

THE VERTEX-CUT-TREE OF GALTON–WATSON TREES CONVERGING TO A STABLE TREE

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We consider a fragmentation of discrete trees where the internal vertices are deleted independently at a rate proportional to their degree. Informally, the associated cut-tree represents the genealogy of the nested connected components created by this process. We essentially work in the setting of Galton–Watson trees with offspring distribution belonging to the domain of attraction of a stable law of index $\alpha \in (1, 2)$. Our main result is that, for a sequence of such trees \mathcal{T}_n conditioned to have size n , the corresponding rescaled cut-trees converge in distribution to the stable tree of index α , in the sense induced by the Gromov–Prokhorov topology. This gives an analogue of a result obtained by Bertoin and Miermont in the case of Galton–Watson trees with finite variance.

1. Introduction and main result. Fragmentations of random trees were first introduced in the work of Meir and Moon [23] as a recursive random edge-deletion process on discrete trees. Since then, it has been recognized that fragmentations of discrete and continuous trees appear in several natural contexts; see, for example, [11, 15] for a connection with forest fire models, [6, 8] for fragmentations of the Brownian tree [5] and its relation to the additive coalescent, and [3, 24, 25] for fragmentations of the stable tree of index $\alpha \in (1, 2)$ [17]. The fragmentations considered in the two last cases, which arise naturally in the setting of Brownian and stable trees, are self-similar fragmentations as studied by Bertoin [9], whose characteristics are explicitly known.

Several recent articles investigated the question of the asymptotic distribution of the number of cuts needed to isolate a specific vertex, for various classes of random trees. In specific cases, Panholzer [26] showed that the Rayleigh distribution arises naturally as a limit in this context, and Janson [21] showed that this limiting result holds for general Galton–Watson trees with a finite variance offspring distribution, using a method of moments. He also established a connection to the Brownian tree, which is natural since the Rayleigh distribution is the law of the distance between two uniformly chosen vertices in the CRT. Later, Addario-Berry, Broutin and Holmgren [4] provided a different proof giving a more concrete connection to the Brownian tree. Bertoin and Miermont [12] then studied the genealogy of the cutting procedure in itself, which is related to the problem of the

Received December 2013; revised May 2014.

MSC2010 subject classifications. 60F05, 60J80.

Key words and phrases. Galton–Watson tree, cut-tree, stable continuous random tree.

isolation of several vertices rather than just the root (certain of these ideas were implicitly present in former papers, including [4, 11]). This allows to code the discrete cutting procedure in terms of a “cut-tree,” whose scaling limit is shown to be a Brownian tree that describes in some sense the genealogy of the Aldous–Pitman fragmentation [6].

Note that the results of [4], by introducing a reversible transformation of the Brownian tree, can be understood as building the “first branch” of the limiting cut-tree, the latter being a kind of iteration *ad libitum* of this transformation. This transformation was extended in [2] in the context of a fragmentation of stable trees. The main goal of the present work is to show that the approach of Bertoin and Miermont [12] can also be adapted to Galton–Watson trees with offspring distribution in the domain of attraction of a non-Gaussian stable law, showing the convergence of the whole discrete cut-tree to a limiting stable tree. This gives in passing a natural definition of the continuum cut-tree for the fragmentation studied in [25].

Let us describe more precisely the result of [12] we are interested in. Consider a sequence of Galton–Watson trees \mathcal{T}_n , conditioned to have exactly n edges, with critical offspring distribution having finite variance σ^2 . The associated cut-trees $\text{Cut}(\mathcal{T}_n)$ describe the genealogy of the fragments obtained by deleting the edges in a uniform random order. It is well known that the rescaled trees $(\sigma/\sqrt{n}) \cdot \mathcal{T}_n$ converge in distribution to the Brownian tree \mathcal{T} ; see [5] for the convergence of the associated contour functions, which implies that this convergence holds for the commonly used Gromov–Hausdorff topology, and for the Gromov–Prokhorov topology. In the present work, we will mainly use the latter. Bertoin and Miermont showed that there is in fact the joint convergence

$$\left(\frac{\sigma}{\sqrt{n}} \mathcal{T}_n, \frac{1}{\sigma\sqrt{n}} \text{Cut}(\mathcal{T}_n) \right) \xrightarrow[n \rightarrow \infty]{(d)} (\mathcal{T}, \text{Cut}(\mathcal{T})),$$

where $\text{Cut}(\mathcal{T})$ is the so-called cut-tree of \mathcal{T} . Informally, $\text{Cut}(\mathcal{T})$ describes the genealogy of the fragments obtained by cutting \mathcal{T} at points chosen according to a Poisson point process on its skeleton. Moreover, $\text{Cut}(\mathcal{T})$ has the same law as \mathcal{T} .

Our goal is to show an analogue result in the case where the \mathcal{T}_n are Galton–Watson trees with offspring distribution belonging to the domain of attraction of a stable law of index $\alpha \in (1, 2)$, and \mathcal{T} is the stable tree of index α . For the stable tree, a self-similar fragmentation arises naturally by splitting at branching points with a rate proportional to their “width,” as shown in [25]. This will lead us to modify the edge-deletion mechanism for the discrete trees, so that the rate at which internal vertices are removed increases with their degree. Therefore, we call *edge-fragmentation* the fragmentation studied in [12], and *vertex-fragmentation* our model. Note that more general fragmentations of the stable tree can be constructed by splitting both at branching points and at uniform points of the skeleton, as in [3]. However, these fragmentations are not self-similar (see [25]), and will not be studied here.

In the rest of the **Introduction**, we will describe our setting more precisely and give the exact definition of the cut-trees, both in the discrete and the continuous cases. This will enable us to state our main results in Section 1.4.

1.1. *Vertex-fragmentation of a discrete tree.* We begin with some notation. Let \mathbb{T} be the set of all finite plane rooted trees. For every $T \in \mathbb{T}$, we call $E(T)$ the set of edges of T , $V(T)$ the set of vertices of T , and $\rho(T)$ the root-vertex of T . For each vertex $v \in V(T)$, $\text{deg}(v, T)$ denotes the number of children of v in T (or $\text{deg } v$, if this notation is not ambiguous), and for each edge $e \in E(T)$, e^- (resp., e^+) denotes the extremity of e which is closest to (resp., furthest away from) the root.

For any tree T with n edges, we label the vertices of T by v_0, v_1, \dots, v_n , and the edges of T by e_1, \dots, e_n , in the depth-first order. Note that the planar structure of T gives an order on the offspring of each vertex, say “from left to right,” hence the depth-first order is well defined. With this notation, we have $v_j = e_j^+$ for all $j \in \{1, \dots, n\}$.

We let $T \in \mathbb{T}$ be a finite tree with n edges. We consider a discrete-time fragmentation on T , which can be described as follows:

- at each step, we mark a vertex of T at random, in such a way that the probability of marking a given vertex v is proportional to $\text{deg } v$;
- when a vertex v is marked, we delete all the edges e such that $e^- = v$.

Note that the total number of steps N is at most n . To keep track of the genealogy induced by this edge-deletion process, we introduce a new structure called the cut-tree of T , denoted by $\text{Cut}_v(T)$.

For all $r \in \{1, \dots, N\}$, we let $v(r)$ be the vertex which receives a mark at step r , $E_r = \{e \in E(T) : e^- = v(r)\}$ be the set of the edges which are deleted at step r , $k_r = |E_r|$, and $D_r = \{i \in \{1, \dots, n\} : e_i \in \bigcup_{r' \leq r} E_{r'}\}$. We say that $j \sim_r j'$ if and only if e_j and $e_{j'}$ are still connected in the forest obtained from T by deleting the edges in D_r . Thus, \sim_r is an equivalence relation on $\{1, \dots, n\} \setminus D_r$. The family of the equivalence classes (without repetition) of the relations \sim_r for $r = 1, \dots, N$ forms the set of internal nodes of $\text{Cut}_v(T)$. The initial block $\{1, \dots, n\}$ is seen as the root, and the leaves of $\text{Cut}_v(T)$ are given by $1, \dots, n$. We stress that we distinguish the leaves i and the internal nodes $\{i\}$.

We now build the cut-tree $\text{Cut}_v(T)$ inductively. At the r th step, we let B be the equivalence class for \sim_{r-1} containing the indices i such that $e_i \in E_r$. Deleting the edges in E_r splits the block B into k'_r equivalence classes $B_1, \dots, B_{k'_r}$ for \sim_r , with $k'_r \leq k_r + 1$. We draw k'_r edges between B and the sets $B_1, \dots, B_{k'_r}$, and k_r edges between B and the leaves i such that $e_i \in E_r$. Thus, the graph-distance between the leaf i and the root in $\text{Cut}_v(T)$ is the number of cuts in the component of T containing the edge e_i before e_i itself is removed. Note that $\text{Cut}_v(T)$ does not have a natural planar structure, but that the actual embedding does not intervene in our work. Figure 1 gives an example of this construction for a tree T with 16 edges.

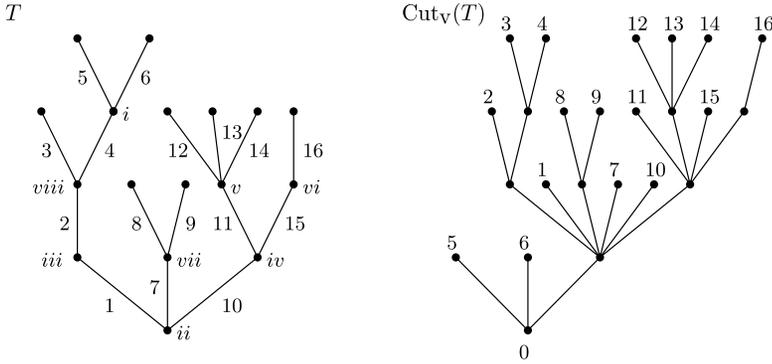


FIG. 1. The cut-tree $\text{Cut}_v(T)$ of a tree T . The order of deletion of the internal vertices of T is indicated in Roman numerals. The correspondence between the edges of T and the leaves of $\text{Cut}_v(T)$ is indicated in Arabic numerals.

If T is a random tree, the fragmentation of T and the cut-tree $\text{Cut}_v(T)$ are defined similarly, by conditioning on T and performing the above construction.

Note that, equivalently, we could mark the edges of T in a uniform random order, and delete all the edges e such that $e^- = e_i^-$, as soon as e_i is marked. The cut-tree $\text{Cut}_v(T)$ would then be obtained by performing the same construction with $E_r = \{e \in E(T) : e^- = e_i^-\}$. This procedure sometimes adds “neutral steps,” which have no effect on the fragmentation, but this does not change the cut-tree. It will sometimes be more convenient to work with this point of view, for example, in Sections 2.1 and 4.

1.2. *Fragmentation and cut-tree of the stable tree of index $\alpha \in (1, 2)$.* Following Duquesne and Le Gall (see, e.g., [18]), we see stable trees as random rooted \mathbb{R} -trees.

DEFINITION 1.1. A metric space (T, d) is an \mathbb{R} -tree if, for every $u, v \in T$:

- There exists a unique isometric map $f_{u,v}$ from $[0, d(u, v)]$ into T such that $f_{u,v}(0) = u$ and $f_{u,v}(d(u, v)) = v$.
- For any continuous injective map f from $[0, 1]$ into T , such that $f(0) = u$ and $f(1) = v$, we have

$$f([0, 1]) = f_{u,v}([0, d(u, v)]) := \llbracket u, v \rrbracket.$$

A rooted \mathbb{R} -tree is an \mathbb{R} -tree (T, d, ρ) with a distinguished point ρ called the root.

The trees we will work with can be seen as \mathbb{R} -trees coded by continuous functions from $[0, 1]$ into \mathbb{R}_+ , as in [18]. In particular, the stable tree (\mathcal{T}, d) of index α is the \mathbb{R} -tree coded by the excursion of length 1 of the height process $H^{(\alpha)}$, defined as follows in [17]. Let $X^{(\alpha)}$ be a stable spectrally positive Lévy process with

parameter α , whose normalization will be prescribed in Section 2.2.1. For every $t > 0$, let $\widehat{X}^{(\alpha,t)}$ be the process defined by

$$\widehat{X}_s^{(\alpha,t)} = \begin{cases} X_t^{(\alpha)} - X_{(t-s)^-}^{(\alpha)}, & \text{if } 0 \leq s < t, \\ X_t^{(\alpha)}, & \text{if } s = t, \end{cases}$$

and write $\widehat{S}_s^{(\alpha,t)} = \sup_{0 \leq r \leq s} \widehat{X}_r^{(\alpha,t)}$ for all $r \in [0, t]$.

DEFINITION 1.2. The height process $H^{(\alpha)}$ is the real-valued process such that $H_0 = 0$ and, for every $t > 0$, H_t is the local time at level 0 at time t of the process $\widehat{X}^{(\alpha,t)} - \widehat{S}^{(\alpha,t)}$.

The normalization of local time, and the proof of the existence of a continuous modification of this process, are given in [17], Section 1.2. This definition of \mathcal{T} allows us to introduce the canonical projection $p : [0, 1] \rightarrow \mathcal{T}$. We endow \mathcal{T} with a probability mass-measure μ defined as the image of the Lebesgue measure on $[0, 1]$ under p , and say that the rot of \mathcal{T} is the unique point which has height 0.

For the fragmentation of the stable tree, we will use a process introduced and studied by Miermont in [25], which consists in deleting the nodes of \mathcal{T} in such a way that the fragmentation is self-similar. We first recall that the multiplicity of a point v in an \mathbb{R} -tree T can be defined as the number of connected components of $T \setminus \{v\}$. To be consistent with the definitions of Section 1.1, we define the degree of a point as its multiplicity minus 1, and say that a branching point of T is a point v such that $\text{deg}(v, T) \geq 2$. Duquesne and Le Gall have shown in [18], Theorem 4.6, that a.s. the branching points in \mathcal{T} form a countable set, and that these branching points have infinite degree. We let \mathcal{B} denote the set of these branching points. For any $b \in \mathcal{B}$, one can define the local time, or width of b as the almost sure limit

$$L(b) = \lim_{\varepsilon \rightarrow 0^+} \varepsilon^{-1} \mu\{v \in \mathcal{T} : b \in \llbracket \rho, v \rrbracket, d(b, v) < \varepsilon\},$$

where ρ is the root of the stable tree \mathcal{T} . The existence of this quantity is justified in [25], Proposition 2, (see also [18]).

We can now describe the fragmentation we are interested in. Conditionally on \mathcal{T} , we let $(t_i, b_i)_{i \in I}$ be the family (indexed by a countable set I) of the atoms of a Poisson point process with intensity $dt \otimes \sum_{b \in \mathcal{B}} L(b) \delta_b(dv)$ on $\mathbb{R}_+ \times \mathcal{B}$. Seeing these atoms as marks on the branching points of \mathcal{T} , we let $\overline{\mathcal{T}}(t) = \mathcal{T} \setminus \{b_i : t_i \leq t\}$.

For every $x \in \mathcal{T}$, we let $\mathcal{T}_x(t)$ be the connected component of $\overline{\mathcal{T}}(t)$ containing x , with the convention that $\mathcal{T}_x(t) = \emptyset$ if $x \notin \overline{\mathcal{T}}(t)$. We also let $\mu_x(t) = \mu(\mathcal{T}_x(t))$. Adding a distinguished point 0 to \mathcal{T} , we define a function δ from $(\mathcal{T} \sqcup \{0\})^2$ into $\mathbb{R}_+ \cup \{\infty\}$, such that for all $x, y \in \mathcal{T}$,

$$\begin{aligned} \delta(0, 0) &= 0, & \delta(0, x) &= \delta(x, 0) = \int_0^\infty \mu_x(t) dt, \\ \delta(x, y) &= \int_{t(x,y)}^\infty (\mu_x(t) + \mu_y(t)) dt, \end{aligned}$$

where $t(x, y) := \inf\{t \in \mathbb{R}_+ : \mathcal{T}_x(t) \neq \mathcal{T}_y(t)\}$ is a.s. finite. We think of δ as our new “distance” in the cut-tree. This definition might seem surprising, but the results of Section 2.1 will show that it provides an analogue of the distance we defined in the discrete case, in terms of number of cuts; as will be explained in Section 3.1, it also has a natural interpretation as a time-change between two fragmentation processes of the stable tree, studied in [24] and [25]. The role of the extra point 0 in our (time-changed) fragmentation will be similar to the role played by the root of \mathcal{T} in the “fragmentation at heights” which will be introduced in Section 3.1.

A first idea would be to build the vertex-cut-tree $\text{Cut}_v(\mathcal{T})$ as a completion of $(\mathcal{T} \sqcup \{0\}, \delta)$. However, making this idea rigorous is difficult, since it is not clear whether δ is a.s. finite, and defines a distance on $\mathcal{T} \sqcup \{0\}$. We will instead use an approach introduced by Aldous, which consists in building a continuous random tree such that the subtrees determined by k randomly chosen leaves have the right distribution. To this end, we use the conditions given by Aldous in [5], Theorem 3.

Set $\xi(0) = 0$, and let $(\xi(i))_{i \in \mathbb{N}}$ be an i.i.d. sequence distributed according to μ , conditionally on \mathcal{T} . The key argument of our construction is the identity in law

$$(\delta(\xi(i), \xi(j)))_{i, j \geq 0} \stackrel{(d)}{=} (d(\xi(i + 1), \xi(j + 1)))_{i, j \geq 0},$$

which will be proven in Section 3.1. In particular, it implies that almost surely, for all $i, j \geq 0$, $\delta(\xi(i), \xi(j))$ is finite, and that δ is a.s. a distance on $\{\xi(i), i \geq 0\}$. This allows us to see the spaces $\mathcal{R}(k) := (\{\xi(i), 0 \leq i \leq k\}, \delta)$, for all $k \in \mathbb{N}$, as random rooted trees with k leaves. Using the terminology of Aldous, $(\mathcal{R}(k), k \in \mathbb{N})$ forms a *consistent* family of random rooted trees which satisfies the *leaf-tight condition*:

$$\min_{1 \leq j \leq k} \delta(\xi(0), \xi(j)) \xrightarrow[k \rightarrow \infty]{\mathbb{P}} 0.$$

Indeed, the second part of Theorem 3 of [5] shows that these conditions hold for the reduced trees $(\{\xi(i), 1 \leq i \leq k + 1\}, d)$. As a consequence, the family $(\mathcal{R}(k), k \in \mathbb{N})$ can be represented as a continuous random tree $\text{Cut}_v(\mathcal{T})$, and $(\delta(\xi(i), \xi(j)))_{i, j \geq 0}$ is the matrix of mutual distances between the points of an i.i.d. sample of $\text{Cut}_v(\mathcal{T})$. This tree $\text{Cut}_v(\mathcal{T})$ is called the cut-tree of \mathcal{T} . Note that $\text{Cut}_v(\mathcal{T})$ depends on \mathcal{T} and on the extra randomness of the Poisson process.

