big|\mu(v+\zeta) + (v^2 + z^2 - u^2)\sigma^2/2$$

$$+ \overline{\overline{o}}(z^2 - u^2) + |\overline{\overline{o}}(uv)| + |\overline{\overline{o}}(v^2)| + \overline{\overline{o}}(z) + \overline{\overline{o}}(1)\big|\mathbf{1}_{\{||u|-\sqrt{z^2+v^2}|> \varepsilon v\}}.$$

Let us estimate the first term in the above:

$$\big|\mu(u+z) + vu\sigma^2 - z\zeta\sigma^2 + \overline{\overline{o}}(vu) + \overline{\overline{o}}(z)$$

$$+ \overline{\overline{o}}(z^2 - u^2) + \mathrm{sign}(z+u)|\overline{\overline{o}}((z+u)^2)|\big|\mathbf{1}_{\{||u|-\sqrt{z^2+v^2}|\leq \varepsilon v\}}$$

$$\geq \big(v|u|\sigma^2 + \mu u + z(\mu - \zeta\sigma^2) + \overline{\overline{o}}(v(|z| + v))$$

$$+ \overline{\overline{o}}(z) + \overline{\overline{o}}(v|z| + v^2) + \mathrm{sign}(z+u)|\overline{\overline{o}}((z+u)^2)|\big)\mathbf{1}_{\{||u|-\sqrt{z^2+v^2}|\leq \varepsilon v, u\geq 0\}}$$

$$+ \big(v|u|\sigma^2 - \mu u - z(\mu - \zeta\sigma^2)$$

$$+ \overline{\overline{o}}(v(|z| + v)) + \overline{\overline{o}}(z) + \overline{\overline{o}}(v|z| + v^2)$$

$$- \mathrm{sign}(z+u)|\overline{\overline{o}}((z+u)^2)|\big)\mathbf{1}_{\{||u|-\sqrt{z^2+v^2}|\leq \varepsilon v, u< 0\}}$$

$$\geq \big(c_3 v(|z| + v) + \mu u + z(\mu - \zeta\sigma^2) + \overline{\overline{o}}(v(|z| + v))$$

$$+ \overline{\overline{o}}(z) + \overline{\overline{o}}(v|z| + v^2) + \overline{\overline{o}}(v^2)\big)\mathbf{1}_{\{||u|-\sqrt{z^2+v^2}|\leq \varepsilon v, u\geq 0\}}$$

$$+ \big(c_4 v(|z| + v) - \mu u - z(\mu - \zeta\sigma^2)$$

$$+ \overline{\overline{o}}(v(|z| + v)) + \overline{\overline{o}}(z) + \overline{\overline{o}}(v|z| + v^2) + \overline{\overline{o}}(v^2)\big)\mathbf{1}_{\{||u|-\sqrt{z^2+v^2}|\leq \varepsilon v, u< 0\}}$$

$$\geq c_1 v(|z| + v)\mathbf{1}_{\{||u|-\sqrt{z^2+v^2}|\leq \varepsilon v\}},$$

where c_i's are positive constants, and we assume that $|z|$ and $v > 0$ are large enough. In the above, we use the fact that if $||u| - \sqrt{z^2 + v^2}| \leq \varepsilon v$, then

$$\frac{1-\varepsilon}{2}(v + |z|) \leq |u| \leq v + (1+\varepsilon)|z|,$$

and if, in addition, $uz < 0$, then

$$||u| - |z|| \leq \varepsilon v + \frac{v^2}{\sqrt{z^2 + v^2} + |z|} \leq (1+\varepsilon)v.$$

Finally, we estimate the second term on the right-hand side of (41):

$$\big|\mu(v+\zeta)+(v^2+z^2-u^2)\sigma^2/2+\bar{\bar{o}}(z^2-u^2)+|\bar{\bar{o}}(uv)|+|\bar{\bar{o}}(v^2)|+\bar{\bar{o}}(z)\big|$$

$$\times \mathbf{1}_{\{||u|-\sqrt{z^2+v^2}|>\varepsilon v\}}$$

$$\geq |u^2-(v^2+z^2)|$$

$$\times \left(\sigma^2/2+\frac{\mu(v+z)+\bar{\bar{o}}(u^2-z^2)+|\bar{\bar{o}}(uv)|+|\bar{\bar{o}}(v^2)|+\bar{\bar{o}}(z)}{|u^2-(v^2+z^2)|}\right)$$

$$\times \mathbf{1}_{\{||u|-\sqrt{z^2+v^2}|>\varepsilon v\}}$$

$$\geq c_2|u^2-(v^2+z^2)|\mathbf{1}_{\{||u|-\sqrt{z^2+v^2}|>\varepsilon v\}},$$

which holds for all large enough $|z|$ and $v>0$, with some positive constants c_i. In the above we make use of the fact that if $||u|-\sqrt{z^2+v^2}|>\varepsilon v$, then

$$|u|\geq (1-\varepsilon)v,$$

$$|u^2-(v^2+z^2)|\geq \varepsilon v(|u|+\sqrt{z^2+v^2}),$$

$$\frac{|u^2-z^2|}{|u^2-(v^2+z^2)|}\leq \frac{v}{v^2-v}\mathbf{1}_{\{|u^2-z^2|\leq v\}}+\left(1+\frac{v^2}{\varepsilon v(|u|+\sqrt{v^2+z^2})}\right)\mathbf{1}_{\{|u^2-z^2|>v\}}$$

$$\leq \frac{1}{v-1}+\left(1+\frac{1}{\varepsilon}\right)\mathbf{1}_{\{|u^2-z^2|>v\}},$$

and hence

$$\frac{\mu(v+z)+\bar{\bar{o}}(u^2-z^2)+|\bar{\bar{o}}(uv)|+|\bar{\bar{o}}(v^2)|+\bar{\bar{o}}(z)}{|u^2-(v^2+z^2)|}$$

can be made arbitrarily small by choosing large enough v. □

PROOF OF LEMMA 5. First, we notice that since ψ is analytic, we have $|\psi(v-iu)|=|\psi(v+iu)|$, and hence it suffices to consider only $u>0$.
Using Lemma 3, we obtain the following inequalities:

$$\big(\mathrm{Re}\big(\psi(v+iu)-\psi(-\zeta-iz)\big)\big)^2$$

$$=\big(\mu(v+\zeta)+(v^2+z^2-u^2)\sigma^2/2+\bar{\bar{o}}(z^2-u^2)$$

$$+\bar{\bar{o}}(uv)+\bar{\bar{o}}(v^2)+\bar{\bar{o}}(z)+\bar{\bar{o}}(1)\big)^2$$

$$\geq \big(v^2+(v+\zeta)2\mu/\sigma^2+z^2-u^2\big)^2\sigma^4/4$$

$$+\big(v^2+z^2-u^2+(v+\zeta)2\mu/\sigma^2\big)$$

$$\times\big(\bar{\bar{o}}(z^2-u^2)+\bar{\bar{o}}(uv)+\bar{\bar{o}}(v^2)+\bar{\bar{o}}(z)\big)\mathbf{1}_{\{|u-|z||\leq\varepsilon\}}$$

