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A GENERALIZED BACKWARD SCHEME FOR SOLVING NONMONOTONIC STOCHASTIC RECURSIONS

BY P. MOYAL

Université de Technologie de Compiègne

We propose an explicit construction of a stationary solution for a stochastic recursion of the form $X \circ \theta = \varphi(X)$ on a partially-ordered Polish space, when the monotonicity of φ is not assumed. Under certain conditions, we show that an extension of the original probability space exists, on which a solution is well defined, and construct explicitly this extension using a randomized contraction technique. We then provide conditions for the existence of a solution on the original space. We finally apply these results to the stability study of two nonmonotonic queuing systems.

1. Introduction. The evolution of a number of dynamical systems depends on punctual, random perturbations which may be assumed to be time-stationary. In such cases, the state of the system can be described in discrete time by a random sequence of the form

$$W_{n+1} = \varphi_n(W_n), \qquad n \geq 0.$$

In the general framework (of crucial interest in the applications), where the sequence of random mappings $\{\varphi_n\}$ driving the recursion is time-stationary but not necessarily independent, we adopt an ergodic-theoretical approach to formally address the central question of *stability*, that is, of existence of an equilibrium state for the recursion.

Since the pioneering works of Loynes (see [16] and among others, [3] and [11]), it is well known that a stationary state exists whenever the random maps φ_n enjoy mild properties, such as (i) monotonicity and continuity, as assumed by Loynes, or (ii) some regenerative property: in many cases, Borovkov's theory of renovating events (introduced in [8]) handles the questions of uniqueness of the stationary state, and coupling with this state; see also [10] and [7]. Notice that the latter framework is also suitable, under certain conditions; for random sequences that are not stochastically recursive, see [7] and [14].

However, many models do not verify such mild assumptions. A classical example is the so-called *Loss queueing system*, addressed in Section 6. It is easy to construct cases in which either none, or several stationary states may exist. For

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this particular model, Neveu [19] and Flipo [12, 13] have shown that the stability problem can be solved, at least on a larger probability space. Their constructions, inspired by skew-product methods used to solve ordinary or partial differential equations, lead to an extension (also called *enrichment*) of the original probability space on which a stationary solution exists; see also Lisek [15] for related developments.

More recently, Anantharam and Konstantopoulos [1, 2] have shown that such extensions exist under various assumptions on the statistics of the recursion, using an approach based on tightness properties. The construction presented in [1, 2], although more general, is less tractable in that the probability measure on the extension (termed *weak* solution) is identified as a weak limit, and is not explicitly defined.

Following the same direction, we aim to identify the conditions of existence of such extensions for a more general class of models. We also propose, under such conditions, a constructive scheme of the enriched probability space; see Theorem 1 below. Returning to the original problem, our framework appears particularly adequate, as it leads to several sufficient conditions of existence on the original probability space; see Theorem 3. Then Loynes's theorem and Borovkov and Foss's theory of renovating events turn out to be particular cases of our result; see Section 4.3. In fact, the three approaches all rely on the same time reversal technique (usually termed *backward scheme*), and on some kind of contraction property exhibited by the φ_n's. We therefore term our construction *generalized backward scheme*.

The outline of this paper is the following. After introducing our main notation and assumptions in Section 2, we give a sufficient existence condition of an extension solving the recursion in Section 3, based on the tightness argument of Anantharam and Konstantopoulos [1, 2]. The main result of this work is presented in Section 4: we explicitly construct the extension and deduce several conditions for solving the original stability problem. We conclude with two case studies: in Section 5 we address in this framework the stability problem for queues with impatient customers, thereby completing the existence result of [17]. In Section 6, we handle the same problem for the loss queue.

2. Preliminaries. Let E be a Polish space and $\|\cdot\|_E$ denote the metric on E. Assume that E is endowed with a partial ordering "$\preceq$". For all $x, y \in E$ such that $x \preceq y$, we denote

$$[x, y] := \{z \in E; x \preceq z \preceq y\}.$$

We assume that E admits a $\preceq$-minimal point 0_E, and that any $\preceq$-bounded and increasing sequence converges (possibly to some point in the adherence of E) for the metric $\|\cdot\|_E$. A typical example is given by $E = \mathbb{R}^N$ for some $N \geq 1$ and the partial coordinatewise ordering "$\prec$": for all $x, y \in \mathbb{R}^N$, $x \prec y$ if $x(i) \leq y(i)$ for all $i = 1, \dots, N$.