1.3. *Fragmentation and cut-tree of the Brownian tree.* We will also work on the Brownian tree $(\mathcal{T}^{\text{br}}, d^{\text{br}}, \rho^{\text{br}})$, which was defined by Aldous (see [5]) as the \mathbb{R} -tree coded by $(H_t)_{0 \leq t \leq 1} = (2B_t)_{0 \leq t \leq 1}$, where B denotes the standard Brownian excursion of length 1. This tree can be seen as the stable tree of index $\alpha = 2$ (up to a scale factor, with the normalization we will use). In particular, we have a probability mass-measure μ^{br} on \mathcal{T}^{br} , defined as the image of the Lebesgue measure on $[0, 1]$ under the canonical projection. We also define a length-measure l on \mathcal{T}^{br} , which is the sigma-finite measure such that, for all $u, v \in \mathcal{T}^{\text{br}}$, $l(\llbracket u, v \rrbracket) = d^{\text{br}}(u, v)$.

The fragmentation of the Brownian tree we consider is the same as in [12]: conditionally on \mathcal{T}^{br} , we let $(t_i, b_i)_{i \in I}$ be the family of the atoms of a Poisson point process with intensity $dt \otimes l(dv)$ on $\mathbb{R}_+ \times \mathcal{T}^{\text{br}}$. As for the stable tree, we let $\mathcal{T}_x^{\text{br}}(t)$ be the connected component of $\mathcal{T}^{\text{br}} \setminus \{b_i : t_i \leq t\}$, and $\mu_x^{\text{br}}(t) = \mu^{\text{br}}(\mathcal{T}_x^{\text{br}}(t))$, for every $x \in \mathcal{T}^{\text{br}}$. Adding a distinguished point 0 to \mathcal{T}^{br} , we define a function δ^{br} on $(\mathcal{T}^{\text{br}} \sqcup \{0\})^2$ such that for all $x, y \in \mathcal{T}^{\text{br}}$,

$$\delta^{\text{br}}(0, 0) = 0, \quad \delta^{\text{br}}(0, x) = \delta^{\text{br}}(x, 0) = \int_0^\infty \mu_x^{\text{br}}(t) dt,$$

$$\delta^{\text{br}}(x, y) = \int_{t^{\text{br}}(x, y)}^\infty (\mu_x^{\text{br}}(t) + \mu_y^{\text{br}}(t)) dt,$$

where $t^{\text{br}}(x, y) := \inf\{t \in \mathbb{R}_+ : \mathcal{T}_x^{\text{br}}(t) \neq \mathcal{T}_y^{\text{br}}(t)\}$ is a.s. finite. As shown in [12], we can define a new tree $\text{Cut}(\mathcal{T}^{\text{br}})$ for which the matrix of mutual distances between the points of an i.i.d. sample of $\text{Cut}(\mathcal{T}^{\text{br}})$ is $(\delta(\xi(i), \xi(j)))_{i, j \geq 0}$, where $\xi(0) = 0$ and $(\xi(i))_{i \in \mathbb{N}}$ is an i.i.d. sequence distributed according to μ^{br} , conditionally on \mathcal{T}^{br} . Moreover, $\text{Cut}(\mathcal{T}^{\text{br}})$ has the same law as \mathcal{T}^{br} .

1.4. *Main results.* As stated in the **Introduction**, we mainly work in the setting of Galton–Watson trees with critical offspring distribution ν , where ν is a probability distribution belonging to the domain of attraction of a stable law of index $\alpha \in (1, 2)$. We shall also assume that ν is aperiodic. Finally, for a technical reason, we will need the additional hypothesis

$$(1) \quad \sup_{r \geq 1} \left(\frac{r \mathbb{P}(\hat{Z} = r)}{\mathbb{P}(\hat{Z} > r)} \right) < \infty,$$

where \hat{Z} is a random variable such that $\mathbb{P}(\hat{Z} = r) = r\nu(\{r\})$. For example, this is the case if $\nu(\{r\})$ is equivalent to $c/r^{\alpha+1}$ as $n \rightarrow \infty$, for a constant $c \in (0, \infty)$. In all our work, we shall implicitly work for values of n such that, for a Galton–Watson tree T with offspring distribution ν , $\mathbb{P}(|E(T)| = n) \neq 0$. We let \mathcal{T}_n be a ν -Galton–Watson tree, conditioned to have exactly n edges. We let δ_n denote the graph-distance on $\{0, 1, \dots, n\}$ induced by $\text{Cut}_\nu(\mathcal{T}_n)$. We will use the notation ρ_n for the root of \mathcal{T}_n , and μ_n for the uniform distribution on $E(\mathcal{T}_n)$ (by slight abuse, μ_n will also sometimes be used for the uniform distribution on $\{1, \dots, n\}$).

Our main goal is to study the asymptotic behavior of $\text{Cut}_\nu(\mathcal{T}_n)$ as $n \rightarrow \infty$. To this end, it will be convenient to see trees as pointed metric measure spaces, and work with the Gromov–Prokhorov topology on the set of (equivalence classes of) such spaces. Let us recall a few definitions and facts on these objects (see, e.g., [19] for details).

A pointed metric measure space is a quadruple (X, D, m, x) , where m is a Borel probability measure on the metric space (X, D) , and x is a point of X . These objects are considered up to a natural notion of isometry-equivalence. One says

that a sequence (X_n, D_n, m_n, x_n) of pointed measure metric spaces converges in the Gromov–Prokhorov sense to $(X_\infty, D_\infty, m_\infty, x_\infty)$ if and only if the following holds: for $n \in \mathbb{N} \cup \{\infty\}$, set $\xi_n(0) = x_n$ and let $\xi_n(1), \xi_n(2), \dots$ be a sequence of i.i.d. random variables with law m_n , then the vector $(D_n(\xi_n(i), \xi_n(j)) : 0 \leq i, j \leq k)$ converges in distribution to $(D_\infty(\xi_\infty(i), \xi_\infty(j)) : 0 \leq i, j \leq k)$ for every $k \geq 1$. The space \mathbb{M} of (isometry-equivalence classes of) pointed measure metric spaces, endowed with the Gromov–Prokhorov topology, is a Polish space.

In this setting, the stable tree \mathcal{T} with index α can be seen as a scaling limit of the Galton–Watson trees $\mathcal{T}_n, n \in \mathbb{N}$. More precisely, we endow the discrete trees \mathcal{T}_n with the associated graph-distance d_n and the uniform distribution m_n on $V(\mathcal{T}_n) \setminus \{\rho_n\}$. Note that m_n is uniform on $\{v_1(\mathcal{T}_n), \dots, v_n(\mathcal{T}_n)\}$; by slight abuse, it will sometimes be identified with the uniform distribution on $\{1, \dots, n\}$. For any pointed metric measure space $\mathbf{X} = (X, D, m, x)$ and any $a \in (0, \infty)$, we let $a\mathbf{X} = (X, aD, m, x)$. With this formalism, there exists a sequence $(a_n)_{n \in \mathbb{N}}$ such that

$$(2) \quad \frac{a_n}{n} \mathcal{T}_n \xrightarrow{(d)} \mathcal{T},$$

in the sense of the Gromov–Prokhorov topology, and $a_n = n^{1/\alpha} f(n)$ for a slowly-varying function f . This is a consequence of the convergence of the contour functions associated with the trees \mathcal{T}_n , shown in [16], Theorem 3.1. We will give a slightly more precise version of this result in Section 2.2.2.

We can now state our main result.

THEOREM 1.3. *Let $(a_n)_{n \in \mathbb{N}}$ be a sequence such that (2) holds. Then we have the following joint convergence in distribution:*

$$\left(\frac{a_n}{n} \mathcal{T}_n, \frac{a_n}{n} \text{Cut}_v(\mathcal{T}_n) \right) \xrightarrow{n \rightarrow \infty} (\mathcal{T}, \text{Cut}_v(\mathcal{T})),$$

where \mathbb{M} is endowed with the Gromov–Prokhorov topology and $\mathbb{M} \times \mathbb{M}$ has the associated product topology. Furthermore, the cut-tree $\text{Cut}_v(\mathcal{T})$ has the same distribution as \mathcal{T} .

Note that this generalizes Proposition 1.4 of [1], which gave the scaling limit of the number of cuts needed to isolate the root in a stable Galton–Watson tree.

In the following sections, we fix the sequence (a_n) . For some of the preliminary results, we will use a particular choice of this sequence, detailed in Section 2.2.1. Nevertheless, it is easy to check that the theorem holds for any equivalent sequence.

To complete this result, we will study the limit of the cut-tree obtained for the vertex-fragmentation, in the case where the offspring distribution ν has finite variance (still assuming that ν is critical and aperiodic). More precisely, we will show the following.

THEOREM 1.4. *If the offspring distribution ν has finite variance σ^2 , then we have the joint convergence in distribution*

$$\left(\frac{\sigma}{\sqrt{n}} \mathcal{T}_n, \frac{1}{\sqrt{n}} \left(\sigma + \frac{1}{\sigma} \right) \text{Cut}_\nu(\mathcal{T}_n) \right) \xrightarrow{n \rightarrow \infty} (\mathcal{T}^{\text{br}}, \text{Cut}(\mathcal{T}^{\text{br}}))$$

in $\mathbb{M} \times \mathbb{M}$.

Let us explain informally why we get a factor $\sigma + 1/\sigma$, instead of the $1/\sigma$ we had in the case of the edge-fragmentation. In the vertex-fragmentation, the average number of deleted edges at each step is roughly $\sum_k k\nu(k) \times k = \sigma^2 + 1$. Thus, the edge-deletions happen $\sigma^2 + 1$ times faster than for the edge-fragmentation. As a consequence, $(1/\sqrt{n}) \cdot \text{Cut}_\nu(\mathcal{T}_n)$ behaves approximatively like $(1/(\sigma^2 + 1)\sqrt{n}) \cdot \text{Cut}(\mathcal{T}_n)$, that is, $(\sigma + 1/\sigma)^{-1} (1/\sigma\sqrt{n}) \cdot \text{Cut}(\mathcal{T}_n)$.

Also note that we would need additional hypotheses to extend this result to the more general case of an offspring distribution belonging to the domain of attraction of a Gaussian distribution. Indeed, as will be seen in the Section 4, the proof of this result relies on the convergence of the coefficients n/a_n^2 : if ν has finite variance, we may and will take $a_n = \sigma\sqrt{n}$, but in the general case, this convergence is not granted.

For both of these theorems, it is known that the first component converges in the stronger sense of the Gromov–Hausdorff–Prokhorov topology. However, as in the case studied by Bertoin and Miermont, the question of whether the joint convergences hold in this sense remains open.

In the following sections, we will first work on the proof of Theorem 1.3: preliminary results will be given in Section 2, and the proof will be completed in Section 3. The global structure of this proof is close to that of [12], although the technical arguments differ, especially in Section 2. Section 4 will be devoted to the study of the finite variance case.

2. Preliminary results.

2.1. *Modified distance on $\text{Cut}_\nu(\mathcal{T}_n)$.* We begin by introducing a new distance δ'_n on $\text{Cut}_\nu(\mathcal{T}_n)$, defined in a similar way as the distance δ for a continuous tree. We show that this distance is “close” enough to $(a_n/n) \cdot \delta_n$, which will enable us to work on the modified cut-tree $\text{Cut}'_\nu(\mathcal{T}_n) := (\text{Cut}_\nu(\mathcal{T}_n), \delta'_n)$.

Recall the fragmentation of \mathcal{T}_n introduced in Section 1.1. We now turn this process into a continuous-time fragmentation, by saying that each vertex $v \in V(T)$ is marked independently, with rate $\text{deg } v/a_n$. Equivalently, this can be seen as marking each edge of T independently with rate $1/a_n$, and deleting all the edges e such that $e^- = e_i^-$ as soon as e_i is marked. Thus, we obtain a forest $\overline{\mathcal{T}}_n(t)$ at time t . For every $i \in \{1, \dots, n\}$, we let $\mathcal{T}_{n,i}(t)$ denote the component of $\overline{\mathcal{T}}_n(t)$ containing the edge e_i , with the convention $\mathcal{T}_{n,i}(t) = \emptyset$ if $e_i \notin \overline{\mathcal{T}}_n(t)$, and $\mu_{n,i}(t) = \mu_n(\mathcal{T}_{n,i}(t))$.

Note that $n\mu_{n,i}(t)$ is the number of edges in $\mathcal{T}_{n,i}(t)$. For all $i, j \in \{1, \dots, n\}$, we now define

$$\delta'_n(0, 0) = 0, \quad \delta'_n(0, i) = \delta'_n(i, 0) = \int_0^\infty \mu_{n,i}(t) dt,$$

$$\delta'_n(i, j) = \int_{t_n(i,j)}^\infty (\mu_{n,i}(t) + \mu_{n,j}(t)) dt,$$

where $t_n(i, j)$ denotes the first time when the components $\mathcal{T}_{n,i}(t)$ and $\mathcal{T}_{n,j}(t)$ become disjoint.

LEMMA 2.1. *For all $i, j \in \{1, \dots, n\}$, we have*

$$\mathbb{E}\left[\left|\frac{a_n}{n}\delta_n(0, i) - \delta'_n(0, i)\right|^2\right] = \frac{a_n}{n}\mathbb{E}[\delta'_n(0, i)]$$

and

$$\mathbb{E}\left[\left|\frac{a_n}{n}\delta_n(i, j) - \delta'_n(i, j)\right|^2\right] \leq \frac{a_n}{n}\mathbb{E}[\delta'_n(0, i) + \delta'_n(0, j)].$$

PROOF. We work conditionally on \mathcal{T}_n . Fix $i \in \{1, \dots, n\}$. For all $t \in \mathbb{R}_+$, we let $N_i(t)$ be the number of cuts happening in the component containing e_i up to time t . Since each edge of \mathcal{T}_n is marked independently with rate $1/a_n$, the process $(M_i(t))_{t \geq 0}$, where

$$M_i(t) := \frac{a_n}{n}N_i(t) - \int_0^t \mu_i(s) ds,$$

is a purely discontinuous martingale. Its predictable quadratic variation can be written as

$$\langle M_i \rangle_t = \frac{a_n}{n} \int_0^t \mu_i(s) ds.$$

As a consequence, we have $\mathbb{E}[|M_i(\infty)|^2] = \mathbb{E}[\langle M_i \rangle_\infty]$. Since

$$\lim_{t \rightarrow \infty} N_i(t) = \delta_n(0, i) \quad \text{and} \quad \lim_{t \rightarrow \infty} \int_0^t \mu_i(s) ds = \delta'_n(0, i),$$

we get

$$\mathbb{E}\left[\left|\frac{a_n}{n}\delta_n(0, i) - \delta'_n(0, i)\right|^2\right] = \frac{a_n}{n}\mathbb{E}[\delta'_n(0, i)].$$

For the second part, we use similar arguments. We fix $i \neq j \in \{1, \dots, n\}$, and we write t_{ij} instead of $t_n(i, j)$. For all $t \geq 0$, let \mathcal{F}_t denote the σ -algebra generated by \mathcal{T}_n and the atoms $\{(t_r, e_{i_r}) : t_r \leq t\}$ of the Poisson point process of marks on the edges introduced in Section 1.1. Conditionally on $\mathcal{F}_{t_{ij}}$,

$$M_{ij}(t) := M_i(t_{ij} + t) - M_i(t_{ij}) + M_j(t_{ij} + t) - M_j(t_{ij})$$

defines a purely discontinuous martingale such that

$$\begin{aligned} \lim_{t \rightarrow \infty} M_{ij}(t) &= \frac{a_n}{n} (\delta_n(b_{ij}, i) + \delta_n(b_{ij}, j)) - \int_{t_{ij}}^{\infty} \mu_i(s) ds - \int_{t_{ij}}^{\infty} \mu_j(s) ds \\ &= \frac{a_n}{n} \delta_n(i, j) - \delta'_n(i, j), \end{aligned}$$

where b_{ij} denotes the most recent common ancestor of the leaves i and j in $\text{Cut}_\nu(\mathcal{T}_n)$. Besides, since the edges of $\mathcal{T}_{n,i}$ and $\mathcal{T}_{n,j}$ are marked independently after time t_{ij} , the predictable quadratic variation of M_{ij} is

$$\langle M_{ij} \rangle_t = \frac{a_n}{n} \mathbb{E} \left[\int_{t_{ij}}^{t_{ij}+t} (\mu_i(s) + \mu_j(s)) ds \right].$$

Since $\delta'_n(i, j) = \delta'_n(0, i) + \delta'_n(0, j) - 2\delta'_n(0, b_{ij})$, this yields

$$\mathbb{E} \left[\left| \frac{a_n}{n} \delta_n(i, j) - \delta'_n(i, j) \right|^2 \right] \leq \frac{a_n}{n} \mathbb{E} [\delta'_n(0, i) + \delta'_n(0, j)]. \quad \square$$

2.2. A first joint convergence. In this section, we first state precisely the convergence theorems we will rely on to prove the following lemmas. To this end, we work in the setting of sums of i.i.d. random variable $S_n = Z_1 + \dots + Z_n$, where the laws of the Z_i are in the domain of attraction of a stable law. Under additional hypotheses, Theorem 2.2 below gives a choice of scaling constants a_n for which S_n/a_n converges in law to a stable variable, and a formulation of Gnedenko’s local limit theorem in this setting. Next, we will recall a result of Duquesne which shows, in particular, the convergence (2). The version we will use is a joint convergence of three functions encoding the trees \mathcal{T}_n and \mathcal{T} . These results will allow us to prove a first joint convergence for the fragmented trees in Proposition 2.5.

2.2.1. Local limit theorem. We say that a measure π on \mathbb{Z} is lattice if there exists integers $b \in \mathbb{Z}$, $d \geq 2$ such that $\text{supp}(\pi) \subset b + d\mathbb{Z}$. We know from our hypotheses that ν is critical, aperiodic, and $\nu(\{0\}) > 0$, and these three conditions imply that ν is nonlattice.

For any $\beta \in (1, 2)$, we let $X^{(\beta)}$ be a stable spectrally positive Lévy process with parameter β , and $p_t^{(\beta)}(x)$ the density of the law of $X_t^{(\beta)}$. Similarly, for $\beta \in (0, 1)$, we let $X^{(\beta)}$ be a stable subordinator with parameter β , and $q_t^{(\beta)}(x)$ be the density of the law of $X_t^{(\beta)}$. We fix the normalization of these processes by setting, for all $\lambda \geq 0$,

$$\begin{aligned} \mathbb{E}[e^{-\lambda X_t^{(\beta)}}] &= e^{t\lambda^\beta} && \text{if } \beta \in (1, 2), \\ \mathbb{E}[e^{-\lambda X_t^{(\beta)}}] &= e^{-t\lambda^\beta} && \text{if } \beta \in (0, 1). \end{aligned}$$

We also introduce the set R_ρ of regularly varying functions with index ρ .