$$+ (v^2 + z^2 - u^2 + (v + \zeta)2\mu/\sigma^2)$$
$$\times (\overline{\overline{o}}(z^2 - u^2) + \overline{\overline{o}}(uv) + \overline{\overline{o}}(v^2) + \overline{\overline{o}}(z))\mathbf{1}_{\{|u - |z|| > \varepsilon\}}$$
$$\geq (v^2 + (v + \zeta)2\mu/\sigma^2)^2 \sigma^4/4$$
$$+ (v^2 + (v + \zeta)2\mu/\sigma^2)(z^2 - u^2)\sigma^4/2 + (z^2 - u^2)^2 \sigma^4/4$$
$$+ (v^2 + u)(\overline{\overline{o}}(z^2 - u^2) + \overline{\overline{o}}(uv) + \overline{\overline{o}}(v^2) + \overline{\overline{o}}(z))\mathbf{1}_{\{|u - |z|| \leq \varepsilon\}}$$
$$+ (v^2 + z^2 - u^2 + (v + \zeta)2\mu/\sigma^2)$$
$$\times (\overline{\overline{o}}(z^2 - u^2) + \overline{\overline{o}}(uv) + \overline{\overline{o}}(v^2) + \overline{\overline{o}}(z))\mathbf{1}_{\{|u - |z|| > \varepsilon\}}$$
$$= (v^2 + (v + \zeta)2\mu/\sigma^2)^2 \sigma^4/4 + (z^2 - u^2)^2 \sigma^4/4$$
$$+ (v^2 + (v + \zeta)2\mu/\sigma^2)z^2\sigma^4/2 - (v^2 + (v + \zeta)2\mu/\sigma^2)u^2\sigma^4/2$$
$$+ (v^2 + u)(u\overline{\overline{o}}(v) + \overline{\overline{o}}(v^2) + \overline{\overline{o}}(u))\mathbf{1}_{\{|u - |z|| \leq \varepsilon\}}$$
$$+ (v^2 + |z^2 - u^2|)(\overline{\overline{o}}(z^2 - u^2) + \overline{\overline{o}}(uv) + \overline{\overline{o}}(v^2) + \overline{\overline{o}}(z))\mathbf{1}_{\{|u - |z|| > \varepsilon\}}$$
$$\geq v^4(1 - \varepsilon')\sigma^4/4 + (z^2 - u^2)^2\sigma^4/4 + v^2 z^2 (1 - \varepsilon')\sigma^4/2$$
$$- u^2 v^2(1 + \varepsilon')\sigma^4/2 + u^2\overline{\overline{o}}(v) + \overline{\overline{o}}(u^2) + u\overline{\overline{o}}(v^3) + v^2\overline{\overline{o}}(u) + \overline{\overline{o}}(v^4)$$
$$+ (v^2 + |z^2 - u^2|)(\overline{\overline{o}}(z^2 - u^2) + \overline{\overline{o}}(uv) + \overline{\overline{o}}(v^2) + \overline{\overline{o}}(z))\mathbf{1}_{\{|u - |z|| > \varepsilon\}},$$

which hold for any (fixed) $\varepsilon, \varepsilon' > 0$ and all large enough u, v and $|z|$.

Next, using Lemmas 1, 2, we obtain the following inequalities:

$$(\mathrm{Im}(\psi(v + iu) - \psi(-\zeta - iz)))^2$$
$$= (\mu(u + z) + uv\sigma^2 - z\zeta\sigma^2 + \overline{\overline{o}}(uv) + \overline{\overline{o}}(z)$$
$$+ \overline{\overline{o}}(z^2 - u^2) + \overline{\overline{o}}((z + u)^2))^2\mathbf{1}_{\{z \leq 0\}}$$
$$+ (\mu(u + z) + uv\sigma^2 - z\zeta\sigma^2 + \overline{\overline{o}}(vu) + \overline{\overline{o}}(z) + |\overline{\overline{o}}(u^2)| + |\overline{\overline{o}}(z^2)|)^2\mathbf{1}_{\{z > 0\}}$$
$$\geq (uv\sigma^2 + \mu(u + z) - z\zeta\sigma^2 + \overline{\overline{o}}(vu) + \overline{\overline{o}}(z))^2$$
$$+ (\mu(u + z) + uv\sigma^2 - z\zeta\sigma^2 + \overline{\overline{o}}(vu) + \overline{\overline{o}}(z))$$
$$\times (\overline{\overline{o}}(z^2 - u^2) + \overline{\overline{o}}((z + u)^2))\mathbf{1}_{\{z \leq 0\}}$$
$$+ (\mu(u + z) + uv\sigma^2 - z\zeta\sigma^2 + \overline{\overline{o}}(vu) + \overline{\overline{o}}(z))(|\overline{\overline{o}}(u^2)| + |\overline{\overline{o}}(z^2)|)\mathbf{1}_{\{z > 0\}}$$
$$\geq u^2 v^2 \sigma^4 + 2uv\sigma^2(\mu(u + z) - z\zeta\sigma^2 + \overline{\overline{o}}(vu) + \overline{\overline{o}}(z))$$
$$+ (\mu(u + z) + uv\sigma^2 - z\zeta\sigma^2 + \overline{\overline{o}}(vu) + \overline{\overline{o}}(z))$$
$$\times (\overline{\overline{o}}(z^2 - u^2) + \overline{\overline{o}}((z + u)^2))\mathbf{1}_{\{z \leq -u - \varepsilon''\}}$$

$$+ v\overline{\overline{o}}(u^2)$$

$$+ (\mu(u+z) + uv\sigma^2 - z\zeta\sigma^2 + \overline{\overline{o}}(vu) + \overline{\overline{o}}(z))\overline{o}(z^2 - u^2)\mathbf{1}_{\{-u+\varepsilon'' \le z \le 0\}}$$

$$+ (uv\sigma^2 + \mu u - z\sigma^2(\zeta - \mu/\sigma^2) + \overline{\overline{o}}(vu) + \overline{\overline{o}}(z))$$

$$\times (\overline{\overline{o}}(u^2) + \overline{\overline{o}}(z^2))\mathbf{1}_{\{z|\zeta - \mu/\sigma^2| > u(v - \varepsilon'')\}},$$

which hold for any (fixed) $\varepsilon'' > 0$ and all large enough u, v and $|z|$. In the above, we also make use of the fact that

$$(\mu(u+z) + uv\sigma^2 - z\zeta\sigma^2 + \overline{\overline{o}}(vu) + \overline{\overline{o}}(z))(\overline{o}(z^2 - u^2) + \overline{\overline{o}}((z+u)^2))$$

$$\times \mathbf{1}_{\{-u-\varepsilon'' \le z \le -u+\varepsilon''\}} = v\overline{\overline{o}}(u^2)$$

and

$$(uv\sigma^2 + \mu u - z\sigma^2(\zeta - \mu/\sigma^2) + \overline{\overline{o}}(vu) + \overline{\overline{o}}(z))(|\overline{\overline{o}}(u^2)| + |\overline{\overline{o}}(z^2)|)$$

$$\times \mathbf{1}_{\{0 \le z|\zeta - \mu/\sigma^2| \le u(v - \varepsilon'')\}} \ge 0$$

hold for all large enough u, v and $|z|$.

