The Annals of Applied Probability
2015, Vol. 25, No. 2, 860–897
DOI: 10.1214/14-AAP1012

FINITENESS OF ENTROPY FOR THE HOMOGENEOUS BOLTZMANN EQUATION WITH MEASURE INITIAL CONDITION

BY NICOLAS FOURNIER

Université Pierre et Marie Curie

We consider the $3D$ spatially homogeneous Boltzmann equation for (true) hard and moderately soft potentials. We assume that the initial condition is a probability measure with finite energy and is not a Dirac mass. For hard potentials, we prove that any reasonable weak solution immediately belongs to some Besov space. For moderately soft potentials, we assume additionally that the initial condition has a moment of sufficiently high order (8 is enough) and prove the existence of a solution that immediately belongs to some Besov space. The considered solutions thus instantaneously become functions with a finite entropy. We also prove that in any case, any weak solution is immediately supported by $\mathbb{R}^3$.

1. Introduction and results.

1.1. *The Boltzmann equation.* We consider a spatially homogeneous gas modeled by the Boltzmann equation: the density $f_t(v)$ of particles with velocity $v \in \mathbb{R}^3$ at time $t \geq 0$ solves

$$\partial_t f_t(v) = \int_{\mathbb{R}^3} dv_* \int_{\mathbb{S}^2} d\sigma\, B(|v - v_*|, \cos\theta)\big[f_t(v') f_t(v'_*) - f_t(v) f_t(v_*)\big], \tag{1.1}$$

where

$$\begin{aligned} v' &= \frac{v + v_*}{2} + \frac{|v - v_*|}{2}\sigma, \qquad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2}\sigma \quad \text{and} \\ \cos\theta &= \left\langle \frac{v - v_*}{|v - v_*|}, \sigma \right\rangle. \end{aligned} \tag{1.2}$$

The cross section $B(|v - v_*|, \cos\theta) \geq 0$ depends on the type of interaction between particles. We refer to the book of Cercignani [7] for a physical reference on the Boltzmann equation and to the review papers of Villani [38] and Alexandre [2] for many details on what is known from the mathematical point of view. Conservation of mass, momentum and kinetic energy hold for reasonable solutions, and we classically may assume without loss of generality that $\int_{\mathbb{R}^3} f_0(v)\, dv = 1$.

Received March 2013.
MSC2010 subject classifications. 82C40, 60J75, 60H30.
Key words and phrases. Kinetic equations, regularization, absolute continuity, entropy, Besov spaces.

1.2. *Assumptions*. We will assume that for some $\gamma \in (-1, 1)$, some $\nu \in (0, 1)$ with $\gamma + \nu > 0$, some measurable function $b : (0, \pi] \mapsto \mathbb{R}_+$,

$$(A_{\gamma,\nu}) \quad \begin{cases} B(|v - v_*|, \cos\theta) \sin\theta = |v - v_*|^\gamma b(\theta), \\ \exists 0 < c_0 < C_0, \qquad \forall \theta \in (0, \pi/2], \qquad c_0\theta^{-1-\nu} \le b(\theta) \le C_0\theta^{-1-\nu}, \\ \forall \theta \in (\pi/2, \pi], \qquad b(\theta) = 0. \end{cases}$$

As noted in the introduction of [3], this last assumption ($b = 0$ on $(\pi/2, \pi]$) is not a restriction since we can always reduce to this case by a symmetry argument. When particles collide by pairs due to a repulsive force proportional to $1/r^s$ for some $s > 2$, then ($A_{\gamma,\nu}$) holds with $\gamma = (s - 5)/(s - 1)$ and $\nu = 2/(s - 1)$. Thus our study includes the case of hard potentials ($s > 5$), Maxwell molecules ($s = 5$) and moderately soft potentials [$s \in (3, 5)$].

1.3. *Functional spaces*. Let us introduce all the functional spaces we will use in this paper:

- $\mathcal{M}(\mathbb{R}^d)$ is the set of nonnegative finite measures on $\mathbb{R}^d$.
- $\mathcal{P}(\mathbb{R}^d)$ is the set of probability measures on $\mathbb{R}^d$.
- $\mathcal{P}_p(\mathbb{R}^d)$ is the set of all $f \in \mathcal{P}(\mathbb{R}^d)$ such that $m_p(f) := \int_{\mathbb{R}^d} |v|^p f(dv) < \infty$.
- $\mathrm{Lip}_b(\mathbb{R}^d)$ is the set of bounded globally Lipschitz-continuous functions.
- $C_b(\mathbb{R}^d)$ is the set of bounded continuous functions.
- $C_0(\mathbb{R}^d)$ is the set of continuous functions vanishing at infinity.
- $C_c^1(\mathbb{R}^d)$ is the set of compactly supported C^1 functions.
- For $\alpha \in (0, 1)$, $C_b^\alpha(\mathbb{R}^d)$ is the set of all functions g such that

$$\|g\|_{C_b^\alpha(\mathbb{R}^d)} := \sup_{x\in\mathbb{R}^d} |g(x)| + \sup_{x,y\in\mathbb{R}^d, x\neq y} \frac{|g(x) - g(y)|}{|x - y|^\alpha} < \infty.$$

- $L^p(\mathbb{R}^d)$ is the usual Lebesgue space with $\|f\|_{L^p(\mathbb{R}^d)} := (\int_{\mathbb{R}^d} |f(x)|^p\,dx)^{1/p}$.
- For $s \in (0, 1)$, the Besov space $B^s_{1,\infty}(\mathbb{R}^d)$ consists of all functions f such that

$$\|f\|_{B^s_{1,\infty}(\mathbb{R}^d)} := \|f\|_{L^1(\mathbb{R}^d)} + \sup_{h\in\mathbb{R}^d, 0<|h|<1} |h|^{-s} \int_{\mathbb{R}^d} |f(x + h) - f(x)|\,dx < \infty.$$

In the whole paper, when a measure $f \in \mathcal{M}(\mathbb{R}^d)$ has a density, we also denote by f this density.

1.4. *Weak solutions*. We will consider weak solutions in the following sense.

DEFINITION 1.1. Assume ($A_{\gamma,\nu}$) for some $\nu \in (0, 1)$ and $\gamma \in (-1, 1)$.

(i) A family $(f_t)_{t\ge0} \subset \mathcal{P}_2(\mathbb{R}^3)$ is a weak solution to (1.1) if for all $t \ge 0$,

$$(1.3) \quad \int_{\mathbb{R}^3} v f_t(dv) = \int_{\mathbb{R}^3} v f_0(dv) \quad \text{and} \quad \int_{\mathbb{R}^3} |v|^2 f_t(dv) = \int_{\mathbb{R}^3} |v|^2 f_0(dv) < \infty$$

and if for any $\phi \in \mathrm{Lip}_b(\mathbb{R}^3)$ and any $t \geq 0$,

$$\int_{\mathbb{R}^3} \phi(v) f_t(dv) = \int_{\mathbb{R}^3} \phi(v) f_0(dv) + \int_0^t \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} L_B\phi(v, v_*) f_s(dv_*) f_s(dv)\, ds, \tag{1.4}$$

where, for $v' = v'(v, v_*, \sigma)$ and $\theta = \theta(v, v_*, \sigma)$ defined in (1.2),

$$L_B\phi(v, v_*) := \int_{\mathbb{S}^2} B(|v - v_*|, \cos\theta)\big[\phi(v') - \phi(v)\big]\, d\sigma. \tag{1.5}$$

The right-hand side of (1.4) is well-defined due to (1.3) and $(A_{\gamma,\nu})$. Indeed, there holds $|v' - v| = |v - v_*|\sqrt{(1-\cos\theta)/2} \leq |v - v_*||\theta|$, so that $|L_B\phi(v, v_*)| \leq C_\phi \int_{\mathbb{S}^2} B(|v - v_*|, \cos\theta)|v - v_*||\theta|\, d\sigma \leq C_\phi |v - v_*|^{1+\gamma} \int_0^{\pi/2} |\theta|^{-\nu}\, d\theta \leq C_\phi(1 + |v|^2 + |v_*|^2)$.

Concerning the well-posedness of (1.1) given $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$, the following results are available.

Hard potentials. Assume $(A_{\gamma,\nu})$ for some $\nu \in (0, 1)$ and $\gamma \in (0, 1)$. Then by Lu–Mouhot [29], there exists a weak solution to (1.1) starting from f_0. This solution furthermore satisfies that $\sup_{[t_0,\infty)} m_p(f_t) < \infty$ for all $t_0 > 0$, all $p \geq 2$. Such a moment production property was discovered by Elmroth [15] and Desvillettes [10]. Two different uniqueness results are available, assuming either that f_0 is regular ($f_0 \in W^{1,1}(\mathbb{R}^3)$ with $\int_{\mathbb{R}^3} (1 + |v|^2)|\nabla f_0(v)|\, dv < \infty$, Desvillettes and Mouhot [13]) or localized ($\int_{\mathbb{R}^3} e^{a|v|^\gamma} f_0(dv) < \infty$ for some $a > 0$, [22]).

Maxwell molecules. Assume $(A_{\gamma,\nu})$ for some $\nu \in (0, 1)$ and with $\gamma = 0$. Then there exists a unique weak solution to (1.1) starting from f_0 due to Toscani and Villani [36].

Moderately soft potentials. Assume $(A_{\gamma,\nu})$ for some $\nu \in (0, 1)$, some $\gamma \in (-1, 0)$ with $\gamma + \nu > 0$. Assume also that f_0 has a density with a finite entropy, that is, $\int_{\mathbb{R}^3} f_0(v)|\log f_0(v)|\, dv < \infty$. Then there exists a weak solution to (1.1) starting from f_0 due to Villani [37]. This solution is unique [22] if $f_0 \in \mathcal{P}_q(\mathbb{R}^3)$ for some $q > \gamma^2/(\gamma + \nu)$.

Very soft potentials. Assume $(A_{\gamma,\nu})$ for some $\nu \in (0, 2)$, some $\gamma \in (-3, 0)$. If f_0 has a density with a finite entropy, there exists a weak solution to (1.1) starting from f_0 due to Villani [37]. Uniqueness holds locally in time [20] provided $f_0 \in L^p(\mathbb{R}^3)$ for some $p > 3/(3 + \gamma)$.

1.5. *Main result*. Let us mention that during the proof, we will check the following property.

THEOREM 1.2. *Assume* $(A_{\gamma,\nu})$ *for some* $\gamma \in (-1,1)$, $\nu \in (0,1)$. *Let also* $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$ *not be a Dirac mass. For any weak solution* $(f_t)_{t\geq 0}$ *to* (1.1) *starting from* f_0, $\mathrm{Supp}\, f_t = \mathbb{R}^3$ *for all* $t > 0$.

The main result of the paper is the following.

THEOREM 1.3. *Assume* $(A_{\gamma,\nu})$ *for some* $\gamma \in (-1,1)$, $\nu \in (0,1)$ *with* $\gamma + \nu > 0$. *Let* $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$ *not be a Dirac mass.*

(i) *If* $\gamma \in (0,1)$, *then any weak solution* $(f_t)_{t\geq 0}$ *to* (1.1) *starting from* f_0 *and such that*

$$\forall t_0 > 0, \forall p \geq 2, \qquad \sup_{t\geq t_0} m_p(f_t) < \infty \tag{1.6}$$

satisfies that $f_t \in B^s_{1,\infty}(\mathbb{R}^3)$ *for all* $t > 0$, *all* $s \in (0, s_\nu)$, *where*

$$\begin{aligned} s_\nu &= \sup_{\alpha\in(0,\nu]} \left(\frac{2\alpha}{1+2\alpha} - \alpha \right) \\ &= \begin{cases} (\nu - 2\nu^2)/(1+2\nu) & \text{if } \nu \in \big(0, (\sqrt{2}-1)/2\big), \\ (\sqrt{2}-1)^2/2 & \text{if } \nu \in \big[(\sqrt{2}-1)/2, 1\big). \end{cases} \end{aligned} \tag{1.7}$$

(ii) *If* $\gamma \in (-1,0]$, *assume also that* $f_0 \in \mathcal{P}_{4+\gamma+4|\gamma|/\nu}(\mathbb{R}^3)$. *There exists a weak solution* $(f_t)_{t\geq 0}$ *to* (1.1) *starting from* f_0 *such that* $f_t \in B^s_{1,\infty}(\mathbb{R}^3)$ *for all* $t > 0$, *all* $s \in (0, s_{\gamma,\nu})$, *where*

$$s_{\gamma,\nu} = \sup_{\alpha\in(0,\nu]} \left(\frac{(2+\gamma/\nu)\alpha}{1+(2+\gamma/\nu)\alpha} - \alpha \right). \tag{1.8}$$

(iii) *In any case*, f_t *has a density satisfying* $\int_{\mathbb{R}^3} f_t(v)|\log f_t(v)|\,dv < \infty$ *as soon as* $t > 0$.

No regularization may hold if f_0 is a Dirac mass, since Dirac masses are stationary solutions to (1.1). In the case of moderately soft potentials ($\gamma \in (-1,0]$ and $\gamma + \nu > 0$), we need a few moments; observe that we always have $4 \leq 4 + \gamma + 4|\gamma|/\nu \leq 8$. Of course, (1.8) can of be made explicit, but the resulting formula is awful. While we show that any solution is regularized for hard potentials, we can only prove that there exists at least one solution enjoying some regularization properties for moderately soft potentials. This is due to our probabilistic interpretation: when $\gamma \in (0,1)$, we can associate a Boltzmann stochastic process to any weak solution, while when $\gamma \in (-1,0]$, we are only able to prove that there exists a Boltzmann stochastic process and that its law is a weak solution.

In [38], Theorem 9(iii), page 95, Villani announces a result very similar to Theorem 1.3. However, he obtains only some gain of integrability, while we obtain some (extremely weak) regularity. We know from a private communication that this work has never been written down.

REMARK 1.4. As can be checked from the proof, the same result as stated in Theorem 1.3(i) holds for regularized hard potentials where $B(|v - v_*|, \cos\theta) = (1 + |v - v_*|^2)^{\gamma/2} b(\theta)$, with $\gamma \in (0, 1)$ and $c_0|\theta|^{-\nu-1} \le b(\theta) \le C_0|\theta|^{-\nu-1}$ for some $\nu \in (0, 1)$.

1.6. *Motivation*. The main interest of Theorem 1.3 is the following: almost all the papers on the Boltzmann equation (concerning, e.g., regularization or large-time behavior) assume that the initial condition has a finite entropy; see the long review paper of Villani [38]. This condition is of course physically reasonnable. Our result shows that it is indeed physically reasonnable, since the entropy *automatically* becomes finite. Consequently, the results assuming the finiteness of the entropy of the initial condition extend to any measure initial data with a finite mass and energy which are not Dirac masses. For example, we deduce from Alexandre et al. [3], Chen and He [8], Desvillettes and Wennberg [14] and Huo et al. [27] that for any (non-Dirac) measure initial condition with finite mass and energy:

- under the assumptions of Theorem 1.3, $(1 + |v|^2)^{\gamma/2}\sqrt{f_t(v)} \in H^{\nu/2}(\mathbb{R}^3)$ for all $t > 0$ by [8];
- for regularized hard potentials, $f_t \in C^\infty(\mathbb{R}^3)$ for all $t > 0$ due to [14, 27].

1.7. *Known regularization results*. In many papers, Grad's cutoff is assumed: the cross section B, which physically satisfies $\int_0^\pi B(|v - v_*|, \cos\theta)\, d\theta = \infty$, is replaced by an integrable cross section. No regularization may arise under Grad's cutoff; see, for example, Mouhot and Villani [31]. The first results about regularization for the homogeneous Boltzmann equation without cutoff are due to Desvillettes [11, 12]. There are now roughly four types of available results.

- General results applying to all *true* physical potentials, relying on the entropy dissipation, providing weak regularity. Under $(A_{\gamma,\nu})$ for some $\nu \in (0, 2)$ and some $\gamma \in (-3, 1)$, when f_0 is a function with finite mass, entropy and energy, it has been shown (among many other things) by Alexandre et al. [3] that $\sqrt{f_t} \in H^{\nu/2}_{\text{loc}}(\mathbb{R}^3)$ for all $t > 0$. This has been recently precised, in the case of hard and moderately soft potentials by Chen and He [8], Theorem 1.3: $(1 + |v|^2)^{\gamma/2}\sqrt{f_t(v)} \in H^{\nu/2}(\mathbb{R}^3)$ for all $t > 0$.
- High regularization for *true* physical potentials assuming that f is already known to be slightly regular. It is proved by Chen and He [8], Theorem 1.5, that for hard and moderately soft potentials, if $f_0 \in H^3(\mathbb{R}^3)$ and $\int_{\mathbb{R}^3} (1 + |v|^q)|\nabla f_0(v)|\, dv < \infty$ for some $q \ge 2$ large enough, then the solution immediately lies in $H^N(\mathbb{R}^3)$ for some N depending on q.

- Full regularization for *regularized* hard potentials, when f_0 is a function with finite mass, entropy and energy. See Desvillettes and Wennberg [14], Alexandre and Elsafadi [4] and Huo et al. [27].
- Very restrictive results when f_0 is a (non-Dirac) probability measure in the $2D$ case: full regularization for Maxwell molecules (see Graham and Méléard [25] and [16]) and weak regularization [5] for a class of hard potentials (applying to interaction forces in $1/r^s$ with $s > 13.75$). All these works use some Malliavin calculus and seem very difficult to extend to the $3D$ case.

Here we deal with *true* physical potentials, for which there are several complications: $|w|^\gamma$ is not bounded below (and vanishes when $\gamma > 0$), which makes ellipticity estimates nontrivial, explodes either at 0 or at infinity and is in any case not smooth at 0. To our knowledge, the only regularization results that concern the homogeneous Boltzmann equation for true physical potentials are those of [3], [8] and [5]. The present result consequently improves on [5] (we treat the $3D$ case, all interaction forces in $1/r^s$ with $s > 3$ and we remove some technical assumptions) and is not in competition with [3] or [8] (the finiteness of the entropy is assumed in [3] and [8]).

1.8. *Known positivity results*. The proof of Theorem 1.2 is very easy, but it seems to be new. The first lower bound of solutions to the Boltzmann equation is due to Carleman [6] in the case of hard spheres ($\gamma = 1$, $b \equiv 1$). In [32], Pulvirenti and Wennberg obtained some Maxwellian lowerbound in the case of hard potentials with cutoff ($\gamma \in (0, 1]$ and $\int_0^\pi b(\theta)\, d\theta < \infty$), assuming that f_0 has a finite entropy. A quantitative version of Theorem 1.2 (for measure solutions) has been proved by Zhang and Zhang [39], still in the case of hard potentials with cutoff. Some positivity results [17] are available for $2D$ Maxwell molecules without cutoff. For general physical potentials without cutoff, some indications concerning the positivity of smooth solutions are given in Villani [38], Sections 6.2 and 6.3. Finally, Mouhot [30] proved some quantitative lower bound in the much more complicated spatially inhomogeneous case without cutoff, but for quite regular solutions [corresponding here, roughly, to the assumption $f \in L^\infty_{\mathrm{loc}}([0, \infty), W^{2,\infty}(\mathbb{R}^3))$].

1.9. *Comments on the method*. The classical way to prove some regularization results by probabilistic methods is to use some Malliavin calculus, based on the famous probabilistic interpretation of the homogeneous Boltzmann equation in terms of a nonlinear jumping stochastic differential equation initiated by Tanaka [35]. Unfortunately, this s.d.e. has regular coefficients only in the $2D$-case and for Maxwell molecules. In the case of $3D$ Maxwell molecules, a sort of Lipschitz property was observed by Tanaka [35] (see Lemma 3.2 below), but we cannot hope for more. This seems to make almost impossible the use of Malliavin calculus to study the $3D$ Boltzmann equation.

Here we use no Malliavin calculus, but a recent method introduced in [23] to prove that stochastic processes with rather irregular coefficients have a density. Recently, Debussche and Romito [9] have considerably improved this method by using Besov spaces, in order to study the regularity of the law of the solution to a $3D$ stochastic Navier–Stokes equation. For example, only $1D$ diffusion processes with diffusion coefficient in $C_b^{1/2+\varepsilon}(\mathbb{R})$ were treated in [23], while some quick computations seem to show that diffusion processes in any dimension and with diffusion coefficient in $C_b^{\varepsilon}(\mathbb{R}^d)$ can be studied using the tools of [9]. As we will see, it also perfectly applies to the s.d.e. associated with the homogeneous Boltzmann equation.

Let us mention that our proof is not *deeply* probabilistic: we use no stopping times, no Malliavin calculus, etc. We believe that a very similar deterministic proof can be written down. The advantage would be to remove Section 9 below, which is long and boring, in which we build the stochastic processes related to Boltzmann's equation. The disadvantage would be that the computations of Section 6 would become awful (and would look completely artificial).

