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GLASSY PHASE AND FREEZING OF LOG-CORRELATED GAUSSIAN POTENTIALS

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In this paper, we consider the Gibbs measure associated to a logarithmically correlated random potential (including two-dimensional free fields) at low temperature. We prove that the energy landscape freezes and enters in the so-called glassy phase. The limiting Gibbs weights are integrated atomic random measures with random intensity expressed in terms of the critical Gaussian multiplicative chaos constructed in [*Ann. Probab.* **42** (2014) 1769–1808 and *Comm. Math. Phys.* (2013) To appear]. This could be seen as a first rigorous step in the renormalization theory of super-critical Gaussian multiplicative chaos.

1. Introduction. Consider a log-correlated random distribution $(X(x))_{x\in\mathbb{R}^{\mathsf{d}}}$ on (a subdomain of) $\mathbb{R}^{\mathsf{d}}$ and apply a cut-off regularization procedure to get a field $(X_t(x))_{x\in\mathbb{R}^{\mathsf{d}}}$ with variance of order t, that is, $\mathbf{E}[X_t(x)^2]\simeq t$ as $t\to\infty$. One may, for instance, think of the convolution of X with a mollifying sequence, the projection of X onto a finite-dimensional set of functions or a white noise decomposition of X. We will be interested in the study of the behaviour of the random measure on the Borel sets of $\mathbb{R}^{\mathsf{d}}$:

$$M_t(dx)=e^{\gamma X_t(x)}\,dx,$$

where $\gamma>0$ is a parameter that stands for the inverse temperature. The high temperature phase is well known since the original work of Kahane [14] where it is proved that for $\gamma^2<2\mathsf{d}$ the renormalized measure

$$e^{-\gamma^2/2t}M_t(dx)$$

almost surely weakly converges toward a nontrivial measure $M_\gamma(dx)$, which is diffuse. At the critical temperature $\gamma^2=2\mathsf{d}$, the renormalized measure

$$\sqrt{t}e^{-\mathsf{d}t}M_t(dx)$$

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weakly converges in probability toward a nontrivial diffuse measure $M'(dx)$, which is called derivative multiplicative chaos [10, 11]. The purpose of this paper is to study the supercritical/low temperature phase $\gamma^2 > 2\mathsf{d}$ and to prove that the renormalized measure

$$t^{3\gamma/(2\sqrt{2\mathsf{d}})} e^{(\gamma\sqrt{2\mathsf{d}}-\mathsf{d})t} M_t(dx) \tag{1.1}$$

weakly converges in law toward a purely atomic stable random measure S_γ with (random intensity) M', up to a deterministic multiplicative constant; call it $C(\gamma)$ (see Section 2 for a rigorous statement).

This is a longstanding problem, which has received much attention by physicists. It was first raised in [9, 18] on dyadic trees, and then followed by [8, 12, 13] for log-correlated Gaussian random fields. Following our notation, these papers essentially derived the statistics of the size ordered atoms of the measure $\frac{M_t(dx)}{M_t([0,1]^{\mathsf{d}})}$, the so-called Poisson–Dirichlet statistics characteristic of stable Lévy processes. However, these papers did not investigate the problem of the localization of these atoms.

A few years later, the mathematical community caught up on this problem. In the context of branching random walks, convergence of the measures (1.1) is investigated in [16, 24]. Built on these works, the limit is identified in [5] and is expressed as a stable transform of the so-called derivative martingale. In the context of log-correlated Gaussian potentials, the authors in [4] conjecture that results similar to branching random walks should hold. The first rigorous and important result for log-correlated Gaussian fields appeared in [2] where the authors established the Poisson–Dirichlet statistics of the limiting measure in dimension 1 (renormalized by its total mass) via spin glass techniques, hence confirming the prediction of [8] (these results were recently extended by the same authors in [3] to cover the case of the discrete GFF in a bounded domain).

Roughly speaking, the terminology freezing comes from the linearization of the free energy of the measure M_t beyond the value $\gamma^2 = 2\mathsf{d}$ (see [8, 9, 12, 13] for further comments). The terminology *glassy phase* comes from the fact that for $\gamma^2 > 2\mathsf{d}$, the measure M_t is essentially dominated by a few points, the local extreme values of the field X_t (along with the neighborhood of these extreme values). Therefore, this paper possesses strong connections with the study of the extreme values of the field X_t. This was conjectured in [11] and important advances on this topic have recently appeared in [6, 7] in the context of the discrete GFF and in [17] for a large class of log-correlated fields. However, the description of the local maxima obtained in [6] is not sufficient to obtain the so-called freezing theorems that will be established in this paper.

Finally, we would like to stress that we will only deal with the case of white noise cut-off of the Gaussian distribution X, building on techniques developed in [17]. We will then extend our results to two-dimensional free fields. It is natural to wonder whether the nature of the cut-off may affect the structure of the limiting measure. We will prove that the freezing theorem does not depend on the chosen cutoff family provided the cutoff is not too far from a white noise decomposition. From a more general angle, we believe that the glassy phase does not depend on the chosen cut-off, except at the level of the multiplicative constant $C(\gamma)$. For instance, given a smooth mollifier θ and setting $\theta_\varepsilon = \frac{1}{\varepsilon^{\mathrm{d}}}\theta(\frac{\cdot}{\varepsilon})$, similar theorems should hold for measures built on approximations of the form $\theta_\varepsilon * X$: in this setting, one would obtain an analog of Theorem 2.2 where the constant $C(\gamma)$ is replaced by a constant $C(\theta,\gamma)$ depending on θ,γ.

2. Setup and main results.

2.1. *Star scale invariant fields*. We denote by $\mathcal{B}_b(\mathbb{R}^{\mathrm{d}})$ the Borel subsets of $\mathbb{R}^{\mathrm{d}}$. Let us introduce a canonical family of log-correlated Gaussian distributions, called star scale invariant, and their cut-off approximations, which we will work with in the first part of the paper. Let us consider a continuous covariance kernel k on $\mathbb{R}^{\mathrm{d}}$ such that we have the following.

ASSUMPTION (A). The kernel k satisfies the following assumptions, for some constant C independent of $x \in \mathbb{R}^{\mathrm{d}}$:

A1. k is of class C^1, nonnegative and normalized by the condition $k(0) = 1$,
A2. k has compact support,
A3. $|k(x) - k(0)| \le C|x|$ for some constant $C := C_k$ independent of $x \in \mathbb{R}^{\mathrm{d}}$.

We set for $t \ge 0$ and $x \in \mathbb{R}^{\mathrm{d}}$

$$K_t(x) = \int_1^{e^t} \frac{k(xu)}{u}\,du. \tag{2.1}$$

We consider a family of centered Gaussian processes $(X_t(x))_{x\in\mathbb{R}^{\mathrm{d}},t\ge 0}$ with covariance kernel given by

$$\forall t, s \ge 0, \qquad \mathbf{E}\big[X_t(x)X_s(y)\big] = K_{t\wedge s}(y-x), \tag{2.2}$$

where $t \wedge s := \min(t,s)$. The construction of such fields is possible via a white noise decomposition as explained in Section 4 (page 762) of [1]. We set

$$\mathcal{F}_t = \sigma\{X_u(x); x \in \mathbb{R}^{\mathrm{d}}, u \le t\}.$$

We stress that, for $s > t$, the field $(X_s(x) - X_t(x))_{x\in\mathbb{R}^{\mathrm{d}}}$ is independent from $\mathcal{F}_t$.

We introduce for $t > 0$ and $\gamma > 0$, the random measures $M_t'(dx)$ and $M_t^\gamma(dx)$

$$\begin{aligned} M_t'(A) &:= \int_A \big(\sqrt{2\mathrm{d}}t - X_t(x)\big)e^{\sqrt{2\mathrm{d}}X_t(x)-\mathrm{d}t}\,dx, \\ M_t^\gamma(A) &:= \int_A e^{\gamma X_t(x)-(\gamma^2/2)t}\,dx \qquad \forall A \in \mathcal{B}_b(\mathbb{R}^{\mathrm{d}}). \end{aligned} \tag{2.3}$$

Recall that (see [11]) we have the following.

THEOREM 2.1. *For each bounded open set $A \subset \mathbb{R}^{\mathsf{d}}$, the martingale $(M'_t(A))_{t\geq 0}$ converges almost surely toward a positive random variable denoted by $M'(A)$.*

Furthermore, the family of random signed measures $(M'_t(dx))_{t\geq 0}$ almost surely weakly converges toward a random measure $M'(dx)$, which is atom-free and has full support.

2.2. *Results for star scale invariant fields.* The main purpose of this paper is to establish the following result which was conjectured in [11].

THEOREM 2.2 (Freezing theorem). *For any $\gamma > \sqrt{2\mathsf{d}}$, there exists a constant $C(\gamma) > 0$ such that for any smooth nonnegative function f on $[0,1]^{\mathsf{d}}$*

$$\begin{aligned}
&\lim_{t\to\infty} \mathbf{E}\Big(\exp\Big(-t^{3\gamma/(2\sqrt{2\mathsf{d}})} e^{t(\gamma/\sqrt{2}-\sqrt{\mathsf{d}})^2} \int_{[0,1]^{\mathsf{d}}} f(x) M_t^{\gamma}(dx)\Big)\Big) \\
&\quad = \mathbf{E}\Big(\exp\Big(-C(\gamma)\int_{[0,1]^{\mathsf{d}}} f(x)^{\sqrt{2\mathsf{d}}/\gamma} M'(dx)\Big)\Big).
\end{aligned} \tag{2.4}$$

As a consequence, we deduce the following.

COROLLARY 2.3. *For any $\gamma > \sqrt{2\mathsf{d}}$, the family of random measures $(t^{3\gamma/(2\sqrt{2\mathsf{d}})} e^{t(\gamma/\sqrt{2}-\sqrt{\mathsf{d}})^2} M_t^{\gamma}(dx))_{t\geq 0}$ weakly converges in law toward a purely atomic random measure denoted by $S_\gamma(dx)$. The law of S_γ can be described as follows: conditionally on M', S_γ is an independently scattered random measure such that*

$$\mathbf{E}\big(\exp(-\theta S_\gamma(A))\big) = \mathbf{E}\big(\exp(-\theta^{\sqrt{2\mathsf{d}}/\gamma} C(\gamma) M'(A))\big) \tag{2.5}$$

for all $\theta \geq 0$ and all Borelian subsets A of $\mathbb{R}^{\mathsf{d}}$.

In other words, S_γ is an integrated α-stable Poisson random measure of spatial intensity given by the derivative martingale M'. Indeed, the law of S_γ may be described as follows. Conditionally on M', consider a Poisson random measure n_γ on $\mathbb{R}^{\mathsf{d}} \times \mathbb{R}_+$ with intensity

$$M'(dx) \otimes \frac{dz}{z^{1+\sqrt{2\mathsf{d}}/\gamma}}.$$

Then the law of S_γ is the same as the purely atomic measure (Γ stands for the function gamma)

$$S_\gamma(A) = c\int_A \int_0^\infty z n_\gamma(dx, dz) \qquad \text{with } c = \Big(C(\gamma)\frac{\sqrt{2\mathsf{d}}}{\gamma\Gamma(1-\sqrt{2\mathsf{d}}/\gamma)}\Big)^{\gamma/\sqrt{2\mathsf{d}}}.$$

From Theorem 2.1, we observe that $M'(\mathcal{O}) > 0$ almost surely for any nonempty open set. By considering this together with Corollary 2.3, it is plain to deduce the following.

COROLLARY 2.4. *For each bounded open set $\mathcal{O}$, the family of random measures $(\frac{M_t(dx\cap\mathcal{O})}{M_t(\mathcal{O})})_t$ converges in law in the sense of weak convergence of measures toward $\frac{S_\gamma(dx)}{S_\gamma(\mathcal{O})}$.*

We point out that the size reordered atoms of the measure $\frac{S_\gamma(dx)}{S_\gamma(\mathcal{O})}$ form the Poisson–Dirichlet process studied in [2, 3]. The interesting point here is that we keep track of the spatial localization of the atoms whereas all this information is lost in the Poisson–Dirichlet approach. Yet, we stress that the methods used in [2, 3] rely on spin glass technics and remain thus quite interesting since far different from those used here.

REMARK 2.5. We stress that Corollary 2.4 also holds for all the examples described below but we will refrain from stating it anymore.

2.3. *Massive free field.* In this section, we extend our results (Theorem 2.2) to kernels with long range correlations, and in particular, we will be interested in the whole plane Massive Free Field (MFF).

The whole plane MFF is a centered Gaussian distribution with covariance kernel given by the Green function of the operator $2\pi(m^2-\triangle)^{-1}$ on $\mathbb{R}^2$, that is, by

$$\forall x,y\in\mathbb{R}^2,\qquad G_m(x,y)=\int_0^\infty e^{-(m^2/2)u-|x-y|^2/(2u)}\frac{du}{2u}. \tag{2.6}$$

The real $m>0$ is called the mass. This kernel is of σ-positive type in the sense of Kahane [14] since we integrate a continuous function of positive type with respect to a positive measure. It is furthermore a star-scale invariant kernel (see [1]): it can be rewritten as

$$G_m(x,y)=\int_1^{+\infty}\frac{k_m(u(x-y))}{u}\,du, \tag{2.7}$$

for some continuous covariance kernel $k_m(z)=\frac{1}{2}\int_0^\infty e^{-m^2/(2v)|z|^2-v/2}\,dv$. We consider a family of centered Gaussian processes $(X_t(x))_{x\in\mathbb{R}^{\mathrm{d}},t\geq 0}$ with covariance kernel given by

$$\forall t,s\geq 0,\qquad \mathbf{E}\big[X_t(x)X_s(y)\big]=G_{m,t\wedge s}(y-x):=\int_1^{t\wedge s}\frac{k_m(u(x-y))}{u}. \tag{2.8}$$

One can construct the derivative martingale M' associated to $(X_t)_{t\geq 0}$ as prescribed in Section D of [10]. Now we claim that our result holds in the case of the MFF for any cut-off family of the MFF uniformly close to $(G_{m,t})_t$.

DEFINITION 2.6. A cut-off family of the MFF is said uniformly close to $(G_{m,t})_t$ if it is a family of stochastically continuous centered Gaussian processes $(X_n(x))_{n\in\mathbb{N},x\in\mathbb{R}^2}$ with respective covariance kernels $(K_n)_n$ satisfying:

– we can find a subsequence $(t_n)_n$ such that $\lim_{n\to\infty} t_n = +\infty$,
– the family $(K_n - G_{m,t_n})_n$ uniformly converges toward 0 over the compact subsets of $\mathbb{R}^2$.

Then we claim the following.

THEOREM 2.7 (Freezing theorem for MFF). *For any $\gamma > 2$, there exists a constant $C(\gamma) > 0$ such that for every cut-off family $(X_n)_n$ of the MFF uniformly close to $(G_{m,t})_t$, the family of random measures $(t_n^{(3\gamma)/4} e^{t_n(\gamma/\sqrt{2}-\sqrt{2})^2} M_n^\gamma(dx))_{n\geq 0}$, where*

$$M_n^\gamma(dx) = e^{\gamma X_n(x) - (\gamma^2/2)\mathbf{E}[X_n(x)^2]}\,dx,$$

weakly converges in law toward a purely atomic random measure denoted by S_γ. The law of S_γ can be described as follows:

$$\mathbf{E}\big(\exp(-S_\gamma(f))\big) = \mathbf{E}\Big(\exp\Big(-C(\gamma)\int_{\mathbb{R}^2} f(x)^{2/\gamma} M'(dx)\Big)\Big) \tag{2.9}$$

for all nonnegative continuous function f with compact support.

The above theorem is a bit flexible in the sense that there is some robustness with respect to the chosen cutoff approximation: among the class of cut-off families of the MFF uniformly close to $(G_{m,t})_t$, the freezing phenomena related to the MFF do not depend on the structure of the chosen cutoff.

2.4. *Gaussian free field on planar bounded domains*. Consider a bounded open domain D of $\mathbb{R}^2$. Formally, a GFF on D is a Gaussian distribution with covariance kernel given by the Green function of the Laplacian on D with prescribed boundary conditions (see [23] for further details). We describe here the case of Dirichlet boundary conditions. The Green function is then given by the formula

$$G_D(x,y) = \pi \int_0^\infty p_D(t,x,y)\,dt, \tag{2.10}$$

where p_D is the (sub-Markovian) semi-group of a Brownian motion B killed upon touching the boundary of D, namely the Radon–Nykodim derivative

$$p_D(t,x,y) = P^x(B_t \in dy, T_D > t)/dy$$

with $T_D = \inf\{t \geq 0, B_t \notin D\}$. Note the factor π, which makes sure that $G_D(x,y)$ takes on the form

$$G_D(x,y) = \ln_+ \frac{1}{|x-y|} + g(x,y),$$

where $\ln_+ = \max(\ln, 0)$ and for some continuous function g on $D \times D$. The most direct way to construct a cutoff family of the GFF on D is then to consider a white noise W distributed on $D \times \mathbb{R}_+$ and define

$$X(x) = \sqrt{\pi} \int_{D\times\mathbb{R}_+} p_D\Big(\frac{s}{2}, x, y\Big) W(dy, ds).$$

One can check that $\mathbf{E}[X(x)X(x')] = \pi \int_0^\infty p_D(s, x, x')\, ds = G_D(x, x')$. The corresponding cut-off approximations are given by

$$X_t(x) = \sqrt{\pi} \int_{D\times[e^{-2t},\infty[} p_D\Big(\frac{s}{2}, x, y\Big) W(dy, ds), \tag{2.11}$$

which has covariance kernel

$$G_{D,t}(x, y) = \pi \int_{e^{-2t}}^{\infty} p_D(r, x, y)\, dr.$$

We define the approximating measures

$$M_t^2(dx) = e^{2X_t(x)-2\mathbf{E}[X_t(x)^2]}\, dx$$

and

$$M_t'(dx) = \big(2\mathbf{E}\big[X_t(x)^2\big] - X_t(x)\big)e^{2X_t(x)-2\mathbf{E}[X_t(x)^2]}\, dx.$$

Let us stress that Theorem 2.1 holds for this family $(X_t)_t$ (see Section D of [10]).

THEOREM 2.8 (Freezing theorem for GFF on planar domains). *For any $\gamma > 2$ and every bounded planar domain $D \subset \mathbb{R}^2$, there exists a constant $C(\gamma) > 0$ such that for every cut-off family $(X_n)_n$ of the GFF uniformly close to $(G_{D,t})_t$, the family of random measures $(t_n^{3\gamma/4} e^{t_n(\gamma/\sqrt{2}-\sqrt{2})^2} M_n^\gamma(dx))_{t\geq 0}$, where*

$$M_n^\gamma(dx) = e^{\gamma X_n(x)-\gamma^2/2 t_n}\, dx,$$

weakly converges in law toward a purely atomic random measure denoted by S_γ. The law of S_γ can be described as follows:

$$\mathbf{E}\big(\exp(-S_\gamma(f))\big) = \mathbf{E}\Big(\exp\Big(-C(\gamma)\int_{\mathbb{R}^2} f(x)^{2/\gamma} C(x, D)^2 M'(dx)\Big)\Big) \tag{2.12}$$

for all nonnegative continuous function f with compact support, where $C(x, D)$ stands for the conformal radius at $x \in D$.

REMARK 2.9. The derivative martingale construction of Theorem 2.1 applies to other cut-offs of the GFF than (2.11). For instance, one can consider the projection of the GFF on the triangular lattice with mesh going to 0 along powers of 2 (in this case, the law on the lattice points of this projection is nothing but the discrete GFF on the triangular lattice). Then the derivative martingale construction of

Theorem 2.1 holds in this context by the methods of [11] since the approximations correspond to adding independent functions; see [23]. Unfortunately, the methods of this paper do not enable to prove an analog of Theorem 2.8 in the context of the projection on the triangular lattice. There are several difficulties to overcome in this context. First, it would be interesting to prove that Seneta–Heyde renormalization of [10] yields the same limit as the derivative martingale in this setting (this is not obvious from the techniques of [10]). By the universality results in [21] (more precisely Theorem 5.13), this would imply that the approximation (2.11) and the projection on the triangular lattice yield the same critical measure M' (in law). Proving an analog of Theorem 2.8 for the triangular lattice would then imply by the above discussion that the renormalized supercritical measures with the triangular lattice cut-off converge in law to the S_γ defined in (2.12) (up to some multiplicative constant).

2.5. *Further generalization.* Our strategy of proofs apply to a more general class of kernels, at least to some extent, in any dimension. There are two main inputs to take care of.

First, we discuss the case of long range correlated star scale invariant kernels. One has to adopt the same strategy as we do for the MFF. Basically, what one really needs is assumptions (B.1) + (B.2) + (B.3) and the Seneta–Heyde norming, whatever the dimension. However, further conditions on the kernel k are required in order to make sure that the Seneta–Heyde norming holds (see [10], Remark 31). One may, for instance, treat in this way the case of covariance kernel given by the Green function of the operator $(m^2 - \triangle)^{\mathrm{d}/2}$ in $\mathbb{R}^{\mathrm{d}}$ provided that $m > 0$.

One may then wish to treat the case of nontranslation invariant fields, for instance, with correlations given by the Green function of $(-\triangle)^{\mathrm{d}/2}$ in a bounded domain of $\mathbb{R}^{\mathrm{d}}$ with appropriate boundary conditions. Then one has to adopt the strategy we use for the GFF on planar domains: just replace the conformal radius by the function

$$F(x, D) = \lim_{t\to\infty} e^{\mathbf{E}[X_t(x)^2]-\ln t}.$$

3. Proofs for star scale invariant fields. In this section, we carry out the main arguments of the proof of Theorem 2.2 with the help of auxiliary results that are gathered in a toolbox in Appendix. Furthermore, from Assumption A2, the covariance kernel k has compact support. Without loss of generality, we will assume that the support of k is contained in the ball centered at 0 with radius 1.

