

SEMI-DISCRETE SEMI-LINEAR PARABOLIC SPDES

BY NICOS GEORGIU¹, MATHEW JOSEPH^{1,3},
DAVAR KHOSHNEVISAN^{2,3} AND SHANG-YUAN SHIU⁴

University of Sussex and University of Utah, University of Sheffield,
University of Utah and National Central University

Consider an infinite system

$$\partial_t u_t(x) = (\mathcal{L}u_t)(x) + \sigma(u_t(x))\partial_t B_t(x)$$

of interacting Itô diffusions, started at a nonnegative deterministic bounded initial profile. We study local and global features of the solution under standard regularity assumptions on the nonlinearity σ . We will show that, locally in time, the solution behaves as a collection of independent diffusions. We prove also that the k th moment Lyapunov exponent is frequently of sharp order k^2 , in contrast to the continuous-space stochastic heat equation whose k th moment Lyapunov exponent can be of sharp order k^3 . When the underlying walk is transient and the noise level is sufficiently low, we prove also that the solution is a.s. uniformly dissipative provided that the initial profile is in $\ell^1(\mathbf{Z}^d)$.

1. Introduction.

Model and motivation. We propose to study the following system of infinitely-many interacting diffusions:

$$(SHE) \quad \frac{du_t(x)}{dt} = (\mathcal{L}u_t)(x) + \sigma(u_t(x))\frac{dB_t(x)}{dt},$$

where $t > 0$ denotes the *time* variable and $x \in \mathbf{Z}^d$ is the *space* variable. In parts of the literature, (SHE) is thought of as a *stochastic heat equation* on $(0, \infty) \times \mathbf{Z}^d$, viewed as a semi-discrete stochastic partial differential equation.

We interpret (SHE) as an infinite-dimensional system of Itô stochastic differential equations, where $\{B(x)\}_{x \in \mathbf{Z}^d}$ denotes a field of independent standard linear

Received November 2013; revised September 2014.

¹Supported in part by the NSF Grant DMS-07-47758.

²Supported in part by the NSF Grant DMS-13-07470.

³Supported in part by the NSF Grant DMS-10-06903.

⁴Supported in part by the NSC Grant 101-2115-M-008-10-MY2 and the NCU Grant 102G607-3.
MSC2010 subject classifications. Primary 60J60, 60K35, 60K37; secondary 47B80, 60H25.

Key words and phrases. The stochastic heat equation, semi-discrete stochastic heat equation, discrete space, parabolic Anderson model, Lyapunov exponents, dissipative behavior, comparison principle, interacting diffusions, BDG inequality.

Brownian motions and $\sigma : \mathbf{R} \rightarrow \mathbf{R}$ is a Lipschitz-continuous nonrandom function with

$$(1.1) \quad \sigma(0) = 0.$$

The drift operator \mathcal{L} acts on the variable x only, and is the generator of a continuous-time random walk $X := \{X_t\}_{t \geq 0} := \{\sum_{j=1}^{N_t} Z_j\}_{t \geq 0}$ on \mathbf{Z}^d where N_t is a Poisson process with jump-rate one and the Z_j 's are i.i.d. random variables with values in \mathbf{Z}^d . We consider only initial values $u_0 : \mathbf{Z}^d \rightarrow \mathbf{R}$ that are nonrandom and satisfy

$$(1.2) \quad u_0(x) \geq 0 \quad \text{for all } x \in \mathbf{Z}^d \quad \text{and} \quad 0 < \sup_{x \in \mathbf{Z}^d} u_0(x) < \infty,$$

though some of the theory developed here applies to more general initial profiles. According to Shiga and Shimizu [37], condition (1.2) ensures that the system (SHE) has an a.s.-unique solution.

Such systems have been studied extensively [12, 13, 18–22, 26, 29, 32, 36, 37], most commonly in the context of well-established models of statistical mechanics, population genetics and related models of infinitely-many interacting diffusion processes. One of the central examples of this literature is the *parabolic Anderson model* [13]—also known as *diffusion in random potential*—which is (SHE) when the function σ is linear. It is not hard to prove that, for the parabolic Anderson model, the k th moment Lyapunov exponent $\gamma_k(u)$ exists and is positive and finite for all real numbers $k \geq 2$, where

$$(1.3) \quad \gamma_k(u) := \lim_{t \rightarrow \infty} t^{-1} \log E(|u_t(x)|^k).$$

[One can prove that, because of (1.2), $\gamma_k(u)$ does not depend on x .] Jensen’s inequality readily implies that $k \mapsto \gamma_k(u)$ is nondecreasing on $[2, \infty)$. In the case that \mathcal{L} denotes the discrete Laplacian on \mathbf{Z} , for example, the theory of Carmona and Molchanov [13] implies that

$$(1.4) \quad k \mapsto \gamma_k(u) \quad \text{is strictly increasing on } [2, \infty).$$

This property is referred to as *intermittency* and suggests that the random function u develops very tall peaks that are distributed over small space–time “islands.” Section 2.4 of Bertini and Cancrini [6] and Section 7.1 of Khoshnevisan [31] describe two heuristic derivations of this “peaking behavior” from intermittency condition (1.4).

In the present nonlinear setup, the Lyapunov exponents do not generally exist. Therefore, one considers instead the (maximal) *bottom* and *top Lyapunov exponents* of the solution u to (SHE); those are, respectively, defined as

$$(1.5) \quad \begin{aligned} \underline{\gamma}_k(u) &:= \liminf_{t \rightarrow \infty} \sup_{x \in \mathbf{Z}^d} t^{-1} \log E(|u_t(x)|^k), \\ \bar{\gamma}_k(u) &:= \limsup_{t \rightarrow \infty} \sup_{x \in \mathbf{Z}^d} t^{-1} \log E(|u_t(x)|^k). \end{aligned}$$

Whenever $\underline{\gamma}_k(u) = \overline{\gamma}_k(u)$, we write $\gamma_k(u)$ for their common value and think of $\gamma_k(u)$ as the *k*th moment Lyapunov exponent of u .

In the present nonlinear setting, intermittency is defined as the property that the functions $k \mapsto k^{-1}\underline{\gamma}_k(u)$ and $k \mapsto k^{-1}\overline{\gamma}_k(u)$ are both strictly increasing on $[2, \infty)$. Because of convexity, one can establish intermittency when $\underline{\gamma}_2(u) > 0$ and $\overline{\gamma}_k(u) < \infty$ for all $k \geq 2$, [31], Proposition 7.2.

We will prove that $\overline{\gamma}_k(u)$ is generically finite for all $k \in [2, \infty)$; see Theorem 2.1. Therefore, as far as matters of intermittency are concerned, it remains to establish the positivity of the bottom Lyapunov exponent. This is a nongeneric property. To wit, when u_0 is a constant and \mathcal{L} is the discrete Laplacian on \mathbf{Z}^d , Carmona and Molchanov [13] have shown that $\gamma_2(u) > 0$ if and only if $d \in \{1, 2\}$.

This paper is concerned with various results that surround this general topic. As a first example of the type of result that we will establish, let us point out the following, which is related to the mentioned peaking property of the parabolic Anderson model: One can apply Theorem 2.1 below to the parabolic Anderson model in order to see that for all $t > 0$ there exist finite and positive constants $a(t)$ and $A(t)$ such that

$$(1.6) \quad a(t)k^2 \leq \log E(|u_t(x)|^k) \leq A(t)k^2,$$

uniformly for all $x \in \mathbf{Z}^d$ and $k \in [2, \infty)$. It is then possible to combine this bound, together with the method of Conus et al. [17], in order to estimate the size of the peaks of the solution relative to the spatial variable x . For example, if \mathcal{L} is finite range, then it is possible to prove that the tall peaks grow at all times as $\exp\{\text{const} \cdot \sqrt{\log \|x\|}\}$ for large values of $\|x\|$ and more precisely, that

$$(1.7) \quad 0 < \limsup_{\|x\| \rightarrow \infty} \frac{\log |u_t(x)|}{\sqrt{\log \|x\|}} < \infty \quad \text{a.s.}$$

We will not establish this fact since it follows fairly readily from (1.6) and the methods of [17]. Instead let us return to (SHE) in its nonlinear form.

In general, properties such as (1.6), whence (1.7), can be shown to fail. This is so, for example, when σ is bounded; see Conus et al. [17] for analogous results. Therefore, in order to prove (1.6) for (SHE) we need to impose some growth conditions on the nonlinearity σ .

Define

$$(1.8) \quad \text{Lip}_\sigma := \sup_{-\infty < a \neq b < \infty} \left| \frac{\sigma(a) - \sigma(b)}{a - b} \right|, \quad \ell_\sigma := \inf_{z \in \mathbf{R}} \left| \frac{\sigma(z)}{z} \right|.$$

Because of (1.1),

$$(1.9) \quad \ell_\sigma \leq \left| \frac{\sigma(z)}{z} \right| \leq \text{Lip}_\sigma \quad \text{for all } z \in \mathbf{R} \setminus \{0\}.$$

Suppose that $\ell_\sigma > 0$ and $\sigma(z) > 0$ for all $z > 0$. Then we will prove that (1.6) holds, and in particular the Lyapunov exponents are always positive and finite; see

Theorem 2.1. Property (1.6) contrasts sharply with the continuous-space analogues of (SHE) wherein typically the k th moment Lyapunov exponents are of sharp order k^3 as $k \rightarrow \infty$ [3, 5–7, 25] and implies that, although the intermittency peaks of semi-discrete stochastic PDEs grow rapidly with time, they grow far less rapidly than those of fully-continuous stochastic PDEs.

Not much else seems to be known about the detailed behavior of the Lyapunov exponents of the solution to (SHE): Ours seem to be the only methods that thus far have succeeded in analyzing the asymptotics of the Lyapunov exponents of the solution to (SHE) when σ is nonlinear and/or \mathcal{L} is nonlocal; Borodin and Corwin have found a few remarkable instances of (SHE) where all integer-moment Lyapunov exponents can be computed precisely.

The second nontrivial contribution of this paper concerns the local behavior of the solution to (SHE). The statement is that there frequently exists a monotone function S such that for all $t > 0$ fixed, η_τ converges to white noise on \mathbf{Z}^d as $\tau \downarrow 0$, where

$$(1.10) \quad \eta_\tau(x) := \frac{S(u_{t+\tau}(x)) - S(u_t(x))}{\sqrt{\tau}} \quad \text{for all } x \in \mathbf{Z}^d.$$

See Theorem 2.2, and especially Theorem 2.5, for more details. One can interpret our result as saying that the function S is the infinite-dimensional analogue of the scale function for a finite-dimensional diffusion. The function S is in fact also an abstract Hopf–Cole transformation for many nonlinear systems of the form (SHE); see [3, 6] for the role of the latter transformation in the continuous-space parabolic Anderson model.

Finally, we state and prove perhaps the most interesting contribution of this paper, namely that the solution to (SHE) is strongly dissipative when (SHE) is weakly disordered. See Theorem 2.7 for a precise statement. The latter theorem implies that (SHE) is a model with hysteresis; see Remark 2.8 for more details. Theorem 2.7 is also significant because it rules out the possibility of the existence of an Anderson mobility edge for the present model (SHE), when (SHE) is weakly disordered. We are aware only of one such nonexistence theorem, the original stationary Anderson model (but on “tree graphs”); see the recent paper by Aizenman and Warzel [2]. Among other things, the proof of Theorem 2.7 relies on a comparison theorem for renewal processes, which we state and prove in the Appendix [see Lemma A.2]. It is likely that our comparison theorem has other uses in applied probability as well.

2. Results. In this section we present the main results of this paper. Let us begin by making the assertions in the Introduction more precise.

THEOREM 2.1. *The nonlinear stochastic heat equation (SHE) has a solution u that is continuous in the variable t , and is unique among all predictable random fields that satisfy $\sup_{t \in [0, T]} \sup_{x \in \mathbf{Z}^d} \mathbb{E}(|u_t(x)|^2) < \infty$ for all $T > 0$. Moreover,*

$$(2.1) \quad \bar{\gamma}_k(u) \leq 8 \text{Lip}_\sigma^2 k^2 \quad \text{for all integers } k \geq 2.$$

Furthermore, $u_t(x) \geq 0$ for all $t \geq 0$ and $x \in \mathbf{Z}^d$ a.s., provided that $u_0(x) \geq 0$ for all $x \in \mathbf{Z}^d$. Finally, if $\ell_\sigma > 0$ and $\sigma(x) > 0$ for all $x > 0$, then for all $\varepsilon \in (0, 1)$,

$$(2.2) \quad \gamma_k(u) \geq (1 - \varepsilon)\ell_\sigma^2 k^2 \quad \text{for every integer } k \geq \varepsilon^{-1} + (\varepsilon\ell_\sigma^2)^{-1}.$$

Our next result shows that, at each point $x \in \mathbf{Z}^d$, the solution behaves locally in time like a Brownian motion. Standard moment methods—which we will have to reproduce here as well—show that $t \mapsto u_t(x)$ is almost surely a Hölder-continuous random function for every Hölder exponent $< \frac{1}{2}$. The following proves that the Hölder exponent $\frac{1}{2}$ is sharp.

THEOREM 2.2 (A Radon–Nikodým property). *For every $t \geq 0$ and $x \in \mathbf{Z}^d$,*

$$(2.3) \quad \lim_{\tau \downarrow 0} \frac{u_{t+\tau}(x) - u_t(x)}{B_{t+\tau}(x) - B_t(x)} = \sigma(u_t(x)) \quad \text{in probability.}$$

In addition,

$$(2.4) \quad \limsup_{\tau \downarrow 0} \frac{u_{t+\tau}(x) - u_t(x)}{\sqrt{2\tau \log \log(1/\tau)}} = - \liminf_{\tau \downarrow 0} \frac{u_{t+\tau}(x) - u_t(x)}{\sqrt{2\tau \log \log(1/\tau)}} = |\sigma(u_t(x))|,$$

almost surely.

Local iterated logarithm laws, such as (2.4), are well known in the context of finite-dimensional diffusions; see, for instance, Anderson [4], Theorem 4.1. The time-change methods employed in the finite-dimensional setting will, however, not work effectively in the present infinite-dimensional context. Here, we obtain (2.4) as a ready consequence of the proof of the “random Radon–Nikodým property” (2.3).

REMARK 2.3. Fix an $x \in \mathbf{Z}^d$ and a $t > 0$, and consider the ratio $R(\tau) := [u_{t+\tau}(x) - u_t(x)]/[B_{t+\tau}(x) - B_t(x)]$; this is a well-defined random variable for every $\tau > 0$, since $B_{t+\tau}(x) - B_t(x) \neq 0$ with probability one for every $\tau > 0$. However, $\{R(\tau)\}_{\tau > 0}$ is not a well-defined stochastic process since there exist random times $\tau > 0$ such that $B_{t+\tau}(x) - B_t(x) = 0$ a.s. Thus one does not expect that the mode of convergence in (2.3) can be improved to almost-sure convergence. This statement can be strengthened further still, but we will not do so here.

REMARK 2.4. According to (2.3), the solution to the (SHE) behaves as the noninteracting system “ $du_t(x) \approx \sigma(u_t(x)) dB_t(x)$ ” of diffusions, locally to first order. This might seem to suggest the [false] assertion that $x \mapsto u_t(x)$ ought to be a sequence of independent random variables. This is not true, as can be seen by looking more closely at the time increments of $t \mapsto u_t(x)$. In fact, our arguments can be extended to show that the spatial correlation structure of u appears at second-order

approximation levels in the sense of the following three-term stochastic Taylor expansion: in the scale $\tau^{1/2}$:

$$(2.5) \quad u_{t+\tau}(x) \simeq u_t(x) + \tau^{1/2}\sigma(u_t(x))Z_1 + \tau Z_2 + \tau^{3/2}U(\tau) \quad \text{as } \tau \downarrow 0,$$

where: (i) “ \simeq ” denotes approximation of distributions; (ii) Z_1 is a standard normal variable independent of $u_t(x)$; (iii) Z_2 is a nontrivial random variable that depends on the entire random field $\{u_s(y)\}_{s \in [0,t], y \in \mathbf{Z}^d}$; (iv) $U(\tau) = O_P(1)$ as $\tau \downarrow 0$ means that $\lim_{m \uparrow \infty} \limsup_{\tau \downarrow 0} P\{|U(\tau)| \geq m\} = 0$. In particular, (2.5) tells us that the temporally-local interactions in the random field $x \mapsto u_t(x)$ are second order in nature.

Rather than prove these refined assertions, we next turn our attention to a different local property of the solution to (SHE) and show that, after a scale change, the local-in-time behavior of the solution to (SHE) is that of spatial white noise.

THEOREM 2.5. *Suppose $\sigma(z) > 0$ for all $z \in \mathbf{R} \setminus \{0\}$, and define*

$$(2.6) \quad S(z) := \int_{z_0}^z \frac{dw}{\sigma(w)} \quad (z \geq 0),$$

where $z_0 \in \mathbf{R} \setminus \{0\}$ is a fixed number. Then, $S(u_t(x)) < \infty$ a.s. for all $t > 0$ and $x \in \mathbf{Z}^d$. Furthermore, if we choose and fix m distinct points $x_1, \dots, x_m \in \mathbf{Z}^d$, then for all $t > 0$ and $q_1, \dots, q_m \in \mathbf{R}$,

$$(2.7) \quad \lim_{\tau \downarrow 0} P\left(\bigcap_{j=1}^m \{S(u_{t+\tau}(x_j)) - S(u_t(x_j)) \leq q_j \sqrt{\tau}\}\right) = \prod_{j=1}^m \Phi(q_j),$$

where $\Phi(q) := (2\pi)^{-1/2} \int_{-\infty}^q \exp(-w^2/2) dw$ denotes the standard Gaussian cumulative distribution function.

The preceding manifests itself in interesting ways for different choices of the nonlinearity coefficient σ . Let us mention the following parabolic Anderson model, which has been a motivating example for us.

EXAMPLE 2.6. Consider the semi-discrete parabolic Anderson model, which is (SHE) with $\sigma(x) \equiv qx$ [for some fixed constant $q > 0$]. In that case, the solution to (SHE) is positive and the “scale function” S is $S(z) = q^{-1} \ln(z/z_0)$ for $z, z_0 > 0$. As such, $\sigma(u_t(x)) = qu_t(x)$ in (SHE), and we find the following log-normal limit law: For every $t > 0$ and $x_1, \dots, x_m \in \mathbf{Z}^d$ fixed,

$$(2.8) \quad \left(\left[\frac{u_{t+\tau}(x_1)}{u_t(x_1)} \right]^{1/\sqrt{\tau}}, \dots, \left[\frac{u_{t+\tau}(x_m)}{u_t(x_m)} \right]^{1/\sqrt{\tau}} \right) \Rightarrow (e^{qN_1}, \dots, e^{qN_m}) \quad \tau \downarrow 0,$$

where N_1, \dots, N_m are i.i.d. standard normal variables, and “ \Rightarrow ” denotes convergence in distribution.

Our final main result is a statement about the large-time behavior of the solution u to (SHE). We prove a rigorous version of the following assertion: “If the random walk X is transient and Lip_σ is sufficiently small—so that (SHE) is not very noisy—then a decay condition such as $u_0 \in \ell^1(\mathbf{Z}^d)$ on the initial profile is enough to ensure that $\sup_{x \in \mathbf{Z}^d} |u_t(x)| \rightarrow 0$ almost surely as $t \rightarrow \infty$.” This is new even for the parabolic Anderson model, where $\sigma(x) \propto x$ and $\mathcal{L} :=$ the generator of the simple walk on \mathbf{Z}^d . In fact, this result gives a partial [though strong] negative answer to an open problem of Carmona and Molchanov [13], page 122, and rules out the existence of [the analogue of] a nontrivial “Anderson mobility edge” in the present nonstationary setting, when $u_0 \in \ell^1(\mathbf{Z}^d)$.

Recall that $X := \{X_t\}_{t \geq 0}$ is a continuous-time random walk on \mathbf{Z}^d with generator \mathcal{L} . Let X' denote an independent copy of X , and define

$$(2.9) \quad \Upsilon(0) := \int_0^\infty \mathbb{P}\{X_t = X'_t\} dt = \mathbb{E} \int_0^\infty \mathbf{1}_{\{0\}}(X_t - X'_t) dt.$$

We can think of $\Upsilon(0)$ as the expected value of the total occupation time of $\{0\}$, as viewed by the symmetrized random walk $X - X'$. Although $\Upsilon(0)$ is always well defined, it is finite if and only if the symmetrized random walk $X - X'$ is transient [14]. We are ready to state our final result.