THEOREM 2.2. *Let $(Z_i, i \in \mathbb{N})$ be an i.i.d. sequence of random variables in $\mathbb{N} \cup \{-1, 0\}$. We denote by Z a random variable having the same law as the Z_i . Suppose that the law of Z belongs to the domain of attraction of a stable law of index $\beta \in (0, 2) \setminus \{1\}$, and is nonlattice. If $\beta \in (1, 2)$, we also suppose that Z is centered. We introduce*

$$S_n = \sum_{i=1}^n Z_i, \quad n \geq 0.$$

Then there exists an increasing function $A \in R_\beta$ and a constant c such that:

(i) *It holds that*

$$(3) \quad \mathbb{P}(Z > r) \sim \frac{c}{A(r)} \quad \text{as } r \rightarrow \infty.$$

(ii) *Letting a be the inverse function of A , and $a_n = a(n)$ for all $n \in \mathbb{N}$, we have*

$$(4) \quad \lim_{n \rightarrow \infty} \sup_{k \in \mathbb{N}} \left| a_n \mathbb{P}(S_n = k) - p_1^{(\beta)} \left(\frac{k}{a_n} \right) \right| = 0.$$

PROOF. Theorem 8.3.1 of [13] shows that, since $Z \geq -1$ a.s., the law of Z belongs to the domain of attraction of a stable law of index β if and only if $\mathbb{P}(Z > r) \in R_{-\beta}$. Using Theorem 1.5.3 of [13], we can take a monotone equivalent of $\mathbb{P}(Z > r)$, hence the existence of A such that (3) holds with a constant c which will be chosen hereafter.

The remarks following Theorem 8.3.1 in [13] give a characterization of the a_n such that S_n/a_n converges in law to a stable variable of index β . In particular, it is enough to take a_n such that $n/A(a_n)$ converges, so $a = A^{-1}$ is a suitable choice. We now choose the constant c such that S_n/a_n converges to $X_1^{(\beta)}$. The second point of the theorem is given by Gnedenko’s local limit theorem (see, e.g., Theorem 4.2.1 of [20]). \square

2.2.2. Coding the trees \mathcal{T}_n and \mathcal{T} . We now recall three classical ways of coding a tree $T \in \mathbb{T}$, namely the associated contour function, height function and Lukasiewicz path. Detailed descriptions and properties of these objects can be found, for example, in [16].

To define the contour function $C^{[n]}$ of \mathcal{T}_n , we see \mathcal{T}_n as the embedded tree in the oriented half-plane, with each edge having length 1. We consider a particle that visits continuously all edges at unit speed, from the left to the right, starting from the root. Then, for every $t \in [0, 2n]$, we let $C_t^{[n]}$ be the height of the particle at time t , that is, its distance to the root. The height function is defined by letting $H_j^{[n]}$ be the height of the vertex v_j . Finally, for all $i \in \{0, \dots, n\}$, we let $Z_{i+1}^{[n]}$ be the number of offspring of the vertex v_i . Then the Lukasiewicz path of \mathcal{T}_n is defined by

$$W_j^{[n]} = \sum_{i=1}^j Z_i^{[n]} - j, \quad j = 0, \dots, n + 1.$$

tion 1.2. We will use the following adaptation of the results shown by Duquesne in [16]:

THEOREM 2.3 (Duquesne). *Consider the rescaled functions $C^{(n)}$, $H^{(n)}$ and $X^{(n)}$, defined by*

$$C_t^{(n)} = \frac{a_n}{n} C_{(2n+2)t}^{[n]}, \quad H_t^{(n)} = \frac{a_n}{n} H_{\lfloor (n+1)t \rfloor}^{[n]}, \quad X_t^{(n)} = \frac{1}{a_n} W_{\lfloor (n+1)t \rfloor}^{[n]}$$

for all $t \in [0, 1]$. If ν is aperiodic and hypothesis (5) holds, then we have the joint convergence

$$(C_t^{(n)}, H_t^{(n)}, X_t^{(n)})_{0 \leq t \leq 1} \xrightarrow[n \rightarrow \infty]{(d)} (H_t, H_t, X_t)_{0 \leq t \leq 1}.$$

Proposition 4.3 of [16] shows the convergence of the corresponding bridges (with a change of index which comes from the fact that we are working on trees conditioned to have n edges instead of n vertices). Using the continuity of the Vervaat transform as in the proof of [16], Theorem 3.1, then gives the result.

The fact that these convergences hold jointly will be used in the proof of Lemma 2.4 below. Apart from this, we will mainly use the convergence of the rescaled Lukasiewicz paths $X^{(n)}$, because of the following link between the rates of our fragmentation and the jumps of $X^{(n)}$. Recall from Section 1.2 that $p : [0, 1] \rightarrow \mathcal{T}$ denotes the canonical projection from $[0, 1]$ onto \mathcal{T} . Now, the set of the branching points of \mathcal{T} is $\{p(t) : t \in [0, 1] \text{ s.t. } \Delta X_t > 0\}$, and the associated local times are $L(p(t)) = \Delta X_t$ (see [18], proof of Theorem 4.7, and [25], Proposition 2). Similarly, we introduce the projection p_n from $K_n := \{1/(n+1), \dots, 1\}$ onto $V(\mathcal{T}_n)$, such that $p_n(j/(n+1))$ is the vertex v_{j-1} of \mathcal{T}_n . Thus, for all $t \in K_n$, we have

$$(6) \quad \Delta X_t^{(n)} = \frac{1}{a_n} (\deg(p_n(t), \mathcal{T}_n) - 1).$$

We conclude this part by showing another result of joint convergence, for the Lukasiewicz paths of two symmetric sequences of trees. For all $n \in \mathbb{N}$, we introduce the symmetrized tree $\tilde{\mathcal{T}}_n$, obtained by reversing the order of the children of each vertex of \mathcal{T}_n . We let $\tilde{W}^{[n]}$ denote the Lukasiewicz path of $\tilde{\mathcal{T}}_n$. (We would obtain the same process by visiting the vertices of \mathcal{T}_n “from right to left” in the depth-first search.) Finally, we define the rescaled process $\tilde{X}^{(n)}$ by

$$\tilde{X}_t^{(n)} = \frac{1}{a_n} \tilde{W}_{\lfloor (n+1)t \rfloor}^{[n]} \quad \forall t \in [0, 1].$$

LEMMA 2.4. *There exists a process $(\tilde{X}_t)_{0 \leq t \leq 1}$ such that there is the joint convergence*

$$(7) \quad (X^{(n)}, \tilde{X}^{(n)}) \xrightarrow[n \rightarrow \infty]{(d)} (X, \tilde{X}).$$

Moreover:

- The processes \tilde{X} and X have the same law.
- For every jump-time t of X ,

$$\Delta \tilde{X}_{1-t-l(t)} = \Delta X_t \quad \text{a.s.,}$$

where $l(t) = \inf\{s > t : X_s = X_{t-}\} - t$.

PROOF. Since \mathcal{T}_n and $\tilde{\mathcal{T}}_n$ have the same law, $\tilde{X}^{(n)}$ converges in distribution to an excursion of the Lévy process $X^{(\alpha)}$ in the Skorokhod space \mathbb{D} . Thus the sequence of the laws of the processes $(X^{(n)}, \tilde{X}^{(n)})$ is tight in $\mathbb{D} \times \mathbb{D}$. Up to extraction, we can assume that $(X^{(n)}, \tilde{X}^{(n)})$ converges in distribution to a couple of processes (X, \tilde{X}) .

For all $n \in \mathbb{N}$, $j \in \{0, \dots, n\}$, a simple computation shows that the vertex $v_j(\mathcal{T}_n)$ corresponds to $v_{\tilde{j}}(\tilde{\mathcal{T}}_n)$, where

$$\tilde{j} = n - j + H_j^{[n]} - D_j^{[n]},$$

and $D_j^{[n]}$ is the number of strict descendants of $v_j(\mathcal{T}_n)$. Note that $D_j^{[n]}$ is the largest integer such that $W_i^{[n]} \geq W_j^{[n]}$ for all $i \in [j, j + D_j^{[n]}]$. Then (6) shows that we have

$$(8) \quad \Delta \tilde{X}_{(n-j+H_j^{[n]}-D_j^{[n]}+1)/(n+1)}^{(n)} = \Delta X_{(j+1)/(n+1)}^{(n)}.$$

For all $n \in \mathbb{N} \cup \{\infty\}$, we let $(s_i^{(n)})_{i \in \mathbb{N}}$ be the sequence of the times where $X^{(n)}$ has a positive jump, ranked in such a way that the sequence of the jumps $(\Delta X_{s_i^{(n)}}^{(n)})_{i \in \mathbb{N}}$ is nonincreasing. We define the $(\tilde{s}_i^{(n)})_{i \in \mathbb{N}}$ in a similar way for the $\tilde{X}^{(n)}$, $n \in \mathbb{N} \cup \{\infty\}$. Fix $i \in \mathbb{N}$. Then (8) can be translated into

$$(9) \quad \tilde{s}_i^{(n)} = 1 - s_i^{(n)} + \frac{1}{n+1} (1 + H_{(n+1)s_i^{(n)}-1}^{[n]} - D_{(n+1)s_i^{(n)}-1}^{[n]}).$$

Using the Skorokhod representation theorem, we now work under the hypothesis

$$(H_t^{(n)}, X_t^{(n)})_{0 \leq t \leq 1} \xrightarrow[n \rightarrow \infty]{} (H_t, X_t)_{0 \leq t \leq 1} \quad \text{a.s.}$$

Then the following convergences hold a.s., for all $i \geq 1$:

$$\begin{aligned} s_i^{(n)} &\xrightarrow[n \rightarrow \infty]{} s_i, \\ \Delta X_{s_i^{(n)}}^{(n)} &\xrightarrow[n \rightarrow \infty]{} \Delta X_{s_i}, \\ \frac{1}{n+1} H_{(n+1)s_i^{(n)}-1}^{[n]} &\xrightarrow[n \rightarrow \infty]{} 0, \\ \frac{1}{n+1} D_{(n+1)s_i^{(n)}-1}^{[n]} &\xrightarrow[n \rightarrow \infty]{} l(s_i). \end{aligned}$$

The first two convergences hold because the ΔX_{s_i} are distinct, and the last one uses the fact that a.s.

$$\inf_{0 \leq u \leq \varepsilon} X_{s_i+l(s_i)+u} < X_{(s_i)^-} \quad \forall \varepsilon > 0.$$

As a consequence, $\tilde{s}_i^{(n)}$ converges a.s. to $1 - s_i - l(s_i)$. Thus, $\tilde{s}_i = 1 - s_i - l(s_i)$ a.s., and $\Delta \tilde{X}_{\tilde{s}_i} = \Delta X_{s_i}$ a.s. (Since the discontinuity points are countable, this holds jointly for all i .)

The Lévy–Itô representation theorem shows that \tilde{X} can be written as a measurable function of $(\tilde{s}_i, \Delta \tilde{X}_{\tilde{s}_i})_{i \in \mathbb{N}}$. This identifies uniquely the law of (X, \tilde{X}) , hence (7). \square

2.2.3. Joint convergence of the subtree sizes. Recall from Section 1.2 that $(\xi(i), i \in \mathbb{N})$ is a sequence of i.i.d. variables in \mathcal{T} , with distribution the mass-measure μ , and $\xi(0) = 0$. For all $n \in \mathbb{N}$, we introduce independent sequences $(\xi_n(i), i \in \mathbb{N})$ of i.i.d. uniform integers in $\{1, \dots, n\}$, and set $\xi_n(0) = 0$. Recalling the notation of Section 2.1, we let $\tau_n(i, j) = t_n(\xi_n(i), \xi_n(j))$ be the first time when the components $\mathcal{T}_{n, \xi_n(i)}(t)$ and $\mathcal{T}_{n, \xi_n(j)}(t)$ become disjoint. Similarly, $\tau(i, j)$ will denote the first time when the components containing $\xi(i)$ and $\xi(j)$ become disjoint in the fragmentation of \mathcal{T} . Our goal is to prove the following result.

PROPOSITION 2.5. *As $n \rightarrow \infty$, we have the following weak convergences*

$$\begin{aligned} \frac{a_n}{n} \mathcal{T}_n &\xrightarrow{(d)} \mathcal{T}, \\ (\tau_n(i, j))_{i, j \in \mathbb{N}} &\xrightarrow{(d)} (\tau(i, j))_{i, j \in \mathbb{N}}, \\ (\mu_{n, \xi_n(i)}(t))_{i \in \mathbb{N}, t \geq 0} &\xrightarrow{(d)} (\mu_{\xi(i)}(t))_{i \in \mathbb{N}, t \geq 0}, \end{aligned}$$

where the three hold jointly.

For the proof of this proposition, it will be convenient to identify the $\xi_n(i)$ with vertices of \mathcal{T}_n instead of edges. As noted in [12], proof of Lemma 2, this makes no difference for the result we seek.

We let

$$t_i^{(n)} = \frac{\xi_n(i) + 1}{n + 1},$$

so that $p_n(t_i^{(n)}) = v_{\xi_n(i)}(\mathcal{T}_n)$. Furthermore, we may and will take $\xi(i) = p(t_i)$, with a sequence $(t_i, i \in \mathbb{N})$ of independent uniform variables in $[0, 1]$. The sequence $(t_i^{(n)}, i \in \mathbb{N})$ converges in distribution to $(t_i, i \in \mathbb{N})$. Since these sequences are independent of the trees \mathcal{T}_n and \mathcal{T} , the Skorokhod representation theorem allows us

to assume

$$(10) \quad \begin{cases} (X^{(n)}, \tilde{X}^{(n)}) \xrightarrow{n \rightarrow \infty} (X, \tilde{X}) & \text{a.s.}, \\ (t_i^{(n)}, i \in \mathbb{N}) \xrightarrow{n \rightarrow \infty} (t_i, i \in \mathbb{N}) & \text{a.s.} \end{cases}$$

We will sometimes write $X_t^{(\infty)}$ and $t_i^{(\infty)}$ for X_t and t_i , when it makes notation easier.

For any two vertices u, v of a discrete tree T , we introduce the notation

$$\llbracket u, v \rrbracket_V = \llbracket u, v \rrbracket \cap V(T) \quad \text{and} \quad \llbracket u, v \rrbracket_V = \llbracket u, v \rrbracket_V \setminus \{u, v\},$$

where $\llbracket u, v \rrbracket$ is the segment between u and v in T (seen as an \mathbb{R} -tree).

DEFINITION 2.6. Fix $T \in \mathbb{T}$. The shape of T is the discrete tree $S(T)$ such that

$$\begin{aligned} V(S(T)) &= \{v \in V(T) : \text{deg } v \neq 1\}, \\ E(S(T)) &= \{\{u, v\} \in V(S(T))^2 : \forall w \in \llbracket u, v \rrbracket_V, \text{deg } w = 1\}. \end{aligned}$$

Note that this definition can easily be extended to the case of an \mathbb{R} -tree (T, d) having a finite number of leaves, by using the ‘‘convention’’ $V(T) = \{v \in T : \text{deg } v \neq 1\}$ in the previous definition.

For all $n, k \in \mathbb{N}$, we let $\mathcal{R}_n(k)$ denote the shape of the subtree of \mathcal{T}_n spanned by the vertices $\xi_n(1), \dots, \xi_n(k)$ and the root. Similarly, $\mathcal{R}_\infty(k)$ will denote the shape of the subtree of \mathcal{T} spanned by $\xi(1), \dots, \xi(k)$ and the root. For all $n \in \mathbb{N} \cup \{\infty\}$, we let $V_n(k)$ be the set of the vertices of $\mathcal{R}_n(k)$, and we identify the edges of $\mathcal{R}_n(k)$ with the corresponding segments in \mathcal{T}_n . In particular, for any edge $e = \{u, v\}$ of $\mathcal{R}_n(k)$, we write $w \in e$ if $w \in \llbracket u, v \rrbracket_V$. We let $L_n(v)$ denote the rate at which a vertex v is deleted in \mathcal{T}_n . Recall from Section 2.1 that $L_n(v) = \text{deg}(v, \mathcal{T}_n)/a_n$.

LEMMA 2.7. Fix $k \in \mathbb{N}$. Under (10), $\mathcal{R}_n(k)$ is a.s. constant for all n large enough (say $n \geq N$). Identifying $V_n(k)$ with $V_\infty(k)$ for all $n \geq N$, we have

$$(L_n(v), v \in V_n(k)) \xrightarrow{n \rightarrow \infty} (L(v), v \in V_\infty(k)) \quad \text{a.s.}$$

The above convergence can be written more rigorously by numbering the vertices of $\mathcal{R}_n(k)$ and $\mathcal{R}_\infty(k)$, and indexing on $i \in \{1, \dots, |V_\infty(k)|\}$, but we keep this form to make the notation easier.

PROOF OF LEMMA 2.7. For all $n \in \mathbb{N} \cup \{\infty\}$, $s < t \in [0, 1]$, we let

$$I_{s,t}^{(n)} = \inf_{s < u < t} X_u^{(n)},$$

and for all $i, j \in \mathbb{N}$,

$$t_{ij}^{(n)} = \sup\{s \in [0, t_i^{(n)} \wedge t_j^{(n)}] : I_{s,t_i}^{(n)} = I_{s,t_j}^{(n)}\}.$$

Note that $p_n(t_{ij}^{(n)})$ is the most recent common ancestor of the vertices $\xi_n(i)$ and $\xi_n(j)$ in \mathcal{T}_n . If, for example, $t_i^{(n)} < t_j^{(n)}$, we can rewrite $t_{ij}^{(n)}$ as

$$\sup\{s \in [0, t_i^{(n)}] : X_{s^-}^{(n)} \leq I_{t_i^{(n)}, t_j^{(n)}}^{(n)}\}.$$

Besides, for $n = \infty$, we can replace the inequality in the broad sense by a strict inequality:

$$t_{ij} = \sup\{s \in [0, t_i] : X_{s^-} < I_{t_i, t_j}\}.$$

With this notation, it is elementary to show that the following properties hold a.s. for all $i, j, i', j' \geq 0$:

- (i) X is continuous at t_i , and $X_{t_i}^{(n)}$ converges to X_{t_i} as $n \rightarrow \infty$.
- (ii) $t_{ij}^{(n)}$ converges to t_{ij} as $n \rightarrow \infty$.
- (iii) $X_{t_{ij}}^{(n)}$ converges to $X_{t_{ij}}$ and $X_{(t_{ij})^-}^{(n)}$ converges to $X_{(t_{ij})^-}$ as $n \rightarrow \infty$.
- (iv) If $t_{ij} = t_{i'j'}$, then $t_{ij}^{(n)} = t_{i'j'}^{(n)}$ for all n large enough.