3.1. *Some further notations.*

Processes and measures. Before proceeding with the proof, we introduce some further notation. We define for all $x \in \mathbb{R}^{\mathrm{d}}$, $t \geq 0$, all Borelian subset A of $\mathbb{R}^{\mathrm{d}}$:

$$Y_t(x) := X_t(x) - \sqrt{2\mathrm{d}}t \quad \text{and} \quad Y_{s,t}(x) := Y_{s+t}(x) - Y_s(x). \tag{3.1}$$

We recall the following scaling property:

$$(Y_{s,t}(x))_{t\in\mathbb{R}^+,x\in\mathbb{R}^{\mathsf{d}}} \overset{\text{(law)}}{=} (Y_t(xe^s))_{t\in\mathbb{R}^+,x\in\mathbb{R}^{\mathsf{d}}}, \tag{3.2}$$

which can be checked with a straightforward computation of covariances. This scaling property is related to the notion of star scale invariance and the reader is referred to [1] for more on this.

The main purpose of Theorem 2.2 will be to establish the convergence of the renormalized measure $t^{3\gamma/(2\sqrt{2\mathsf{d}})}e^{t(\gamma/\sqrt{2}-\sqrt{\mathsf{d}})^2}M_t^\gamma(dx)$ and it will thus be convenient to shortcut this expression as

$$\tilde{M}_t^\gamma(dx) := t^{3\gamma/(2\sqrt{2\mathsf{d}})}e^{t(\gamma/\sqrt{2}-\sqrt{\mathsf{d}})^2}M_t^\gamma(dx). \tag{3.3}$$

We will denote by $|A|$ the Lebesgue measure of a measurable set $A \subset \mathbb{R}^{\mathsf{d}}$.

Regularity, spaces of functions. We denote by $\mathcal{C}(D,\mathbb{R}^p)$ the space of continuous functions from D (a subset of $\mathbb{R}^{\mathsf{d}}$) into $\mathbb{R}^p$. $B(x,r)$ stand for the ball centered at x with radius r.

For any domain $D \subset \mathbb{R}^d$, any continuous function $f \in \mathcal{C}(D,\mathbb{R})$ and $\delta > 0$, we consider the two following modulus of regularity of f:

$$w_f^{(D)}(\delta) := \sup_{x,y\in D,|x-y|\le\delta} |f(x)-f(y)|$$

and

$$w_f^{(D,1/3)}(\delta) := \sup_{x,y\in D,|x-y|\le\delta} \frac{|f(x)-f(y)|}{|x-y|^{1/3}}.$$

When $D = [0,R]^{\mathsf{d}}$ ($R > 0$), we will use $w_f^{(R)}(\delta)$ and $w_f^{(R,1/3)}(\delta)$ instead of respectively $w_f^{([0,R]^{\mathsf{d}})}(\delta)$ and $w_f^{([0,R]^{\mathsf{d}},1/3)}(\delta)$. Similarly, when $D = B(0,b)$ for some $b > 0$, we denote $w_f^{(0,b)}(\delta) := w_f^{(B(0,b))}(\delta)$. For any $a, b, t, R > 0$, we define

$$\begin{aligned}\mathcal{C}_R(t,a,b) = \Big\{ f : [0,R]^{\mathsf{d}} \to \mathbb{R};\ & w_f^{(R,1/3)}(t^{-1}) \le 1, \\ & \min_{y\in[0,R]^{\mathsf{d}}} f(y) > a \text{ and } \max_{y\in[0,R]^{\mathsf{d}}} f(y) < b \Big\}.\end{aligned} \tag{3.4}$$

Constants. We also set for $z, t \ge 0$

$$\kappa_{\mathsf{d}} = \frac{1}{8\sqrt{2\mathsf{d}}} \quad \text{and} \quad a_t := -\frac{3}{2\sqrt{2\mathsf{d}}}\ln t. \tag{3.5}$$

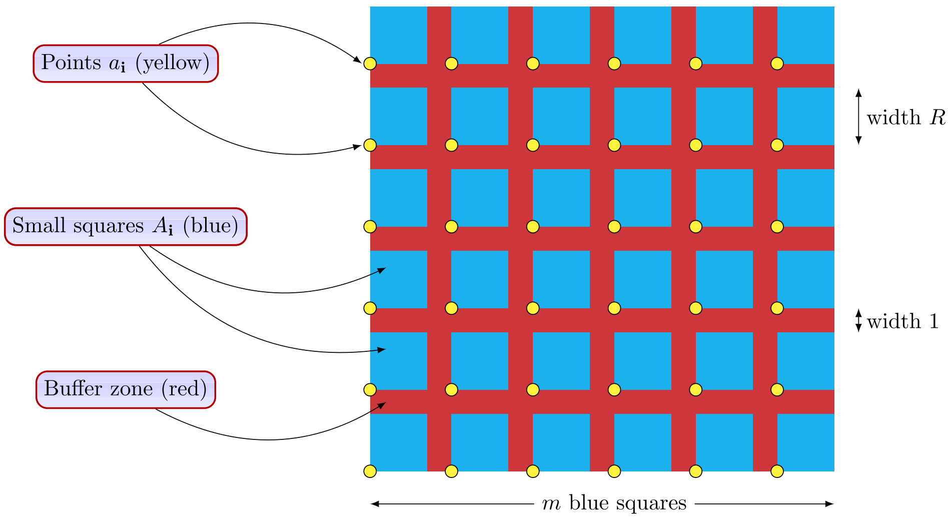

FIG. 1. *Decomposition of the cube* $[0, e^{t'}]^d$.

3.2. *A decomposition.* Before proceeding with the proof of Theorem 2.2, we first explain a decomposition of the cube $[0, e^{t'}]^{\mathsf{d}}$ with $t' > 0$ that will be used throughout the proof of Theorem 2.2. We will divide this cube into several smaller cubes of size $R > 0$, all of these smaller cubes being at distance greater than 1 from each other. To understand more easily our notation, the reader may keep in mind the picture of Figure 1. We assume that R, t' are such that

$$m := \frac{e^{t'} + 1}{(R+1)} \in \mathbb{N}^*.$$

The integer m stands for the number of small squares of size R that one meets along an edge of the cube. The basis of each small square will be indexed with a d-uplet

$$\mathbf{i} = (i_1, \dots, i_{\mathsf{d}}) \in \{1, \dots, m\}^{\mathsf{d}}.$$

The basis of the square $A_{\mathbf{i}}$ is then located at

$$a_{\mathbf{i}} := (R+1)\big((i_1 - 1), \dots, (i_d - 1)\big) \in [0, e^{t'}]^{\mathsf{d}}$$

in such a way that

$$A_{\mathbf{i}} := a_{\mathbf{i}} + [0, R]^{\mathsf{d}}.$$

One may observe on Figure 1 that all the squares $A_{\mathbf{i}}$ are separated from each other by a fishnet shaped buffer zone (red), which is precisely

$$\mathrm{BZ}_{R,t'} := [0, e^{t'}]^{\mathsf{d}} \Big\backslash \bigcup_{\mathbf{i} \in \{1, \dots, m\}^{\mathsf{d}}} A_{\mathbf{i}}.$$

The terminology "buffer zone" is used because this is the minimal area needed to make sure that the values taken by the process Y_t inside each (blue square) $A_{\mathbf{i}}$ are independent of its values on all other $A_{\mathbf{j}}$ for $\mathbf{j} \neq \mathbf{i}$.

3.3. *Main frame of the proof of Theorem* 2.2. This subsection is devoted to the proof of Theorem 2.2 up to admitting a few auxiliary results, which will be proved later.

We have to study the Laplace transform of $\int_{[0,1]^{\mathsf{d}}} f(x)\tilde{M}_t^{\gamma}(dx)$ for all continuous function f on $[0,1]^{\mathsf{d}}$. It is not difficult to see that it is enough to prove the result for all f that are continuous on $[0,1]^{\mathsf{d}}$ and strictly positive. The proof that we develop below works for all such functions but for the sake of clarity and simplicity, we write the proof when f is the characteristic function of the set $[0,1]^{\mathsf{d}}$. The reader may check that the proof is easily adapted to the case $f>0$ and continuous.

We fix $\varepsilon > 0$ and $\theta > 0$. For $R > 0$ and $t' > 0$ such that $\frac{e^{t'}+1}{R+1} \in \mathbb{N}^*$, we define the set (recall the decomposition in Section 3.2)

$$\begin{aligned}
\mathcal{Y}_{R,\theta}(t') := \Big\{ & w^{(1,1/3)}_{Y_{t'}(\cdot)}\Big(\frac{1}{t'}e^{-t'}\Big) \le e^{t'/3}, \\
& \big|\gamma^{-1}\ln\theta\big| M_{t'}^{\sqrt{2\mathsf{d}}}([0,1]^{\mathsf{d}}) + \big|M'_{t'}(e^{-t'}\mathrm{BZ}_{R,t'})\big| \le \varepsilon\theta^{-\sqrt{2\mathsf{d}}/\gamma}, \\
& \forall x \in [0,1]^{\mathsf{d}}, -10\sqrt{2\mathsf{d}}t' \le Y_{t'}(x) \le -\kappa_{\mathsf{d}} \ln t' \Big\}.
\end{aligned} \tag{3.6}$$

Now we consider t, t' such that $t \ge e^{t'}$. We have

$$\begin{aligned}
& \mathbf{E}\big(e^{-\theta\tilde{M}_t^{\gamma}([0,1]^{\mathsf{d}})}; \mathcal{Y}_{R,\theta}(t')\big) \\
& \quad \le \mathbf{E}\big(e^{-\theta\tilde{M}_t^{\gamma}([0,1]^{\mathsf{d}})}\big) \\
& \quad \le \mathbf{E}\big(e^{-\theta\tilde{M}_t^{\gamma}([0,1]^{\mathsf{d}})}; \mathcal{Y}_{R,\theta}(t')\big) + \mathbf{P}\big(\mathcal{Y}_{R,\theta}(t')^c\big).
\end{aligned} \tag{3.7}$$

We estimate now the left-hand side of this relation. Because

$$\tilde{M}_t^{\gamma}([0,1]^{\mathsf{d}}) = \tilde{M}_t^{\gamma}(e^{-t'}\mathrm{BZ}_{R,t'}) + \tilde{M}_t^{\gamma}(e^{-t'} \cup A_{\mathbf{i}}),$$

we can use the relation $uv \ge u+v-1$ for $u, v \in [0,1]$ to get

$$e^{-\theta\tilde{M}_t^{\gamma}([0,1]^{\mathsf{d}})} \ge e^{-\theta\tilde{M}_t^{\gamma}(e^{-t'}\mathrm{BZ}_{R,t'})} - 1 + e^{-\theta\tilde{M}_t^{\gamma}(e^{-t'}\cup A_{\mathbf{i}})}.$$

We deduce from (3.7)

$$\begin{aligned}
& \mathbf{E}\big(e^{-\theta\tilde{M}_t^{\gamma}(e^{-t'}\mathrm{BZ}_{R,t'})} - 1; \mathcal{Y}_{R,\theta}(t')\big) + \mathbf{E}\big(e^{-\theta\tilde{M}_t^{\gamma}(e^{-t'}\cup A_{\mathbf{i}})}; \mathcal{Y}_{R,\theta}(t')\big) \\
& \quad \le \mathbf{E}\big(e^{-\theta\tilde{M}_t^{\gamma}([0,1]^{\mathsf{d}})}\big) \\
& \quad \le \mathbf{E}\big(e^{-\theta\tilde{M}_t^{\gamma}(e^{-t'}\cup A_{\mathbf{i}})}; \mathcal{Y}_{R,\theta}(t')\big) + \mathbf{P}\big(\mathcal{Y}_{R,\theta}(t')^c\big).
\end{aligned} \tag{3.8}$$

Now we claim:

LEMMA 3.1. *The following convergences hold for each $t' \geq 0$:*

$$\limsup_{t\to\infty} \mathbf{E}\big[1 - e^{-\theta \tilde{M}_t^\gamma(e^{-t'}\mathrm{BZ}_{R,t'})}; \mathcal{Y}_{R,\theta}(t')\big] \leq \varepsilon, \tag{3.9}$$

$$\begin{aligned}&\limsup_{t\to\infty} \mathbf{E}\big[e^{-\theta \tilde{M}_t^\gamma(e^{-t'}\cup A_{\mathrm{i}})}; \mathcal{Y}_{R,\theta}(t')\big] \\ &\qquad \leq \mathbf{E}\big[\exp\big(-(C(\gamma)-\varepsilon)\theta^{\sqrt{2\mathrm{d}}/\gamma} M'([0,1]^{\mathrm{d}}) + 2\varepsilon(C(\gamma)-\varepsilon)\big)\big]\end{aligned} \tag{3.10}$$

and a lower bound similar to (3.10) *with a* $\liminf_{t\to\infty}$ *in the left-hand side.*

By taking the $\limsup_{t\to\infty}$ in (3.8) and by using Lemma 3.1, we get

$$\begin{aligned}&\limsup_{t\to\infty} \mathbf{E}\big[e^{-\theta \tilde{M}_t^\gamma([0,1]^{\mathrm{d}})}\big] \\ &\qquad \leq \mathbf{E}\big[\exp\big(-(C(\gamma)-\varepsilon)\theta^{\sqrt{2\mathrm{d}}/\gamma} M'([0,1]^{\mathrm{d}}) + 2\varepsilon(C(\gamma)-\varepsilon)\big)\big] + \mathbf{P}\big(\mathcal{Y}_{R,\theta}(t')^c\big).\end{aligned}$$

From Lemma A.2, we have $\limsup_{t'\to\infty} \mathbf{P}(\mathcal{Y}_{R,\theta}(t')^c) \leq \varepsilon$. We deduce

$$\begin{aligned}&\limsup_{t\to\infty} \mathbf{E}\big[e^{-\theta \tilde{M}_t^\gamma([0,1]^{\mathrm{d}})}\big] \\ &\qquad \leq \mathbf{E}\big[\exp\big(-(C(\gamma)-\varepsilon)\theta^{\sqrt{2\mathrm{d}}/\gamma} M'([0,1]^{\mathrm{d}}) + 2\varepsilon(C(\gamma)-\varepsilon)\big)\big] + \varepsilon.\end{aligned}$$

We can proceed in the same way for the lower bound. Since ε can be chosen arbitrarily close to 0, the proof of Theorem 2.2 follows, provided that we prove the above lemma.

To prove Lemma 3.1, we need the following proposition, which can actually be seen as the key tool of this subsection. Its proof requires some additional material and is carried out in Section 5. In what follows, for any function $\chi(\cdot) \in \mathcal{C}_R(t', \kappa_{\mathrm{d}} \ln t', \ln t)$, we set

$$\begin{aligned}\mathtt{I}(\chi,\theta,\gamma) &= \theta^{\sqrt{2\mathrm{d}}/\gamma} \int_{[0,R]^{\mathrm{d}}} \left(\chi(x) - \frac{\ln\theta}{\gamma}\right) e^{-\sqrt{2\mathrm{d}}\chi(x)}\, dx \\ &\qquad \big[\text{when } \theta = 1 \text{ we will denote } \mathtt{I}(\chi) = \mathtt{I}(\chi,1,\gamma)\big].\end{aligned} \tag{3.11}$$

PROPOSITION 3.2. *Let $\gamma > \sqrt{2\mathrm{d}}$. There exists a constant $C(\gamma) > 0$ such that for all $R \geq 1$, $\theta > 0$ and $\varepsilon > 0$, we can find $t_0 > 0$ such that for all $t' > t_0$ satisfying $\frac{e^{t'}+1}{R+1} \in \mathbb{N}^*$, there exists $T > 0$, such that for any $t > T$*

$$\begin{aligned}&\left|\mathbf{E}\left[\exp\left(-\theta \int_{[0,R]^{\mathrm{d}}} e^{\gamma[Y_t(x) - a_t - \chi(x)] + \mathrm{d}t}\, dx\right) - 1\right] + C(\gamma)\mathtt{I}(\chi,\theta,\gamma)\right| \\ &\qquad \leq \varepsilon \mathtt{I}(\chi,\theta,\gamma).\end{aligned} \tag{3.12}$$

PROOF OF LEMMA 3.1. We first prove the first relation (3.9). By the Markov property at time t' and the scaling property (3.2) applied on the set $\mathrm{BZ}_{R,t'}$ we get that

$$\begin{aligned}
&\mathbf{E}[1-e^{-\theta\tilde{M}_t^{\gamma}(e^{-t'}\mathrm{BZ}_{R,t'})};\mathcal{Y}_{R,\theta}(t')]\\
&\quad=\mathbf{E}\biggl[\mathbf{E}\biggl[1-\exp\biggl(-\theta\int_{e^{-t'}\mathrm{BZ}_{R,t'}}e^{\gamma[Y_{t',t-t'}(x)-a_t-\chi(x)]+\mathsf{d}t}\,dx\biggr)\biggr]_{|\chi(\cdot)=-Y_{t'}(\cdot)};\\
&\qquad\mathcal{Y}_{R,\theta}(t')\biggr] \qquad (3.13)\\
&\quad=\mathbf{E}\biggl[\mathbf{E}\biggl[1-\exp\biggl(-\theta\int_{\mathrm{BZ}_{R,t'}}e^{\gamma[Y_{t-t'}(x)-a_t-\chi(x)]+\mathsf{d}(t-t')}\,dx\biggr)\biggr]_{|\chi(\cdot)=-Y_{t'}(\cdot/e^{t'})};\\
&\qquad\mathcal{Y}_{R,\theta}(t')\biggr].
\end{aligned}$$

We can find a finite collection of points in $[0,e^{t'}]^{\mathsf{d}}$, call it $(y_j)_{j\in J}$, such that:

– for any distinct $j_1,\dots,j_{\mathsf{d}+2}\in J$, $\bigcap_{k=1}^{\mathsf{d}+2}(y_{j_k}+[0,1]^{\mathsf{d}})=\varnothing$,
– the set $\bigcup_{j\in J}(y_j+[0,1]^{\mathsf{d}})$ is contained in the closure $\overline{\mathrm{BZ}_{R,t'}}$ of $\mathrm{BZ}_{R,t'}$.

We do not detail the construction of these points but this is rather elementary: basically, you have to cover the red area in Figure 1 with closed squares of side length 1 (which corresponds to the width of the red strips). Of course, the squares that you choose may overlap but if this covering is made efficiently enough, they will not overlap too much in such a way that any intersection of $\mathsf{d}+2$ such squares will be empty.

By using in turn the elementary inequality $1-\prod_{j\in J}u_j\le\sum_{j\in J}1-u_j$ for $(u_j)_j\in[0,1]^J$ and then invariance by translation, we get

$$\begin{aligned}
&\mathbf{E}\biggl[1-\exp\biggl(-\theta\int_{\mathrm{BZ}_{R,t'}}e^{\gamma[Y_{t-t'}(x)-a_t-\chi(x)]+\mathsf{d}(t-t')}\,dx\biggr)\biggr]_{|\chi(\cdot)=-Y_{t'}(e^{-t'}\cdot)}\\
&\quad\le\mathbf{E}\biggl[1\\
&\qquad-\prod_{j\in J}\exp\biggl(-\theta\int_{y_j+[0,1]^{\mathsf{d}}}e^{\gamma[Y_{t-t'}(x)-a_t-\chi(x)]+\mathsf{d}(t-t')}\,dx\biggr)\biggr]_{|\chi(\cdot)=-Y_{t'}(e^{-t'}\cdot)} \qquad (3.14)\\
&\quad\le\sum_{j\in J}\mathbf{E}\biggl[1\\
&\qquad-\exp\biggl(-\theta\int_{[0,1]^{\mathsf{d}}}e^{\gamma[Y_{t-t'}(x)-a_t-\chi(x)]+\mathsf{d}(t-t')}\,dx\biggr)\biggr]_{|\chi(\cdot)=-Y_{t'}(y_j+e^{-t'}\cdot)}.
\end{aligned}$$

Moreover, on $\mathcal{Y}_{R,\theta}(t')$, the function $x \in [0,1]^{\mathsf{d}} \mapsto -Y_{t'}(y_j + e^{-t'}x)$ belongs to $\mathcal{C}_1(t', \kappa_{\mathsf{d}} \ln t', \ln t)$ as soon as $\ln t > 10\sqrt{2\mathsf{d}}t'$. So, by Proposition 3.2 with $\varepsilon = 1$, we can find t_0 such that for any $t' > t_0$ satisfying $\frac{e^{t'}+1}{R+1} \in \mathbb{N}^*$ there exists $T > 0$ such that for any $t > T$ and on $\mathcal{Y}_{R,\theta}(t')$

$$\begin{aligned}
&\mathbf{E}\Big[1 - \exp\Big(-\theta \int_{[0,1]^{\mathsf{d}}} e^{\gamma[Y_{t-t'}(x) - a_t - \chi(x)] + \mathsf{d}(t-t')}\,dx\Big)\Big]_{|\chi(\cdot) = -Y_{t'}(y_j + e^{-t'}\cdot)} \\
&\quad \le \big(C(\gamma)+1\big) \int_{y_j+[0,1]^{\mathsf{d}}} \Big(-\frac{1}{\gamma}\ln\theta - Y_{t'}\big(xe^{-t'}\big)\Big) \\
&\qquad \times e^{\sqrt{2\mathsf{d}}(Y_{t'}(xe^{-t'}) + (1/\gamma)\ln\theta)}\,dx
\end{aligned}$$

for any $j \in J$ (notice that $a_t - a_{t-t'} \to 0$ when t goes to infinity so that we can replace a_t by $a_{t-t'}$ in the above expression in order to be in position to apply Proposition 3.2). Plugging this estimate into (3.14) yields

$$\begin{aligned}
&\mathbf{E}\Big[1 - \exp\Big(-\theta \int_{\mathrm{BZ}_{R,t'}} e^{\gamma[Y_{t-t'}(x) - a_t - \chi(x)] + \mathsf{d}(t-t')}\,dx\Big)\Big]_{|\chi(\cdot) = -Y_{t'}(e^{-t'}\cdot)} \\
&\quad \le \sum_{j \in J} \big(C(\gamma)+1\big) \\
&\qquad \times \int_{y_j+[0,1]^{\mathsf{d}}} \Big(-\frac{1}{\gamma}\ln\theta - Y_{t'}\big(xe^{-t'}\big)\Big) e^{\sqrt{2\mathsf{d}}(Y_{t'}(xe^{-t'}) + (1/\gamma)\ln\theta)}\,dx.
\end{aligned} \tag{3.15}$$

Now we may assume that $\kappa_{\mathsf{d}} \ln t' > \frac{1}{\gamma}\ln\theta$ in such a way that, on $\mathcal{Y}_{R,\theta}(t')$, we have $(-\frac{1}{\gamma}\ln\theta - Y_{t'}(xe^{-t'})) \ge 0$ for $x \in [0, e^{t'}]^{\mathsf{d}}$. Furthermore, the relation $\bigcap_{k=1}^{\mathsf{d}+2}(y_{j_k} + [0,1]^{\mathsf{d}}) = \varnothing$ (valid for all families of distinct indices) entails that $\sum_{j\in J} \mathbb{1}_{\{y_{j_k}+[0,1]^{\mathsf{d}}\}} \le (d+2)\mathbb{1}_{\overline{\mathrm{BZ}_{R,t'}}}$. Hence, on $\mathcal{Y}_{R,\theta}(t')$

$$\begin{aligned}
&\mathbf{E}\Big[1 - \exp\Big(-\theta \int_{\mathrm{BZ}_{R,t'}} e^{\gamma[Y_{t-t'}(x) - a_t - \chi(x)] + \mathsf{d}(t-t')}\,dx\Big)\Big]_{|\chi(\cdot) = -Y_{t'}(e^{-t'}\cdot)} \\
&\quad \le (\mathsf{d}+2)\big(C(\gamma)+1\big) \\
&\qquad \times \int_{\overline{\mathrm{BZ}_{R,t'}}} \Big(-\frac{1}{\gamma}\ln\theta - Y_{t'}\big(xe^{-t'}\big)\Big) e^{\sqrt{2\mathsf{d}}(Y_{t'}(xe^{-t'}) + (1/\gamma)\ln\theta)}\,dx \\
&\quad \le (\mathsf{d}+2)\big(C(\gamma)+1\big) \\
&\qquad \times \int_{e^{-t'}\mathrm{BZ}_{R,t'}} \Big(-\frac{1}{\gamma}\ln\theta - Y_{t'}(x)\Big) e^{\sqrt{2\mathsf{d}}(Y_{t'}(x) + (1/\gamma)\ln\theta) + \mathsf{d}t'}\,dx.
\end{aligned} \tag{3.16}$$

The last inequality results from the change of variables $xe^{-t'} \to x$. We recognize the expressions of the martingales $M_{t'}^{\sqrt{2\mathsf{d}}}$ and $M'_{t'}$ as in (2.3). By gathering (3.13) and the above relation, we deduce

$$\begin{aligned}
&\mathbf{E}\big[1-e^{-\theta \tilde{M}_t^{\gamma}(e^{-t'}\mathrm{BZ}_{R,t'})};\mathcal{Y}_{R,\theta}(t')\big]\\
&\quad\le (\mathsf{d}+2)\big(C(\gamma)+1\big)\mathbf{E}\big[\theta^{\sqrt{2\mathsf{d}}/\gamma}\big(-\gamma^{-1}\ln\theta M_{t'}^{\sqrt{2\mathsf{d}}}\big(e^{-t'}\mathrm{BZ}_{R,t'}\big)\\
&\qquad + M'_{t'}\big(e^{-t'}\mathrm{BZ}_{R,t'}\big)\big);\mathcal{Y}_{R,\theta}(t')\big].
\end{aligned}\tag{3.17}$$

By using the definition (3.6) of $\mathcal{Y}_{R,\theta}(t')$, we see that this latter quantity is less than $\varepsilon(\mathsf{d}+2)(C(\gamma)+1)$. By choosing ε as small as we please, we complete the proof of the first relation (3.9).