Finally, choosing $\varepsilon = \varepsilon' = \varepsilon'' \in (0, 1)$, we collect the above to obtain

$$|\psi(v+iu) - \psi(-\zeta - iz)|^2$$

$$= (\operatorname{Re}(\psi(v+iu) - \psi(-\zeta - iz)))^2 + (\operatorname{Im}(\psi(v+iu) - \psi(-\zeta - iz)))^2$$

$$\ge v^4(1-\varepsilon)\sigma^4/4 + (z^2 - u^2)^2\sigma^4/4 + v^2z^2(1-\varepsilon)\sigma^4/2 - u^2v^2(1+\varepsilon)\sigma^4/2$$

$$+ u^2v^2\sigma^4 + 2uv\sigma^2(\mu(u+z) - z\zeta\sigma^2 + \overline{\overline{o}}(vu) + \overline{\overline{o}}(z)) + v\overline{\overline{o}}(u^2)$$

$$+ u^2\overline{\overline{o}}(v) + \overline{\overline{o}}(u^2) + u\overline{\overline{o}}(v^3) + v^2\overline{\overline{o}}(u) + \overline{\overline{o}}(v^4)$$

$$+ (v^2 + |z^2 - u^2|)(\overline{\overline{o}}(z^2 - u^2) + \overline{\overline{o}}(uv) + \overline{\overline{o}}(v^2) + \overline{\overline{o}}(z))\mathbf{1}_{\{|u - |z|| > \varepsilon\}}$$

$$+ (\mu(u+z) + uv\sigma^2 - z\zeta\sigma^2 + \overline{\overline{o}}(vu) + \overline{\overline{o}}(z))$$

$$\times (\overline{\overline{o}}(z^2 - u^2) + \overline{\overline{o}}((z+u)^2))\mathbf{1}_{\{z \le -u-\varepsilon\}}$$

$$+ (\mu(u+z) + uv\sigma^2 - z\zeta\sigma^2 + \overline{\overline{o}}(vu) + \overline{\overline{o}}(z))\overline{o}(z^2 - u^2)\mathbf{1}_{\{-u+\varepsilon \le z \le 0\}}$$

$$+ (uv\sigma^2 + \mu u - z\sigma^2(\zeta - \mu/\sigma^2) + \overline{\overline{o}}(vu) + \overline{\overline{o}}(z))$$

$$\times (\overline{\overline{o}}(u^2) + \overline{\overline{o}}(z^2))\mathbf{1}_{\{z|\zeta - \mu/\sigma^2| > u(v - \varepsilon)\}}$$

$$\ge v^4(1-\varepsilon)\sigma^4/4 + (z^2 - u^2)^2\sigma^4/20 + v^2z^2(1-\varepsilon)\sigma^4/2$$

$$+ u^2v^2(1-\varepsilon)\sigma^4/2 + 2u^2v\mu\sigma^2 + 2uvz\sigma^2(\mu - \zeta\sigma^2)$$

$$+ \overline{\overline{o}}((vu)^2) + uv\overline{\overline{o}}(z) + v\overline{\overline{o}}(u^2)$$

$$+ u^2\overline{\overline{o}}(v) + u\overline{\overline{o}}(v^3) + v^2\overline{\overline{o}}(u) + \overline{\overline{o}}(v^4)$$

$$+ (z^2 - u^2)^2 \frac{\sigma^4}{20}$$

$$+ (v^2 + |z^2 - u^2|)\big(\overline{\overline{o}}(z^2 - u^2) + \overline{\overline{o}}(uv) + \overline{\overline{o}}(v^2) + \overline{\overline{o}}(z)\big)\mathbf{1}_{\{|u-|z||>\varepsilon\}}$$

$$+ (z^2 - u^2)^2 \frac{\sigma^4}{20} + \big(\mu(u + z) + uv\sigma^2 - z\zeta\sigma^2 + \overline{\overline{o}}(vu) + \overline{\overline{o}}(z)\big)$$

$$\times \big(\overline{\overline{o}}(z^2 - u^2) + \overline{\overline{o}}((z + u)^2)\big)\mathbf{1}_{\{z \le -u-\varepsilon\}}$$

$$+ (z^2 - u^2)^2 \frac{\sigma^4}{20}$$

$$+ \big(\mu(u + z) + uv\sigma^2 - z\zeta\sigma^2 + \overline{\overline{o}}(vu) + \overline{\overline{o}}(z)\big)\overline{\overline{o}}(z^2 - u^2)\mathbf{1}_{\{-u+\varepsilon \le z \le 0\}}$$

$$+ (z^2 - u^2)^2 \frac{\sigma^4}{20} + \big(uv\sigma^2 + \mu u - z\sigma^2(\zeta - \mu/\sigma^2) + \overline{\overline{o}}(vu) + \overline{\overline{o}}(z)\big)$$

$$\times \big(\overline{\overline{o}}(u^2) + \overline{\overline{o}}(z^2)\big)\mathbf{1}_{\{z|\zeta - \mu/\sigma^2| > u(v-\varepsilon)\}}.$$

Let us estimate the above terms separately:

$$v^4(1 - \varepsilon)\sigma^4/4 + (z^2 - u^2)^2\sigma^4/20 + v^2 z^2(1 - \varepsilon)\sigma^4/2 + u^2 v^2(1 - \varepsilon)\sigma^4/2$$

$$+ 2u^2 v\mu\sigma^2 + 2uvz\sigma^2(\mu - \zeta\sigma^2) + \overline{\overline{o}}((vu)^2) + uv\overline{\overline{o}}(z) + v\overline{o}(u^2)$$

$$+ u^2\overline{\overline{o}}(v) + u\overline{\overline{o}}(v^3) + v^2\overline{\overline{o}}(u) + \overline{\overline{o}}(v^4)$$

$$\ge v^4(1 - \varepsilon - \delta)\sigma^4/4 + (z^2 - u^2)^2\sigma^4/20 + v^2 z^2(1 - \varepsilon)\sigma^4/2$$

$$+ u^2 v^2(1 - \varepsilon - \delta)\sigma^4/2 + u\overline{\overline{o}}(v^3) - c_1 uv|z|,$$

$$(z^2 - u^2)^2 \frac{\sigma^4}{20} + (v^2 + |z^2 - u^2|)$$

$$\times \big(\overline{\overline{o}}(z^2 - u^2) + \overline{\overline{o}}(uv) + \overline{\overline{o}}(v^2) + \overline{\overline{o}}(z)\big)\mathbf{1}_{\{|u-|z||>\varepsilon\}}$$

$$(42) \quad \ge (z^2 - u^2)^2 \bigg(\frac{\sigma^4}{20} + \bigg(\frac{v^2}{|z| + u} + 1\bigg)\frac{\overline{\overline{o}}(|z| + u) + \overline{\overline{o}}(uv) + \overline{\overline{o}}(v^2) + \overline{\overline{o}}(z)}{|z| + u}\bigg),$$

$$(z^2 - u^2)^2 \frac{\sigma^4}{20} + \big(\mu(u + z) + uv\sigma^2 - z\zeta\sigma^2 + \overline{\overline{o}}(vu) + \overline{\overline{o}}(z)\big)$$