We now fix $k \in \mathbb{N}$. We introduce the set

$$B_n(k) = \{t_i^{(n)} : i \in \{1, \dots, k\}\} \cup \{t_{ij}^{(n)} : i, j \in \{1, \dots, k\}\} \cup \{0\}$$

of the times coding the vertices of $\mathcal{R}_n(k)$. We let $N_n(k)$ be the number of elements of $B_n(k)$, and $b_i^{(n,k)}$ be the i th element of $B_n(k)$. Properties (i)–(iv) can be translated into the a.s. properties:

- (i)' For n large enough, $N_n(k)$ is constant.
- (ii)' For all $i \in \{1, \dots, N_\infty(k)\}$,

$$b_i^{(n,k)} \xrightarrow{n \rightarrow \infty} b_i^{(\infty,k)},$$

$$X_{b_i^{(n,k)}}^{(n)} \xrightarrow{n \rightarrow \infty} X_{b_i^{(\infty,k)}},$$

$$X_{(b_i^{(n,k)})^-}^{(n)} \xrightarrow{n \rightarrow \infty} X_{(b_i^{(\infty,k)})^-}.$$

Moreover, $\mathcal{R}_n(k)$ and the $L_n(v)$, $v \in V_n(k)$, can be recovered in a simple way using $B_n(k)$ and the $X_b^{(n)}$, $b \in B_n(k)$:

- Construct a graph with vertices labeled by $B_n(k)$, the root having label 0.
- For every $b \in B_n(k) \setminus \{0\}$, let b' denote the largest $b'' < b$ such that $b'' \in B_n(k)$ and $X_{b''}^{(n)} \leq X_b^{(n)}$, then draw an edge between the vertices labelled b and b' .

- For each vertex v labeled by $b \in B_n(k)$, let $L_n(v) = \Delta X_b^{(n)} + 1/a_n$.

This entails the lemma. \square

This first lemma allows us to control the rate at which fragmentations happen at the vertices of $\mathcal{R}_n(k)$. We now need another quantity for the fragmentations happening “on the branches” of $\mathcal{R}_n(k)$, that is, at vertices $v \in V(\mathcal{T}_n) \setminus V_n(k)$. For every $n \in \mathbb{N} \cup \{\infty\}$, we let

$$\sigma_n(t) = \sum_{\substack{0 < s < t \\ X_{s^-}^{(n)} < I_{s,t}^{(n)}}} \Delta X_s^{(n)} \quad \forall t \in [0, 1].$$

If $n \in \mathbb{N}$, the quantity $a_n \sigma_n(t)$ is the sum of the quantities $\deg v - 1$ over all strict ancestors $v \neq \rho_n$ of $p_n(t)$ in \mathcal{T}_n . Similarly, $\sigma(t)$ is the (infinite) sum of the $L(v)$ for all branching points v of \mathcal{T} that are on the path $\llbracket p(t), \rho \rrbracket$.

LEMMA 2.8. *With the preceding notation, in the setting of (10), for all $i \in 1, \dots, N(k)$, we have the convergence*

$$\sigma_n(b_i^{(n,k)}) \xrightarrow{n \rightarrow \infty} \sigma_\infty(b_i^{(\infty,k)}) \quad a.s.$$

PROOF. We fix $i \in \mathbb{N}$, and let $b_n = b_i^{(n,k)}$ to simplify the notation. For all $n \in \mathbb{N} \cup \{\infty\}$, we write $\sigma_n(t) = \sigma_n^-(t) + \sigma_n^+(t)$, where

$$\begin{aligned} \sigma_n^+(t) &= \sum_{\substack{0 < s < t \\ X_{s^-}^{(n)} < I_{s,t}^{(n)}}} (X_s^{(n)} - I_{s,t}^{(n)}), \\ \sigma_n^-(t) &= \sum_{\substack{0 < s < t \\ X_{s^-}^{(n)} < I_{s,t}^{(n)}}} (I_{s,t}^{(n)} - X_{s^-}^{(n)}). \end{aligned}$$

For any s, t such that $0 < s < t$ and $X_{s^-}^{(n)} < I_{s,t}^{(n)}$, the term $a_n(X_s^{(n)} - I_{s,t}^{(n)})$ corresponds to the number of children of $p_n(s)$ that are visited before $p_n(t)$ in the depth-first search, and $a_n(I_{s,t}^{(n)} - X_{s^-}^{(n)})$ is the number of children of $p_n(s)$ that are visited after $p_n(t)$. Writing the same decomposition $\tilde{\sigma}_n(t) = \tilde{\sigma}_n^-(t) + \tilde{\sigma}_n^+(t)$ for the trees $\tilde{\mathcal{T}}_n$, and recalling (9), we thus get

$$\sigma_n^+(b_n) = \tilde{\sigma}_n^-(\tilde{b}_n),$$

where

$$\tilde{b}_n = 1 - b_n + \frac{1}{n+1} (1 + H_{(n+1)b_n-1}^{[n]} - D_{(n+1)b_n-1}^{[n]}).$$

Now we note that for all $t \geq 0$, we have $\sigma_n^-(t) = X_{t^-}^{(n)}$ and $\sigma_\infty^-(t) = X_{t^-}$. As a consequence, using (10), we get

$$\sigma_n^-(b_n) \xrightarrow[n \rightarrow \infty]{} X_{b^-} \quad \text{a.s.}$$

The same relation for $\tilde{\sigma}_n^-$ and $\tilde{X}^{(n)}$, and the fact that \tilde{b}_n converges a.s. to $\tilde{b} := 1 - b - l(b)$, show that

$$\sigma_n^+(b_n) = \tilde{\sigma}_n^-(\tilde{b}_n) \xrightarrow[n \rightarrow \infty]{} \tilde{X}_{\tilde{b}^-} \quad \text{a.s.}$$

Thus, $\sigma_n(b_n)$ converges a.s. to $\sigma_\infty^-(b) + \tilde{\sigma}_\infty^-(\tilde{b})$. To show that this quantity is equal to $\sigma_\infty(b)$, we introduce the “truncated” sums $\sigma_{n,\varepsilon}(t)$, $\sigma_{n,\varepsilon}^+(t)$, $\sigma_{n,\varepsilon}^-(t)$, obtained by taking into account only the $s \in (0, t)$ such that $X_{s^-}^{(n)} < I_{s,t}^{(n)}$ and $\Delta X_s^{(n)} > \varepsilon$. For all $n \in \mathbb{N} \cup \{\infty\}$, these quantities are finite sums. Therefore, the a.s. convergence (10) implies that for all $\varepsilon > 0$,

$$\sigma_{\infty,\varepsilon}^+(b) = \lim_{n \rightarrow \infty} \sigma_{n,\varepsilon}^+(b_n) = \lim_{n \rightarrow \infty} \tilde{\sigma}_{n,\varepsilon}^-(\tilde{b}_n) = \tilde{\sigma}_{\infty,\varepsilon}^-(\tilde{b}).$$

Thus, $\sigma_{\infty,\varepsilon}(b) = \sigma_{\infty,\varepsilon}^-(b) + \tilde{\sigma}_{\infty,\varepsilon}^-(\tilde{b})$. By letting $\varepsilon \rightarrow 0$, we get $\sigma_\infty(b) = \sigma_\infty^-(b) + \tilde{\sigma}_\infty^-(\tilde{b})$. \square

We now come back to the proof of Proposition 2.5.

PROOF OF PROPOSITION 2.5. For all $n \in \mathbb{N} \cup \{\infty\}$, we add edge-lengths to the discrete tree $\mathcal{R}_n(k)$ by letting

$$\begin{aligned} \ell_n(\{u, v\}) &= d_n(u, v) & \text{if } n \in \mathbb{N}, \\ \ell_\infty(\{u, v\}) &= d(u, v), \end{aligned}$$

for every edge $\{u, v\}$. Let $\mathcal{R}'_n(t)$ denote the resulting tree with edge-lengths. We now write $\mathcal{R}_n(k, t)$ for the tree $\mathcal{R}'_n(t)$ endowed with point processes of marks on its edges and vertices, defined as follows:

- The marks on the vertices of $\mathcal{R}_n(k)$ appear at the same time as the marks on the corresponding vertices of \mathcal{T}_n .
- Each edge receives a mark at its midpoint at the first time when a vertex v of \mathcal{T}_n such that $v \in e$ is marked in \mathcal{T}_n .

For each n , these two point processes are independent, and their rates are the following:

- Each vertex $v \in V_n(k)$ is marked at rate $L_n(v)$, independently of the other vertices.

- For each edge e of $\mathcal{R}_n(k)$, letting b, b' denote the points of $B_n(k)$ corresponding to e^-, e^+ (as explained in the proof of Lemma 2.7), the edge e is marked at rate $\Sigma L_n(e)$, independently of the other edges, with

$$\begin{aligned} \Sigma L_n(e) &= \sum_{v \in V(\mathcal{T}_n) \cap e} L_n(v) \\ &= \sigma_n(b') - \sigma_n(b) + \frac{n}{a_n^2} (H_{(b')^-}^{(n)} - H_{b^-}^{(n)}) - L_n(e^-) \end{aligned}$$

if $n \in \mathbb{N}$, and

$$\Sigma L_\infty(e) = \Sigma L(e) = \sum_{v \in V(\mathcal{T}) \cap e} L(v) = \sigma_\infty(b') - \sigma_\infty(b) - L(e^-).$$

Now Lemmas 2.7 and 2.8 show that $L_n(v)$ and $\Sigma L_n(e)$ converge to $L(v)$ and $\Sigma L(e)$ (resp.) as $n \rightarrow \infty$. Therefore, we have the convergence

$$(11) \quad \left(\frac{a_n}{n} \mathcal{R}_n(k, t), t \geq 0 \right) \xrightarrow[n \rightarrow \infty]{(d)} (\mathcal{R}_\infty(k, t), t \geq 0),$$

where $(a_n/n) \cdot \mathcal{R}_n(k, t)$ and $\mathcal{R}_\infty(k, t)$ can be seen as random variables in $\mathbb{T} \times (\mathbb{R}_+ \cup \{-1\})^{\mathbb{N}} \times \{-1, 0, 1\}^{\mathbb{N}^2}$, for example,

$$(a_n/n) \cdot \mathcal{R}_n(k, t) = (\mathcal{R}_n(k), (l_i)_{i \geq 1}, (\delta_V(i, t))_{i \geq 0}, (\delta_E(i, t))_{i \geq 1}),$$

where

$$\begin{aligned} l_i &= \begin{cases} (a_n/n) \cdot \ell(e_i(\mathcal{R}_n(k))), & \text{if } i < N_n(k), \\ -1, & \text{if } i \geq N_n(k), \end{cases} \\ \delta_V(i, t) &= \begin{cases} 1, & \text{if } i < N_n(k) \text{ and the vertex } v_i(\mathcal{R}_n(k)) \\ & \text{has been marked before time } t, \\ 0, & \text{if } i < N_n(k) \text{ and the vertex } v_i(\mathcal{R}_n(k)) \\ & \text{has not been marked before time } t, \\ -1, & \text{if } i \geq N_n(k), \end{cases} \\ \delta_E(i, t) &= \begin{cases} 1, & \text{if } i < N_n(k) \text{ and the edge } e_i(\mathcal{R}_n(k)) \\ & \text{has been marked before time } t, \\ 0, & \text{if } i < N_n(k) \text{ and the edge } e_i(\mathcal{R}_n(k)) \\ & \text{has not been marked before time } t, \\ -1, & \text{if } i \geq N_n(k) \end{cases} \end{aligned}$$

[recall that $N_n(k)$ is the number of vertices of $\mathcal{R}_n(k)$]. Note that we could keep working under (10) to get an a.s. convergence, but this is no longer necessary.

The rest of the proof goes as in [12]. For every $i \in \mathbb{N}$, we let $\eta_n(k, i, t)$ denote the number of vertices among $\xi_n(1), \dots, \xi_n(k)$ in the component of $\mathcal{R}_n(k)$ containing $\xi_n(i)$ at time t . Similarly, denote by $\eta_\infty(k, i, t)$ the number of vertices among

$\xi(1), \dots, \xi(k)$ in the component of $\mathcal{R}_\infty(k)$ containing $\xi(i)$ at time t . It follows from (11) that we have the joint convergences

$$\begin{aligned} \frac{a_n}{n} \mathcal{T}_n &\xrightarrow{(d)} \mathcal{T}, \\ (\eta_n(k, i, t))_{t \geq 0, i \in \mathbb{N}} &\xrightarrow{(d)} (\eta_\infty(k, i, t))_{t \geq 0, i \in \mathbb{N}}, \\ (\tau_n(i, j))_{i, j \in \mathbb{N}} &\xrightarrow{(d)} (\tau(i, j))_{i, j \in \mathbb{N}}. \end{aligned}$$

Besides, the law of large numbers gives that for each $i \in \mathbb{N}$ and $t \geq 0$,

$$\frac{1}{k} \eta_\infty(k, i, t) \xrightarrow{n \rightarrow \infty} \mu_{\xi(i)}(t) \quad \text{a.s.}$$

Thus, for every fixed integer l and times $0 \leq t_1 \leq \dots \leq t_l$, we can construct a sequence $k_n \rightarrow \infty$ sufficiently slowly, such that

$$\left(\frac{1}{k_n} \eta_n(k_n, i, t_j) \right)_{i, j \in \{1, \dots, l\}} \xrightarrow{(d)} (\mu_{\xi(i)}(t_j))_{i, j \in \{1, \dots, l\}},$$

or equivalently (see [6], Lemma 11)

$$(\mu_{n, \xi_n(i)}(t_j))_{i, j \in \{1, \dots, l\}} \xrightarrow{(d)} (\mu_{\xi(i)}(t_j))_{i, j \in \{1, \dots, l\}},$$

both holding jointly with the preceding convergences. This entails the proposition. □

2.3. Upper bound for the expected component mass. To get the convergence of $(\mathcal{T}_n, \text{Cut}_v(\mathcal{T}_n))$, we will finally need to control the quantities

$$\mathbb{E} \left[\int_{2^l}^\infty \mu_{n, \xi_n}(t) dt \right],$$

where ξ_n is a uniform random integer in $\{1, \dots, n\}$. Our main goal is to show that these quantities converge to 0 as l tends to ∞ , uniformly in n , as stated in Corollary 2.15.

To this end, we will sometimes work under the size-biased measure GW^* , defined as follows. We recall that a pointed tree is a pair (T, v) , where T is a rooted planar tree and v is a vertex of T . The measure GW^* is the sigma-finite measure such that, for every pointed tree (T, v) ,

$$\text{GW}^*(T, v) = \mathbb{P}(\mathbf{T} = T),$$

where \mathbf{T} is a Galton–Watson tree with offspring distribution v . We let \mathbb{E}^* denote the expectation under this “law.” In particular, the conditional law GW^* given $|V(T)| = n + 1$ is well-defined, and corresponds to the distribution of a pair (\mathcal{T}_n, v) where given \mathcal{T}_n , v is a uniform random vertex of \mathcal{T}_n . Hereafter, T will denote a ν -Galton–Watson tree, whose expectation will either be taken under the unbiased

law or under a conditioned version of the law GW^* . Recall that we only consider values of n such that $P_n = \mathbb{P}(|V(T)| = n + 1) \neq 0$.

For all $m, n \in \mathbb{N}$ such that $m \leq n$ and $P_m \neq 0$, for all $t \in \mathbb{R}_+$, we define

$$(12) \quad E_{m,n}(t) = \frac{1}{m} \mathbb{E} \left[\sum_{e \in E(\mathcal{T}_m)} \exp \left(- \sum_{u \in \llbracket \rho_m, e^- \rrbracket_V} \text{deg}(u, \mathcal{T}_m) \frac{t}{a_n} \right) \right],$$

and $E_n(t) = E_{n,n}(t)$. Equivalently, we can write

$$E_{m,n}(t) = \frac{1}{m} \mathbb{E}^* \left[\sum_{e \in E(T)} \exp \left(- \sum_{u \in \llbracket \rho(T), e^- \rrbracket_V} \text{deg}(u, T) \frac{t}{a_n} \right) \middle| |V(T)| = m + 1 \right].$$

For all $m < n$, we also use the notation

$$P_{m,n}^* := \mathbb{P}^*(|V(T_v)| = m + 1 \mid |V(T)| = n + 1),$$

where T_v denotes the tree formed by v and its descendants. Our first step is to show the following.

LEMMA 2.9. *Let ξ_n be a uniform random edge of \mathcal{T}_n . Using the previous notation, we have*

$$(13) \quad \mathbb{E}[\mu_{n,\xi_n}(t)] \leq \frac{1}{n} e^{-t/a_n} + 2 \left(E_n(t) + \sum_{\substack{m=1 \\ P_m \neq 0}}^{n-1} P_{m,n}^* \frac{m}{n} E_{m,n}(t) \right).$$

The proof of this lemma will use Proposition 2.10 below. Let us first introduce some notation. For all $v \in V(T)$, we let T^v be the subtree obtained by deleting all the strict descendants of v in T , and as before, T_v be the tree formed by v and its descendants. We define a new tree $\hat{T}^{\hat{v}}$, constructed by taking T^v and modifying it as follows:

- we remove the edge $e(v)$ between v and $p(v)$;
- we add a new child \hat{v} to the root, and let $\hat{e}_{\hat{v}}$ denote the edge between \hat{v} and the root;
- we reroot the tree at $p(v)$.

An example of this construction is given in Figure 3. Note that we have natural bijective correspondences between $V(T)$, $(V(T^v) \setminus \{v\}) \sqcup V(T_v)$ and $(V(\hat{T}^{\hat{v}}) \setminus \{\hat{v}\}) \sqcup V(T_v)$, and between $E(T)$, $E(T^v) \sqcup E(T_v)$ and $E(\hat{T}^{\hat{v}}) \sqcup E(T_v)$. Furthermore, one can easily check that for all $u \in V(\hat{T}^{\hat{v}}) \setminus \{\hat{v}\}$, we have $\text{deg}(u, \hat{T}^{\hat{v}}) = \text{deg}(u, T)$, and for all $u \in V(T_v)$, $\text{deg}(u, T_v) = \text{deg}(u, T)$.

This transformation is the same as in [12], page 21, except that we work with rooted trees instead of planted trees. In our case, adding the edge $\hat{e}_{\hat{v}}$ and deleting $e(v)$ mimics the existence of a base edge. Thus, we can use Proposition 2 of [12].

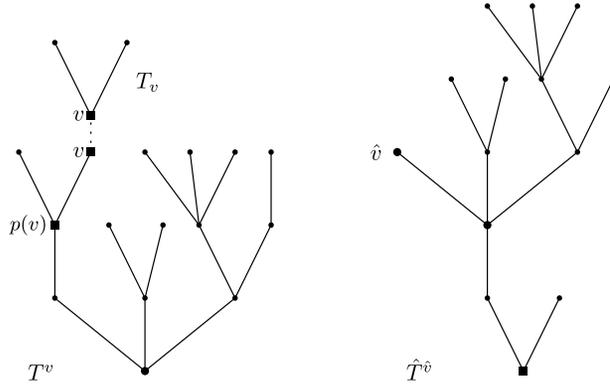


FIG. 3. The trees T_v , T^v and $\hat{T}^{\hat{v}}$ obtained from a pointed tree (T, v) .