Now we prove (3.10). As previously mentioned, we first apply the Markov property at time t' and the scaling property (3.2).

$$\begin{aligned}
&\mathbf{E}\big[e^{-\theta\tilde{M}_t^{\gamma}(e^{-t'}\cup A_{\mathbf{i}})};\mathcal{Y}_{R,\theta}(t')\big]\\
&\quad=\mathbf{E}\Big[\mathbf{E}\Big[\exp\Big(-\theta\int_{\cup A_{\mathbf{i}}} e^{\gamma[Y_{t-t'}(x)-a_t-\chi(x)]+\mathsf{d}(t-t')}\,dx\Big)\Big]_{|\chi(\cdot)=-Y_{t'}(\cdot e^{-t'})};\\
&\qquad\mathcal{Y}_{R,\theta}(t')\Big].
\end{aligned}\tag{3.18}$$

The important point here is to see that for any $t\ge 0$, the process $(Y_t(x))_{x\in\mathbb{R}^d}$ is decorrelated at distance 1 [recall that k has compact support in the ball $B(0,1)$]. Therefore, the random variables $(\int_{A_{\mathbf{i}}} e^{\gamma[Y_{t-t'}(x)-a_t-\chi(x)]+\mathsf{d}(t-t')}\,dx)_{\mathbf{i}}$ appearing in the latter expectation are independent since $\mathrm{dist}(A_{\mathbf{i}},A_{\mathrm{j}})\ge 1$ for any $\mathbf{i}\ne\mathbf{j}$. We deduce that

$$\begin{aligned}
&\mathbf{E}\big[e^{-\theta\tilde{M}_t^{\gamma}(e^{-t'}\cup A_{\mathbf{i}})};\mathcal{Y}_{R,\theta}(t')\big]\\
&\quad=\mathbf{E}\Big[\prod_{\mathbf{i}\in\{1,\dots,m\}^{\mathsf{d}}}\mathbf{E}\Big[\exp\Big(-\theta\\
&\qquad\times\int_{A_{\mathbf{i}}} e^{\gamma[Y_{t-t'}(x)-a_t-\chi(x)]+\mathsf{d}(t-t')}\,dx\Big)\Big]_{|\chi(\cdot)=-Y_{t'}(\cdot e^{-t'})};\mathcal{Y}_{R,\theta}(t')\Big].
\end{aligned}\tag{3.19}$$

As previously mentioned, we can choose t sufficiently large so that, on $\mathcal{Y}_{R,\theta}(t')$ and for any $j\in J$, the function $x\in[0,R]^{\mathsf{d}}\mapsto -Y_{t'}(e^{-t'}(x+a_{\mathbf{i}}))$ belongs to $\mathcal{C}_R(t',\kappa_{\mathsf{d}}\ln t',\ln t)$. We can then apply Proposition 3.2 once again and get some $t_0>0$ such that for all $t'>t_0$ (with $\frac{e^{t'}+1}{R+1}\in\mathbb{N}^*$) there exists $T>0$ such that for all $t\ge T$ and all $\mathbf{i}$,

$$\begin{aligned}
&\Big|\mathbf{E}\Big[\exp\Big(-\theta\int_{A_{\mathbf{i}}} e^{\gamma[Y_{t-t'}(x)-a_t-\chi(x)]+\mathsf{d}(t-t')}\,dx\Big)\Big]-1+C(\gamma)\mathtt{I}(\chi,\theta,\gamma)\Big|\\
&\quad\le\varepsilon\mathtt{I}(\chi,\theta,\gamma),
\end{aligned}$$

with $\chi(x) = -Y_{t'}(e^{-t'}x)$. By plugging this estimate into (3.19) and by making a change of variables $xe^{-t'} \to x$, we obtain (once again by identifying $M_{t'}^{\sqrt{2\mathsf{d}}}$ and $M'_{t'}$)

$$\begin{aligned}
&\mathbf{E}\big[e^{-\theta \tilde{M}_t^\gamma(\cup A_{\mathbf{i}})}; \mathcal{Y}_{R,\theta}(t')\big] \\
&\quad \le \mathbf{E}\bigg[\prod_{\mathbf{i}\in\{1,\dots,m\}^{\mathsf{d}}} \big(1 - (C(\gamma) - \varepsilon)\theta^{\sqrt{2\mathsf{d}}/\gamma} \\
&\qquad \times \big[-\gamma^{-1}\ln\theta M_{t'}^{\sqrt{2\mathsf{d}}}(e^{-t'}A_{\mathbf{i}}) + M'_{t'}(e^{-t'}A_{\mathbf{i}})\big]\big); \mathcal{Y}_{R,\theta}(t')\bigg].
\end{aligned}$$

On $\mathcal{Y}_{R,\theta}(t')$, $\forall \mathbf{i} \in \{1,\dots,m\}^{\mathsf{d}}$ we have $|\gamma^{-1}\ln\theta M_{t'}^{\sqrt{2\mathsf{d}}}(e^{-t'}A_{\mathbf{i}})| + |M'_{t'}(e^{-t'}A_{\mathbf{i}})| \le c\frac{\ln t'}{(t')^a} \le \varepsilon$ for any $a < \frac{\kappa_d\sqrt{2\mathsf{d}}}{2}$ and t' large enough. Indeed, on $\mathcal{Y}_{R,\theta}(t')$, $\forall x \in [0,1]^{\mathsf{d}}$, $\kappa_{\mathsf{d}} \ln t' \le -Y_{t'}(x)$, thus

$$\begin{aligned}
M_{t'}^{\sqrt{2\mathsf{d}}}(e^{-t'}A_{\mathbf{i}}) \le M'_{t'}(e^{-t'}A_{\mathbf{i}}) &= \int_{e^{-t'}A_{\mathbf{i}}} -Y_{t'}(x)e^{\sqrt{2\mathsf{d}}Y_{t'}(x)+\mathsf{d}t'}\,dx \\
&\le R^{\mathsf{d}} \sup_{u\ge\kappa_{\mathsf{d}}\ln t'} ue^{-\sqrt{2\mathsf{d}}u}.
\end{aligned}$$

Then, by using the inequality $\prod_{i\in I}(1-u_i) \le e^{-\sum_{i\in I}u_i}$ for $u_i \in [0,1]$, we obtain

$$\begin{aligned}
&\mathbf{E}\big[e^{-\theta \tilde{M}_t^\gamma(\cup A_{\mathbf{i}})}; \mathcal{Y}_{R,\theta}(t')\big] \\
&\quad \le \mathbf{E}\bigg[\exp\bigg(-(C(\gamma) - \varepsilon)\theta^{\sqrt{2\mathsf{d}}/\gamma} \\
&\qquad \times \bigg[M'_{t'}\Big(e^{-t'}\bigcup_{\mathbf{i}} A_{\mathbf{i}}\Big) - \gamma^{-1}\ln\theta M_{t'}^{\sqrt{2\mathsf{d}}}\Big(e^{-t'}\bigcup_{\mathbf{i}} A_{\mathbf{i}}\Big)\bigg]\bigg); \mathcal{Y}_{R,\theta}(t')\bigg].
\end{aligned}$$

Recall that, on $\mathcal{Y}_{R,\theta}(t')$,

$$\begin{aligned}
&\Big|M'_{t'}\Big(e^{-t'}\bigcup_{\mathbf{i}} A_{\mathbf{i}}\Big) - M'_{t'}([0,1]^{\mathsf{d}})\Big| + \Big|\gamma^{-1}\ln\theta M_{t'}^{\sqrt{2\mathsf{d}}}\Big(e^{-t'}\bigcup_{\mathbf{i}} A_{\mathbf{i}}\Big)\Big| \\
&\quad \le |\gamma^{-1}\ln\theta M_{t'}^{\sqrt{2\mathsf{d}}}([0,1]^{\mathsf{d}})| + |M'_{t'}(e^{-t'}\mathrm{BZ}_{R,t'})| \le \varepsilon\theta^{-\sqrt{2\mathsf{d}}/\gamma}.
\end{aligned}$$

So $\limsup_{t\to\infty}\mathbf{E}(e^{-\theta\tilde{M}_t^\gamma(\cup A_{\mathbf{i}})}; \mathcal{Y}_{R,\theta}(t')) \le \mathbf{E}(\exp(-(C(\gamma) - \varepsilon)\theta^{\sqrt{2\mathsf{d}}/\gamma} \times M'([0,1]^{\mathsf{d}}) + 2\varepsilon(C(\gamma) - \varepsilon)))$. The lower bound of (3.10) can be derived in the same way. $\square$

3.4. *Proof of Corollary* 2.3. Here, we assume that Theorem 2.2 holds and we show that this implies convergence in law in the sense of weak convergence of measures. For $a > 0$, let us denote by C_a the cube $[-a, a]^{\mathsf{d}}$. Since for all bounded continuous function f compactly supported in C_R, we have

$$0 \le \int_{C_a} f(x)\tilde{M}_t^{\gamma}(dx) \le \|f\|_\infty \tilde{M}_t^{\gamma}(C_a)$$

and since the right-hand side is tight, this ensures that the family of random measures $(\tilde{M}_t^{\gamma}(dx))_t$ is tight for the weak convergence of measures on C_a. Since we can find a sequence $(f_n)_n$ of smooth strictly positive functions on C_a that is dense in the set of nonnegative continuous compactly supported functions in C_a for the uniform topology, uniqueness in law then results from Theorem 2.2. As it is rather a standard argument of functional analysis, we let the reader check the details, if need be.

4. Estimation on the tail of distribution of $\tilde{M}_t^{\gamma}([0, R]^{\mathsf{d}})$. In this section, we will identify the path configurations $t \mapsto Y_t(x)$ that really contribute to the behaviour of the measure $\tilde{M}_t^{\gamma}$. We will show that, for these paths, $Y_t(x)$ typically goes faster than $a_t = -\frac{3}{2\sqrt{2\mathsf{d}}}\ln t$.

To quantify the above rough claim, we will establish the following.

PROPOSITION 4.1. *Let $R, \varepsilon > 0$. There exists a constant $A > 0$ such that for any t', T large enough we have*

$$\begin{aligned}\tag{4.1} &\mathbf{E}\Big[1 - \exp\Big(-\int_{[0,R]^{\mathsf{d}}} e^{\gamma[Y_t(x)-a_t-\chi(x)]+\mathsf{d}t}\mathbb{1}_{\{Y_t(x)\le a_t+\chi(x)-A\}}\,dx\Big)\Big] \\ &\quad\le \varepsilon \int_{[0,R]^{\mathsf{d}}} \chi(x)e^{-\sqrt{2\mathsf{d}}\chi(x)}\,dx,\end{aligned}$$

for any $t \ge T$ and $\chi(\cdot) \in \mathcal{C}_R(t', \kappa_{\mathsf{d}} \ln t', \ln t)$.

Then we focus on the shape of the tail distribution of $\tilde{M}_t^{\gamma}([0, R]^{\mathsf{d}})$. For instance, it is well known in Tauberian theory that an estimate of the type

$$\tag{4.2} C^{-1}xe^{-\sqrt{2\mathsf{d}}x} \le 1 - \mathbf{E}\big[e^{-e^{-\gamma x}\tilde{M}_t^{\gamma}([0,R]^{\mathsf{d}})}\big] \le Cxe^{-\sqrt{2\mathsf{d}}x}$$

valid for $x > 0$ gives you a tail estimate for $\tilde{M}_t^{\gamma}([0, R]^{\mathsf{d}})$ of the type

$$\mathbf{P}\big(\tilde{M}_t^{\gamma}([0, R]^{\mathsf{d}}) > e^{\gamma x}\big) \asymp xe^{-\sqrt{2\mathsf{d}}x}$$

as $x \to \infty$. Basically, the following proposition is a functional version of (4.2), meaning that we will replace the variable x by some function χ. Hence, we will establish the following.

PROPOSITION 4.2. *There exist c_1, c_2, such that for any $t' > 0$ there exists $T > 0$ such that for any $R \in [1, \ln t']$:*

- *for any* $t \geq T$ *and any* $\chi(\cdot) \in \mathcal{C}_R(t', \kappa_{\mathsf{d}} \ln t', \ln t)$,

$$\mathbf{E}\Big[1 - \exp\Big(-\int_{[0,R]^{\mathsf{d}}} e^{\gamma[Y_t(x) - a_t - \chi(x)] + \mathsf{d}t}\,dx\Big)\Big] \leq c_2 \int_{[0,R]^{\mathsf{d}}} \chi(x) e^{-\sqrt{2\mathsf{d}}\chi(x)}\,dx.$$

- *for any* $\chi(\cdot) \in \mathcal{C}_R(t', \kappa_{\mathsf{d}} \ln t', +\infty)$

$$c_1 \int_{[0,R]^{\mathsf{d}}} \chi(x) e^{-\sqrt{2\mathsf{d}}\chi(x)}\,dx \leq \liminf_{t\to\infty} \mathbf{E}\Big[1 - \exp\Big(-\int_{[0,R]^{\mathsf{d}}} e^{\gamma[Y_t(x) - a_t - \chi(x)] + \mathsf{d}t}\,dx\Big)\Big].$$

The two following subsections are devoted to the proofs of Propositions 4.1 and 4.2.

4.1. *Proof of Proposition* 4.1. Fix $\varepsilon > 0$. We consider $t' > 0$ and $R \geq 1$ such that $\frac{e^{t'}+1}{R+1} \in \mathbb{N}^*$. We have for $t > e^{t'}$

$$\begin{aligned}
&\mathbf{E}\Big[1 - \exp\Big(-\int_{[0,R]^{\mathsf{d}}} e^{\gamma[Y_t(x) - a_t - \chi(x)] + \mathsf{d}t} \mathbb{1}_{\{Y_t(x) \leq a_t + \chi(x) - A\}}\,dx\Big)\Big] \\
&\quad \leq \mathbf{E}\Big[1 - \exp\Big(-\int_{[0,R]^{\mathsf{d}}} e^{\gamma[Y_t(x) - a_t - \chi(x)] + \mathsf{d}t} \\
&\qquad \times \mathbb{1}_{\{\sup_{s\in[\ln t', t]} Y_s(x) \leq \chi(x), Y_t(x) \leq a_t + \chi(x) - A\}}\,dx\Big)\Big] \\
&\qquad + \mathbf{P}\Big(\sup_{x\in[0,R]^{\mathsf{d}}} \sup_{s\in[\ln t', \infty[} Y_s(x) \geq \chi(x)\Big).
\end{aligned} \tag{4.3}$$

If $\chi(\cdot) \in \mathcal{C}_R(t', \kappa_{\mathsf{d}} \ln t', \ln t)$ (with t' large enough so as to make $\kappa_{\mathsf{d}} \ln t' > 10$), we can estimate the probability in the right-hand side with the help of Lemma A.3. If t' is again large enough, we have

$$(\ln t')^{3/8} + \chi(x)^{3/4} \leq \frac{\varepsilon}{2} \kappa_{\mathsf{d}} \ln t' + \chi(x)^{3/4} \leq \varepsilon \chi(x)$$

in such a way that

$$\mathbf{P}\Big(\sup_{x\in[0,R]^{\mathsf{d}}} \sup_{s\in[\ln t', \infty[} Y_s(x) \geq \chi(x)\Big) \leq \varepsilon \int_{[0,R]^{\mathsf{d}}} \chi(x) e^{-\sqrt{2\mathsf{d}}\chi(x)}\,dx. \tag{4.4}$$

So we need to bound the first term in the right-hand side of (4.3). To this purpose, we will use Lemma A.4 as follows. First, observe that we can partition the whole space of events Ω as

$$\begin{aligned}
\Omega \subset & \Big\{ \sup_{s\in[t/2,t]} Y_s(x) \le a_t + \chi(x) + L \Big\} \\
& \times \bigcup_{j\ge L+1} \Big\{ \sup_{s\in[t/2,t]} Y_s(x) - a_t - \chi(x) \in [j-1, j] \Big\} \\
\subset & \Big\{ \sup_{s\in[t/2,t]} Y_s(x) \le a_t + \chi(x) + L \Big\} \\
& \times \bigcup_{j\ge L+1} \Big\{ \sup_{s\in[t/2,t-\nu_j]} Y_s(x) - a_t - \chi(x) \in [j-1, j], \\
& \quad \sup_{s\in[t-\nu_j,t]} Y_s(x) \le a_t + \chi(x) + j \Big\} \\
& \times \bigcup_{j\ge L+1} \Big\{ \sup_{s\in[t/2,t-\nu_j]} Y_s(x) \le a_t + \chi(x) + j, \\
& \quad \sup_{s\in[t-\nu_j,t]} Y_s(x) - a_t - \chi(x) \in [j-1, j] \Big\}
\end{aligned}$$

for all family $(\nu_j)_j$ such that $0 \le \nu_j \le t/2$ for all j. We deduce the relation

$$\mathbb{1}_{\{\sup_{s\in[\ln t',t]} Y_s(x) \le \chi(x), Y_t(x) \le a_t + \chi(x) - A\}} \le \mathbb{1}_{E^1_{t',t}(x)} + \mathbb{1}_{E^2_{t',t}(x)} + \mathbb{1}_{E^3_{t',t}(x)},$$

where the set $E^1_{t',t}(x)$, $E^2_{t',t}(x)$, $E^3_{t',t}(x)$ are defined as follows: we consider the constants c_4, c_5 of Lemma A.4 and for any $j \ge 1$, we define $\nu_j := e^{(c_5/2)j}$. Then we set

$$\begin{aligned}
E^1_{t',t}(x) := & \Big\{ \sup_{s\in[\ln t',t]} Y_s(x) \le \chi(x), \\
& \sup_{s\in[t/2,t]} Y_s(x) \le a_t + \chi(x) + L, Y_t(x) \le a_t + \chi(x) - A \Big\}, \\
E^2_{t',t}(x) := & \bigcup_{j\ge L+1} \Big\{ \sup_{s\in[\ln t',t]} Y_s(x) \le \chi(x), \\
& \sup_{s\in[t/2,t-\nu_j]} Y_s(x) - a_t - \chi(x) \in [j-1, j], \\
& \sup_{s\in[t-\nu_j,t]} Y_s(x) \le a_t + \chi(x) + j, Y_t(x) \le a_t + \chi(x) - A \Big\}, \\
E^3_{t',t}(x) := & \bigcup_{j\ge L+1} \Big\{ \sup_{s\in[\ln t',t]} Y_s(x) \le \chi(x), \sup_{s\in[t/2,t-\nu_j]} Y_s(x) \le a_t + \chi(x) + j, \\
& \sup_{s\in[t-\nu_j,t]} Y_s(x) - a_t - \chi(x) \in [j-1, j] \Big\}.
\end{aligned}$$

According to Lemma A.4, there exists $T > 0$ such that for all $t > T$ and $\chi(\cdot) \in \mathcal{C}_R(t', 10, +\infty)$

$$
\begin{aligned}
&\mathbf{P}\Big(\sup_{x\in[0,R]^{\mathsf{d}}} \mathbb{1}_{E^3_{t',t}(x)} = 1\Big) \\
&\qquad \le \sum_{j\ge L+1} c_4(1+\nu_j)e^{-c_5 j}\int_{[0,R]^{\mathsf{d}}} \big(\sqrt{\ln t'} + \chi(x)\big)e^{-\sqrt{2\mathsf{d}}\chi(x)}\,dx \qquad (4.5)\\
&\qquad \le c e^{-(c_5/2)L}\int_{[0,R]^{\mathsf{d}}} \big(\sqrt{\ln t'} + \chi(x)\big)e^{-\sqrt{2\mathsf{d}}\chi(x)}\,dx.
\end{aligned}
$$