$$\times \big(\overline{\overline{o}}(z^2 - u^2) + \overline{\overline{o}}((z + u)^2)\big)\mathbf{1}_{\{z \le -u-\varepsilon\}}$$

$$(43) \quad \ge (z^2 - u^2)^2 \bigg(\frac{\sigma^4}{20} + \frac{\mu(u + z) + uv\sigma^2 - z\zeta\sigma^2 + \overline{\overline{o}}(vu) + \overline{\overline{o}}(z)}{u + |z|}$$

$$\times \frac{\overline{\overline{o}}(z^2 - u^2) + \overline{\overline{o}}((|z| - u)^2)}{(|z| - u)|z^2 - u^2|}\bigg),$$

$$(z^2 - u^2)^2 \frac{\sigma^4}{20} + \big(\mu(u+z) + uv\sigma^2 - z\zeta\sigma^2 + \overline{\overline{o}}(vu) + \overline{\overline{o}}(z)\big)$$

$$\times \overline{\overline{o}}(z^2 - u^2)\mathbf{1}_{\{-u+\varepsilon \le z \le 0\}}$$

$$(44) \quad \ge (z^2 - u^2)^2$$

$$\times \left(\frac{\sigma^4}{20} + \frac{(\mu(u+z) + uv\sigma^2 - z\zeta\sigma^2 + \overline{\overline{o}}(vu) + \overline{\overline{o}}(z))}{|z|+u} \frac{\overline{\overline{o}}(|z|+u)}{(u-|z|)^2} \right),$$

$$(z^2 - u^2)^2 \frac{\sigma^4}{20} + \big(uv\sigma^2 + \mu u - z\sigma^2(\zeta - \mu/\sigma^2) + \overline{\overline{o}}(vu) + \overline{\overline{o}}(z)\big)$$

$$\times \big(\overline{\overline{o}}(u^2) + \overline{\overline{o}}(z^2)\big)\mathbf{1}_{\{z|\zeta-\mu/\sigma^2|>u(v-\varepsilon)\}}$$

$$(45) \quad \ge (z^2 - u^2)^2$$

$$\times \left(\frac{\sigma^4}{20} + \frac{uv\sigma^2 + \mu u - z\sigma^2(\zeta - \mu/\sigma^2) + \overline{\overline{o}}(vu) + \overline{\overline{o}}(z)}{|z|+u} \frac{\overline{\overline{o}}(z^2)}{z^2} \right),$$

where we fix arbitrary $\delta \in (0, 1-\varepsilon)$ and assume that u, v and $|z|$ are large enough.

It only remans to notice that for any $v > 0$, there exist $R_1 > 0$ and $R_2 > 0$, such that for all $|z| > R_1$ and $u > R_2$, the right-hand sides of (42)–(45) are nonnegative, and in addition,

$$u^2 v^2 (1 - \varepsilon - \delta)\sigma^4/2 + u\overline{\overline{o}}(v^3) \ge u^2 v^2 (1 - \varepsilon - \delta)\sigma^4/4.$$

Thus we conclude that for any large enough $v > 0$, there exist $R_1 > 0$, $R_2 > 0$ and $\{c_i > 0\}$, such that, for all $|z| > R_1$ and $u > R_2$, the following holds:

$$\big|\psi(v+iu) - \psi(-\zeta - iz)\big|^2$$

$$\ge c_2 v^4 + c_3 (z^2 - u^2)^2 + 2c_4 v^2 z^2 + c_5 u^2 v^2 - c_1 uv|z|$$

$$= c_2 v^4 + c_3 (z^2 - u^2)^2 + c_4 v^2 z^2 + \big(\sqrt{c_4} v|z| - \sqrt{c_5} uv\big)^2$$

$$+ uv|z|(2\sqrt{c_4 c_5} v - c_1)$$

$$\ge c_6 \big((z^2 - u^2)^2 + z^2\big). \qquad \qquad \square$$

APPENDIX B

PROOF OF THEOREM 1. The proof consists of four steps. In step 1, using only Assumptions 1 and 3, we construct the generalized function $g \in \mathcal{D}^*$ as a limit of g_r, to show that it has support in $[0, \infty)$ and to establish the rate of convergence (28). In step 2, assuming in addition that $\hat{h} \in \mathbb{L}^1(\mathbb{R})$, we show that g coincides with a continuous function in $(0, \infty)$, and that g_r converges pointwise, with the rate of convergence given in the theorem. In step 3, we make the additional assumption (30) to prove that g is locally integrable and that $\mathbb{E}g(X_t) = \mathbb{E}h(X_t)$ for all $t > 0$. Finally, in step 4, we use the results of steps 2 and 3, to show that, even in

the absence of additional assumptions on $\hat{h}$ (i.e., using only Assumptions 1 and 3), the generalized function $g \in \mathcal{D}^*$, constructed in step 1, satisfies $\langle g, p_t \rangle = \mathbb{E}h(X_t)$, for all $t > 0$.

Step 1. First, for any $w \geq \gamma$, we introduce

$$(46) \quad g_r^w(x) = \frac{1}{2\pi i} \int_{\mathcal{G}_r^w} e^{\lambda x} \int_{\mathbb{R}} \left(\frac{\psi'(\lambda)}{\psi(\lambda) - \psi(-\zeta - iz)} - \frac{1}{\lambda + \zeta + iz} \right) \hat{h}(z) \, dz \, d\lambda,$$

where we denote by $\mathcal{G}_r^w$ the vertical interval $[w - ir, w + ir]$. Notice that $g_r = g_r^\gamma$. Let us show that g_r^w has a weak limit $g \in \mathcal{D}^*$, as $r \to \infty$, which is independent of w. Fix any $r' > r$, and any test function $\phi \in \mathcal{D}$, and proceed as follows:

$$(47) \quad \begin{aligned} &|\langle g_{r'}^w - g_r^w, \phi \rangle| \\ &\leq \frac{1}{2\pi} \int_{r < |u| < r'} \left| \int_{\mathbb{R}} e^{(w+iu)x} \phi(x) \, dx \right| \\ &\qquad \times \left| \int_{\mathbb{R}} \left(\frac{\psi'(w+iu)}{\psi(w+iu) - \psi(-\zeta - iz)} \right. \right. \\ &\qquad\qquad\qquad \left. \left. - \frac{1}{w + iu + \zeta + iz} \right) \hat{h}(z) \, dz \right| du. \end{aligned}$$

Integrating by parts repeatedly, we obtain

$$(48) \qquad \left| \int_{\mathbb{R}} e^{(w+iu)x} \phi(x) \, dx \right| = |w + iu|^{-3} \int_{\mathbb{R}} e^{wx} |\phi'''(x)| \, dx.$$

Making use of (21), we obtain

$$|\langle g_{r'}^w - g_r^w, \phi \rangle| \leq \frac{c_1}{\pi} (1/r - 1/r') (\|\hat{h}\|_{\mathbb{L}^1(\mathbb{R})} \wedge \|\hat{h}\|_{\mathbb{L}^2(\mathbb{R})}) \int_{\mathbb{R}} e^{wx} |\phi'''(x)| \, dx.$$