Any subset $A \subset E$ is said *locally finite* if for any compact subset $C \subset E$, $A \cap C$ is of finite cardinality. We equip E with its Borel σ-algebra $\mathcal{E}$.

Let $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{N}^*$ denote the sets of integers, nonnegative integers and positive integers, respectively. We denote for any $x, y \in \mathbb{R}$, $x \vee y = \max(x, y)$, $x \wedge y = \min(x, y)$ and $x^+ = x \vee 0$. For any real number x, let $\lfloor x \rfloor$ the integer part of x and $\lceil x \rceil = \lfloor x \rfloor + 1$.

Consider a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, furnished with the measurable bijective flow θ (denote θ^{-1}, its measurable inverse). Suppose that $\mathbf{P}$ is stationary and ergodic under θ, that is, for all $\mathcal{A} \in \mathcal{F}$, $\mathbf{P}[\theta^{-1}\mathcal{A}] = \mathbf{P}[\mathcal{A}]$ and all $\mathcal{A}$ that are θ-invariant (i.e., such that $\theta\mathcal{A} = \mathcal{A}$) are of probability 0 or 1. Note that according to these axioms, all θ-contracting events (such that $\mathbf{P}[\mathcal{A}^c \cap \theta^{-1}\mathcal{A}] = 0$) are of probability 0 or 1. We denote for all $n \in \mathbb{N}^*$, $\theta^n = \theta \circ \theta \circ \cdots \circ \theta$ and $\theta^{-n} = \theta^{-1} \circ \theta^{-1} \circ \cdots \circ \theta^{-1}$. Except when explicitly mentioned, throughout all the random variables (r.v.'s for short) are defined on $(\Omega, \mathcal{F}, \mathbf{P})$. Under such conditions, we term the quadruple $(\Omega, \mathcal{F}, \mathbf{P}, \theta)$ a *stationary ergodic dynamical system*.

We denote $\mathcal{M}(E)$ the set of measurable mappings from E into itself, equipped with the Borel σ-algebra $\tilde{\mathcal{E}}$. For any $\mathcal{M}(E)$-valued r.v. φ, for any $x \in E$, let $\varphi_\omega(x)$ be the image of x by φ for the sample ω. For any $f \in \mathcal{M}(E)$ and any subset $B \subset E$, we denote $f(B) = \{f(x); x \in B\}$, and accordingly for any $\mathcal{M}(E)$-valued r.v. φ and all samples ω, $\varphi_\omega(B) = \{\varphi_\omega(x); x \in B\}$. We adopt the convention that $\varphi_\omega(\varnothing) = \varnothing$ a.s.

Fix a $E \times \mathcal{M}(E)$-valued r.v. (X, φ) [where $E \times \mathcal{M}(E)$ is equipped with the product σ-algebra]. The *stochastic recursion* $\{W_n^{[X]}\}_{n \in \mathbb{N}}$ started at X and driven by φ is the E-valued sequence defined by

$$\begin{cases} W_0^{[X]} = X; \\ W_{n+1}^{[X]} = \varphi \circ \theta^n\big(W_n^{[X]}\big), \qquad n \in \mathbb{N}. \end{cases}$$

Define for all samples ω, all n and $x \in E$,

$$\Phi_\omega^n(x) := W_n^{[x]}\big(\theta^{-n}\omega\big) = \varphi_{\theta^{-1}\omega} \circ \varphi_{\theta^{-2}\omega} \circ \cdots \circ \varphi_{\theta^{-n}\omega}(x).$$

The r.v. $\Phi^n(x)$ represents the value of the recursion driven by φ at index 0, when setting to x the value at $-n$. In other words, $\Phi_\omega^n(x) = W_n^{[x]} \circ \theta^{-n}$.

This paper is devoted to the study of existence of a stationary version for the sequence $\{W_n^{[X]}\}_{n \in \mathbb{N}}$, that is, of a E-valued r.v. X such that $W_n^{[X]} = X \circ \theta^n$ for all $n \in \mathbb{N}$. It is then easily seen that X solves the functional equation

$$X \circ \theta = \varphi(X) \qquad \text{a.s.} \tag{1}$$

As stated by Loynes's theorem, (1) admits a solution whenever φ is a.s. monotonic and continuous. Hereafter, we aim to solve (1), possibly on an extension of Ω, under various weaker conditions on φ, which are listed below.