If we further impose $\chi(\cdot) \in \mathcal{C}_R(t', \kappa_{\mathsf{d}} \ln t', +\infty)$ while choosing t' large enough so as to make the term $\sqrt{\ln t'}$ smaller than $\varepsilon\kappa_{\mathsf{d}} \ln t'$ (and therefore less than $\varepsilon\chi$) as well as choosing L large enough to have $ce^{-(c_5/2)L} \le \varepsilon$, we deduce

$$
\begin{aligned}
&\mathbf{E}\Big[1-\exp\Big(-\int_{[0,R]^{\mathsf{d}}} e^{\gamma[Y_t(x)-a_t-\chi(x)]+\mathsf{d}t}\mathbb{1}_{E^3_{t',t}(x)}\,dx\Big)\Big] \qquad (4.6)\\
&\qquad \le 2\varepsilon\int_{[0,R]^{\mathsf{d}}} \chi(x)e^{-\sqrt{2\mathsf{d}}\chi(x)}\,dx.
\end{aligned}
$$

Now we focus on $E^1_{t',t}(x)$. By partitioning the event $\{Y_t(x) \le a_t + \chi(x) - A\}$ as

$$
\big\{Y_t(x) \le a_t + \chi(x) - A\big\} = \bigcup_{p\ge 0}\big\{Y_t(x) - a_t - \chi(x) + A \in [-p-1, -p]\big\}
$$

and by using the relation $1 - e^{-u} \le u$ for $u \ge 0$, we obtain

$$
\begin{aligned}
&\mathbf{E}\Big[1-\exp\Big(-\int_{[0,R]^{\mathsf{d}}} e^{\gamma[Y_t(x)-a_t-\chi(x)]+\mathsf{d}t}\mathbb{1}_{E^1_{t',t}(x)}\,dx\Big)\Big]\\
&\qquad \le \mathbf{E}\Big[\int_{[0,R]^{\mathsf{d}}} e^{\gamma[Y_t(x)-a_t-\chi(x)]+\mathsf{d}t}\mathbb{1}_{E^1_{t',t}(x)}\,dx\Big] \qquad (4.7)\\
&\qquad \le e^{-\gamma A}\sum_{p\ge 0} e^{-\gamma p}e^{\mathsf{d}t}\\
&\qquad\quad \times \int_{[0,R]^{\mathsf{d}}} \mathbf{P}\big(E^1_{t',t}(x), Y_t(x) - a_t - \chi(x) + A \in [-p-1,-p]\big)\,dx.
\end{aligned}
$$

By Girsanov's transform (with density $e^{\sqrt{2\mathsf{d}}Y_t(x)+\mathsf{d}t}$), we obtain for any $x \in [0, R]^{\mathsf{d}}$ and $p \ge 0$,

$$
\begin{aligned}
&\mathbf{P}\big(E^1_{t',t}(x), Y_t(x) - a_t - \chi(x) + A \in [-p-1,-p]\big)\\
&\qquad \le e^{-\sqrt{2\mathsf{d}}[a_t+\chi(x)-A-p-1]-\mathsf{d}t}\mathbf{P}_{-\chi(x)}\Big(\sup_{s\in[\ln t',t]} B_s \le 0, \qquad (4.8)\\
&\qquad\qquad \sup_{s\in[t/2,t]} B_s \le a_t + L, B_t - a_t + A \in [-p-1,-p]\Big),
\end{aligned}
$$

where, under $\mathbf{P}_{-\chi(x)}$, the process B is a standard Brownian motion starting from $-\chi(x)$. At this step, we observe that similar quantities have been treated in [17]. More precisely, (A.9) shows that, for some constant $\bar{c} > 0$ (which does not depend on relevant quantities)

$$\begin{aligned}&\mathbf{P}_{-\chi(x)}\Big(\sup_{s\in[\ln t',t]} B_s \le 0, \sup_{s\in[t/2,t]} B_s \le a_t + L, B_t - a_t + A \in [-p-1,-p]\Big)\\ &\quad\le t^{-3/2}\bar{c}(L+A+p)\mathbf{E}\big[\big(B_{\ln t'} + \chi(x)\big)\mathbb{1}_{\{B_{\ln t'}+\chi(x)\ge 0\}}\big].\end{aligned} \tag{4.9}$$

Finally, by combining (4.7) + (4.8) + (4.9), we get

$$\begin{aligned}&\mathbf{E}\Big[1-\exp\Big(-\int_{[0,R]^{\mathsf{d}}} e^{\gamma[Y_t(x)-a_t-\chi(x)]+\mathsf{d}t}\mathbb{1}_{E^1_{t',t}(x)}\,dx\Big)\Big]\\ &\quad\le e^{-(\gamma-\sqrt{2\mathsf{d}})A}\sum_{p\ge 0}\bar{c}(L+A+p)e^{-(\gamma-\sqrt{2\mathsf{d}})p}\int_{[0,R]^{\mathsf{d}}}\chi(x)e^{-\sqrt{2\mathsf{d}}\chi(x)}\,dx\\ &\quad\le (L+A)e^{-(\gamma-\sqrt{2\mathsf{d}})A}c\int_{[0,R]^{\mathsf{d}}}\chi(x)e^{-\sqrt{2\mathsf{d}}\chi(x)}\,dx,\end{aligned} \tag{4.10}$$

where we took, for instance, $c = \bar{c}\sum_{p\ge 0}(1+p)e^{-(\gamma-\sqrt{2\mathsf{d}})p}$.

Finally, we treat the contribution of the term $E^2_{t',t}(x)$. First, we can follow the same argument as for $E^1_{t',t}(x)$ to get

$$\begin{aligned}&\mathbf{E}\Big[1-\exp\Big(-\int_{[0,R]^{\mathsf{d}}} e^{\gamma[Y_t(x)-a_t-\chi(x)]+\mathsf{d}t}\mathbb{1}_{E^2_{t',t}(x)}\,dx\Big)\Big]\\ &\quad\le \sum_{j\ge L+1} e^{-\gamma A}\sum_{p\ge 0} e^{-\gamma p}e^{\mathsf{d}t}\\ &\qquad\times\int_{[0,R]^{\mathsf{d}}}\mathbf{P}\big(E^2_{t',t}(x), Y_t(x)-a_t-\chi(x)+A\in[-p-1,-p]\big)\,dx.\end{aligned} \tag{4.11}$$

By Girsanov's transform again (with density $e^{\sqrt{2\mathsf{d}}Y_t(x)+\mathsf{d}t}$), we can estimate the probability in (4.11) by

$$\begin{aligned}&\mathbf{P}\big(E^2_{t',t}(x), Y_t(x)-a_t-\chi(x)+A\in[-p-1,-p]\big)\\ &\quad\le e^{-\sqrt{2\mathsf{d}}[a_t+\chi(x)-A-p-1]-\mathsf{d}t}\mathbf{P}_{-\chi(x)}\Big(\sup_{s\in[\ln t',t]} B_s\le 0,\\ &\qquad\sup_{s\in[t/2,t-\nu_j]} B_s - a_t\in[j-1,j],\\ &\qquad\sup_{s\in[t-\nu_j,t]} B_s\le a_t+j, B_t-a_t+A\in[-p-1,-p]\Big).\end{aligned} \tag{4.12}$$

Now we use (A.8) to see that this latter quantity is smaller than

$$ce^{-\mathsf{d}t}e^{\sqrt{2\mathsf{d}}(A+p+1)}e^{-\sqrt{2\mathsf{d}}\chi(x)}(1+j+A+p)\nu_j^{-1/2}\chi(x). \tag{4.13}$$

By recalling that $\nu_j = e^{(c_5/2)j}$ and by combining (4.11) + (4.12) + (4.13), we get

$$
\begin{aligned}
&\mathbf{E}\Big[1-\exp\Big(-\int_{[0,R]^{\mathsf{d}}} e^{\gamma[Y_t(x)-a_t-\chi(x)]+\mathsf{d}t}\mathbb{1}_{E^2_{t',t}(x)}\,dx\Big)\Big]\\
&\quad\le ce^{-(\gamma-\sqrt{2\mathsf{d}})A}\sum_{j\ge L+1} e^{-(c_5/4)j}\sum_{p\ge 0}(1+j+A+p)e^{-(\gamma-\sqrt{2\mathsf{d}})p}\\
&\qquad\times\int_{[0,R]^{\mathsf{d}}} e^{-\sqrt{2\mathsf{d}}\chi(x)}\chi(x)\,dx\\
&\quad\le ce^{-(c_5/8)L}Ae^{-(\gamma-\sqrt{2\mathsf{d}})A}\int_{[0,R]^{\mathsf{d}}} e^{-\sqrt{2\mathsf{d}}\chi(x)}\chi(x)\,dx.
\end{aligned}
\tag{4.14}
$$

Now recall that our purpose is to estimate the right-hand side in (4.3). The expectation in this right-hand side is estimated by combining (4.6) + (4.10) + (4.14) in such a way that

$$
\begin{aligned}
&\mathbf{E}\Big[1-\exp\Big(-\int_{[0,R]^{\mathsf{d}}} e^{\gamma[Y_t(x)-a_t-\chi(x)]+\mathsf{d}t}\mathbb{1}_{\{Y_t(x)\le a_t+\chi(x)-A\}}\,dx\Big)\Big]\\
&\quad\le c\big(e^{-[\gamma-\sqrt{2\mathsf{d}}]A}\big[(L+A)+e^{-(c_5/8)L}A\big]+2\varepsilon\big)\\
&\qquad\times\int_{[0,R]^{\mathsf{d}}} e^{-\sqrt{2\mathsf{d}}\chi(x)}\chi(x)\,dx.
\end{aligned}
\tag{4.15}
$$

So it suffices to choose A large enough such that $ce^{-[\gamma-\sqrt{2\mathsf{d}}]A}[(L+A)+e^{-(c_5/8)L}A]\le\varepsilon$ to complete the proof of Proposition 4.1.

4.2. *Proof of Proposition* 4.2. The first relation of Proposition 4.2 is an easy consequence of Lemma A.5 and (4.15). Indeed by using the relation $1-e^{-(u+v)}\le(1-e^{-u})+(1-e^{-v})$ for $u, v\ge 0$ and by applying (4.15) with $A=1$ and L chosen with $\varepsilon=1$ [see the relation $ce^{-(c_5/2)L}\le 1$ just before (4.6)], we obtain

$$
\begin{aligned}
&\mathbf{E}\Big[1-\exp\Big(-\int_{[0,R]^{\mathsf{d}}} e^{\gamma[Y_t(x)-a_t-\chi(x)]+\mathsf{d}t}\,dx\Big)\Big]\\
&\quad\le\mathbf{E}\Big[1-\exp\Big(-\int_{[0,R]^{\mathsf{d}}} e^{\gamma[Y_t(x)-a_t-\chi(x)]+\mathsf{d}t}\mathbb{1}_{\{Y_t(x)\ge a_t+\chi(x)-1\}}\,dx\Big)\Big]\\
&\qquad+\mathbf{E}\Big[1-\exp\Big(-\int_{[0,R]^{\mathsf{d}}} e^{\gamma[Y_t(x)-a_t-\chi(x)]+\mathsf{d}t}\mathbb{1}_{\{Y_t(x)\le a_t+\chi(x)-1\}}\,dx\Big)\Big]\\
&\quad\le\mathbf{P}\big(\exists x\in[0,R]^{\mathsf{d}}, Y_t(x)\ge a_t+\chi(x)-1\big)\\
&\qquad+c\big(e^{-[\gamma-\sqrt{2\mathsf{d}}]A}\big[(L+A)+e^{-(c_5/8)L}A\big]+2\big)\int_{[0,R]^{\mathsf{d}}}\chi(x)e^{-\sqrt{2\mathsf{d}}\chi(x)}\,dx\\
&\quad\le c'\int_{[0,R]^{\mathsf{d}}}\chi(x)e^{-\sqrt{2\mathsf{d}}\chi(x)}\,dx,
\end{aligned}
\tag{4.16}
$$

with $c' := c_1 + c((L+1) + e^{-(c_5/8)L} + 2)$ where c_2 is the constant appearing in Lemma A.5.

Now we prove the second inequality. For each $t \geq 0$, we introduce a tiling process Y_t^{tiling} as follows. We consider a partition of the cube $[0,R]^{\mathsf{d}}$ with cubes of edge size e^{-t}. The amount of cubes in such a partition is of order $(Re^t)^{\mathsf{d}}$. For each point $z \in [0,R]^{\mathsf{d}}$, there exists a unique cube denoted $C(z)$ in the partition such that $z \in C(z)$. Let us consider the center c_z of such a cube. We define $Y_t^{\text{tiling}}(z) = Y_t(c_z)$. To sum up, the process Y_t^{tiling} is constant over each cube in the partition and takes the value of the process Y_t at the center of this cube. Because of assumption (A.3), it is plain to check that there is a fixed constant D such that for all $R > 0$, all $x, x' \in [0,R]^{\mathsf{d}}$, all $t \geq 0$

$$K_t(x - x') - D \leq \mathbf{E}\big[Y_t^{\text{tiling}}(x) Y_t^{\text{tiling}}(x')\big] \leq K_t(x - x') + D. \tag{4.17}$$

With the help of this covariance inequality, we can use Kahane's convexity inequality (see Kahane's original paper [14] or [21], Theorem 2.1, for an english statement) to the concave function $x \mapsto 1 - e^{-x}$ to get

$$\begin{aligned}
&\mathbf{E}\Big[1 - \exp\Big(-\int_{[0,R]^{\mathsf{d}}} e^{\gamma[Y_t(x) - a_t - \chi(x)] + \mathsf{d}t}\,dx\Big)\Big] \\
&\qquad \geq \mathbf{E}\Big[1 - \exp\Big(-e^{\gamma Z - \gamma^2 D/2} \int_{[0,R]^{\mathsf{d}}} e^{\gamma[Y_t^{\text{tiling}}(x) - a_t - \chi(x)] + \mathsf{d}t}\,dx\Big)\Big]
\end{aligned}$$

for some centered Gaussian random variable Z with variance D independent of everything.

Let us define the event

$$E(R, t, \chi) = \big\{\exists x \in [0,R]^{\mathsf{d}}, Y_t^{\text{tiling}}(x) \geq a_t + \chi(x)\big\}.$$

According to the definition (4.65) of h_{good} in [17], it is clear that $\{h_{\text{good}} \geq 1\} \subset E(R, t, \chi)$. Moreover, with the Paley–Zygmund inequality page 32 in [17], there exists $c_2 > 0$ such that for any $t' \geq 2$, there exists $T > 0$ such that for any $R \in [1, \ln t']$ and $t \geq T$,

$$\mathbf{P}\big(E(R, t, \chi)\big) \geq \mathbf{P}(h_{\text{good}} \geq 1) \geq c_2 \int_{[0,R]^{\mathsf{d}}} \chi(x) e^{-\sqrt{2\mathsf{d}}\chi(x)}\,dx \tag{4.18}$$

for any function $\chi \in \mathcal{C}_R(t', \kappa_{\mathsf{d}} \ln t', \ln t)$. On $E(R, t, \chi)$, let us choose any x_0 belonging to $\{x \in [0,R]^{\mathsf{d}}, Y_t^{\text{tiling}}(x) \geq a_t + \chi(x)\}$. Then we observe that

$$\begin{aligned}
&\mathbf{E}\Big[1 - \exp\Big(-e^{\gamma Z - \gamma^2 D/2} \int_{[0,R]^{\mathsf{d}}} e^{\gamma[Y_t^{\text{tiling}}(x) - a_t - \chi(x)] + \mathsf{d}t}\,dx\Big)\Big] \\
&\quad \geq \mathbf{E}\Big[\Big(1 - \exp\Big(-e^{\gamma Z - \gamma^2 D/2} \int_{[0,R]^{\mathsf{d}}} e^{\gamma[Y_t^{\text{tiling}}(x) - a_t - \chi(x)] + \mathsf{d}t}\,dx\Big)\Big)\mathbb{1}_{E(R,t,\chi)}\Big] \\
&\quad \geq \mathbf{E}\Big[\Big(1 - \exp\Big(-e^{\gamma Z - \gamma^2 D/2} \int_{C(x_0)} e^{\gamma[Y_t^{\text{tiling}}(x) - a_t - \chi(x)] + \mathsf{d}t}\,dx\Big)\Big)\mathbb{1}_{E(R,t,\chi)}\Big].
\end{aligned}$$

Now we use the fact that $Y_t^{\mathrm{tiling}}(x) = Y_t^{\mathrm{tiling}}(x_0)$ on $C(x_0)$ and the relation $Y_t(x_0) - a_t - \chi(x_0) \geq 0$ to deduce

$$\begin{aligned}\mathbf{E}\Big[1 - \exp\Big(-\int_{[0,R]^{\mathrm{d}}} e^{\gamma[Y_t(x)-a_t-\chi(x)]+\mathrm{d}t}\,dx\Big)\Big] &\geq \mathbf{E}\big[\big(1-\exp\big(-e^{\gamma Z-\gamma^2 D/2}\big)\big)\mathbb{1}_{E(R,t,\chi)}\big] \\ &= \mathbf{E}\big[1-\exp\big(-e^{\gamma Z-\gamma^2 D/2}\big)\big]\mathbf{P}\big(E(R,t,\chi)\big).\end{aligned}$$

We complete the proof with (4.18).

5. Proof of Proposition 3.2. Our aim is to study for t, t' large and $\chi(\cdot) \in \mathcal{C}_R(t', \kappa_{\mathrm{d}} \ln t', \ln t)$,

$$\mathbf{E}\Big[\exp\Big(-\int_{[0,R]^{\mathrm{d}}} e^{\gamma[Y_t(x)-a_t-\chi(x)]+\mathrm{d}t}\,dx\Big)\Big]. \tag{5.1}$$

According to Proposition 4.1, for A large enough, we can restrain our study to the expectation of

$$\begin{aligned}&\Phi^{(A)}\big(\chi(\cdot), t\big) \\ &\quad := \exp\Big(-\int_{[0,R]^{\mathrm{d}}} e^{\gamma[Y_t(x)-a_t-\chi(x)]+\mathrm{d}t}\mathbb{1}_{\{Y_t(x)-a_t-\chi(x)\geq -A\}}\,dx\Big).\end{aligned} \tag{5.2}$$

Throughout this section, keep in mind that the function $\Phi^{(A)}(\chi(\cdot), t)$ is bounded by 1. We fix $R, A, \varepsilon > 0$. We stick to the notation introduced in [17] page 33 relations (5.1), (5.2) and (5.3): we define

$$\begin{aligned}M_{t,\chi} &:= \sup_{y\in[0,R]^{\mathrm{d}}} \big(Y_t(y) - \chi(y)\big), \\ \mathfrak{O}_{t,\chi} &:= \big\{y \in [0,R]^{\mathrm{d}}, Y_t(y) \geq a_t + \chi(y) - 1\big\},\end{aligned} \tag{5.3}$$

$$\begin{aligned}M_{t,\chi}(x,b) &:= \sup_{y\in B(x,e^{b-t})} \big(Y_t(y) - \chi(y)\big), \\ \mathfrak{O}_{t,\chi}(x,b) &:= \big\{y \in B\big(x, e^{b-t}\big), Y_t(y) \geq a_t + \chi(y) - 1\big\},\end{aligned} \tag{5.4}$$

$$\mathbb{R}_t := \big[e^{-t/2}, R - e^{-t/2}\big]^{\mathrm{d}}. \tag{5.5}$$

Observe that on the set $\{M_{t,\chi-A} < a_t\}$, we have $1 - \Phi^{(A)}(\chi(\cdot), t) = 0$. Moreover, for any $t > 0$, because of the continuity of the function $x \mapsto Y_t(x) - \chi(x)$, the random variables $|\mathfrak{O}_{t,\chi-A}|$ and $|\mathfrak{O}_{t,\chi-A}(x,b)|$ (recall that $|B|$ stands for the Lebesgue measure of the set $B \subset \mathbb{R}^{\mathrm{d}}$) are strictly positive respectively on

$\{M_{t,\chi-A} \geq a_t\}$ and $\{M_{t,\chi-A}(x,b) \geq a_t\}$. Therefore, for any $L \geq 1$,

$$
\begin{aligned}
&\mathbf{E}\left[1-\exp\left(-\int_{[0,R]^{\mathsf{d}}} e^{\gamma[Y_t(x)-a_t-\chi(x)]+\mathsf{d}t}\mathbb{1}_{\{Y_t(x)-a_t-\chi(x)\geq -A\}}\,dx\right)\right]\\
&\quad=\mathbf{E}\big[1-\phi^{(A)}\big(\chi(\cdot),t\big)\big];\, M_{t,\chi-A}\geq a_t\big]\\
&\quad=\mathbf{E}\left[\int_{[0,R]^{\mathsf{d}}}\big[1-\phi^{(A)}\big(\chi(\cdot),t\big)\big]\frac{\mathbb{1}_{\{m\in\mathfrak{O}_{t,\chi-A}\}}\mathbb{1}_{\{M_{t,\chi-A}\geq a_t\}}}{|\mathfrak{O}_{t,\chi-A}|}\,dm\right]\\
&\quad:=\mathbf{E}_{(5.6)}.
\end{aligned}
\tag{5.6}
$$

Now we want to exclude the particles $m \in \mathfrak{O}_{t,\chi-A}$ such that their paths $Y_{\cdot}(m)$ are unlikely. We set

$$
\begin{aligned}
\rhd_t^{\alpha,A,L} := \Big\{&(f_s)_{s\geq 0},\ \sup_{s\in[0,t]} f(s)\leq \alpha,\\
&\sup_{s\in[t/2,t]} f(s)\leq a_t+\alpha+L,\ f_t\geq a_t+\alpha-A-1\Big\}
\end{aligned}
\tag{5.7}
$$

$$\forall L,\alpha,t>0.$$

Observe that $\rhd_t^{\alpha,0,L} =: \rhd_t^{\alpha,L}$ which is introduced in (1.15) in [17].