The above estimate shows that $g^w = \lim_{r \to \infty} g_r^w$ is well defined as an element of $\mathcal{D}^*$. To see that g^w is independent of w, consider arbitrary $w' > w$ and connect the two intervals of integration $\mathcal{G}_r^w$ and $\mathcal{G}_r^{w'}$ by the two horizontal parts: $\mathcal{C}_r^1 = [w - ir, w' - ir]$ and $\mathcal{C}_r^2 = [w + ir, w' + ir]$. Since the integrand in (46) is analytic with respect to λ, the integral over the closed contour (with appropriately chosen directions on each part) is zero. Thus we only need to show that the integrals over $\mathcal{C}_r^1$ and $\mathcal{C}_r^2$ vanish, as $r \to \infty$,

$$\begin{aligned} &|\langle g_r^{w'} - g_r^w, \phi \rangle| \\ &\leq \frac{1}{2\pi} \sum_{u = -r, r} \int_w^{w'} \left| \int_{\mathbb{R}} e^{(v+iu)x} \phi(x) \, dx \right| \\ &\qquad \times \left| \int_{\mathbb{R}} \left(\frac{\psi'(v+iu)}{\psi(v+iu) - \psi(-\zeta - iz)} \right. \right. \\ &\qquad\qquad\qquad \left. \left. - \frac{1}{v + iu + \zeta + iz} \right) \hat{h}(z) \, dz \right| dv. \end{aligned}$$

Estimates (48) and (21) imply that the right-hand side of the above vanishes as $r \to \infty$. Thus g^w is independent of w, and we denote it by g. Let us show that g has support in $[0, \infty)$. Choose an arbitrary $\phi \in \mathcal{D}$, such that $\operatorname{supp}(\phi) \subset (-\infty, 0)$, and consider $\langle g_r^w, \phi \rangle$. Equation (48), in this case, becomes

$$\left| \int_{-\infty}^0 e^{(w+iu)y} \phi(y) \, dy \right| = |w + iu|^{-3} \int_{-\infty}^0 e^{wy} |\phi'''(y)| \, dy \leq c_2 |w + iu|^{-3},$$

which holds uniformly over all $u \in \mathbb{R}$ and $w \geq \gamma$. Thus we can close the contour of integration in the integral representation of $\langle g_r^w, \phi \rangle$ [cf. (47)] by a semicircle (on the right-hand side), and using the above estimate, along with (21) and the analyticity of the integrand in H_R (cf. Corollary 1), conclude that $\langle g_r^w, \phi \rangle \to 0$, as $r \to \infty$. To obtain (28), we recall (47) and the fact that

$$\int_{\mathbb{R}} e^{(w+iu)x} p_t(x) \, dx = \exp(t \psi(w + iu)).$$

Then (28) follows from (21) and (27).

Step 2. Next, under the additional assumption that $\hat{h}$ is absolutely integrable, we show that g coincides with a continuous function in $(0, \infty)$, and that $g_r(x)$ converges to $g(x)$ for every $x > 0$. Applying Fubini's theorem and integration by parts we obtain

$$
\begin{aligned}
& 2\pi i g_r(x) \\
&= \frac{1}{2\pi i} \int_{\mathcal{G}_r} e^{\lambda x} \int_{\mathbb{R}} \left(\frac{\psi'(\lambda)}{\psi(\lambda) - \psi(-\zeta - iz)} - \frac{1}{\lambda + \zeta + iz} \right) \hat{h}(z) \, dz \, d\lambda \\
&= \int_{\mathcal{G}_r} \int_{\mathbb{R}} e^{\lambda x} \frac{\psi'(\lambda)}{\psi(\lambda) - \psi(-\zeta - iz)} \hat{h}(z) \, dz \, d\lambda \\
&\quad - \int_{\mathcal{G}_r} \int_{\mathbb{R}} e^{\lambda x} \frac{1}{\lambda + \zeta + iz} \hat{h}(z) \, dz \, d\lambda \\
&= \frac{1}{x} \int_{\mathbb{R}} \int_{\mathcal{G}_r} (e^{\lambda x})' \frac{\psi'(\lambda)}{\psi(\lambda) - \psi(-\zeta - iz)} \, d\lambda \hat{h}(z) \, dz \\
&\quad - \frac{1}{x} \int_{\mathbb{R}} \int_{\mathcal{G}_r} (e^{\lambda x})' \frac{1}{\lambda + \zeta + iz} \, d\lambda \hat{h}(z) \, dz \\
&= \frac{1}{x} \int_{\mathbb{R}} \left(\frac{\exp((\gamma + ir)x) \psi'(\gamma + ir)}{\psi(\gamma + ir) - \psi(-\zeta - iz)} - \frac{\exp((\gamma - ir)x) \psi'(\gamma - ir)}{\psi(\gamma - ir) - \psi(-\zeta - iz)} \right) \hat{h}(z) \, dz \\
&\quad + \frac{1}{x} \int_{\mathbb{R}} \left(\frac{\exp((\gamma - ir)x)}{\gamma + \zeta - ir + iz} - \frac{\exp((\gamma + ir)x)}{\gamma + \zeta + ir + iz} \right) \hat{h}(z) \, dz \\
&\quad - \frac{1}{x} \int_{\mathbb{R}} \int_{\mathcal{G}_r} \frac{\exp(\lambda x) \psi''(\lambda)}{\psi(\lambda) - \psi(-\zeta - iz)} \, d\lambda \hat{h}(z) \, dz \\
&\quad + \frac{1}{x} \int_{\mathbb{R}} \int_{\mathcal{G}_r} \frac{\exp(\lambda x)(\psi'(\lambda))^2}{(\psi(\lambda) - \psi(-\zeta - iz))^2} \, d\lambda \hat{h}(z) \, dz
\end{aligned}
\tag{49}
$$

$$- \frac{1}{x} \int_{\mathbb{R}} \int_{\mathcal{G}_r} \frac{\exp(\lambda x)}{(\lambda + \zeta + iz)^2} \, d\lambda \hat{h}(z) \, dz.$$

Let us show that the first integral on the right-hand side of (49) converges to zero, as $r \to \infty$. Due to (23), for any $R > 0$, there exist $c_1, c_2, c_3 > 0$ and $r' > 0$, such that, for all $r > r'$, we have

$$\left| \int_{\mathbb{R}} \left(\frac{\exp((\gamma + ir)x)\psi'(\gamma + ir)}{\psi(\gamma + ir) - \psi(-\zeta - iz)} - \frac{\exp((\gamma - ir)x)\psi'(\gamma - ir)}{\psi(\gamma - ir) - \psi(-\zeta - iz)} \right) \hat{h}(z) \, dz \right|$$

$$\le c_1 e^{\gamma x} \int_{|z| \le R} \frac{r}{r^2 - c_2} |\hat{h}(z)| \, dz$$

(50)

$$+ c_3 e^{\gamma x} \int_{|z| > R} \left(\frac{r}{|\psi(\gamma + ir) - \psi(-\zeta - iz)|} \right.$$

$$\left. + \frac{r}{|\psi(\gamma - ir) - \psi(-\zeta - iz)|} \right) |\hat{h}(z)| \, dz.$$