PROPOSITION 2.10. Under GW^* , $(\hat{T}^{\hat{v}}, T_v)$ and (T^v, T_v) have the same “law,” and the trees T^v and T_v are independent, with T_v being a Galton–Watson tree.

PROOF OF LEMMA 2.9. In this proof, we identify ξ_n with the edge e_{ξ_n} , to make notation easier. We first note that for each edge $e \in E(\mathcal{T}_n)$, e belongs to the component $\mathcal{T}_{n, \xi_n}(t)$ if and only if no vertex on the path $\llbracket e^-, \xi_n^- \rrbracket_V$ has been removed at time t . Given \mathcal{T}_n and ξ_n , this happens with probability

$$\exp\left(- \sum_{u \in \llbracket e^-, \xi_n^- \rrbracket_V} \text{deg } u \cdot \frac{t}{a_n}\right)$$

[for any vertex u , at time t , u has been deleted from the initial tree with probability $1 - \exp(- \text{deg } u \cdot t/a_n)$]. Thus,

$$\begin{aligned} \mathbb{E}[n\mu_{n, \xi_n}] &= \mathbb{E}\left[\sum_{e \in E(\mathcal{T}_n)} \mathbb{1}_{e \in \mathcal{T}_{n, \xi_n}(t)}\right] \\ &= \mathbb{E}\left[\sum_{e \in E(\mathcal{T}_n)} \exp\left(- \sum_{u \in \llbracket e^-, \xi_n^- \rrbracket_V} \text{deg } u \cdot \frac{t}{a_n}\right)\right]. \end{aligned}$$

Since the edge ξ_n is chosen uniformly in $E(\mathcal{T}_n)$, this yields

$$\begin{aligned} \mathbb{E}[n\mu_{n, \xi_n}] &= \frac{1}{n} \mathbb{E}\left[\sum_{e, \xi \in E(\mathcal{T}_n)} \exp\left(- \sum_{u \in \llbracket e^-, \xi^- \rrbracket_V} \text{deg } u \frac{t}{a_n}\right)\right] \\ &= \frac{1}{n} \mathbb{E}\left[\sum_{v \in V(\mathcal{T}_n)} \mathbb{1}_{v \neq \rho(\mathcal{T}_n)} \sum_{e \in E(\mathcal{T}_n)} \exp\left(- \sum_{u \in \llbracket e^-, p(v) \rrbracket_V} \text{deg } u \frac{t}{a_n}\right)\right], \end{aligned}$$

where $p(v)$ denotes the parent of vertex v . Hence, calling $A_n(T)$ the event $\{|V(T)| = n + 1\}$,

$$\mathbb{E}[n\mu_n, \xi_n] = \frac{n + 1}{n} \mathbb{E}^* \left[\mathbb{1}_{v \neq \rho(T)} \sum_{e \in E(T)} \exp \left(- \sum_{u \in \llbracket e^-, p(v) \rrbracket_V} \deg u \frac{t}{a_n} \right) \middle| A_n(T) \right].$$

Distinguishing the cases for which $e \in E(T_v)$, $e \in E(T^v) \setminus \{e(v)\}$ and $e = e(v)$, we split this quantity into three terms:

$$(14) \quad \mathbb{E}[n\mu_n, \xi_n] = \left(1 + \frac{1}{n} \right) (\Sigma_v + \Sigma^v + \varepsilon_v),$$

where

$$\Sigma_v = \mathbb{E}^* \left[\mathbb{1}_{v \neq \rho(T)} \sum_{e \in E(T_v)} \exp \left(- \sum_{u \in \llbracket e^-, v \rrbracket_V} (\deg(u, T_v) + \deg p(v)) \frac{t}{a_n} \right) \middle| A_n(T) \right],$$

$$\Sigma^v = \mathbb{E}^* \left[\mathbb{1}_{v \neq \rho(T)} \sum_{e \in E(T^v) \setminus \{e(v)\}} \exp \left(- \sum_{u \in \llbracket e^-, p(v) \rrbracket_V} \deg(u, T^v) \frac{t}{a_n} \right) \middle| A_n(T) \right]$$

and

$$\varepsilon_v = \mathbb{E}^* \left[\mathbb{1}_{v \neq \rho(T)} \exp \left(- \deg p(v) \frac{t}{a_n} \right) \middle| A_n(T) \right].$$

For the first term, we have

$$\Sigma_v \leq \mathbb{E}^* \left[\mathbb{1}_{v \neq \rho(T)} \sum_{e \in E(T_v)} \exp \left(- \sum_{u \in \llbracket \rho(T_v), e^- \rrbracket_V} \deg(u, T_v) \frac{t}{a_n} \right) \middle| A_n(T) \right].$$

Since $|V(T)| = |V(T_v)| + |V(T^v)| - 1$, this gives

$$\Sigma_v \leq \sum_{\substack{m=1 \\ P_m \neq 0}}^{n-1} P_{m,n}^* \mathbb{E}^* \left[\sum_{e \in E(T_v)} \exp \left(- \sum_{u \in \llbracket \rho(T_v), e^- \rrbracket_V} \deg(u, T_v) \frac{t}{a_n} \right) \middle| \begin{array}{l} |V(T_v)| = m + 1, \\ |V(T^v)| = n - m + 1 \end{array} \right]$$

$[m = n$ would correspond to the case where $v = \rho(T)$, and $m = 0$ to the case where $E(T_v) = \emptyset$]. Proposition 2.10 gives that the trees T_v and T^v are independent, with T_v being a Galton–Watson tree. Hence,

$$(15) \quad \begin{aligned} \Sigma_v &\leq \sum_{\substack{m=1 \\ P_m \neq 0}}^{n-1} P_{m,n}^* \mathbb{E}^* \left[\sum_{e \in E(T)} \exp \left(- \sum_{u \in \llbracket \rho(T), e^- \rrbracket_V} \deg(u, T) \frac{t}{a_n} \right) \middle| A_m(T) \right] \\ &\leq \sum_{\substack{m=1 \\ P_m \neq 0}}^{n-1} P_{m,n}^* m E_{m,n}(t). \end{aligned}$$

For the second term, we use the correspondence between $E(T^v) \setminus \{e(v)\}$ and $E(\hat{T}^{\hat{v}}) \setminus \{\hat{e}_{\hat{v}}\}$, and the fact that $\rho(\hat{T}^{\hat{v}}) = p(v)$:

$$\Sigma^v = \mathbb{E}^* \left[\mathbb{1}_{v \neq \rho(T)} \sum_{e \in E(\hat{T}^{\hat{v}}) \setminus \{\hat{e}_{\hat{v}}\}} \exp \left(- \sum_{u \in \llbracket \rho(\hat{T}^{\hat{v}}), e^- \rrbracket_V} \deg(u, \hat{T}^{\hat{v}}) \frac{t}{a_n} \right) \middle| A_n(T) \right].$$

This gives

$$\Sigma^v \leq \mathbb{E}^* \left[\sum_{e \in E(\hat{T}^{\hat{v}})} \exp \left(- \sum_{u \in \llbracket \rho(\hat{T}^{\hat{v}}), e^- \rrbracket_V} \deg(u, \hat{T}^{\hat{v}}) \frac{t}{a_n} \right) \middle| A_n(T) \right].$$

Using the fact that T^v and $\hat{T}^{\hat{v}}$ have the same law under GW^* , we get

$$\Sigma^v \leq \mathbb{E}^* \left[\sum_{e \in E(T^v)} \exp \left(- \sum_{u \in \llbracket \rho(T^v), e^- \rrbracket_V} \deg(u, T^v) \frac{t}{a_n} \right) \middle| A_n(T) \right].$$

Seeing $E(T^v)$ as a subset of $E(T)$, we can write

$$(16) \quad \Sigma^v \leq \mathbb{E}^* \left[\sum_{e \in E(T)} \exp \left(- \sum_{u \in \llbracket \rho(T), e^- \rrbracket_V} \deg(u, T) \frac{t}{a_n} \right) \middle| A_n(T) \right] = n E_n(t).$$

For the third term, we simply notice that

$$(17) \quad \varepsilon_v \leq \frac{n}{n+1} e^{-t/a_n}.$$

Putting together (15), (16) and (17) into (14), we finally get

$$\mathbb{E}[n\mu_{n,\xi_n}(t)] \leq e^{-t/a_n} + \left(1 + \frac{1}{n}\right) \left(n E_n(t) + \sum_{\substack{m=1 \\ P_m \neq 0}}^{n-1} P_{m,n}^* m E_{m,n}(t) \right).$$

Thus,

$$\mathbb{E}[\mu_{n,\xi_n}(t)] \leq \frac{1}{n} e^{-t/a_n} + \left(1 + \frac{1}{n}\right) \left(E_n(t) + \sum_{\substack{m=1 \\ P_m \neq 0}}^{n-1} P_{m,n}^* \frac{m}{n} E_{m,n}(t) \right). \quad \square$$

Next, we compute $E_{m,n}(t)$. To this end, we introduce two new independent sequences of i.i.d. variables:

- $(\hat{Z}_i)_{i \geq 1}$ with law \hat{v} , where \hat{v} is the size-biased version of v ;
- $(N_i)_{i \geq 1}$, with same law as the number of vertices of a Galton–Watson tree with offspring distribution v .

For all $k, h \in \mathbb{N}$, we also write

$$\hat{S}_h = \sum_{i=1}^h \hat{Z}_i \quad \text{and} \quad Y_k = \sum_{i=1}^k N_i.$$

LEMMA 2.11. For every $m, n \in \mathbb{N}$ such that $m \leq n$ and $P_m \neq 0$, one has

$$(18) \quad E_{m,n}(t) = \frac{1}{mP_m} \sum_{1 \leq h \leq k \leq m} e^{-kt/a_n} \mathbb{P}(\hat{S}_h = k) \mathbb{P}(Y_{k-h+1} = m - h + 1).$$

PROOF. We first note that relation (12) can be written otherwise, using the one-to-one correspondence $e \mapsto e^+$ between $E(T)$ and $V(T) \setminus \{\rho(T)\}$:

$$E_{m,n}(t) = \frac{1}{m} \mathbb{E} \left[\sum_{v \in V(T) \setminus \rho(T)} \exp \left(- \sum_{u \in \llbracket \rho(T), p(v) \rrbracket_V} \deg(u, T) \frac{t}{a_n} \right) \middle| |E(T)| = m \right].$$

We thus have

$$\begin{aligned} E_{m,n}(t) &= \frac{1}{mP_m} \mathbb{E} \left[\sum_{v \in V(T) \setminus \rho(T)} \exp \left(- \sum_{u \in \llbracket \rho(T), p(v) \rrbracket_V} \deg(u, T) \frac{t}{a_n} \right), |E(T)| = m \right] \\ &= \frac{1}{mP_m} \mathbb{E}^* \left[\mathbb{1}_{v \neq \rho(T)} \exp \left(- \sum_{u \in \llbracket \rho(T), p(v) \rrbracket_V} \deg(u, T) \frac{t}{a_n} \right), |E(T)| = m \right]. \end{aligned}$$

We now use the following description of a typical pointed tree (T, v) under GW^* (see the proof of Proposition 2 of [12] and [22]):

- The “law” under GW^* of the distance $h(v)$ of the pointed vertex v to the root is the counting measure on $\mathbb{N} \cup \{0\}$.
- Conditionally on $h(v) = h$, the subtrees T_v and T^v are independent, with T_v being a Galton–Watson tree with offspring distribution ν , and T^v having GW_h^* law, which can be described as follows. T^v has a distinguished branch $B = \{u_1 = \rho(T^v), u_2, \dots, u_{h+1} = v\}$ of length h . Every vertex of T^v has an offspring that is distributed independently of the other vertices, with offspring distribution ν for the vertices in $V(T^v) \setminus B$, $\hat{\nu}$ for the vertices u_1, \dots, u_h , and u_{h+1} having no descendants. The tree T^v can thus be constructed inductively from the root u_1 , by choosing the i th vertex u_i of the distinguished branch uniformly at random from the children of u_{i-1} .

In this representation, conditionally on having $h(v) = h$, $\llbracket \rho(T), p(v) \rrbracket_V$ equals $\{u_1, \dots, u_h\}$ and, for every $i \in \{1, \dots, h\}$,

$$\deg(u_i, T) = \hat{Z}_i.$$

Besides, the total number of vertices of T is the sum of the number of vertices h of $B \setminus \{v\}$, of $|V(T_v)|$, and of the $|V(T_u)|$ for u such that $p(u) \in B \setminus \{v\}$ and $u \notin B$. There are $\sum_{i=1}^h (\hat{Z}_i - 1)$ such trees T_u . Hence, under GW^* :

$$|E(T)| = |V(T)| - 1 \stackrel{(d)}{=} Y_{\sum_{i=1}^h (\hat{Z}_i - 1) + 1} + h - 1.$$

Thus,

$$\begin{aligned}
 E_{m,n}(t) &= \frac{1}{mP_m} \sum_{1 \leq h} \mathbb{E} \left[\exp \left(- \sum_{i=1}^h \hat{Z}_i \frac{t}{a_n} \right), Y_{\sum_{i=1}^h \hat{Z}_i - h + 1} = m - h + 1 \right] \\
 &= \frac{1}{mP_m} \sum_{1 \leq h \leq k \leq m} e^{-kt/a_n} \mathbb{P}(\hat{S}_h = k) \mathbb{P}(Y_{k-h+1} = m - h + 1). \quad \square
 \end{aligned}$$

We now compute upper bounds for the terms $\mathbb{P}(Y_{k-h+1} = m - h + 1)$, $\mathbb{P}(\hat{S}_h = k)$ and $(mP_m)^{-1}$.

Upper bound for $\mathbb{P}(Y_{k-h+1} = m - h + 1)$. Recalling the notation of Section 2.2.2, we have

$$\begin{aligned}
 \mathbb{P}(Y_k = n) &= \mathbb{P}(W_n = -k \text{ and, } \forall p < n, W_p > -k) \\
 &= \frac{k}{n} \mathbb{P}(W_n = -k).
 \end{aligned}$$

The second equality is given by the cyclic lemma (see [27], Lemma 6.1). We will now use the fact, given by Theorem 2.2, that

$$(19) \quad \lim_{n \rightarrow \infty} \sup_{k \in \mathbb{N}} \left| a_n \mathbb{P}(W_n = -k) - p_1^{(\alpha)} \left(-\frac{k}{a_n} \right) \right| = 0.$$

For all $s, x \in (0, \infty)$, we have

$$xp_s^{(\alpha)}(-x) = sq_x^{(1/\alpha)}(s)$$

(see, e.g., [7], Corollary VII.1.3). Taking $s = 1$ and $x = k/a_n$, this gives

$$\frac{k}{a_n} p_1^{(\alpha)} \left(-\frac{k}{a_n} \right) = q_{k/a_n}^{(1/\alpha)}(1).$$

Thus,

$$n\mathbb{P}(Y_n = k) - q_{k/a_n}^{(1/\alpha)}(1) = \frac{k}{a_n} \left(a_n \mathbb{P}(W_n = -k) - p_1^{(\alpha)} \left(-\frac{k}{a_n} \right) \right),$$

and we get

$$\begin{aligned}
 \mathbb{P}(Y_k = n) &\leq \frac{1}{n} (|n\mathbb{P}(Y_n = k) - q_{k/a_n}^{(1/\alpha)}(1)| + q_{k/a_n}^{(1/\alpha)}(1)) \\
 &\leq \frac{k}{na_n} \left(\left| a_n \mathbb{P}(W_n = -k) - p_1^{(\alpha)} \left(-\frac{k}{a_n} \right) \right| + p_1^{(\alpha)} \left(-\frac{k}{a_n} \right) \right).
 \end{aligned}$$

Since $p_1^{(\alpha)}$ is bounded and (19) holds, there exists a constant $M \in (0, \infty)$ such that, for all $k, n \in \mathbb{N}$,

$$\mathbb{P}(Y_k = n) \leq \frac{k}{na_n} M.$$

Thus, we have the following upper bound:

$$(20) \quad \mathbb{P}(Y_{k-h+1} = m - h + 1) \leq \frac{k - h + 1}{(m - h + 1)a_{m-h+1}} M.$$

Upper bound for $\mathbb{P}(\hat{S}_h = k)$. We use Theorem 2.2 for the i.i.d. variables $(\hat{Z}_i)_{i \in \mathbb{N}}$. Let $\hat{A} \in R_{\alpha-1}$ be an increasing function given by (i), such that

$$\mathbb{P}(\hat{Z}_1 > r) \sim \frac{1}{\hat{A}(r)},$$

and \hat{a} be the inverse function of \hat{A} . Then

$$\lim_{h \rightarrow \infty} \sup_{k \in \mathbb{N}} \left| \hat{a}_h \mathbb{P}(\hat{S}_h = k) - q_1^{(\alpha-1)} \left(\frac{k}{\hat{a}_h} \right) \right| = 0.$$

Using the fact that $q_1^{(\alpha-1)}$ is bounded, and writing

$$\mathbb{P}(\hat{S}_h = k) \leq \frac{1}{\hat{a}_h} \left(\left| \hat{a}_h \mathbb{P}(\hat{S}_h = k) - q_1^{(\alpha-1)} \left(\frac{k}{\hat{a}_h} \right) \right| + q_1^{(\alpha-1)} \left(\frac{k}{\hat{a}_h} \right) \right),$$

we get the existence of a constant $M' \in (0, \infty)$ such that, for all $h, k \in \mathbb{N}$,

$$(21) \quad \mathbb{P}(\hat{S}_h = k) \leq \frac{M'}{\hat{a}_h}.$$

Furthermore, when h is small enough, we have a better bound for $\mathbb{P}(\hat{S}_h = k)$:

LEMMA 2.12. *Using the previous notation, if hypothesis (1) holds, then there exist constants B, C such that for all $k \in \mathbb{N}$, for all h such that $k/\hat{a}_h \geq B$,*

$$\mathbb{P}(\hat{S}_h = k) \leq C \frac{h}{k \hat{A}(k)}.$$

This result is an adaptation of a theorem by Doney [14]. The main ideas of the proof, which is rather technical, will be given in the [Appendix](#).

Besides, using the fact that A is regularly varying and an Abel transformation of $\mathbb{P}(\hat{Z} > r)$, we get that

$$(22) \quad \frac{1}{\hat{A}(r)} \sim \frac{\alpha r}{A(r)} \quad \text{as } r \rightarrow \infty.$$

Upper bound for $(mP_m)^{-1}$. We have

$$P_m = \mathbb{P}(|E(\mathcal{T})| = m) \sim \frac{p_1^{(\alpha)}(0)}{ma_m}$$

(this is a straightforward consequence of the cyclic lemma and the local limit theorem). This gives the existence of a constant $K \in (0, \infty)$ which verifies, for all m such that $P_m \neq 0$,

$$(23) \quad \frac{1}{mP_m} \leq Ka_m.$$

Before coming back to the proof of Corollary 2.15, we give another useful result on regularly varying functions.