LEMMA 5.1. *For any $A,\varepsilon>0$ there exists $L>0$ such that for any $t',T>0$ large enough we have for any $t\geq T$, $\chi\in\mathcal{C}_R(t',\kappa_{\mathsf{d}}\ln t',\ln t)$,*

$$
\begin{aligned}
&\mathbf{P}\big(\exists m\in\mathfrak{O}_{t,\chi-A}\cap[0,R]^{\mathsf{d}},\ Y_{\cdot}(m)\notin\rhd_t^{\chi(m),A,L}\big)\leq\varepsilon\mathtt{I}(\chi),\\
&\qquad\mathbf{P}\big(\exists m\in[0,R]^{\mathsf{d}}/_{\mathbb{R}_t},\ m\in\mathfrak{O}_{t,\chi-A}\big)\leq\varepsilon\mathtt{I}(\chi).
\end{aligned}
\tag{5.8}
$$

With Proposition 4.4 [17] and the arguments used to bound (3) and obtain (5.11) in [17], the inequalities of (5.8) are proved for $A=0$, but it does not make any difficulties to extend for any fixed $A>0$. Thus, we do not detail the proof of Lemma 5.1.

Recalling (5.6), from Lemma 5.1, we deduce that, for any A, there exist $L>0$, $t_0>0$ such that for any $t'\geq t_0$ there exists $T>0$ such that $\forall t\geq T,\ \chi(\cdot)\in\mathcal{C}_R(t',\kappa_{\mathsf{d}}\ln t',\ln t)$,

$$
\begin{aligned}
&\Bigg|\mathbf{E}\Bigg[\int_{\mathbb{R}_t}\frac{\mathbb{1}_{\{m\in\mathfrak{O}_{t,\chi-A},\,Y_{\cdot}(m)\in\rhd_t^{\chi(m),A,L}\}}\mathbb{1}_{\{M_{t,\chi-A}\geq a_t\}}}{|\mathfrak{O}_{t,\chi-A}|}\\
&\qquad\times\big[1-\phi^{(A)}\big(\chi(\cdot),t\big)\big]\,dm\Bigg]-\mathbf{E}_{(5.6)}\Bigg|\leq\varepsilon\mathtt{I}(\chi).
\end{aligned}
\tag{5.9}
$$

Now the constant L is also fixed.

For any $t > b \geq 0$, let us introduce:

$$\begin{aligned}(5.10)\quad &\Xi_{\chi-A,t}(b,m)\\ &\quad = \{\exists y \in [0,R]^{\mathsf{d}}, |y-m| \geq e^{b-t}, Y_t(y) \geq a_t + \chi(y) - A - 1\}.\end{aligned}$$

On the complement of $\Xi_{\chi-A,t}(b,m)$, we have [just observe that everything happens inside the ball $B(m,e^{b-t})$]

$$\frac{\mathbb{1}_{\{M_{t,\chi-A} \geq a_t\}}}{|\mathfrak{O}_{t,\chi-A}|} = \frac{\mathbb{1}_{\{M_{t,\chi-A}(m,b) \geq a_t\}}}{|\mathfrak{O}_{t,\chi-A}(m,b)|}.$$

Also, still on the complement of $\Xi_{\chi-A,t}(b,m)$, the function $[1-\phi^{(A)}(\chi(\cdot),t)]$ is equal to

$$\begin{aligned}(5.11)\quad &1 - \exp\biggl(-\int_{B(m,e^{b-t})} e^{\gamma[Y_t(x)-a_t-\chi(x)]+\mathsf{d}t}\mathbb{1}_{\{Y_t(x)-a_t-\chi(x)\geq -A\}}\,dx\biggr)\\ &\quad := 1 - \phi^{(A,b)}\bigl(\chi(\cdot),t,m\bigr).\end{aligned}$$

Therefore, for any $b \geq 1$, $m \in \mathbb{R}_t$ we can write

$$\begin{aligned}&[1-\phi^{(A)}(\chi(\cdot),t)]\frac{\mathbb{1}_{\{M_{t,\chi-A}\geq a_t\}}}{|\mathfrak{O}_{t,\chi-A}|}\\ &\quad = \bigl[1-\phi^{(A)}\bigl(\chi(\cdot),t\bigr)\bigr]\frac{\mathbb{1}_{\{M_{t,\chi-A}\geq a_t\}}}{|\mathfrak{O}_{t,\chi-A}|}(\mathbb{1}_{\{\Xi_{\chi-A,t}(b,m)^c\}} + \mathbb{1}_{\{\Xi_{\chi-A,t}(b,m)\}})\\ &\quad = \bigl(1-\phi^{(A,b)}\bigl(\chi(\cdot),t,m\bigr)\bigr)\frac{\mathbb{1}_{\{M_{t,\chi-A}(m,b)\geq a_t\}}}{|\mathfrak{O}_{t,\chi-A}(m,b)|}\mathbb{1}_{\{\Xi_{\chi-A,t}(b,m)^c\}}\\ &\qquad + \bigl[1-\phi^{(A)}\bigl(\chi(\cdot),t\bigr)\bigr]\frac{\mathbb{1}_{\{M_{t,\chi-A}\geq a_t\}}}{|\mathfrak{O}_{t,\chi-A}|}\mathbb{1}_{\{\Xi_{\chi-A,t}(b,m)\}}\\ &\quad = \bigl(1-\phi^{(A,b)}\bigl(\chi(\cdot),t,m\bigr)\bigr)\frac{\mathbb{1}_{\{M_{t,\chi-A}(m,b)\geq a_t\}}}{|\mathfrak{O}_{t,\chi-A}(m,b)|}\\ &\qquad - \bigl(1-\phi^{(A,b)}\bigl(\chi(\cdot),t,m\bigr)\bigr)\frac{\mathbb{1}_{\{M_{t,\chi-A}(m,b)\geq a_t\}}}{|\mathfrak{O}_{t,\chi-A}(m,b)|}\mathbb{1}_{\{\Xi_{\chi-A,t}(b,m)\}}\\ &\qquad + \bigl[1-\phi^{(A)}\bigl(\chi(\cdot),t\bigr)\bigr]\frac{\mathbb{1}_{\{M_{t,\chi-A}\geq a_t\}}}{|\mathfrak{O}_{t,\chi-A}|}\mathbb{1}_{\{\Xi_{\chi-A,t}(b,m)\}}.\end{aligned}$$

Following this decomposition, the first expectation in (5.9) is equal to the sum of (we use the fact that $\{m \in \mathfrak{O}_{t,\chi-A}\} \subset \{Y_\cdot(m) \in \rhd_t^{\chi(m),A,L}\}$)

$$\begin{aligned}(5.12)\quad &(1)_{A,L,b} := \mathbf{E}\biggl[\int_{\mathbb{R}_t}\bigl[1-\phi^{(A,b)}\bigl(\chi(\cdot),t,m\bigr)\bigr]\\ &\qquad\qquad \times \frac{\mathbb{1}_{\{Y_\cdot(m)\in\rhd_t^{\chi(m),A,L}\}}\mathbb{1}_{\{M_{t,\chi-A}(m,b)\geq a_t\}}}{|\mathfrak{O}_{t,\chi-A}(m,b)|}\,dm\biggr],\end{aligned}$$

$$(2)_{A,L,b} := \mathbf{E}\bigg[\int_{\mathbb{R}_t} [1-\phi^{(A)}(\chi(\cdot),t)] \times \mathbb{1}_{\Xi_{\chi-A,t}(b,m)} \frac{\mathbb{1}_{\{Y_\cdot(m)\in \rhd_t^{\chi(m),A,L}\}}\mathbb{1}_{\{M_{t,\chi-A}\geq a_t\}}}{|\mathfrak{O}_{t,\chi-A}|}\,dm\bigg], \tag{5.13}$$

$$(3)_{A,L,b} := -\mathbf{E}\bigg[\int_{\mathbb{R}_t} [1-\phi^{(A,b)}(\chi(\cdot),t,m)] \times \mathbb{1}_{\Xi_{\chi-A,t}(b,m)} \frac{\mathbb{1}_{\{Y_\cdot(m)\in \rhd_t^{\chi(m),A,L}\}}\mathbb{1}_{\{M_{t,\chi-A}(m,b)\geq a_t\}}}{|\mathfrak{O}_{t,\chi-A}(m,b)|}\,dm\bigg]. \tag{5.14}$$

LEMMA 5.2. *For any $A, L, \varepsilon > 0$, there exists b_0, t_0 large enough such that for any $t' \geq t_0$, $b \geq b_0$, $\exists T > 0$ such that for any $t \geq T$, $\chi \in \mathcal{C}_R(t', \kappa_{\mathsf{d}} \ln t', \ln t)$ we have*

$$\big|(2)_{A,L,b}\big| + \big|(3)_{A,L,b}\big| \leq \varepsilon \mathtt{I}(\chi). \tag{5.15}$$

We do not detail the proof of Lemma 5.2 but, recalling that $|1-\phi^{(A,b)}(\chi(\cdot), t,m)|$ and $|1-\phi^{(A)}(\chi(\cdot),t)|$ are bounded by 1, we just remark that the amounts $(2)_{A,L,b}$ and $(3)_{A,L,b}$ are very similar to $(2)_{L,b}$ and $(3)_{L,b}$ defined in (5.15) and (5.16) of [17]. Then Lemma 5.2 is a minor adaptation of the proofs of Lemmas 5.1 and 5.2 in [17] (in [17] $A = 0$, whereas here A is a fixed positive constant).

Thus, combining Lemma 5.2 and (5.9), we deduce that there exist b and $t_0 > 0$, such that for any $t' > t_0$ there exists $T > 0$ such that $\forall t \geq T, \chi(\cdot) \in \mathcal{C}_R(t', \kappa_{\mathsf{d}} \ln t', \ln t)$,

$$\big|\mathbf{E}_{(5.6)} - (1)_{A,L,b}\big| \leq 2\varepsilon \int_{[0,R]^{\mathsf{d}}} \chi(x) e^{-\sqrt{2\mathsf{d}}\chi(x)}\,dx. \tag{5.16}$$

Therefore, we can restrain our study to $(1)_{A,L,b}$ (with A, L, b fixed). The Markov property at time $t_b = t - b$ and the invariance by translation of $(Y_s(x))_{s\geq 0, x\in\mathbb{R}^{\mathsf{d}}}$ give

$$\begin{aligned}(1)_{A,L,b} &= \mathbf{E}\bigg[\int_{\mathbb{R}_t} \frac{\mathbb{1}_{\{Y_\cdot(m)\in \rhd_t^{\chi(m),A,L}, m\in\mathfrak{O}_{t,\chi-A}\}}\mathbb{1}_{\{M_{t,\chi-A}(m,b)\geq a_t\}}}{|\mathfrak{O}_{t,\chi-A}(m,b)|} \\ &\qquad \times [1-\phi^{(A,b)}(\chi(\cdot),t,m)]\,dm\bigg] \\ &= \int_{\mathbb{R}_t} \mathbf{E}\big[\mathbb{1}_{\{\sup_{s\in[0,t_b]} Y_s(m)\leq \chi(m), \sup_{s\in[t/2,t_b]} Y_s(m)\leq a_t+\chi(m)+L\}} D_{m,t}^{A,L,b}\big]\,dm,\end{aligned} \tag{5.17}$$

where

$$D_{m,t}^{A,L,b} := \mathbf{E}\bigg[\frac{\mathbb{1}_{\{\sup_{s\in[0,b]} Y_{t_b,s}(0)+\bar{z}\leq 0, Y_{t_b,b}(0)+\bar{z}\geq -L-A-1, \exists y\in B(0,e^{b-t}), Y_{t_b,b}(y)+\bar{z}\geq -L-A-g(y)\}}}{|B(0,e^{b-t})\cap\{y : Y_{t_b,b}(y)+\bar{z}\geq -L-A-1-g(y)\}|}$$

$$\times \biggl(1-\exp\biggl\{-\int_{B(0,e^{b-t})} e^{\gamma[Y_{t_b,b}(y)+\bar z+g(y)+L]+\mathsf{d}t} \times \mathbb{1}_{\{Y_{t_b,b}(y)+\bar z\geq -A-L-g(y)\}}\,dy\biggr\}\biggr)\biggr],$$

with

$$g(y)=Y_{t_b}(m+y)-Y_{t_b}(m)-\bigl(\chi(m+y)-\chi(m)\bigr),$$
$$\bar z=Y_{t_b}(m)-a_t-\chi(m)-L.$$

In the following, we will denote

$$\forall m\in\mathbb{R}_t,\qquad \chi_m(\cdot):=\chi(m+\cdot)-\chi(m). \tag{5.18}$$

According to the scaling property

$$\bigl(Y_{t_b,s}(y)\bigr)_{s\leq b,y\in B(0,e^{b-t})}\overset{(d)}{=}\bigl(Y_s\bigl(ye^{t-b}\bigr)\bigr)_{s\leq b,y\in B(0,e^{b-t})},$$

thus we can rewrite $D_{m,t}^{A,L,b}$ as

$$e^{\mathsf{d}t_b}\mathbf{E}_{\bar z}\biggl[\frac{\mathbb{1}_{\{\sup_{s\in[0,b]}Y_s(0)\leq 0,Y_b(0)\geq -L-A-1\}}\mathbb{1}_{\{\exists y\in B(0,1),Y_b(y)\geq -L-A-g(ye^{b-t})\}}}{|B(0,1)\cap\{y:Y_b(y)\geq -L-A-1-g(ye^{b-t})\}|} \times\biggl(1-\exp\biggl(-\int_{B(0,1)}e^{\gamma[Y_b(y)+g(ye^{b-t})+L]}\mathbb{1}_{\{Y_b(y)\geq -A-L-g(ye^{b-t})\}}\,dy\biggr)\biggr)\biggr],$$

where we have used the convention: for any $z\in\mathbb{R}$, $(Y_s(x))_{s\geq 0,x\in\mathbb{R}^{\mathsf{d}}}$ under $\mathbf{P}_z$ has the law $(z+Y_s(x))_{s\geq 0,x\in\mathbb{R}^{\mathsf{d}}}$ under $\mathbf{P}$. By applying Lemma A.1 to the process $g(y)$ and the Girsanov transformation to the process $(Y_s(m))_{s\leq t_b}$, we get

$$\begin{aligned}(1)_{A,L,b}&=\int_{\mathbb{R}_t}\mathbf{E}\bigl[e^{\sqrt{2\mathsf{d}}Y_{t_b}(m)+\mathsf{d}t_b}\mathbb{1}_{\{\sup_{s\in[0,t_b]}Y_s(m)\leq\chi(m),\sup_{s\in[t/2,t_b]}Y_s(m)\leq a_t+\chi(m)+L\}}\\&\qquad\times e^{-\sqrt{2\mathsf{d}}Y_{t_b}(m)-\mathsf{d}t_b}e^{\mathsf{d}t_b}D_{m,t}^{A,L,b}\bigr]\,dm\\&=\int_{\mathbb{R}_t}e^{-\sqrt{2\mathsf{d}}\chi(m)}t^{3/2}\mathbf{E}_{-\chi(m)}\bigl[\mathbb{1}_{\{\sup_{s\in[0,t_b]}B_s\leq 0,\sup_{s\in[t/2,t_b]}B_s\leq a_t+L\}}\\&\qquad\times F_{A,L,b}\bigl(B_{t_b}-a_t-L,\mathfrak{G}_{t,b}^{\chi_m}\bigr)\bigr]\,dm,\end{aligned}$$

where B a standard Brownian motion and, for $g\in\mathcal{C}(B(0,1),\mathbb{R})$, $z\in\mathbb{R}$,

$$\begin{aligned}&F_{A,L,b}(z,g)\\&\quad:=e^{-\sqrt{2\mathsf{d}}(z+L)}\\&\qquad\times\mathbf{E}_z\biggl[\frac{\mathbb{1}_{\{\sup_{s\in[0,b]}Y_s(0)\leq 0,Y_b(0)\geq -L-A-1\}}\mathbb{1}_{\{\exists y\in B(0,1),Y_b(y)\geq -L-A-g(ye^{b})\}}}{|B(0,1)\cap\{y:Y_b(y)\geq -L-A-1-g(ye^{b})\}|}\\&\qquad\times\biggl(1-\exp\biggl(-\int_{B(0,1)}e^{\gamma[Y_b(y)+g(ye^{b})+L]}\mathbb{1}_{\{Y_b(y)\geq -A-L-g(ye^{b})\}}\,dy\biggr)\biggr)\biggr],\end{aligned}\tag{5.19}$$

and for any $\Psi \in \mathcal{C}_R(B(0,e^b),\mathbb{R})$,

$$\begin{aligned}\mathfrak{G}^{\Psi}_{t,b} : B(0,e^b) \ni y \\ \mapsto -\int_0^{t_b} (1-k(e^{s-t}y))\,dB_s - \zeta_t(ye^{-t}) + Z^0_{t_b}(ye^{-t}) - \Psi(ye^{-t}).\end{aligned} \tag{5.20}$$

For $\Psi = 0$, we denote $\mathfrak{G}^0_{t,b} = \mathfrak{G}_{t,b}$. In passing, we define for any $\sigma \in [0,t_b]$,

$$\begin{aligned}\mathfrak{G}_{t,b,\sigma} : B(0,e^b) \ni y \\ \mapsto -\int_{t_b-\sigma}^{t_b} (1-k(e^{s-t}y))(e^{s-t}y)\,dB_s - \zeta_t(ye^{-t}) + Z^0_{t_b}(ye^{-t})\end{aligned} \tag{5.21}$$

and the processes ζ, Z are defined in Lemma A.1. Note that $Z^0_{t_b}(\cdot)$ is a centered Gaussian process, independent of B, which has the covariances as in [17], equation (2.6). Furthermore, by [17], Proposition 2.4, for any $b>0$, the Gaussian process $B(0,e^b) \ni y \mapsto Z^0_{t-b}(ye^{-t}) - \zeta_{t-b}(ye^{-t})$, converges in law to $B(0,e^b) \ni y \mapsto Z(ye^{-b}) - \zeta(ye^{-b})$ when t goes to infinity.

Finally, with our new notation, we have to study for any $m \in \mathbb{R}_t$,

$$\mathbf{E}_{-\chi(m)}\big(\mathbb{1}_{\{\sup_{s\in[0,t_b]} B_s \le 0, \sup_{s\in[t/2,t_b]} B_s \le a_t + L\}} F_{A,L,b}\big(B_{t_b} - a_t - L, \mathfrak{G}^{\chi_m}_{t,b}\big)\big).$$

Recalling Proposition 3.2, our goal is to prove that this quantity is equivalent to a constant times $t^{-3/2}\chi(m)$, when t goes to infinity. To do this, we need a renewal theorem proved in [17].

DEFINITION 5.3. A continuous function $F : \mathbb{R} \times \mathcal{C}(B(0,e^b),\mathbb{R}) \to \mathbb{R}^+$ is "*b regular*" if there exists two functions $h : \mathbb{R} \to \mathbb{R}_+$ and $F^* : \mathcal{C}(B(0,e^b)) \to \mathbb{R}^+$ satisfying:

(i)

$$\sup_{x\in\mathbb{R}} h(x) < +\infty \quad \text{and} \quad h(x) \underset{x\to-\infty}{=} O(e^x). \tag{5.22}$$

(ii) There exists $c>0$ such that for any $\delta \in (0,1)$, $g \in \mathcal{C}(B(0,e^b),\mathbb{R})$ with $w^{(0,1)}_{g(\cdot e^b)}(\delta) \le \frac{1}{4}$,

$$F^*(g) \le c\delta^{-10}. \tag{5.23}$$

(iii) For any $z \in \mathbb{R}$, $g \in \mathcal{C}(B(0,e^b),\mathbb{R})$, $F(z,g) \le h(z)F^*(g)$.

(iv) There exists $c>0$ such that for any $z \in \mathbb{R}$, $g_1, g_2 \in \mathcal{C}(B(0,e^b),\mathbb{R})$ with $\|g_1 - g_2\|_\infty \le \frac{1}{8}$,

$$\big|F(z,g_1) - F(z,g_2)\big| \le c\|g_1-g_2\|_\infty^{1/4} h(z) F^*(g_1). \tag{5.24}$$

DEFINITION 5.4. For any $M \ge 0$ and F a function *b regular*, we define

$$F^{(M)}(x,g) := \big(F(x,g)\wedge M\big)\mathbb{1}_{\{x\ge -M\}}. \tag{5.25}$$

For any $\gamma \in \mathbb{R}$, let $T_\gamma := \inf\{s \geq 0, B_s = \gamma\}$ and let $(\mathtt{R}_s)_{s\geq 0}$ be a three-dimensional Bessel process starting from 0.

THEOREM 5.5 ([17] Theorem 5.6). *Let $b > 0$ and $F : \mathbb{R} \times \mathcal{C}(B(0, e^b), \mathbb{R}) \to \mathbb{R}^+$ be a function b regular. For any $\varepsilon > 0$, there exist $M, \sigma, t', T > 0$ large enough such that for any $t \geq T$, $\chi(\cdot) \in \mathcal{C}_R(t', \kappa_d \ln t', \ln t)$, $\mathtt{z} \in [1, (\ln t)^{30}]$,*

$$\begin{aligned}\Bigg| \int_{\mathbb{R}_t} t^{3/2} e^{-\sqrt{2\mathsf{d}}\chi(x)} \mathbf{E}_{-\chi(x)} \big(\mathbb{1}_{\{\sup_{s\in[0,t_b]} B_s \leq 0, \sup_{s\in[t/2,t_b]} B_s \leq -\mathtt{z}\}} \\ \times F\big(B_{t_b} + \mathtt{z}, \mathfrak{G}^{\chi_x}_{t,b}\big)\big)\, dx - C_{M,\sigma}(F)\mathtt{I}(\chi) \Bigg| \leq \varepsilon \mathtt{I}(\chi),\end{aligned} \tag{5.26}$$

with

$$\begin{aligned} C_{M,\sigma}(F) := \sqrt{\frac{2}{\pi}} \int_0^M \int_0^u \mathbf{E}\Big(F^{(M)}\Big(-u, y \mapsto Z\big(ye^{-b}\big) - \zeta\big(ye^{-b}\big) \\ - \int_0^{T_{-\gamma}\wedge\sigma} \big(1 - k\big(e^{-s} y e^{-b}\big)\big)\, dB_s \\ - \int_{T_{-\gamma}\wedge\sigma}^{\sigma} \big(1 - k\big(e^{-s} y e^{-b}\big)\big)\, d\mathtt{R}_{s-T_{-\gamma}} \Big)\Big)\, d\gamma\, du. \end{aligned} \tag{5.27}$$

The proof of the following lemma is postponed until Section 5.1.