1.10. *Heuristics.* Let us say a word about the reasons for regularization. Consider an initial velocity distribution f_0, possibly very singular. Pick at random a particle in the initial system, and call V_t its velocity at time t. Observe that the law of V_t is f_t for all $t \geq 0$. This particle collides, at time $t \geq 0$, at rate $\int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(|V_t - v_*|, \cos\theta)\, d\sigma\, f_t(dv_*)$. In the case without cutoff, this rate is thus infinite: the particle is subjected to infinitely many collisions on each finite time interval. Furthermore, at each collision, some randomness is added, since v_* and σ are chosen at random. Hence, we expect that for each $t > 0$, our particle has been subjected to infinitely many collisions on the time interval $[0, t]$, each of these collisions producing some randomness. Consequently, V_t will be much more random than V_0, so that its law should be much more regular.

Conversely, in the case with cutoff where the rate of collision of our particle is finite, we expect that $V_t = V_0$ during some (random) positive time, so that the solution f_t will contain all the singularities of f_0, at least for small times.

1.11. *Plan of the paper.* In the next section, we state the main lemma we will use, which is due to Debussche and Romito [9] and we give an elementary proof. In Section 3, we rewrite in an adequate way the weak formulation of (1.1) and prove a few properties of weak solutions. Section 4 is devoted to the proof of Theorem 1.2 and to some slightly more quantitative lower bound. Then we adapt the probabilistic interpretation of Tanaka [35] to hard and moderately soft potentials in Section 5. The proof of the existence of the Boltzmann process lies at the end of the paper (Section 9). Then the strategy of the proof is the following: we approximate the Boltzmann process by a Lévy process (Section 6) and study the regularity of the law of the approximating Lévy process (Section 7). Using that the approximating process has a regular law and that the true Boltzmann process is close to the approximating process, we conclude in Section 8.

1.12. *Notation.* We will write C for a (large) finite constant and c for a (small) positive constant, whose values may change from line to line and which depend only on ν, γ, c_0, C_0 [recall $(A_{\gamma,\nu})$] and on the weak solution $(f_t)_{t\ge 0}$. We write in index all the additional dependence of constants.

2. Main lemma. Our study is based on the following result due to Debussche and Romito [9], End of the proof of Theorem 5.1.

LEMMA 2.1. *Let $g \in \mathcal{M}(\mathbb{R}^d)$. Assume that there are $0 < \alpha < a < 1$ and a constant κ such that for all function $\phi \in C_b^\alpha(\mathbb{R}^d)$, all $h \in \mathbb{R}^d$ with $|h| \le 1$,*

$$\left|\int_{\mathbb{R}^d} [\phi(x+h) - \phi(x)] g(dx)\right| \le \kappa \|\phi\|_{C_b^\alpha(\mathbb{R}^d)} |h|^a. \tag{2.1}$$

Then g has a density in $B_{1,\infty}^{a-\alpha}(\mathbb{R}^d)$ and $\|g\|_{B_{1,\infty}^{a-\alpha}(\mathbb{R}^d)} \le g(\mathbb{R}^d) + C_{d,a,\alpha}\kappa$.

Actually, the result in [9] is more general. The proof in [9] relies on several theorems of functional analysis. We present here an *elementary* (though longer) proof.

PROOF OF LEMMA 2.1. We divide the proof into four steps.

Step 1: Preliminaries. For $r > 0$, consider the function $\chi_r(x) = (v_d r^d)^{-1} \times \mathbb{1}_{\{|x|<r\}}$, where v_d is the volume of the unit ball in $\mathbb{R}^d$. An easy computation shows that for all $x, y \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} |\chi_r(x-z) - \chi_r(y-z)|\, dz \le C_d \min(1, |x-y|/r). \tag{2.2}$$

For $\psi \in L^\infty(\mathbb{R}^d)$ and $r \in (0,1]$, $\psi \star \chi_r$ belongs to $C_b^\alpha(\mathbb{R}^d)$ (it is actually Lipschitz-continuous) and

$$\|\psi \star \chi_r\|_{C_b^\alpha(\mathbb{R}^d)} \le C_d \|\psi\|_{L^\infty(\mathbb{R}^d)} r^{-\alpha}. \tag{2.3}$$

Indeed, it obviously holds that $\|\psi \star \chi_r\|_{L^\infty(\mathbb{R}^d)} \le \|\psi\|_{L^\infty(\mathbb{R}^d)}$ and for $x \ne y$, we deduce from (2.2) that $|\psi \star \chi_r(x) - \psi \star \chi_r(y)| \le C_d \|\psi\|_{L^\infty(\mathbb{R}^d)} \min(1, |x-y|/r) \le C_d \|\psi\|_{L^\infty(\mathbb{R}^d)} r^{-\alpha} |x-y|^\alpha$.

Step 2. Next we prove that for any $r \in (0,1]$, any $|h| \le 1$,

$$\int_{\mathbb{R}^d} |g \star \chi_r(x+h) - g \star \chi_r(x)|\, dx \le C_d \kappa |h|^a r^{-\alpha}.$$

It suffices to prove that for any $\psi \in L^\infty(\mathbb{R}^d)$, $I_r(h,\psi) := |\int_{\mathbb{R}^d} \psi(x)[g \star \chi_r(x+h) - g \star \chi_r(x)]\, dx| \le C_d \kappa \|\psi\|_{L^\infty(\mathbb{R}^d)} |h|^a r^{-\alpha}$. But using (2.1) and (2.3), we get

$$\begin{aligned} I_r(h,\psi) &= \left|\int_{\mathbb{R}^d} [\psi \star \chi_r(y-h) - \psi \star \chi_r(y)] g(dy)\right| \le \kappa \|\psi \star \chi_r\|_{C_b^\alpha(\mathbb{R}^d)} |h|^a \\ &\le C_d \kappa \|\psi\|_{L^\infty(\mathbb{R}^d)} |h|^a r^{-\alpha}. \end{aligned}$$

Step 3. Here we assume additionally that g has a density in $C^1(\mathbb{R}^d)$ satisfying $\int_{\mathbb{R}^d} |\nabla g(x)|\,dx < \infty$ (which implies that all the computations below are licit), and we check that

$$\sup_{|h|\le 1} |h|^{\alpha-a} \int_{\mathbb{R}^d} \big|g(x+h) - g(x)\big|\,dx \le C_{d,a,\alpha}\kappa.$$

To this end, we first write, using Step 2, for all $|h| \le 1$, all $r \in (0, 1]$,

$$\begin{aligned}
&\int_{\mathbb{R}^d} \big|g(x+h) - g(x)\big|\,dx \\
&\quad\le \int_{\mathbb{R}^d} \big|g \star \chi_r(x+h) - g \star \chi_r(x)\big|\,dx + 2\int_{\mathbb{R}^d} \big|g \star \chi_r(x) - g(x)\big|\,dx \\
&\quad\le C_d\kappa|h|^a r^{-\alpha} + \frac{2}{v_d r^d} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \big|g(y) - g(x)\big| \mathbb{1}_{\{|x-y|<r\}}\,dx\,dy \\
&\quad= C_d\kappa|h|^a r^{-\alpha} + \frac{2}{v_d r^d} \int_{|u|<r} du \int_{\mathbb{R}^d} dx \big|g(x+u) - g(x)\big|.
\end{aligned}$$

Thus, setting $I_t := \sup_{|h|=t} \int_{\mathbb{R}^d} |g(x+h) - g(x)|\,dx$ and $S_t = \sup_{s\in(0,t]} s^{\alpha-a} I_s$, we deduce that for all $t \in (0, 1]$, all $r \in (0, 1]$ (below, the variable u belongs to $\mathbb{R}^d$),

$$\begin{aligned}
t^{\alpha-a} I_t &\le C_d\kappa(t/r)^\alpha + \frac{2t^{\alpha-a}}{v_d r^d} \int_{|u|<r} |u|^{a-\alpha} S_{|u|}\,du \\
&\le C_d\kappa(t/r)^\alpha + \frac{2t^{\alpha-a}}{v_d r^d} S_1 r^{a-\alpha} v_d r^d \\
&\le C_d\kappa(t/r)^\alpha + 2(r/t)^{a-\alpha} S_1.
\end{aligned}$$

Choosing $r = 4^{-1/(a-\alpha)}t$, we deduce that for all $t \in (0, 1]$, $t^{\alpha-a} I_t \le 4^{\alpha/(a-\alpha)} \times C_d\kappa + S_1/2$. This implies $S_1 \le 4^{\alpha/(a-\alpha)} C_d\kappa + S_1/2$ and finally $S_1 \le 2.4^{\alpha/(a-\alpha)} \times C_d\kappa$ as desired.

Step 4. Consider now g as in the statement. For $n \ge 1$, put $g_n = g \star G_n$, where $G_n(x) = (n/\pi)^{d/2} e^{-n|x|^2}$. Then $g_n \in C^1(\mathbb{R}^d)$, $\int_{\mathbb{R}^d} g_n(x)\,dx = g(\mathbb{R}^d)$ and $\int_{\mathbb{R}^d} |\nabla g_n(x)|\,dx < \infty$. Furthermore, one easily checks that g_n satisfies (2.1) with the same constant κ as g. Thus we can apply Step 3 and deduce that $\sup_{|h|\le 1} |h|^{\alpha-a} \int_{\mathbb{R}^d} |g_n(x+h) - g_n(x)|\,dx \le C_{d,a,\alpha}\kappa$ for all $n \ge 1$, whence $\|g_n\|_{B^{a-\alpha}_{1,\infty}} \le g(\mathbb{R}^d) + C_{d,a,\alpha}\kappa$ (recall Section 1.3). Consequently, the sequence g_n is strongly compact in $L^1(\mathbb{R}^d)$ (because the balls of $B^s_{1,\infty}(\mathbb{R}^d)$ are compact in $L^1(\mathbb{R}^d)$ for all $s > 0$; see, e.g., [33]). But g_n tends weakly (in the sense of measures) to g. We deduce that $g \in L^1(\mathbb{R}^d)$ and that we can find a subsequence such that $\lim_k \|g_{n_k} - g\|_{L^1(\mathbb{R}^d)} = 0$. One easily concludes that for all $|h| \le 1$, $\int_{\mathbb{R}^d} |g(x+h) - g(x)|\,dx = \lim_k \int_{\mathbb{R}^d} |g_{n_k}(x+h) - g_{n_k}(x)|\,dx \le C_{d,a,\alpha}\kappa|h|^{a-\alpha}$. We deduce that $\|g\|_{B^{a-\alpha}_{1,\infty}(\mathbb{R}^d)} \le g(\mathbb{R}^d) + C_{d,a,\alpha}\kappa$. □

3. Weak solutions. First, we parameterize (1.2) as in [21]. For each $X \in \mathbb{R}^3 \setminus \{0\}$, we introduce $I(X), J(X) \in \mathbb{R}^3$ such that $(\frac{X}{|X|}, \frac{I(X)}{|X|}, \frac{J(X)}{|X|})$ is an orthonormal basis of $\mathbb{R}^3$, in such a way that $X \mapsto (I(X), J(X))$ is measurable. We also put $I(0) = J(0) = 0$. For $X, v, v_* \in \mathbb{R}^3$, $\theta \in [0, \pi)$ and $\varphi \in [0, 2\pi)$, we set

$$
\begin{cases} \Gamma(X,\varphi) := (\cos\varphi) I(X) + (\sin\varphi) J(X), \\ v'(v, v_*, \theta, \varphi) := v - \dfrac{1-\cos\theta}{2}(v - v_*) + \dfrac{\sin\theta}{2}\Gamma(v - v_*, \varphi), \\ a(v, v_*, \theta, \varphi) := v'(v, v_*, \theta, \varphi) - v. \end{cases} \tag{3.1}
$$

The choice of $(I(X), J(X))$ does not matter. The important thing is that for any reasonable $F : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \times [0, \pi) \mapsto \mathbb{R}$, any $v, v_* \in \mathbb{R}^3$,

$$
\int_0^\pi \int_0^{2\pi} F\big(v, v_*, v'(v, v_*, \theta, \varphi), \theta\big) \sin\theta \, d\varphi \, d\theta = \int_{\mathbb{S}^2} F\big(v, v_*, v', \theta\big) \, d\sigma,
$$

where on the right-hand side, $v' = v'(v, v_*, \sigma)$ and $\theta = \theta(v, v_*, \sigma) \in (0, \pi)$ are defined by (1.2). This in particular implies that for all $\phi \in \mathrm{Lip}_b(\mathbb{R}^3)$, recalling (1.5) and then $(A_{\gamma,\nu})$,

$$
\begin{aligned} L_B\phi(v, v_*) & \\ &= \int_0^\pi \int_0^{2\pi} \big[\phi\big(v + a(v, v_*, \theta, \varphi)\big) - \phi(v)\big] B\big(|v - v_*|, \cos\theta\big) \sin\theta \, d\varphi \, d\theta \qquad (3.2) \\ &= |v - v_*|^\gamma \int_0^{\pi/2} \int_0^{2\pi} \big[\phi\big(v + a(v, v_*, \theta, \varphi)\big) - \phi(v)\big] b(\theta) \, d\varphi \, d\theta. \qquad (3.3) \end{aligned}
$$

We will frequently use that, by a straightforward computation,

$$
\big|a(v, v_*, \theta, \varphi)\big| = \sqrt{\frac{1-\cos\theta}{2}} |v - v_*| \le \frac{1}{2}\theta |v - v_*|. \tag{3.4}
$$

We will also need the following remark, corresponding to the $2D$ equality $\langle \xi, X^\perp \rangle = \pm \langle \xi^\perp, X \rangle$.

REMARK 3.1. For any measurable nonnegative function $F : \mathbb{R} \mapsto \mathbb{R}$, any $X \in \mathbb{R}^3$, any $\xi \in \mathbb{R}^3$,

$$
\int_0^{2\pi} F\big(\langle \xi, \Gamma(X, \varphi)\rangle\big) \, d\varphi = \int_0^{2\pi} F\big(\langle X, \Gamma(\xi, \varphi)\rangle\big) \, d\varphi.
$$

PROOF. Recall that these integrals do not depend on the choice of $(I(X), J(X))$ and $(I(\xi), J(\xi))$ [as soon as $(\frac{X}{|X|}, \frac{I(X)}{|X|}, \frac{J(X)}{|X|})$ and $(\frac{\xi}{|\xi|}, \frac{I(\xi)}{|\xi|}, \frac{J(\xi)}{|\xi|})$ are orthonormal bases of $\mathbb{R}^3$]. If X and ξ are colinear $\langle \xi, \Gamma(X, \varphi)\rangle = \langle X, \Gamma(\xi, \varphi)\rangle = 0$ for all φ and the result follows. Otherwise, choose $(I(X), J(X))$ and $(I(\xi), J(\xi))$ such that $X, \xi, I(X), I(\xi)$ are in the same plane and such that $\langle X, I(\xi)\rangle = \langle \xi, I(X)\rangle$, which implies that $\langle \xi, \Gamma(X, \varphi)\rangle = \langle X, \Gamma(\xi, \varphi)\rangle$ for all φ. □

Unfortunately, it is not possible to build I in such a way that $X \mapsto I(X)$ is smooth. Tanaka [35] found a way to overcome this difficulty, which was slightly precised in [21], Lemma 2.6.

LEMMA 3.2. *There exists a measurable function* $\varphi_0 : \mathbb{R}^3 \times \mathbb{R}^3 \mapsto [0, 2\pi)$, *such that for all* $v, v_*, w, w_* \in \mathbb{R}^3$, *all* $\theta \in [0, \pi)$ *and all* $\varphi \in [0, 2\pi)$,

$$\left|a(v, v_*, \theta, \varphi) - a\big(w, w_*, \theta, \varphi + \varphi_0(v - v_*, w - w_*)\big)\right| \le 2\theta\big(|v - w| + |v_* - w_*|\big).$$

We conclude this section with a useful time-regularity property of weak solutions. This must be more or less classical; see, for example, Gamba, Panferov and Villani [24] for a stronger result in the case of cutoff hard potentials, but we found no precise reference in the present setting.

LEMMA 3.3. *Let* $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$. *Assume* $(A_{\gamma,\nu})$ *for some* $\gamma \in (-1, 1)$, $\nu \in (0, 1)$. *Consider any weak solution* $(f_t)_{t\ge 0}$ *to* (1.1) *starting from* f_0. *Then for any* $\phi \in \mathrm{Lip}_b(\mathbb{R}^3)$, $L_B\phi$ *is continuous on* $\mathbb{R}^3 \times \mathbb{R}^3$ *and the map* $t \mapsto \int_{\mathbb{R}^3} \phi(v) f_t(dv)$ *belongs to* $C^1([0, \infty))$.

PROOF. Recall (1.4): to show that $t \mapsto \int_{\mathbb{R}^3} \phi(v) f_t(dv)$ is of class $C^1([0, \infty))$, it suffices to check that $t \mapsto \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} L_B\phi(v, v_*) f_t(dv_*) f_t(dv)$ is continuous on $[0, \infty)$.

Step 1. For $\phi \in \mathrm{Lip}_b(\mathbb{R}^3)$, $|L_B\phi(v, v_*)| \le C_\phi |v - v_*|^{\gamma+1} \le C_\phi(1 + |v|^2 + |v_*|^2)$ by (3.3), (3.4) and since $\int_0^{\pi/2} \theta b(\theta)\, d\theta < \infty$ by $(A_{\gamma,\nu})$. By (1.3), we deduce that $\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} L_B\phi(v, v_*) f_t(dv_*) f_t(dv)$ is bounded, so that $t \mapsto \int_{\mathbb{R}^3} \phi(v) f_t(dv)$ is continuous on $[0, \infty)$ by (1.4). The Portemanteau theorem thus implies that $t \mapsto f_t$ is weakly continuous, which classically implies that $t \mapsto f_t \otimes f_t$ is weakly continuous: for all $\phi \in C_b(\mathbb{R}^3 \times \mathbb{R}^3)$, $t \mapsto \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \phi(v, v_*) f_t(dv) f_t(dv_*)$ is continuous on $[0, \infty)$.

Step 2. Recall that $B(|v - v_*|, \cos\theta) \sin\theta = |v - v_*|^\gamma b(\theta)$ by $(A_{\gamma,\nu})$ and define, for $k \ge 1$, $B_k(|v - v_*|, \cos\theta) \sin\theta = (|v - v_*|^\gamma \wedge k) b(\theta) \mathbb{1}_{\{\theta > 1/k\}}$. It is immediately checked that $L_{B_k}\phi \in C_b(\mathbb{R}^3 \times \mathbb{R}^3)$ for any $\phi \in \mathrm{Lip}_b(\mathbb{R}^3)$. By Step 1, we deduce that $t \mapsto \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} L_{B_k}\phi(v, v_*) f_t(dv_*) f_t(dv)$ is continuous on $[0, \infty)$.

Step 3. We claim that $|(L_B - L_{B_k})\phi(v, v_*)| \le C_\phi(1 + |v|^2 + |v_*|^2) k^{-\kappa}$ for some $\kappa = \kappa(\gamma, \nu) > 0$, for all $\phi \in \mathrm{Lip}_b(\mathbb{R}^3)$. Using (3.3), (3.4) and then $(A_{\gamma,\nu})$, we get

$$\begin{aligned}
&\big|(L_B - L_{B_k})\phi(v, v_*)\big| \\
&\quad\le C_\phi |v - v_*|^\gamma \int_0^{\pi/2} \int_0^{2\pi} \theta |v - v_*| (\mathbb{1}_{\{|v - v_*|^\gamma > k\}} + \mathbb{1}_{\{\theta \le 1/k\}})\, d\varphi\, b(\theta)\, d\theta \\
&\quad\le C_\phi |v - v_*|^{\gamma+1} \mathbb{1}_{\{|v - v_*|^\gamma > k\}} + C_\phi |v - v_*|^{\gamma+1} k^{\nu-1} \\
&\quad\le C_\phi |v - v_*|^{\gamma+1} \mathbb{1}_{\{|v - v_*|^\gamma > k\}} + C_\phi\big(1 + |v|^2 + |v_*|^2\big) k^{\nu-1}.
\end{aligned}$$

If $\gamma \in (0,1)$, we write $|v-v_*|^{\gamma+1}\mathbb{1}_{\{|v-v_*|^\gamma>k\}} \le k^{1-1/\gamma}|v-v_*|^2$ and conclude with $\kappa=(1/\gamma-1)\wedge(1-\nu)$. If $\gamma=0$, $|v-v_*|^\gamma>k$ never happens (since $k\ge 1$), whence the claim with $\kappa=1-\nu$. If $\gamma\in(-1,0)$, $|v-v_*|^\gamma>k$ implies $|v-v_*|<k^{-1/|\gamma|}$ and we conclude with $\kappa=((\gamma+1)/|\gamma|)\wedge(1-\nu)$.