LEMMA 2.13. *Fix $\beta \in (0, \infty)$. Let f be a positive increasing function in R_β on \mathbb{R}_+ , and x_0 a positive constant. For every $\delta \in (0, \beta)$, there exists a constant $C_\delta \in (0, \infty)$ such that, for all $x' \geq x \geq x_0$,*

$$C_\delta^{-1} \left(\frac{x'}{x}\right)^{\beta-\delta} \leq \frac{f(x')}{f(x)} \leq C_\delta \left(\frac{x'}{x}\right)^{\beta+\delta}.$$

This result is a consequence of the Potter bounds (see, e.g., Theorem 1.5.6 of Bingham et al. [13]). In particular, it implies that for all x bounded away from 0, for all $z \geq 1$,

$$(24) \quad C_\delta^{-1} z^{\beta-\delta} \leq \frac{f(xz)}{f(x)} \leq C_\delta z^{\beta+\delta},$$

and likewise, for all $x \in (0, \infty)$, $z \leq 1$ such that xz is bounded away from 0,

$$(25) \quad C_\delta^{-1} z^{\beta+\delta} \leq \frac{f(xz)}{f(x)} = \frac{f(xz)}{f(xzz^{-1})} \leq C_\delta z^{\beta-\delta}.$$

We can finally state the following.

LEMMA 2.14. *We have*

$$(26) \quad \lim_{l \rightarrow \infty} \sup_{n \in \mathbb{N}} \int_{2^l}^\infty E_n(t) dt = 0$$

and

$$\lim_{l \rightarrow \infty} \sup_{n \in \mathbb{N}} \sup_{\substack{1 \leq m \leq n \\ P_m \neq 0}} \int_{2^l}^\infty \frac{m}{n} E_{m,n}(t) dt = 0.$$

PROOF. For every $n, l \in \mathbb{N}$, we let

$$I_{n,l} = \int_{2^l}^\infty E_n(t) dt.$$

Putting together (18) and (23), we have

$$E_n(t) \leq Ka_n \sum_{k=1}^n \sum_{h=1}^k e^{-kt/a_n} \mathbb{P}(\hat{S}_h = k) \mathbb{P}(Y_{k-h+1} = n - h + 1).$$

This yields

$$I_{n,l} \leq K a_n^2 \sum_{k=1}^n \sum_{h=1}^k \frac{1}{k} e^{-2^l k/a_n} \mathbb{P}(\hat{S}_h = k) \mathbb{P}(Y_{k-h+1} = n - h + 1).$$

Writing $h(n, k) = \hat{A}(k/B) \wedge \lfloor n/2 \rfloor$ and $h'(n, k) = k \wedge \lfloor n/2 \rfloor$, we split this sum into three parts:

$$\begin{aligned} I_{n,l}^1 &= a_n^2 \sum_{k=1}^n \sum_{h=1}^{h(n,k)} \frac{1}{k} e^{-2^l k/a_n} \mathbb{P}(\hat{S}_h = k) \mathbb{P}(Y_{k-h+1} = n - h + 1), \\ I_{n,l}^2 &= a_n^2 \sum_{k=1}^n \sum_{h=h(n,k)+1}^{h'(n,k)} \frac{1}{k} e^{-2^l k/a_n} \mathbb{P}(\hat{S}_h = k) \mathbb{P}(Y_{k-h+1} = n - h + 1), \\ I_{n,l}^3 &= a_n^2 \sum_{k=1}^n \sum_{h=h'(n,k)+1}^k \frac{1}{k} e^{-2^l k/a_n} \mathbb{P}(\hat{S}_h = k) \mathbb{P}(Y_{k-h+1} = n - h + 1). \end{aligned}$$

Our first goal is to show that, for $i = 1, 2, 3$,

$$\lim_{l \rightarrow \infty} \sup_{n \in \mathbb{N}} I_{n,l}^i = 0.$$

Let us first examine $I_{n,l}^1$. Since a is increasing, the upper bound (20) gives, for $n - h + 1 \geq n/2$,

$$\begin{aligned} \mathbb{P}(Y_{k-h+1} = n - h + 1) &\leq M \frac{k - h + 1}{(n - k + 1)a_{n-k+1}} \\ (27) \qquad \qquad \qquad &\leq 2M \frac{k}{na_{n/2}}. \end{aligned}$$

Thus, we have

$$I_{n,l}^1 \leq 2M \frac{a_n^2}{na_{n/2}} \sum_{k=1}^n e^{-2^l k/a_n} \sum_{h=1}^{h(n,k)} \mathbb{P}(\hat{S}_h = k).$$

Turning the first sum into an integral, and using the substitution $y' = y/a_n$, we get

$$\begin{aligned} I_{n,l}^1 &\leq 2M \frac{a_n^2}{na_{n/2}} \int_1^\infty dy e^{-2^l \lfloor y \rfloor / a_n} \left(\sum_{h=1}^{h(n, \lfloor y \rfloor)} \mathbb{P}(\hat{S}_h = \lfloor y \rfloor) \right) \\ &= 2M \frac{a_n^3}{na_{n/2}} \int_{1/a_n}^\infty dy e^{-2^l \lfloor a_n y \rfloor / a_n} \left(\sum_{h=1}^{h(n, \lfloor a_n y \rfloor)} \mathbb{P}(\hat{S}_h = \lfloor a_n y \rfloor) \right). \end{aligned}$$

Since \hat{a} is increasing, for all $h \leq h(n, k)$, we have $\hat{a}_h \leq k/B$. Therefore, Lemma 2.12 gives

$$\mathbb{P}(\hat{S}_h = k) \leq C \frac{h}{k \hat{A}(k)}.$$

This yields

$$\begin{aligned}
 I_{n,l}^1 &\leq 2CM \frac{a_n^3}{na_{n/2}} \int_{1/a_n}^\infty dy e^{-2^l \lfloor a_n y \rfloor / a_n} \left(\sum_{h=1}^{h(n, \lfloor a_n y \rfloor)} \frac{h}{a_n y \hat{A}(a_n y)} \right) \\
 &\leq 2CM \frac{a_n^3}{na_{n/2}} \int_{1/a_n}^\infty dy e^{-2^l \lfloor a_n y \rfloor / a_n} \left(\frac{\hat{A}(\lfloor a_n y \rfloor / B)^2}{\lfloor a_n y \rfloor \hat{A}(\lfloor a_n y \rfloor)} \right).
 \end{aligned}$$

We fix $\delta \in (0, (\alpha - 1) \wedge (2 - \alpha))$. Since \hat{A} is regularly varying with index $\alpha - 1$, for all $y \geq 1/a_n$, we have

$$\frac{\hat{A}(\lfloor a_n y \rfloor / B)}{\hat{A}(\lfloor a_n y \rfloor)} \leq \frac{C_\delta^{-1}}{B^{\alpha-1-\delta}}$$

[we can use (24) because $\lfloor a_n y \rfloor / B \geq 1/B$ for all $y \in (1/a_n, \infty)$, $n \in \mathbb{N}$]. As a consequence, there exists a positive constant K_1 such that

$$I_{n,l}^1 \leq K_1 \frac{a_n^3}{na_{n/2}} \int_{1/a_n}^\infty dy e^{-2^l \lfloor a_n y \rfloor / a_n} \left(\frac{\hat{A}(\lfloor a_n y \rfloor)}{\lfloor a_n y \rfloor} \right) = K_1 J_{n,l}.$$

Therefore, it suffices to show that

$$(28) \quad \limsup_{l \rightarrow \infty} \sup_{n \in \mathbb{N}} J_{n,l} = 0.$$

To this end, we use the upper bounds (24) and (25), with $x = a_n$ and $y = \lfloor a_n y \rfloor / a_n$ (x and xy being, resp., greater than a_0 and 1):

$$\frac{\hat{A}(\lfloor a_n y \rfloor)}{\hat{A}(a_n)} \leq C_\delta \left(\left(\frac{\lfloor a_n y \rfloor}{a_n} \right)^{\alpha-1+\delta} \vee \left(\frac{\lfloor a_n y \rfloor}{a_n} \right)^{\alpha-1-\delta} \right).$$

Thus,

$$J_{n,l} \leq \frac{a_n^2 \hat{A}(a_n)}{na_{n/2}} \int_{1/a_n}^\infty dy e^{-2^l \lfloor a_n y \rfloor / a_n} \left(\left(\frac{a_n}{\lfloor a_n y \rfloor} \right)^{2-\alpha-\delta} \vee \left(\frac{a_n}{\lfloor a_n y \rfloor} \right)^{2-\alpha+\delta} \right).$$

Using the fact that $\lfloor a_n y \rfloor \geq a_n y - 1$, and the change of variable $y' = y - 1/a_n$, we get

$$J_{n,l} \leq \frac{a_n^2 \hat{A}(a_n)}{na_{n/2}} \int_0^\infty dy e^{-2^l y} \left(\frac{1}{y^{2-\alpha-\delta}} \vee \frac{1}{y^{2-\alpha+\delta}} \right).$$

Now (22) gives that $\hat{A}(a_n)/n = \hat{A}(a_n)/A(a_n) \sim 1/\alpha a_n$, so we have

$$\frac{a_n^2 \hat{A}(a_n)}{na_{n/2}} \sim \frac{a_n}{\alpha a_{n/2}}.$$

Since a is regularly varying with index $1/\alpha$, the right-hand term has a finite limit as n goes to infinity. Therefore, $a_n^2 \hat{A}(a_n)/na_{n/2}$ is bounded uniformly in n . Hence, there exists a constant $K \in (0, \infty)$ such that

$$\sup_{n \in \mathbb{N}} J_{n,l} \leq K \int_0^\infty dy e^{-2^l y} \left(\frac{1}{y^{2-\alpha-\delta}} \vee \frac{1}{y^{2-\alpha+\delta}} \right).$$

This yields (28) by taking the limit as l goes to infinity.

For the second part, we can still use (27). As in the first step, we get

$$I_{n,l}^2 \leq 2M \frac{a_n^3}{na_{n/2}} \int_{1/a_n}^\infty dy e^{-2^l \lfloor a_n y \rfloor / a_n} \left(\sum_{h=h(n, \lfloor a_n y \rfloor)+1}^{h'(n, \lfloor a_n y \rfloor)} \mathbb{P}(\hat{S}_h = \lfloor a_n y \rfloor) \right).$$

Since the sum is null if $\hat{A}(\lfloor a_n y \rfloor / B) > \lfloor n/2 \rfloor$, we have

$$I_{n,l}^2 \leq 2M \frac{a_n^3}{na_{n/2}} \int_{1/a_n}^\infty dy e^{-2^l \lfloor a_n y \rfloor / a_n} \left(\sum_{h=\hat{A}(\lfloor a_n y \rfloor / B)+1}^\infty \mathbb{P}(\hat{S}_h = \lfloor a_n y \rfloor) \right).$$

We now turn the remaining sum into an integral:

$$I_{n,l}^2 \leq 2M \frac{a_n^3}{na_{n/2}} \int_{1/a_n}^\infty dy e^{-2^l \lfloor a_n y \rfloor / a_n} \int_{\hat{A}(\lfloor a_n y \rfloor / B)}^\infty dx \mathbb{P}(\hat{S}_{\lfloor x+1 \rfloor} = \lfloor a_n y \rfloor).$$

Using the change of variable $x' = \hat{A}(\lfloor a_n y \rfloor / B)x$ and the upper bound (21), this gives

$$I_{n,l}^2 \leq 2MM' \frac{a_n^3}{na_{n/2}} \int_{1/a_n}^\infty dy e^{-2^l \lfloor a_n y \rfloor / a_n} \int_1^\infty dx \frac{\hat{A}(\lfloor a_n y \rfloor / B)}{\hat{a}(\lfloor \hat{A}(\lfloor a_n y \rfloor / B)x + 1 \rfloor)}.$$

Since \hat{a} is increasing, for all x, y , we have

$$\hat{a}(\lfloor \hat{A}(\lfloor a_n y \rfloor / B)x + 1 \rfloor) \geq \hat{a}(\hat{A}(\lfloor a_n y \rfloor / B)x).$$

Fix $\delta \in (0, 1/(\alpha - 1) - 1)$. Inequality (24) then gives, for all $x \geq 1, y \geq 1/a_n$,

$$\begin{aligned} \hat{a}(\lfloor \hat{A}(\lfloor a_n y \rfloor / B)x + 1 \rfloor) &\geq c_\delta^{-1} \hat{a}(\hat{A}(\lfloor a_n y \rfloor / B))x^{1/(\alpha-1)-\delta} \\ &= c_\delta^{-1} \frac{\lfloor a_n y \rfloor}{B} x^{1/(\alpha-1)-\delta}. \end{aligned}$$

Thus, there exist constants $K_2, K_2' \in (0, \infty)$ such that

$$I_{n,l}^2 \leq K_2 \frac{a_n^3}{na_{n/2}} \int_{1/a_n}^\infty dy e^{-2^l \lfloor a_n y \rfloor / a_n} \frac{\hat{A}(\lfloor a_n y \rfloor / B)}{\lfloor a_n y \rfloor} \int_1^\infty \frac{dx}{x^{1/(\alpha-1)-\delta}} = K_2' J_{n,l},$$

and (28) also gives the conclusion.

For the third part, since the terms with indices $k \leq \lfloor n/2 \rfloor$ are null, we simply use the bounds $\mathbb{P}(Y_{k-h+1} = n - h + 1) \leq 1$ and $\mathbb{P}(\hat{S}_h = k) \leq 1$:

$$\begin{aligned} I_{n,l}^3 &\leq a_n^2 \sum_{k=\lfloor n/2 \rfloor + 1}^n \sum_{h=1}^k \frac{1}{k} e^{-2^l k/a_n} \\ &\leq a_n^2 e^{-n2^l/2a_n} \sum_{k=\lfloor n/2 \rfloor + 1}^n 1 \\ &\leq na_n^2 e^{-n2^l/2a_n}. \end{aligned}$$

This quantity tends to 0 as l goes to infinity, uniformly in n . Indeed, for any $\kappa > 0$, the function $g_\kappa : x \mapsto x^\kappa e^{-x}$ is bounded by a constant G_κ , hence

$$I_{n,l}^3 \leq G_\kappa \frac{2^\kappa a_n^{2+\kappa}}{n^{\kappa-1}} \cdot 2^{-l\kappa}.$$

For any $\varepsilon > 0$, there exists a constant C_ε such that $a_n \leq C_\varepsilon n^{1/\alpha+\varepsilon}$ for all $n \in \mathbb{N}$. Therefore, the quantity $a_n^{2+\kappa}/n^{\kappa-1}$ is bounded as soon as $\kappa > (2 + \alpha)/(\alpha - 1)$. This completes the proof of (26).

For the second limit, we note that (18) yields

$$\int_{2^l}^\infty E_{m,n}(t) dt = \frac{a_n}{a_m} \int_{2^l}^\infty E_m(t) dt,$$

for all $m \leq n$ such that $P_m \neq 0$. Thus,

$$\sup_{\substack{n \in \mathbb{N} \\ P_m \neq 0}} \sup_{1 \leq m \leq n} \int_{2^l}^\infty \frac{m}{n} E_{m,n}(t) dt = \sup_{n \in \mathbb{N}} \sup_{\substack{1 \leq m \leq n \\ P_m \neq 0}} \frac{ma_n}{na_m} I_{m,l}.$$

As a consequence, it is enough to show that ma_n/na_m is bounded over $\{(m, n) \in \mathbb{N}^2 : m \leq n\}$. Now,

$$\begin{aligned} \sup \left\{ \frac{ma_n}{na_m} : m, n \in \mathbb{N}, m \leq n \right\} &\leq \sup \left\{ \frac{ma_{\lambda m}}{\lambda ma_m} : m \in \mathbb{N}, \lambda \in (1, \infty) \right\} \\ &\leq \sup \left\{ \frac{a_{\lambda m}}{\lambda a_m} : m \in \mathbb{N}, \lambda \in (1, \infty) \right\}. \end{aligned}$$

Fix $\delta \in (0, 1 - 1/\alpha)$. Since a is a positive increasing function in $R_{1/\alpha}$, Lemma 2.13 shows the existence of a constant such that, for all $m \in \mathbb{N}, \lambda \in (1, \infty)$,

$$\frac{a_{\lambda m}}{a_m} \leq C_\delta \lambda^{1/\alpha+\delta}.$$

Hence, for all $\lambda \in (1, \infty)$,

$$\sup_{m \in \mathbb{N}} \frac{a_{\lambda m}}{\lambda a_m} \leq C_\delta \lambda^{1/\alpha+\delta-1} \leq C_\delta. \quad \square$$

Key estimates for the proof of Theorem 1.3. We conclude this section by giving two consequences of Lemma 2.14 which will be used in the proof of Theorem 1.3.