LEMMA 5.6 (Control of $F_{A,L,b}$). *For any $A, L, b > 0$, the function $F_{A,L,b}$ defined in (5.19) is b regular.*

Now we are in position to complete the proof Proposition 3.2. Indeed by combining Proposition 4.1, inequalities (5.9), (5.16), Lemma 5.6 and Theorem 5.5 we deduce that: $\forall \varepsilon > 0$ *there exist $A, L, b, M, \sigma > 0$ such that for t', $T > 0$ large enough we have*: *for any $t \geq T$, $\chi(\cdot) \in \mathcal{C}_R(t', \kappa_{\mathsf{d}} \ln t', \ln t)$,*

$$\begin{aligned} \Bigg| \mathbf{E}\Bigg[1 - \exp\Bigg(- \int_{[0,R]^{\mathsf{d}}} e^{\gamma[Y_t(x) - a_t - \chi(x)] + \mathsf{d}t}\, dx \Bigg)\Bigg] \\ - C_{M,\sigma}(F_{A,L,b}) \int_{[0,R]^{\mathsf{d}}} \chi(x) e^{-\sqrt{2\mathsf{d}}\chi(x)}\, dx \Bigg| \\ \leq \varepsilon \int_{[0,R]^{\mathsf{d}}} \chi(x) e^{-\sqrt{2\mathsf{d}}\chi(x)}\, dx. \end{aligned} \tag{5.28}$$

In addition, by Proposition 4.2, there exist $c_2 > 0$ and $t', T > 0$ large enough such that for any $t \geq T$ and $\rho(\cdot) \in \mathcal{C}_R(t', \kappa_{\mathsf{d}} \ln t', \ln t)$,

$$\begin{aligned} \mathbf{E}\Bigg[1 - \exp\Bigg(- \int_{[0,R]^{\mathsf{d}}} e^{\gamma[Y_t(x) - a_t - \chi(x)] + \mathsf{d}t}\, dx \Bigg)\Bigg] \\ \leq c_2 \int_{[0,R]^{\mathsf{d}}} \chi(x) e^{-\sqrt{2\mathsf{d}}\chi(x)}\, dx. \end{aligned} \tag{5.29}$$

For any $n > 0$, let $(A_n, L_n, b_n, M_n, \sigma_n)$ such that (5.28) is true with $\varepsilon = \frac{1}{n}$. Clearly $C_n := C_{M_n,\sigma_n}(F_{A_n,L_n,b_n}) \in [0, 2c_2]$ for any $n \in \mathbb{N}$. Let $\phi : \mathbb{N} \to \mathbb{N}$ strictly increasing such that $C_{\phi(n)} \to C(\gamma) \in [0, 2c_2]$ as $n \to \infty$.

Now we fix $\varepsilon > 0$. Let $N_0 > 0$ such that for any $n \geq N_0$, $|C_{\phi(n)} - C(\gamma)| \leq \varepsilon$. Then we choose $N_1 > N_0$ such that $n \geq N_1$ implies $\frac{1}{\phi(n)} \leq \varepsilon$. Finally, there exist [according to (5.28)] $t'(= t'(N_1))$ and $T(= T(N_1)) > 0$ such that for any $t \geq T$, $\chi(\cdot) \in \mathcal{C}_R(t', \kappa_{\mathsf{d}} \ln t', \ln t)$,

$$\begin{aligned}\bigg|\mathbf{E}\bigg[1 - \exp\bigg(-\int_{[0,R]^{\mathsf{d}}} e^{\gamma[Y_t(x)-a_t-\chi(x)]+\mathsf{d}t}\,dx\bigg)\bigg] \\ - C(\gamma)\int_{[0,R]^{\mathsf{d}}} \chi(x)e^{-\sqrt{2\mathsf{d}}\chi(x)}\,dx\bigg| \\ \leq \varepsilon\int_{[0,R]^{\mathsf{d}}} \chi(x)e^{-\sqrt{2\mathsf{d}}\chi(x)}\,dx.\end{aligned}$$

To complete the proof of Proposition 3.2, it remains to prove that $C(\gamma) > 0$. It is a consequence of Proposition 4.2. Indeed let $t' > 0$ large and $\chi \in \mathcal{C}_R(t', \kappa_d \ln t', +\infty)$ such that for any $t > T$,

$$\begin{aligned}\bigg|\mathbf{E}\bigg[1 - \exp\bigg(-\int_{[0,R]^{\mathsf{d}}} e^{\gamma[Y_t(x)-a_t-\chi(x)]+\mathsf{d}t}\,dx\bigg)\bigg] \\ - C(\gamma)\int_{[0,R]^{\mathsf{d}}} \chi(x)e^{-\sqrt{2\mathsf{d}}\chi(x)}\,dx\bigg| \\ \leq \frac{c_1}{2}\int_{[0,R]^{\mathsf{d}}} \chi(x)e^{-\sqrt{2\mathsf{d}}\chi(x)}\,dx,\end{aligned}$$

with c_1 the constant defined in Proposition 4.2. From Proposition 4.2, we have

$$\begin{aligned}\liminf_{t\to\infty}\mathbf{E}\bigg[1 - \exp\bigg(-\int_{[0,R]^{\mathsf{d}}} e^{\gamma[Y_t(x)-a_t-\chi(x)]+\mathsf{d}t}\,dx\bigg)\bigg] \\ \geq c_1\int_{[0,R]^{\mathsf{d}}} \chi(x)e^{-\sqrt{2\mathsf{d}}\chi(x)}\,dx,\end{aligned}$$

then it is plain to deduce $C(\gamma) \geq \frac{c_1}{2} > 0$, which completes the proof of Proposition 3.2.

5.1. *Proof of Lemma* 5.6. Recall the convention: for any $z \in \mathbb{R}$, $(Y_s(x))_{s\geq 0, x\in\mathbb{R}^{\mathsf{d}}}$ under $\mathbf{P}_z$ has the law $(z + Y_s(x))_{s\geq 0, x\in\mathbb{R}^{\mathsf{d}}}$ under $\mathbf{P}$.

PROOFS OF LEMMA 5.6. Fix $A, L, b > 1$, recall (5.19) for the definition of $F_{A,L,b}$. We shall prove that $F_{A,L,b}$ is b *regular* with

$$h = h_{L,b}(z) := e^{-\sqrt{2\mathsf{d}}(z+L)}\mathbf{P}_{z+L+1}\big(Y_b(0) \geq 0\big)^{1/2}, \tag{5.30}$$

$$F^* = F_b^*(g) := \sup_{z\in\mathbb{R}}\mathbf{E}_z\bigg[\frac{\mathbb{1}_{\{\exists y\in B(0,1), Y_b(y)\geq -g(ye^b)\}}}{|B(0,1)\cap\{y, Y_b(y)\geq -g(ye^b)-1/2\}|^8}\bigg]^{1/4}. \tag{5.31}$$

Check (i) is an elementary computation whereas (iii) stems from the Cauchy–Schwarz inequality. Let us start by showing that F_b^* satisfies (5.23). Let $g \in \mathcal{C}(B(0,e^b),\mathbb{R})$ such that $w^{(0,1)}_{g(\cdot e^b)}(\delta) \le \frac{1}{4}$. We define

$$\Lambda = \big|B(0,1) \cap \{y, Y_b(y) \ge -g(ye^b) - \tfrac{1}{2}\}\big|.$$

On the set $\{\exists y \in B(0,1), Y_b(y) \ge -g(ye^b)\}$, we introduce

$$\begin{aligned}\mathbf{r} = \sup\big\{&\mathbf{s}; \exists x_{\mathbf{s}} \text{ with } B(x_{\mathbf{s}},\mathbf{s}) \subset B(0,1), \exists z_{\mathbf{s}} \in B(x_{\mathbf{s}},\mathbf{s}) \text{ with } Y_b(z_{\mathbf{s}}) \ge g(z_{\mathbf{s}}e^b),\\ &\forall y \in B(x_{\mathbf{s}},\mathbf{s}), Y_b(y) \ge -g(ye^b) - \tfrac{1}{2}\big\}.\end{aligned}$$

With S the volume of the unit ball, observe that

$$\begin{aligned}F_b^*(g)^4 &= \sup_{x\in\mathbb{R}} \mathbf{E}_x\Big[\frac{\mathbb{1}_{\{\exists y\in B(0,1), Y_b(y)\ge -g(ye^b)\}}}{\Lambda^8}\Big]\\ &\le S^{-8}(e^b/\delta)^8 + \sum_{k=e^b/\delta}^{\infty} S^{-8}(k+1)^8\\ &\quad\times \sup_{x\in\mathbb{R}} \mathbf{E}_x[\mathbb{1}_{\{\exists y\in B(0,1), Y_b(y)\ge -g(ye^b)\}}\mathbb{1}_{\{S/(k+1)\le\Lambda\le S/k\}}].\end{aligned}$$

Clearly, $\Lambda \le S(\frac{1}{k})^{\mathsf{d}}$ implies $\mathbf{r} \le \frac{1}{k}$ and $\{\mathbf{r} \le \frac{1}{k} < \delta\}$ implies

$$\Big\{\sup_{\substack{x,y\in B(0,1)\\|x-y|\le 1/k}} |Y_b(x) - Y_b(y)| \ge \frac{1}{2} - w^{(0,1)}_{g(.e^b)}(\delta)\Big\}.$$

Thus by recalling that $w^{(0,1)}_{g(\cdot e^b)}(\delta) \le \frac{1}{4}$, one has

$$\begin{aligned}\mathbf{P}\Big(\mathbf{r} \le \frac{1}{k} < \delta\Big) &\le \mathbf{P}\Big(\sup_{\substack{x,y\in B(0,1)\\|x-y|\le 1/k}} |Y_b(x) - Y_b(y)| \ge \frac{1}{2} - w^{(0,1)}_{g(.e^b)}(\delta)\Big)\\ &\le \mathbf{P}\Big(\sup_{\substack{x,y\in B(0,1)\\|x-y|\le 1/k}} |Y_b(x) - Y_b(y)| \ge \frac{1}{4}\Big).\end{aligned}$$

From [17], equation (3.10) (with $h = \frac{1}{k}, m = 2k, p = 2, t' = b$ and $x = ce^{-b}k$), we have

$$\begin{aligned}\sup_{z\in\mathbb{R}} \mathbf{P}_z\Big(\sup_{\substack{x,y\in B(0,1)\\|x-y|\le 1/k}} |Y_b(x) - Y_b(y)| \ge \frac{1}{4}\Big) &= \mathbf{P}_0\Big(\sup_{\substack{x,y\in B(0,1)\\|x-y|\le 1/k}} |Y_b(x) - Y_b(y)| \ge 1/4\Big)\\ &\le c' e^{-1/c'' e^{-b}k}.\end{aligned}$$

Finally, $F_b^*(g)^4 \le S^{-8}e^{8b}/\delta^8 + \sum_{k=1+e^b/\delta}^{\infty} S^{-8}(k+1)^8 c e^{-1/c'' e^{-b}k} \le c(b)\delta^{-8}$, which suffices to prove (5.23).

Now it remains to prove (5.24). Let g_1, g_2 two continuous functions from $B(0, e^b) \to \mathbb{R}$ such that $\|g_1 - g_2\|_\infty = \delta < \frac{1}{8}$. Let us define (uniquely for this proof) $\forall g \in \mathcal{C}(B(0, e^b), \mathbb{R})$ and $\gamma \in \mathbb{R}$:

$$M(g) := \sup_{y \in B(0,1)} \big(Y_b(y) + g(ye^b)\big),$$

$$\Lambda_g(\gamma) := \big|B(0,1) \cap \{y, Y_b(y) \geq -g(ye^b) + \gamma\}\big|.$$

With the triangular inequality then twice the Cauchy–Schwarz inequality we obtain

$$\begin{aligned}
&\big|F_{A,L,b}(z, g_1) - F_{A,L,b}(z, g_2)\big| \\
&\quad \leq e^{-\sqrt{2\mathsf{d}}(z+L)} \mathbf{E}_{z+L+1}\left[\mathbb{1}_{\{Y_b(0) \geq 0\}} \left| \frac{\mathbb{1}_{\{M(g_1) \geq 1\}}}{\Lambda_{g_1}(0)} - \frac{\mathbb{1}_{\{M(g_2) \geq 1\}}}{\Lambda_{g_2}(0)} \right| \right] \\
&\qquad + h_{L,b}(z) F_b^*(g_1) \mathbf{E}_{z+L+1}\big[\big(\Delta(g_1, g_2)\big)^8\big],
\end{aligned} \tag{5.32}$$

with

$$\begin{aligned}
\Delta(g_1, g_2) &:= e^{-\int_{B(0,1)} e^{\gamma[Y_b(y) + g_1(ye^b) - 1]} \mathbb{1}_{\{Y_b(y) + A \geq 1 - g_1(ye^b)\}} \, dy} \\
&\qquad - e^{-\int_{B(0,1)} e^{\gamma[Y_b(y) + g_2(ye^b) - 1]} \mathbb{1}_{\{Y_b(y) + A \geq 1 - g_2(ye^b)\}} \, dy}.
\end{aligned} \tag{5.33}$$

Let us treat the first term of (5.32). From [20], Theorem 3.1, as $\mathrm{Var}(Y_b(y)) = b \geq 1$, $\forall y \in B(0,1)$, we can affirm that there exists $c > 0$ such that for any $\delta \in (0,1)$, $g \in \mathcal{C}(B(0, e^b), \mathbb{R})$,

$$\sup_{z \in \mathbb{R}} \mathbf{P}\big(M(g) \in [z - \delta, z + \delta]\big) \leq c\delta. \tag{5.34}$$

Thus, the first term in (5.32) is smaller than

$$\begin{aligned}
&\leq \mathbf{E}_{z+L+1}\left[\frac{\mathbb{1}_{\{Y_b(0) \geq 0, M(g_1) \in [1-\delta, 1+\delta]\}}}{\Lambda_{g_1}(0)}\right] \\
&\quad + \mathbf{E}_{z+L+1}\left[\mathbb{1}_{\{Y_b(0) \geq 0, M(g_2) \geq 1\}} \frac{\Lambda_{g_1}(-\delta) - \Lambda_{g_1}(\delta)}{\Lambda_{g_1}(0)\Lambda_{g_2}(0)}\right] \\
&:= (A) + (B).
\end{aligned}$$

By applying twice the Cauchy–Schwarz inequality to (A), we get that

$$\begin{aligned}
(A) &\leq \mathbf{P}_{z+L+1}\big(Y_b(0) \geq 0\big)^{1/2} \times \mathbf{E}_{z+L+1}\left[\frac{\mathbb{1}_{\{M(g_1) \geq 1-\delta\}}}{\Lambda_{g_1}(0)^4}\right]^{1/4} \\
&\quad \times \mathbf{P}_{z+L+1}\big(M(g_1) \in [1-\delta, 1+\delta]\big)^{1/4}.
\end{aligned}$$

Now by applying (5.34) to the last term we obtain

$$
\begin{aligned}
(A) &\le c\mathbf{P}_{z+L+1}\big(Y_b(0)\ge 0\big)^{1/2} \times \mathbf{E}_{z+L+\delta}\bigg[\frac{\mathbb{1}_{\{M(g_1)\ge 0\}}}{\Lambda_{g_1}^4(\delta-1)}\bigg]^{1/4}\delta^{1/4} \\
&\le c\|g_1-g_2\|_\infty^{1/4} h_{L,b}(z)F_b^*(g_1) \qquad \big(\text{as } \delta-1\le -\tfrac{1}{2}\big).
\end{aligned}
\tag{5.35}
$$

Similarly, observing that $\min(\Lambda_{g_1}(0),\Lambda_{g_2}(0))\ge \Lambda_{g_1}(\frac{1}{4})$, we deduce that

$$
\begin{aligned}
(B) &= \int_{B(0,1)} \mathbf{E}_{z+L+1}\bigg[\frac{\mathbb{1}_{\{Y_b(0)\ge 0, M(g_2)\ge 1\}}}{\Lambda_{g_1}(0)\Lambda_{g_2}(0)}\mathbb{1}_{\{Y_b(x)+g_1(xe^b)\in[-\delta,\delta]\}}\bigg]\,dx \\
&\le \mathbf{P}_{z+L+1}\big(Y_b(0)\ge 0\big)^{1/2}\mathbf{E}_{z+L+1}\bigg[\frac{\mathbb{1}_{\{M(g_1)\ge 1-\delta\}}}{[\Lambda_{g_1}(1/4)]^8}\bigg]^{1/4} \\
&\quad \times \int_{B(0,1)} \mathbf{P}_{z+L+1+g_1(xe^b)}\big(Y_b(x)\in[-\delta,\delta]\big)^{1/4}\,dx \\
&\le c\mathbf{P}_{z+L+1}\big(Y_b(0)\ge 0\big)^{1/2} \\
&\quad \times \mathbf{E}_{z+L+1+\delta}\bigg[\frac{\mathbb{1}_{\{\exists y\in B(0,1), Y_b(y)\ge -g_1(ye^b)+1\}}}{[\Lambda_{g_1}(1/4+\delta)]^8}\bigg]^{1/4}\delta^{1/4} \\
&\le c\|g_1-g_2\|_\infty^{1/4} h_{L,b}(z)F_b^*(g_1).
\end{aligned}
\tag{5.36}
$$

So we are done with the study of the first term of (5.32). Now we treat the second term. By the triangular inequality, $|\Delta(g_1,g_2)|$ is smaller than $(1)+(2)$ with

$$
\begin{aligned}
(1) := \bigg|&\exp\bigg(-\int_{B(0,1)} e^{\gamma[Y_b(y)+g_1(ye^b)-1]}\mathbb{1}_{\{Y_b(y)+A\ge 1-g_1(ye^b)\}}\,dy\bigg) \\
&- \exp\bigg(-\int_{B(0,1)} e^{\gamma[Y_b(y)+g_2(ye^b)-1]}\mathbb{1}_{\{Y_b(y)+A\ge 1-g_1(ye^b)\}}\,dy\bigg)\bigg|,
\end{aligned}
\tag{5.37}
$$

$$
\begin{aligned}
(2) := \bigg|&\exp\bigg(-\int_{B(0,1)} e^{\gamma[Y_b(y)+g_2(ye^b)-1]}\mathbb{1}_{\{Y_b(y)+A\ge 1-g_1(ye^b)\}}\,dy\bigg) \\
&- \exp\bigg(-\int_{B(0,1)} e^{\gamma[Y_b(y)+g_2(ye^b)-1]}\mathbb{1}_{\{Y_b(y)+A\ge 1-g_2(ye^b)\}}\,dy\bigg)\bigg|.
\end{aligned}
\tag{5.38}
$$

Recalling that $\|g_1-g_2\|_\infty = \sup_{x\in B(0,e^b)}|g_1(x)-g_2(x)| := \delta$, in (5.37) by forcing the factorization by

$$
\exp(-U*) := \exp\bigg(-\int_{B(0,1)} e^{\gamma[Y_b(y)+g_1(ye^b)-1]}\mathbb{1}_{\{Y_b(y)+A\ge 1-g_1(ye^b)\}}\,dy\bigg),
$$

we have

$$
(1) \le e^{-U^*}\big(e^{(e^{\gamma\delta}-e^{-\gamma\delta})U^*}-1\big) \le e^{\gamma\delta}-e^{-\gamma\delta}.
\tag{5.39}
$$

Similarly, by some elementary computations we get

$$\begin{aligned}(2) \leq \Big[1-\exp\Big(-\int_{B(0,1)} & e^{\gamma[Y_b(y)+g_2(ye^b)-1]} \\ & \times \mathbb{1}_{\{\min(-g_1(ye^b),-g_2(ye^b))\geq Y_b(y)+A-1\leq \max(-g_1(ye^b),-g_2(ye^b))\}}\,dy\Big)\Big] \\ \leq \int_{B(0,1)} & \mathbb{1}_{\{\min(-g_1(ye^b),-g_2(ye^b))\leq Y_b(y)+A-1\leq \max(-g_1(ye^b),-g_2(ye^b))\}}\,dy \\ \leq \int_{B(0,1)} & \mathbb{1}_{\{\min(-g_1(ye^b),-g_2(ye^b))\leq Y_b(y)+A-1\leq \min(-g_1(ye^b),-g_2(ye^b))+\delta\}}\,dy \\ & \text{(for } \delta \leq 1).\end{aligned}$$

By the Jensen inequality and recalling that $\sup_{y\in B(0,1)}\sup_{z\in\mathbb{R}}\mathbf{P}(Y_b(y)\in[z,z+\delta])\leq\frac{\delta}{\sqrt{b}}$, we deduce that the expectation of $[(1)+(2)]^8$ is smaller than $c\delta$. Combining this inequality with (5.39) yields

$$\sup_{z\in\mathbb{R}}\mathbf{E}_{z+L+1}\big[\big|\Delta(g_1,g_2)\big|^8\big]\leq c\|g_1-g_2\|_\infty. \tag{5.40}$$

Finally, by combining (5.35), 5.36) and (5.40), we deduce (5.24). □

6. Proofs for two-dimensional free fields.

6.1. *Proof of Theorem* 2.7. Before proceeding with the proof, let us make a few observations. First, we stress that the kernel k_m satisfies:

B.1 k_m is nonnegative, of class C^1 and $k(0)=1$.
B.2 k_m is Lipschitz at 0, that is, $|k_m(0)-k_m(x)|\leq C|x|$ for all $x\in\mathbb{R}^2$
B.3 k_m satisfies the integrability condition $\sup_{|e|=1}\int_1^\infty\frac{k_m(ue)}{u}\,du<+\infty$.

We stick to the notation of Section 2 so that we set for $t\geq 0$ and $x\in\mathbb{R}^{\mathsf{d}}$

$$G_{m,t}(x)=\int_1^{e^t}\frac{k_m(xu)}{u}\,du. \tag{6.1}$$

In [10], Theorem 5, it is proved that we have the following.