Step 4. Let $\phi\in \mathrm{Lip}_b(\mathbb{R}^3)$. By Step 2, $L_{B_k}\phi\in C_b(\mathbb{R}^3\times\mathbb{R}^3)$ and Step 3 implies that $L_{B_k}\phi$ tends to $L_B\phi$ uniformly on compacts, whence $L_B\phi$ is continuous. Next, Step 3 and (1.3) show that $\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} L_{B_k}\phi(v,v_*)f_t(dv_*)f_t(dv)$ goes to $\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} L_B\phi(v,v_*)f_t(dv_*)f_t(dv)$ uniformly for $t\in|0,\infty)$. Using Step 2, we conclude that $t\mapsto\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} L_B\phi(v,v_*)f_t(dv_*)f_t(dv)$ is continuous on $[0,\infty)$. □

4. Lowerbound.

The aim of this section is to prove Theorem 1.2 and to deduce some lowerbounds of weak solutions. For $x\in\mathbb{R}^3$ and $r>0$, we denote by $\mathcal{B}(x,r):=\{y\in\mathbb{R}^3:|y-x|<r\}$ and by $\mathcal{S}(x,r):=\{y\in\mathbb{R}^3:|y-x|=r\}$. We start with the following preliminary result.

LEMMA 4.1. *Consider $g\in\mathcal{P}(\mathbb{R}^3)$ enjoying the following property*: $v_1,v_2\in \operatorname{Supp} g$ *implies that* $\mathcal{S}((v_1+v_2)/2,|v_1-v_2|/2)\subset\operatorname{Supp} g$. *If g is not a Dirac mass, then* $\operatorname{Supp} g=\mathbb{R}^3$.

PROOF. We first claim that for any $x\in\mathbb{R}^3$, any $r>0$, $\mathcal{S}(x,r)\subset\operatorname{Supp} g$ implies $\bar{\mathcal{B}}(x,\sqrt{2}r)\subset\operatorname{Supp} g$. Due to our assumption, it suffices to show that for any $v\in\bar{\mathcal{B}}(x,\sqrt{2}r)$, there exists $v_1,v_2\in\mathcal{S}(x,r)$ such that $v\in\mathcal{S}((v_1+v_2)/2,|v_1-v_2|/2)$. This is not hard: write $v=x+\alpha r\sigma$, for some $\sigma\in\mathbb{S}^2$ and some $\alpha\in[0,\sqrt{2}]$, consider any $\tau\in\mathbb{S}^2$ orthogonal to σ and choose $v_1=x+r[(\alpha+\sqrt{2-\alpha^2})\sigma+(\alpha-\sqrt{2-\alpha^2})\tau]/2$ and $v_2=x+r[(\alpha+\sqrt{2-\alpha^2})\sigma-(\alpha-\sqrt{2-\alpha^2})\tau]/2$.

Since g is not a Dirac mass, we can find $v_1\neq v_2$ in $\operatorname{Supp} g$. Put $x_0=(v_1+v_2)/2$ and $r_0=|v_1-v_2|/2>0$. By assumption, $\mathcal{S}(x_0,r_0)\subset\operatorname{Supp} g$, whence $\bar{\mathcal{B}}(x_0,\sqrt{2}r_0)\subset\operatorname{Supp} g$. Thus in particular, $\mathcal{S}(x_0,\sqrt{2}r_0)\subset\operatorname{Supp} g$, whence $\bar{\mathcal{B}}(x_0,2r_0)\subset\operatorname{Supp} g$, and so on. We find that $\bar{\mathcal{B}}(x_0,2^{n/2}r_0)\subset\operatorname{Supp} g$ for any $n\ge1$, which ends the proof. □

We can now give the proof of Theorem 1.2. Let us mention that Step 2 below is inspired by Villani [38], Chapter 3, Section 6.2.

PROOF OF THEOREM 1.2. We thus assume ($A_{\gamma,\nu}$) for some $\gamma\in(-1,1)$, $\nu\in(0,1)$ and consider a weak solution $(f_t)_{t\ge0}$ to (1.1) starting from some non-Dirac initial condition $f_0\in\mathcal{P}_2(\mathbb{R}^3)$.

Step 1. For all $t>0$, f_t is not a Dirac mass. This is immediate from the conservations of momentum and energy (1.3) and the fact that f_0 is not a Dirac mass: for all $t\ge0$, all $v_0\in\mathbb{R}^3$,

$$\int_{\mathbb{R}^3}|v-v_0|^2 f_t(dv)=\int_{\mathbb{R}^3}|v-v_0|^2 f_0(dv)>0.$$

Step 2. Here we prove that for any $t > 0$, any $v_0 \in \mathbb{R}^3$, any $\varepsilon > 0$, [recall that $v' = v'(v, v_*, \sigma)$ and $\theta = \theta(v, v_*, \sigma)$ were defined in (1.2)]

$$\begin{aligned} &f_t\big(\mathcal{B}(v_0, \varepsilon)\big) = 0 \\ &\quad\Longrightarrow \quad \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} \mathbb{1}_{\{v'(v,v_*,\sigma)\in\mathcal{B}(v_0,\varepsilon)\}} \\ &\qquad\qquad\qquad \times \mathbb{1}_{\{v\neq v_*, \theta(v,v_*,\sigma)\in(0,\pi/2)\}}\, d\sigma\, f_t(dv_*)\, f_t(dv) = 0. \end{aligned}$$

Assume thus that $f_t(\mathcal{B}(v_0, \varepsilon)) = 0$ and consider $\phi_{\varepsilon,v_0} \in \mathrm{Lip}_b(\mathbb{R}^3)$, strictly positive on $\mathcal{B}(v_0, \varepsilon)$ and vanishing outside $\mathcal{B}(v_0, \varepsilon)$. By Lemma 3.3, $s \mapsto \int_{\mathbb{R}^3} \phi_{\varepsilon,v_0}(v) \times f_s(dv)$ belongs to $C^1([0,\infty))$. Since it is nonnegative and vanishes at $t > 0$, its derivative also vanishes at t. Consequently, by (1.4),

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B\big(|v - v_*|, \cos\theta\big)\big[\phi_{\varepsilon,v_0}\big(v'\big) - \phi_{\varepsilon,v_0}(v)\big]\, d\sigma\, f_t(dv_*)\, f_t(dv) = 0.$$

But $f_t(\mathcal{B}(v_0, \varepsilon)) = 0$ and $\mathrm{Supp}\, \phi_{\varepsilon,v_0} \subset \mathcal{B}(v_0, \varepsilon)$, so that

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B\big(|v - v_*|, \cos\theta\big)\phi_{\varepsilon,v_0}\big(v'\big)\, d\sigma\, f_t(dv_*)\, f_t(dv) = 0.$$

This implies the result, since $\phi_{\varepsilon,v_0}(v')B(|v - v_*|, \cos\theta) > 0$ as soon as $v' \in \mathcal{B}(v_0, \varepsilon)$, $v \neq v_*$ and $\theta \in (0, \pi/2)$ due to $(A_{\gamma,\nu})$.

Step 3. We now show that for any $t > 0$, $v_1, v_2 \in \mathrm{Supp}\, f_t$ implies $\mathcal{S}((v_1 + v_2)/2, |v_1 - v_2|/2) \subset \mathrm{Supp}\, f_t$. We can assume that $v_1 \neq v_2$, because else, $\mathcal{S}((v_1 + v_2)/2, |v_1 - v_2|/2) = \{v_1\}$ and the result is obvious. Observe that $\mathcal{S}((v_1 + v_2)/2, |v_1 - v_2|/2)$ is the closure of $\Delta_{v_1,v_2} \cup \Delta_{v_2,v_1}$, where

$$\Delta_{v_1,v_2} := \big\{v'(v_1, v_2, \sigma) : \sigma \in \mathbb{S}^2, \theta(v_1, v_2, \sigma) \in (0, \pi/2)\big\}.$$

Since $\mathrm{Supp}\, f_t$ is closed, it suffices to prove that $\Delta_{v_1,v_2} \cup \Delta_{v_2,v_1} \subset \mathrm{Supp}\, f_t$. Let thus, for example, $v_0 \in \Delta_{v_1,v_2}$. Then $v_0 = v'(v_1, v_2, \sigma_0)$ for some $\sigma_0 \in \mathbb{S}^2$ with $\theta_0 = \theta(v_1, v_2, \sigma_0) \in (0, \pi/2)$. Thus for all $v \simeq v_1$, all $v_* \simeq v_2$, all $\sigma \simeq \sigma_0$, we have $v'(v, v_*, \sigma) \simeq v_0$, $v \neq v_*$ and $\theta(v, v_*, \sigma) \in (0, \pi/2)$. Since $v_1 \in \mathrm{Supp}\, f_t(dv)$ and $v_2 \in \mathrm{Supp}\, f_t(dv_*)$, we conclude that for any $\varepsilon > 0$,

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} \mathbb{1}_{\{v'(v,v_*,\sigma)\in\mathcal{B}(v_0,\varepsilon)\}}\mathbb{1}_{\{v\neq v_*,\theta(v,v_*,\sigma)\in(0,\pi/2)\}}\, d\sigma\, f_t(dv_*)\, f_t(dv) > 0.$$

This implies that $f_t(\mathcal{B}(v_0, \varepsilon)) > 0$ for all $\varepsilon > 0$ by Step 2.

Step 4. We conclude from Lemma 4.1 and Steps 1 and 3 that for all $t > 0$, $\mathrm{Supp}\, f_t = \mathbb{R}^3$. $\square$

We finally check the following estimate.

PROPOSITION 4.2. *Assume* $(A_{\gamma,\nu})$ *for some* $\gamma \in (-1, 1)$, $\nu \in (0, 1)$. *Let also* $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$ *not be a Dirac mass. Consider any weak solution* $(f_t)_{t\geq 0}$ *to* (1.1) *starting from* f_0. *For all* $0 < t_0 < t_1$,

$$q_{t_0,t_1} := \inf_{t\in[t_0,t_1], w\in\mathbb{R}^3, \zeta\in\mathbb{R}^3} f_t\big(K(w,\zeta)\big) > 0,$$

where $K(w,\zeta) := \{v \in \mathbb{R}^3 : |v| \le 3, |v-w| \ge 1, |\langle v-w,\zeta\rangle| \ge |\zeta|\}$.

PROOF. We divide the proof into three steps.

Step 1. We first prove that for any $0 < t_0 < t_1$, $\inf_{t\in[t_0,t_1],x\in\mathcal{S}(0,2)} f_t(\mathcal{B}(x,1)) > 0$. To this end, consider $\phi \in \mathrm{Lip}_b(\mathbb{R}^3)$ such that $\mathbb{1}_{\mathcal{B}(0,1/2)} \le \phi \le \mathbb{1}_{\mathcal{B}(0,1)}$. Define $F(t,x) = \int_{\mathbb{R}^3} \phi(v-x) f_t(dv)$. We know from Lemma 3.3 that $t \mapsto F(t,x)$ is continuous for each $x \in \mathbb{R}^3$. Furthermore, denoting by C the Lipschitz constant of ϕ, we have $\sup_{t\ge 0} |F(t,x) - F(t,y)| \le C|x-y|$. All this implies that F is continuous on $[0,\infty) \times \mathbb{R}^3$. Since $F(t,x) \ge f_t(\mathcal{B}(x,1/2))$, we deduce from Theorem 1.2 that $F(t,x) > 0$ for all $t > 0$, all $x \in \mathbb{R}^3$. The continuity of F and the compactness of $[t_1,t_2] \times \mathcal{S}(0,2)$ imply that $\inf_{[t_1,t_2]\times\mathcal{S}(0,2)} F > 0$. This ends the step, because $f_t(\mathcal{B}(x,1)) \ge F(t,x)$.

Step 2. Here we check that for any $w \in \mathbb{R}^3$, any $\zeta \in \mathbb{R}^3$ we can find $x_{w,\zeta} \in \mathcal{S}(0,2)$ such that $\mathcal{B}(x_{w,\zeta},1) \subset K(w,\zeta)$. We may assume that $\zeta \ne 0$ [because $K(w,\zeta) \subset K(w,0)$ for any $\zeta \ne 0$]. Put $\mathrm{sg}(y) = 1$ for $y \ge 0$ and $\mathrm{sg}(y) = -1$ for $y < 0$. Choose $x_{w,\zeta} = -2\,\mathrm{sg}(\langle w,\zeta\rangle)\zeta/|\zeta| \in \mathcal{S}(0,2)$. It remains to prove that $\mathcal{B}(x_{w,\zeta},1) \subset K(w,\zeta)$. Let thus $v \in \mathcal{B}(x_{w,\zeta},1)$.

(a) First, $|v| \le |x_{w,\zeta}| + 1 = 3$.

(b) Next, observe that $|w - x_{w,\zeta}| = |w + 2\,\mathrm{sg}(\langle w,\zeta\rangle)\zeta/|\zeta|| \ge \sqrt{|w|^2+4} \ge 2$, so that

$$|w-v| \ge |w - x_{w,\zeta}| - |x_{w,\zeta} - v| \ge 2 - 1 = 1.$$

(c) Finally, using that $|\langle w - x_{w,\zeta},\zeta\rangle| = |\langle w,\zeta\rangle + 2\,\mathrm{sg}(\langle w,\zeta\rangle)|\zeta|| \ge 2|\zeta|$, we see that

$$\big|\langle w-v,\zeta\rangle\big| \ge \big|\langle w - x_{w,\zeta},\zeta\rangle\big| - \big|\langle x_{w,\zeta} - v,\zeta\rangle\big| \ge 2|\zeta| - |\zeta| = |\zeta|.$$

All this shows that $v \in K(w,\zeta)$ as desired.

Step 3. By Step 2, we have

$$\inf_{t\in[t_0,t_1], w\in\mathbb{R}^3, \zeta\in\mathbb{R}^3} f_t\big(K(w,\zeta)\big) \ge \inf_{t\in[t_0,t_1], x\in\mathcal{S}(0,2)} f_t\big(\mathcal{B}(x,1)\big).$$

This last quantity is positive if $0 < t_0 < t_1$ by Step 1. □

5. Probabilistic interpretation. We write down the probabilistic interpretation of (1.1) initiated by Tanaka [35] in the case of Maxwell molecules.

PROPOSITION 5.1. *Let* $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$. *Assume* $(A_{\gamma,\nu})$ *for some* $\gamma \in (-1,1)$, $\nu \in (0,1)$.

(i) *Assume first that* $\gamma \in (0,1)$. *Then for any weak solution* $(f_t)_{t\ge 0}$ *to* (1.1) *starting from* f_0 *and satisfying* (1.6), *there exist*, *on some probability space*

$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \Pr)$, *a* $\mathcal{F}_0$*-measurable random variable* V_0 *with law* f_0, *a* $(\mathcal{F}_t)_{t\geq 0}$*-Poisson measure* $N(ds, dv, d\theta, d\varphi, du)$ *on* $[0,\infty) \times \mathbb{R}^3 \times (0, \pi/2] \times [0, 2\pi) \times [0,\infty)$ *with intensity* $ds f_s(dv) b(\theta)\, d\theta\, d\varphi\, du$ *and a càdlàg* $(\mathcal{F}_t)_{t\geq 0}$*-adapted* $\mathbb{R}^3$*-valued process* $(V_t)_{t\geq 0}$ *satisfying* $\mathcal{L}(V_t) = f_t$ *for all* $t \geq 0$ *and solving*

$$V_t = V_0 + \int_0^t \int_{\mathbb{R}^3} \int_0^{\pi/2} \int_0^{2\pi} \int_0^\infty a(V_{s-}, v, \theta, \varphi) \times \mathbb{1}_{\{u \leq |V_{s-} - v|^\gamma\}} N(ds, dv, d\theta, d\varphi, du). \tag{5.1}$$

(ii) *Assume next that* $\gamma \in (-1, 0]$ *and that* $f_0 \in \mathcal{P}_p(\mathbb{R}^3)$ *for some* $p > 2$. *There exists a weak solution* $(f_t)_{t\geq 0}$ *to* (1.1) *starting from* f_0 *satisfying*

$$\forall T > 0, \qquad \sup_{[0,T]} m_p(f_t) \leq C_{T,p} \tag{5.2}$$

and such that there exist, on some probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \Pr)$, *a* $\mathcal{F}_0$*-measurable random variable* V_0 *with law* f_0, *a* $(\mathcal{F}_t)_{t\geq 0}$*-Poisson measure* $N(ds, dv, d\theta, d\varphi, du)$ *on* $[0,\infty) \times \mathbb{R}^3 \times (0, \pi/2] \times [0, 2\pi) \times [0,\infty)$ *with intensity* $ds f_s(dv) b(\theta)\, d\theta\, d\varphi\, du$ *and a càdlàg* $(\mathcal{F}_t)_{t\geq 0}$*-adapted* $\mathbb{R}^3$*-valued process* $(V_t)_{t\geq 0}$ *solving* (5.1) *and satisfying* $\mathcal{L}(V_t) = f_t$ *for all* $t \geq 0$.

The proof of this result is fastidious and not very interesting, so we will give at the end of the paper. In the sequel, $(V_t)_{t\geq 0}$ will be called Boltzmann process.

6. Approximation. We now wish to approximate the Boltzmann process $(V_t)_{t\geq 0}$ by a process $(V_t^\varepsilon)_{t\geq 0}$ of which we can more easily study the law. We essentially freeze the integrand in the Poisson integral during a small time interval $[t - \varepsilon, t]$, so that the resulting process V_t^ε becomes a Lévy process conditionally on $\mathcal{F}_{t-\varepsilon}$. The advantage of Lévy processes is that we can easily study their laws through their Fourier transforms. Due to the lack of regularity of the function a, we have to make use of φ_0 introduced in Lemma 3.2.

PROPOSITION 6.1. *Assume* $(A_{\gamma,\nu})$ *for some* $\gamma \in (-1, 1)$, $\nu \in (0, 1)$ *with* $\gamma + \nu > 0$. *Consider a Boltzmann process* $(V_t)_{t\geq 0}$ *built with a Poisson measure* N *as in Proposition* 5.1. *For* $\varepsilon \in (0, t \wedge 1)$, *set*

$$V_t^\varepsilon := V_{t-\varepsilon} + \int_{t-\varepsilon}^t \int_{\mathbb{R}^3} \int_0^{\pi/2} \int_0^{2\pi} \int_0^\infty a\big(V_{t-\varepsilon}, v, \theta, \varphi + \varphi_0(V_{s-} - v, V_{t-\varepsilon} - v)\big) \times \mathbb{1}_{\{u \leq |V_{t-\varepsilon} - v|^\gamma\}} N(ds, dv, d\theta, d\varphi, du). \tag{6.1}$$

(i) *If* $\gamma \in (0, 1)$, *then for any* $0 < t_0 \leq t - \varepsilon \leq t$ *with* $\varepsilon \in (0, 1)$ *and any* $\eta \in (0, 2)$,

$$\mathbb{E}\big[|V_t - V_t^\varepsilon|^\nu\big] \leq C_{t_0,\eta} \varepsilon^{2-\eta}.$$

(ii) *If* $\gamma \in (-1, 0]$, *then for any* $0 \le t - \varepsilon \le t$ *with* $\varepsilon \in (0, 1)$ *and any* $\eta \in (0, 2 + \gamma/\nu)$,

$$\mathbb{E}[|V_t - V_t^\varepsilon|^\nu] \le C_\eta \varepsilon^{2+\gamma/\nu-\eta}.$$

We will use that for $a, b > 0$, there are some constants $0 < c_{a,b} < C_{a,b}$ such that

$$\forall x, y > 0, \qquad c_{a,b}|x^{a+b} - y^{a+b}| \le (x^a + y^a)|x^b - y^b| \le C_{a,b}|x^{a+b} - y^{a+b}|. \tag{6.2}$$

PROOF. We divide the proof into several steps.

Step 1. Here we check that for all $\beta \in (\nu, 1)$ and all $0 \le s \le t$, $\mathbb{E}[|V_t - V_s|^\beta] \le C_\beta(t - s)$ in both cases (i) and (ii). Using the subadditivity of $x \mapsto x^\beta$, we deduce from (5.1) that

$$|V_t - V_s|^\beta \le \int_s^t \int_{\mathbb{R}^3} \int_0^{\pi/2} \int_0^{2\pi} \int_0^\infty |a(V_{r-}, v, \theta, \varphi)|^\beta \times \mathbb{1}_{\{u \le |V_{r-} - v|^\gamma\}} N(dr, dv, d\theta, d\varphi, du).$$

Taking expectations, integrating in u and using (3.4), we obtain

$$\begin{aligned}\mathbb{E}[|V_t - V_s|^\beta] &\le \mathbb{E}\Big[\int_s^t \int_{\mathbb{R}^3} \int_0^{\pi/2} \int_0^{2\pi} \int_0^\infty |a(V_r, v, \theta, \varphi)|^\beta \\ &\qquad \times \mathbb{1}_{\{u \le |V_r - v|^\gamma\}}\, du\, d\varphi b(\theta)\, d\theta f_r(dv)\, dr\Big] \\ &\le \int_s^t \int_{\mathbb{R}^3} \int_0^{\pi/2} \int_0^{2\pi} \theta^\beta \mathbb{E}[|V_r - v|^{\gamma+\beta}]\, d\varphi b(\theta)\, d\theta f_r(dv)\, dr \\ &\le C_\beta(t - s).\end{aligned}$$

We used that $\beta > \nu$, whence $\int_0^{\pi/2} \theta^\beta b(\theta)\, d\theta \le C_0 \int_0^{\pi/2} \theta^{\beta-1-\nu}\, d\theta < \infty$ by $(A_{\gamma,\nu})$, that $|V_r - v|^{\gamma+\beta} \le C(1 + |V_r|^2 + |v|^2)$ [because $\gamma + \beta \in (0, 2)$] and that $\int_{\mathbb{R}^3} \mathbb{E}(1 + |v|^2 + |V_r|^2) f_r(dv) = 1 + 2m_2(f_r) = C$ by (1.3) [recall that $\mathcal{L}(V_t) = f_t$].