We choose R to be large enough, so that the estimate in Lemma 5 can be applied for all $|z| \ge R$ and $|u| = r$,

$$\frac{1}{|\psi(\gamma + ir) - \psi(-\zeta - iz)|} + \frac{1}{|\psi(\gamma - ir) - \psi(-\zeta - iz)|}$$

$$\le c_4 \frac{1}{\sqrt{(z^2 - r^2)^2 + z^2}}.$$

Thus, the first integral on the right-hand side of (49) is estimated from above by

$$c_1 e^{\gamma x} \int_{|z| \le N} \frac{r}{r^2 - c_2} |\hat{h}(z)| \, dz + c_3 c_4 e^{\gamma x} \int_{|z| > N} \frac{r}{\sqrt{(z^2 - r^2)^2 + z^2}} |\hat{h}(z)| \, dz$$

(51)

$$\le c_1 e^{\gamma x} \int_{|z| \le N} \frac{r}{r^2 - c_2} |\hat{h}(z)| \, dz + c_3 c_4 e^{\gamma x} \int_{|z| > N, |z| \le r/2} \frac{2}{r} |\hat{h}(z)| \, dz$$

$$+ 2 c_3 c_4 e^{\gamma x} \int_{|z| > N, |z| > r/2} |\hat{h}(z)| \, dz,$$

which vanishes, as $r \to \infty$. Similarly, we proceed with the second integral on the right-hand side of (49):

$$\left| \int_{\mathbb{R}} \left(\frac{\exp((\gamma - ir)x)}{\gamma + \zeta - ir + iz} - \frac{\exp((\gamma + ir)x)}{\gamma + \zeta + ir + iz} \right) \hat{h}(z) \, dz \right|$$

(52)

$$\le 2 e^{\gamma x} \int_{\mathbb{R}} \frac{1}{\sqrt{(\gamma + \zeta)^2 + (z - r)^2}} |\hat{h}(z)| \, dz$$

$$\le 2 e^{\gamma x} \int_{|z| \le r/2} \frac{2}{r} |\hat{h}(z)| \, dz + 2 e^{\gamma x} \int_{|z| > r/2} \frac{1}{\gamma + \zeta} |\hat{h}(z)| \, dz.$$

As $r \to \infty$, the third, fourth and fifth integrals on the right-hand side of (49) converge uniformly over x, changing on any compact in $(0, \infty)$. Let us prove the convergence of the third integral. We consider arbitrary, large enough, $r' > r$ and proceed as follows:

$$
\left| \int_{\mathbb{R}} \int_{\mathcal{G}_{r'} \backslash \mathcal{G}_r} \frac{\exp(\lambda x) \psi''(\lambda)}{\psi(\lambda) - \psi(-\zeta - iz)} \, d\lambda \hat{h}(z) \, dz \right|
$$

(53)
$$
\leq 2 e^{\gamma x} \int_{\mathbb{R}} \int_{|u| \in [r, r']} \frac{|\psi''(\gamma + iu)|}{|\psi(\gamma + iu) - \psi(-\zeta - iz)|} \, du \, |\hat{h}(z)| \, dz
$$

$$
\leq c_5 e^{\gamma x} \int_{-R}^{R} |\hat{h}(z)| \, dz \int_{r}^{r'} \frac{1}{u^2 - c_6} \, du
$$

$$
+ c_5 e^{\gamma x} \int_{|z| > R} \int_{r}^{r'} \left| \frac{1}{\psi(\gamma + iu) - \psi(-\zeta - iz)} \right| \, du \, |\hat{h}(z)| \, dz,
$$

where, again, we choose $R > 0$ to be large enough, so that Lemma 5 can be applied for all $|z| \geq R$ and all $|u| \geq r$. It is easy to see that the first term on the right-hand side of (53) vanishes, as $r, r' \to \infty$. To show that the second term on the right-hand side of (53) vanishes, as $r, r' \to \infty$, we make use of Lemma 5 to obtain

$$
\int_{|z| > R} \int_{r}^{r'} \left| \frac{1}{\psi(\gamma + iu) - \psi(-\zeta - iz)} \right| \, du \, |\hat{h}(z)| \, dz
$$

$$
\leq c_7 \int_{|z| > R} \int_{r}^{r'} \frac{1}{\sqrt{(u^2 - z^2)^2 + z^2}} \, du \, |\hat{h}(z)| \, dz.
$$

Thus the right-hand side of (53) is bounded from above by the following expression:

$$
c_8 e^{\gamma x} \frac{1}{r} \int_{-R}^{R} |\hat{h}(z)| \, dz + c_9 e^{\gamma x} \int_{|z| > R} \int_{r}^{\infty} \frac{1}{\sqrt{(u^2 - z^2)^2 + z^2}} \, du \, |\hat{h}(z)| \, dz
$$

(54)
$$
\leq c_8 e^{\gamma x} \frac{1}{r} \int_{-R}^{R} |\hat{h}(z)| \, dz
$$

$$
+ c_9 e^{\gamma x} \int_{R < |z| \leq r/2} \int_{r}^{\infty} \frac{1}{(u - r/2)^2} \, du \, |\hat{h}(z)| \, dz
$$

$$
+ c_9 e^{\gamma x} \int_{|z| > R, |z| > r/2} \int_{0}^{\infty} \frac{1}{\sqrt{u^4 + 1}} \, du \, |\hat{h}(z)| \, dz,
$$

where we make use of

(55)
$$
\left| \int_{r}^{r'} \frac{1}{\sqrt{(u^2 - z^2)^2 + z^2}} \, du \right| \leq \int_{\mathbb{R}} \frac{1}{\sqrt{u^4 + 1}} \, du < \infty.
$$

Next, we use Lemma 5 to prove the convergence of the fourth integral on the right-hand side of (49):

$$\int_{\mathbb{R}} \int_r^{r'} \frac{|\exp((\gamma + iu)x)||\psi'(\gamma + iu)|^2}{|\psi(\gamma + iu) - \psi(-\zeta - iz)|^2} \, du |\hat{h}(z)| \, dz$$

(56)
$$\leq c_{10} e^{\gamma x} \int_{|z| \leq R} \int_r^{r'} \frac{u^2}{|u^2 - c_{11}|^2} \, du |\hat{h}(z)| \, dz$$

$$+ c_{12} e^{\gamma x} \int_{|z| > R} \int_r^{r'} \frac{u^2}{(u^2 - z^2)^2 + z^2} \, du |\hat{h}(z)| \, dz.$$

The right-hand side of the above is bounded by

$$c_{13} e^{\gamma x} \frac{1}{r} \int_{|z| \leq R} |\hat{h}(z)| \, dz + c_{12} e^{\gamma x} \int_{R < |z| \leq r/2} \int_r^{\infty} \frac{u^2}{(u - r/2)^4} \, du |\hat{h}(z)| \, dz$$

(57)
$$+ c_{12} e^{\gamma x} \int_{R \langle |z| \rangle r/2} \left(2(1 + 1/R)^2 + \int_{|u| > 1} \frac{1}{u^2} \, du \right) |\hat{h}(z)| \, dz,$$

where we make use of

$$\left| \int_0^{\infty} \frac{u^2}{(u^2 - z^2)^2 + z^2} \, du \right|$$

$$\leq \int_{z-1}^{z+1} \frac{u^2}{(u^2 - z^2)^2 + z^2} \, du + \int_{u \in [0, z-1] \cup [z+1, \infty)} \frac{u^2}{(u^2 - z^2)^2 + z^2} \, du$$