THEOREM 2.7 (Dissipation). *Suppose that*

$$(2.10) \quad \text{Lip}_\sigma < [\Upsilon(0)]^{-1/2},$$

and that there exists $\alpha \in (1, \infty)$ such that

$$(2.11) \quad \mathbb{P}\{X_t = X'_t\} = O(t^{-\alpha}) \quad (t \rightarrow \infty),$$

where X' denotes an independent copy of X . If, in addition, $u_0 \in \ell^1(\mathbf{Z}^d)$ and the underlying probability space is complete, then

$$(2.12) \quad \lim_{t \rightarrow \infty} \sup_{x \in \mathbf{Z}^d} |u_t(x)| = \lim_{t \rightarrow \infty} \sum_{x \in \mathbf{Z}^d} |u_t(x)|^2 = 0 \quad \text{almost surely.}$$

REMARK 2.8 (Hysteresis). Consider the parabolic Anderson model [$\sigma(x) \propto x$], where the underlying symmetrized walk $X - X'$ is transient, the noise level is small and u_0 is a constant. It is well known that under these conditions $u_t(x)$ converges weakly as $t \rightarrow \infty$ to a nondegenerate random variable $u_\infty(x)$ for every $x \in \mathbf{Z}^d$. See, for example, Greven and den Hollander [29], Theorem 1.4, Cox and Greven [19], and Shiga [36]. These results provide a partial affirmative answer to a question of Carmona and Molchanov [13], page 122, about the existence of long-term invariant laws in the low-noise regime of the transient parabolic Anderson model, in particular. By contrast, Theorem 2.7 shows that if u_0 is far from stationary (here, it decays at infinity), then the system is very strongly dissipative in the low-noise regime. This result implies that *the parabolic Anderson model remembers its initial state forever*.

EXAMPLE 2.9. Continuous-time walks that have property (2.11) include all transient finite-variance centered random walks on \mathbf{Z}^d [$d > 2$, necessarily]. For those walks, $\alpha := d/2$, thanks to the local central limit theorem. There are more interesting examples as well. For instance, suppose $t^{-1/p}X_t$ converges in distribution to a stable random variable S as $t \rightarrow \infty$; see Gnedenko and Kolmogorov [28], Section 35, for necessary and sufficient conditions. Then S is necessarily stable with index p , $p \in (0, 2]$, and $t^{-1/p}(X_t - X'_t)$ converges in law to a symmetric stable random variable S with stability index p . If, in addition, the group of all possible values of $X_t - X'_t$ generates all of \mathbf{Z}^d , then a theorem of Gnedenko [28], page 236, ensures that $t^{1/p}\mathbb{P}\{X_t = X'_t\}$ converges to $f(0) < \infty$, where f denotes the probability density function of S , as long as $p \in (0, 1)$.

Organization of the paper. In Section 3 we introduce the mild solution to (SHE) and state the version of Burkholder–Davis–Gundy inequality that we will use throughout the paper.

In Section 4 we prove Theorem 2.1: Section 4.1 includes proof of upper bounds for the Lyapunov exponents, from which existence of the solution follows; Section 4.2 contains a comparison principle for (SHE), together with proof of lower bounds for the Lyapunov exponents.

Section 5 has some results on the local (in time) behavior of the solution. These results are used in Section 6 in order to prove Theorem 2.2.

Section 7 contains a proof of Theorem 2.5. That proof hinges on Theorem 2.2 and the fact that the solution to (SHE) immediately becomes strictly positive everywhere (Proposition 7.2).

Section 8 explains how $\|u_t\|_{\ell^2(\mathbf{Z}^d)}^2$ is connected to the intersection local times of two independent continuous-time random walks with common generator \mathcal{L} . The results of Section 8 are then used in Section 9 in order to prove Theorem 2.7.

3. Preliminaries. In this small section we collect some preliminary facts about SPDEs interacting diffusion processes and BDG-type martingale inequalities. These facts are used throughout the rest of the paper.

3.1. *The mild solution.* Recall that the convolution on \mathbf{Z}^d is defined by

$$(3.1) \quad (f * g)(x) := \sum_{y \in \mathbf{Z}^d} f(x - y)g(y) \quad (x \in \mathbf{Z}^d).$$

For every function $h : \mathbf{Z}^d \rightarrow \mathbf{R}$ we define a new function \tilde{h} ,

$$(3.2) \quad \tilde{h}(x) := h(-x) \quad (x \in \mathbf{Z}^d),$$

as the *reflection* of h .

By a “solution” to (SHE) we mean a solution in integrated—or “mild”—form. That is, a predictable process $t \mapsto u_t$, with values in $\mathbf{R}^{\mathbf{Z}^d}$, that solves the following infinite system of Itô SDEs:

$$(3.3) \quad u_t(x) = (\tilde{p}_t * u_0)(x) + \sum_{y \in \mathbf{Z}^d} \int_0^t p_{t-s}(y-x) \sigma(u_s(y)) \, dB_s(y),$$

where $p_t(x) := P\{X_t = x\}$.

It might be helpful to note also that $(P_t \phi)(x) := (\tilde{p}_t * \phi)(x)$ defines the semi-group of the random walk X via the identity $(P_t \phi)(x) = E\phi(x + X_t)$. Thus we can write (3.3) in the following, perhaps more familiar, form:

$$(3.4) \quad u_t(x) = (P_t u_0)(x) + \sum_{y \in \mathbf{Z}^d} \int_0^t p_{t-s}(y-x) \sigma(u_s(y)) \, dB_s(y).$$

3.2. *A BDG inequality.* We begin this subsection with some background on Burkholder’s constants which will give us the best constants in Lemma 3.1. According to the Burkholder–Davis–Gundy inequality [8–10],

$$(3.5) \quad z_p := \sup_x \sup_{t > 0} \left[\frac{E(|x_t|^p)}{E(\langle x \rangle_t^{p/2})} \right]^{1/p} < \infty,$$

where the supremum “ \sup_x ” is taken over all nonzero martingales $x := \{x_t\}_{t \geq 0}$ that have continuous trajectories and are in $L^2(\mathbf{P})$ at all times, $\langle x \rangle_t$ denotes the quadratic variation of x at time t and $0/0 := \infty/\infty := 0$. Davis [23] has computed the numerical value of z_p in terms of zeroes of special functions. When $p \geq 2$ an integer, Davis’s theorem implies that z_p is equal to the largest positive root of the modified Hermite polynomial He_p . Thus, for example, we obtain the following from direct evaluation of the zeros:

$$(3.6) \quad \begin{aligned} z_2 &= 1, & z_3 &= \sqrt{3}, & z_4 &= \sqrt{3 + \sqrt{6}} \approx 2.334, \\ z_5 &= \sqrt{5 + \sqrt{10}} \approx 2.857, & z_6 &\approx 3.324, \dots \end{aligned}$$

It is known that $z_p \sim 2\sqrt{p}$ as $p \rightarrow \infty$, and $\sup_{p \geq 2} (z_p/\sqrt{p}) = 2$; see Carlen and Krée [11], Appendix.

Suppose $Z := \{Z_t(x)\}_{t \geq 0, x \in \mathbf{Z}^d}$ is a predictable random field, with respect to the infinite-dimensional Brownian motion $\{B_t(\bullet)\}_{t \geq 0}$, that satisfies the moment bound $E \int_0^t \|Z_s\|_{\ell^2(\mathbf{Z}^d)}^2 \, ds < \infty$. Then the Itô integral process defined by

$$(3.7) \quad \int_0^t Z_s \cdot dB_s := \sum_{y \in \mathbf{Z}^d} \int_0^t Z_s(y) \, dB_s(y) \quad (t \geq 0)$$

exists and defines a continuous $L^2(\mathbf{P})$ martingale. See, for example, Prévôt and Röckner [34]. The following variation of the Burkholder–Davis–Gundy inequality yields moment bounds for this martingale.

LEMMA 3.1 (BDG lemma). *For all finite real numbers $k \geq 2$ and $t \geq 0$,*

$$(3.8) \quad \mathbb{E} \left(\left| \int_0^t Z_s \cdot dB_s \right|^k \right) \leq \left| 4k \sum_{y \in \mathbb{Z}^d} \int_0^t \{ \mathbb{E}(|Z_s(y)|^k) \}^{2/k} ds \right|^{k/2}.$$

PROOF. We follow a method of Foondun and Khoshnevisan [25].

A standard approximation argument tells us that it suffices to consider the case where $y \mapsto Z_s(y)$ has finite support. Let $F \subset \mathbb{Z}^d$ be a finite set of cardinality $m \geq 1$, and suppose $Z_s(y) = 0$ for all $y \notin F$. Consider the (standard, finite-dimensional) Itô integral process $\int_0^t Z_s \cdot dB_s := \sum_{y \in F} \int_0^t Z_s(y) dB_s(y)$. According to Davis’s [23] form of the Burkholder–Davis–Gundy inequality m -dimensional Brownian motion [8–10],

$$(3.9) \quad \mathbb{E} \left(\left| \int_0^t Z_s \cdot dB_s \right|^k \right) \leq z_k^k \mathbb{E} \left(\left| \sum_{y \in F} \int_0^t [Z_s(y)]^2 ds \right|^{k/2} \right).$$

Finally, we use the Carlen–Krée bound $z_k \leq 2\sqrt{k}$ [11] together with the Minkowski inequality to finish the proof in the case where F is finite. A standard finite-dimensional approximation completes the proof. \square

4. Proof of Theorem 2.1.

4.1. *Bounds for the upper Lyapunov exponents.* Existence and uniqueness, and also continuity, of the solution are dealt with extensively in the literature and are well known; see, for example, Shiga and Shimizu [37] and the general theory of Prévôt and Röckner [34] for some of the latest developments. However, in order to derive our estimates of the Lyapunov exponents we will need a priori estimates which will also yield existence and uniqueness. Therefore, in this section, we hash out some—though not all—of the details.

Let us proceed by applying Picard iteration. Let $u_t^{(0)}(x) := u_0(x)$, and then define iteratively for all $n \geq 0$,

$$(4.1) \quad u_t^{(n+1)}(x) := (\tilde{p}_t * u_0)(x) + \sum_{y \in \mathbb{Z}^d} \int_0^t p_{t-s}(y-x) \sigma(u_s^{(n)}(y)) dB_s(y).$$

It follows from the properties of the Itô integral that

$$(4.2) \quad M_t^{(n+1)} := \sup_{x \in \mathbb{Z}^d} \mathbb{E}(|u_t^{(n+1)}(x)|^k) \leq 2^{k-1} \sup_{x \in \mathbb{Z}^d} (I_x + J_x),$$

where

$$(4.3) \quad \begin{aligned} I_x &:= |(\tilde{p}_t * u_0)(x)|^k, \\ J_x &:= \mathbb{E} \left(\left| \sum_{y \in \mathbb{Z}^d} \int_0^t p_{t-s}(y-x) \sigma(u_s^{(n)}(y)) dB_s(y) \right|^k \right). \end{aligned}$$

The first term is easy to bound:

$$(4.4) \quad \sup_{x \in \mathbf{Z}^d} I_x \leq \|u_0\|_{\ell^\infty(\mathbf{Z}^d)}^k,$$

since $\sum_x p_t(x) = 1$. Next we bound J_x .

Because σ is Lipschitz continuous and $\sigma(0) = 0$, we can see that $|\sigma(z)| \leq \text{Lip}_\sigma |z|$ for all $z \in \mathbf{R}$. Thus we may use the BDG lemma (Lemma 3.1) in order to see that

$$(4.5) \quad J_x^{2/k} \leq 4k \text{Lip}_\sigma^2 \sum_{y \in \mathbf{Z}^d} \int_0^t [p_{t-s}(y-x)]^2 \{E(|u_s^{(n)}(y)|^k)\}^{2/k} ds.$$

Therefore, we may recall the inductive definition (4.2) of M to see that

$$(4.6) \quad \begin{aligned} J_x^{2/k} &\leq 4k \text{Lip}_\sigma^2 \sum_{y \in \mathbf{Z}^d} \int_0^t [p_{t-s}(y-x)]^2 (M_s^{(n)})^{2/k} ds \\ &\leq 4k \text{Lip}_\sigma^2 \int_0^t (M_s^{(n)})^{2/k} ds, \end{aligned}$$

since

$$(4.7) \quad \sum_{z \in \mathbf{Z}^d} [p_r(z)]^2 = P\{X_r = X'_r\} \leq 1,$$

where X' denotes an independent copy of X . (This last bound might appear to be quite crude, and it is when r is large. However, it turns out that the behavior of r near zero matters more to us. Therefore, the inequality is tight in the regime $r \approx 0$ of interest to us.)

We may combine (4.2), (4.4) and (4.6) in order to see that for all $\beta, t > 0$,

$$(4.8) \quad \begin{aligned} e^{-\beta t} M_t^{(n+1)} &\leq 2^{k-1} \|u_0\|_{\ell^\infty(\mathbf{Z}^d)}^k + (16k \text{Lip}_\sigma^2)^{k/2} \left| \int_0^t e^{-2\beta(t-s)/k} (e^{-\beta s} M_s^{(n)})^{2/k} ds \right|^{k/2}. \end{aligned}$$

Consequently, the sequence defined by

$$(4.9) \quad N_\beta^{(m)} := \sup_{t \geq 0} (e^{-\beta t} M_t^{(m)}) \quad (m \geq 0)$$

satisfies the recursive inequality

$$(4.10) \quad \begin{aligned} N_\beta^{(n+1)} &\leq 2^{k-1} \|u_0\|_{\ell^\infty(\mathbf{Z}^d)}^k + (16k \text{Lip}_\sigma^2)^{k/2} \left| \int_0^t e^{-2\beta s/k} ds \right|^{k/2} N_\beta^{(n)} \\ &\leq 2^{k-1} \|u_0\|_{\ell^\infty(\mathbf{Z}^d)}^k + \left(\frac{8k^2 \text{Lip}_\sigma^2}{\beta} \right)^{k/2} N_\beta^{(n)}. \end{aligned}$$

In particular, if we denote (temporarily for this proof)

$$(4.11) \quad \alpha := 8(1 + \delta) \text{Lip}_\sigma^2,$$

where $\delta > 0$ is fixed but arbitrary, then

$$(4.12) \quad N_{\alpha k^2}^{(n+1)} \leq 2^{k-1} \|u_0\|_{\ell^\infty(\mathbf{Z}^d)}^k + (1 + \delta)^{-k/2} N_{\alpha k^2}^{(n)}.$$

We may apply induction on n now in order to see that $\sup_{n \geq 0} N_{\alpha k^2}^{(n)} < \infty$; equivalently, for all $k \geq 2$ there exists $c_k \in (0, \infty)$ such that

$$(4.13) \quad \sup_{x \in \mathbf{Z}^d} \mathbb{E}(|u_t^{(n)}(x)|^k) \leq c_k e^{8(1+\delta) \text{Lip}_\sigma^2 k^2 t} \quad \text{for all } t \geq 0.$$

Similarly,

$$(4.14) \quad \begin{aligned} & \mathbb{E}(|u_t^{(n+1)}(x) - u_t^{(n)}(x)|^k) \\ &= \mathbb{E}\left(\left|\sum_{y \in \mathbf{Z}^d} \int_0^t p_{t-s}(y-x) \{\sigma(u_s^{(n)}(y)) - \sigma(u_s^{(n-1)}(y))\} dB_s(y)\right|^k\right) \\ &\leq (4k \text{Lip}_\sigma^2)^{k/2} \mathbb{E}\left(\left|\sum_{y \in \mathbf{Z}^d} \int_0^t [p_{t-s}(y-x)]^2 \{u_s^{(n)}(y) - u_s^{(n-1)}(y)\}^2 ds\right|^{k/2}\right). \end{aligned}$$

Define

$$(4.15) \quad L_t^{(n+1)} := \sup_{x \in \mathbf{Z}^d} \mathbb{E}(|u_t^{(n+1)}(x) - u_t^{(n)}(x)|^k)$$

to deduce from the preceding, (4.7) and Minkowski's inequality that

$$(4.16) \quad \begin{aligned} L_t^{(n+1)} &\leq (4k \text{Lip}_\sigma^2)^{k/2} \left(\sum_{y \in \mathbf{Z}^d} \int_0^t [p_{t-s}(y-x)]^2 (L_s^{(n)})^{2/k} ds\right)^{k/2} \\ &\leq (4k \text{Lip}_\sigma^2)^{k/2} \left(\int_0^t (L_s^{(n)})^{2/k} ds\right)^{k/2}. \end{aligned}$$

Therefore,

$$(4.17) \quad K_{\alpha k^2}^{(m)} = \sup_{t \geq 0} (e^{-\alpha k^2 t} L_t^{(m)})$$

satisfies

$$(4.18) \quad \begin{aligned} K_{\alpha k^2}^{(n+1)} &\leq (4k \text{Lip}_\sigma^2)^{k/2} \left(\int_0^t e^{-2\alpha k(t-s)} ds\right)^{k/2} K_{\alpha k^2}^{(n)} \\ &\leq \left(\frac{4 \text{Lip}_\sigma^2}{2\alpha}\right)^{k/2} K_{\alpha k^2}^{(n)} \\ &\leq 2^{-k} K_{\alpha k^2}^{(n)}. \end{aligned}$$

From this we conclude that $\sum_{n=0}^\infty K_{\alpha k^2}^{(n)} < \infty$. Therefore, there exists a random field $u_t(x)$ such that $\lim_{n \rightarrow \infty} u_t^{(n)}(x) = u_t(x)$ in $L^k(\mathbb{P})$. It follows readily that u solves (SHE), and u satisfies (2.1) by (4.13) and Fatou’s lemma. Uniqueness is proved by similar means, and we skip the details.

4.2. *Bounds for the lower Lyapunov exponents.* We start the section with a truncation error estimate for the nonlinearity σ . This will be needed to use the results of Cox, Fleischmann and Greven [18] on comparison of moments for interacting diffusions. We can then reduce our problem to the case of $\sigma(x) = \ell_\sigma x$.

LEMMA 4.1. *Define $\sigma^{(N)}$ by $\sigma^{(N)} := \sigma$ on $(-N, N)$, $\sigma^{(N)} := 0$ on $[-N - 1, N + 1]^c$, and defined by linear interpolation on $[-N - 1, -N] \cup [N, N + 1]$. Let $U_t^{(N)}(x)$ denote the a.s.-unique solution to (SHE) where σ is replaced by $\sigma^{(N)}$. Then, $\lim_{N \rightarrow \infty} U_t^{(N)}(x) = u_t(x)$ a.s. and in $L^k(\mathbb{P})$ for all $k \geq 2, t \geq 0$ and $x \in \mathbb{Z}^d$.*

PROOF. Since $\sigma^{(N)}$ is Lipschitz continuous, Theorem 2.1 ensures the existence and uniqueness of $U^{(N)}$ for every $N \geq 1$. Then by (3.3)

$$(4.19) \quad u_t(x) - U_t^{(N)}(x) = T_1 + T_2,$$

where

$$(4.20) \quad \begin{aligned} T_1 &:= \sum_{y \in \mathbb{Z}^d} \int_0^t p_{t-s}(y-x) \{ \sigma(u_s(y)) - \sigma^{(N)}(u_s(y)) \} dB_s(y); \\ T_2 &:= \sum_{y \in \mathbb{Z}^d} \int_0^t p_{t-s}(y-x) \{ \sigma^{(N)}(u_s(y)) - \sigma^{(N)}(U_s^{(N)}(y)) \} dB_s(y). \end{aligned}$$

Because $|\sigma(z)| \leq \text{Lip}_\sigma |z|$, Lemma 3.1 implies that $\{E(|T_1|^k)\}^{2/k}$ is at most

$$(4.21) \quad 4k \text{Lip}_\sigma^2 \sum_{y \in \mathbb{Z}^d} \int_0^t [p_{t-s}(y-x)]^2 \{E(|u_s(y)|^k; |u_s(y)| \geq N)\}^{2/k} ds.$$

We have $E(|Y|^k; |Y| \geq N) \leq N^{-k} E(Y^{2k})$, valid for all $Y \in L^{2k}(\Omega)$. Therefore,

$$(4.22) \quad \{E(|T_1|^k)\}^{2/k} \leq \frac{4k \text{Lip}_\sigma^2}{N^2} \sum_{y \in \mathbb{Z}^d} \int_0^t [p_{t-s}(y-x)]^2 \{E(|u_s(y)|^{2k})\}^{2/k} ds.$$

Because $\sum_{y \in \mathbb{Z}^d} [p_{t-s}(y-x)]^2 \leq 1$ —see (4.7)—the already-proved bound (2.1) tells us that

$$(4.23) \quad \{E(|T_1|^k)\}^{2/k} \leq \frac{a_k}{N^2} \int_0^t e^{128 \text{Lip}_\sigma^2 ks} ds \leq \frac{A a_k e^{Akt}}{N^2},$$

where a_k and A are uninteresting finite and positive constants; moreover, a_k depends only on k . This estimates the norm of T_1 .