THEOREM 6.1. *We set $M_t^\gamma(dx)=e^{\gamma X_t(x)-(\gamma^2/2)\mathbf{E}[X_t(x)^2]}\,dx$. The family $(\sqrt{t}M_t^{\sqrt{2\mathsf{d}}})_t$ weakly converges in probability as $t\to\infty$ toward a nontrivial limit, which turns out to be the same, up to a multiplicative constant, as the limit of the derivative martingale. More precisely, we have*

$$\sqrt{t}M_t^{\sqrt{2\mathsf{d}}}(dx)\to\sqrt{\frac{2}{\pi}}M'(dx)\qquad\textit{in probability as } t\to\infty.$$

Now we begin the proof and we first treat the case when the cut-off family of the MFF is $(X_t)_t$. We will see thereafter that the general case [i.e., any other cut-off uniformly close to $(G_{m,t})_t$] is a straightforward consequence. The problem to face is that the covariance kernel of the family $(X_t)_t$ does not possess a compact support so that Theorem 2.2 does not apply as it is. We split the proof into two levels: a main level along which we explain the main steps of the proof relying on a few lemmas and a second level in which we prove these auxiliary lemmas.

Main level: Let us consider any nonnegative smooth function $\rho : \mathbb{R}^2 \to \mathbb{R}_+$ such that: $\int_{\mathbb{R}^2} \rho^2(y)\, dy = 1$, ρ is isotropic and has compact support. We set

$$\varphi(x) = \int_{\mathbb{R}^2} \rho(y+x)\rho(y)\, dy.$$

Under the above assumptions on ρ, it is plain to see that φ is nonnegative, smooth, positive definite, with compact support, $\varphi(0) = 1$ and isotropic. For each $\varepsilon > 0$, let us define the function

$$\forall x \in \mathbb{R}^2, \qquad \varphi_\varepsilon(x) = \varphi(\varepsilon x).$$

It is straightforward to check that the family $(\varphi_\varepsilon)_\varepsilon$ uniformly converges toward 1 over the compact subsets of $\mathbb{R}^2$ as $\varepsilon \to 0$. For $\varepsilon > 0$, we further define

$$k_\varepsilon(x) = k_m(x)\varphi_\varepsilon(x), \qquad K_t^\varepsilon(x) = \int_1^{e^t} \frac{k_\varepsilon(ux)}{u}\, du.$$

Observe that k_ε satisfies Assumption (A) in Section 2.1 (it is positive definite because it is the Fourier transform of the convolution of the spectrum of k_m and that of φ_ε). For each $\varepsilon > 0$, we follow Section 2 to introduce all the objects related to the kernel k_ε and add an extra superscript ε in the notation to indicate the relation to k_ε [i.e., we introduce $(X_t^\varepsilon(x))_{t,x}$, $M_t^{\prime,\varepsilon}$, $M_t^{\gamma,\varepsilon}$, $M^{\prime,\varepsilon}$].

Now we claim the following.

LEMMA 6.2. *For each $\delta > 0$, there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ and all $x \in \mathbb{R}^2$ and all $t \geq 0$:*

$$K_t^\varepsilon(x) \leq G_{m,t}(x) \leq K_t^\varepsilon(x) + \delta. \tag{6.2}$$

This lemma will help us to see the family $(X_t^\varepsilon)_t$ as a rather good approximation of the family $(X_t)_t$ as $\varepsilon \to 0$. Because k_ε satisfies Assumption (A), Theorem 2.2 holds for the family $(X_t^\varepsilon)_t$ for any $\gamma > 2$. The conclusion of this theorem involves some constant $C_\varepsilon(\gamma)$, which may depend on ε. Fortunately, we claim the following.

LEMMA 6.3. *For each fixed $\gamma > 2$, the family $(C_\varepsilon(\gamma))_\varepsilon$ converges as $\varepsilon \to 0$ toward some constant denoted by $C(\gamma)$.*

Then we use Lemmas 6.2 and 6.3 to prove the following.

LEMMA 6.4. *For any $\gamma > 2$ and for any continuous nonnegative function f with compact support, we have*

$$\lim_{t\to\infty} \mathbf{E}[\exp(-t^{(3\gamma)/4} e^{t(\gamma/\sqrt{2}-\sqrt{2})^2} M_t^\gamma(f))]$$
$$= \lim_{\varepsilon\to 0} \mathbf{E}\left[\exp\left(-C(\gamma)\int_{\mathbb{R}^2} f(x)^{2/\gamma} M'^{,\varepsilon}(dx)\right)\right].$$

It thus remains to compute the limit in the above right-hand side.

LEMMA 6.5. *For any $\gamma > 2$ and for any continuous negative function f with compact support, we have*

$$\lim_{\varepsilon\to 0} \mathbf{E}\left[\exp\left(-C(\gamma)\int_{\mathbb{R}^2} f(x)^{2/\gamma} M'^{,\varepsilon}(dx)\right)\right]$$
$$= \mathbf{E}\left[\exp\left(-C(\gamma)\int_{\mathbb{R}^2} f(x)^{2/\gamma} M'(dx)\right)\right].$$

We are now done with the proof of Theorem 2.7. It just remains to prove the four above lemmas.

PROOFS OF AUXILIARY LEMMAS. Most of the forthcoming proofs will heavily rely on Kahane's convexity inequalities (KCI for short) so that the reader is referred to Kahane's original paper [14] (or [21, 22] for an English statement).

PROOF OF LEMMA 6.2. Let us fix $\delta > 0$. In what follows and when considering an isotropic function f, we will identify, with a slight abuse of notation, the function $f : \mathbb{R}^2 \to \mathbb{R}$ with the function $f : \mathbb{R}_+ \to \mathbb{R}$ through the relation $f(y) = f(|y|)$ for $y \in \mathbb{R}^2$. Observe that

$$\begin{aligned}
|K_t^\varepsilon(x) - G_{m,t}(x)| &= \left|\int_1^{e^t} \frac{k_m(u|x|) - k_\varepsilon(u|x|)}{u}\, du\right| \\
&\le \int_1^\infty \frac{|k_m(u|x|) - k_\varepsilon(u|x|)|}{u}\, du \\
&\le \int_0^\infty \frac{|k_m(v) - k_\varepsilon(v)|}{v}\, dv.
\end{aligned}$$

We prove now that we can get the above quantities arbitrarily close to 0. We fix $R > 1$ such that $\int_R^\infty \frac{|k_m(v)|}{v}\, dv \le \delta/4$. Since $\varphi_\varepsilon(y) \le \varphi_\varepsilon(0) = 1$ (by positive definiteness), we also have

$$\int_R^\infty \frac{|k_\varepsilon(v)|}{v}\, dv \le \int_R^\infty \frac{|k_m(v)|}{v}\, dv \le \delta/4.$$

On $[1, R]$, we use the fact that the family $(\varphi_\varepsilon)_\varepsilon$ uniformly converges toward 1 over compact sets to deduce that for some ε_0 and all $\varepsilon < \varepsilon_0$, we have

$$\int_1^R \frac{|k_m(v) - k_\varepsilon(v)|}{v}\, dv \le \delta/4.$$

It remains to treat the interval $[0, 1]$. Since φ is smooth, it is locally Lipschitz at 0, meaning that we can find a constant C such that $|1 - \varphi(x)| \le C|x|$ for all x belonging to some ball centered at 0, say $B(0, 1)$. Furthermore, $|k_m(v)| \le k_m(0) = 1$. We deduce

$$\begin{aligned}\int_0^1 \frac{|k_m(v) - k_\varepsilon(v)|}{v}\, dv &= \int_0^1 \frac{|k_m(v)||1 - \varphi(\varepsilon v)|}{v}\, dv \\ &\le C\varepsilon.\end{aligned}$$

For ε small enough, this quantity can be made less than $\delta/4$. $\square$

PROOF OF LEMMA 6.5. Let us fix $\delta > 0$. Because of the convexity of the function $x \mapsto e^{-x}$ and the covariance inequality of Lemma 6.2 for ε small enough, we can apply KCI to get for all $\theta > 0$ and some standard Gaussian random variable $\mathcal{N}$ independent of everything

$$\begin{aligned}&\mathbf{E}\Big[\exp\Big(-\theta\sqrt{t}\int_{\mathbb{R}^2} f(x)^{2/\gamma} M_t^{2,\varepsilon}(dx)\Big)\Big] \\ &\quad \le \mathbf{E}\Big[\exp\Big(-\theta\sqrt{t}\int_{\mathbb{R}^2} f(x)^{2/\gamma} M_t^{2}(dx)\Big)\Big], \\ &\mathbf{E}\Big[\exp\Big(-\theta\sqrt{t}\int_{\mathbb{R}^2} f(x)^{2/\gamma} M_t^{2}(dx)\Big)\Big] \\ &\quad \le \mathbf{E}\Big[\exp\Big(-e^{\sqrt{\delta}\mathcal{N}-\delta/2}\theta\sqrt{t}\int_{\mathbb{R}^2} f(x)^{2/\gamma} M_t^{2,\varepsilon}(dx)\Big)\Big].\end{aligned}$$

By taking the limit as $t \to \infty$ and by using Theorem 6.1, we obtain for all $\theta \ge 0$

$$\begin{aligned}\mathbf{E}[\exp(-\theta M'^{,\varepsilon}(f^{2/\gamma}))] &\le \mathbf{E}[\exp(-\theta M'(f^{2/\gamma}))] \\ &\le \mathbf{E}[\exp(-\theta e^{\sqrt{\delta}\mathcal{N}-\delta/2} M'^{,\varepsilon}(f^{2/\gamma}))].\end{aligned}$$

It is then straightforward to deduce that

$$\lim_{\varepsilon\to 0} \mathbf{E}[\exp(-\theta M'^{,\varepsilon}(f^{2/\gamma}))] = \mathbf{E}[\exp(-\theta M'(f^{2/\gamma}))]. \qquad \square$$

PROOF OF LEMMAS 6.3 AND 6.4. First, recall that the family $(t^{(3\gamma)/4} \times e^{t(\gamma/\sqrt{2}-\sqrt{2})^2} M_t^\gamma(f))_t$ is tight and every possible converging limit (in law) is nontrivial [11], Section 4.3, provided that f is nontrivial.

Then, for any $\varepsilon > 0$, we have from Theorem 2.2

$$\lim_{t\to\infty} \mathbf{E}\big[\exp\big(-t^{(3\gamma)/4} e^{t(\gamma/\sqrt{2}-\sqrt{2})^2} M_t^{\gamma,\varepsilon}(f)\big)\big]$$
$$= \mathbf{E}\bigg[\exp\bigg(-C_\varepsilon(\gamma)\int_{\mathbb{R}^2} f(x)^{2/\gamma} M'^{,\varepsilon}(dx)\bigg)\bigg].$$

Furthermore, for each $\delta > 0$ and ε small enough, we have at our disposal the inequality $K_t^\varepsilon(x) \le G_{m,t}(x) \le K_t^\varepsilon(x) + \delta$ and a convex function $x \mapsto e^{-x}$. So, denoting by $\mathcal{N}$ a standard Gaussian random variable, we can apply KCI to get for all $\delta > 0$, ε large enough and all $\theta \ge 0$

$$\begin{aligned}&\mathbf{E}\big[\exp\big(-\theta t^{3\gamma/4} e^{t(\gamma/\sqrt{2}-\sqrt{2})^2} M_t^{\gamma,\varepsilon}(f)\big)\big]\\ &\quad\le \mathbf{E}\big[\exp\big(-\theta t^{(3\gamma)/4} e^{t(\gamma/\sqrt{2}-\sqrt{2})^2} M_t^{\gamma}(f)\big)\big],\end{aligned} \tag{6.3}$$

$$\begin{aligned}&\mathbf{E}\big[\exp\big(-\theta t^{3\gamma/4} e^{t(\gamma/\sqrt{2}-\sqrt{2})^2} M_t^{\gamma}(f)\big)\big]\\ &\quad\le \mathbf{E}\big[\exp\big(-\theta e^{\sqrt{\delta}\mathcal{N}-\delta/2} t^{3\gamma/4} e^{t(\gamma/\sqrt{2}-\sqrt{2})^2} M_t^{\gamma,\varepsilon}(f)\big)\big].\end{aligned} \tag{6.4}$$

Consider a possible limit Z of some subsequence of the family $(t^{3\gamma/4} \times e^{t(\gamma/\sqrt{2}-\sqrt{2})^2} M_t^{\gamma}(f))_t$. By taking the limit as $t \to \infty$ along the proper subsequence in (6.3) + (6.4), we get for all $\theta \ge 0$

$$\mathbf{E}\bigg[\exp\bigg(-\theta C_\varepsilon(\gamma)\int_{\mathbb{R}^2} f(x)^{2/\gamma} M'^{,\varepsilon}(dx)\bigg)\bigg] \le \mathbf{E}\big[\exp(-\theta Z)\big] \tag{6.5}$$

and for $\eta > 0$

$$\begin{aligned}\mathbf{E}\big[\exp(-\theta Z)\big] &\le \mathbf{E}\bigg[\exp\bigg(-\theta e^{\sqrt{\delta}\mathcal{N}-\delta/2} C_\varepsilon(\gamma)\int_{\mathbb{R}^2} f(x)^{2/\gamma} M'^{,\varepsilon}(dx)\bigg)\bigg]\\ &\le \mathbf{E}\bigg[\exp\bigg(-\theta(1-\eta) C_\varepsilon(\gamma)\int_{\mathbb{R}^2} f(x)^{2/\gamma} M'^{,\varepsilon}(dx)\bigg)\bigg]\\ &\quad + \mathbf{P}\big(e^{\sqrt{\delta}\mathcal{N}-\delta/2} \le 1-\eta\big).\end{aligned} \tag{6.6}$$

By taking the $\limsup_{\varepsilon\to 0}$, then $\liminf_{\varepsilon\to 0}$ and finally $\lim_{\delta\to 0}$, we deduce that for all $\theta \ge 0$ and $\eta > 0$

$$\limsup_{\varepsilon\to 0} \mathbf{E}\bigg[\exp\bigg(-\theta C_\varepsilon(\gamma)\int_{\mathbb{R}^2} f(x)^{2/\gamma} M'^{,\varepsilon}(dx)\bigg)\bigg] \le \mathbf{E}\big[\exp(-\theta Z)\big], \tag{6.7}$$

$$\begin{aligned}&\liminf_{\varepsilon\to 0} \mathbf{E}\bigg[\exp\bigg(-\theta C_\varepsilon(\gamma)\int_{\mathbb{R}^2} f(x)^{2/\gamma} M'^{,\varepsilon}(dx)\bigg)\bigg]\\ &\quad\ge \mathbf{E}\big[\exp\big(-\theta(1-\eta)^{-1} Z\big)\big].\end{aligned} \tag{6.8}$$

Now we can take the limit as $\eta \to 0$ and get

$$\lim_{\varepsilon\to 0} \mathbf{E}\bigg[\exp\bigg(-\theta C_\varepsilon(\gamma)\int_{\mathbb{R}^2} f(x)^{2/\gamma} M'^{,\varepsilon}(dx)\bigg)\bigg] = \mathbf{E}\big[\exp(-\theta Z)\big]. \tag{6.9}$$

Therefore, the family $(C_\varepsilon(\gamma)\int_{\mathbb{R}^2} f(x)^{2/\gamma} M'^{,\varepsilon}(dx))_\varepsilon$ converges in law toward Z. As a by-product, this shows that the law of Z does not depend on the chosen subsequence along which the family $(t^{3\gamma/4}e^{t(\gamma/\sqrt{2}-\sqrt{2})^2}M_t^\gamma(f))_t$ converges in law. Thus, the whole family converges in law toward a nontrivial random variable Z. Furthermore, Lemma 6.5 shows that the family $(\int_{\mathbb{R}^2} f(x)^{2/\gamma} M'^{,\varepsilon}(dx))_\varepsilon$ converges in law as $\varepsilon \to 0$ toward $\int_{\mathbb{R}^2} f(x)^{2/\gamma} M'(dx)$ which is almost surely strictly positive because f is not trivial. This comes from the fact that M' has full support [11]. It is then straightforward to deduce that the family $(C_\varepsilon(\gamma))_\varepsilon$ converges as $\varepsilon \to 0$. $\square$

General case: Now, we consider a general cut-off family $(X_n)_n$ of the MFF uniformly close to $(G_{m,t})_t$. By assumption, this family satisfies Lemma 6.2 with K_n instead of K_t^ε and G_{m,t_n} instead of $G_{m,t}$. We can then control the kernel K_n in terms of G_{m,t_n}. Furthermore, we now that the freezing theorem holds for the family $(G_{m,t_n})_n$ with some fixed constant $C(\gamma)$: this was the difficult part that we have handled above. Now we can use the same strategy of using KCI to transfer the freezing theorem to the family $(X_n)_n$. Details are quite the same as those we have just developed and are thus left to the reader. $\square$

6.2. *Proof of Theorem* 2.8. In what follows, $(X_t)_t$ is the family defined by (2.11) and

$$M_t^\gamma(dx) = e^{\gamma X_t - (\gamma^2/2)t}\,dx,$$

$$M'(dx) = \lim_{t\to\infty}\big(2\mathbf{E}\big[X_t(x)^2\big] - X_t(x)\big)e^{2X_t(x) - 2\mathbf{E}[X_t(x)^2]}\,dx.$$

For $t_0 > 0$, we will also consider

$$M'_{t_0,\infty}(dx) = \lim_{t\to\infty}\big(2\mathbf{E}\big[(X_t - X_{t_0})(x)^2\big] - X_t(x) + X_{t_0}(x)\big) \times e^{2(X_t - X_{t_0})(x) - 2\mathbf{E}[(X_t - X_{t_0})(x)^2]}\,dx.$$

For each $t_0 > 0$, we consider the MFF like fields [constructed in the same way as in Section 2.3 and assumed to be independent of the family $(X_t)_t$, e.g., by considering a white noise W independent of that involved in the construction of $(X_t)_t$]

$$X_{t_0,t}^{\mathrm{MFF}}(x) = \sqrt{\pi}\int_{\mathbb{R}^2\times[e^{-2t},e^{-2t_0}[} p\left(\frac{s}{2}, x, y\right) W(dy, ds)$$

with covariance kernel

$$G_{t_0,t}(x,y) = \int_{e^{-2t}}^{e^{-2t_0}} p(s,x,y)\,ds.$$

We further introduce the corresponding measures for $\gamma > 2$

$$M_{t_0,t}^{\gamma,\mathrm{MFF}}(dx) = e^{\gamma X_{t_0,t}^{\mathrm{MFF}}(x) - (\gamma^2/2)\mathbf{E}[X_{t_0,t}^{\mathrm{MFF}}(x)^2]}\,dx$$

and the derivative multiplicative chaos

$$M'^{,\mathrm{MFF}}_{t_0,\infty}(dx) = \lim_{t\to\infty}\big(2t - 2t_0 - X^{\mathrm{MFF}}_{t_0,t}(x)\big)e^{2X^{\mathrm{MFF}}_{t_0,t}(x)-2\mathbf{E}[(X^{\mathrm{MFF}}_{t_0,t}(x))^2]}\,dx.$$

The strategy that we followed to prove Theorem 2.7 for the MFF works for these MFF like fields as well

$$\tag{6.10}\begin{aligned}&\lim_{t\to\infty}\mathbf{E}\big[\exp\big(-t^{3\gamma/4}e^{(t-t_0)(\gamma/\sqrt{2}-\sqrt{2})^2}M^{\gamma,\mathrm{MFF}}_{t_0,t}(f)\big)\big]\\&\quad=\mathbf{E}\bigg[\exp\bigg(-C_{t_0}(\gamma)\int_{\mathbb{R}^2}f(x)^{2/\gamma}M'^{,\mathrm{MFF}}_{t_0,\infty}(dx)\bigg)\bigg],\end{aligned}$$

for some constant $C_{t_0}(\gamma)$, which depends on t_0. Indeed, observe that the covariance kernel of $X^{\mathrm{MFF}}_{t_0,t}$ is the same as X^{MFF}_t up to a multiplicative change of spatial coordinates so that this variation in the covariance structure should affect $C_{t_0}(\gamma)$. Actually we can even explicitly calculate this dependence.

LEMMA 6.6. *Let us set $C(\gamma) := C_{t_0=0}(\gamma)$. The constant $C_{t_0}(\gamma)$ satisfies*

$$\tag{6.11} C_{t_0}(\gamma)e^{-2t_0+(4/\gamma)t_0} = C(\gamma) \qquad \forall t_0 \geq 0.$$

PROOF. It suffices to apply (6.10) at two different scales t_0 and t_0+s. Then in the relation corresponding to t_0+s, we replace the function f by $e^{s(\gamma/\sqrt{2}-\sqrt{2})^2}e^{\gamma X^{\mathrm{MFF}}_{t_0,t_0+s}(x)-(\gamma^2/2)\mathbf{E}[X^{\mathrm{MFF}}_{t_0,t_0+s}(x)^2]}f(x)$, which remains a compactly supported continuous function. It is random but independent of the measure $M^{\gamma,\mathrm{MFF}}_{t_0+s,t}(dx)$. By identification of both limits, we get the relation $C_{t_0+s}(\gamma)\times e^{-2s+(4/\gamma)s} = C_{t_0}(\gamma)$. $\square$

Equipped with this relation, we will now try to apply the freezing theorem to a process that we call *switch process*. Basically the switch process is a Gaussian interpolation between the MFF and the GFF. We will plug this switch process in (6.10) in order to transfer by interpolation the property (6.10) to the GFF. For $t_0 \leq t$, the switch process is defined by

$$S_{t_0,t}(x) = X_{t_0}(x) + X^{\mathrm{MFF}}_{t_0,t}(x)$$

(keep in mind that this is a sum of two independent processes) and we also consider the associated measure

$$M^{\gamma,\mathrm{switch}}_{t_0,t}(dx) = e^{\gamma S_{t_0,t}(x)-(\gamma^2/2)t}\,dx.$$

To evaluate to which extent the switch process is a good interpolation between the MFF and the GFF, we need to evaluate how the covariance kernel of the switch process evolves with t_0. To this purpose, we set

$$\forall x,y\in D,\qquad G_{D,t_0,t}(x,y) = G_{D,t}(x,y) - G_{D,t_0}(x,y).$$

Consider a domain $D' \subset D$ such that $\operatorname{dist}(D', D^c) > 0$. We have

$$\lim_{t,t_0\to\infty(t_0\le t)} \sup_{x,y\in D'} \left|G_{D,t_0,t}(x,y) - G_{t_0,t}(x,y)\right| = 0. \tag{6.12}$$

This comes from the following lemma, the proof of which is postponed to the end of this subsection.