(58)
$$\leq \int_{1-1/z}^{1+1/z} \frac{zu^2}{z^2(u^2 - 1)^2 + 1} \, du$$

$$+ \int_{u \in [0, z-1] \cup [z+1, \infty)} \frac{u^2}{(u - z)^2(u + z)^2} \, du$$

$$\leq 2(1 + 1/R)^2 + \int_{|u| > 1} \frac{1}{u^2} \, du < \infty.$$

To show that the last integral on the right-hand side of (49) converges as $r \to \infty$, we notice that

$$\left| \int_{\mathbb{R}} \int_{\mathcal{G}_{r'} \setminus \mathcal{G}_r} \frac{\exp(\lambda x)}{(\lambda + \zeta + iz)^2} \, d\lambda \hat{h}(z) \, dz \right|$$

$$\leq e^{\gamma x} \int_{\mathbb{R}} \int_r^{r'} \frac{1}{(\gamma + \zeta)^2 + (z + u)^2} \, du |\hat{h}(z)| \, dz$$

(59)
$$\leq e^{\gamma x} \int_{|z| \leq r/2} \int_r^{\infty} \frac{1}{(u - r/2)^2} \, du |\hat{h}(z)| \, dz$$

$$+ e^{\gamma x} \int_{|z| > r/2} \int_{\mathbb{R}} \frac{1}{(\gamma + \zeta)^2 + u^2} \, du |\hat{h}(z)| \, dz.$$

E. BAYRAKTAR AND S. NADTOCHIY

Thus we have shown that for $x > 0$, $g_r(x)$ converges to $g(x)$, as $r \to \infty$. Moreover, estimates (51), (52), (54), (57) and (59) imply the desired rate of convergence (29). To see that the limiting function $g(x)$ is continuous for $x \in (0, \infty)$, we notice that, due to (29), the convergence is uniform over x changing on any compact in $(0, \infty)$. Similarly, (29) implies that g has at most exponential growth at infinity.

Step 3. Under the additional assumption that $\hat{h}$ is absolutely integrable and that (30) holds, let us show that for every $w \geq \gamma$, the function

$$F^w : u \mapsto \int_{\mathbb{R}} \left(\frac{\psi'(w + iu)}{\psi(w + iu) - \psi(-\zeta - iz)} - \frac{1}{wa + iu + \zeta + iz} \right) \hat{h}(z) \, dz$$

is square integrable over $\mathbb{R}$. Notice that F^w is continuous (cf. Corollary 1), and therefore, it suffices to estimate $|F^w(u)|$ for large $|u|$. Let us choose a large enough $R > 0$, for which there exists a constant $\beta \in (0, 1)$, such that

$$|\psi(w + iu)| \geq 2|\psi(-\zeta - iz)|$$

holds for all $|z| \leq \beta|u|$ and all $|u| > R$. Notice that such R and β do exist, due to the inequalities (23). Increasing R, if necessary, we ensure that Lemma 5 can be applied for all $|u| > R$ and all $|z| > \beta R$. Finally, for any $|u| > R$, we integrate by parts to obtain

$$\left| F^w(u) \right| \leq \int_{|z| \leq \beta |u|} \left| \frac{\psi'(w + iu)}{\psi(w + iu) - \psi(-\zeta - iz)} - \frac{1}{w + iu + \zeta + iz} \right| |\hat{h}(z)| \, dz$$

$$+ \sum_{z = -\beta u, \beta u} \left| \frac{\psi'(w + iu)}{\psi(w + iu) - \psi(-\zeta - iz)} \right.$$

$$\left. - \frac{1}{w + iu + \zeta + iz} \right| \left| \int_{|z'| > \beta |u|} |\hat{h}(z')| \, dz' \right.$$

$$+ \int_{|z| > \beta |u|} \left| \frac{\psi'(-\zeta - iz)\psi'(w + iu)}{(\psi(w + iu) - \psi(-\zeta - iz))^2} \right.$$

$$\left. - \frac{1}{(w + iu + \zeta + iz)^2} \right| \int_{|z'| > |z|} |\hat{h}(z')| \, dz' \, dz$$

$$\leq \frac{c_1}{|u|} \int_{\mathbb{R}} |\hat{h}(z)| \, dz$$

$$+ c_2 \int_{|z'| > \beta |u|} |\hat{h}(z')| \, dz' \int_{|z| > \beta |u|} \left(\frac{|z||u|}{(z^2 - u^2)^2 + z^2} \right.$$

$$\left. + \frac{1}{(w + \zeta)^2 + (z + u)^2} \right) dz$$

$$\leq \frac{c_1}{|u|} \int_{\mathbb{R}} |\hat{h}(z)| \, dz + c_3 \int_{|z'| > \beta |u|} |\hat{h}(z')| \, dz' \left(\int_0^\infty \frac{z^2}{(z^2 - u^2)^2 + u^2} \, dz + c_4 \right)$$

$$\leq \frac{c_1}{|u|} \int_{\mathbb{R}} |\hat{h}(z)| \, dz + c_5 \int_{|z'| > \beta |u|} |\hat{h}(z')| \, dz',$$

where we made use of (58). Using the above estimate and (30), we conclude that F^w belongs to $\mathbb{L}^2(\mathbb{R})$. Then the standard properties of Fourier transform yield that F^w is a Fourier transform of some $\tilde{g}^w \in \mathbb{L}^2(\mathbb{R})$. Moreover, $\tilde{g}^w$ can be obtained as the $\mathbb{L}^2(\mathbb{R})$-limit of functions $x \mapsto e^{-wx} g_r^w(x)$, as $r \to \infty$. Since we showed in step 1 that g_r^w converges weakly to g (which is independent of w), we conclude that $g(x)$ is locally integrable, and $e^{-wx} g(x)$ is square integrable over $x \in \mathbb{R}$, for any $w \geq \gamma$. In addition, the Fourier transform of $x \mapsto e^{-wx} g(x)$ is F^w, which implies

$$\lim_{R \to \infty} \int_{-R}^{R} e^{-(w+iu)x} g(x)\, dx = F^w(u),$$

where the convergence is understood in an $\mathbb{L}^2(\mathbb{R})$ sense. Moreover, since $x \mapsto e^{-wx} g(x)$ is square integrable over $x \in \mathbb{R}$ and g has support in $[0, \infty)$, it is easy to deduce that the left-hand side of the above equation converges point wise to a continuous function of $w + iu$, for all $u \in \mathbb{R}$ and $w \geq \gamma$. This, along with Corollary 1, implies

$$\int_0^{\infty} e^{-wx} g(x)\, dx = F^w(0)$$
$$= \int_{\mathbb{R}} \left(\frac{\psi'(w)}{\psi(w) - \psi(-\zeta - iz)} - \frac{1}{w + \zeta + iz} \right) \hat{h}(z)\, dz$$
$$= \psi'(w) \Upsilon(\psi(w)),$$

for all $w \geq \gamma$. As shown in Section 2.1, a change of variables turns the left and the right-hand sides of the above equation into the Laplace transforms of $\mathbb{E}g(X_t)$ and $\mathbb{E}h(X_t)$, respectively. Due to the uniqueness of the Laplace inverse, the expectations have to coincide for all $t > 0$.