As for T_2 , we use the simple inequality $|\sigma^{(N)}(r) - \sigma^{(N)}(\rho)| \leq C|r - \rho|$, together with the BDG Lemma 3.1 in order to find that

$$(4.24) \quad \{E(|T_2|^k)\}^{2/k} \leq b_k \int_0^t \sup_{y \in \mathbf{Z}^d} \{E(|u_s(y) - U_s^{(N)}(y)|^k)\}^{2/k} ds,$$

where b_k is a constant dependent on σ and k . Together, the preceding moment bounds for T_1 and T_2 imply that

$$(4.25) \quad D_t^{(N)} := \sup_{x \in \mathbf{Z}^d} \{E(|u_t(x) - U_t^{(N)}(x)|^k)\}^{2/k}$$

satisfies the recursion

$$(4.26) \quad D_t^{(N)} \leq \frac{\tilde{a}_k e^{\tilde{A}k^2 t}}{N^2} + \tilde{b}_k \int_0^t D_s^{(N)} ds,$$

where \tilde{a}_k , \tilde{b}_k and \tilde{A} are positive and finite constants, and the first two depend only on k (whereas the latter is universal). An application of the Gronwall inequality shows that $\sup_{t \in [0, T]} D_t^{(N)} = O(N^{-2})$ as $N \rightarrow \infty$, for every fixed value $T \in (0, \infty)$. This is enough to yield the lemma. \square

We complete the proof of Theorem 2.1 by verifying the two remaining assertions of that theorem: (i) The solution is nonnegative because $u_0(x) \geq 0$ and $\sigma(0) = 0$; and (ii) The lower bound (2.2) for the lower Lyapunov exponent holds. We keep the two parts separate, as they use different ideas.

THEOREM 4.2 (Comparison principle). *Suppose u and v are the solutions to (SHE) with respective initial functions u_0 and v_0 . If $u_0(x) \geq v_0(x)$ for all $x \in \mathbf{Z}^d$, then $u_t(x) \geq v_t(x)$ for all $t \geq 0$ and $x \in \mathbf{Z}^d$ a.s.*

The nonnegativity assertion of Theorem 2.1 is well known [36], but also follows from the preceding comparison principle. This is because condition (1.1) implies that $v_t(x) \equiv 0$ is the unique solution to (SHE) with initial condition $v_0(x) \equiv 0$. Therefore, the comparison principle yields $u_t(x) \geq v_t(x) = 0$ a.s.

PROOF OF THEOREM 4.2. Consider the following infinite dimensional SDE:

$$(4.27) \quad w_t(x) = w_0(x) + \int_0^t (\mathcal{L}w_s)(x) ds + \int_0^t \sigma(w_s(x)) dB_s(x) \quad (x \in \mathbf{Z}^d).$$

It is a well-known fact that the mild solution to (SHE) is also a solution in the weak sense. See, for example, Theorem 3.1 of Iwata [30] and its proof. Therefore, $u_t(x)$ and $v_t(x)$, respectively, solve (4.27) with initial conditions $u_0(x)$ and $v_0(x)$.

Let $\{S_n\}_{n=1}^\infty$ denote a growing sequence of finite subsets of \mathbf{Z}^d that exhaust all of \mathbf{Z}^d . Consider, for every $n \geq 1$, the stochastic integral equation,

$$(4.28) \quad \begin{cases} u_t^{(n)}(x) = u_0(x) + \int_0^t (\mathcal{L}u_s^{(n)})(x) \, ds \\ \qquad \qquad \qquad + \int_0^t \sigma(u_s^{(n)}(x)) \, dB_s(x), & \text{if } x \in S_n; \\ u_t^{(n)}(x) = u_0(x), & \text{if } x \notin S_n. \end{cases}$$

Similarly, we let $v^{(n)}$ solve the same equation, but start it from $v_0(x)$.

Each of these equations is in fact a finite-dimensional SDE, and has a unique strong solution, by Itô’s theory. Moreover, Shiga and Shimizu’s proof of their Theorem 2.1 [37] shows that, for every $x \in \mathbf{Z}^d$ and $t > 0$, there exists a subsequence $\{n_k\}_{k=1}^\infty$ of increasing integers such that

$$(4.29) \quad u_t^{(n_k)}(x) \xrightarrow{P} u_t(x) \quad \text{and} \quad v_t^{(n_k)}(x) \xrightarrow{P} v_t(x),$$

as $k \rightarrow \infty$. Therefore, we may appeal to a comparison principle for finite-dimensional SDEs, such as that of Geiss and Manthey [27], Theorem 1.2, in order to conclude the result; the quasi-monotonicity condition of [27] is met simply because \mathcal{L} is the generator of a Markov chain. The verification of that detail is left to the interested reader. \square

We are now in position to establish the lower bound (2.2) on the lower Lyapunov exponent of the solution to (SHE).

PROOF OF THEOREM 2.1: VERIFICATION OF (2.2). Let v solve the stochastic heat equation

$$(4.30) \quad dv_t(x) = (\mathcal{L}v_t)(x) \, dt + \ell_\sigma v_t(x) \, dB_t(x),$$

subject to $v_0(x) := u_0(x)$. Also define $V^{(N)}$ to be the solution to

$$(4.31) \quad dV_t^{(N)}(x) = (\mathcal{L}v_t)(x) \, dt + \zeta^{(N)}(V_t^{(N)}(x)) \, dB_t(x),$$

where $\zeta^{(N)}(x) := \ell_\sigma x$ on $(-N, N)$, $\zeta^{(N)}(x) := 0$ when $|x| \geq N + 1$, and $\zeta^{(N)}$ is defined by linear interpolation everywhere else.

Define $\sigma^{(N)}$ and $U^{(N)}$ as in Lemma 4.1. Because $\sigma^{(N)} \geq \zeta^{(N)}$ everywhere on \mathbf{R}_+ , and since both $U^{(N)}$ and $V^{(N)}$ are ≥ 0 a.s. and pointwise, the comparison theorem of Cox, Fleischmann and Greven [18], Theorem 1, shows us that

$$(4.32) \quad E(|V_t^{(N)}(x)|^k) \leq E(|U_t^{(N)}(x)|^k),$$

for all $t \geq 0$, $x \in \mathbf{Z}^d$, $k \geq 2$ and $N \geq 1$. Let $N \rightarrow \infty$, and apply Lemma 4.1 to find that $V_t^{(N)}(x) \rightarrow v_t(x)$ and $U_t^{(N)}(x) \rightarrow u_t(x)$ in $L^k(P)$ for all $k \geq 2$. As a result, one can let $N \rightarrow \infty$ in the preceding display in order to deduce the following:

$$(4.33) \quad E(|v_t(x)|^k) \leq E(|u_t(x)|^k).$$

Therefore, it remains to bound $\underline{\gamma}_k(v)$ from below.

Let $\{X^{(i)}\}_{i=1}^k$ denote k independent copies of the random walk X . It is possible to prove that

$$(4.34) \quad \mathbb{E}(|v_t(x)|^k) = \mathbb{E}\left(\prod_{j=1}^k u_0(X_t^{(j)} + x) \cdot e^{M_k(t)}\right),$$

where $M_k(t)$ denotes the ‘‘multiple collision local time,’’

$$(4.35) \quad M_k(t) := 2\ell_\sigma^2 \sum_{1 \leq i < j \leq k} \int_0^t \mathbf{1}_{\{0\}}(X_s^{(i)} - X_s^{(j)}) \, ds.$$

When X is the continuous-time simple random walk on \mathbf{Z}^d , this is a well-known consequence of a Feynman–Kac formula; see, for instance, Carmona and Molchanov [13], page 19. When X is replaced by a Lévy process, Conus [15] has found an elegant derivation of this formula. The class of all Lévy processes includes that of continuous-time random walks, whence follows (4.34).

Note that a.s. on the event that none of the walks $X^{(1)}, \dots, X^{(k)}$ jump in the time interval $[0, t]$,

$$(4.36) \quad \prod_{j=1}^k u_0(X^{(j)} + x) e^{M_k(t)} \geq [u_0(x)]^k e^{k(k-1)\ell_\sigma^2 t}.$$

Since the probability is $\exp(-t)$ that $X^{(j)}$ does not jump in $[0, t]$, it follows from the independence of $X^{(1)}, \dots, X^{(k)}$ that

$$(4.37) \quad \mathbb{E}(|v_t(x)|^k) \geq [u_0(x)]^k \exp\{[k(k-1)\ell_\sigma^2 - k]t\}.$$

Because u_0 is not identically zero, it follows that

$$(4.38) \quad \underline{\gamma}_k(u) \geq \underline{\gamma}_k(v) \geq k(k-1)\ell_\sigma^2 - k.$$

The preceding is $\geq (1 - \varepsilon)k^2\ell_\sigma^2$ when $k \geq \varepsilon^{-1} + (\varepsilon\ell_\sigma^2)^{-1}$. This completes the proof of the theorem. \square

5. A local approximation theorem. In this section we develop a description of the local dynamics of the random field $t \mapsto u_t(\bullet)$ in the form of several approximation results.

Our first approximation lemma is a standard sample-function continuity result; it states basically that outside a single null set,

$$(5.1) \quad u_{t+\tau}(x) = u_t(x) + O(\tau^{(1+o(1))/2}) \quad \text{as } \tau \rightarrow 0, \text{ for all } t \geq 0 \text{ and } x \in \mathbf{Z}^d.$$

The result is well known, but we need to be cautious with various constants that crop up in the proof. Therefore, we include the details to account for the dependencies of the implied constants.

LEMMA 5.1. *There exists a version of u that is a.s. continuous in t with critical Hölder exponent $\geq \frac{1}{2}$. In fact, for every $T \geq 1$, $\varepsilon \in (0, 1)$ and $k \geq 2$,*

$$(5.2) \quad \sup_{x \in \mathbf{Z}^d} \sup_I \mathbb{E} \left(\sup_{\substack{s, t \in I \\ s \neq t}} \left[\frac{|u_t(x) - u_s(x)|}{|t - s|^{(1-\varepsilon)/2}} \right]^k \right) < \infty,$$

where “ \sup_I ” denotes the supremum over all closed subintervals I of $[0, T]$ that have length ≤ 1 .

PROOF. Minkowski’s inequality gives

$$(5.3) \quad [\mathbb{E}(|u_{t+\tau}(x) - u_t(x)|^k)]^{1/k} \leq |Q_1| + Q_2 + Q_3,$$

where

$$(5.4) \quad \begin{aligned} Q_1 &:= (\tilde{p}_{t+\tau} * u_0)(x) - (\tilde{p}_t * u_0)(x), \\ Q_2 &:= \left[\mathbb{E} \left(\left| \sum_{y \in \mathbf{Z}^d} \int_0^t [p_{t+\tau-s}(y-x) - p_{t-s}(y-x)] \sigma(u_s(y)) \, dB_s(y) \right|^k \right) \right]^{1/k}, \\ Q_3 &:= \left[\mathbb{E} \left(\left| \sum_{y \in \mathbf{Z}^d} \int_t^{t+\tau} p_{t+\tau-s}(y-x) \sigma(u_s(y)) \, dB_s(y) \right|^k \right) \right]^{1/k}. \end{aligned}$$

We estimate each item in turn.

Let $J_{t,t+\tau}$ denote the event that the random walk X jumps some time during the time interval $(t, t + \tau)$. Because

$$(5.5) \quad \begin{aligned} \sum_{x \in \mathbf{Z}^d} |p_{t+\tau}(x) - p_t(x)| &= \sum_{x \in \mathbf{Z}^d} |\mathbb{E}(\mathbf{1}_{\{X_{t+\tau}=x\}} - \mathbf{1}_{\{X_t=x\}}; J_{t,t+\tau})| \\ &\leq 2\mathbb{P}(J_{t,t+\tau}) = 2(1 - e^{-\tau}) \leq 2\tau, \end{aligned}$$

we obtain the following estimate for $|Q_1|$:

$$(5.6) \quad |Q_1| \leq 2\|u_0\|_{\ell^\infty(\mathbf{Z}^d)} \tau.$$

By BDG Lemma 3.1,

$$(5.7) \quad \begin{aligned} Q_2^2 &\leq 4k \sum_{y \in \mathbf{Z}^d} \int_0^t [p_{t+\tau-s}(y-x) - p_{t-s}(y-x)]^2 \{ \mathbb{E}(|\sigma(u_s(y))|^k) \}^{2/k} \, ds \\ &\leq 4k \int_0^t \mathcal{Q}(s) \sup_{y \in \mathbf{Z}^d} \{ \mathbb{E}(|\sigma(u_s(y))|^k) \}^{2/k} \, ds, \end{aligned}$$

where

$$(5.8) \quad \mathcal{Q}(s) := \sum_{z \in \mathbf{Z}^d} |p_{t+\tau-s}(z) - p_{t-s}(z)|^2 \quad (0 < s < t).$$

Note that $\mathcal{Q}(s) \leq [\sum_z |p_{t+\tau-s}(z) - p_{t-s}|]^2 \leq 4\tau^2$ uniformly for $s \in (0, t)$ from (5.5). This shows that

$$Q_2^2 \leq 16k\tau^2 \int_0^t \sup_{y \in \mathbf{Z}^d} \{E(|\sigma(u_s(y))|^k)\}^{2/k} ds.$$

Because $|\sigma(z)| \leq \text{Lip}_\sigma |z|$ for all $z \in \mathbf{R}$, the already-proved bound (2.1) tells us that there exist constants $c, c_k \in (0, \infty)$ [$k \geq 2$] such that

$$(5.9) \quad \sup_{y \in \mathbf{Z}^d} E(|\sigma(u_s(y))|^k) \leq c_k^k e^{ck^2s} \quad \text{for all integers } k \geq 2 \text{ and } s \geq 0.$$

Therefore,

$$(5.10) \quad Q_2^2 \leq 8c^{-1} c_k^2 e^{2ckt} \tau^2.$$

Finally, we apply BDG Lemma 3.1 to see that

$$(5.11) \quad \begin{aligned} Q_3^2 &\leq 4k \sum_{y \in \mathbf{Z}^d} \int_t^{t+\tau} [p_{t+\tau-s}(y-x)]^2 \{E(|\sigma(u_s(y))|^k)\}^{2/k} ds \\ &\leq 4kc_k^2 \sum_{y \in \mathbf{Z}^d} \int_t^{t+\tau} [p_{t+\tau-s}(y-x)]^2 e^{2cks} ds, \end{aligned}$$

owing to (5.9). Because $\sum_{y \in \mathbf{Z}^d} [p_h(y-x)]^2 \leq 1$ for all $h \geq 0$, we find that

$$(5.12) \quad Q_3^2 \leq 4c^{-1} c_k^2 e^{2ck(t+\tau)} \tau.$$

We combine (5.6), (5.10) and (5.12) and find that for all integers $k \geq 2$, there exists a finite and positive constant $\tilde{a} := \tilde{a}(T, k)$ such that for every $\tau \in (0, 1)$,

$$(5.13) \quad \sup_{x \in \mathbf{Z}^d} \sup_{t \in (0, T)} E(|u_{t+\tau}(x) - u_t(x)|^k) \leq \tilde{a} e^{\tilde{a}T} \tau^{k/2}.$$

The lemma follows from this bound, and an application of a quantitative form of the Kolmogorov continuity theorem [35], Theorem 2.1, page 25. We omit the remaining details, as they are nowadays standard. \square

Our next approximation result is the highlight of this section, and refines (5.1) by inspecting more closely the main contribution to the $O(\tau^{(1+o(1))/2})$ error term in (5.1). In order to describe the next approximation result, we first define for every fixed $t \geq 0$ an infinite-dimensional Brownian motion $B^{(t)}$ as follows:

$$(5.14) \quad B_\tau^{(t)}(x) := B_{\tau+t}(x) - B_t(x) \quad (x \in \mathbf{Z}^d, \tau \geq 0).$$

If we continue to hold t fixed, then it is easy to see that $\{B_\bullet^{(t)}(x)\}_{x \in \mathbf{Z}^d}$ is a collection of independent d -dimensional Brownian motions. Furthermore, the entire process $B^{(t)}$ is independent of the infinite-dimensional random variable $u_t(\bullet)$, since it is

easy to see from the proof of the first part of Theorem 2.1 that u_t is a measurable function of $\{B_s(y)\}_{s \in [0,t], y \in \mathbb{Z}^d}$, which is therefore independent of $B^{(t)}$ by the Markov property of B . Now for every fixed $t \geq 0$ and $x \in \mathbb{Z}^d$, consider the solution $u_\bullet^{(t)}(x)$ to the following (autonomous/noninteracting) Itô stochastic differential equation:

$$(5.15) \quad \begin{cases} \frac{du_\tau^{(t)}(x)}{d\tau} = \frac{d(\tilde{p}_\tau * u_t)(x)}{d\tau} + \sigma(u_\tau^{(t)}(x)) \frac{dB_\tau^{(t)}(x)}{d\tau}, \\ \text{subject to} & u_0^{(t)}(x) = u_t(x). \end{cases}$$

Note, once again, that $B^{(t)}$ is independent of $u_t(\bullet)$. Moreover,

$$(5.16) \quad \sup_{\tau > 0} E(|(\tilde{p}_\tau * u_t)(x)|^2) \leq \sup_{y \in \mathbb{Z}^d} E(|u_t(y)|^2) < \infty,$$

thanks to the already-proved bound (2.1) and the Cauchy–Schwarz inequality. Therefore, (5.15) is a standard Itô-type SDE and hence has a unique strong solution.

THEOREM 5.2 (The local-diffusion property). *For every $t \geq 0$, the following holds a.s. for all $x \in \mathbb{Z}^d$:*

$$(5.17) \quad u_{t+\tau}(x) = u_\tau^{(t)}(x) + O(\tau^{(3/2)+o(1)}) \quad \text{as } \tau \downarrow 0.$$

The proof of Theorem 5.2 hinges on three technical lemmas that we state next.

LEMMA 5.3. *Choose and fix $t \geq 0$, $\tau \in [0, 1]$, and $x \in \mathbb{Z}^d$, and define*

$$(5.18) \quad \begin{aligned} \mathcal{A} &:= \sum_{y \in \mathbb{Z}^d} \int_t^{t+\tau} p_{t+\tau-s}(y-x) \sigma(u_s(y)) dB_s(y), \\ \mathcal{B} &:= \int_t^{t+\tau} \sigma(u_s(x)) dB_s(x). \end{aligned}$$

Then, for all real numbers $k \geq 2$ there exist a finite constant $C_k > 0$ —depending on k but not on (t, τ, x) —and a finite constant $C > 0$ —not depending on (t, τ, x, k) —such that

$$(5.19) \quad E(|\mathcal{A} - \mathcal{B}|^k) \leq C_k e^{Ck^2(t+1)} \tau^{3k/2}.$$

LEMMA 5.4. *For every $k \geq 2$ and $T \geq 1$, there exists a finite constant $C(k, T)$ such that for every $\tau \in (0, 1]$,*

$$(5.20) \quad \sup_{t \in [0, T]} \sup_{x \in \mathbb{Z}^d} E(|u_{t+\tau}(x) - u_\tau^{(t)}(x)|^k) \leq C(k, T) \tau^{3k/2}.$$

LEMMA 5.5. *There exists a version of $u^{(\bullet)}$ that is a.s. continuous in (t, τ) . Moreover, for every $T \geq 1$, $\varepsilon \in (0, 1)$ and $k \geq 2$,*

$$(5.21) \quad \sup_{t \in [0, T]} \sup_{x \in \mathbf{Z}^d} \sup_I \sup_{\substack{v, \mu \in I \\ v \neq \mu}} \mathbb{E} \left(\sup_{v, \mu \in I} \left[\frac{|u_v^{(t)}(x) - u_\mu^{(t)}(x)|}{|v - \mu|^{(1-\varepsilon)/2}} \right]^k \right) < \infty,$$

where “ \sup_I ” denotes the supremum over all closed subintervals I of $[0, T]$ that have length ≤ 1 .