Step 2. In this step we prove that for all $\beta \in (\nu, 1)$ and all $0 \le t - \varepsilon \le t$, in cases (i) and (ii),

$$\mathbb{E}[|V_t - V_t^\varepsilon|^\beta] \le C_\beta \int_{t-\varepsilon}^t \int_{\mathbb{R}^3} \mathbb{E}[A_s^{1,\beta,\varepsilon}(v) + A_s^{2,\beta,\varepsilon}(v) + A_s^{3,\beta,\varepsilon}(v)] f_s(dv)\, ds,$$

where, using the notation $x_+ = x \vee 0$,

$$\begin{aligned}A_s^{1,\beta,\varepsilon}(v) &:= (|V_{t-\varepsilon} - v|^\gamma \wedge |V_s - v|^\gamma) \\ &\qquad \times (|V_s - V_{t-\varepsilon}|^\beta \wedge [|V_{t-\varepsilon} - v|^\beta + |V_s - v|^\beta]), \\ A_s^{2,\beta,\varepsilon}(v) &:= (|V_{t-\varepsilon} - v|^\gamma - |V_s - v|^\gamma)_+ |V_{t-\varepsilon} - v|^\beta, \\ A_s^{3,\beta,\varepsilon}(v) &:= (|V_s - v|^\gamma - |V_{t-\varepsilon} - v|^\gamma)_+ |V_s - v|^\beta.\end{aligned}$$

Exactly as in Step 1, we obtain

$$\mathbb{E}\big[|V_t - V_t^\varepsilon|^\beta\big] \le \mathbb{E}\bigg[\int_{t-\varepsilon}^t \int_{\mathbb{R}^3}\int_0^{\pi/2}\int_0^{2\pi}\int_0^\infty \big|a(V_s, v, \theta, \varphi)\mathbb{1}_{\{u\le |V_s-v|^\gamma\}} - a\big(V_{t-\varepsilon}, v, \theta, \varphi+\varphi_0(V_s - v, V_{t-\varepsilon}-v)\big) \times \mathbb{1}_{\{u\le |V_{t-\varepsilon}-v|^\gamma\}}\big|^\beta \,du\,d\varphi b(\theta)\,d\theta f_s(dv)\,ds\bigg].$$

Integrating in u, we get $\mathbb{E}[|V_t - V_t^\varepsilon|^\beta] \le \int_{t-\varepsilon}^t \int_{\mathbb{R}^3} \mathbb{E}[B_s^{1,\beta,\varepsilon}(v) + B_s^{2,\beta,\varepsilon}(v) + B_s^{2,\beta,\varepsilon}(v)] f_s(dv)\,ds$, where

$$B_s^{1,\beta,\varepsilon}(v) := \int_0^{\pi/2}\int_0^{2\pi} \big(|V_{t-\varepsilon} - v|^\gamma \wedge |V_s - v|^\gamma\big) \times \big|a(V_s, v, \theta, \varphi) - a\big(V_{t-\varepsilon}, v, \theta, \varphi+\varphi_0(V_s - v, V_{t-\varepsilon}-v)\big)\big|^\beta \,d\varphi b(\theta)\,d\theta,$$

$$B_s^{2,\beta,\varepsilon}(v) := \int_0^{\pi/2}\int_0^{2\pi} \big(|V_{t-\varepsilon} - v|^\gamma - |V_s - v|^\gamma\big)_+ \times \big|a\big(V_{t-\varepsilon}, v, \theta, \varphi+\varphi_0(V_s - v, V_{t-\varepsilon}-v)\big)\big|^\beta \,d\varphi b(\theta)\,d\theta,$$

$$B_s^{3,\beta,\varepsilon}(v) := \int_0^{\pi/2}\int_0^{2\pi} \big(|V_s - v|^\gamma - |V_{t-\varepsilon} - v|^\gamma\big)_+ \big|a(V_s, v, \theta, \varphi)\big|^\beta \,d\varphi b(\theta)\,d\theta.$$

Using Lemma 3.2 and (3.4), we realize that

$$\big|a(V_s, v, \theta, \varphi) - a\big(V_{t-\varepsilon}, v, \theta, \varphi+\varphi_0(V_s - v, V_{t-\varepsilon}-v)\big)\big| \le 2\theta\big(|V_s - V_{t-\varepsilon}| \wedge \big[|V_{t-\varepsilon} - v| + |V_s - v|\big]\big).$$

Since $\int_0^{\pi/2} \theta^\beta b(\theta)\,d\theta < \infty$, we deduce that $B_s^{1,\beta,\varepsilon}(v) \le C_\beta A_s^{1,\beta,\varepsilon}(v)$. Using (3.4), we get $B_s^{2,\beta,\varepsilon}(v) \le C_\beta A_s^{2,\beta,\varepsilon}(v)$ and $B_s^{3,\beta,\varepsilon}(v) \le C_\beta A_s^{3,\beta,\varepsilon}(v)$, which completes the step.

Step 3. Here we conclude the proof of (i). We thus assume that $\gamma \in (0,1)$ and fix $0 < t_0 \le t - \varepsilon \le t$ with $\varepsilon \in (0,1)$. We also fix $\beta \in (\nu, 1)$ and apply Step 2. We first observe that

$$A_s^{1,\beta,\varepsilon}(v) \le C\big(|v|^\gamma + |V_{t-\varepsilon}|^\gamma + |V_s|^\gamma\big)|V_s - V_{t-\varepsilon}|^\beta.$$

We next use twice (6.2) (with $a = \gamma$ and $b = \beta$) to deduce that

$$\begin{aligned} A_s^{2,\beta,\varepsilon}(v) + A_s^{3,\beta,\varepsilon}(v) &\le \big(|V_{t-\varepsilon} - v|^\beta + |V_s - v|^\beta\big)\big||V_{t-\varepsilon} - v|^\gamma - |V_s - v|^\gamma\big| \\ &\le C_\beta\big||V_{t-\varepsilon} - v|^{\beta+\gamma} - |V_s - v|^{\beta+\gamma}\big| \\ &\le C_\beta\big(|V_{t-\varepsilon} - v|^\gamma + |V_s - v|^\gamma\big)\big||V_{t-\varepsilon} - v|^\beta - |V_s - v|^\beta\big| \end{aligned}$$

$$\leq C_\beta\big(|V_{t-\varepsilon}-v|^\gamma+|V_s-v|^\gamma\big)|V_s-V_{t-\varepsilon}|^\beta$$
$$\leq C_\beta\big(|v|^\gamma+|V_{t-\varepsilon}|^\gamma+|V_s|^\gamma\big)|V_s-V_{t-\varepsilon}|^\beta.$$

We thus have

$$\mathbb{E}\big[\big|V_t-V_t^\varepsilon\big|^\beta\big]\leq C_\beta\int_{t-\varepsilon}^t\int_{\mathbb{R}^3}\mathbb{E}\big[|V_s-V_{t-\varepsilon}|^\beta\big(|v|^\gamma+|V_{t-\varepsilon}|^\gamma+|V_s|^\gamma\big)\big]f_s(dv)\,ds$$
$$\leq C_\beta\int_{t-\varepsilon}^t\mathbb{E}\big[|V_s-V_{t-\varepsilon}|^\beta\big(1+|V_{t-\varepsilon}|^\gamma+|V_s|^\gamma\big)\big]\,ds,$$

since $\int_{\mathbb{R}^3}|v|^\gamma f_s(dv)\leq\int_{\mathbb{R}^3}(1+|v|^2)f_t(dv)\leq C$ by (1.3). We now consider $\delta\in(0,1-\beta)$ and apply the Hölder inequality [with $p=1/(1-\delta)$ and $q=1/\delta$]:

$$\mathbb{E}\big[\big|V_t-V_t^\varepsilon\big|^\beta\big]\leq C_\beta\int_{t-\varepsilon}^t\mathbb{E}\big[|V_s-V_{t-\varepsilon}|^{\beta/(1-\delta)}\big]^{1-\delta}$$
$$\times\,\mathbb{E}\big[\big(1+|V_{t-\varepsilon}|^\gamma+|V_s|^\gamma\big)^{1/\delta}\big]^\delta\,ds.$$

By Step 1 [observe that $\beta/(1-\delta)\in(\nu,1)$], we have $\mathbb{E}[|V_s-V_{t-\varepsilon}|^{\beta/(1-\delta)}]\leq C_{\beta,\delta}\varepsilon$ for all $s\in[t-\varepsilon,t]$. Using (1.6) [recall that $\mathcal{L}(V_s)=f_s$ for all $s\geq0$], we see that $\mathbb{E}[(1+|V_{t-\varepsilon}|^\gamma+|V_s|^\gamma)^{1/\delta}]\leq C_{t_0,\delta}$ (because $s\geq t-\varepsilon\geq t_0>0$). Thus

$$\mathbb{E}\big[\big|V_t-V_t^\varepsilon\big|^\beta\big]\leq C_{\beta,\delta,t_0}\int_{t-\varepsilon}^t\varepsilon^{1-\delta}\,ds\leq C_{\beta,\delta,t_0}\varepsilon^{2-\delta}.$$

Using finally the Hölder inequality, we deduce that for all $\beta\in(\nu,1)$ and all $\delta\in(0,1-\beta)$, $\mathbb{E}[|V_t-V_t^\varepsilon|^\nu]\leq\mathbb{E}[|V_t-V_t^\varepsilon|^\beta]^{\nu/\beta}\leq C_{\beta,\delta,t_0}\varepsilon^{(2-\delta)\nu/\beta}$. Since we can choose $\beta\in(\nu,1)$ arbitrarily close to ν and $\delta\in(0,1-\beta)$ arbitrarily close to 0, it holds that $(2-\delta)\nu/\beta\in(0,2)$ is arbitrarily close to 2, which ends the proof of (i).

Step 4. We finally check (ii). We thus assume that $\gamma\in(-1,0]$, that $\gamma+\nu>0$ and we fix $0\leq t-\varepsilon\leq t$ with $\varepsilon\in(0,1)$. We also fix $\beta\in(\nu,1)$ and apply Step 2. First, since $|\gamma|/\beta\in(0,1)$,

$$A_s^{1,\beta,\varepsilon}(v)\leq\big(|V_{t-\varepsilon}-v|^\gamma\wedge|V_s-v|^\gamma\big)|V_s-V_{t-\varepsilon}|^{\beta(1-|\gamma|/\beta)}$$
$$\times\big(|V_{t-\varepsilon}-v|^\beta+|V_s-v|^\beta\big)^{|\gamma|/\beta}$$
$$\leq\big(|V_{t-\varepsilon}-v|^\gamma\wedge|V_s-v|^\gamma\big)\big(|V_{t-\varepsilon}-v|^{|\gamma|}+|V_s-v|^{|\gamma|}\big)|V_s-V_{t-\varepsilon}|^{\beta+\gamma}$$
$$\leq 2|V_s-V_{t-\varepsilon}|^{\beta+\gamma}.$$

Next, using twice (6.2) with $a=|\gamma|$ and $b=\beta+\gamma$ (lines 2 and 4),

$$A_s^{2,\beta,\varepsilon}(v)=\mathbb{1}_{\{|V_{t-\varepsilon}-v|<|V_s-v|\}}\big(|V_s-v|^{|\gamma|}-|V_{t-\varepsilon}-v|^{|\gamma|}\big)|V_{t-\varepsilon}-v|^{\beta+\gamma}|V_s-v|^\gamma$$
$$\leq C_\beta\mathbb{1}_{\{|V_{t-\varepsilon}-v|<|V_s-v|\}}\big(|V_s-v|^\beta-|V_{t-\varepsilon}-v|^\beta\big)|V_s-v|^\gamma$$
$$\leq C_\beta\mathbb{1}_{\{|V_{t-\varepsilon}-v|<|V_s-v|\}}\frac{|V_s-v|^\beta-|V_{t-\varepsilon}-v|^\beta}{|V_s-v|^{|\gamma|}+|V_{t-\varepsilon}-v|^{|\gamma|}}$$

$$\leq C_\beta\big(|V_s - v|^{\beta+\gamma} - |V_{t-\varepsilon} - v|^{\beta+\gamma}\big)$$
$$\leq C_\beta |V_s - V_{t-\varepsilon}|^{\beta+\gamma},$$

where we finally used that $0 < \beta + \gamma < 1$. Treating $A_s^{3,\beta,\varepsilon}(v)$ similarly, we finally get

$$\mathbb{E}\big[\big|V_t - V_t^\varepsilon\big|^\beta\big] \leq C_\beta \int_{t-\varepsilon}^t \int_{\mathbb{R}^3} \mathbb{E}\big[|V_s - V_{t-\varepsilon}|^{\beta+\gamma}\big] f_s(dv)\, ds$$
$$\leq C_\beta \int_{t-\varepsilon}^t \mathbb{E}\big[|V_s - V_{t-\varepsilon}|^{\beta+\gamma}\big]\, ds.$$

Using the Hölder inequality (recall that $0 < \beta + \gamma < \beta$) and Step 1, we obtain

$$\mathbb{E}\big[\big|V_t - V_t^\varepsilon\big|^\beta\big] \leq C_\beta \int_{t-\varepsilon}^t \mathbb{E}\big[|V_s - V_{t-\varepsilon}|^\beta\big]^{1+\gamma/\beta}\, ds \leq C_\beta \varepsilon^{2+\gamma/\beta},$$

whence $\mathbb{E}[|V_t - V_t^\varepsilon|^\nu] \leq \mathbb{E}[|V_t - V_t^\varepsilon|^\beta]^{\nu/\beta} \leq C_\beta \varepsilon^{(2+\gamma/\beta)\nu/\beta}$. Since we can choose $\beta \in (\nu, 1)$ arbitrarily close to ν it holds that $(2 + \gamma/\beta)\nu/\beta \in (0, 2 + \gamma/\nu)$ is arbitrarily close to $2 + \gamma/\nu$, which completes the proof of (ii). □

7. Density estimate for the approximate process. The aim of this section, strongly inspired by Schilling, Sztonyk and Wang [34], Propositions 2.1, 2.2, 2.3, is to prove that V_t^ε has a regular law in some sense, with some precise estimates in terms of ε.

PROPOSITION 7.1. *Assume* $(A_{\gamma,\nu})$ *for some* $\gamma \in (-1, 1)$, $\nu \in (0, 1)$. *Let* $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$ *not be a Dirac mass. If* $\gamma \in (-1, 0]$, *assume additionally that* $f_0 \in \mathcal{P}_{4+\gamma+4|\gamma|/\nu}(\mathbb{R}^3)$. *Consider the approximate Boltzmann process* V_t^ε *defined in Proposition 6.1 associated with a weak solution* $(f_t)_{t\geq 0}$ *to* (1.1) *starting from* f_0. *For all* $h \in \mathbb{R}^d$, *all* $\phi \in L^\infty(\mathbb{R}^3)$, *all* $0 < t_0 \leq t - \varepsilon < t \leq t_1$ *with* $\varepsilon \in (0, 1)$,

$$\big|\mathbb{E}\big[\phi\big(V_t^\varepsilon + h\big) - \phi\big(V_t^\varepsilon\big)\big]\big| \leq C_{t_0,t_1} \|\phi\|_{L^\infty(\mathbb{R}^3)} \frac{|h|}{\varepsilon^{1/\nu}}.$$

We will use the following easy estimate, which resembles [34], Proposition 2.1: it is much less general, but sharper.

LEMMA 7.2. *Let* λ *be a nonnegative measure on* $\mathbb{R}^3$ *such that* $\int_{\mathbb{R}^3} |y|\lambda(dy) < \infty$ *and consider the infinitely divisible distribution* k *with Fourier transform*

$$\hat{k}(\xi) := \int_{\mathbb{R}^3} e^{i\langle \xi, x\rangle} k(dx) = \exp\big(-\Phi(\xi)\big) \qquad \text{with } \Phi(\xi) = \int_{\mathbb{R}^3} \big(1 - e^{i\langle \xi, y\rangle}\big)\lambda(dy).$$

If the right-hand side of the following inequality is finite, then k *has a density* (*still denoted by* k) *and*

$$\|\nabla k\|_{L^1(\mathbb{R}^3)} \leq C\big(1 + m_1^4(\lambda) + m_4(\lambda)\big) \int_{\mathbb{R}^3} e^{-\operatorname{Re}\Phi(\xi)}\big(1 + |\xi|\big)\, d\xi,$$

where $m_n(\lambda) = \int_{\mathbb{R}^3} |y|^n \lambda(dy)$ *and* C *is a universal constant.*

PROOF. The proof is quite similar to [34], Proposition 2.1. We will show that

$$\|\nabla k\|_{L^\infty(\mathbb{R}^3)} \le C \int_{\mathbb{R}^3} e^{-\operatorname{Re}\Phi(\xi)}|\xi|\,d\xi, \tag{7.1}$$

$$\||x|^4\nabla k(x)\|_{L^\infty(\mathbb{R}^3)} \le C\big(1+m_1^4(\lambda)+m_4(\lambda)\big)\int_{\mathbb{R}^3} e^{-\operatorname{Re}\Phi(\xi)}\big(1+|\xi|\big)\,d\xi, \tag{7.2}$$

from which the result follows, since $(1+|x|)^{-4} \in L^1(\mathbb{R}^3)$. First,

$$\begin{aligned}\|\nabla k\|_{L^\infty(\mathbb{R}^3)} &\le (2\pi)^{-3}\|\widehat{\nabla k}\|_{L^1(\mathbb{R}^3)} = (2\pi)^{-3}\|\xi\hat{k}(\xi)\|_{L^1(\mathbb{R}^3)}\\ &= (2\pi)^{-3}\int_{\mathbb{R}^3} e^{-\operatorname{Re}\Phi(\xi)}|\xi|\,d\xi,\end{aligned}$$

whence (7.1). To check (7.2), we start with

$$\||x|^4\nabla k(x)\|_{L^\infty(\mathbb{R}^3)} \le (2\pi)^{-3}\|\Delta^2(\widehat{\nabla k})\|_{L^1(\mathbb{R}^3)} \le C\|D^4(\xi\hat{k}(\xi))\|_{L^1(\mathbb{R}^3)}.$$

A tedious computation recalling that $\hat{k}(\xi) = e^{-\Phi(\xi)}$ shows that

$$\begin{aligned}&|D^4(\xi\hat{k}(\xi))|\\ &\quad\le C\big(1+|\xi|\big)|e^{-\Phi(\xi)}|\\ &\qquad\times\big(|D^4\Phi(\xi)|+|D^3\Phi(\xi)D\Phi(\xi)|+|D^2\Phi(\xi)|^2+|D\Phi(\xi)|^2|D^2\Phi(\xi)|\\ &\qquad\qquad+|D\Phi(\xi)|^4+|D^3\Phi(\xi)|+|D\Phi(\xi)||D^2\Phi(\xi)|+|D\Phi(\xi)|^3\big).\end{aligned}$$

But from the expression of Φ, we see that $|D^n\Phi(\xi)| \le m_n(\lambda)$ for all $n \ge 1$. Since $|e^{-\Phi(\xi)}| = e^{-\operatorname{Re}\Phi(\xi)}$, we get, setting $m_n = m_n(\lambda)$ for simplicity,

$$\begin{aligned}|D^4(\xi\hat{k}(\xi))| &\le C\big(1+|\xi|\big)e^{-\operatorname{Re}\Phi(\xi)}\\ &\qquad\times\big(m_4+m_3m_1+m_2^2+m_1^2m_2+m_1^4+m_3+m_1m_2+m_1^3\big)\\ &\le C\big(1+|\xi|\big)e^{-\operatorname{Re}\Phi(\xi)}\big(1+m_4+m_3^{4/3}+m_2^2+m_1^4\big),\end{aligned}$$

where we used the Young inequality. To complete the proof of (7.2), it only remains to check that $m_3^{4/3}+m_2^2 \le C(m_4+m_1^4)$, which is not hard by the Hölder and Young inequalities. □

Unfortunately, applying directly Lemma 7.2 to the law of V_t^ε does not give the correct power of ε. We thus use the same trick as in [34]: we only consider the part of V_t^ε corresponding to small values of θ (grazing collisions), in such a way that it does not affect the estimate from below of $\operatorname{Re}\Phi(\xi)$, but which makes consequently decrease the moment estimates [of $m_1^4(\lambda)+m_4(\lambda)$].

We start with the following remark.