Step 4. It only remains to show that even without the additional assumptions, $\hat{h} \in \mathbb{L}^1(\mathbb{R})$ and (30), the action of the generalized function $g \in \mathcal{D}^*$ (constructed in step 1) on p_t coincides with $\mathbb{E}h(X_t)$, for all $t > 0$. Since $\hat{h} \in \mathbb{L}^1(\mathbb{R})$ [resp., $\hat{h} \in \mathbb{L}^2(\mathbb{R})$], there exists a sequence of functions $\hat{h}^n$, such that every $\hat{h}^n$ is infinitely smooth, with compact support in $(-\infty, 0)$, and $\hat{h}^n$ converges to $\hat{h}$ in $\mathbb{L}^1(\mathbb{R})$ [resp., $\mathbb{L}^2(\mathbb{R})$]. Denote by $e^{\zeta x} h^n(x)$ the Fourier transform of $\hat{h}^n$. Then $e^{\zeta x} h^n(x)$ converges to $e^{\zeta x} h(x)$ in $\mathbb{L}^{\infty}(\mathbb{R})$ [resp., $\mathbb{L}^2(\mathbb{R})$]. Notice that every $\hat{h}^n$ belongs to $\mathbb{L}^1(\mathbb{R}) \cap \mathbb{L}^2(\mathbb{R})$ and satisfies (30). Then as shown in steps 2 and 3, there exists a locally integrable function g^n, with at most exponential growth at infinity, which is the weak symmetry image of h^n. Namely, it satisfies

$$\langle g^n, p_t \rangle = \mathbb{E}g^n(X_t) = \mathbb{E}h^n(X_t) \qquad \forall t > 0.$$

Notice that, since X_t has a continuous density and since g^n is locally integrable with at most exponential growth, the expectation of $g^n(X_t)$ is well defined. Due to

Lemma 7 and the choice of h^n, we obtain

$$\mathbb{E}\big|h(X_t) - h^n(X_t)\big|$$

$$= \int_{\mathbb{R}} \big|h(x) - h^n(x)\big| e^{\zeta x} e^{-\zeta x} p_t(x)\, dx$$

$$\leq \min\big(\|(h - h^n)e^{\zeta \cdot}\|_{\mathbb{L}^\infty(\mathbb{R})} \|p_t e^{-\zeta \cdot}\|_{\mathbb{L}^1(\mathbb{R})}, \|(h - h^n)e^{\zeta \cdot}\|_{\mathbb{L}^2(\mathbb{R})} \|p_t e^{-\zeta \cdot}\|_{\mathbb{L}^2(\mathbb{R})}\big)$$

$$\to 0,$$

as $n \to \infty$. Let us show that $\langle g^n, p_t \rangle \to \langle g, p_t \rangle$:

$$\big|\langle g - g^n, p_t \rangle\big|$$

$$\leq \frac{1}{2\pi} \int_{\mathbb{R}} \left|\int_{\mathbb{R}} e^{(\gamma + iu)y} p_t(y)\, dy\right|$$

$$\times \left|\int_{\mathbb{R}} \left(\frac{\psi'(\gamma + iu)}{\psi(\gamma + iu) - \psi(-\zeta - iz)} - \frac{1}{\gamma + iu + \zeta + iz}\right)\right|$$

$$\times \big|\hat{h}(z) - \hat{h}^n(z)\big|\, dz\, du.$$

Making use of (48) and (21), we conclude that

$$\big|\langle g - g^n, p_t \rangle\big| \leq c_1\big(\|\hat{h} - \hat{h}^n\|_{\mathbb{L}^1(\mathbb{R})} \wedge \|\hat{h} - \hat{h}^n\|_{\mathbb{L}^2(\mathbb{R})}\big) \to 0,$$

as $n \to \infty$, which implies $\langle g, p_t \rangle = \mathbb{E}h(X_t)$, for all $t > 0$, and completes the proof of the theorem. $\quad\square$

PROOF OF COROLLARY 2.   First, we notice that

$$\big|g(x) - g_{r,R}(x)\big| \leq \big|g(x) - g_r(x)\big| + \big|g_r(x) - g_{r,R}(x)\big|.$$

The first term on the right-hand side of the above is bounded by the right-hand side of (34) due to Theorem 1. To analyze the second term, we only need to estimate the right-hand side of (49), with the integration over $z \in \mathbb{R}$ replaced by the integration over $|z| > R$. We will refer to it as the modified right-hand side of (49). The estimation is done as in step 2 in the proof of Theorem 1, with the exception that in the present case, all the terms vanish, as $r, R \to \infty$. Following the derivation of (51), we conclude that the first integral on the modified right-hand side of (49) is estimated from above by

$$c_1 e^{\gamma x} \int_{|z| > R} \frac{r}{\sqrt{(z^2 - r^2)^2 + z^2}} \big|\hat{h}(z)\big|\, dz$$

$$\leq c_1 e^{\gamma x} \int_{R < |z| \leq r/2} \frac{2}{r} \big|\hat{h}(z)\big|\, dz + 2c_1 e^{\gamma x} \int_{|z| > R, |z| > r/2} \big|\hat{h}(z)\big|\, dz.$$

Similarly to (52), the second integral on the modified right-hand side of (49) is bounded by

$$2e^{\gamma x} \int_{R < |z| \leq r/2} \frac{2}{r} \big|\hat{h}(z)\big|\, dz + 2e^{\gamma x} \int_{|z| > R, |z| > r/2} \frac{1}{\gamma + \zeta} \big|\hat{h}(z)\big|\, dz.$$

Following (54), we estimate the third integral on the modified right-hand side of (49) via

$$c_2 e^{\gamma x} \int_{R < |z| \le r/2} \int_r^\infty \frac{1}{(u - r/2)^2} \, du \big| \hat{h}(z) \big| \, dz$$

$$+ c_3 e^{\gamma x} \int_{|z| > R, |z| > r/2} \int_0^\infty \frac{1}{\sqrt{u^4 + 1}} \, du \big| \hat{h}(z) \big| \, dz.$$

Similarly to (57), we find the upper bound for the fourth integral on the modified right-hand side of (49):

$$c_4 e^{\gamma x} \int_{R < |z| \le r/2} \int_r^\infty \frac{u^2}{(u - r/2)^4} \, du \big| \hat{h}(z) \big| \, dz$$

$$+ c_4 e^{\gamma x} \int_{|z| > R, |z| > r/2} \left(2(1 + 1/N)^2 + \int_{|u| > 1} \frac{1}{u^2} \, du \right) \big| \hat{h}(z) \big| \, dz.$$

Finally, we obtain the estimate of the fifth integral on the modified right-hand side of (49), following the derivation of (59):

$$e^{\gamma x} \int_{R < |z| \le r/2} \int_r^\infty \frac{1}{(u - r/2)^2} \, du \big| \hat{h}(z) \big| \, dz$$

$$+ e^{\gamma x} \int_{|z| > R, |z| > r/2} \int_{\mathbb{R}} \frac{1}{(\gamma + \zeta)^2 + u^2} \, du \big| \hat{h}(z) \big| \, dz.$$

Collecting the above estimates, we obtain (34). □

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