In order to maintain the flow of the discussion, we prove Theorem 5.2 first. Then we conclude this section by establishing the three supporting lemmas mentioned above.

PROOF OF THEOREM 5.2. Throughout the proof we choose and fix some $t \in [0, T]$ and $x \in \mathbf{Z}^d$.

Our plan is to prove that for all $\delta \in (0, \frac{1}{2})$,

$$(5.22) \quad u_{t+\tau}(x) - u_\tau^{(t)}(x) = O(\tau^{(3/2)-\delta}) \quad \text{as } \tau \downarrow 0, \text{ a.s.}$$

Henceforth, we choose and fix some $\delta \in (0, \frac{1}{2})$, and denote by A_k, A'_k, A''_k , etc. finite constants that depend only on a parameter $k \geq 2$ that will be selected later, during the course of the proof.

Thanks to Lemma 5.4, for all $k \geq 2$ and $\tau \in [0, 1]$,

$$(5.23) \quad \mathbb{P}\{|u_{t+\tau}(x) - u_\tau^{(t)}(x)| \geq \frac{1}{3}\tau^{(3/2)-\delta}\} \leq C(k, T)\tau^{\delta k}.$$

We can choose k large enough and then apply the Borel–Cantelli lemma in order to deduce that with probability one,

$$(5.24) \quad |u_{t+\tau_n}(x) - u_{\tau_n}^{(t)}(x)| < \tau_n^{(3/2)-\delta} \quad \text{for all but a finite number of } n\text{'s,}$$

where $\tau_n := n^{\delta-(1/2)}$. Because $\tau_n - \tau_{n+1} \sim \text{const} \times n^{-1}\tau_n$ as $n \rightarrow \infty$, Hölder continuity ensures the following (Lemmas 5.1 and 5.5): Uniformly for all $\tau \in [\tau_{n+1}, \tau_n]$,

$$(5.25) \quad \begin{aligned} |u_{t+\tau}(x) - u_{t+\tau_n}(x)| + |u_{\tau_n}^{(t)}(x) - u_\tau^{(t)}(x)| &= O([\tau_n/n]^{(1/2)-\delta}) \quad \text{a.s.} \\ &= O(\tau_n^{(3/2)-\delta}), \end{aligned}$$

by the particular choice of the sequence $\{\tau_n\}_{n=1}^\infty$. The preceding two displays can now be combined to imply (5.17). \square

PROOF OF LEMMA 5.3. We may rewrite \mathcal{B} as follows:

$$(5.26) \quad \mathcal{B} = \sum_{y \in \mathbf{Z}^d} \int_t^{t+\tau} \mathbf{1}_{\{0\}}(y-x)\sigma(u_s(y)) \, dB_s(y).$$

Therefore, BDG Lemma 3.1 can be used to show that

$$\begin{aligned}
 & \{E(|\mathcal{A} - \mathcal{B}|^k)\}^{2/k} \\
 & \leq 4k \sum_{y \in \mathbf{Z}^d} \int_t^{t+\tau} [p_{t+\tau-s}(y-x) - \mathbf{1}_{\{0\}}(y-x)]^2 \{E(|\sigma(u_s(y))|^k)\}^{2/k} ds \\
 (5.27) \quad & \leq 4kc_k^2 e^{2ck(t+1)} \left(\sum_{y \in \mathbf{Z}^d \setminus \{0\}} \int_0^\tau [p_s(y)]^2 ds + \int_0^\tau [1 - p_s(0)]^2 ds \right) \\
 & \leq 4kc_k^2 e^{2ck(t+1)} 2 \int_0^\tau [1 - p_s(0)]^2 ds,
 \end{aligned}$$

where c, c_k appear in (5.9). Observe that $p_s(0) = P\{X_s = 0\} \geq P\{N_s = 0\} = e^{-s}$, where $\{N_s\}_{s \geq 0}$ denotes the underlying Poisson clock. Therefore, we obtain $\int_0^\tau [1 - p_s(0)]^2 ds \leq (1/3)\tau^3$, and hence

$$(5.28) \quad E(|\mathcal{A} - \mathcal{B}|^k) \leq (8/3)^{k/2} k^{k/2} c_k^k e^{ck^2(t+1)} \tau^{3k/2}.$$

This implies the lemma. \square

PROOF OF LEMMA 5.4. In accord with (3.3), we may write $u_{t+\tau}(x)$ as

$$(5.29) \quad (\tilde{p}_{t+\tau} * u_0)(x) + \sum_{y \in \mathbf{Z}^d} \int_0^t p_{t+\tau-s}(y-x) \sigma(u_s(y)) dB_s(y) + \mathcal{A},$$

where \mathcal{A} was defined in Lemma 5.3.

By the Chapman–Kolmogorov property of the transition functions $\{p_t\}_{t \geq 0}$,

$$\begin{aligned}
 (5.30) \quad & (\tilde{p}_\tau * u_t)(x) \\
 & = (\tilde{p}_{t+\tau} * u_0)(x) + \sum_{y \in \mathbf{Z}^d} \int_0^t p_{t+\tau-s}(y-x) \sigma(u_s(y)) dB_s(y).
 \end{aligned}$$

The exchange of summation with stochastic integration can be justified, using the already-proved moment bound (2.1) of Theorem 2.1; we omit the details. Instead, let us apply this in (5.29) to see that

$$\begin{aligned}
 (5.31) \quad & u_{t+\tau}(x) = (\tilde{p}_\tau * u_t)(x) + \int_t^{t+\tau} \sigma(u_s(x)) dB_s(x) + (\mathcal{A} - \mathcal{B}) \\
 & = (\tilde{p}_\tau * u_t)(x) + \int_0^\tau \sigma(u_{t+s}(x)) d_s B_s^{(t)}(x) + (\mathcal{A} - \mathcal{B}).
 \end{aligned}$$

Lemma 5.3 implies that for all $k \geq 2, t, \tau \geq 0$ and $x \in \mathbf{Z}^d$,

$$\begin{aligned}
 (5.32) \quad & E\left(\left| u_{t+\tau}(x) - (\tilde{p}_\tau * u_t)(x) - \int_0^\tau \sigma(u_{t+s}(x)) d_s B_s^{(t)}(x) \right|^k \right) \\
 & \leq a_k e^{ak^2(t+1)} \tau^{3k/2},
 \end{aligned}$$

where $a \in (0, \infty)$ is universal and $a_k \in (0, \infty)$ depends only on k . On the other hand,

$$(5.33) \quad u_\tau^{(t)}(x) - (\tilde{p}_\tau * u_t)(x) - \int_0^\tau \sigma(u_s^{(t)}(x)) dB_s^{(t)}(x) = 0 \quad \text{a.s.},$$

by the very definition of $u^{(t)}$, and thanks to the fact that $u_0^{(t)}(y) = u_t(y)$. The preceding two displays and Minkowski’s inequality that

$$(5.34) \quad \psi(\tau) := \{E(|u_{t+\tau}(x) - u_\tau^{(t)}(x)|^k)\}^{1/k} \leq a_k^{1/k} e^{ak(t+1)} \tau^{3/2} + Q,$$

where

$$(5.35) \quad Q := \left\{ E \left(\left| \int_0^\tau [\sigma(u_{t+s}(x)) - \sigma(u_s^{(t)}(x))] d_s B_s^{(t)}(x) \right|^k \right) \right\}^{1/k}.$$

According to BDG Lemma 3.1 (actually we need a one-dimensional version of that lemma only), and since $|\sigma(r) - \sigma(\rho)| \leq \text{Lip}_\sigma |r - \rho|$,

$$(5.36) \quad \begin{aligned} Q^2 &\leq 4k \text{Lip}_\sigma^2 \int_0^\tau \{E(|u_{t+s}(x) - u_s^{(t)}(x)|^k)\}^{2/k} ds \\ &= 4k \text{Lip}_\sigma^2 \int_0^\tau [\psi(s)]^2 ds. \end{aligned}$$

Thus we find that

$$(5.37) \quad [\psi(\tau)]^2 \leq 2a_k^{2/k} e^{2ak(t+1)} \tau^3 + 8k \text{Lip}_\sigma^2 \int_0^\tau [\psi(s)]^2 ds$$

for all $0 \leq \tau \leq 1$.

The lemma follows from this and an application of Gronwall’s lemma. \square

PROOF OF LEMMA 5.5. One can model closely a proof after that of Lemma 5.1. However, we omit the details, since this is a result about finite-dimensional diffusions and as such simpler than Lemma 5.1. \square

We conclude this section with a final approximation lemma. The next assertion shows that the solution to (SHE) depends continuously on its initial function (in a suitable topology).

LEMMA 5.6. *Let u and v denote the unique solutions to (SHE), corresponding, respectively, to initial functions u_0 and v_0 . Then*

$$(5.38) \quad \sup_{x \in \mathbf{Z}^d} E(|u_t(x) - v_t(x)|^2) \leq \|u_0 - v_0\|_{\ell^\infty(\mathbf{Z}^d)}^2 e^{\text{Lip}_\sigma^2 t} \quad \text{for all } t \geq 0.$$

PROOF. Choose and fix $t \geq 0$. The fact that $\sum_{y \in \mathbf{Z}^d} p_t(y) = 1$ alone ensures that

$$(5.39) \quad \sup_{x \in \mathbf{Z}^d} |(\tilde{p}_t * u_0)(x) - (\tilde{p}_t * v_0)(x)| \leq \|u_0 - v_0\|_{\ell^\infty(\mathbf{Z}^d)}.$$

Therefore, (3.3) and Itô's isometry together imply that

$$(5.40) \quad \begin{aligned} & \mathbb{E}(|u_t(x) - v_t(x)|^2) \\ & \leq \|u_0 - v_0\|_{\ell^\infty(\mathbf{Z}^d)}^2 \\ & \quad + \text{Lip}_\sigma^2 \int_0^t \|p_s\|_{\ell^2(\mathbf{Z}^d)}^2 \cdot \sup_{y \in \mathbf{Z}^d} \mathbb{E}(|u_s(y) - v_s(y)|^2) ds. \end{aligned}$$

Since $\|p_s\|_{\ell^2(\mathbf{Z}^d)}^2 = \mathbb{P}\{X_s = X'_s\} \leq 1$, where X' is an independent copy of X , we may conclude that $f(t) := \sup_{x \in \mathbf{Z}^d} \mathbb{E}(|u_t(x) - v_t(x)|^2)$ satisfies

$$(5.41) \quad f(t) \leq \|u_0 - v_0\|_{\ell^\infty(\mathbf{Z}^d)}^2 + \text{Lip}_\sigma^2 \int_0^t f(s) ds.$$

Therefore, the lemma follows from Gronwall's inequality. \square

6. Proof of Theorem 2.2. Theorem 2.2 is a consequence of the following result.

PROPOSITION 6.1. *For every $t \geq 0$, the following holds a.s. for all $x \in \mathbf{Z}^d$:*

$$(6.1) \quad u_{t+\tau}(x) - u_t(x) = \sigma(u_t(x))\{B_{t+\tau}(x) - B_t(x)\} + o(\tau^{1+o(1)})$$

as $\tau \downarrow 0$.

Indeed, we obtain (2.3) from this proposition, simply because well-known properties of Brownian motion imply that for all $\varepsilon \in (0, \frac{1}{2})$ and $t \geq 0$,

$$(6.2) \quad \lim_{\tau \downarrow 0} \frac{\tau^{1-\varepsilon}}{B_{t+\tau}(x) - B_t(x)} = 0 \quad \text{in probability.}$$

Moreover, (2.4) follows from the local law of the iterated logarithm for Brownian motion. It remains to prove Proposition 6.1.

PROOF. According to (5.32), for every integer $k \geq 2$, and all $t, \tau \geq 0$ and $x \in \mathbf{Z}^d$,

$$(6.3) \quad \begin{aligned} & \mathbb{E}\left(\left|u_{t+\tau}(x) - u_t(x) - \int_0^\tau \sigma(u_{t+s}(x)) d_s B_s^{(t)}(x)\right|^k\right) \\ & \leq 2^{k-1} [a_k e^{ak^2(t+1)} \tau^{3k/2} + \mathbb{E}(|u_t(x) - (\tilde{p}_\tau * u_t)(x)|^k)]. \end{aligned}$$

We may write

$$\begin{aligned}
 & \mathbb{E}(|u_t(x) - (\tilde{p}_\tau * u_t)(x)|^k) \\
 (6.4) \quad &= \mathbb{E}\left(\left|u_t(x) - \sum_{y \in \mathbb{Z}^d} p_\tau(y-x)u_t(y)\right|^k\right) \\
 &= \mathbb{E}\left(\left|u_t(x)\mathbb{P}\{X_\tau \neq 0\} - \sum_{y \in \mathbb{Z}^d \setminus \{x\}} p_\tau(y-x)u_t(y)\right|^k\right).
 \end{aligned}$$

Because $\mathbb{P}\{X_\tau \neq 0\} = 1 - \exp(-\tau) \leq \tau$, Minkowski’s inequality shows that

$$\begin{aligned}
 & \{\mathbb{E}(|u_t(x) - (\tilde{p}_\tau * u_t)(x)|^k)\}^{1/k} \\
 (6.5) \quad &\leq \tau \{\mathbb{E}(|u_t(x)|^k)\}^{1/k} + \sum_{y \in \mathbb{Z}^d \setminus \{x\}} p_\tau(y-x) \{\mathbb{E}(|u_t(y)|^k)\}^{1/k} \\
 &\leq 2\tau \sup_{y \in \mathbb{Z}^d} \{\mathbb{E}(|u_t(y)|^k)\}^{1/k}.
 \end{aligned}$$

We can conclude from this development and from Theorem 2.1 that there exists $A_k < \infty$, depending only on k , and a universal $A < \infty$ such that

$$\begin{aligned}
 & \mathbb{E}\left(\left|u_{t+\tau}(x) - u_t(x) - \int_0^\tau \sigma(u_{t+s}(x)) \, d_s B_s^{(t)}(x)\right|^k\right) \\
 (6.6) \quad &\leq A_k e^{Ak^2(t+1)} [\tau^{3k/2} + \tau^k] \leq A_k e^{Ak^2(t+1)} \tau^k,
 \end{aligned}$$

for all $\tau \in [0, 1]$. Now, we may apply BDG Lemma 3.1 in order to see that

$$\begin{aligned}
 & \left[\mathbb{E}\left(\left|\int_0^\tau \sigma(u_{t+s}(x)) \, d_s B_s^{(t)}(x) - \sigma(u_t(x))\{B_{t+\tau}(x) - B_t(x)\}\right|^k\right)\right]^{2/k} \\
 &= \left[\mathbb{E}\left(\left|\int_0^\tau [\sigma(u_{t+s}(x)) - \sigma(u_t(x))] \, d_s B_s^{(t)}(x)\right|^k\right)\right]^{2/k} \\
 (6.7) \quad &\leq 4k \text{Lip}_\sigma^2 \int_0^\tau [\mathbb{E}(|u_{t+s}(x) - u_t(x)|^k)]^{2/k} \, ds \\
 &\leq \tilde{a}_k e^{\tilde{a}t} \int_0^\tau s \, ds \leq \text{const} \cdot \tau^2,
 \end{aligned}$$

using (5.13). Therefore, we can deduce from (6.6) that

$$(6.8) \quad \mathbb{E}(|D(\tau, x)|^k) \leq c_{k,t} \tau^k \quad (0 \leq \tau \leq 1),$$

where we defined

$$(6.9) \quad D(\tau, x) := u_{t+\tau}(x) - u_t(x) - \sigma(u_t(x))\{B_{t+\tau}(x) - B_t(x)\},$$

and $c_{k,t}$ is a finite constant that depends only on k and t ; in particular, $c_{k,t}$ does not depend on τ . Choose and fix some $\eta > \xi > 0$ such that $\eta + \xi < \frac{1}{2}$, and then apply the Chebyshev inequality, and the preceding with any choice of integer $k > \xi^{-1}$, in order to see that $\sum_{n=1}^{\infty} \mathbb{P}\{|D(n^{-\eta}, x)| > n^{-(\eta-\xi)}\} \leq c_{k,t} \sum_{n=1}^{\infty} n^{-\xi k} < \infty$. Thus

$$(6.10) \quad D(n^{-\eta}, x) = O(n^{-(\eta-\xi)}) \quad \text{a.s.}$$

by the Borel–Cantelli lemma. Because $n^{-\eta} - (n+1)^{-\eta} = O(n^{-1-\eta})$, the modulus of continuity of Brownian motion, together with Lemma 5.1, imply that

$$(6.11) \quad \sup_{(n+1)^{-\eta} \leq \tau \leq n^{-\eta}} |D(n^{-\eta}, x) - D(\tau, x)| = O(n^{-1/2}) = o(n^{-(\eta+\xi)}) \quad \text{a.s.}$$

Therefore a standard monotonicity argument and (6.10) together reveal that $D(t, x) = O(t^{(\eta-\xi)/\eta})$ as $t \downarrow 0$, a.s. Since $\eta > \xi$ are arbitrary positive numbers, it follows that $\limsup_{t \downarrow 0} (\log D(t, x) / \log t) \leq 1$ a.s. This is another way to state the result. \square

7. Proof of Theorem 2.5. First we prove a preliminary lemma that guarantees strict positivity of the solution to the (SHE). We follow the method described in Conus, Joseph and Khoshnevisan [16], Theorem 5.1, which in turn borrowed heavily from ideas of Mueller [32] and Mueller and Nualart [33].

LEMMA 7.1. $\inf_{0 \leq t \leq T} u_t(x) > 0$ a.s. for every $T \in (0, \infty)$ and all $x \in \mathbf{Z}^d$ that satisfy $u_0(x) > 0$.

PROOF. We are going to prove that if $u_0(x_0) > 0$ for a fixed $x_0 \in \mathbf{Z}^d$, then there exist finite and positive constants A and C such that

$$(7.1) \quad \mathbb{P}\left\{\inf_{0 < s < t} u_s(x_0) \leq \varepsilon\right\} \leq A\varepsilon^{C \log |\log \varepsilon|},$$

for that same point x_0 , uniformly for all $\varepsilon \in (0, 1)$. It turns out to be convenient to prove the following equivalent formulation of the preceding:

$$(7.2) \quad \mathbb{P}\left\{\inf_{0 < s < t} u_s(x_0) \leq e^{-n}\right\} \leq An^{-Cn},$$

simultaneously for all $n \geq 1$, after a possible relabeling of the constants $A, C \in (0, \infty)$. If so, then we can simply let $n \rightarrow \infty$ and deduce the lemma.

Without loss of generality we assume that $u_0(0) > 0$, and we aim to prove (7.2) with $x_0 = 0$. In fact, we will simplify the exposition further and establish (7.2) when $u_0(0) = 1$; the general case follows from this one and scaling. Finally, we appeal to a comparison principle (Theorem 4.2) in order to reduce our problem further to the following special case:

$$(7.3) \quad u_0(x) = \delta_0(x) \quad \text{for all } x \in \mathbf{Z}^d.$$

Thus we consider this case only from now on.

Let $\mathcal{F}_t := \sigma\{B_s(x) : x \in \mathbf{Z}^d, 0 < s \leq t\}$ describe the filtration generated by time t by all the Brownian motions, enlarged so that $t \mapsto u_t$ is a $C(\mathbf{R})$ -valued (strong) Markov process. Set $T_0 := 0$, and define iteratively for $k \geq 0$ the sequence of $\{\mathcal{F}_t\}_{t>0}$ -stopping times

$$(7.4) \quad T_{k+1} := \inf\{s > T_k : u_s(0) \leq e^{-k-1}\},$$

using the usual convention that $\inf \emptyset := \infty$. We may observe that the preceding definitions imply that, almost surely on $\{T_k < \infty\}$,

$$(7.5) \quad u_{T_k}(x) \geq e^{-k} \delta_0(x) \quad \text{for all } x \in \mathbf{Z}^d.$$

We plan to apply the strong Markov property. In order to do that, we first define $u^{(k+1)}$ to be the unique continuous solution to the (SHE) (for same Brownian motions, pathwise), with initial data $u_0^{(k+1)}(x) := e^{-k} \delta_0(x)$. Next we note that, for every $k \geq 0$, the random field

$$(7.6) \quad w_t^{(k+1)}(x) := e^k u_t^{(k+1)}(x)$$

solves the system

$$(7.7) \quad \begin{cases} \frac{dw_t^{(k+1)}(x)}{dt} = (\mathcal{L}w_t^{(k+1)})(x) + \sigma_k(w_t^{(k+1)}(x)) \frac{dB_t(x)}{dt}, \\ w_0^{(k+1)}(x) = \delta_0(x), \end{cases}$$

where $\sigma_k(y) := e^k \sigma(e^{-k}y)$. Because $\sigma(0) = 0$, we have $\text{Lip}_{\sigma_k} = \text{Lip}_\sigma$, uniformly for all $k \geq 1$. Thus we can keep track of the constants in the proof of Lemma 5.1, in order to deduce the existence of a finite constant $K := K(\varepsilon)$ so that for all t, s with $|t - s| < 1$,

$$(7.8) \quad \mathbb{E} \left(\sup_{0 < |t-s| < 1} \frac{|w_t^{(k+1)}(0) - w_s^{(k+1)}(0)|^m}{|t - s|^{m(1-\varepsilon)/2}} \right) \leq Km^2 e^{Km^2},$$

for all real numbers $m \geq 2$.