LEMMA 7.3. *Adopt the notation and assumptions of Proposition 7.1. Let $\varepsilon \in (0, t\wedge 1)$ be fixed.*

(i) *We can find a* $(\mathcal{F}_t)_{t\geq 0}$*-Poisson measure* M *with the same intensity as* N (*see Proposition* 5.1) *such that*

$$V_t^\varepsilon := V_{t-\varepsilon} + \int_{t-\varepsilon}^t \int_{\mathbb{R}^3} \int_0^{\pi/2} \int_0^{2\pi} \int_0^\infty a(V_{t-\varepsilon}, v, \theta, \varphi) \times \mathbb{1}_{\{u \leq |V_{t-\varepsilon}-v|^\gamma\}} M(ds, dv, d\theta, d\varphi, du). \tag{7.3}$$

(ii) *We write* $V_t^\varepsilon = U_t^\varepsilon + W_t^\varepsilon$ *with*

$$U_t^\varepsilon := \int_{t-\varepsilon}^t \int_{\mathbb{R}^3} \int_0^{\pi/2} \int_0^{2\pi} \int_0^\infty a(V_{t-\varepsilon}, v, \theta, \varphi) \times \mathbb{1}_{\{u \leq |V_{t-\varepsilon}-v|^\gamma\}} \mathbb{1}_{\{\theta < \varepsilon^{1/\nu}\}} M(ds, dv, d\theta, d\varphi, du),$$

$$W_t^\varepsilon := V_{t-\varepsilon} + \int_{t-\varepsilon}^t \int_{\mathbb{R}^3} \int_0^{\pi/2} \int_0^{2\pi} \int_0^\infty a(V_{t-\varepsilon}, v, \theta, \varphi) \mathbb{1}_{\{u \leq |V_{t-\varepsilon}-v|^\gamma\}} \times \mathbb{1}_{\{\theta \geq \varepsilon^{1/\nu}\}} M(ds, dv, d\theta, d\varphi, du),$$

so that U_t^ε *and* W_t^ε *are independent conditionally on* $\mathcal{F}_{t-\varepsilon}$.

(iii) *For all* $\xi \in \mathbb{R}^3$, $\mathbb{E}[e^{i\langle \xi, U_t^\varepsilon\rangle} | \mathcal{F}_{t-\varepsilon}] = \exp(-\Psi_{\varepsilon,t,V_{t-\varepsilon}}(\xi))$, *where, for* $v_0 \in \mathbb{R}^3$,

$$\Psi_{\varepsilon,t,v_0}(\xi) = \int_{t-\varepsilon}^t \int_{\mathbb{R}^3} \int_0^{\varepsilon^{1/\nu}} \int_0^{2\pi} \big(1 - e^{i\langle \xi, a(v_0,v,\theta,\varphi)\rangle}\big) |v - v_0|^\gamma \, d\varphi b(\theta) \, d\theta f_s(dv) \, ds.$$

PROOF. To prove point (i), define M as the image measure of N by the $(\mathcal{F}_t)_{t\geq 0}$-predictable map $(s, v, \theta, \varphi, u) \mapsto (s, v, \theta, \varphi + \varphi_0(V_{s-} - v, V_{t-\varepsilon} - v)$ modulo $2\pi, u)$. Then (6.1) obviously rewrites as (7.3). The fact that M is a $(\mathcal{F}_t)_{t\geq 0}$-Poisson measure with the same intensity as N is due to the fact that the Lebesgue measure on $[0, 2\pi)$ is invariant by translation (modulo 2π). This was already noticed by Tanaka [35]; see [21], Lemma 4.7, for a very similar statement. Points (ii) and (iii) follow from standard properties of Poisson measures, because in U_t^ε and W_t^ε, the integrands are $\mathcal{F}_{t-\varepsilon}$-measurable and the Poisson integrals concern the time interval $[t - \varepsilon, t]$. □

We next estimate the Fourier transform of the law of U_t^ε.

LEMMA 7.4. *Adopt the notation and assumptions of Proposition* 7.1. *Recall that* Ψ_{ε,t,v_0} *was defined in Lemma* 7.3. *For all* $\xi \in \mathbb{R}^3$, *all* $0 < t_0 \leq t - \varepsilon < t \leq t_1$ *with* $\varepsilon \in (0, 1)$,

$$\operatorname{Re} \Psi_{\varepsilon,t,v_0}(\varepsilon^{-1/\nu}\xi) \geq \begin{cases} c_{t_0,t_1}\big(|\xi|^2 \wedge |\xi|^\nu\big) & \text{if } \gamma \in (0, 1), \\ c_{t_0,t_1}\big(1 + |v_0|\big)^\gamma \big(|\xi|^2 \wedge |\xi|^\nu\big) & \text{if } \gamma \in (-1, 0]. \end{cases}$$

PROOF. We divide the proof into three steps.

Step 1. Here we assume that $\gamma \in (-1, 1)$. We have

$$\operatorname{Re}\Psi_{\varepsilon,t,v_0}(\varepsilon^{-1/\nu}\xi) = \int_{t-\varepsilon}^{t}\int_{\mathbb{R}^3}\int_0^{\varepsilon^{1/\nu}}\int_0^{2\pi}\big(1-\cos(\varepsilon^{-1/\nu}\langle\xi, a(v_0, v, \theta, \varphi)\rangle)\big) \times |v-v_0|^\gamma\, d\varphi b(\theta)\, d\theta f_s(dv)\, ds.$$

By (3.1), $\langle\xi, a(v_0, v, \theta, \varphi)\rangle = (\cos\theta - 1)\langle\xi, v_0 - v\rangle/2 + \sin\theta\langle\xi, \Gamma(v_0 - v, \varphi)\rangle/2$. Hence

$$\begin{aligned}
&\int_0^{2\pi}\big(1-\cos(\varepsilon^{-1/\nu}\langle\xi, a(v_0, v, \theta, \varphi)\rangle)\big)\, d\varphi \\
&\quad = \int_0^{2\pi}\big(1-\cos(\varepsilon^{-1/\nu}(\cos\theta - 1)\langle\xi, v_0 - v\rangle/2) \\
&\qquad\qquad \times \cos(\varepsilon^{-1/\nu}\sin\theta\langle\xi, \Gamma(v_0 - v, \varphi)\rangle/2) \\
&\qquad\quad + \sin(\varepsilon^{-1/\nu}(\cos\theta - 1)\langle\xi, v_0 - v\rangle/2) \\
&\qquad\qquad \times \sin(\varepsilon^{-1/\nu}\sin\theta\langle\xi, \Gamma(v_0 - v, \varphi)\rangle/2)\big)\, d\varphi \\
&\quad = \int_0^{2\pi}\big(1-\cos(\varepsilon^{-1/\nu}(\cos\theta - 1)\langle\xi, v_0 - v\rangle/2) \\
&\qquad\qquad \times \cos(\varepsilon^{-1/\nu}\sin\theta\langle\xi, \Gamma(v_0 - v, \varphi)\rangle/2)\big)\, d\varphi \\
&\quad \ge \int_0^{2\pi}\big(1-|\cos(\varepsilon^{-1/\nu}\sin\theta\langle\xi, \Gamma(v_0 - v, \varphi)\rangle/2)|\big)\, d\varphi.
\end{aligned}$$

Since $1-\cos x \ge x^2/4$ and $|\sin x| \ge |x|/2$ for $x \in [-1, 1]$ and since $|\sin x| \le |x|$ for all $x \in \mathbb{R}$ (recall that $\theta \le \varepsilon^{1/\nu} \le 1$),

$$\begin{aligned}
&\int_0^{2\pi}\big(1-\cos(\varepsilon^{-1/\nu}\langle\xi, a(v_0, v, \theta, \varphi)\rangle)\big)\, d\varphi \\
&\quad \ge \int_0^{2\pi}\frac{\varepsilon^{-2/\nu}\sin^2\theta\langle\xi, \Gamma(v_0 - v, \varphi)\rangle^2}{16}\mathbb{1}_{\{|\langle\xi, \Gamma(v_0 - v, \varphi)\rangle\sin\theta| \le 2\varepsilon^{1/\nu}\}}\, d\varphi \\
&\quad \ge \int_0^{2\pi}\frac{\varepsilon^{-2/\nu}\theta^2\langle\xi, \Gamma(v_0 - v, \varphi)\rangle^2}{64}\mathbb{1}_{\{|\theta| \le 2\varepsilon^{1/\nu}/|\langle\xi, \Gamma(v_0 - v, \varphi)\rangle|\}}\, d\varphi.
\end{aligned}$$

Using the lower bound of b given by ($A_{\gamma,\nu}$) and then integrating in θ, we obtain

$$\begin{aligned}
&\operatorname{Re}\Psi_{\varepsilon,t,v_0}(\varepsilon^{-1/\nu}\xi) \\
&\quad \ge \frac{c}{\varepsilon^{2/\nu}}\int_{t-\varepsilon}^{t}\int_{\mathbb{R}^3}\int_0^{\varepsilon^{1/\nu}}\int_0^{2\pi}\theta^2\langle\xi, \Gamma(v_0 - v, \varphi)\rangle^2 \\
&\qquad\qquad \times \mathbb{1}_{\{|\theta| \le 2\varepsilon^{1/\nu}/|\langle\xi, \Gamma(v_0 - v, \varphi)\rangle|\}} \\
&\qquad\qquad \times |v-v_0|^\gamma\, d\varphi\theta^{-1-\nu}\, d\theta f_s(dv)\, ds
\end{aligned}$$

$$= \frac{c}{\varepsilon^{2/\nu}} \int_{t-\varepsilon}^{t} \int_{\mathbb{R}^3} \int_0^{2\pi} \langle \xi, \Gamma(v_0 - v, \varphi)\rangle^2 \left[\varepsilon^{1/\nu} \wedge \frac{2\varepsilon^{1/\nu}}{|\langle \xi, \Gamma(v_0 - v, \varphi)\rangle|}\right]^{2-\nu} \times |v - v_0|^\gamma f_s(dv)\, d\varphi\, ds$$

$$\geq \frac{c}{\varepsilon} \int_{t-\varepsilon}^{t} \int_{\mathbb{R}^3} \int_0^{2\pi} \left[\langle \xi, \Gamma(v_0 - v, \varphi)\rangle^2 \wedge \left|\langle \xi, \Gamma(v_0 - v, \varphi)\rangle\right|^\nu\right] \times |v - v_0|^\gamma f_s(dv)\, d\varphi\, ds$$

$$= \frac{c}{\varepsilon} \int_{t-\varepsilon}^{t} \int_{\mathbb{R}^3} \int_0^{2\pi} \left[\langle v_0 - v, \Gamma(\xi, \varphi)\rangle^2 \wedge \left|\langle v_0 - v, \Gamma(\xi, \varphi)\rangle\right|^\nu\right] \times |v - v_0|^\gamma f_s(dv)\, d\varphi\, ds,$$

where we finally used Remark 3.1.

Step 2. We now assume that $\gamma \in (0, 1)$. Recall Proposition 4.2 [and the fact that $|\Gamma(\xi, \varphi)| = |\xi|$, see (3.1)]: for any $v_0, \xi \in \mathbb{R}^3$, any $\varphi \in [0, 2\pi)$, any $v \in K(v_0, \Gamma(\xi, \varphi))$, we have $|v - v_0| \geq 1$ and $|\langle v_0 - v, \Gamma(\xi, \varphi)\rangle| \geq |\Gamma(\xi, \varphi)| = |\xi|$. Thus, using that $f_s(K(v_0, \Gamma(\xi, \varphi))) \geq q_{t_0,t_1} > 0$ for all $0 < t_0 \leq t - \varepsilon \leq s \leq t \leq t_1$, we get

$$\operatorname{Re} \Psi_{\varepsilon,t,v_0}(\varepsilon^{-1/\nu}\xi) \geq \frac{c}{\varepsilon} \int_{t-\varepsilon}^{t} \int_0^{2\pi} \left[|\xi|^2 \wedge |\xi|^\nu\right] f_s\big(K\big(v_0, \Gamma(\xi, \varphi)\big)\big)\, d\varphi\, ds$$
$$\geq c q_{t_0,t_1}\left[|\xi|^2 \wedge |\xi|^\nu\right].$$

Step 3. We finally assume that $\gamma \in (-1, 0]$. Recall again Proposition 4.2 and that $|\Gamma(\xi, \varphi)| = |\xi|$: for any $v_0, \xi \in \mathbb{R}^3$, any $\varphi \in [0, 2\pi)$, any $v \in K(v_0, \Gamma(\xi, \varphi))$, we have $|v - v_0| \leq |v| + |v_0| \leq 3 + |v_0|$ [so that $|v - v_0|^\gamma \geq 3^\gamma(1 + |v_0|)^\gamma$] and $|\langle v_0 - v, \Gamma(\xi, \varphi)\rangle| \geq |\Gamma(\xi, \varphi)| = |\xi|$. Thus, using that $f_s(K(v_0, \Gamma(\xi, \varphi))) \geq q_{t_0,t_1} > 0$ for all $0 < t_0 \leq t - \varepsilon \leq s \leq t \leq t_1$, we get

$$\operatorname{Re} \Psi_{\varepsilon,t,v_0}(\varepsilon^{-1/\nu}\xi) \geq \frac{c}{\varepsilon} \int_{t-\varepsilon}^{t} \int_0^{2\pi} \left[|\xi|^2 \wedge |\xi|^\nu\right](1 + |v_0|)^\gamma f_s\big(K\big(v_0, \Gamma(\xi, \varphi)\big)\big)\, d\varphi\, ds$$
$$\geq c q_{t_0,t_1}(1 + |v_0|)^\gamma\left[|\xi|^2 \wedge |\xi|^\nu\right],$$

which completes the proof. □

We now estimate the regularity of the law of U_t^ε.

LEMMA 7.5. *Adopt the notation and assumptions of Proposition 7.1. Recall that Ψ_{ε,t,v_0} was defined in Lemma 7.3. Consider $g_{\varepsilon,t,v_0} \in \mathcal{P}(\mathbb{R}^3)$ such that $\widehat{g_{\varepsilon,t,v_0}}(\xi) = \exp(-\Psi_{\varepsilon,t,v_0}(\xi))$. If $0 < t_0 \leq t - \varepsilon < t \leq t_1$ and $\varepsilon \in (0, 1)$, g_{ε,t,v_0} has a density and*

$$\|\nabla g_{\varepsilon,t,v_0}\|_{L^1(\mathbb{R}^3)} \leq \begin{cases} C_{t_0,t_1}\varepsilon^{-1/\nu}(1 + |v_0|)^{4\gamma+4} & \text{if } \gamma \in (0, 1), \\ C_{t_0,t_1}\varepsilon^{-1/\nu}(1 + |v_0|)^{4+\gamma+4|\gamma|/\nu} & \text{if } \gamma \in (-1, 0]. \end{cases}$$

PROOF. We introduce, for X_{ε,t,v_0} a g_{ε,t,v_0}-distributed random variable, $Y_{\varepsilon,t,v_0} := \varepsilon^{-1/\nu} X_{\varepsilon,t,v_0}$. Then the law k_{ε,t,v_0} of Y_{ε,t,v_0} satisfies $\widehat{k_{\varepsilon,t,v_0}}(\xi) = \widehat{g_{\varepsilon,t,v_0}}(\varepsilon^{-1/\nu}\xi) = \exp(-\Psi_{\varepsilon,t,v_0}(\varepsilon^{-1/\nu}\xi))$ and $k_{\varepsilon,t,v_0}(x) = \varepsilon^{3/\nu} g_{\varepsilon,t,v_0}(\varepsilon^{1/\nu}x)$. Observe that

$$\|\nabla g_{\varepsilon,t,v_0}\|_{L^1(\mathbb{R}^3)} = \varepsilon^{-1/\nu} \|\nabla k_{\varepsilon,t,v_0}\|_{L^1(\mathbb{R}^3)}. \tag{7.4}$$

Step 1. We want to apply Lemma 7.2. We have $\widehat{k_{\varepsilon,t,v_0}}(\xi) = \exp(-\Phi_{\varepsilon,t,v_0}(\xi))$, where $\Phi_{\varepsilon,t,v_0}(\xi) = \Psi_{\varepsilon,t,v_0}(\varepsilon^{-1/\nu}\xi)$, whence

$$\begin{aligned}\Phi_{\varepsilon,t,v_0}(\xi) &= \int_{t-\varepsilon}^{t}\int_{\mathbb{R}^3}\int_0^{\varepsilon^{1/\nu}}\int_0^{2\pi}\bigl(1 - e^{i\langle \xi, \varepsilon^{-1/\nu} a(v_0,v,\theta,\varphi)\rangle}\bigr) \\ &\qquad\qquad \times |v-v_0|^\gamma \, d\varphi b(\theta)\, d\theta f_s(dv)\, ds \\ &= \int_{\mathbb{R}^3} \bigl(1 - e^{i\langle \xi, z\rangle}\bigr)\lambda_{t,\varepsilon,v_0}(dz),\end{aligned}$$

the measure $\lambda_{t,\varepsilon,v_0}$ being defined by

$$\begin{aligned}&\int_{\mathbb{R}^3} F(z)\lambda_{t,\varepsilon,v_0}(dz) \\ &\quad = \int_{t-\varepsilon}^{t}\int_{\mathbb{R}^3}\int_0^{\varepsilon^{1/\nu}}\int_0^{2\pi} F\biggl(\frac{a(v_0,v,\theta,\varphi)}{\varepsilon^{1/\nu}}\biggr)|v-v_0|^\gamma \, d\varphi b(\theta)\, d\theta f_s(dv)\, ds\end{aligned}$$

for all nonnegative measurable $F : \mathbb{R}^3 \mapsto \mathbb{R}$. Lemma 7.2 thus implies

$$\begin{aligned}\|\nabla k_{\varepsilon,t,v_0}\|_{L^1(\mathbb{R}^3)} &\le C\bigl(1 + m_1^4(\lambda_{t,\varepsilon,v_0}) + m_4(\lambda_{t,\varepsilon,v_0})\bigr) \\ &\qquad \times \int_{\mathbb{R}^3} e^{-\operatorname{Re}\Phi_{\varepsilon,t,v_0}(\xi)}\bigl(1+|\xi|\bigr)\, d\xi \\ &\le C\bigl(1 + m_1^4(\lambda_{t,\varepsilon,v_0}) + m_4(\lambda_{t,\varepsilon,v_0})\bigr) \\ &\qquad \times \biggl(1 + \int_{|\xi|\ge 1} e^{-\operatorname{Re}\Psi_{\varepsilon,t,v_0}(\varepsilon^{-1/\nu}\xi)}|\xi|\, d\xi\biggr).\end{aligned} \tag{7.5}$$

A simple computation using (3.4) and $(A_{\gamma,\nu})$ shows that for $n = 1, 4$,

$$\begin{aligned}m_n(\lambda_{t,\varepsilon,v_0}) &\le \int_{t-\varepsilon}^{t}\int_{\mathbb{R}^3}\int_0^{\varepsilon^{1/\nu}}\int_0^{2\pi} \frac{|\theta|^n |v-v_0|^n}{2^n \varepsilon^{n/\nu}} |v-v_0|^\gamma \, d\varphi b(\theta)\, d\theta f_s(dv)\, ds \\ &\le C \int_{t-\varepsilon}^{t}\int_{\mathbb{R}^3}\int_0^{\varepsilon^{1/\nu}} \frac{|\theta|^{n-1-\nu}|v-v_0|^{n+\gamma}}{\varepsilon^{n/\nu}}\, d\theta f_s(dv)\, ds \\ &\le C \int_{t-\varepsilon}^{t}\int_{\mathbb{R}^3} \bigl(|v|^{\gamma+n} + |v_0|^{\gamma+n}\bigr)\frac{\varepsilon^{(n-\nu)/\nu}}{\varepsilon^{n/\nu}} f_s(dv)\, ds \\ &\le C \sup_{s\in[t-\varepsilon,t]} \int_{\mathbb{R}^3} \bigl(|v|^{\gamma+n} + |v_0|^{\gamma+n}\bigr) f_s(dv).\end{aligned} \tag{7.6}$$

Step 2. Here we conclude when $\gamma \in (0,1)$. Let thus $0 < t_0 \le t-\varepsilon \le t \le t_1$ with $\varepsilon \in (0,1)$. Using (1.3), we deduce that $\sup_{s\in[t-\varepsilon,t]} \int_{\mathbb{R}^3} (|v|^{\gamma+1} + |v_0|^{\gamma+1}) f_s(dv) \le C(1+|v_0|^{\gamma+1})$ and by (1.6), $\sup_{s\in[t-\varepsilon,t]} \int_{\mathbb{R}^3} (|v|^{\gamma+4} + |v_0|^{\gamma+4}) f_s(dv) \le C_{t_0}(1+|v_0|^{\gamma+4})$. Hence $m_1^4(\lambda_{t,\varepsilon,v_0}) + m_4(\lambda_{t,\varepsilon,v_0}) \le C_{t_0}(1+|v_0|^{4\gamma+4})$. By Lemma 7.4, $\int_{|\xi|\ge 1} e^{-\operatorname{Re}\Psi_{\varepsilon,t,v_0}(\varepsilon^{-1/\nu}\xi)}|\xi|\,d\xi \le C_{t_0,t_1}$. Recalling (7.5), we finally find that $\|\nabla k_{\varepsilon,t,v_0}\|_{L^1(\mathbb{R}^3)} \le C_{t_0,t_1}(1+|v_0|^{4\gamma+4})$, whence the result by (7.4).