COROLLARY 2.15. *It holds that*

$$\lim_{l \rightarrow \infty} \sup_{n \in \mathbb{N}} \mathbb{E} \left[\int_{2^l}^{\infty} \mu_{n, \xi_n}(t) dt \right] = 0.$$

PROOF. Using (13), we get

$$\begin{aligned} \sup_{n \in \mathbb{N}} \mathbb{E} \left[\int_{2^l}^{\infty} \mu_{n, \xi_n}(t) dt \right] &\leq \sup_{n \in \mathbb{N}} \frac{a_n}{n} e^{-2^l/a_n} + 2 \sup_{n \in \mathbb{N}} \int_{2^l}^{\infty} E_n(t) dt \\ &\quad + 2 \sup_{n \in \mathbb{N}} \sup_{1 \leq m \leq n} \int_{2^l}^{\infty} \frac{m}{n} E_{m,n}(t) dt. \end{aligned}$$

Lemma 2.14 shows that the last two terms tend to 0 as l goes to infinity. For the first term, we use again the fact that for any $\kappa > 0$, the function $g_\kappa : x \mapsto x^\kappa e^{-x}$ is bounded by a constant G_κ . Hence, for all $n \in \mathbb{N}$,

$$\frac{a_n}{n} e^{-2^l/a_n} \leq G_\kappa \frac{a_n^{\kappa+1}}{n} \cdot 2^{-\kappa l}.$$

Taking $\kappa < \alpha - 1$, we get that $a_n^{\kappa+1}/n$ is bounded, which completes the proof. \square

COROLLARY 2.16. *There exists a constant C such that, for all $n \in \mathbb{N}$,*

$$\mathbb{E}[\delta'_n(0, \xi_n)] \leq C.$$

PROOF. Recalling the definition of δ'_n , we get

$$\mathbb{E}[\delta'_n(0, \xi_n)] = \mathbb{E} \left[\int_0^{\infty} \mu_{n, \xi_n}(t) dt \right].$$

Now the upper bound (13) gives

$$\begin{aligned} \mathbb{E}[\delta'_n(0, \xi_n)] &\leq 1 + \mathbb{E} \left[\int_1^{\infty} \mu_{n, \xi_n}(t) dt \right] \\ &\leq 1 + \frac{a_n}{n} e^{-1/a_n} + 2 \int_1^{\infty} E_n(t) dt + 2 \sup_{1 \leq m \leq n} \int_1^{\infty} \frac{m}{n} E_{m,n}(t) dt. \end{aligned}$$

The second term is bounded as $n \rightarrow \infty$. Recall from the proof of Lemma 2.14 that

$$\int_1^{\infty} E_n(t) dt = I_{n,0} \leq I_{n,0}^1 + I_{n,0}^2 + I_{n,0}^3 \leq (K_1 + K'_2) J_{n,0} + I_{n,0}^3.$$

Moreover, we have seen that for any $\delta > 0$, there exists a constant K such that

$$\sup_{n \in \mathbb{N}} J_{n,0} \leq K \int_0^{\infty} dy e^{-y} \left(\frac{1}{y^{2-\alpha-\delta}} \wedge \frac{1}{y^{2-\alpha+\delta}} \right) < \infty,$$

and

$$I_{n,0}^3 \leq 2na_n^2 e^{-n/a_n}$$

is bounded as $n \rightarrow \infty$. Since we have seen at the end of the proof of Lemma 2.9 that there exists a constant K' such that for all $n \in \mathbb{N}$, $m \leq n$ such that $P_m \neq 0$,

$$\int_1^\infty \frac{m}{n} E_{m,n}(t) dt \leq K' \int_1^\infty E_m(t) dt,$$

this implies the corollary. \square

3. Proof of Theorem 1.3.

3.1. *Identity in law between $\text{Cut}_v(\mathcal{T})$ and \mathcal{T} .* In this section, we show that the semi-infinite matrices of the mutual distance of uniformly sampled points in \mathcal{T} and $\text{Cut}_v(\mathcal{T})$ have the same law. This justifies the existence of $\text{Cut}_v(\mathcal{T})$, as explained in Section 1.2, and shows the identity in law between \mathcal{T} and $\text{Cut}_v(\mathcal{T})$. The structure of the proof will be similar to that of Lemma 4 in [12]. Precise descriptions of the fragmentation processes we consider can be found in [24] and [25].

Recall that $(\xi(i))_{i \in \mathbb{N}}$ is a sequence of i.i.d. random variables in \mathcal{T} , with law μ , and $\xi(0) = 0$. Since the law of \mathcal{T} is invariant under uniform rerooting (see, e.g., [18], Proposition 4.8), and the definition of δ does not depend on the choice of the root of \mathcal{T} , we may assume that $\xi(1) = \rho$.

PROPOSITION 3.1. *It holds that*

$$(\delta(\xi(i), \xi(j)))_{i,j \geq 0} \stackrel{(d)}{=} (d(\xi(i+1), \xi(j+1)))_{i,j \geq 0}.$$

PROOF. Here, it is convenient to work on fragmentation processes taking values in the set of the partitions of \mathbb{N} .

First, we introduce a process Π which corresponds to our fragmentation of \mathcal{T} by saying that $i, j \in \mathbb{N}$ belong to the same block of $\Pi(t)$ if and only if the path $[[\xi(i), \xi(j)]]_V$ does not intersect the set $\{b_k : k \in I, t_k \leq t\}$ of the points marked before time t . For every $i \in \mathbb{N}$, we let $B_i(t)$ be the block of the partition $\Pi(t)$ containing i . Note that the partitions $\Pi(t)$ are exchangeable, which justifies the existence of the asymptotic frequencies $\lambda(B_i(t))$ of the blocks $B_i(t)$, where

$$\lambda(B) = \lim_{n \rightarrow \infty} \frac{1}{n} |B \cap \{1, \dots, n\}|.$$

Then we define

$$\sigma_i(t) = \inf \left\{ u \geq 0 : \int_0^u \lambda(B_i(s)) ds > t \right\}.$$

We use σ_i as a time-change, letting $\Pi'(t)$ be the partition whose blocks are the sets $B_i(\sigma_i(t))$ for $i \in \mathbb{N}$. Note that this is possible because $B_i(\sigma_i(t))$ and $B_j(\sigma_j(t))$ are either equal or disjoint.

We define a second fragmentation Γ , which results from cutting the stable tree \mathcal{T} at its heights. For every $x, y \in \mathcal{T}$, we let $x \wedge y$ denote the branch-point between x and y , that is, the unique point such that $\llbracket \rho, x \wedge y \rrbracket_V = \llbracket \rho, x \rrbracket_V \cap \llbracket \rho, y \rrbracket_V$. With this notation, we say that $i, j \in \mathbb{N}$ belong to the same block of $\Gamma(t)$ if and only if $d(\rho, \xi(i+1) \wedge \xi(j+1)) > t$.

Then we have the following link between the two fragmentations.

LEMMA 3.2. *The fragmentation processes Π' and Γ have the same law.*

PROOF. Miermont has shown in [25], Theorem 1, that the process Π is a self-similar fragmentation with index $1/\alpha$, erosion coefficient 0 and dislocation measure Δ_α known explicitly. Applying Theorem 3.3 in [10], we get that the time-changed fragmentation Π' is still self-similar, with index $1/\alpha - 1$, erosion coefficient 0 and the same dislocation measure Δ_α . Now the process Γ is also self-similar, with the same characteristics as Π' (see [24], Proposition 1, Theorem 1). Thus, Γ and Π' have the same law. \square

Using the law of large numbers, we note that $\lambda(B_i(s)) = \mu_{\xi(i)}(s)$ almost surely. As a consequence, $\sigma_i(t) = \infty$ for $t = \int_0^\infty \lambda(B_i(s)) ds = \delta(0, \xi(i))$, which means that $\delta(0, \xi(i))$ can be seen as the first time when the singleton $\{i\}$ is a block of Π' . Recalling that $d(\rho, \xi(i+1)) = d(\xi(1), \xi(i+1))$ is the first time when $\{i\}$ is a block of Γ , we get

$$(29) \quad (\delta(0, \xi(i)))_{i \geq 1} \stackrel{(d)}{=} (d(\xi(1), \xi(i+1)))_{i \geq 1}.$$

Similarly, for any $i \neq j \in \mathbb{N}$,

$$\begin{aligned} \delta(0, \xi(i) \wedge \xi(j)) &= \frac{1}{2}(\delta(0, \xi(i)) + \delta(0, \xi(j)) - \delta(\xi(i), \xi(j))) \\ &= \int_0^{\tau(i,j)} \lambda(B_i(s)) ds, \end{aligned}$$

where $\tau(i, j)$ denotes the first time when a mark appears on the segment $\llbracket \xi(i), \xi(j) \rrbracket_V$. Thus, $\delta(0, \xi(i) \wedge \xi(j))$ is the first time when the blocks containing i and j are separated in Π' . In terms of the fragmentation Γ , this corresponds to $d(\rho, \xi(i+1) \wedge \xi(j+1))$. Hence,

$$(\delta(0, \xi(i) \wedge \xi(j)))_{i, j \geq 1} \stackrel{(d)}{=} (d(\xi(1), \xi(i+1) \wedge \xi(j+1)))_{i, j \geq 1},$$

and this holds jointly with (29). This entails the proposition. \square

3.2. *Weak convergence.* We first establish the convergence for the cut-tree $\text{Cut}'_v(\mathcal{T}_n)$ endowed with the modified distance δ'_n , as defined in Section 2.1.

PROPOSITION 3.3. *There is the joint convergence*

$$\left(\frac{a_n}{n} \mathcal{T}_n, \text{Cut}'_v(\mathcal{T}_n)\right) \xrightarrow[n \rightarrow \infty]{(d)} (\mathcal{T}, \text{Cut}_v(\mathcal{T}))$$

in $\mathbb{M} \times \mathbb{M}$.

PROOF. Proposition 2.5 shows that for every fixed integer l , there is the joint convergence

$$\begin{aligned} & \frac{a_n}{n} \mathcal{T}_n \xrightarrow[n \rightarrow \infty]{(d)} \mathcal{T}, \\ & \left(2^{-l} \sum_{j=1}^{4^l} \mu_{n, \xi_n(i)}(j2^{-l})\right)_{i \in \mathbb{N}} \xrightarrow[n \rightarrow \infty]{(d)} \left(2^{-l} \sum_{j=1}^{4^l} \mu_{\xi(i)}(j2^{-l})\right)_{i \in \mathbb{N}}. \end{aligned}$$

Let

$$\Delta_{n,l}(i) = \mathbb{E} \left[\left| \int_0^\infty \mu_{n, \xi_n(i)}(t) dt - 2^{-l} \sum_{j=1}^{4^l} \mu_{n, \xi_n(i)}(j2^{-l}) \right| \right].$$

For any nonincreasing function $f : \mathbb{R}_+ \rightarrow [0, 1]$, we have the upper bound

$$(30) \quad \left| \int_0^\infty f(t) dt - 2^{-l} \sum_{j=1}^{4^l} f(j2^{-l}) \right| \leq 2^{-l} + \int_{2^l}^\infty f(t) dt.$$

Applying this inequality to $\mu_{n, \xi_n(i)}$ yields

$$\Delta_{n,l}(i) \leq 2^{-l} + \mathbb{E} \left[\int_{2^l}^\infty \mu_{n, \xi_n(i)}(t) dt \right].$$

Corollary 2.15 now shows that

$$\lim_{l \rightarrow \infty} \sup_{n \in \mathbb{N}} \Delta_{n,l}(i) = 0,$$

and $\Delta_{n,l}(i)$ does not depend on i . Besides, Proposition 3.1 shows that

$$\delta(0, \xi(i)) = \int_0^\infty \mu_{\xi(i)}(t) dt$$

has the same law as $d(0, \xi(i))$ and, therefore, has finite mean. As a consequence,

$$\begin{aligned} & \mathbb{E} \left[\left| \int_0^\infty \mu_{\xi(i)}(t) dt - 2^{-l} \sum_{j=1}^{4^l} \mu_{\xi(i)}(j2^{-l}) \right| \right] \\ & \leq 2^{-l} + \mathbb{E} \left[\int_{2^l}^\infty \mu_{\xi(i)}(t) dt \right] \\ & \xrightarrow[l \rightarrow \infty]{} 0, \end{aligned}$$

and the left-hand side does not depend on i . We conclude that

$$(\delta'_n(0, \xi_n(i)))_{i \in \mathbb{N}} \xrightarrow[n \rightarrow \infty]{(d)} (\delta(0, \xi(i)))_{i \in \mathbb{N}},$$

jointly with $(a_n/n) \cdot \mathcal{T}_n \xrightarrow{(d)} \mathcal{T}$.

Using in addition the convergence of the $\tau_n(i, j)$ shown in Proposition 2.5, a similar argument shows that the preceding convergences also hold jointly with

$$(\delta'_n(\xi_n(i), \xi_n(j)))_{i, j \in \mathbb{N}} \xrightarrow[n \rightarrow \infty]{(d)} (\delta(\xi(i), \xi(j)))_{i, j \in \mathbb{N}}.$$

This entails the proposition. \square

The convergence stated in Theorem 1.3 now follows immediately. Indeed, Lemma 2.1 and Corollary 2.16 show that

$$\mathbb{E} \left[\left| \frac{a_n}{n} \delta_n(i, j) - \delta'_n(i, j) \right|^2 \right] \leq \frac{2Ca_n}{n}$$

for all $i, j \geq 0$ [recalling that $\xi_n(0) = 0$]. Thus, the preceding proposition gives the joint convergence

$$\left(\frac{a_n}{n} \mathcal{T}_n, \frac{a_n}{n} \text{Cut}_v(\mathcal{T}_n) \right) \xrightarrow[n \rightarrow \infty]{(d)} (\mathcal{T}, \text{Cut}_v(\mathcal{T})).$$

4. The finite variance case. In this section, we assume that the offspring distribution ν of the Galton–Watson trees \mathcal{T}_n has finite variance σ^2 . Theorem 23 of [5] shows that $(\sigma/\sqrt{n}) \cdot \mathcal{T}_n$ converges to the Brownian tree \mathcal{T}^{br} . More precisely, still using the three processes described in Section 2.2.2 to encode the trees \mathcal{T}_n , the joint convergence stated in Theorem 2.3 holds with $a_n = \sigma\sqrt{n}$, and limit processes defined by $X_t = B_t$ and $H_t = 2B_t$ for all $t \in [0, 1]$. (Recall that B denotes the excursion of length 1 of the standard Brownian motion.) Note that the normalization of X is not exactly the same as the one we used for the stable tree, since the Laplace transform of a standard Brownian motion B' is $\mathbb{E}[e^{-\lambda B'_t}] = e^{\lambda^2 t/2}$. The fact that the height process H is equal to $2X$ can be seen from the definition of H as a local time, as explained in [17], Section 1.2.

Given these results, the proof of Theorem 1.4 follows the same structure as that of the main theorem. We first note that the results on the modified distance, introduced in Section 2.1, still hold. In the next two sections, we will see that we also have analogues for Proposition 2.5, and Corollaries 2.15 and 2.16.

4.1. *Convergence of the component masses.* We use the same notation as in Section 2.2. Recall in particular that $\mu_{n, \xi_n(i)}$ denotes the mass of the component $\mathcal{T}_{n, \xi_n(i)}(t)$, and that $\tau_n(i, j)$ denotes the first time when the components $\mathcal{T}_{n, \xi_n(i)}(t)$ and $\mathcal{T}_{n, \xi_n(j)}(t)$ become disjoint. To simplify, we drop the superscript br for the

quantities associated to the Brownian tree (e.g., the mass-measure, the mass of a component, etc.), keeping the notation we used in the case of the stable tree. Our first step is to prove the following result.

PROPOSITION 4.1. *As $n \rightarrow \infty$, we have the following weak convergences:*

$$\begin{aligned} \frac{\sigma}{\sqrt{n}} \mathcal{T}_n &\xrightarrow{(d)} \mathcal{T}^{\text{br}}, \\ (\tau_n(i, j))_{i, j \geq 0} &\xrightarrow{(d)} \left(\left(1 + \frac{1}{\sigma^2} \right)^{-1} \tau(i, j) \right)_{i, j \geq 0}, \\ (\mu_{n, \xi_n(i)}(t))_{i \geq 0, t \geq 0} &\xrightarrow{(d)} \left(\mu_{\xi(i)} \left(\left(1 + \frac{1}{\sigma^2} \right) t \right) \right)_{i \geq 0, t \geq 0}, \end{aligned}$$

where the three hold jointly.

We begin by showing the same kind of property as in Lemma 2.4. For all $n \in \mathbb{N}$, we let $\tilde{X}^{(n)}$ and $\tilde{C}^{(n)}$ denote the rescaled Lukasiewicz path and contour function of the symmetrized tree $\tilde{\mathcal{T}}_n$.

LEMMA 4.2. *We have the joint convergence*

$$(X^{(n)}, C^{(n)}, \tilde{X}^{(n)}, \tilde{C}^{(n)}) \xrightarrow[n \rightarrow \infty]{(d)} (X, H, \tilde{X}, \tilde{H}),$$

where $\tilde{H}_t = H_{1-t}$ and $\tilde{X}_t = \tilde{H}_t/2$ for all $t \in [0, 1]$.

PROOF. Since \mathcal{T}_n and $\tilde{\mathcal{T}}_n$ have the same law, $(\tilde{X}^{(n)}, \tilde{C}^{(n)})$ converges in distribution to a couple of processes having the same law as (X, H) in $\mathbb{D} \times \mathbb{D}$. Thus, the sequence of the laws of the processes $(X^{(n)}, C^{(n)}, \tilde{X}^{(n)}, \tilde{C}^{(n)})$ is tight in \mathbb{D}^4 . Up to extraction, we can assume that $(X^{(n)}, C^{(n)}, \tilde{X}^{(n)}, \tilde{C}^{(n)})$ converges in distribution to $(X, H, \tilde{X}, \tilde{H})$.

Fix $t \in [0, 1]$. The definition of the contour function shows that for all $n \in \mathbb{N}$, we have $\tilde{C}_t^{(n)} = C_{1-t}^{(n)}$. Since H and \tilde{H} are a.s. continuous, taking the limit yields $\tilde{H}_t = H_{1-t}$ almost surely. Besides, since (X, H) and (\tilde{X}, \tilde{H}) have the same law, we have $\tilde{X}_t = \tilde{H}_t/2$ a.s. for all $t \in [0, 1]$.

These equalities also hold a.s., simultaneously for a countable number of times t , and the continuity of H, X, \tilde{H} and \tilde{X} give that a.s., they hold for all $t \in [0, 1]$. This identifies uniquely the law of $(X, H, \tilde{X}, \tilde{H})$, hence the lemma. \square

This lemma shows that we can still work in the setting of

$$(31) \quad \begin{cases} (X^{(n)}, \tilde{X}^{(n)}) \xrightarrow[n \rightarrow \infty]{} (X, \tilde{X}) & \text{a.s.,} \\ (t_i^{(n)}, i \in \mathbb{N}) \xrightarrow[n \rightarrow \infty]{} (t_i, i \in \mathbb{N}) & \text{a.s.,} \end{cases}$$

where $t_i^{(n)} = (\xi_n(i) + 1)/(n + 1)$ for all $n \in \mathbb{N}, i \geq 0$, and $(t_i, i \in \mathbb{N})$ is a sequence of independent uniform variables in $[0, 1]$ such that $\xi(i) = p(t_i)$.

Recall the notation $\mathcal{R}_n(k)$ for the shape of the subtree of \mathcal{T}_n (or \mathcal{T}^{br} if $n = \infty$) spanned by the root and the vertices $\xi_n(1), \dots, \xi_n(k)$ [or $\xi(1), \dots, \xi(k)$ if $n = \infty$]. We also keep the notation $L_n(v) = \text{deg}(v, \mathcal{T}_n)/a_n$ for the rate at which a vertex v is deleted in \mathcal{T}_n (if $n \in \mathbb{N}$), and

$$\sigma_n(t) = \sum_{\substack{0 < s < t \\ X_{s^-}^{(n)} < I_{s,t}^{(n)}}} \Delta X_s^{(n)} \quad \forall t \in [0, 1],$$

where $I_{s,t}^{(n)} = \inf_{s < u < t} X_u^{(n)}$, and $X^{(\infty)} = X$.

As in Section 2.2, we state two lemmas which allow us two control the rates at which the fragmentations happen on the vertices and the edges of $\mathcal{R}_n(k)$.