For each $k \geq 0$ let us define

$$(7.9) \quad T_1^{(k+1)} = \inf\{t > 0 : w_t^{(k+1)}(0) \leq e^{-1}\}.$$

Equation (7.5), the strong Markov property and the comparison principle (Theorem 4.2) together imply that outside of a null set, the solution to the revised SPDE (7.7) satisfies

$$(7.10) \quad e^{-k} w_t^{(k+1)}(x) \leq u_{T_k+t}(x).$$

Therefore, in particular,

$$(7.11) \quad T_1^{(k+1)} \leq T_{k+1} - T_k,$$

and the stopping times $T_1^{(k+1)}$ and $T_1^{(\ell+1)}$ are independent if $k \neq \ell$. For all real numbers $t \in (0, 1)$ and $m \geq 2$,

$$(7.12) \quad \begin{aligned} \mathbb{P}\{T_1^{(k+1)} \leq t\} &\leq \mathbb{P}\left\{\sup_{0 < s < t} |w_t^{(k+1)}(0) - w_s^{(k+1)}(0)| \geq 1 - e^{-1}\right\} \\ &\leq Km^2 e^{Km^2} (1 - e^{-1})^{-m} t^{(1-\varepsilon)m/2}, \end{aligned}$$

where the last inequality follows by Chebyshev’s inequality and (7.8) and is valid for all $0 < \varepsilon < 1$. Let us emphasize that the constant of the bound in (7.12) does not depend on the parameter k which appears in the superscript of the random variable $T_1^{(k+1)}$. Now we compute

$$(7.13) \quad \begin{aligned} \mathbb{P}\left\{\inf_{0 < s \leq t} u_s(0) \leq e^{-n}\right\} &= \mathbb{P}\{T_n \leq t\} \\ &= \mathbb{P}\{(T_n - T_{n-1}) + \dots + (T_1 - T_0) \leq t\} \\ &\leq \mathbb{P}\{T_1^{(n)} + T_1^{(n-1)} + \dots + T_1^{(1)} \leq t\}, \end{aligned}$$

owing to (7.11).

The terms $T_1^{(n)}, \dots, T_1^{(1)}$ that appear in the ultimate line of (7.13) are independent nonnegative random variables. By the triangle inequality, if the sum of those terms is at most t , then certainly it must be that at least $n/2$ of those terms are at most $2t/n$. (This application of the triangle inequality is also known as the pigeon-hole principle.) If n is an even integer, larger than $t > 2$, then a simple union bound on (7.13) and (7.12) yields

$$(7.14) \quad \begin{aligned} &\mathbb{P}\left\{\inf_{0 < s \leq t} u_s(0) \leq e^{-n}\right\} \\ &\leq \binom{n}{n/2} K^{n/2} m^n e^{Km^2 n/2} (1 - e^{-1})^{-mn/2} (2t/n)^{(1-\varepsilon)mn/4} \\ &\leq \tilde{K}^n m^n e^{Km^2 n/2} (1 - e^{-1})^{-mn/2} t^{(1-\varepsilon)mn/4} n^{-(1-\varepsilon)mn/4} 2^{n(1+m(1-\varepsilon)/4)}, \end{aligned}$$

uniformly for all real numbers $m \geq 2$. Now we set $m := \log n / \log \log n$ in (7.14) in order to deduce (7.2) for $x_0 = 0$ and every $n \geq 1$ sufficiently large. This readily yields (7.2). \square

Next we show that if we start with an initial profile u_0 such that $u_0(x) > 0$ for at least one point $x \in \mathbf{Z}^d$, then $u_t(z) > 0$ for all $z \in \mathbf{Z}^d$ and $t > 0$ a.s. Because we are interested in establishing a lower bound, we may apply scaling and a comparison theorem (Theorem 4.2) in order to reduce our problem to the following special case:

$$(7.15) \quad u_0 = \delta_0.$$

In this way, we are led to the following representation of the solution:

$$(7.16) \quad u_t(x) = p_t(x) + \int_0^t \sum_{y \in \mathbf{Z}^d} p_{t-s}(y-x) \sigma(u_s(y)) dB_s(y).$$

PROPOSITION 7.2. *If $u_0 = \delta_0$, then $u_t(x) > 0$ for all $x \in \mathbf{Z}^d$ and $t > 0$ a.s.*

Proposition 7.2 follows from a few preparatory lemmas.

LEMMA 7.3. *If $u_0 = \delta_0$, then*

$$(7.17) \quad E(|u_t(x)|^2) \leq \exp(\text{Lip}_\sigma^2 t) \cdot [p_t(x)]^2 \quad \text{for all } t > 0 \text{ and } x \in \mathbf{Z}^d.$$

PROOF. We begin with representation (7.16) of the solution u , in integral form, and appeal to Picard’s iteration in order to prove the lemma.

Let $u_t^{(0)}(x) := 1$ for all $t \geq 0, x \in \mathbf{Z}^d$, and then let $\{u^{(n+1)}\}_{n \geq 0}$ be defined iteratively by

$$(7.18) \quad u_t^{(n+1)}(x) := p_t(x) + \int_0^t \sum_{y \in \mathbf{Z}^d} p_{t-s}(y-x) \sigma(u_s^{(n)}(y)) dB_s(y).$$

Let us define

$$(7.19) \quad M_t^{(k)} := \sup_{x \in \mathbf{Z}^d} E\left(\left|\frac{u_t^{(n+1)}(x)}{p_t(x)}\right|^2\right),$$

and apply Itô’s isometry in order to deduce the recursive inequality for the $M^{(k)}$ ’s,

$$(7.20) \quad M_t^{(n+1)} \leq 1 + \text{Lip}_\sigma^2 \cdot \sup_{x \in \mathbf{Z}^d} \int_0^t \sum_y \left[\frac{p_{t-s}(y-x)p_s(y)}{p_t(x)}\right]^2 M_s^{(n)} ds.$$

Because $\sum_{y \in \mathbf{Z}^d} [f(y)]^2 \leq [\sum_{y \in \mathbf{Z}^d} f(y)]^2$ for all $f : \mathbf{Z}^d \rightarrow \mathbf{R}_+$, the semigroup property of $\{p_t\}_{t > 0}$ yields the bound

$$(7.21) \quad \sum_{y \in \mathbf{Z}^d} [p_{t-s}(y-x)p_s(y)]^2 \leq [p_t(x)]^2,$$

whence $M_t^{(n+1)} \leq 1 + \text{Lip}_\sigma^2 \cdot \int_0^t M_s^{(n)} ds$ for all $t > 0$ and $n \geq 0$. It follows readily from this that $M_t^{(n)} \leq \exp(\text{Lip}_\sigma^2 t)$, uniformly for all $n \geq 0$ and $t > 0$; equivalently,

$$(7.22) \quad E(|u_t^{(n)}(x)|^2) \leq e^{\text{Lip}_\sigma^2 t} [p_t(x)]^2,$$

uniformly for all $n \geq 0, x \in \mathbf{Z}^d$ and $t > 0$. The lemma follows from this and Fatou’s lemma, since $u_t^{(n)}(x) \rightarrow u_t(x)$ in $L^2(\mathbf{P})$ as $n \rightarrow \infty$. \square

Our next lemma shows that the random term on the right-hand side of (7.16) is small, for small time, as compared with the nonrandom term in (7.16).

LEMMA 7.4. Assume the conditions of Proposition 7.2. Then there exists a finite constant $C > 0$ such that for all $t \in (0, 1)$,

$$(7.23) \quad \sup_{x \in \mathbf{Z}^d} \mathbb{P} \left\{ \left| \int_0^t \sum_{y \in \mathbf{Z}^d} p_{t-s}(y-x) \sigma(u_s(y)) dB_s(y) \right| > \frac{p_t(x)}{2} \right\} \leq Ct.$$

PROOF. By Lemma 7.3 and Itô’s isometry,

$$(7.24) \quad \begin{aligned} & \mathbb{E} \left(\left| \int_0^t \sum_{y \in \mathbf{Z}^d} p_{t-s}(y-x) \sigma(u_s(y)) dB_s(y) \right|^2 \right) \\ & \leq \text{Lip}_\sigma^2 \cdot \int_0^t \sum_{y \in \mathbf{Z}^d} [p_{t-s}(y-x) p_s(y)]^2 e^{\text{Lip}_\sigma^2 s} ds \\ & \leq \text{Lip}_\sigma^2 [p_t(x)]^2 \cdot \int_0^t e^{\text{Lip}_\sigma^2 s} ds, \end{aligned}$$

where we have used (7.21) in the last inequality. Because $\int_0^t \exp(\text{Lip}_\sigma^2 s) ds \leq ct$ for all $t \in (0, 1)$ with $c := \exp(\text{Lip}_\sigma^2)$, the lemma follows from Chebyshev’s inequality. \square

Now we can establish Proposition 7.2.

PROOF OF PROPOSITION 7.2. Let us choose and fix an arbitrary $x \in \mathbf{Z}^d$. By the strong Markov property of the solution, and thanks to Lemma 7.1, we know that once the solution becomes positive at a point, it remains positive at that point at all future times, almost surely. Thus it suffices to show that $u_t(x) > 0$ for all times of the form $t = 2^{-k}$, when k is a large enough integer. This is immediate from (7.16) and (7.23), thanks to the Borel–Cantelli lemma. \square

The preceding lemmas lay the groundwork for the proof of Theorem 2.5. We now proceed with the main proof.

PROOF OF THEOREM 2.5. Let us first consider the case $m = 1$ and, without loss of generality, $x_1 = 0$. In this case, we may write

$$(7.25) \quad \begin{aligned} & \lim_{\tau \downarrow 0} \mathbb{P} \{ S(u_{t+\tau}(0)) - S(u_t(0)) \leq q\sqrt{\tau} \} \\ & = \lim_{\tau \downarrow 0} \mathbb{P} \left\{ \int_{u_t(0)}^{u_{t+\tau}(0)} \frac{dy}{\sigma(y)} \leq q\sqrt{\tau} \right\} \\ & = \lim_{\tau \downarrow 0} \mathbb{P} \left\{ \int_{u_t(0)}^{u_{t+\tau}(0)} \left(\frac{1}{\sigma(y)} - \frac{1}{\sigma(u_t(0))} \right) dy + \frac{u_{t+\tau}(0) - u_t(0)}{\sigma(u_t(0))} \leq q\sqrt{\tau} \right\}. \end{aligned}$$

Lemma 7.1 and the positivity condition on σ ensure that $\sigma(u_t(0)) > 0$ a.s. Therefore, the theorem follows from Theorem 2.2 if we were to show that

$$(7.26) \quad \frac{1}{\sqrt{\tau}} \int_{u_t(0)}^{u_{t+\tau}(0)} \left(\frac{1}{\sigma(y)} - \frac{1}{\sigma(u_t(0))} \right) dy \rightarrow 0 \quad \text{almost surely, as } \tau \downarrow 0.$$

Let $\mathcal{I}(t, t + \tau)$ denote the random closed interval with endpoints $u_t(0)$ and $u_{t+\tau}(0)$. Our strict positivity result (Lemma 7.1) implies that

$$(7.27) \quad \mathcal{I}(t, t + \tau) \subset (0, \infty) \quad \text{for all } t, \tau > 0 \text{ a.s.,}$$

and thus paves way for the a.s. bounds

$$\begin{aligned} \left| \int_{u_t(0)}^{u_{t+\tau}(0)} \left(\frac{1}{\sigma(y)} - \frac{1}{\sigma(u_t(0))} \right) dy \right| &\leq \text{Lip}_\sigma \cdot \frac{|u_t(0) - u_{t+\tau}(0)|^2}{\inf_{y \in \mathcal{I}(t, t+\tau)} |\sigma(y)|^2} \\ &= O(\tau \log |\log \tau|) \quad (\tau \downarrow 0); \end{aligned}$$

see (2.4) for the last part. This implies (7.26) and thus completes our proof for $m = 1$. The proof for general m is an easy adaption since $\{B(x_j)\}_{j=1}^m$ are i.i.d. Brownian motions. \square

8. Preliminaries for the proof of Theorem 2.7. The following function will play a prominent role in the ensuing analysis:

$$(8.1) \quad \bar{P}(\tau) := \|p_\tau\|_{\ell^2(\mathbf{Z}^d)}^2 = \sum_{x \in \mathbf{Z}^d} [p_\tau(x)]^2 \quad \text{for all } \tau \geq 0.$$

Because of the Chapman–Kolmogorov property, we can also think of \bar{P} as

$$(8.2) \quad \bar{P}(\tau) := \mathbb{P}\{X_\tau - X'_\tau = 0\},$$

where X' is an independent copy of X . There is another useful way to think of \bar{P} as well. Using the fact that

$$(8.3) \quad \mathbb{E}e^{i\xi \cdot X_t} = e^{-t(1-\varphi(\xi))} \quad \text{for all } \xi \in \mathbf{R}^d \text{ and } t \geq 0$$

and the Plancherel theorem, we see that

$$(8.4) \quad \begin{aligned} \bar{P}(\tau) &= (2\pi)^{-d} \int_{(-\pi, \pi)^d} |\mathbb{E} \exp(i\xi \cdot X_\tau)|^2 d\xi \\ &= (2\pi)^{-d} \int_{(-\pi, \pi)^d} e^{-2\tau(1-\text{Re} \varphi(\xi))} d\xi, \end{aligned}$$

where $\varphi(\xi) = \mathbb{E}[\exp(i\xi \cdot Z_1)]$; recall that Z_1 is the distribution of jump size. Therefore, in particular, the Laplace transform of \bar{P} is

$$(8.5) \quad \begin{aligned} \Upsilon(\beta) &:= \int_0^\infty e^{-\beta\tau} \bar{P}(\tau) d\tau \quad (\beta \geq 0) \\ &= (2\pi)^{-d} \int_{(-\pi, \pi)^d} \frac{d\xi}{\beta + 2(1 - \text{Re} \varphi(\xi))}. \end{aligned}$$

The interchange of the integrals is justified by Tonelli’s theorem, since $1 - \operatorname{Re} \varphi(\xi) \geq 0$.

Note that $\Upsilon(0)$ agrees with (2.9). Also, the classical theory of random walks tells us that $X - X'$ is transient if and only if $\Upsilon(0) = \int_0^\infty \bar{P}(\tau) \, d\tau < \infty$, which is in turn equivalent to the condition

$$(8.6) \quad \int_{(-\pi, \pi)^d} \frac{d\xi}{1 - \operatorname{Re} \varphi(\xi)} < \infty;$$

this is the Chung–Fuchs theorem [14], transliterated to the setting of continuous-time symmetric random walks, thanks to a standard Poissonization argument which we feel free to omit.

LEMMA 8.1. *If $u_0 \in \ell^2(\mathbf{Z}^d)$, then $u_t \in \ell^2(\mathbf{Z}^d)$ a.s. for all $t \geq 0$. Moreover, for every $\beta \geq 0$ such that $\operatorname{Lip}_\sigma^2 \Upsilon(\beta) < 1$,*

$$(8.7) \quad \mathbb{E}(\|u_t\|_{\ell^2(\mathbf{Z}^d)}^2) \leq \frac{\|u_0\|_{\ell^2(\mathbf{Z}^d)}^2 e^{\beta t}}{1 - \operatorname{Lip}_\sigma^2 \Upsilon(\beta)} \quad \text{for all } t \geq 0.$$

PROOF. Let $u_t^{(0)}(x) := u_0(x)$ for all $t \geq 0$ and $x \in \mathbf{Z}^d$, and define $u^{(k)}$ to be the resulting k th-step approximation to u via Picard iteration. It follows that

$$(8.8) \quad \begin{aligned} & \mathbb{E}(|u_t^{(n+1)}(x)|^2) \\ &= |(\tilde{p}_t * u_0)(x)|^2 + \sum_{y \in \mathbf{Z}^d} \int_0^t [p_{t-s}(y-x)]^2 \mathbb{E}(|\sigma(u_s^{(n)}(y))|^2) \, ds \\ &\leq |(\tilde{p}_t * u_0)(x)|^2 + \operatorname{Lip}_\sigma^2 \sum_{y \in \mathbf{Z}^d} \int_0^t [p_{t-s}(y-x)]^2 \mathbb{E}(|u_s^{(n)}(y)|^2) \, ds. \end{aligned}$$

We may add over all $x \in \mathbf{Z}^d$ to deduce from this and Young’s inequality that

$$(8.9) \quad \mathbb{E}(\|u_t^{(n+1)}\|_{\ell^2(\mathbf{Z}^d)}^2) \leq \|u_0\|_{\ell^2(\mathbf{Z}^d)}^2 + \operatorname{Lip}_\sigma^2 \int_0^t \bar{P}(t-s) \mathbb{E}(\|u_s^{(n)}\|_{\ell^2(\mathbf{Z}^d)}^2) \, ds.$$

Since $\Upsilon(\beta) = \beta^{-1} \int_0^\infty \exp(-s) \bar{P}(s/\beta) \, ds \leq \beta^{-1} < \infty$, we can find $\beta > 0$ large enough to guarantee that $\operatorname{Lip}_\sigma^2 \Upsilon(\beta) < 1$.

We multiply both sides of (8.9) by $\exp(-\beta t)$ —for this choice of β —and notice from (8.9) that

$$(8.10) \quad A_k := \sup_{t \geq 0} [e^{-\beta t} \mathbb{E}(\|u_t^{(k)}\|_{\ell^2(\mathbf{Z}^d)}^2)] \quad (k \geq 0)$$

satisfies

$$(8.11) \quad A_{n+1} \leq \|u_0\|_{\ell^2(\mathbf{Z}^d)}^2 + \operatorname{Lip}_\sigma^2 \Upsilon(\beta) A_n \quad \text{for all } n \geq 0.$$

Since $A_0 = \|u_0\|_{\ell^2(\mathbf{Z}^d)}^2$, the preceding shows that $\sup_{n \geq 0} A_n$ is bounded above by $(1 - \text{Lip}_\sigma^2 \Upsilon(\beta))^{-1} \|u_0\|_{\ell^2(\mathbf{Z}^d)}^2$. \square

PROPOSITION 8.2. *If $u_0 \in \ell^1(\mathbf{Z}^d)$, then for every $\beta \geq 0$ such that $\text{Lip}_\sigma^2 \Upsilon(\beta) < 1$,*

$$(8.12) \quad \int_0^\infty e^{-\beta t} \mathbf{E}(\|u_t\|_{\ell^2(\mathbf{Z}^d)}^2) dt \leq \frac{\|u_0\|_{\ell^1(\mathbf{Z}^d)}^2 \Upsilon(\beta)}{1 - \text{Lip}_\sigma^2 \Upsilon(\beta)}.$$

Moreover,

$$(8.13) \quad \int_0^\infty e^{-\beta t} \mathbf{E}(\|u_t\|_{\ell^2(\mathbf{Z}^d)}^2) dt = \infty,$$

for all $\beta \geq 0$ such that $\ell_\sigma^2 \Upsilon(\beta) \geq 1$.