Step 3. We finally conclude when $\gamma \in (-1,0]$. Let thus $0 < t_0 \le t-\varepsilon \le t \le t_1$ with $\varepsilon \in (0,1)$. Using (1.3), we deduce that $\sup_{s\in[t-\varepsilon,t]} \int_{\mathbb{R}^3} (|v|^{\gamma+1} + |v_0|^{\gamma+1}) f_s(dv) \le C(1+|v_0|^{\gamma+1})$. By (5.2) and since $f_0 \in \mathcal{P}_{4+\gamma+4|\gamma|/\nu}(\mathbb{R}^3) \subset \mathcal{P}_{4+\gamma}(\mathbb{R}^3)$, we deduce that $\sup_{s\in[t-\varepsilon,t]} \int_{\mathbb{R}^3} (|v|^{\gamma+4} + |v_0|^{\gamma+4}) f_s(dv) \le C_{t_1}(1+|v_0|^{\gamma+4})$. Hence $m_1^4(\lambda_{t,\varepsilon,v_0}) + m_4(\lambda_{t,\varepsilon,v_0}) \le C_{t_1}(1+|v_0|^{\gamma+4})$. By Lemma 7.4, $\int_{|\xi|\ge 1} e^{-\operatorname{Re}\Psi_{\varepsilon,t,v_0}(\varepsilon^{-1/\nu}\xi)}|\xi|\,d\xi \le \int_{|\xi|\ge 1} e^{-c_{t_0,t_1}(1+|v_0|)^{\gamma}|\xi|^{\nu}}|\xi|\,d\xi \le C_{t_0,t_1}(1+|v_0|)^{4|\gamma|/\nu}$. Recalling (7.5), we finally get $\|\nabla k_{\varepsilon,t,v_0}\|_{L^1(\mathbb{R}^3)} \le C_{t_0,t_1}(1+|v_0|^{\gamma+4}) \times (1+|v_0|)^{4|\gamma|/\nu} \le C_{t_0,t_1}(1+|v_0|)^{4+\gamma+4|\gamma|/\nu}$, whence the result by (7.4). $\square$

We finally have all the weapons to give the following:

PROOF OF LEMMA 7.1. Let thus $t_0 \le t-\varepsilon \le t \le t_1$ with $\varepsilon \in (0,1)$, and let $\phi \in L^\infty(\mathbb{R}^3)$. Recall the notation introduced in Lemma 7.3. Write, using that W_t^ε and U_t^ε are independent conditionally on $\mathcal{F}_{t-\varepsilon}$ and that the law of U_t^ε conditionally on $\mathcal{F}_{t-\varepsilon}$ is $g_{\varepsilon,t,V_{t-\varepsilon}}$ (see Lemma 7.5)

$$\begin{aligned}
&\big|\mathbb{E}[\phi(V_t^\varepsilon + h) - \phi(V_t^\varepsilon)]\big| \\
&\quad = \big|\mathbb{E}[\phi(U_t^\varepsilon + W_t^\varepsilon + h) - \phi(U_t^\varepsilon + W_t^\varepsilon)]\big| \\
&\quad = \big|\mathbb{E}[\mathbb{E}(\phi(U_t^\varepsilon + W_t^\varepsilon + h) - \phi(U_t^\varepsilon + W_t^\varepsilon)|\mathcal{F}_{t-\varepsilon})]\big| \\
&\quad = \left|\mathbb{E}\left[\int_{\mathbb{R}^3} [\phi(x + W_t^\varepsilon + h) - \phi(x + W_t^\varepsilon)] g_{\varepsilon,t,V_{t-\varepsilon}}(x)\,dx\right]\right| \\
&\quad = \left|\mathbb{E}\left[\int_{\mathbb{R}^3} \phi(x + W_t^\varepsilon)\big[g_{\varepsilon,t,V_{t-\varepsilon}}(x-h) - g_{\varepsilon,t,V_{t-\varepsilon}}(x)\big]\,dx\right]\right| \\
&\quad \le \|\phi\|_{L^\infty(\mathbb{R}^3)}|h|\mathbb{E}\big[\|\nabla g_{\varepsilon,t,V_{t-\varepsilon}}\|_{L^1(\mathbb{R}^3)}\big].
\end{aligned}$$

We used that $\int_{\mathbb{R}^3} |g(x-h) - g(x)|\,dx \le \int_{\mathbb{R}^3}\int_0^1 |h.\nabla g(x-uh)|\,du\,dx \le |h| \times \int_0^1 \|\nabla g(\cdot - uh)\|_{L^1(\mathbb{R}^3)}\,du = |h|\|\nabla g\|_{L^1(\mathbb{R}^3)}$.

Assume first that $\gamma \in (0,1)$. Using Lemma 7.5, we get

$$\big|\mathbb{E}[\phi(V_t^\varepsilon + h) - \phi(V_t^\varepsilon)]\big| \le C_{t_0,t_1}\|\phi\|_{L^\infty(\mathbb{R}^3)}|h|\varepsilon^{-1/\nu}\mathbb{E}\big[(1+|V_{t-\varepsilon}|)^{4\gamma+4}\big].$$

The conclusion follows, since

$$\mathbb{E}\big[|V_{t-\varepsilon}|^{4\gamma+4}\big] = m_{4\gamma+4}(f_{t-\varepsilon}) \le \sup_{s\ge t_0} m_{4\gamma+4}(f_s) < \infty$$

by (1.6).

Assume next that $\gamma \in (-1, 0]$. In this case, Lemma 7.5 gives

$$\big|\mathbb{E}\big[\phi\big(V_t^\varepsilon + h\big) - \phi\big(V_t^\varepsilon\big)\big]\big| \le C_{t_0,t_1} \|\phi\|_{L^\infty(\mathbb{R}^3)} |h| \varepsilon^{-1/\nu} \mathbb{E}\big[\big(1 + |V_{t-\varepsilon}|\big)^{4+\gamma+4|\gamma|/\nu}\big].$$

But since $f_0 \in \mathcal{P}_{4+\gamma+4|\gamma|/\nu}(\mathbb{R}^3)$ and $0 \le t - \varepsilon \le t_1$, (5.2) implies that $\mathbb{E}[|V_{t-\varepsilon}|^{4+\gamma+4|\gamma|/\nu}] = m_{4+\gamma+4|\gamma|/\nu}(f_{t-\varepsilon}) \le C_{t_1}$, which completes the proof. □

8. Conclusion. We finally can give the following:

PROOF OF THEOREM 1.3. We thus assume $(A_{\gamma,\nu})$ for some $\gamma \in (-1, 1)$, $\nu \in (0, 1)$ such that $\gamma + \nu > 0$. We also consider $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$ such that f_0 is not a Dirac mass. If $\gamma \in (0, 1)$, we consider any weak solution $(f_t)_{t\ge 0}$ to (1.1) starting from f_0 and satisfying (1.6) and we consider the associated Boltzmann process $(V_t)_{t\ge 0}$ built in Proposition 5.1(ii). If $\gamma \in (-1, 0]$, we assume additionally that $f_0 \in \mathcal{P}_{4+\gamma+4|\gamma|/\nu}(\mathbb{R}^3)$, and we consider the weak solution $(f_t)_{t\ge 0}$ to (1.1) starting from f_0 and the associated Boltzmann process $(V_t)_{t\ge 0}$ built in Proposition 5.1(ii). From now on, we fix $t > 0$.

We wish to apply Lemma 2.1. Let thus $h \in \mathbb{R}^3$ such that $|h| \le 1$ and $\phi \in C_b^\alpha(\mathbb{R}^3)$ for some $\alpha \in (0, 1)$. Let us define

$$I_{t,h}^\phi = \left|\int_{\mathbb{R}^3} \big(\phi(v + h) - \phi(v)\big) f_t(dv)\right| = \big|\mathbb{E}\big[\phi(V_t + h) - \phi(V_t)\big]\big|.$$

For $\varepsilon \in (0, (t/2) \wedge 1)$, we write, recalling that the approximate Boltzmann process V_t^ε was defined in Lemma 6.1,

$$\begin{aligned} I_{t,h}^\phi &\le \big|\mathbb{E}\big[\phi(V_t + h) - \phi\big(V_t^\varepsilon + h\big)\big]\big| + \big|\mathbb{E}\big[\phi(V_t) - \phi\big(V_t^\varepsilon\big)\big]\big| \\ &\quad + \big|\mathbb{E}\big[\phi\big(V_t^\varepsilon + h\big) - \phi\big(V_t^\varepsilon\big)\big]\big| \\ &\le 2\|\phi\|_{C_b^\alpha(\mathbb{R}^3)} \mathbb{E}\big[\big|V_t - V_t^\varepsilon\big|^\alpha\big] + C_t \|\phi\|_\infty \varepsilon^{-1/\nu} |h| \\ &\le C_t \|\phi\|_{C_b^\alpha(\mathbb{R}^3)} \big[\mathbb{E}\big[\big|V_t - V_t^\varepsilon\big|^\alpha\big] + \varepsilon^{-1/\nu} |h|\big], \end{aligned}$$

where we used Lemma 7.1 (with $t_0 = t/2$ and $t_1 = t$) and that $\|\phi\|_{L^\infty(\mathbb{R}^3)} \le \|\phi\|_{C_b^\alpha(\mathbb{R}^3)}$.

Point (i). We assume here that $\gamma \in (0, 1)$. We consider $\alpha \in (0, \nu]$, and we apply Proposition 6.1(i): for any $\eta \in (0, 2)$, we write $\mathbb{E}[|V_t - V_t^\varepsilon|^\alpha] \le \mathbb{E}[|V_t - V_t^\varepsilon|^\nu]^{\alpha/\nu} \le C_{t,\eta} \varepsilon^{(2-\eta)\alpha/\nu}$. We have proved that for all $\eta \in (0, 2)$, all $\varepsilon \in (0, (t/2) \wedge 1)$,

$$I_{t,h}^\phi \le C_{t,\eta} \|\phi\|_{C_b^\alpha(\mathbb{R}^3)} \big[\varepsilon^{(2-\eta)\alpha/\nu} + \varepsilon^{-1/\nu} |h|\big].$$

Choosing $\varepsilon = (1 \wedge (t/2))|h|^{\nu/(1+(2-\eta)\alpha)}$, we obtain $I_{t,h}^\phi \le C_{t,\eta} \|\phi\|_{C_b^\alpha(\mathbb{R}^3)} \times |h|^{(2-\eta)\alpha/(1+(2-\eta)\alpha)}$. For $\alpha \in (0, \nu]$ small enough and $\eta \in (0, 2)$ small enough, it holds that $\frac{(2-\eta)\alpha}{1+(2-\eta)\alpha} > \alpha$. Applying Lemma 2.1, we deduce that f_t has a density with furthermore $f_t \in B^s_{1,\infty}(\mathbb{R}^3)$ for any $s \in (0, s_\nu)$, where

$$s_\nu = \sup\left\{\frac{(2-\eta)\alpha}{1+(2-\eta)\alpha} - \alpha : \alpha \in (0,\nu], \eta \in (0,2)\right\}.$$

It is easily checked that s_ν is given by (1.7).

Point (ii). We next assume that $\gamma \in (-1,0]$ and that $\gamma + \nu > 0$. We consider $\alpha \in (0,\nu]$ and we apply Proposition 6.1(ii): for any $\eta \in (0, 2+\gamma/\nu)$, $\mathbb{E}[|V_t - V_t^\varepsilon|^\alpha] \le \mathbb{E}[|V_t - V_t^\varepsilon|^\nu]^{\alpha/\nu} \le C_{t,\eta}\varepsilon^{(2+\gamma/\nu-\eta)\alpha/\nu}$. Hence for all $\eta \in (0, 2+\gamma/\nu)$, all $\varepsilon \in (0, (t/2) \wedge 1)$,

$$I^\phi_{t,h} \le C_{t,\eta}\|\phi\|_{C^\alpha_b(\mathbb{R}^3)}\big[\varepsilon^{(2+\gamma/\nu-\eta)\alpha/\nu} + \varepsilon^{-1/\nu}|h|\big].$$

Choosing $\varepsilon = (1 \wedge (t/2))|h|^{\nu/(1+(2+\gamma/\nu-\eta)\alpha)}$, we obtain $I^\phi_{t,h} \le C_{t,\eta}\|\phi\|_{C^\alpha_b(\mathbb{R}^3)} \times |h|^{(2+\gamma/\nu-\eta)\alpha/(1+(2+\gamma/\nu-\eta)\alpha)}$. For $\alpha \in (0,\nu]$ small enough and $\eta \in (0, 2+2\gamma/\nu)$ small enough, it holds that $\frac{(2+\gamma/\nu-\eta)\alpha}{1+(2+\gamma/\nu-\eta)\alpha} > \alpha$ (because $2+\gamma/\nu > 1$). Applying Lemma 2.1, we deduce that f_t has a density with furthermore $f_t \in B^s_{1,\infty}(\mathbb{R}^3)$ for any $s \in (0, s_{\gamma,\nu})$, where

$$s_{\gamma,\nu} = \sup\left\{\frac{(2+\gamma/\nu-\eta)\alpha}{1+(2+\gamma/\nu-\eta)\alpha} - \alpha : \alpha \in (0,\nu], \eta \in (0, 2+\gamma/\nu)\right\}.$$

It is easily checked that $s_{\gamma,\nu}$ is given by (1.8).

Point (iii). In any case, we thus have $f_t \in B^s_{1,\infty}(\mathbb{R}^3)$ for some $s > 0$. This implies that $f_t \in L^p(\mathbb{R}^3)$ for all $p \in (1, 3/(3-s))$; see, for example, [33], Corollary 2(ii), page 36. The facts that $f_t \in \mathcal{P}_2(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$ for some $p > 1$ classically imply that $\int_{\mathbb{R}^3} f_t(v)|\log f_t(v)|\,dv < \infty$. $\square$

9. Existence of the Boltzmann process. It remains to prove Proposition 5.1. We have already checked very similar results in several closely related situations, but always with some restrictions (in the $2D$-case or for bounded velocity cross sections or assuming conditions on the initial data that guarantees uniqueness of the solution). We thus give a rather complete proof. Unfortunately, we have to treat separately the case of hard and moderately soft potentials: for hard potentials, we associate a Boltzmann process to any weak solution, while for moderately soft potentials, we can only build one Boltzmann process, which corresponds to one weak solution. Thus the proofs really differ.

9.1. *Moderately soft potentials.* In the whole subsection, we assume $(A_{\gamma,\nu})$ for some $\gamma \in (-1,0]$, $\nu \in (0,1)$, and we consider $f_0 \in \mathcal{P}_p(\mathbb{R}^3)$ for some $p > 2$. We want to prove Proposition 5.1(ii). Recall that L_B was defined in (1.5) and rewritten in (3.2).

DEFINITION 9.1. Let $B(|v - v_*|, \cos\theta)$ be a given cross section. A càdlàg adapted process $(V_t)_{t\geq 0}$ on some probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \Pr)$ is said to solve the martingale problem $\mathrm{MP}(f_0, B)$ if:

(a) $\mathcal{L}(V_0) = f_0$,

(b) for all $t \geq 0$, $\mathbb{E}[V_t] = \int_{\mathbb{R}^3} v f_0(dv)$ and $\mathbb{E}[|V_t|^2] = \int_{\mathbb{R}^3} |v|^2 f_0(dv)$,

(c) for all $\phi \in \mathrm{Lip}_b(\mathbb{R}^3)$, $(M_t^\phi)_{t\geq 0}$ is a $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \Pr)$-martingale, where $M_t^\phi := \phi(V_t) - \int_0^t \int_{\mathbb{R}^3} L_B\phi(V_s, v) f_s(dv)\, ds$ and where $f_t := \mathcal{L}(V_t)$.

The following remarks are classical.

REMARK 9.2. (i) A càdlàg adapted process $(V_t)_{t\geq 0}$ on some probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \Pr)$ is a solution to $\mathrm{MP}(f_0, B)$ if and only if it satisfies point (a) and (b) of the above definition and if there exists, on a possibly enlarged probability space, a $(\mathcal{F}_t)_{t\geq 0}$-Poisson measure $N(ds, dv, d\theta, d\varphi, du)$ on $[0,\infty) \times \mathbb{R}^3 \times (0, \pi/2] \times [0, 2\pi) \times [0, \infty)$ with intensity $ds f_s(dv) b(\theta)\, d\theta\, d\varphi\, du$ [where $f_t := \mathcal{L}(V_t)$] such that $(V_t)_{t\geq 0}$ solves (5.1).

(ii) If $(V_t)_{t\geq 0}$ solves $\mathrm{MP}(f_0, B)$ and if $f_t := \mathcal{L}(V_t)$, then $(f_t)_{t\geq 0}$ is a weak solution to (1.1) starting from f_0.

See, for example, Tanaka [35], Section 4, for (i). Point (ii) is obvious: use that for $\phi \in \mathrm{Lip}_b(\mathbb{R}^3)$, for $t \geq 0$, $\mathbb{E}[M_t^\phi] = \mathbb{E}[M_0^\phi] = \mathbb{E}[\phi(V_0)]$.

We start with the following statement.

REMARK 9.3. Let B be a cross section satisfying $(A_{\gamma,\nu})$ for some $\gamma \in (-1, 0]$, $\nu \in (0,1)$. For $k \geq 1$, define $B_k(|v - v_*|, \cos\theta)\sin\theta = (|v - v_*|^\gamma \wedge k) b(\theta) \mathbb{1}_{\{\theta > 1/k\}}$. There exists a (unique in law) solution to $(V_t^k)_{t\geq 0}$ to $\mathrm{MP}(f_0, B_k)$.

This result can be checked easily, because $\int_0^{\pi/2} b(\theta)\mathbb{1}_{\{\theta>1/k\}}\, d\theta < \infty$ and because $(|z|^\gamma \wedge k)$ is bounded. For example, one can use a perfect simulation algorithm, see, for example, [19] for a very similar result concerning the Smoluchowski equation.

Below, $\mathbb{D}([0,\infty), \mathbb{R}^3)$ stands for the set of $\mathbb{R}^3$-valued càdlàg functions, which we endow with the Skorokhod topology; see, for example, Jacod and Shiryaev [28].

LEMMA 9.4. *Adopt the assumptions and notation of Remark 9.3 and recall that* $f_0 \in \mathcal{P}_p(\mathbb{R}^3)$ *for some* $p > 2$:

(i) *for all* $T > 0$, $\sup_{k\geq 1} \mathbb{E}[\sup_{[0,T]} |V_t^k|^p] \leq C_{T,p}$;

(ii) *the family* $((V_t^k)_{t\geq 0})_{k\geq 1}$ *is tight in* $\mathbb{D}([0,\infty), \mathbb{R}^3)$ *and any limit process* $(V_t)_{t\geq 0}$ *satisfies* $\Pr(V_t \neq V_{t-}) = 0$ *for all* $t \geq 0$;

(iii) *any limit* $(V_t)_{t\geq 0}$ *solves* $\mathrm{MP}(f_0, B)$ *and verifies* $\mathbb{E}[\sup_{[0,T]} |V_t|^p] \leq C_{T,p}$ *for all* $T > 0$.

PROOF. We start with (i). Set $f_t^k := \mathcal{L}(V_t^k)$. As in Remark 9.2(i), there is a Poisson measure $N_k(ds, dv, d\theta, d\varphi, du)$ on $[0, \infty) \times \mathbb{R}^3 \times (0, \pi/2] \times [0, 2\pi) \times [0, \infty)$ with intensity $ds f_s^k(dv) b(\theta)\, d\theta\, d\varphi\, du$ such that

$$V_t^k = V_0^k + \int_0^t \int_{\mathbb{R}^3} \int_0^{\pi/2} \int_0^{2\pi} \int_0^\infty a(V_{s-}^k, v, \theta, \varphi) \times \mathbb{1}_{\{u \le |V_{s-}^k - v|^\gamma \wedge k\}} \times \mathbb{1}_{\{\theta > 1/k\}} N_k(ds, dv, d\theta, d\varphi, du).$$

Observe now that due to (3.4),

$$\begin{aligned} &\big||V_{s-}^k + a(V_{s-}^k, v, \theta, \varphi)|^p - |V_{s-}^k|^p\big| \\ &\quad \le C_p(|V_{s-}^k|^{p-1} + |a(V_{s-}^k, v, \theta, \varphi)|^{p-1})|a(V_{s-}^k, v, \theta, \varphi)| \\ &\quad \le C_p(1 + |V_{s-}^k|^{p-1} + |v|^{p-1})|V_{s-}^k - v|\theta \end{aligned}$$

so that, using the Itô formula for jump process (see, e.g., Jacod and Shiryaev [28], Theorem 4.57, page 56),

$$\begin{aligned} \sup_{[0,t]} |V_r^k|^p &\le |V_0^k|^p + C_p \int_0^t \int_{\mathbb{R}^3} \int_0^{\pi/2} \int_0^{2\pi} \int_0^\infty (1 + |V_{s-}^k|^{p-1} + |v|^{p-1})|V_{s-}^k - v|\theta \\ &\qquad \times \mathbb{1}_{\{u \le |V_{s-}^k - v|^\gamma\}} N_k(ds, dv, d\theta, d\varphi, du). \end{aligned}$$

Taking expectations and using that $\int_0^{\pi/2} \theta b(\theta)\, d\theta < \infty$ by $(A_{\gamma,\nu})$, we get

$$\begin{aligned} \mathbb{E}\Big(\sup_{[0,t]} |V_r^k|^p\Big) &\le \mathbb{E}(|V_0^k|^p) \\ &\quad + C_p \int_0^t \int_{\mathbb{R}^3} \mathbb{E}[(1 + |V_s^k|^{p-1} + |v|^{p-1})|V_s^k - v|^{1+\gamma}] f_s^k(dv)\, ds. \end{aligned}$$

Since $\gamma + 1 \in (0, 1]$ and $f_t^k = \mathcal{L}(V_t^k)$,

$$\begin{aligned} \mathbb{E}\Big(\sup_{[0,t]} |V_r^k|^p\Big) &\le \mathbb{E}(|V_0^k|^p) + C_p \int_0^t \int_{\mathbb{R}^3} \mathbb{E}[1 + |V_s^k|^p + |v|^p] f_s^k(dv)\, ds \\ &\le \mathbb{E}(|V_0^k|^p) + C_p \int_0^t \mathbb{E}[1 + |V_s^k|^p]\, ds. \end{aligned}$$

Finally, $\mathbb{E}(|V_0^k|^p) = m_p(f_0) < \infty$ does not depend on k and we conclude with the Grönwall lemma.