LEMMA 4.3. *Fix $k \in \mathbb{N}$. Under (31), $\mathcal{R}_n(k)$ is a.s. constant for all n large enough (say $n \geq N$). Identifying the vertices of $\mathcal{R}_n(k)$ with $\mathcal{R}_\infty(k)$ for all $n \geq N$, we have the a.s. convergence*

$$L_n(v) \xrightarrow[n \rightarrow \infty]{} 0 \quad \forall v \in V(\mathcal{R}_\infty(k)).$$

PROOF. The proof is the same as that of Lemma 2.7. In particular, we get that if the $b^{(n,k)}$ are the times encoding the “same” vertex v of $\mathcal{R}_n(k)$, for $n \geq N$, then we have the a.s. convergences

$$\begin{aligned} b^{(n,k)} &\xrightarrow[n \rightarrow \infty]{} b^{(\infty,k)}, \\ X_{b^{(n,k)}}^{(n)} &\xrightarrow[n \rightarrow \infty]{} X_{b^{(\infty,k)}}, \\ X_{(b^{(n,k)})^-}^{(n)} &\xrightarrow[n \rightarrow \infty]{} X_{(b^{(\infty,k)})^-}. \end{aligned}$$

Since X is now continuous, this yields

$$L_n(v) = \Delta X_{b^{(n,k)}}^{(n)} + \frac{1}{a_n} \xrightarrow[n \rightarrow \infty]{} \Delta X_{b^{(\infty,k)}} = 0. \quad \square$$

LEMMA 4.4. *Let $(b_n)_{n \geq 1} \in [0, 1]^{\mathbb{N}}$ be a converging sequence in $[0, 1]$, and let b denote its limit. Then*

$$\sigma_n(b_n) \xrightarrow[n \rightarrow \infty]{} H_b \quad \text{a.s.}$$

PROOF. As in the proof of Lemma 2.8, for all $n \in \mathbb{N} \cup \{\infty\}$, we write $\sigma_n(t) = \sigma_n^-(t) + \sigma_n^+(t)$, where

$$\sigma_n^+(t) = \sum_{\substack{0 < s < t \\ X_{s^-}^{(n)} < I_{s,t}^{(n)}}} (X_s^{(n)} - I_{s,t}^{(n)}) \quad \text{and} \quad \sigma_n^-(t) = \sum_{\substack{0 < s < t \\ X_{s^-}^{(n)} < I_{s,t}^{(n)}}} (I_{s,t}^{(n)} - X_{s^-}^{(n)}).$$

For all $t \geq 0, n \in \mathbb{N}$, we have $\sigma_n^-(t) = X_t^{(n)}$. As a consequence, (31) gives

$$\sigma_n^-(b_n) \xrightarrow[n \rightarrow \infty]{} X_b \quad \text{a.s.}$$

Besides, we still have $\sigma_n^+(b_n) = \tilde{\sigma}_n^-(\tilde{b}_n)$, with

$$\tilde{b}_n = 1 - b_n + \frac{1}{n+1} (1 + H_{(n+1)b_n-1}^{[n]} - D_{(n+1)b_n-1}^{[n]}).$$

Now

$$\tilde{b}_n \xrightarrow[n \rightarrow \infty]{} 1 - b - l(b),$$

where $l(b) = \inf\{s > b : X_s = X_b\} - b$. Using (31) again, we get

$$\sigma_n^+(b_n) \xrightarrow[n \rightarrow \infty]{} \tilde{X}_{1-b-l(b)} = X_{b+l(b)} = X_b \quad \text{a.s.}$$

Thus, we have the a.s. convergence

$$\sigma_n(b_n) \xrightarrow[n \rightarrow \infty]{} 2X_b = H_b. \quad \square$$

We can now give the proof of Proposition 4.1.

PROOF OF PROPOSITION 4.1. Fix $n \in \mathbb{N} \cup \{\infty\}$. As in the proof of Proposition 2.5, we write $\mathcal{R}_n(k, t)$ for the reduced tree with edge-lengths, endowed with point processes of marks on its edges and vertices such that:

- The marks on the vertices of $\mathcal{R}_n(k)$ appear at the same time as the marks on the corresponding vertices of \mathcal{T}_n .
- Each edge receives a mark at its midpoint at the first time when a vertex v of \mathcal{T}_n such that $v \in e$ is marked in \mathcal{T}_n .

These two point processes are independent, and their rates are the following:

- If $n \in \mathbb{N}$, each vertex v of $\mathcal{R}_n(k)$ is marked at rate $L_n(v)$, independently of the other vertices. If $n = \infty$, there are no marks on the vertices.
- For each edge e of $\mathcal{R}_n(k)$, letting b, b' denote the points of $B_n(k)$ corresponding to e^-, e^+ , the edge e is marked at rate $\Sigma L_n(e)$, independently of the other edges, with

$$\begin{aligned} \Sigma L_n(e) &= \sum_{v \in V(\mathcal{T}_n) \cap e} L_n(v) \\ &= \sigma_n(b') - \sigma_n(b) + \frac{n}{a_n^2} (H_{(b')^-}^{(n)} - H_{b^-}^{(n)}) - L_n(e^-) \end{aligned}$$

if $n \in \mathbb{N}$, and

$$\Sigma L_\infty(e) = H_{b'} - H_b.$$

We see from Lemmas 4.3 and 4.4 that $L_n(v)$ converges to 0 as $n \rightarrow \infty$, and that

$$\Sigma L_n(e) \xrightarrow{n \rightarrow \infty} \left(1 + \frac{1}{\sigma^2}\right) \Sigma L_\infty(e).$$

As a consequence, we have the convergence

$$(32) \quad \left(\frac{a_n}{n} \mathcal{R}_n(k, t), t \geq 0\right) \xrightarrow{n \rightarrow \infty}^{(d)} \left(\mathcal{R}_\infty\left(k, \left(1 + \frac{1}{\sigma^2}\right)t\right), t \geq 0\right).$$

[As in the case $\alpha \in (1, 2)$, $(a_n/n) \cdot \mathcal{R}_n(k, t)$ and $\mathcal{R}_\infty(k, t)$ can be seen as random variables in $\mathbb{T} \times (\mathbb{R}_+ \cup \{-1\})^\mathbb{N} \times \{-1, 0, 1\}^{\mathbb{N}^2}$.]

For all $i \in \mathbb{N}$, we let $\eta_n(k, i, t)$ denote the number of vertices among $\xi_n(1), \dots, \xi_n(k)$ in the component of $\mathcal{R}_n(k)$ containing $\xi_n(i)$ at time t , and similarly $\eta_\infty(k, i, t)$ the number of vertices among $\xi(1), \dots, \xi(k)$ in the component of $\mathcal{R}_\infty(k)$ containing $\xi(i)$ at time t . It follows from (32) that we have the joint convergences

$$\begin{aligned} \frac{a_n}{n} \mathcal{T}_n &\xrightarrow{(d)} \mathcal{T}^{\text{br}}, \\ (\eta_n(k, i, t))_{t \geq 0, i \in \mathbb{N}} &\xrightarrow{(d)} \left(\eta_\infty\left(k, i, \left(1 + \frac{1}{\sigma^2}\right)t\right)\right)_{t \geq 0, i \in \mathbb{N}}, \\ (\tau_n(i, j))_{i, j \in \mathbb{N}} &\xrightarrow{(d)} \left(\left(1 + \frac{1}{\sigma^2}\right)^{-1} \tau(i, j)\right)_{i, j \in \mathbb{N}}. \end{aligned}$$

The end of the proof is the same as for Proposition 2.5. \square

4.2. *Upper bound for the expected component mass.* The second step is to show that, as in Section 2.3, the following properties hold.

LEMMA 4.5. *It holds that*

$$\lim_{l \rightarrow \infty} \sup_{n \in \mathbb{N}} \mathbb{E} \left[\int_{2^l}^\infty \mu_{n, \xi_n}(t) dt \right] = 0.$$

Besides, there exists a constant C such that, for all $n \in \mathbb{N}$,

$$\mathbb{E}[\delta'_n(0, \xi_n)] \leq C.$$

PROOF. We use the fact that there exists a natural coupling between the edge-fragmentation and the vertex-fragmentation of \mathcal{T}_n . Indeed, both can be obtained by a deterministic procedure, given \mathcal{T}_n and a uniform permutation (i_1, \dots, i_n) of $\{1, \dots, n\}$. More precisely, in the edge-fragmentation, we delete the edge e_{i_k} at each step k , thus splitting \mathcal{T}_n into at most two connected components, whereas in the vertex fragmentation, we delete all the edges such that $e^- = e_{i_k}^-$. Thus, at each step, the connected component containing a given edge e for the

vertex-fragmentation is included in the component containing e for the edge-fragmentation.

Now consider the continuous-time versions of these fragmentations: each edge is marked independently with rate $a_n/n = \sigma/\sqrt{n}$ in our case, and $1/\sqrt{n}$ in [12]. We let $\mathcal{T}_{n,i}^E(t)$ and $\mathcal{T}_{n,i}^V(t)$ denote the connected components containing the edge e_i at time t , respectively, for the edge-fragmentation and the vertex-fragmentation. Then the preceding remark shows that there exists a coupling such that $\mathcal{T}_{n,i}^V(t) \subset \mathcal{T}_{n,i}^E(\sigma t)$ a.s., and thus $\mu_n(\mathcal{T}_{n,i}^V(t)) \leq \mu_n(\mathcal{T}_{n,i}^E(\sigma t))$ almost surely.

Lemma 3 and Corollary 1 of [12] show that the two announced properties hold for the case of the edge-fragmentation. Therefore, they also hold for the vertex-fragmentation. \square

4.3. *Proof of Theorem 1.4.* As before, the proof of Theorem 1.4 now relies on showing a joint convergence for the rescaled versions of \mathcal{T}_n and the modified cut-tree $\text{Cut}_v(\mathcal{T}_n)$:

$$(33) \quad \left(\frac{a_n}{n} \mathcal{T}_n, \left(1 + \frac{1}{\sigma^2} \right) \text{Cut}'_v(\mathcal{T}_n) \right) \xrightarrow[n \rightarrow \infty]{(d)} (\mathcal{T}^{\text{br}}, \text{Cut}(\mathcal{T}^{\text{br}}))$$

in $\mathbb{M} \times \mathbb{M}$. Indeed, Lemma 2.1 and the second part of Lemma 4.5 show that

$$\mathbb{E} \left[\left| \frac{a_n}{n} \delta_n(i, j) - \delta'_n(i, j) \right|^2 \right] \leq \frac{2Ca_n}{n}$$

for all $i, j \geq 0$. Thus, (33) entails the joint convergence

$$\left(\frac{a_n}{n} \mathcal{T}_n, \frac{a_n}{n} \left(1 + \frac{1}{\sigma^2} \right) \text{Cut}_v(\mathcal{T}_n) \right) \xrightarrow[n \rightarrow \infty]{(d)} (\mathcal{T}, \text{Cut}_v(\mathcal{T})).$$

Since $a_n = \sigma\sqrt{n}$, this gives Theorem 1.4.

Let us finally justify why (33) holds. Proposition 4.1 shows that for every fixed integer l , there is the joint convergence

$$\begin{aligned} & \frac{a_n}{n} \mathcal{T}_n \xrightarrow[n \rightarrow \infty]{(d)} \mathcal{T}^{\text{br}}, \\ & \left(2^{-l} \sum_{j=1}^{4^l} \mu_{n, \xi_n(i)}(j2^{-l}) \right)_{i \in \mathbb{N}} \xrightarrow[n \rightarrow \infty]{(d)} \left(2^{-l} \sum_{j=1}^{4^l} \mu_{\xi(i)}(C_\sigma j2^{-l}) \right)_{i \in \mathbb{N}}, \end{aligned}$$

where $C_\sigma = 1 + 1/\sigma^2$. Using the upper bound (30) and the first part of Lemma 4.5, we get that

$$\limsup_{l \rightarrow \infty} \sup_{n \in \mathbb{N}} \mathbb{E} \left[\left| \int_0^\infty \mu_{n, \xi_n(i)}(t) dt - 2^{-l} \sum_{j=1}^{4^l} \mu_{n, \xi_n(i)}(j2^{-l}) \right| \right] = 0,$$

and these expectations do not depend on i . Proposition 3.1 of [12] shows that $\delta(0, \xi(i))$ has the same law as $d(0, \xi(i))$ and, therefore, has finite mean. Thus,

$$\left| \int_0^\infty \mu_{\xi(i)}(C_\sigma t) dt - 2^{-l} \sum_{j=1}^{4^l} \mu_{\xi(i)}(C_\sigma j 2^{-l}) \right| \leq 2^{-l} + \underbrace{\mathbb{E} \left[\int_{2^l}^\infty \mu_{\xi(i)}(C_\sigma t) dt \right]}_{\xrightarrow{l \rightarrow \infty} 0},$$

and the left-hand side does not depend on i . Since

$$\int_0^\infty \mu_{\xi(i)}(C_\sigma t) dt = C_\sigma^{-1} \int_0^\infty \mu_{\xi(i)}(t) dt = C_\sigma^{-1} \delta(0, \xi(i)),$$

we conclude that

$$(C_\sigma \delta'_n(0, \xi_n(i)))_{i \in \mathbb{N}} \xrightarrow[n \rightarrow \infty]{(d)} (\delta(0, \xi(i)))_{i \in \mathbb{N}},$$

jointly with $(a_n/n) \cdot \mathcal{T}_n \xrightarrow{(d)} \mathcal{T}$. Using in addition the convergence of the $\tau_n(i, j)$ shown in Proposition 2.5, we see that the preceding convergences also hold jointly with

$$(C_\sigma \delta'_n(\xi_n(i), \xi_n(j)))_{i, j \in \mathbb{N}} \xrightarrow[n \rightarrow \infty]{(d)} (\delta(\xi(i), \xi(j)))_{i, j \in \mathbb{N}},$$

and this gives the convergence (33).

APPENDIX: ADAPTATION OF DONEY’S RESULT

We rephrase Lemma 2.12 using the notation of [14].

LEMMA A.1. *Let $(X_i)_{i \in \mathbb{N}}$ be a sequence of i.i.d. variables in $\mathbb{N} \cup \{0\}$, whose law belongs to the domain of attraction of a stable law of index $\hat{\alpha} \in (0, 1)$, and $S_n = X_1 + \dots + X_n$. We also let $A \in R_{\hat{\alpha}}$ be a positive increasing function such that*

$$(34) \quad \mathbb{P}(X > r) \sim \frac{1}{A(r)},$$

and a the inverse function of A . Besides, we suppose that the additional hypothesis

$$(35) \quad \sup_{r \geq 1} \left(\frac{r \mathbb{P}(X = r)}{\mathbb{P}(X > r)} \right) < \infty$$

holds. Then there exist constants B, C such that for all $r \in \mathbb{N}$, for all n such that $r/a_n \geq B$,

$$\mathbb{P}(S_n = r) \leq C \frac{n}{r A(r)}.$$

This result is an adaptation of a theorem shown by Doney in [14], which gives an equivalent for $\mathbb{P}(S_n = r)$ as $n \rightarrow \infty$, uniformly in n such that $r/a_n \rightarrow \infty$, using the slightly stronger hypothesis

$$\mathbb{P}(X = r) \sim \frac{1}{rA(r)} \quad \text{as } r \rightarrow \infty$$

instead of (35).

SKETCH OF THE PROOF. The main idea is to split up $\mathbb{P}(S_n = r)$ into four terms, depending upon the values taken by $M_n = \max\{X_i : i = 1, \dots, n\}$ and $N_n = |\{m \leq n : X_m > z\}|$. More precisely, letting η and γ be constants in $(0, 1)$, $w = r/a_n$ and $z = a_n w^\gamma$, we have

$$\mathbb{P}(S_n = r) = \sum_{i=0}^3 \mathbb{P}(\{S_n = r\} \cap A_i),$$

where $A_i = \{M_n \leq \eta r, N_n = i\}$ for $i = 0, 1$, $A_2 = \{M_n \leq \eta r, N_n \geq 2\}$ and $A_3 = \{M_n > \eta r\}$. For our purposes, it is enough to show that there exist constants c_i such that

$$q_i := \mathbb{P}(\{S_n = r\} \cap A_i) \leq c_i \frac{n}{rA(r)} \quad \forall i \in \{0, 1, 2, 3\}.$$

The constants γ and η are fixed, with conditions that will be given later (see the detailed version of the proof for explicit conditions). In the whole proof, we suppose that $w \geq B$, for B large enough (possibly depending on the values of η and γ). Note that hypotheses (34) and (35) imply the existence of a constant c such that

$$(36) \quad p_r = \mathbb{P}(X = r) \leq \frac{c}{rA(r)} \quad \text{and} \quad \bar{F}(r) = \mathbb{P}(X > r) \leq \frac{c}{A(r)}.$$

The first calculations of [14] show that we have the following inequalities:

$$\begin{aligned} q_3 &\leq n \sup_{l > \eta r} p_l, \\ q_2 &\leq \frac{1}{2} n^2 \bar{F}(z) \sup_{l > z} p_l, \\ q_1 &\leq n \mathbb{P}(M_{n-1} \leq z, S_{n-1} > (1 - \eta)r) \sup_{l > z} p_l. \end{aligned}$$

We now use (36), and apply Lemma 2.13 for the regularly varying function A . The first inequality thus yields the existence of a constant c_3 which only depends on the value of η . Similarly, the second inequality gives the existence of c_2 , provided γ is large enough (independently of B) and $B \geq 1$.

To get the existence of c_1 , we first apply Lemma 2 of [14], which gives an upper bound for the quantity $\mathbb{P}(M_{n-1} \leq z, S_{n-1} > (1 - \eta)r)$ provided z is large enough

and $(1 - \eta)r \geq z$. Since $a_1 w^\gamma \leq z \leq r/w^{1-\gamma}$, these conditions can be achieved by taking B large enough. The lemma gives

$$q_1 \leq c \frac{n}{zA(z)} \cdot \left(\frac{c'z}{(1-\eta)r} \right)^{(1-\eta)r/z},$$

where c' is a constant. Now, applying Lemma 2.13, we get the existence of a constant c'_1 such that

$$q_1 \leq c'_1 \frac{n}{rA(r)} \cdot w^\kappa,$$

where κ depends on the values of η , γ and B . For a given choice of η and γ , and for B large enough, κ is negative, hence the existence of c_1 .

For q_0 , getting the upper bound goes by first showing that we can work under the hypotheses $r \leq nz$ and $r \leq na_n/2$ (instead of the hypotheses $n \rightarrow \infty$ and $r/na_n \rightarrow 0$ of [14]). Indeed, if $r > nz$, then $q_0 = 0$, and if $r > na_n/2$, another application of Lemma 2 of [14] and of Lemma 2.13 yields the result. The rest of the proof relies on replacing the X_i by truncated variables \widehat{X}_i , and using an exponentially biased probability law. This last part is long and technical, but it is rather easy to check that each step still holds with our hypotheses, for B large enough and with an appropriate choice of η (independently of B). \square

Acknowledgement. I would like to thank Grégory Miermont for many insightful comments and very thorough proofreadings.

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