PROOF. We proceed as we did for Lemma 8.1. But instead of deducing (8.9) from (8.8), we use a different bound for $\|\tilde{p}_t * u_0\|_{\ell^2(\mathbf{Z}^d)}$

$$(8.14) \quad \begin{aligned} & \mathbf{E}(\|u_t^{(n+1)}\|_{\ell^2(\mathbf{Z}^d)}^2) \\ & \leq \|p_t\|_{\ell^2(\mathbf{Z}^d)}^2 \|u_0\|_{\ell^1(\mathbf{Z}^d)}^2 + \text{Lip}_\sigma^2 \int_0^t \bar{P}(t-s) \mathbf{E}(\|u_s^{(n)}\|_{\ell^2(\mathbf{Z}^d)}^2) ds \\ & = \bar{P}(t) \|u_0\|_{\ell^1(\mathbf{Z}^d)}^2 + \text{Lip}_\sigma^2 \int_0^t \bar{P}(t-s) \mathbf{E}(\|u_s^{(n)}\|_{\ell^2(\mathbf{Z}^d)}^2) ds, \end{aligned}$$

thanks to a slightly different application of Young’s inequality. If we integrate both sides $[\exp(-\beta t) dt]$, then we find that

$$(8.15) \quad I_k := \int_0^\infty e^{-\beta t} \mathbf{E}(\|u_t^{(k)}\|_{\ell^2(\mathbf{Z}^d)}^2) dt \quad (k \geq 0)$$

satisfies

$$(8.16) \quad \begin{aligned} I_{n+1} & \leq \|u_0\|_{\ell^1(\mathbf{Z}^d)}^2 \int_0^\infty e^{-\beta t} \bar{P}(t) dt + I_n \times \text{Lip}_\sigma^2 \int_0^\infty e^{-\beta t} \bar{P}(t) dt \\ & = \|u_0\|_{\ell^1(\mathbf{Z}^d)}^2 \Upsilon(\beta) + I_n \text{Lip}_\sigma^2 \Upsilon(\beta); \end{aligned}$$

see (8.5). The first portion of the lemma follows from this, induction and Fatou’s lemma since $\text{Lip}_\sigma^2 \Upsilon(\beta) < 1$.

Next, let us suppose that $\ell_\sigma^2 \Upsilon(\beta) \geq 1$. The following complimentary form of (8.14) holds [for the same reasons that (8.14) held]:

$$(8.17) \quad \mathbf{E}(\|u_t\|_{\ell^2(\mathbf{Z}^d)}^2) \geq \|\tilde{p}_t * u_0\|_{\ell^2(\mathbf{Z}^d)}^2 + \ell_\sigma^2 \int_0^t \bar{P}(t-s) \mathbf{E}(\|u_s\|_{\ell^2(\mathbf{Z}^d)}^2) ds.$$

It is not hard to verify directly that

$$(8.18) \quad \|\tilde{p}_t * u_0\|_{\ell^2(\mathbf{Z}^d)}^2 \geq u_0^2(x_0) \|p_t\|_{\ell^2(\mathbf{Z}^d)}^2,$$

whence, by $u_0(x_0) > 0$ for some $x_0 > 0$, it follows that

$$(8.19) \quad F(t) := \mathbb{E}(\|u_t\|_{\ell^2(\mathbf{Z}^d)}^2) \quad (t \geq 0)$$

solves the renewal inequality

$$(8.20) \quad F(t) \geq u_0^2(x_0)\bar{P}(t) + \ell_\sigma^2 \int_0^t \bar{P}(t-s)F(s) \, ds.$$

Therefore, $\tilde{F}(\beta) := \int_0^\infty \exp(-\beta t)F(t) \, dt$ satisfies

$$(8.21) \quad \tilde{F}(\beta) \geq u_0^2(x_0)\Upsilon(\beta) + \ell_\sigma^2 \Upsilon(\beta)\tilde{F}(\beta).$$

Since $u_0(x_0) > 0$ and $\Upsilon(\beta) > 0$ for all $\beta \geq 0$, it follows that $\tilde{F}(\beta) = \infty$ whenever $\ell_\sigma^2 \Upsilon(\beta) \geq 1$. \square

PROPOSITION 8.3. *If $u_0 \in \ell^1(\mathbf{Z}^d)$, then*

$$(8.22) \quad \sup_{t \geq 0} \sup_{x \in \mathbf{Z}^d} u_t(x) < \infty, \quad \sum_{y \in \mathbf{Z}^d} \int_0^\infty |\sigma(u_s(y))|^2 \, ds < \infty \quad \text{a.s.}$$

Moreover: (i) *If, in addition, $q := \text{Lip}_\sigma^2 \Upsilon(0) < 1$, then*

$$(8.23) \quad \begin{aligned} \mathbb{E}\left(\sup_{t \geq 0} \sup_{x \in \mathbf{Z}^d} |u_t(x)|^2\right) &\leq \mathbb{E}\left(\sup_{t \geq 0} \|u_t\|_{\ell^1(\mathbf{Z}^d)}^2\right) \\ &\leq 2\|u_0\|_{\ell^1(\mathbf{Z}^d)}^2 + \frac{8q \cdot \|u_0\|_{\ell^1(\mathbf{Z}^d)}^2}{1-q}. \end{aligned}$$

(ii) *If, in addition, $\ell_\sigma^2 \Upsilon(0) \geq 1$, then*

$$(8.24) \quad \mathbb{E}\left(\sup_{t \geq 0} \|u_t\|_{\ell^1(\mathbf{Z}^d)}^2\right) = \int_0^\infty \mathbb{E}(\|u_s\|_{\ell^2(\mathbf{Z}^d)}^2) \, ds = \infty.$$

REMARK 8.4. Clearly, (8.22) implies that if $u_0 \in \ell^1(\mathbf{Z}^d)$, then

$$(8.25) \quad \liminf_{t \rightarrow \infty} \sup_{x \in \mathbf{Z}^d} |\sigma(u_t(x))|^2 \leq \liminf_{t \rightarrow \infty} \sum_{x \in \mathbf{Z}^d} |\sigma(u_t(x))|^2 = 0 \quad \text{a.s.}$$

If, in addition, $\ell_\sigma > 0$ (say), then we can deduce from the preceding fact that $\liminf_{t \rightarrow \infty} \sup_{x \in \mathbf{Z}^d} |u_t(x)| = 0$ a.s.

Recall that $X - X'$ is transient if and only if $\Upsilon(0) < \infty$. Therefore, in order for the condition $\text{Lip}_\sigma^2 \Upsilon(0) < 1$ to hold, it is necessary—though not sufficient—that $X - X'$ be transient.

PROOF OF PROPOSITION 8.3. First of all, Theorem 2.1 assures us that $u_t(x) \geq 0$ a.s., and hence $\|u_t\|_{\ell^1(\mathbf{Z}^d)} = \sum_{x \in \mathbf{Z}^d} u_t(x)$. Therefore, if we add both sides of (3.3), then we find that

$$(8.26) \quad \|u_t\|_{\ell^1(\mathbf{Z}^d)} = \|u_0\|_{\ell^1(\mathbf{Z}^d)} + \sum_{y \in \mathbf{Z}^d} \int_0^t \sigma(u_s(y)) \, dB_s(y).$$

(It is easy to apply the moment bound of Theorem 2.1 to justify the interchange of the sum and the stochastic integral.) In particular, it follows that

$$(8.27) \quad M_t := \|u_t\|_{\ell^1(\mathbf{Z}^d)} \quad (t \geq 0)$$

defines a nonnegative continuous martingale with mean $\|u_0\|_{\ell^1(\mathbf{Z}^d)}$. Its quadratic variation satisfies the following relations:

$$(8.28) \quad \langle M \rangle_t = \sum_{y \in \mathbf{Z}^d} \int_0^t |\sigma(u_s(y))|^2 \, ds \leq \text{Lip}_\sigma^2 \int_0^t \|u_s\|_{\ell^2(\mathbf{Z}^d)}^2 \, ds.$$

Bound (2.1) of Theorem 2.1 is more than enough to show that $M := \{M_t\}_{t \geq 0}$ is a continuous $L^2(\mathbb{P})$ martingale. Since $M_t \geq 0$ a.s. (Theorem 2.1) it follows from the martingale convergence theorem that $\lim_{t \rightarrow \infty} M_t$ exists a.s. and is finite a.s., which proves the first part of (8.22). And therefore, $\langle M \rangle_\infty = \sum_{y \in \mathbf{Z}^d} \int_0^\infty |\sigma(u_s(y))|^2 \, ds$ has to be also a.s. finite., since we can realize M_t as $W(\langle M \rangle_t)$ for some Brownian motion W , thanks to the Dubins, Dambis-Schwartz representation theorem [35], page 170.

(i) If we know also that $\text{Lip}_\sigma^2 \Upsilon(0) < 1$, then Proposition 8.2 guarantees that $\mathbb{E}\langle M \rangle_\infty$ is bounded from above by $(1 - \text{Lip}_\sigma^2 \Upsilon(0))^{-1} \text{Lip}_\sigma^2 \Upsilon(0) \|u_0\|_{\ell^1(\mathbf{Z}^d)}^2 < \infty$, whence it follows that $M := \{M_t\}_{t \geq 0}$ is a continuous $L^2(\mathbb{P})$ -bounded martingale with

$$(8.29) \quad \mathbb{E}\left(\sup_{t \geq 0} M_t^2\right) \leq 2\|u_0\|_{\ell^1(\mathbf{Z}^d)}^2 + \frac{8 \text{Lip}_\sigma^2 \Upsilon(0) \cdot \|u_0\|_{\ell^1(\mathbf{Z}^d)}^2}{1 - \text{Lip}_\sigma^2 \Upsilon(0)},$$

thanks to Doob’s maximal inequality. This proves part (i) because $\|u_t\|_{\ell^\infty(\mathbf{Z}^d)}$ is bounded above by $\|u_t\|_{\ell^1(\mathbf{Z}^d)}$.

(ii) Finally consider the case $\ell_\sigma^2 \Upsilon(0) \geq 1$. Since

$$(8.30) \quad \begin{aligned} \mathbb{E}(\|u_t\|_{\ell^1(\mathbf{Z}^d)}^2) &= \mathbb{E}(M_t^2) = \|u_0\|_{\ell^1(\mathbf{Z}^d)}^2 + \sum_{y \in \mathbf{Z}^d} \int_0^t \mathbb{E}(|\sigma(u_s(y))|^2) \, ds \\ &\geq \|u_0\|_{\ell^1(\mathbf{Z}^d)}^2 + \ell_\sigma^2 \int_0^t \mathbb{E}(\|u_s\|_{\ell^2(\mathbf{Z}^d)}^2) \, ds, \end{aligned}$$

it suffices to show that this final integral is unbounded (as a function of t), which follows from the second part of Proposition 8.2. \square

COROLLARY 8.5. *If $u_0 \in \ell^1(\mathbf{Z}^d)$, then the following is a P-null set:*

$$(8.31) \quad \left\{ \omega : \limsup_{t \rightarrow \infty} \sup_{x \in \mathbf{Z}^d} |u_t(x)(\omega)| = 0 \right\} \Delta \left\{ \omega : \lim_{t \rightarrow \infty} \|u_t\|_{\ell^2(\mathbf{Z}^d)}(\omega) = 0 \right\}.$$

PROOF. Let E_1 denote the event that $\lim_{t \rightarrow \infty} \sup_{x \in \mathbf{Z}^d} |u_t(x)| = 0$ and E_2 the event that $\lim_{t \rightarrow \infty} \|u_t\|_{\ell^2(\mathbf{Z}^d)} = 0$. Because of the real-variable bounds, $\|u_t\|_{\ell^\infty(\mathbf{Z}^d)}^2 \leq \|u_t\|_{\ell^2(\mathbf{Z}^d)}^2 \leq \|u_t\|_{\ell^\infty(\mathbf{Z}^d)} \cdot \|u_t\|_{\ell^1(\mathbf{Z}^d)}$, we have

$$(8.32) \quad E_1 \Delta E_2 \subseteq \left\{ \omega : \limsup_{t \rightarrow \infty} \|u_t\|_{\ell^1(\mathbf{Z}^d)}(\omega) = \infty \right\}.$$

We have already noted, however, that $M_t := \|u_t\|_{\ell^1(\mathbf{Z}^d)}$ defines a nonnegative martingale, under the conditions of this corollary. Therefore, the final event in (8.32) is P-null, thanks to Doob’s martingale convergence theorem. Thus we find that $E_1 \Delta E_2$ is a measurable subset of a P-null set, and is hence P-null. \square

PROPOSITION 8.6. *Suppose $u_0 \in \ell^1(\mathbf{Z}^d)$ and the random walk X is transient; that is, $\Upsilon(0) < \infty$. Then*

$$(8.33) \quad \limsup_{t \rightarrow \infty} \frac{1}{t} \log E(\|u_t\|_{\ell^1(\mathbf{Z}^d)}^2) \leq \inf\{\beta > 0 : \text{Lip}_\sigma^2 \Upsilon(\beta) < 1\} < \infty.$$

If, in addition, $\ell_\sigma^2 \Upsilon(0) > 1$, then

$$(8.34) \quad \liminf_{t \rightarrow \infty} \frac{1}{t} \log E(\|u_t\|_{\ell^2(\mathbf{Z}^d)}^2) \geq \inf\{\beta > 0 : \ell_\sigma^2 \Upsilon(\beta) < 1\} > 0.$$

PROOF. We have already proved a slightly weaker version of (8.33). Indeed, since $\ell^1(\mathbf{Z}^d) \subset \ell^2(\mathbf{Z}^d)$, (8.7) implies that

$$(8.35) \quad \limsup_{t \rightarrow \infty} \frac{1}{t} \log E(\|u_t\|_{\ell^2(\mathbf{Z}^d)}^2) \leq \inf\{\beta > 0 : \text{Lip}_\sigma^2 \Upsilon(\beta) < 1\}.$$

Then (8.26) and (8.35) together tell us that for every $C > \inf\{\beta > 0 : \text{Lip}_\sigma^2 \Upsilon(\beta) < 1\}$, there exists $K = K(C) \in (0, \infty)$ such that

$$(8.36) \quad \begin{aligned} E(\|u_t\|_{\ell^1(\mathbf{Z}^d)}^2) &\leq \|u_0\|_{\ell^1(\mathbf{Z}^d)}^2 + \text{Lip}_\sigma^2 \int_0^t E(\|u_s\|_{\ell^2(\mathbf{Z}^d)}^2) ds \\ &\leq \|u_0\|_{\ell^1(\mathbf{Z}^d)}^2 + K \int_0^t e^{Cs} ds = O(e^{(C+o(1))t}) \quad \text{as } t \rightarrow \infty. \end{aligned}$$

Thus follows the first bound of the proposition.

Because of (8.17) and (8.18), we find that

$$(8.37) \quad F(t) := E(\|u_t\|_{\ell^2(\mathbf{Z}^d)}^2) \quad (t \geq 0)$$

solves the renewal inequality

$$(8.38) \quad F(t) \geq g(t) + \int_0^t h(t-s)F(s) ds \quad (t \geq 0),$$

where

$$(8.39) \quad g(t) := u_0^2(x_0)\bar{P}(t), \quad h(t) := \ell_\sigma^2 \bar{P}(t) \quad (t \geq 0).$$

A comparison result (Lemma A.2) tells us that $F(t) \geq f(t)$ for all $t \geq 0$, where f is the solution to the renewal equation

$$(8.40) \quad f(t) = g(t) + \int_0^t h(t-s)f(s) \, ds \quad (t \geq 0).$$

The condition that $\ell_\sigma^2 \Upsilon(0) > 1$ is equivalent to $\int_0^\infty h(t) \, dt > 1$. Because of transience [$\Upsilon(0) < \infty$] and the fact that $\Upsilon(\beta)$ is strictly decreasing and continuous, we can find $\beta^* > 0$ such that $\int_0^\infty \exp(-\beta^*t)h(t) \, dt = 1$. Note that $f_{\beta^*}(t) := \exp(-\beta^*t)f(t)$ solves the renewal equation

$$(8.41) \quad f_{\beta^*}(t) = g_{\beta^*}(t) + \int_0^t h_{\beta^*}(t-s)f_{\beta^*}(s) \, ds \quad (t \geq 0),$$

where $g_{\beta^*}(t) := \exp(-\beta^*t)g(t)$ and $h_{\beta^*}(t) := \exp(-\beta^*t)h(t)$. Since h_{β^*} is a probability density function, and g_{β^*} is nonincreasing [see (8.4)], Blackwell’s key renewal theorem [24] implies that

$$(8.42) \quad \begin{aligned} \liminf_{t \rightarrow \infty} e^{-\beta^*t} F(t) &\geq \lim_{t \rightarrow \infty} f_{\beta^*}(t) = \left(\int_0^\infty s h_{\beta^*}(s) \, ds \right)^{-1} \cdot \int_0^\infty g_{\beta^*}(s) \, ds \\ &= u_0^2(x_0)\ell_\sigma^{-2} \left(\int_0^\infty s e^{-\beta^*s} \bar{P}(s) \, ds \right)^{-1} \cdot \Upsilon(\beta^*). \end{aligned}$$

Since $\bar{P}(s) \leq 1$, the right-most quantity is at least $u_0^2(x_0)\ell_\sigma^{-2}(\beta^*)^2\Upsilon(\beta^*) > 0$. This completes the proof of (8.34). Note that we have used the fact that $\Upsilon(\beta)$ is continuous in β and strictly decreasing, so that $\beta^* = \inf\{\beta > 0: \ell_\sigma^2 \Upsilon(\beta) < 1\} > 0$. □

PROPOSITION 8.7. *If $u_0 \in \ell^1(\mathbf{Z}^d)$ and $\text{Lip}_\sigma^2 \Upsilon(0) < 1$, then $\lim_{t \rightarrow \infty} E(\|u_t\|_{\ell^2(\mathbf{Z}^d)}^2) = 0$. Furthermore, as $t \rightarrow \infty$,*

$$(8.43) \quad \begin{aligned} \bar{P}(t) &= O(E(\|u_t\|_{\ell^2(\mathbf{Z}^d)}^2)) \quad \text{and} \\ E(\|u_t\|_{\ell^2(\mathbf{Z}^d)}^2) &= O(t^{-\alpha}) \quad \text{for all } \alpha \geq 0 \text{ such that } \bar{P}(t) = O(t^{-\alpha}). \end{aligned}$$

PROOF. The first assertion of (8.43) is simple to prove; in fact, $E(\|u_t\|_{\ell^2(\mathbf{Z}^d)}^2) \geq [u_0(x_0)]^2 \bar{P}(t)$ ($t \geq 0$) for any $x_0 \in \mathbf{Z}^d$ and all $t > 0$; see (8.17) and (8.18). We concentrate our efforts on the remaining statements.

Thanks to (8.14),

$$(8.44) \quad E(\|u_t\|_{\ell^2(\mathbf{Z}^d)}^2) \leq \bar{P}(t)\|u_0\|_{\ell^1(\mathbf{Z}^d)}^2 + \text{Lip}_\sigma^2 \int_0^t \bar{P}(t-s)E(\|u_s\|_{\ell^2(\mathbf{Z}^d)}^2) \, ds.$$

That is, $F(t) := E(\|u_t\|_{\ell^2(\mathbf{Z}^d)}^2)$ is a sub solution to a renewal equation; namely,

$$(8.45) \quad F(t) \leq g(t) + \int_0^t h(t-s)F(s) \, ds \quad (t \geq 0),$$

for

$$(8.46) \quad g(t) := \bar{P}(t)\|u_0\|_{\ell^1(\mathbf{Z}^d)}^2, \quad h(t) := \text{Lip}_\sigma^2 \bar{P}(t).$$

A comparison lemma (Lemma A.2) shows that $0 \leq F(t) \leq f(t)$ for all $t \geq 0$, where

$$(8.47) \quad f(t) = g(t) + \int_0^t h(t-s)f(s) \, ds \quad (t \geq 0).$$

Therefore, it remains to prove that $f(t) \rightarrow 0$ as $t \rightarrow \infty$. It is easy, as well as classical, that we can write f in terms of the renewal function of h ; that is,

$$(8.48) \quad f(t) = g(t) + \sum_{n=0}^\infty \int_0^t h^{*(n)}(s)g(t-s) \, ds \quad (t \geq 0),$$

where $h^{*(1)}(t) := \int_0^t h(t-s)h(s) \, ds$ denotes the convolution of h with itself, and $h^{*(k+1)}(t) := \int_0^t h^{*(k)}(t-s)h(s) \, ds$ for all $k \geq 0$. We might note that $g(t) \leq g(0) = \|u_0\|_{\ell^2(\mathbf{Z}^d)}^2$ because \bar{P} is nonincreasing [see (8.4)] and one at zero. Therefore,

$$(8.49) \quad \begin{aligned} 0 &\leq \int_0^t h^{*(n)}(s)g(t-s) \, ds \leq \|u_0\|_{\ell^2(\mathbf{Z}^d)}^2 \int_0^\infty h^{*(n)}(s) \, ds \\ &\leq \|u_0\|_{\ell^2(\mathbf{Z}^d)}^2 \left(\int_0^\infty h(s) \, ds \right)^{n+1} \quad \text{[Young's inequality]} \\ &= \|u_0\|_{\ell^2(\mathbf{Z}^d)}^2 (\text{Lip}_\sigma^2 \Upsilon(0))^{n+1}. \quad \square \end{aligned}$$

It is not hard to see that $\lim_{t \rightarrow \infty} g(t) = \lim_{t \rightarrow \infty} \bar{P}(t) = 0$; this follows from (8.4) and the monotone convergence theorem. Because $\text{Lip}_\sigma^2 \Upsilon(0) < 1$, we can deduce from (8.49) and (8.48), in conjunction with the dominated convergence theorem, that $f(t)$ —hence $F(t) = E(\|u_t\|_{\ell^2(\mathbf{Z}^d)}^2)$ —converges to zero as $t \rightarrow \infty$.