To check (ii), we use the Aldous [1] criterion (which shows both tightness and that any limit process has no fixed discontinuity); see also [28], page 321. Due to (i), it suffices that for all $T > 0$,

$$\lim_{\delta \to 0} \sup_{k \ge 1} \sup_{(S,S') \in \mathcal{S}_T(\delta)} \mathbb{E}[|V_{S'}^k - V_S^k|] = 0, \tag{9.1}$$

the set $\mathcal{S}_T(\delta)$ consisting of all pairs (S, S') of stopping times satisfying $0 \le S \le S' \le S + \delta \le T$. Let thus $T > 0$, $\delta > 0$, $(S, S') \in \mathcal{S}_T(\delta)$ and $k \ge 1$ be fixed. Using the s.d.e. satisfied by $(V_t^k)_{t\ge 0}$, we immediately get

$$\begin{aligned}\mathbb{E}[|V_{S'}^k - V_S^k|] \\ &\le \mathbb{E}\biggl[\int_S^{S+\delta}\int_{\mathbb{R}^3}\int_0^{\pi/2}\int_0^{2\pi}|a(V_s, v, \theta, \varphi)||V_s^k - v|^\gamma \, d\varphi b(\theta)\, d\theta\, d\varphi f_s^k(dv)\, ds\biggr].\end{aligned}$$

Using (3.4), that $\int_0^{\pi/2}\theta b(\theta)\,d\theta < \infty$ by $(A_{\gamma,\nu})$ and that $\int_{\mathbb{R}^3}|v|^{\gamma+1} f_s^k(dv) = \mathbb{E}[|V_s^k|^{\gamma+1}]$ is bounded for $s \in [0, T]$ due to (i), this gives

$$\begin{aligned}\mathbb{E}[|V_{S'}^k - V_S^k|] &\le C\mathbb{E}\biggl[\int_S^{S+\delta}\int_{\mathbb{R}^3}|V_s^k - v|^{\gamma+1} f_s^k(dv)\, ds\biggr] \\ &\le C_T\mathbb{E}\biggl[\int_S^{S+\delta}(1 + |V_s^k|)^{\gamma+1}\, ds\biggr].\end{aligned}$$

Finally,

$$\mathbb{E}[|V_{S'}^k - V_S^k|] \le C_T\mathbb{E}\Bigl[\delta \sup_{[0,T]}(1 + |V_s^k|)^{\gamma+1}\Bigr] \le C_T\delta$$

by point (i), whence (9.1).

We finally check (iii). Let thus $(V_t)_{t\ge 0}$ be the limit in law of a (not relabelled) subsequence of $(V_t^k)_{t\ge 0}$. Write $f_t := \mathcal{L}(V_t)$ and $f_t^k := \mathcal{L}(V_t^k)$. First, we obviously have $\mathcal{L}(V_0) = f_0$, since $\mathcal{L}(V_0^k) = f_0$ for all $k \ge 1$. We also have $\mathbb{E}[\sup_{[0,T]}|V_t|^p] \le C_{T,p}$ for all $T > 0$ thanks to point (i). Since we have $\mathbb{E}[V_t^k] = \int_{\mathbb{R}^3} v f_0(dv)$ and $\mathbb{E}[|V_t^k|^2] = \int_{\mathbb{R}^3}|v|^2 f_0(dv)$ for all $k \ge 1$ and all $t \ge 0$, we easily deduce from (i) (recall that $p > 2$) that $\mathbb{E}[V_t] = \int_{\mathbb{R}^3} v f_0(dv)$ and $\mathbb{E}[|V_t|^2] = \int_{\mathbb{R}^3}|v|^2 f_0(dv)$ for all $t \ge 0$. It only remains to check that for all $\phi \in \mathrm{Lip}_b(\mathbb{R}^3)$, $(M_t^\phi)_{t\ge 0}$ is a martingale, where $M_t^\phi := \phi(V_t) - \int_0^t\int_{\mathbb{R}^3} L_B\phi(V_s, v) f_s(dv)\, ds$. To do so, consider $n \ge 1$, $0 \le t_1 \le \cdots \le t_n \le s \le t$ and a family of continuous bounded functions $\phi_1, \dots, \phi_n$ on $\mathbb{R}^3$. We have to prove that $\mathbb{E}[\Psi_{B,f}(V)] = 0$, where, for $x \in \mathbb{D}([0,\infty), \mathbb{R}^3)$,

$$\Psi_{B,f}(x) = \prod_{i=1}^n \phi_i(x_{t_i})\biggl(\phi(x_t) - \phi(x_s) - \int_s^t\int_{\mathbb{R}^3} L_B\phi(x_r, v) f_r(dv)\, dr\biggr).$$

Since $(V_t^k)_{t\ge 0}$ solves $\mathrm{MP}(f_0, B_k)$, we know that $\mathbb{E}[\Psi_{B_k,f^k}(V^k)] = 0$, where Ψ_{B_k,f^k} is defined as $\Psi_{B,f}$, with L_B replaced by L_{B_k} and f_r replaced by f_r^k. Thus we just have to prove that $\lim_k \mathbb{E}[\Psi_{B_k,f^k}(V^k)] = \mathbb{E}[\Psi_{B,f}(V)]$. First, we know from Lemma 3.3 that $L_B\phi$ is continuous on $\mathbb{R}^3 \times \mathbb{R}^3$. We deduce that $\Psi_{B,f}$ is continuous at each $x \in \mathbb{D}([0,\infty), \mathbb{R}^3)$ such that x has no jump at $t_1, \dots, t_n, s, t$. But V has a.s. no jump at fixed points by (ii). Since V^k goes in law to V and since f_r^k tends weakly to f_r for each r (because V^k goes in law to V and since V has no fixed discontinuity), we deduce that $\Psi_{B,f^k}(V^k)$ goes in law to $\Psi_{B,f}(V)$. Using that the family $(\Psi_{B,f^k}(V^k))_{k\geq 1}$ is uniformly integrable [because $|\Psi_{B,f^k}(V^k)| \leq C_\Psi(1+\int_s^t\int_{\mathbb{R}^3}|V_r^k-v|^{\gamma+1}f_r^k(dv)\,dr) \leq C_{t,\Psi}(1+\sup_{[0,t]}|V_r^k|^{\gamma+1})$ and due to (i)], we conclude that $\lim_k \mathbb{E}[\Psi_{B,f^k}(V^k)] = \mathbb{E}[\Psi_{B,f}(V)]$. Hence it only remains to check that $\lim_k \mathbb{E}[|\Psi_{B_k,f^k}(V^k) - \Psi_{B,f^k}(V^k)|] = 0$. Using point (i) and that $|(L_B - L_{B_k})\phi(v,v_*)| \leq C_\phi k^{-\kappa}(1+|v|^2+|v_*|^2)$ for some $\kappa > 0$ (see the proof of Lemma 3.3), one easily concludes. □

We finally may give the following:

PROOF OF PROPOSITION 5.1(ii). We thus assume $(A_{\gamma,\nu})$ for some $\gamma \in (-1,0]$ and some $\nu \in (0,1)$ and consider $f_0 \in \mathcal{P}_p(\mathbb{R}^3)$ for some $p > 2$. We know from Lemma 9.4 that there exists a solution $(V_t)_{t\geq 0}$ to $\mathrm{MP}(f_0, B)$ and that $\mathbb{E}[\sup_{[0,T]}|V_t|^p] \leq C_{T,p}$ for all $T > 0$. For $t \geq 0$, set $f_t = \mathcal{L}(V_t)$. Then (5.2) obviously holds, since $m_p(f_t) = \mathbb{E}[|V_t|^p]$. Finally, Remark 9.2 ensures us that $(V_t)_{t\geq 0}$ solves (5.1) and that $(f_t)_{t\geq 0}$ is a weak solution to (1.1) starting from f_0. □

9.2. *Hard potentials*. We still have to prove Proposition 5.1(i). We use very similar arguments as in [18], Proof of Proposition 3.4, concerning the $3D$ Boltzmann equation without cutoff with velocity cross section $\min(|v-v_*|^\gamma, k)$.

In the whole subsection, we assume $(A_{\gamma,\nu})$ for some $\gamma \in (0,1)$, $\nu \in (0,1)$. A weak solution $(f_t)_{t\geq 0}$ to (1.1) starting from $f_0 \in \mathcal{P}_2(\mathbb{R}^3)$ satisfying (1.6) is fixed.

For $t \geq 0$, we introduce A_t defined, for $\phi \in \mathrm{Lip}_b(\mathbb{R}^3)$ and $v \in \mathbb{R}^3$, by [recall (1.5) and (3.2)]

$$
\begin{aligned}
A_t\phi(v) &= \int_{\mathbb{R}^3} L_B\phi(v,v_*) f_t(dv_*) \\
&= \int_{\mathbb{R}^3}\int_0^{\pi/2}\int_0^{2\pi} |v-v_*|^\gamma \\
&\qquad \times \big[\phi\big(v + a(v,v_*,\theta,\varphi)\big) - \phi(v)\big] b(\theta)\,d\varphi\,d\theta\, f_t(dv_*),
\end{aligned} \tag{9.2}
$$

where a was defined in (3.1). We define similarly, for $k \geq 1$, setting $H_k(v) = \frac{|v|\wedge k}{|v|} v$,

$$
A_t^k\phi(v) = \int_{\mathbb{R}^3}\int_0^{\pi/2}\int_0^{2\pi} |H_k(v)-v_*|^\gamma \times \big[\phi\big(v + a\big(H_k(v),v_*,\theta,\varphi\big)\big) - \phi(v)\big] b(\theta)\,d\varphi\,d\theta\, f_t(dv_*).
$$

DEFINITION 9.5. (i) Let $t_0 \geq 0$ and $\mu \in \mathcal{P}(\mathbb{R}^3)$ be fixed. A càdlàg adapted process $(V_t)_{t\geq t_0}$ on some probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathrm{Pr})$ solves the martingale problem $\mathrm{MP}(\mu, t_0, (A_t)_{t\geq t_0}, C_c^1(\mathbb{R}^3))$ if $\mathcal{L}(V_{t_0}) = \mu$ and if for all $\phi \in C_c^1(\mathbb{R}^3)$, $(M_t^\phi)_{t\geq t_0}$ is a $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq t_0}, \mathrm{Pr})$-martingale, where $M_t^\phi := \phi(V_t) - \int_{t_0}^t A_s\phi(V_s)\,ds$.

(ii) For $t_0 \geq 0$, $\mu \in \mathcal{P}(\mathbb{R}^3)$ and $k \geq 1$, the martingale problem MP$(\mu, t_0, (A_t^k)_{t\geq t_0}, C_c^1(\mathbb{R}^3))$ is defined similarly.

The following remark is classical; see, for example, Tanaka [35], Section 4.

REMARK 9.6. (i) A process $(V_t)_{t\geq t_0}$ on some probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \Pr)$ is solution to MP$(\mu, t_0, (A_t)_{t\geq t_0}, C_c^1(\mathbb{R}^3))$ if and only if $\mathcal{L}(V_{t_0}) = \mu$ and if there exists, on a possibly enlarged probability space, a $(\mathcal{F}_t)_{t\geq 0}$-Poisson measure $N(ds, dv, d\theta, d\varphi, du)$ on $[0,\infty)\times\mathbb{R}^3\times(0,\pi/2]\times[0,2\pi)\times[0,\infty)$ with intensity $ds f_s(dv) b(\theta)\, d\theta\, d\varphi\, du$ such that for all $t \geq t_0$,

$$V_t = V_{t_0} + \int_{t_0}^t \int_{\mathbb{R}^3} \int_0^{\pi/2} \int_0^{2\pi} \int_0^\infty a(V_{s-}, v, \theta, \varphi) \times \mathbb{1}_{\{u \leq |V_{s-} - v|^\gamma\}} N(ds, dv, d\theta, d\varphi, du). \tag{9.3}$$

(ii) Similarly, a process $(V_t^k)_{t\geq t_0}$ solves MP$(\mu, t_0, (A_t^k)_{t\geq t_0}, C_c^1(\mathbb{R}^3))$ if and only if $\mathcal{L}(V_{t_0}) = \mu$ and if it solves

$$V_t^k = V_{t_0} + \int_{t_0}^t \int_{\mathbb{R}^3} \int_0^{\pi/2} \int_0^{2\pi} \int_0^\infty a(H_k(V_{s-}^k), v, \theta, \varphi) \times \mathbb{1}_{\{u \leq |H_k(V_{s-}^k) - v|^\gamma\}} N(ds, dv, d\theta, d\varphi, du). \tag{9.4}$$

We start with the following statement.

REMARK 9.7. For any $t_0 \geq 0$, any $\mu \in \mathcal{P}_2(\mathbb{R}^3)$ and any $k \geq 1$, there exists a unique (in law) solution $(V_t^k)_{t\geq t_0}$ to MP$(\mu, t_0, (A_t^k)_{t\geq t_0}, C_c^1(\mathbb{R}^3))$.

This can be proved exactly as in [18], Proof of Proposition 3.4, Steps 1 to 7. We have checked all the details and omit the proof. Let us only mention that we have to use the following estimates: (i) $\int_{\mathbb{R}^3} f_s(dv_*)(|H_k(v) - v_*|^\gamma + |H_k(v) - v_*|^{\gamma+1}) \leq C_k$, (ii) $\int_{\mathbb{R}^3} f_s(dv_*)|H_k(v) - v_*|^\gamma |H_k(v) - H_k(\tilde{v})| \leq C_k|v - \tilde{v}|$, (iii) $\int_{\mathbb{R}^3} f_s(dv_*)|H_k(v) - v_*|||H_k(v) - v_*|^\gamma - |H_k(\tilde{v}) - v_*|^\gamma| \leq C_k|v - \tilde{v}|$. Points (i) and (ii) are easily checked and use only that $H_k \in \mathrm{Lip}_b(\mathbb{R}^3)$ and that $\int_{\mathbb{R}^3} f_s(dv_*)(1 + |v_*|^\gamma + |v_*|^{\gamma+1}) \leq \int_{\mathbb{R}^3} f_s(dv_*)(3 + |v_*|^2) \leq C$ by (1.3). Point (iii) uses additionally (6.2).

To make tend k to infinity, we will need the following uniform (in k) moment estimates.

LEMMA 9.8. *Consider the solution $(V_t^k)_{t\geq t_0}$ to* MP$(\mu, t_0, (A_t^k)_{t\geq t_0}, C_c^1(\mathbb{R}^3))$, *for some $t_0 > 0$ and some $\mu \in \mathcal{P}_2(\mathbb{R}^3)$. For any $T > t_0$, we have*

(i) $\sup_{[t_0,T]} \mathbb{E}[|V_t^k|^2] \leq C_{t_0,T,\mu}$,

(ii) $\mathbb{E}[\sup_{[t_0,T]} |V_t^k|] \leq C_{t_0,T,\mu}$.

PROOF. We start with (i). Using (9.4), the Itô formula for jump processes (see, e.g., Jacod and Shiryaev [28], Theorem 4.57, page 56), taking expectations and integrating in u, we get, for $t \geq t_0$,

$$\begin{aligned}\mathbb{E}[|V_t^k|^2] = \mathbb{E}[|V_{t_0}^k|^2] \\ + \mathbb{E}\Big[\int_{t_0}^t \int_{\mathbb{R}^3} \int_0^{\pi/2} \int_0^{2\pi} \big(|a(H_k(V_s^k), v, \theta, \varphi)|^2 \\ + 2\langle V_s^k, a(H_k(V_s^k), v, \theta, \varphi)\rangle\big) \\ \times |H_k(V_s^k) - v|^\gamma b(\theta)\, d\varphi\, d\theta f_s(dv)\, ds\Big].\end{aligned}$$

After some explicit computation using (3.1) and (3.4), this yields

$$\begin{aligned}\mathbb{E}[|V_t^k|^2] = \int_{\mathbb{R}^3} |v|^2 \mu(dx) \\ + \mathbb{E}\Big[\int_{t_0}^t \int_{\mathbb{R}^3} \int_0^{\pi/2} \big(|H_k(V_s^k) - v|^2 - 2\langle V_s^k, H_k(V_s^k) - v\rangle\big) \\ \times \pi |H_k(V_s^k) - v|^\gamma (1 - \cos\theta) b(\theta)\, d\theta f_s(dv)\, ds\Big].\end{aligned}$$

Observe that $(1 - \cos\theta)b(\theta)$ is integrable due to $(A_{\gamma,\nu})$. Next, we have $\langle V_s^k, H_k(V_s^k)\rangle \geq |H_k(V_s^k)|^2$ and $|H_k(V_s^k)| \leq |V_s^k|$, from which we deduce $|H_k(V_s^k) - v|^2 - 2\langle V_s^k, H_k(V_s^k) - v\rangle \leq |v|^2 + 2\langle V_s^k - H_k(V_s^k), v\rangle \leq |v|^2 + 2|V_s^k||v|$. We also have $|H_k(V_s^k) - v|^\gamma \leq C(1 + |H_k(V_s^k)| + |v|) \leq C(1 + |V_s^k| + |v|)$. We finally find that $(|H_k(V_s^k) - v|^2 - 2\langle V_s^k, H_k(V_s^k) - v\rangle)|H_k(V_s^k) - v|^\gamma \leq C(|v|^2 + |V_s^k||v|)(1 + |V_s^k| + |v|) \leq C(1 + |v|^3)(1 + |V_s^k|^2)$. Thus

$$\begin{aligned}\mathbb{E}[|V_t^k|^2] &\leq C_\mu + C\mathbb{E}\Big[\int_{t_0}^t \int_{\mathbb{R}^3} (1 + |v|^3)(1 + |V_s^k|^2) f_s(dv)\, ds\Big] \\ &\leq C_\mu + C_{t_0} \int_{t_0}^t \mathbb{E}[1 + |V_s^k|^2]\, ds.\end{aligned}$$

We used that, since $t_0 > 0$, $\sup_{t \geq t_0} m_3(f_s) < \infty$ by (1.6). The Grönwall lemma thus implies $\sup_{[t_0,T]} \mathbb{E}[|V_t^k|^2] \leq C_{t_0,T,\mu}$ as desired.

Point (ii) easily follows, since

$$\begin{aligned}\mathbb{E}\Big[\sup_{[t_0,T]} |V_s^k|\Big] \leq \mathbb{E}[|V_{t_0}^k|] \\ + \mathbb{E}\Big[\int_{t_0}^T \int_{\mathbb{R}^3} \int_0^{\pi/2} \int_0^{2\pi} |a(H_k(V_s^k), v, \theta, \varphi)||H_k(V_s^k) - v|^\gamma \\ \times b(\theta)\, d\varphi\, d\theta f_s(dv)\, ds\Big],\end{aligned}$$

so that using (3.4) and that $\theta b(\theta)$ is integrable by $(A_{\gamma,\nu})$,

$$\mathbb{E}\Big[\sup_{[t_0,T]}|V_s^k|\Big] \le \int_{\mathbb{R}^3}|v|\mu(dv) + C\mathbb{E}\Big[\int_{t_0}^T\int_{\mathbb{R}^3}|H_k(V_s^k)-v|^{\gamma+1}f_s(dv)\,ds\Big]$$
$$\le C_\mu + C\int_{t_0}^T\int_{\mathbb{R}^3}\big(1+\mathbb{E}\big[|V_s^k|^2\big]+|v|^2\big)f_s(dv)\,ds \le C_{t_0,T,\mu}$$

by (i) and (1.3). □

We deduce the well-posedness of $\mathrm{MP}(\mu, t_0, (A_t)_{t\ge t_0}, C_c^1(\mathbb{R}^3))$ when $t_0 > 0$.