LEMMA 6.7. *For all subset D' of D such that* $\operatorname{dist}(D', D^c) > 0$, *the following convergence holds uniformly on $D' \times D'$*:

$$\lim_{t\to 0}\left|p_D(t,\cdot,\cdot) - p(t,\cdot,\cdot)\right| = 0,$$

where $p(t,x,y)$ stands for the transition densities of the whole planar Brownian motion (*i.e.*, *not killed on the boundary of D*).

Let us now tackle the interpolation procedure. By independence of X_{t_0} and $X^{\mathrm{MFF}}_{t_0,t}$, we can apply (6.10) to the function

$$f_{(6.10)}(x) = f(x)e^{t_0(\gamma/\sqrt{2}-\sqrt{2})^2} e^{\gamma X_{t_0}(x)-(\gamma^2/2)t_0}$$

and get after a straightforward calculation involving (6.11)

$$\begin{aligned}
&\lim_{t\to\infty} \mathbf{E}\left[\exp\left(-t^{(3\gamma)/4} e^{t(\gamma/\sqrt{2}-\sqrt{2})^2}\int_{\mathbb{R}^2} f(x) M^{\gamma,\mathrm{switch}}_{t_0,t}(dx)\right)\right] \\
&\quad = \mathbf{E}\left[\exp\left(-C(\gamma)\int_{\mathbb{R}^2} f(x)^{2/\gamma} e^{2X_{t_0}(x)-2t_0} M'^{,\mathrm{MFF}}_{t_0,\infty}(dx)\right)\right].
\end{aligned} \tag{6.13}$$

Let $\varepsilon > 0$ be fixed. From (6.12), we can choose T such that for all $T \le t_0 \le t$,

$$\sup_{x,y\in D'} \left|G_{D,t_0,t}(x,y) - G_{t_0,t}(x,y)\right| \le \varepsilon. \tag{6.14}$$

Let us set $g_{t_0,t}(x) = e^{(\gamma^2/2)(\mathbf{E}[(X_t(x)-X_{t_0}(x))^2]-(t-t_0))}$. From (6.14), we have $e^{-\gamma^2/2\varepsilon} \le g_{t_0,t}(x) \le e^{(\gamma^2/2)\varepsilon}$ for all $T \le t_0 \le t$. We will use this relation in the forthcoming lines. By Kahane's convexity inequalities and (6.14), we have for all $T \le t_0 \le t$

$$\begin{aligned}
&\mathbf{E}\left[\exp\left(-t^{3\gamma/4} e^{t(\gamma/\sqrt{2}-\sqrt{2})^2}\int_{\mathbb{R}^2} f(x) M^{\gamma}_t(dx)\right)\right] \\
&\quad \le \mathbf{E}\left[\exp\left(-t^{3\gamma/4} e^{t(\gamma/\sqrt{2}-\sqrt{2})^2} e^{\varepsilon^{1/2}Z-\varepsilon/2}\int_{\mathbb{R}^2} f(x) g_{t_0,t}(x) M^{\gamma,\mathrm{switch}}_{t_0,t}(dx)\right)\right] \\
&\quad \le \mathbf{E}\left[\exp\left(-t^{3\gamma/4} e^{t(\gamma/\sqrt{2}-\sqrt{2})^2} e^{\varepsilon^{1/2}Z-\varepsilon/2} e^{-(\gamma^2/2)\varepsilon}\int_{\mathbb{R}^2} f(x) M^{\gamma,\mathrm{switch}}_{t_0,t}(dx)\right)\right]
\end{aligned}$$

for some standard Gaussian random variable Z independent of everything. We just explain some subtlety: observe that the definition of M^{γ}_t does not involve a renormalization by the variance $\mathbf{E}[X_t(x)^2]$ but t instead. To apply KCI, one needs to compare measure involving a renormalization by the variance. So the function $g_{t_0,t}(x)$ appearing in the first inequality just results from the switching of variance required to apply KCI.

By taking the lim sup as $t \to \infty$ in the above relation and by using (6.13), we deduce

$$\begin{aligned}
&\limsup_{t\to\infty} \mathbf{E}\Big[\exp\Big(-t^{(3\gamma)/4} e^{t(\gamma/\sqrt{2}-\sqrt{2})^2} \int_{\mathbb{R}^2} f(x) M_t^{\gamma}(dx)\Big)\Big] \\
&\quad \le \limsup_{t\to\infty} \mathbf{E}\Big[\exp\Big(-t^{(3\gamma)/4} e^{t(\gamma/\sqrt{2}-\sqrt{2})^2} e^{\varepsilon^{1/2}Z-\varepsilon/2} e^{-\gamma^2/2\varepsilon} \\
&\qquad \times \int_{\mathbb{R}^2} f(x) M_{t_0,t}^{\gamma,\text{switch}}(dx)\Big)\Big] \\
&\quad = \mathbf{E}\Big[\exp\Big(-C(\gamma) e^{2\varepsilon^{1/2}Z/\gamma-\varepsilon/\gamma-\gamma\varepsilon} \int_{\mathbb{R}^2} f(x)^{2/\gamma} e^{2X_{t_0}(x)-2t_0} M_{t_0,\infty}^{\prime,\text{MFF}}(dx)\Big)\Big].
\end{aligned}$$

Now we want to apply once again KCI to the derivative martingale to replace the $M_{t_0,\infty}^{\prime,\text{MFF}}$ part by $M'_{t_0,\infty}$. Recall that this is possible because we know that the Seneta–Heyde norming [10] holds for both of these measures. The control of covariance kernels is provided by (6.12) (notice that the uniform control w.r.t. t is necessary to apply KCI for $t = \infty$). We get

$$\begin{aligned}
&\limsup_{t\to\infty} \mathbf{E}\Big[\exp\Big(-t^{(3\gamma)/4} e^{t(\gamma/\sqrt{2}-\sqrt{2})^2} \int_{\mathbb{R}^2} f(x) M_t^{\gamma}(dx)\Big)\Big] \\
&\quad \le \mathbf{E}\Big[\exp\Big(-C(\gamma) e^{2\varepsilon^{1/2}Z/\gamma-\varepsilon/\gamma-\gamma\varepsilon+\varepsilon^{1/2}Z'-\varepsilon/2} \\
&\qquad \times \int_{\mathbb{R}^2} f(x)^{2/\gamma} e^{2X_{t_0}(x)-2t_0} M'_{t_0,\infty}(dx)\Big)\Big],
\end{aligned} \tag{6.15}$$

for some other standard Gaussian random variable Z' independent of everything. By using the lognormal $\star$-scale invariance stated in [10], Theorem 4, we have the relation

$$e^{2X_{t_0}(x)-2\mathbf{E}[X_{t_0}]} M'_{t_0,\infty}(dx) = M'(dx),$$

hence we see that (6.15) can be reformulated as

$$\begin{aligned}
&\limsup_{t\to\infty} \mathbf{E}\Big[\exp\Big(-t^{(3\gamma)/4} e^{t(\gamma/\sqrt{2}-\sqrt{2})^2} \int_{\mathbb{R}^2} f(x) M_t^{\gamma}(dx)\Big)\Big] \\
&\quad \le \mathbf{E}\Big[\exp\Big(-C(\gamma) e^{2\varepsilon^{1/2}Z/\gamma+\varepsilon^{1/2}Z'-\varepsilon(1\gamma+\gamma+1/2)} \\
&\qquad \times \int_{\mathbb{R}^2} f(x)^{2/\gamma} e^{2\mathbf{E}[X_{t_0}]-2t_0} M'(dx)\Big)\Big].
\end{aligned} \tag{6.16}$$

By using the uniform convergence on D' of $(\mathbf{E}[X_t(x)^2]-t)_t$ as $t\to\infty$ toward the conformal radius $\ln C(x,D)$ (see [15], Lemma 6.1), we deduce

$$\begin{aligned}\limsup_{t\to\infty}\mathbf{E}\Big[\exp\Big(-t^{(3\gamma)/4}e^{t(\gamma/\sqrt{2}-\sqrt{2})^2}\int_{\mathbb{R}^2}f(x)M_t^\gamma(dx)\Big)\Big]\\ \le\mathbf{E}\Big[\exp\Big(-C(\gamma)e^{2\varepsilon^{1/2}Z/\gamma+\varepsilon^{1/2}Z'-\varepsilon(1\gamma+\gamma+1/2)}\\ \times\int_{\mathbb{R}^2}f(x)^{2/\gamma}C(x,D)^2M'(dx)\Big)\Big].\end{aligned}$$

Since ε can be chosen arbitrarily small, the upper bound for the limit in Theorem 2.8 when the cut off family $(X_t)_t$ has covariance $G_{D,t}$ is proved. The lower bound follows from a similar argument. Then we can use the same arguments as in the case of the MFF to extend the convergence to cut-off families uniformly close to $(G_{D,t})_t$.

PROOF OF LEMMA 6.7. Recall the standard formula ([19], Section 3.3)

$$\begin{aligned}\Delta(s,x,y)&:=p(s,x,y)-p_D(s,x,y)\\ &=\mathbf{E}^x\Big[1_{\{T_D^x\le s\}}\frac{1}{2\pi(s-T_D^x)}e^{-|B^x_{T_D^x}-y|^2/(2(s-T_D^x))}\Big],\end{aligned}$$

where B_t^x is a standard Brownian motion starting from x and $T_D^x=\inf\{t\ge0,B_t^x\notin D\}$. If we denote $\delta=\operatorname{dist}(D',D^c)>0$, we deduce

$$\Delta(s,x,y)\le\mathbf{E}^x\Big[1_{\{T_D^x\le s\}}\frac{1}{2\pi(s-T_D^x)}e^{-\delta^2/(2(s-T_D^x))}\Big].$$

Now observe that the mapping $u\mapsto ue^{-u}$ is decreasing for $u\ge1$. Therefore, for $s\le\delta^2/2$, we have

$$\Delta(s,x,y)\le\frac{1}{2\pi s}e^{-\delta^2/(2s)},$$

which obviously completes the proof of the lemma. □

APPENDIX: TOOLBOX OF TECHNICAL RESULTS

In this appendix, we gather some results in [10, 11, 17] in order to have a paper self-contained, at least as much as possible.

We first recall a lemma that can be found in [11] (see Lemma 16, page 17).

LEMMA A.1. *For any fixed $u\neq x$, the process $(Y_t(u))_{t\ge0}$ can be decomposed as*

$$Y_t(u)=P_t^x(u)+Z_t^x(u)-\zeta_t^x(u)\qquad\forall t>0,$$

where, for $t>0$:

– $\zeta_t^x(u) := \sqrt{2\mathsf{d}}t - \sqrt{2\mathsf{d}}\int_0^t k(e^s(x-u))\,ds$,
– $P_t^x(u) := \int_0^t k(e^s(x-u))\,dY_s(x)$ *is measurable with respect to the σ-algebra generated by* $(Y_t(x))_{t\geq 0}$,
– $(Z_t^x(u))_{t\geq 0}$ *is a centered Gaussian process independent of* $(Y_t(x))_{t\geq 0}$ *with covariance kernel*:

$$\mathbf{E}\big(Z_t^x(u)Z_{t'}^x(v)\big) := \int_0^{t\wedge t'} \big[k\big(e^s(u-v)\big) - k\big(e^s(x-u)\big)k\big(e^s(x-v)\big)\big]\,ds \quad \forall x,u,v\in\mathbb{R}^{\mathsf{d}}. \tag{A.1}$$

The following lemma can be found in [17] (we refer to the Lemmas 3.1, 3.2 and 3.3).

LEMMA A.2. *For any $\theta\in\mathbb{R}_+^*$ and $\varepsilon>0$*

$$\begin{aligned}\lim_{t',R\to\infty,(e^{t'}+1)/(R+1)\in\mathbb{N}^*} & \mathbf{P}\big(|\gamma^{-1}\ln\theta| M_{t'}^{\sqrt{2\mathsf{d}}}([0,1]^{\mathsf{d}}) \geq \varepsilon\theta^{-\sqrt{2\mathsf{d}}/\gamma}\big) \\ & + \mathbf{P}\big(M'_{t'}\big(e^{-t'}\mathrm{BZ}_{R,t'}\big) \geq \varepsilon\theta^{-\sqrt{2\mathsf{d}}/\gamma}\big) \leq \varepsilon,\end{aligned} \tag{A.2}$$

$$\begin{aligned}\lim_{t\to\infty} & \mathbf{P}\bigg(w_{Y_t(\cdot)}^{(1,1/3)}\bigg(\frac{1}{t}e^{-t}\bigg) \geq e^{t/3}\bigg) \\ & = \lim_{t\to\infty}\mathbf{P}\bigg(\sup_{x,y\in[0,e^t]^{\mathsf{d}},|x-y|\leq 1/t}\frac{|Y_t(x/e^t)-Y_t(y/e^t)|}{|x-y|^{1/3}} \geq 1\bigg) = 0,\end{aligned} \tag{A.3}$$

$$\lim_{t\to\infty}\mathbf{P}\big(\forall x\in[0,1]^{\mathsf{d}}, -10\sqrt{2\mathsf{d}}t \leq Y_t(x) \leq -\kappa_{\mathsf{d}}\ln t\big) = 1. \tag{A.4}$$

In this section, we will use the following two lemmas from [17] (see Lemma 4.2 in [17] and (4.26) in [17]).

LEMMA A.3. *We can find a constant $c_3>0$ such that for any $t'>2$ and $R\geq 1$ such that $\frac{e^{t'}+1}{R+1}\in\mathbb{N}^*$*

$$\begin{aligned}\mathbf{P}\Big(&\sup_{x\in[0,R]^{\mathsf{d}}}\sup_{s\in[\ln t',\infty)} Y_s(x) \geq \chi(x)\Big) \\ &\leq c_3\int_{[0,R]^{\mathsf{d}}}\big((\ln t')^{3/8} + \chi(x)^{3/4}\big)e^{-\sqrt{2\mathsf{d}}\chi(x)}\,dx\end{aligned} \tag{A.5}$$

for any $\chi(\cdot)\in\mathcal{C}_R(t',10,+\infty)$.

LEMMA A.4. *We can find two constants $c_4,c_5>0$ such that for any $t'\geq 2$, there exists $T(t')>0$ such that for any $L>0$, $R\geq 1$, $\chi(\cdot)\in\mathcal{C}_R(t',10,+\infty)$, $t\geq t'$*

and $a \leq \frac{t}{2}$,

$$
\begin{aligned}
&\mathbf{P}\Big(\exists x \in [0,R]^{\mathsf{d}}, \sup_{s\in[\ln t',t]} Y_s(x) \leq \chi(x), \\
&\quad \sup_{s\in[t/2,t-a]} Y_s(x) \leq a_t + \chi(x) + L - 1, \\
&\quad \sup_{[t-a,t]} Y_s(x) \in a_t + \chi(x) + L + [-1,0]\Big) \\
&\qquad \leq c_4(1+a)e^{-c_5 L} \int_{[0,R]^{\mathsf{d}}} \big(\sqrt{\ln t'} + \chi(x)\big) e^{-\sqrt{2\mathsf{d}}\chi(x)}\, dx.
\end{aligned}
\tag{A.6}
$$

Here, we reproduce [17], Proposition 4.1.

LEMMA A.5. *There exist two constants* $c_1, c_2 > 0$ *such that for any* $t' \geq 2$, *there exists* $T > 0$ *such that for any* $R \in [1, \ln t']$ *and* $t \geq T$

$$
\begin{aligned}
c_1 \int_{[0,R]^{\mathsf{d}}} \chi(x) e^{-\sqrt{2\mathsf{d}}\chi(x)}\, dx &\geq \mathbf{P}\big(\exists x \in [0,R]^{\mathsf{d}}, Y_t(x) \geq a_t + \chi(x)\big) \\
&\geq c_2 \int_{[0,R]^{\mathsf{d}}} \chi(x) e^{-\sqrt{2\mathsf{d}}\chi(x)}\, dx
\end{aligned}
\tag{A.7}
$$

for any function $\chi \in \mathcal{C}_R(t', \kappa_{\mathsf{d}} \ln t', \ln t)$.

Here, we reproduce the inequalities (B.3) and (B.6) in [17], Lemma B.2 [strictly speaking (A.8) and (A.9) are very slight extensions of (B.3) and (B.6)].

LEMMA A.6. (i) *For any* $a, t', z, j, p > 1$ *and* $\frac{t}{3} \geq a + t' + 1$,

$$
\begin{aligned}
&t^{3/2}\mathbf{P}_{-z}\Big(\sup_{s\in[\ln t',t]} B_s \leq 0, \sup_{s\in[t/2,t-a]} B_s - a_t \in [j-1,j], \\
&\quad \sup_{s\in[t-a,t]} B_s - a_t \leq j, B_t - a_t \in [-p-1,-p]\Big) \\
&\qquad \leq c_{12}\mathbf{E}_z(B_{\ln t'} \mathbb{1}_{\{B_{\ln t'} \geq 0\}})(1+j+p)a^{-1/2}.
\end{aligned}
\tag{A.8}
$$

(ii) *For any* $t', z, L, p \geq 1, t \geq t' + 1$,

$$
\begin{aligned}
&t^{3/2}\mathbf{P}_{-z}\Big(\sup_{s\in[\ln t',t]} B_s \leq 0, \sup_{s\in[t/2,t]} B_s - a_t \leq L, B_t - a_t \in [-p-1,-p]\Big) \\
&\quad \leq c_{12}\mathbf{E}_z(B_{\ln t'} \mathbb{1}_{\{B_{\ln t'} \geq 0\}})(1+L+p).
\end{aligned}
\tag{A.9}
$$

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REFERENCES

[1] ALLEZ, R., RHODES, R. and VARGAS, V. (2013). Lognormal ⋆-scale invariant random measures. *Probab. Theory Related Fields* **155** 751–788. MR3034792

[2] ARGUIN, L.-P. and ZINDY, O. (2014). Poisson–Dirichlet statistics for the extremes of a log-correlated Gaussian field. *Ann. Appl. Probab.* **24** 1446–1481. MR3211001

[3] ARGUIN, L.-P. and ZINDY, O. (2015). Poisson–Dirichlet statistics for the extremes of the two-dimensional discrete Gaussian free field. *Electron. J. Probab.* **20** no. 59, 19. MR3354619

[4] BARRAL, J., JIN, X., RHODES, R. and VARGAS, V. (2013). Gaussian multiplicative chaos and KPZ duality. *Comm. Math. Phys.* **323** 451–485. MR3096527

[5] BARRAL, J., RHODES, R. and VARGAS, V. (2012). Limiting laws of supercritical branching random walks. *C. R. Math. Acad. Sci. Paris* **350** 535–538. MR2929063

[6] BISKUP, M. and LOUIDOR, O. (2013). Extreme local extrema of the two-dimensional discrete Gaussian free field. Preprint. Available at arXiv:1306.2602.

[7] BRAMSON, M., DING, J. and ZEITOUNI, O. (2015). Convergence in law of the maximum of the two-dimensional discrete Gaussian free field. Preprint. Available at arXiv:1301.6669.

[8] CARPENTIER, D. and LE DOUSSAL, P. (2001). Glass transition of a particle in a random potential, front selection in nonlinear RG and entropic phenomena in Liouville and sinh-Gordon models. *Phys. Rev. E* **63** 026110.

[9] DERRIDA, B. and SPOHN, H. (1988). Polymers on disordered trees, spin glasses, and traveling waves. *J. Stat. Phys.* **51** 817–840. MR0971033

[10] DUPLANTIER, B., RHODES, R., SHEFFIELD, S. and VARGAS, V. (2014). Renormalization of critical Gaussian multiplicative chaos and KPZ formula. *Comm. Math. Phys.* **330** 283–330.

[11] DUPLANTIER, B., RHODES, R., SHEFFIELD, S. and VARGAS, V. (2014). Critical Gaussian multiplicative chaos: Convergence of the derivative martingale. *Ann. Probab.* **42** 1769–1808. MR3262492

[12] FYODOROV, Y., LE DOUSSAL, P. and ROSSO, A. (2009). Statistical mechanics of logarithmic REM: Duality, freezing and extreme value statistics of $1/f$ noises generated by Gaussian free fields. *J. Stat. Mech.* P10005.

[13] FYODOROV, Y. V. and BOUCHAUD, J.-P. (2008). Freezing and extreme-value statistics in a random energy model with logarithmically correlated potential. *J. Phys. A* **41** 372001, 12. MR2430565

[14] KAHANE, J.-P. (1985). Sur le chaos multiplicatif. *Ann. Sci. Math. Québec* **9** 105–150. MR0829798

[15] LACOIN, H., RHODES, R. and VARGAS, V. (2015). Complex Gaussian multiplicative chaos. *Comm. Math. Phys.* **337** 569–632. MR3339158

[16] MADAULE, T. (2015). Convergence in law for the branching random walk seen from its tip. Preprint. Available at arXiv:1107.2543v5.

[17] MADAULE, T. (2015). Maximum of a log-correlated Gaussian field. *Ann. Inst. Henri Poincaré Probab. Stat.* **51** 1369–1431.

[18] MANDELBROT, B. (1974). Multiplications aléatoires itérées et distributions invariantes par moyenne pondérée aléatoire. *C. R. Acad. Sci. Paris Sér. A* **278** 289–292. MR0431351

[19] MÖRTERS, P. and PERES, Y. (2010). *Brownian Motion*. Cambridge Univ. Press, Cambridge. MR2604525

[20] PITT, L. D. and TRAN, L. T. (1979). Local sample path properties of Gaussian fields. *Ann. Probab.* **7** 477–493. MR0528325

[21] RHODES, R. and VARGAS, V. (2014). Gaussian multiplicative chaos and applications: A review. *Probab. Surv.* **11** 315–392. MR3274356

[22] ROBERT, R. and VARGAS, V. (2010). Gaussian multiplicative chaos revisited. *Ann. Probab.* **38** 605–631. MR2642887

[23] SHEFFIELD, S. (2007). Gaussian free fields for mathematicians. *Probab. Theory Related Fields* **139** 521–541. MR2322706

[24] WEBB, C. (2011). Exact asymptotics of the freezing transition of a logarithmically correlated random energy model. *J. Stat. Phys.* **145** 1595–1619. MR2863721

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