It remains to prove the second assertion in (8.43). With this in mind, let us suppose \bar{P} satisfies the following: There exists $c \in (0, \infty)$ and $\alpha \in [0, \infty)$ such that

$$(8.50) \quad \bar{P}(t) \leq c(1+t)^{-\alpha},$$

for there is nothing to consider otherwise. We aim to prove that

$$(8.51) \quad E(\|u_t\|_{\ell^2(\mathbf{Z}^d)}^2) \leq \text{const} \cdot (1+t)^{-\alpha},$$

for some finite constant that does not depend on t . This proves the proposition.

Define $F_k(t) := E(\|u_t^{(k)}\|_{\ell^2(\mathbf{Z}^d)}^2)$, where $u^{(k)}$ denotes the k th approximation to u via Picard’s iteration (4.1), starting at $u_t^{(0)}(x) \equiv 0$. We can write (8.14), in short hand, as follows:

$$(8.52) \quad F_{n+1}(t) \leq \bar{P}(t)\|u_0\|_{\ell^1(\mathbf{Z}^d)}^2 + \text{Lip}_\sigma^2 \int_0^t \bar{P}(t-s)F_n(s) \, ds.$$

Now let us choose and fix $\varepsilon \in (0, 1)$ and write

$$(8.53) \quad \begin{aligned} & \int_0^t \bar{P}(t-s)F_n(s) \, ds \\ &= \int_{t\varepsilon}^t \bar{P}(s)F_n(t-s) \, ds + \int_{t(1-\varepsilon)}^t \bar{P}(t-s)F_n(s) \, ds \\ &\leq c \int_{t\varepsilon}^t \frac{F_n(t-s)}{(1+s)^\alpha} \, ds + \sup_{w \geq 0} [(1+w)^\alpha F_n(w)] \int_{t(1-\varepsilon)}^t \frac{\bar{P}(t-s)}{(1+s)^\alpha} \, ds \\ &\leq \frac{c}{\varepsilon^\alpha(1+t)^\alpha} \int_0^\infty F_n(s) \, ds + \sup_{w \geq 0} [(1+w)^\alpha F_n(w)] \frac{\Upsilon(0)}{(1-\varepsilon)^\alpha(1+t)^\alpha}. \end{aligned}$$

The proof of Proposition 8.2 shows that

$$(8.54) \quad \sup_{n \geq 0} \int_0^\infty F_n(s) \, ds \leq \frac{\|u_0\|_{\ell^1(\mathbf{Z}^d)}^2 \Upsilon(0)}{1 - \text{Lip}_\sigma^2 \Upsilon(0)}.$$

Consequently,

$$(8.55) \quad R_k := \sup_{w \geq 0} [(1+w)^\alpha F_k(w)] \quad (k \geq 0)$$

satisfies

$$(8.56) \quad R_{n+1} \leq A + R_n \frac{\text{Lip}_\sigma^2 \Upsilon(0)}{(1-\varepsilon)^\alpha} \quad \text{for all } n \geq 0,$$

where

$$(8.57) \quad A = A(\varepsilon) := c\|u_0\|_{\ell^1(\mathbf{Z}^d)}^2 + \frac{c\|u_0\|_{\ell^1(\mathbf{Z}^d)}^2 \text{Lip}_\sigma^2 \Upsilon(0)}{\varepsilon^\alpha(1 - \text{Lip}_\sigma^2 \Upsilon(0))}.$$

Since $\text{Lip}_\sigma^2 \Upsilon(0) < 1$, we can choose ε sufficiently close to zero to ensure that $\text{Lip}_\sigma^2 \Upsilon(0) < (1-\varepsilon)^{1+\alpha}$. For this particular ε , we find that $R_{n+1} \leq A + (1-\varepsilon)R_n$ for all n . Since $R_0 = 0$, this proves that $\sup_{n \geq 0} R_n \leq A/\varepsilon$. Equation (8.51)—whence the proposition—follows from the latter inequality and Fatou’s lemma.

9. Proof of Theorem 2.7. Let us begin with an elementary real-variable inequality.

LEMMA 9.1. For all real numbers $k \geq 2$ and $x, y, \delta > 0$,

$$(9.1) \quad (x + y)^k \leq (1 + \delta)^{k-1} x^k + \left(\frac{1 + \delta}{\delta}\right)^{k-1} y^k.$$

This is a consequence of Jensen’s inequality when $\delta = 1$. We are interested in the regime $\delta \ll 1$.

PROOF. The function $f(z) := (z + 1)^k - (1 + \delta)^{k-1} z^k$ ($z > 0$) is maximized at $z_* := \delta^{-1}$, and $\max_z f(z) = f(z_*) = \{(1 + \delta)/\delta\}^{k-1}$; that is, $f(x) \leq \{(1 + \delta)/\delta\}^{k-1}$ for all $x > 0$. This is the desired result when $y = 1$. We can factor the variable y from both sides of (9.1) in order to reduce the problem to the previously proved case $y = 1$. \square

LEMMA 9.2. $\int_0^\infty \|p_{s+\tau} - p_s\|_{\ell^2(\mathbf{Z}^d)}^2 ds \leq 4\Upsilon(0)\tau^2$ for all $\tau \geq 0$.

PROOF. We apply the Plancherel theorem and (8.3) in order to deduce that

$$(9.2) \quad \begin{aligned} \|p_{s+\tau} - p_s\|_{\ell^2(\mathbf{Z}^d)}^2 &= (2\pi)^{-d} \int_{(-\pi, \pi)^d} |e^{-(s+\tau)(1-\varphi(\xi))} - e^{-s(1-\varphi(\xi))}|^2 d\xi \\ &= (2\pi)^{-d} \int_{(-\pi, \pi)^d} e^{-2s(1-\operatorname{Re}\varphi(\xi))} |1 - e^{-\tau(1-\varphi(\xi))}|^2 d\xi \\ &\leq \frac{4\tau^2}{(2\pi)^d} \int_{(-\pi, \pi)^d} e^{-2s(1-\operatorname{Re}\varphi(\xi))} d\xi. \end{aligned}$$

Integrate $[ds]$ to finish; compare with (8.5). \square

Recall that z_k denotes the optimal constant in BDG inequality (3.9).

LEMMA 9.3. If $k \in (2, \infty)$ satisfies $z_k \operatorname{Lip}_\sigma \sqrt{\Upsilon(0)} < (1 + \delta)^{-(k-1)/k}$ for some $\delta > 0$, then

$$(9.3) \quad \sup_{t \geq 0} \mathbb{E} \left(\sup_{x \in \mathbf{Z}^d} |u_t(x)|^k \right) \leq \sup_{t \geq 0} \mathbb{E} (\|u_t\|_{\ell^k(\mathbf{Z}^d)}^k) < \infty.$$

PROOF. Let $u_t^{(0)}(x) := u_0(x)$, and define $u^{(n)}$ to be the n th step Picard approximation to u , as in (4.1). Define

$$(9.4) \quad \bar{M}_t^{(n)} := \mathbb{E} (\|u_t^{(n)}\|_{\ell^k(\mathbf{Z}^d)}^k) \quad \text{for all } t \geq 0 \text{ and } k \geq 1.$$

Then we can apply Lemma 9.1 and write

$$(9.5) \quad \bar{M}_t^{(n+1)} \leq \left(\frac{1 + \delta}{\delta}\right)^{k-1} \sum_{x \in \mathbf{Z}^d} I_x + (1 + \delta)^{k-1} \sum_{x \in \mathbf{Z}^d} J_x,$$

where I_x and J_x were defined earlier in (4.3). One estimates $\sum_{x \in \mathbf{Z}^d} I_x$ via Jensen’s inequality, using $p_t(\bullet - x)$ as the base measure, in order to find that

$$(9.6) \quad \sum_{x \in \mathbf{Z}^d} I_x \leq \|u_0\|_{\ell^k(\mathbf{Z}^d)}^k.$$

In order to estimate $\sum_{x \in \mathbf{Z}^d} J_x$, we define—for all $(t, x) \in \mathbf{R}_+ \times \mathbf{Z}^d$ —a Borel measure $\rho_{t,x}$ on $\mathbf{R}_+ \times \mathbf{Z}^d$ as follows:

$$(9.7) \quad \rho_{t,x}(ds dy) := [p_{t-s}(y - x)]^2 \mathbf{1}_{[0,t]}(s) ds \chi(dy);$$

where χ denotes the counting measure on \mathbf{Z}^d . Because of the transience of $X - X'$, the measure $\rho_{t,x}$ is finite; in fact,

$$(9.8) \quad \rho_{t,x}(\mathbf{R}_+ \times \mathbf{Z}^d) = \int_0^t \|p_s\|_{\ell^2(\mathbf{Z}^d)}^2 ds = \int_0^t \bar{P}(s) ds \leq \Upsilon(0).$$

Therefore, we apply (4.5) and Jensen’s inequality, in conjunction, in order to see that

$$(9.9) \quad \begin{aligned} J_x &\leq z_k^k \left(\text{Lip}_\sigma^2 \int_{[0,t] \times \mathbf{Z}^d} \{E(|u_s^{(n)}(y)|^k)\}^{2/k} \rho_{t,x}(ds dy) \right)^{k/2} \\ &\leq (z_k \text{Lip}_\sigma)^k [\Upsilon(0)]^{(k-2)/2} \int_{[0,t] \times \mathbf{Z}^d} E(|u_s^{(n)}(y)|^k) \rho_{t,x}(ds dy). \end{aligned}$$

Thus

$$(9.10) \quad \begin{aligned} \sum_{x \in \mathbf{Z}^d} J_x &\leq (z_k \text{Lip}_\sigma)^k [\Upsilon(0)]^{(k-2)/2} \int_0^t \bar{P}(t - s) E(\|u_s^{(n)}\|_{\ell^k(\mathbf{Z}^d)}^k) ds \\ &\leq (z_k \text{Lip}_\sigma \sqrt{\Upsilon(0)})^k \cdot \sup_{r \geq 0} E(\|u_r^{(n)}\|_{\ell^k(\mathbf{Z}^d)}^k), \end{aligned}$$

thanks to (8.5).

In summary, (9.5) has the following consequence: For all $n \geq 0$,

$$(9.11) \quad \begin{aligned} \sup_{t \geq 0} \bar{M}_t^{(n+1)} &\leq \left(\frac{1 + \delta}{\delta}\right)^{k-1} \|u_0\|_{\ell^k(\mathbf{Z}^d)}^k + (1 + \delta)^{k-1} (z_k \text{Lip}_\sigma \sqrt{\Upsilon(0)})^k \sup_{t \geq 0} \bar{M}_t^{(n)}. \end{aligned}$$

Since $(1 + \delta)^{k-1} (z_k \text{Lip}_\sigma \sqrt{\Upsilon(0)})^k < 1$ and $\sup_{t \geq 0} \bar{M}_t^{(0)} = \|u_0\|_{\ell^k(\mathbf{Z}^d)}^k$, this shows that $C := \sup_{n \geq 0} \sup_{t \geq 0} \bar{M}_t^{(n)} < \infty$. Fatou’s lemma now implies half of the result,

since it shows that $E(\|u_t\|_{\ell^k(\mathbf{Z}^d)}^k) \leq \liminf_{n \rightarrow \infty} E(\|u_t^{(n)}\|_{\ell^k(\mathbf{Z}^d)}^k) \leq C$. The remainder of the proposition follows simply because $\|\bullet\|_{\ell^\infty(\mathbf{Z}^d)} \leq \|\bullet\|_{\ell^k(\mathbf{Z}^d)}$. \square

PROPOSITION 9.4. *Assume that $\limsup_{t \rightarrow \infty} t^\alpha P\{X_t = X'_t\} < 1$ for some $\alpha > 1$, where X and X' are two independent random walks with generator \mathcal{L} . If $k \in (2, \infty)$ satisfies $z_k \text{Lip}_\sigma \sqrt{\Upsilon(0)} < (1 + \delta)^{-(k-1)/k}$ for some $\delta > 0$, then there exists a finite constant A —depending only on $\delta, \text{Lip}_\sigma, \Upsilon(0)$ and $\|u_0\|_{\ell^1(\mathbf{Z}^d)}$ —such that*

$$(9.12) \quad E(\|u_{t+\tau} - u_t\|_{\ell^k(\mathbf{Z}^d)}^k) \leq \frac{A\tau^{k/2}}{(1+t)^\alpha} \quad \text{for every } t, \tau \geq 0.$$

Consequently, there exists a Hölder-continuous modification of the process $t \mapsto u_t(\bullet)$ with values in $\ell^\infty(\mathbf{Z}^d)$. Moreover, for this modification, there is a finite constant A' —depending only on $\delta, \text{Lip}_\sigma, \Upsilon(0)$ and $\|u_0\|_{\ell^1(\mathbf{Z}^d)}$ —such that

$$(9.13) \quad E\left(\sup_{s \neq r \in [t, t+1]} \sup_{x \in \mathbf{Z}^d} \left| \frac{u_r(x) - u_s(x)}{|r - s|^\eta} \right|^k\right) \leq \frac{A'}{(1+t)^\alpha},$$

as long as $0 \leq \eta < (k - 2)/(2k)$.

PROOF. Thanks to Lemma 9.3, $\|u_t\|_{\ell^k(\mathbf{Z}^d)}$ has a finite k th moment. This observation justifies the use of these moments in the ensuing discussion. Now we begin our proof in earnest.

The proof requires us to make a few small adjustments to the derivation of Lemma 5.1; specifically we now incorporate the fact that $\text{Lip}_\sigma^2 \Upsilon(0) < 1$ into that proof. Therefore, we mention only the required changes.

We use the notation of the proof of Lemma 5.1 and write

$$(9.14) \quad \{E(|u_{t+\tau}(x) - u_t(x)|^k)\}^{1/k} \leq |Q_1| + Q_2 + Q_3,$$

whence

$$(9.15) \quad E(|u_{t+\tau}(x) - u_t(x)|^k) \leq 3^{k-1}(|Q_1|^k + Q_2^k + Q_3^k).$$

Note that

$$(9.16) \quad \begin{aligned} & \sum_{x \in \mathbf{Z}^d} |Q_1|^k \\ & \leq \sum_{x \in \mathbf{Z}^d} \left(\sum_{y \in \mathbf{Z}^d} u_0(y) |p_{t+\tau}(y-x) - p_t(y-x)| \right)^k \\ & \leq \|u_0\|_{\ell^1(\mathbf{Z}^d)}^{k-1} \cdot \sum_{x \in \mathbf{Z}^d} \sum_{y \in \mathbf{Z}^d} u_0(y) |p_{t+\tau}(y-x) - p_t(y-x)|^k \\ & = \|u_0\|_{\ell^1(\mathbf{Z}^d)}^k \cdot \|p_{t+\tau} - p_t\|_{\ell^k(\mathbf{Z}^d)}^k \leq \|u_0\|_{\ell^1(\mathbf{Z}^d)}^k \cdot \|p_{t+\tau} - p_t\|_{\ell^2(\mathbf{Z}^d)}^k, \end{aligned}$$

thanks to Jensen’s inequality. We observe that

$$\begin{aligned}
 & \|p_{t+\tau} - p_t\|_{\ell^2(\mathbf{Z}^d)}^2 \\
 &= (2\pi)^{-d} \int_{[-\pi, \pi]^d} |e^{-t(1-\varphi(\xi))}|^2 |e^{-\tau(1-\varphi(\xi))} - 1|^2 d\xi \\
 (9.17) \quad &\leq \text{const} \cdot \tau^2 \int_{[-\pi, \pi]^d} |e^{-t(1-\varphi(\xi))}|^2 d\xi = \text{const} \cdot \tau^2 \mathbb{P}\{X_t = X'_t\} \\
 &\leq \text{const} \cdot \frac{\tau^2}{(1+t)^\alpha}.
 \end{aligned}$$

Consequently,

$$(9.18) \quad \sum_{x \in \mathbf{Z}^d} |Q_1|^k \leq \frac{\text{const} \cdot \tau^k}{(1+t)^{\alpha k/2}}.$$

We estimate Q_2 slightly differently from the proof of Lemma 5.1 as well.

For every $(t, x) \in \mathbf{R}_+ \times \mathbf{Z}^d$, let us define a similar Borel measure $R_{t,x}$ to $\rho_{t,x}$ [see (9.7)] as follows:

$$(9.19) \quad R_{t,x}(ds dy) := [p_{t+\tau-s}(y-x) - p_{t-s}(y-x)]^2 \mathbf{1}_{[0,t]}(s) ds \chi(dy).$$

Now we reexamine the first line of (5.7) and note that

$$\begin{aligned}
 Q_2^2 &\leq (z_k \text{Lip}_\sigma)^2 \sum_{y \in \mathbf{Z}^d} \int_0^t [p_{t+\tau-s}(y-x) - p_{t-s}(y-x)]^2 \{E(|u_s(y)|^k)\}^{2/k} ds \\
 (9.20) \quad &= (z_k \text{Lip}_\sigma)^2 \int_{\mathbf{R}_+ \times \mathbf{Z}^d} \{E(|u_s(y)|^k)\}^{2/k} R_{t,x}(ds dy).
 \end{aligned}$$

This follows from (1.1) and (5.7), but we use the optimal constant z_k in place of the slightly weaker $2\sqrt{k}$ that came from Lemma 3.1.

Lemma 9.2 implies that $R_{t,x}(\mathbf{R}_+ \times \mathbf{Z}^d) = \int_0^t \|p_{s+\tau} - p_s\|_{\ell^2(\mathbf{Z}^d)}^2 ds \leq 4\Upsilon(0)\tau^2$.