LEMMA 9.9. *Let $t_0 > 0$ and $\mu \in \mathcal{P}_2(\mathbb{R}^3)$ be fixed. There exists a unique* (*in law*) *solution $(V_t)_{t\ge t_0}$ to* $\mathrm{MP}(\mu, t_0, (A_t)_{t\ge t_0}, C_c^1(\mathbb{R}^3))$.

PROOF. We only sketch the proof, since it is tedious but rather standard.

Uniqueness. Consider $(V_t)_{t\ge t_0}$ solving $\mathrm{MP}(\mu, t_0, (A_t)_{t\ge t_0}, C_c^1(\mathbb{R}^3))$. Introduce, for $k \ge 1$, $\tau_k = \inf\{t \ge t_0 : |V_t| \ge k\}$ (with the convention that $\tau_k = t_0$ if this set is empty). Since $(V_t)_{t\ge t_0}$ is càdlàg by assumption, it is locally bounded, whence $\tau_k \to \infty$ a.s. as $k \to \infty$. For $k \ge 1$, observe that V solves $\mathrm{MP}(\mu, t_0, (A_t^k)_{t\ge t_0}, C_c^1(\mathbb{R}^3))$ until τ_k (because $v = H_k(v)$ if $|v| \le k$ and because $|V_t| < k$ for all $t \in [t_0, \tau_k)$). By uniqueness for $\mathrm{MP}(\mu, t_0, (A_t^k)_{t\ge t_0}, C_c^1(\mathbb{R}^3))$, we deduce that for any $T > 0$, any $k \ge 1$, the law of $(V_t)_{t\in[t_0,T]}$ knowing $\tau_k > T$ is entirely determined. Using that $\tau_k \to \infty$ a.s. as $k \to \infty$, we easily conclude.

Existence. One way to prove such an existence result is to use a tightness argument as in Lemma 9.4 above. Another way is the following. Consider $T > t_0$ arbitrarily large. Roughly, if k is very large, then a solution $(V_t^k)_{t\ge t_0}$ to $\mathrm{MP}(\mu, t_0, (A_t^k)_{t\ge t_0}, C_c^1(\mathbb{R}^3))$ will not reach k before T with a high probability [due to Lemma 9.8(ii)], so that it actually also solves $\mathrm{MP}(\mu, t_0, (A_t)_{t\ge t_0}, C_c^1(\mathbb{R}^3))$ during $[t_0, T]$ [because as previously, $v = H_k(v)$ for $|v| \le k$]. □

The last preliminary will be useful to show that the law of V_t is indeed f_t.

LEMMA 9.10. *Let $t_0 > 0$ and $\mu \in \mathcal{P}(\mathbb{R}^3)$ be fixed. There exists at most one family $(\mu_t)_{t\ge 0} \subset \mathcal{P}(\mathbb{R}^3)$ such that for all $\phi \in C_c^1(\mathbb{R}^3)$, all $t \ge t_0$,*

$$\int_{\mathbb{R}^3}\phi(v)\mu_t(dv) = \int_{\mathbb{R}^3}\phi(v)\mu(dv) + \int_{t_0}^t\int_{\mathbb{R}^3}A_s\phi(v)\mu_s(dv)\,ds.$$

PROOF. This will follow from Horowitz and Karandikar [26], Theorem B1, if we check the following points:

(a) $C_c^1(\mathbb{R}^3)$ is dense in $C_0(\mathbb{R}^3)$ for the uniform convergence topology;
(b) $(t, v) \mapsto A_t\phi(v)$ is measurable for all $\phi \in C_c^1(\mathbb{R}^3)$;
(c) for each $t \ge 0$, A_t satisfies the maximum principle;

(d) there exists a countable subset $\{\phi_k\} \subset C_c^1(\mathbb{R}^3)$ such that for all $t \geq t_0$, the closure of $\{(\phi_k, A_t\phi_k) : k \geq 1\} \subset C_c^1(\mathbb{R}^3)$ for the bounded-pointwise convergence is $\{(\phi, A_t\phi) : \phi \in C_c^1(\mathbb{R}^3)\}$;

(e) for all $v_0 \in \mathbb{R}^3$, $\mathrm{MP}(\delta_{v_0}, t_0, (A_t)_{t\geq t_0}, C_c^1(\mathbb{R}^3))$ is well posed.

First, (a) and (b) are clear, and (e) follows from Lemma 9.9. Next, (c) is obvious from (9.2): if ϕ attains its maximum at some $v_0 \in \mathbb{R}^3$, $A_t\phi(v_0) \leq 0$. The only delicate point is (d). Consider a countable family $\{\phi_k\}_{k\geq 1} \subset C_c^1(\mathbb{R}^3)$ dense in $C_c^1(\mathbb{R}^3)$ in the following sense: for all $\phi \in C_c^1(\mathbb{R}^3)$ such that $\operatorname{Supp}\phi \subset \mathcal{B}(0, R)$, there is a subsequence ϕ_{k_n} such that $\operatorname{Supp}\phi_{k_n} \subset \mathcal{B}(0, R+1)$ and $\|\phi - \phi_{k_n}\|_{L^\infty(\mathbb{R}^3)} + \|\nabla(\phi - \phi_{k_n})\|_{L^\infty(\mathbb{R}^3)} \to 0$. We have to prove that $(\phi_{k_n}, A_t\phi_{k_n})$ goes to $(\phi, A_t\phi)$ bounded-pointwise. We obviously have that $\phi_{k_n} \to \phi$ bounded-pointwise. An immediate computation using (3.4), ($A_{\gamma,\nu}$) and (1.3) shows that for all $v \in \mathbb{R}^3$, $|A_t\phi_{k_n}(v) - A_t\phi(v)| \leq C\|\nabla(\phi - \phi_{k_n})\|_{L^\infty(\mathbb{R}^3)} \int_{\mathbb{R}^3} \theta|v - v_*|^{\gamma+1} b(\theta)\, d\theta f_t(dv_*) \leq C\|\nabla(\phi - \phi_{k_n})\|_{L^\infty(\mathbb{R}^3)}(1 + |v|^2) \to 0$. It only remains to prove that $\sup_{v\in\mathbb{R}^3} \sup_{n\geq 1} |A_t\phi_{k_n}(v)| < \infty$.

To this end, it suffices to check that for $\phi \in C_c^1(\mathbb{R}^3)$ with $\|\phi\|_{L^\infty(\mathbb{R}^3)} + \|\nabla\phi\|_{L^\infty(\mathbb{R}^3)} \leq K$ and $\operatorname{Supp}\phi \subset \mathcal{B}(0, R)$, we have $\|A_t\phi\|_{L^\infty(\mathbb{R}^3)} \leq C_{K,R}$.

First consider $v \in \mathbb{R}^3$ such that $|v| \leq 5R$. Then using (3.4), ($A_{\gamma,\nu}$) and (1.3), we obtain $|A_t\phi(v)| \leq K \int_{\mathbb{R}^3} \theta|v - v_*|^{\gamma+1} b(\theta)\, d\theta f_t(dv_*) \leq CK(1 + |v|^{\gamma+1}) \leq CK(1 + R^{\gamma+1})$.

Next, consider $v \in \mathbb{R}^3$ such that $|v| \geq 5R$. Then we have $\phi(v) = 0$, so that $|\phi(v + a(v, v_*, \theta, \varphi)) - \phi(v)| \leq K|a(v, v_*, \theta, \varphi)|\mathbb{1}_{\{|v+a(v,v_*,\theta,\varphi)|<R\}}$. But $|v + a(v, v_*, \theta, \varphi)| < R$ implies $|a(v, v_*, \theta, \varphi)| > |v| - R \geq 4|v|/5$, whence [recall (3.4)] $\sqrt{1 - \cos\theta}|v - v_*| > 4\sqrt{2}|v|/5$, from which (recall that $\theta \in (0, \pi/2]$) $|v| + |v_*| > 4\sqrt{2}|v|/5$ and finally $|v_*| > (4\sqrt{2}/5 - 1)|v| > |v|/10$. We thus get $|\phi(v + a(v, v_*, \theta, \varphi)) - \phi(v)| \leq K|a(v, v_*, \theta, \varphi)|\mathbb{1}_{\{|v_*|>|v|/10\}} \leq K\theta|v - v_*|\mathbb{1}_{\{|v_*|>|v|/10\}}$ by (3.4), whence

$$|A_t\phi(v)| \leq K \int_{\mathbb{R}^3} \int_0^{\pi/2} \int_0^{2\pi} \theta|v - v_*|^{1+\gamma} \mathbb{1}_{\{|v_*|>|v|/10\}} b(\theta)\, d\theta\, d\varphi f_t(dv_*).$$

Using ($A_{\gamma,\nu}$) and then (1.3), we deduce that

$$\begin{aligned}|A_t\phi(v)| &\leq K \int_{\mathbb{R}^3} |v - v_*|^{1+\gamma} \mathbb{1}_{\{|v_*|>|v|/10\}} f_t(dv_*)\\ &\leq K \int_{\mathbb{R}^3} (11|v_*|)^{\gamma+1} f_t(dv_*) \leq CK.\end{aligned}$$

We finally have checked that for any $v \in \mathbb{R}^3$, $|A_t\phi(v)| \leq CK(1 + R^{\gamma+1})$. □

We finally may give the

PROOF OF PROPOSITION 5.1(i). We divide the proof into two steps.

Step 1. For $t_0 > 0$, let $(V_t)_{t\geq t_0}$ be the unique (in law) solution to $\mathrm{MP}(f_{t_0}, t_0, (A_t)_{t\geq t_0}, C_c^1(\mathbb{R}^3))$. The aim of this step is to prove that $\mathcal{L}(V_t) = f_t$ for all $t \geq t_0$. To this end, put $\mu_t = \mathcal{L}(V_t)$. For any $\phi \in C_c^1(\mathbb{R}^3)$ and any $t \geq t_0$, we know that $\phi(V_t) - \int_{t_0}^t A_s\phi(V_s)\,ds$ is a martingale, whence $\mathbb{E}[\phi(V_t) - \int_{t_0}^t A_s\phi(V_s)\,ds] = \mathbb{E}[\phi(V_{t_0})]$, which yields

$$\int_{\mathbb{R}^3} \phi(v)\mu_t(dv) = \int_{\mathbb{R}^3} \phi(v) f_{t_0}(dv) + \int_{t_0}^t \int_{\mathbb{R}^3} A_s\phi(v)\mu_s(dv)\,ds.$$

But $(f_t)_{t\geq 0}$ is a weak solution to (1.1), whence, for $\phi \in C_c^1(\mathbb{R}^3) \subset \mathrm{Lip}_b(\mathbb{R}^3)$ and $t \geq t_0$,

$$\begin{aligned}\int_{\mathbb{R}^3} \phi(v) f_t(dv) &= \int_{\mathbb{R}^3} \phi(v) f_{t_0}(dv) + \int_{t_0}^t \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} L_B\phi(v, v_*) f_s(dv_*) f_s(dv)\,ds \\ &= \int_{\mathbb{R}^3} \phi(v) f_{t_0}(dv) + \int_{t_0}^t \int_{\mathbb{R}^3} A_s\phi(v) f_s(dv)\,ds.\end{aligned}$$

Lemma 9.10 implies that $\mu_t = f_t$ for all $t \geq t_0$.

Step 2. We deduce from Step 1 that if $(V_t^{t_0})_{t\geq t_0}$ solves $\mathrm{MP}(f_{t_0}, t_0, (A_t)_{t\geq t_0}, C_c^1(\mathbb{R}^3))$, then for any $t_1 > t_0$, $(V_t^{t_0})_{t\geq t_1}$ solves $\mathrm{MP}(f_{t_1}, t_1, (A_t)_{t\geq t_1}, C_c^1(\mathbb{R}^3))$. This compatibility property [recall that uniqueness holds for $\mathrm{MP}(f_{t_0}, t_0, (A_t)_{t\geq t_0}, C_c^1(\mathbb{R}^3))$ for any $t_0 > 0$ by Lemma 9.9] implies, by the Kolmogorov theorem, that there exists a process $(V_t)_{t\geq 0}$ such that for all $t_0 > 0$, $(V_t)_{t\geq t_0}$ solves $\mathrm{MP}(f_{t_0}, t_0, (A_t)_{t\geq t_0}, C_c^1(\mathbb{R}^3))$. In particular, we have $\mathcal{L}(V_t) = f_t$ for all $t > 0$ by Step 1. Since now f_{t_0} tends weakly to f_0 as $t_0 \to 0$ [use, e.g., Lemma 3.3], we easily deduce that $(V_t)_{t\geq 0}$ solves $\mathrm{MP}(f_0, 0, (A_t)_{t\geq 0}, C_c^1(\mathbb{R}^3))$. Due to Remark 9.6(i), this ends the proof. □

REFERENCES

[1] ALDOUS, D. (1978). Stopping times and tightness. *Ann. Probability* **6** 335–340. MR0474446

[2] ALEXANDRE, R. (2009). A review of Boltzmann equation with singular kernels. *Kinet. Relat. Models* **2** 551–646. MR2556715

[3] ALEXANDRE, R., DESVILLETTES, L., VILLANI, C. and WENNBERG, B. (2000). Entropy dissipation and long-range interactions. *Arch. Ration. Mech. Anal.* **152** 327–355. MR1765272

[4] ALEXANDRE, R. and ELSAFADI, M. (2009). Littlewood–Paley theory and regularity issues in Boltzmann homogeneous equations. II. Non cutoff case and non Maxwellian molecules. *Discrete Contin. Dyn. Syst.* **24** 1–11. MR2476677

[5] BALLY, V. and FOURNIER, N. (2011). Regularization properties of the 2D homogeneous Boltzmann equation without cutoff. *Probab. Theory Related Fields* **151** 659–704. MR2851696

[6] CARLEMAN, T. (1933). Sur la théorie de l'équation intégrodifférentielle de Boltzmann. *Acta Math.* **60** 91–146. MR1555365

[7] CERCIGNANI, C. (1988). *The Boltzmann Equation and Its Applications*. *Applied Mathematical Sciences* **67**. Springer, New York. MR1313028

[8] CHEN, Y. and HE, L. (2011). Smoothing estimates for Boltzmann equation with full-range interactions: Spatially homogeneous case. *Arch. Ration. Mech. Anal.* **201** 501–548. MR2820356

[9] DEBUSSCHE, A. and ROMITO, M. (2014). Existence of densities for the 3D Navier–Stokes equations driven by Gaussian noise. *Probab. Theory Related Fields* **158** 575–596. MR3176359

[10] DESVILLETTES, L. (1993). Some applications of the method of moments for the homogeneous Boltzmann and Kac equations. *Arch. Rational Mech. Anal.* **123** 387–404. MR1233644

[11] DESVILLETTES, L. (1995). About the regularizing properties of the non-cut-off Kac equation. *Comm. Math. Phys.* **168** 417–440. MR1324404

[12] DESVILLETTES, L. (1997). Regularization properties of the 2-dimensional non-radially symmetric non-cutoff spatially homogeneous Boltzmann equation for Maxwellian molecules. *Transport Theory Statist. Phys.* **26** 341–357. MR1475459

[13] DESVILLETTES, L. and MOUHOT, C. (2009). Stability and uniqueness for the spatially homogeneous Boltzmann equation with long-range interactions. *Arch. Ration. Mech. Anal.* **193** 227–253. MR2525118

[14] DESVILLETTES, L. and WENNBERG, B. (2004). Smoothness of the solution of the spatially homogeneous Boltzmann equation without cutoff. *Comm. Partial Differential Equations* **29** 133–155. MR2038147

[15] ELMROTH, T. (1983). Global boundedness of moments of solutions of the Boltzmann equation for forces of infinite range. *Arch. Rational Mech. Anal.* **82** 1–12. MR0684411

[16] FOURNIER, N. (2000). Existence and regularity study for two-dimensional Kac equation without cutoff by a probabilistic approach. *Ann. Appl. Probab.* **10** 434–462. MR1768239

[17] FOURNIER, N. (2001). Strict positivity of the solution to a 2-dimensional spatially homogeneous Boltzmann equation without cutoff. *Ann. Inst. Henri Poincaré Probab. Stat.* **37** 481–502. MR1876840

[18] FOURNIER, N. (2006). Uniqueness for a class of spatially homogeneous Boltzmann equations without angular cutoff. *J. Stat. Phys.* **125** 927–946. MR2283785

[19] FOURNIER, N. and GIET, J.-S. (2004). Exact simulation of nonlinear coagulation processes. *Monte Carlo Methods Appl.* **10** 95–106. MR2096252

[20] FOURNIER, N. and GUÉRIN, H. (2008). On the uniqueness for the spatially homogeneous Boltzmann equation with a strong angular singularity. *J. Stat. Phys.* **131** 749–781. MR2398952

[21] FOURNIER, N. and MÉLÉARD, S. (2002). A stochastic particle numerical method for 3D Boltzmann equations without cutoff. *Math. Comp.* **71** 583–604 (electronic). MR1885616

[22] FOURNIER, N. and MOUHOT, C. (2009). On the well-posedness of the spatially homogeneous Boltzmann equation with a moderate angular singularity. *Comm. Math. Phys.* **289** 803–824. MR2511651

[23] FOURNIER, N. and PRINTEMS, J. (2010). Absolute continuity for some one-dimensional processes. *Bernoulli* **16** 343–360. MR2668905

[24] GAMBA, I. M., PANFEROV, V. and VILLANI, C. (2009). Upper Maxwellian bounds for the spatially homogeneous Boltzmann equation. *Arch. Ration. Mech. Anal.* **194** 253–282. MR2533928

[25] GRAHAM, C. and MÉLÉARD, S. (1999). Existence and regularity of a solution of a Kac equation without cutoff using the stochastic calculus of variations. *Comm. Math. Phys.* **205** 551–569. MR1711273

[26] HOROWITZ, J. and KARANDIKAR, R. L. (1990). Martingale problems associated with the Boltzmann equation. In *Seminar on Stochastic Processes*, 1989 (*San Diego*, *CA*, 1989). *Progress in Probability* **18** 75–122. Birkhäuser, Boston, MA. MR1042343

[27] HUO, Z., MORIMOTO, Y., UKAI, S. and YANG, T. (2008). Regularity of solutions for spatially homogeneous Boltzmann equation without angular cutoff. *Kinet. Relat. Models* **1** 453–489. MR2425608

[28] JACOD, J. and SHIRYAEV, A. N. (2003). *Limit Theorems for Stochastic Processes*, 2nd ed. *Grundlehren der Mathematischen Wissenschaften* [*Fundamental Principles of Mathematical Sciences*] **288**. Springer, Berlin. MR1943877

[29] LU, X. and MOUHOT, C. (2012). On measure solutions of the Boltzmann equation, part I: Moment production and stability estimates. *J. Differential Equations* **252** 3305–3363. MR2871802

[30] MOUHOT, C. (2005). Quantitative lower bounds for the full Boltzmann equation. I. Periodic boundary conditions. *Comm. Partial Differential Equations* **30** 881–917. MR2153518

[31] MOUHOT, C. and VILLANI, C. (2004). Regularity theory for the spatially homogeneous Boltzmann equation with cut-off. *Arch. Ration. Mech. Anal.* **173** 169–212. MR2081030

[32] PULVIRENTI, A. and WENNBERG, B. (1997). A Maxwellian lower bound for solutions to the Boltzmann equation. *Comm. Math. Phys.* **183** 145–160. MR1461954

[33] RUNST, T. and SICKEL, W. (1996). *Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Equations. de Gruyter Series in Nonlinear Analysis and Applications* **3**. de Gruyter, Berlin. MR1419319

[34] SCHILLING, R. L., SZTONYK, P. and WANG, J. (2012). Coupling property and gradient estimates of Lévy processes via the symbol. *Bernoulli* **18** 1128–1149. MR2995789

[35] TANAKA, H. (1978/79). Probabilistic treatment of the Boltzmann equation of Maxwellian molecules. *Z. Wahrsch. Verw. Gebiete* **46** 67–105. MR0512334

[36] TOSCANI, G. and VILLANI, C. (1999). Probability metrics and uniqueness of the solution to the Boltzmann equation for a Maxwell gas. *J. Stat. Phys.* **94** 619–637. MR1675367

[37] VILLANI, C. (1998). On a new class of weak solutions to the spatially homogeneous Boltzmann and Landau equations. *Arch. Rational Mech. Anal.* **143** 273–307. MR1650006

[38] VILLANI, C. (2002). A review of mathematical topics in collisional kinetic theory. In *Handbook of Mathematical Fluid Dynamics, Vol. I* 71–305. North-Holland, Amsterdam. MR1942465

[39] ZHANG, X. and ZHANG, X. (2006). Supports of measure solutions for spatially homogeneous Boltzmann equations. *J. Stat. Phys.* **124** 485–495. MR2264616

LABORATOIRE DE PROBABILITÉS ET MODÈLES ALÉATOIRES
UMR 7599
UNIVERSITÉ PIERRE ET MARIE CURIE
CASE 188, 4 PLACE JUSSIEU
F-75252 PARIS CEDEX 5
FRANCE
E-MAIL: nicolas.fournier@upmc.fr