This bound and Jensen’s inequality together show that $\sum_{x \in \mathbf{Z}^d} Q_2^k$ is bounded from above by

$$\begin{aligned}
 & (z_k \text{Lip}_\sigma)^k (4\Upsilon(0)\tau^2)^{(k-2)/2} \sum_{x \in \mathbf{Z}^d} \int_{\mathbf{R}_+ \times \mathbf{Z}^d} E(|u_s(y)|^k) R_{t,x}(ds dy) \\
 (9.21) \quad &= (z_k \text{Lip}_\sigma)^k (4\Upsilon(0)\tau^2)^{(k-2)/2} \int_0^t \|p_{t+\tau-s} - p_{t-s}\|_{\ell^2(\mathbf{Z}^d)}^2 E(\|u_s\|_{\ell^k(\mathbf{Z}^d)}^k) ds.
 \end{aligned}$$

By an argument similar to the one used in Proposition 8.7, one is able to show that $E(\|u_s\|_{\ell^k(\mathbf{Z}^d)}^k) \leq \text{const} \cdot (1+s)^{-\alpha}$. Here is an outline of the proof: We can follow the proof of Lemma 9.3, but derive a better bound on $\sum_{x \in \mathbf{Z}^d} I_x \leq$

$\bar{P}(t)\|u_0\|_{\ell^1(\mathbf{Z}^d)}^k$, in order to obtain

$$\begin{aligned}
 & \mathbb{E}(\|u_t^{(n+1)}\|_{\ell^k(\mathbf{Z}^d)}^k) \\
 (9.22) \quad & \leq \left(\frac{1+\delta}{\delta}\right)^{k-1} \bar{P}(t)\|u_0\|_{\ell^1(\mathbf{Z}^d)}^k \\
 & \quad + (1+\delta)^{k-1}(z_k \text{Lip}_\sigma)^k[\Upsilon(0)]^{(k-2)/2} \int_0^t \bar{P}(t-s)\mathbb{E}(\|u_s^{(n)}\|_{\ell^k(\mathbf{Z}^d)}^k) ds.
 \end{aligned}$$

From here, we proceed along similar lines, as we did from (8.52) onward. We follow the proof of Proposition 8.2, using (9.22), in order to derive the following analog of (8.54):

$$(9.23) \quad \sup_{n \geq 0} \int_0^\infty F_n(s) ds \leq \frac{((1+\delta)/\delta)^{k-1}\|u_0\|_{\ell^1(\mathbf{Z}^d)}^k \Upsilon(0)}{1 - (1+\delta)^{k-1}(z_k \text{Lip}_\sigma \Upsilon(0))^k},$$

where $F_n(t) := \mathbb{E}(\|u_t^{(n+1)}\|_{\ell^k(\mathbf{Z}^d)}^k)$. In this way, we can obtain the bound $\mathbb{E}(\|u_s\|_{\ell^k(\mathbf{Z}^d)}^k) \leq \text{const} \cdot (1+s)^{-\alpha}$, as was needed. We use this bound, as well as (9.17) in (9.21), and split the integral into two parts (0 to $t/2$ and $t/2$ to t), in order to obtain the following:

$$(9.24) \quad \sum_{x \in \mathbf{Z}^d} Q_2^k \leq \frac{\text{const} \cdot \tau^k}{(1+t)^\alpha}.$$

Finally we estimate $\sum_{x \in \mathbf{Z}^d} Q_3^k$ by first modifying (5.11) as follows:

$$\begin{aligned}
 & Q_3^2 \leq (z_k \text{Lip}_\sigma)^2 \sum_{y \in \mathbf{Z}^d} \int_t^{t+\tau} [p_{t+\tau-s}(y-x)]^2 \{\mathbb{E}(|u_s(y)|^k)\}^{2/k} ds \\
 (9.25) \quad & = (z_k \text{Lip}_\sigma)^2 \int_{\mathbf{R}_+ \times \mathbf{Z}^d} \{\mathbb{E}(|u_s(y)|^k)\}^{2/k} \mathcal{R}_{t,\tau,x}(ds dy),
 \end{aligned}$$

where the Borel measures $\mathcal{R}_{t,\tau,x}$ are defined in a similar manner as in (9.7); that is,

$$(9.26) \quad \mathcal{R}_{t,\tau,x}(ds dy) := \sum_{y \in \mathbf{Z}^d} [p_{t+\tau-s}(y-x)]^2 \mathbf{1}_{[t,t+\tau]}(s) ds \chi(dy).$$

Because $\mathcal{R}_{t,\tau,x}(\mathbf{R}_+ \times \mathbf{Z}^d) = \int_0^\tau \bar{P}(s) ds \leq \tau$, Jensen’s inequality assures us that

$$\begin{aligned}
 & \sum_{x \in \mathbf{Z}^d} Q_3^k \leq (z_k \text{Lip}_\sigma)^k \tau^{(k-2)/2} \sum_{x \in \mathbf{Z}^d} \int_{\mathbf{R}_+ \times \mathbf{Z}^d} \mathbb{E}(|u_s(y)|^k) \mathcal{R}_{t,\tau,x}(ds dy) \\
 (9.27) \quad & = (z_k \text{Lip}_\sigma)^k \tau^{(k-2)/2} \int_t^{t+\tau} \bar{P}(t+\tau-s)\mathbb{E}(\|u_s\|_{\ell^k(\mathbf{Z}^d)}^k) ds \\
 & \leq \frac{\text{const} \cdot \tau^{k/2}}{(1+t)^\alpha},
 \end{aligned}$$

thanks to the bounds $E(\|u_s\|_{\ell^k(\mathbf{Z}^d)}^k) \leq \text{const} \cdot (1+s)^{-\alpha}$ and $\bar{P}(t+\tau-s) \leq 1$. Since $\|u_0\|_{\ell^k(\mathbf{Z}^d)} \leq \|u_0\|_{\ell^1(\mathbf{Z}^d)}$, displays (9.18), (9.24) and (9.27) together imply (9.12). This yields the first estimate of the proposition. The remaining assertions follow (9.12), using a suitable form of the Kolmogorov continuity theorem [35], Theorem 2.1, page 25, and the fact that $\sup_{x \in \mathbf{Z}^d} |u_t(x) - u_s(x)| \leq \|u_t - u_s\|_{\ell^k(\mathbf{Z}^d)}$. \square

PROOF OF THEOREM 2.7. We apply Proposition 8.7 and Chebyshev’s inequality in conjunction in order to see that

$$(9.28) \quad \sum_{n=1}^{\infty} \mathbb{P} \left\{ \sup_{x \in \mathbf{Z}^d} |u_n(x)| > \varepsilon \right\} \leq \frac{1}{\varepsilon^2} \sum_{n=1}^{\infty} E(\|u_n\|_{\ell^2(\mathbf{Z}^d)}^2) \leq \frac{\text{const}}{\varepsilon^2} \cdot \sum_{n=1}^{\infty} n^{-\alpha} < \infty.$$

Therefore, the Borel–Cantelli lemma implies that

$$(9.29) \quad \lim_{n \rightarrow \infty} \sup_{x \in \mathbf{Z}^d} |u_n(x)| = 0 \quad \text{a.s.}$$

We next note that the Burkholder’s constants z_k vary continuously for $k \geq 2$, and $z_2 = 1$ is the minimum; see Davis [23]. Davis [23] obtains z_k as the largest positive zero of the parabolic cylinder function of parameter k and this varies continuously in k ; see Abramowitz and Stegun [1].

If $\text{Lip}_\sigma \sqrt{\Upsilon(0)} < 1$, we can find $k > 2$ and $\delta > 0$ such that

$$(9.30) \quad z_k \text{Lip}_\sigma \sqrt{\Upsilon(0)} < (1 + \delta)^{-(k-1)/k}.$$

We can now use Proposition 9.4 (with $\eta = 0$) along with Chebyshev’s inequality to control the spacings

$$(9.31) \quad \mathbb{P} \left\{ \sup_{s \in [n, n+1]} \sup_{x \in \mathbf{Z}^d} |u_s(x) - u_n(x)| > \varepsilon \right\} \leq \frac{1}{\varepsilon^k} E \left(\sup_{s \in [n, n+1]} \sup_{x \in \mathbf{Z}^d} |u_t(x) - u_s(x)|^k \right) = O(n^{-\alpha}) \quad \text{as } n \rightarrow \infty.$$

We may use the Borel–Cantelli lemma and (9.29) in order to deduce that $\lim_{t \rightarrow \infty} \sup_{x \in \mathbf{Z}^d} |u_t(x)| = 0$ a.s. Thanks to this fact, Corollary 8.5 implies the seemingly stronger assertion that $\lim_{t \rightarrow \infty} \|u_t\|_{\ell^2(\mathbf{Z}^d)}^2 = 0$ a.s., and completes the proof. \square

APPENDIX: SOME RENEWAL THEORY

In this appendix we state and prove a few facts from (linear) renewal theory. These facts ought to be well known, but we have not succeeded to find concrete references, and so will describe them in some detail.

Let us suppose that the functions $h, g : (0, \infty) \rightarrow \mathbf{R}_+$ are locally integrable (say) and pre-defined, and let us look for a measurable solution $f : (0, \infty) \rightarrow \mathbf{R}_+$ to the renewal equation

$$(A.1) \quad f(t) = g(t) + \int_0^t h(t-s)f(s) \, ds \quad (t \geq 0).$$

If $h \in L^1(0, \infty)$, then this is a classical subject [24]. For a more general treatment, we may proceed with Picard's iteration: Let $f^{(0)}(t) : (0, \infty) \rightarrow \mathbf{R}_+$ be a fixed measurable function, and iteratively define

$$(A.2) \quad f^{(n+1)}(t) := g(t) + \int_0^t h(t-s)f^{(n)}(s) \, ds \quad (t > 0, n \geq 0).$$

LEMMA A.1. *Suppose that there exists a constant $\beta \in \mathbf{R}$ that satisfies the following three conditions: (i) $\gamma := \sup_{t \geq 0} [\exp(-\beta t)g(t)] < \infty$; (ii) $\rho := \int_0^\infty \exp(-\beta t)h(t) \, dt < 1$; (iii) $\sup_{t \geq 0} [\exp(-\beta t)f^{(0)}(t)] < \infty$. Then (A.1) has a unique nonnegative solution f that satisfies the following:*

$$(A.3) \quad f(t) \leq \frac{\gamma e^{\beta t}}{1 - \rho} \quad (t \geq 0).$$

Moreover, $\lim_{n \rightarrow \infty} \sup_{t \geq 0} (e^{-\beta t} |f^{(n)}(t) - f(t)|) = 0$.

PROOF. Choose such a $\beta \in \mathbf{R}$, and define

$$(A.4) \quad \gamma := \sup_{t \geq 0} [e^{-\beta t} g(t)], \quad \rho := \int_0^\infty e^{-\beta t} h(t) \, dt < 1$$

and

$$(A.5) \quad C_k := \sup_{t \geq 0} (e^{-\beta t} f^{(k)}(t)), \quad D_k := \sup_{t \geq 0} (e^{-\beta t} |f^{(k)}(t) - f^{(k-1)}(t)|),$$

for integers $k \geq 1$. Thanks to the definition of the $f^{(k)}$'s,

$$(A.6) \quad C_{n+1} \leq \gamma + \rho C_n, \quad D_{n+1} \leq \rho D_n \quad (n \geq 0).$$

Consequently, $\sup_{n \geq 0} C_n \leq \gamma(1 - \rho)^{-1}$ and $D_n = O(\rho^n)$. Since $\sum_{n=0}^\infty D_n < \infty$, it follows that there exists a function f such that $\sup_{t \geq 0} (e^{-\beta t} |f^{(n)}(t) - f(t)|) \rightarrow 0$ as $n \rightarrow \infty$, and $\sup_{t \geq 0} (e^{-\beta t} f(t)) \leq \sup_{n \geq 0} C_n$. These observations together prove the lemma. \square

The following is the main result of this appendix.

LEMMA A.2 (Comparison lemma). *Suppose there exists $\beta \in \mathbf{R}$ such that: (i) $\gamma := \sup_{t \geq 0} [\exp(-\beta t)g(t)] < \infty$ and (ii) $\rho := \int_0^\infty \exp(-\beta t)h(t) \, dt < 1$; and*

let f denote the unique nonnegative solution to (A.1) that satisfies (A.3). If $F : \mathbf{R}_+ \rightarrow \mathbf{R}_+$ satisfies: (a) $\sup_{t \geq 0} [\exp(-\beta t) F(t)] < \infty$ and (b)

$$(A.7) \quad F(t) \geq g(t) + \int_0^t h(t-s)F(s) \, ds \quad (t \geq 0),$$

then $f(t) \leq F(t)$ for all $t \geq 0$. Finally, if we replace condition (A.7) by

$$(A.8) \quad F(t) \leq g(t) + \int_0^t h(t-s)F(s) \, ds \quad (t \geq 0),$$

then $f(t) \geq F(t)$ for all $t \geq 0$.

PROOF. We will prove (A.7); (A.8) is proved similarly.

We apply Picard's iteration with initial function $f^{(0)} := F$ and note that

$$(A.9) \quad f^{(1)}(t) = g(t) + \int_0^t h(t-s)F(s) \, ds \leq F(t) \quad (t \geq 0).$$

This and induction together show that $f^{(n+1)}(t) \leq f^{(n)}(t)$ for all $t \geq 0$ and $n \geq 0$. Let $n \rightarrow \infty$ to deduce the lemma from Lemma A.1. \square

Acknowledgments. We thank an anonymous referee and an Associate Editor whose remarks and suggestions have helped improve the presentation of this paper.

REFERENCES

- [1] ABRAMOWITZ, M. and STEGUN, I. A., eds. (1992). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. Dover, New York. [MR1225604](#)
- [2] AIZENMAN, M. and WARZEL, S. (2011). Absence of mobility edge for the Anderson random potential on tree graphs at weak disorder. *European Physics Letters* **96** 37004.
- [3] AMIR, G., CORWIN, I. and QUASTEL, J. (2011). Probability distribution of the free energy of the continuum directed random polymer in $1 + 1$ dimensions. *Comm. Pure Appl. Math.* **64** 466–537. [MR2796514](#)
- [4] ANDERSON, W. J. (1972). Local behaviour of solutions of stochastic integral equations. *Trans. Amer. Math. Soc.* **164** 309–321. [MR0297031](#)
- [5] BALÁZS, M., QUASTEL, J. and SEPPÄLÄINEN, T. (2011). Fluctuation exponent of the KPZ/stochastic Burgers equation. *J. Amer. Math. Soc.* **24** 683–708. [MR2784327](#)
- [6] BERTINI, L. and CANCRINI, N. (1995). The stochastic heat equation: Feynman–Kac formula and intermittence. *J. Stat. Phys.* **78** 1377–1401. [MR1316109](#)
- [7] BORODIN, A. and CORWIN, I. (2014). Moments and Lyapunov exponents for the parabolic Anderson model. *Ann. Appl. Probab.* **24** 1172–1198. [MR3199983](#)
- [8] BURKHOLDER, D. L. (1966). Martingale transforms. *Ann. Math. Statist.* **37** 1494–1504. [MR0208647](#)
- [9] BURKHOLDER, D. L., DAVIS, B. J. and GUNDY, R. F. (1972). Integral inequalities for convex functions of operators on martingales. In *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability (Univ. California, Berkeley, Calif., 1970/1971), Vol. II: Probability Theory* 223–240. Univ. California Press, Berkeley, CA. [MR0400380](#)
- [10] BURKHOLDER, D. L. and GUNDY, R. F. (1970). Extrapolation and interpolation of quasi-linear operators on martingales. *Acta Math.* **124** 249–304. [MR0440695](#)

- [11] CARLEN, E. and KRÉE, P. (1991). L^p estimates on iterated stochastic integrals. *Ann. Probab.* **19** 354–368. [MR1085341](#)
- [12] CARMONA, R., KORALOV, L. and MOLCHANOV, S. (2001). Asymptotics for the almost sure Lyapunov exponent for the solution of the parabolic Anderson problem. *Random Oper. Stoch. Equ.* **9** 77–86. [MR1910468](#)
- [13] CARMONA, R. A. and MOLCHANOV, S. A. (1994). Parabolic Anderson problem and intermittency. *Mem. Amer. Math. Soc.* **108** viii+125. [MR1185878](#)
- [14] CHUNG, K. L. and FUCHS, W. H. J. (1951). On the distribution of values of sums of random variables. *Mem. Amer. Math. Soc.* **1951** 12. [MR0040610](#)
- [15] CONUS, D. (2013). Moments for the parabolic Anderson model: On a result by Hu and Nualart. *Commun. Stoch. Anal.* **7** 125–152. [MR3080991](#)
- [16] CONUS, D., JOSEPH, M. and KHOSHNEVISAN, D. (2012). Correlation-length bounds, and estimates for intermittent islands in parabolic SPDEs. *Electron. J. Probab.* **17** 15. [MR3005720](#)
- [17] CONUS, D., JOSEPH, M. and KHOSHNEVISAN, D. (2013). On the chaotic character of the stochastic heat equation, before the onset of intermittency. *Ann. Probab.* **41** 2225–2260. [MR3098071](#)
- [18] COX, J. T., FLEISCHMANN, K. and GREVEN, A. (1996). Comparison of interacting diffusions and an application to their ergodic theory. *Probab. Theory Related Fields* **105** 513–528. [MR1402655](#)
- [19] COX, J. T. and GREVEN, A. (1994). Ergodic theorems for infinite systems of locally interacting diffusions. *Ann. Probab.* **22** 833–853. [MR1288134](#)
- [20] CRANSTON, M., GAUTHIER, D. and MOUNTFORD, T. S. (2010). On large deviations for the parabolic Anderson model. *Probab. Theory Related Fields* **147** 349–378. [MR2594357](#)
- [21] CRANSTON, M. and MOLCHANOV, S. (2007). Quenched to annealed transition in the parabolic Anderson problem. *Probab. Theory Related Fields* **138** 177–193. [MR2288068](#)
- [22] CRANSTON, M., MOUNTFORD, T. S. and SHIGA, T. (2002). Lyapunov exponents for the parabolic Anderson model. *Acta Math. Univ. Comenian. (N.S.)* **71** 163–188. [MR1980378](#)
- [23] DAVIS, B. (1976). On the L^p norms of stochastic integrals and other martingales. *Duke Math. J.* **43** 697–704. [MR0418219](#)
- [24] FELLER, W. (1966). *An Introduction to Probability Theory and Its Applications. Vol. II.* Wiley, New York. [MR0210154](#)
- [25] FOONDUN, M. and KHOSHNEVISAN, D. (2009). Intermittence and nonlinear parabolic stochastic partial differential equations. *Electron. J. Probab.* **14** 548–568. [MR2480553](#)
- [26] FUNAKI, T. (1983). Random motion of strings and related stochastic evolution equations. *Nagoya Math. J.* **89** 129–193. [MR0692348](#)
- [27] GEISS, C. and MANTHEY, R. (1994). Comparison theorems for stochastic differential equations in finite and infinite dimensions. *Stochastic Process. Appl.* **53** 23–35. [MR1290705](#)
- [28] GNEDENKO, B. V. and KOLMOGOROV, A. N. (1968). *Limit Distributions for Sums of Independent Random Variables.* Addison-Wesley, Reading, MA. [MR0233400](#)
- [29] GREVEN, A. and DEN HOLLANDER, F. (2007). Phase transitions for the long-time behavior of interacting diffusions. *Ann. Probab.* **35** 1250–1306. [MR2330971](#)
- [30] IWATA, K. (1987). An infinite-dimensional stochastic differential equation with state space $C(\mathbf{R})$. *Probab. Theory Related Fields* **74** 141–159. [MR0863723](#)
- [31] KHOSHNEVISAN, D. (2014). *Analysis of Stochastic Partial Differential Equations.* CBMS Regional Conference Series in Mathematics **119**. Amer. Math. Soc., Providence, RI. [MR3222416](#)
- [32] MUELLER, C. (1991). On the support of solutions to the heat equation with noise. *Stochastics Stochastics Rep.* **37** 225–245. [MR1149348](#)
- [33] MUELLER, C. and NUALART, D. (2008). Regularity of the density for the stochastic heat equation. *Electron. J. Probab.* **13** 2248–2258. [MR2469610](#)

- [34] PRÉVÔT, C. and RÖCKNER, M. (2007). *A Concise Course on Stochastic Partial Differential Equations. Lecture Notes in Math.* **1905**. Springer, Berlin. [MR2329435](#)
- [35] REVUZ, D. and YOR, M. (1991). *Continuous Martingales and Brownian Motion. Grundlehren der mathematischen Wissenschaften* **293**. Springer, Berlin. [MR1083357](#)
- [36] SHIGA, T. (1992). Ergodic theorems and exponential decay of sample paths for certain interacting diffusion systems. *Osaka J. Math.* **29** 789–807. [MR1192741](#)
- [37] SHIGA, T. and SHIMIZU, A. (1980). Infinite-dimensional stochastic differential equations and their applications. *J. Math. Kyoto Univ.* **20** 395–416. [MR0591802](#)

N. GEORGIU
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SUSSEX
PEVENSEY 3 5C15
BRIGHTON, BN1 9RH
UNITED KINGDOM
E-MAIL: N.Georgiou@sussex.ac.uk

D. KHOSHNEVISAN
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF UTAH
SALT LAKE CITY, UTAH 84112-0090
USA
E-MAIL: davar@math.utah.edu
URL: <http://www.math.utah.edu/~davar/>

M. JOSEPH
DEPARTMENT OF PROBABILITY AND STATISTICS
UNIVERSITY OF SHEFFIELD
SHEFFIELD, ENGLAND, S3 7RH
UNITED KINGDOM
E-MAIL: m.joseph@shef.ac.uk
URL: <http://mathew-joseph.staff.shef.ac.uk/>

S.-Y. SHIU
DEPARTMENT OF MATHEMATICS
NATIONAL CENTRAL UNIVERSITY
JHONGLI CITY, TAoyUAN COUNTY 32001
TAIWAN
E-MAIL: shiu@math.ncu.edu.tw