(H1) For some $\mathcal{M}(E)$-valued r.v. ψ:

– for all $x \in E$, $0_E \preceq \varphi(x) \preceq \psi(x)$ a.s.;
– ψ is a.s. $\preceq$-nondecreasing and continuous;
– the following recursion admits at least one E-valued solution:

$$Y \circ \theta = \psi(Y). \tag{2}$$

(H2) φ is a.s. continuous.
(H3) There exists a locally finite set L such that $0_E \in L \subset E$, and that is $\mathbf{P}$-a.s. stable by φ, that is, for all $x \in L$, $\varphi(x) \in L$ a.s.

3. Existence result. Let us assume in this section that the couple $(\Omega, \mathcal{F})$ is Polish (i.e., Ω is Polish, and $\mathcal{F}$ is a sub-σ-algebra of the Borel σ-algebra of Ω). We start by the following existence result, using a tightness argument due to Anantharam and Konstantopoulos [1, 2]. Define:

- $\bar{\Omega} := \Omega \times E$;
- $\bar{\mathcal{F}} := \mathcal{F} \otimes \mathcal{E}$;
- for all $(\omega, x) \in \bar{\Omega}$, $\bar{\theta}(\omega, x) = (\theta\omega, \varphi_\omega(x))$.

We have the following result.

PROPOSITION 1. *Under conditions* (H1) *and* (H2), *there exists a probability measure* $\bar{\mathbf{P}}$ *on* $\bar{\Omega}$ *such that* $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbf{P}}, \bar{\theta})$ *is an extension of* $(\Omega, \mathcal{F}, \mathbf{P}, \theta)$, *and such that*:

- $\bar{\mathbf{P}}$ *is a* $\bar{\theta}$*-invariant probability on* $\bar{\Omega}$ *whose* Ω*-marginal is* $\mathbf{P}$;
- *there exists a* $E \times \mathcal{M}(E)$*-valued r.v.* $(\bar{X}, \bar{\varphi})$, *defined on* $\bar{\Omega}$ *by* (3), *such that the* Ω*-marginal of* $\bar{\varphi}$ *is the distribution of* φ, *and satisfying*

$$\bar{X} \circ \bar{\theta} = \bar{\varphi}(\bar{X}), \qquad \bar{\mathbf{P}}\text{-}a.s.$$

PROOF. This result is a consequence of Theorem 1 in [1], and of the corrections of [2], pages 271–272. Let $\{Y_n^{[0_E]}\}_{n\in\mathbb{N}}$ be the stochastic recursion started at 0_E and driven by ψ. As an immediate consequence of Loynes's theorem for stochastic recursions (see [16] and [3], Section 2.5), there exists a (possibly infinite) solution, say Y_∞, to (2). The r.v. Y_∞, which is given by the almost sure limit of $\{Y_n^{[0_E]} \circ \theta^{-n}\}_{n\in\mathbb{N}}$, is $\preceq$-minimal among all the solutions of (2), and the last assertion of (H1) entails that Y_∞ is E-valued. The sequence $\{Y_n^{[0_E]}\}_{n\in\mathbb{N}}$ tends weakly to Y_∞. It is thus tight, so it follows readily from the first assertion of (H1) and an immediate induction, that $\{W_n^{[0_E]}\}_{n\in\mathbb{N}}$ is tight as well.

It is now easily checked, as is done on page 2 of [1], that the probability distributions $\{(\mathbf{P} \otimes \delta_{0_E}) \circ \tilde{\theta}^{-n}\}_{n\in\mathbb{N}}$ on $\bar{\Omega}$ have Ω-marginal $\mathbf{P}$ and E-marginals, the distributions of $\{W_n^{[0_E]}\}_{n\in\mathbb{N}}$, which form a tight sequence. The sequence $\{(\mathbf{P} \otimes \delta_{0_E}) \circ \tilde{\theta}^{-n}\}_{n\in\mathbb{N}}$ is thus tight. Therefore, any sub-sequential limit is a good candidate for $\bar{\mathbf{P}}$ provided that it is $\bar{\theta}$-invariant. This is true if $\bar{\theta}$ is continuous from

$\Omega \times E$ into itself (condition (A1) in [2]), which clearly holds true under (H2). Finally, define on $\bar{\Omega}$ the random variables

$$\bar{X}(\omega, x) := x, \qquad \bar{\varphi}_{\omega,x} := \varphi_{\omega}. \tag{3}$$

Then $\bar{X}$ is a proper solution to (1) on $\bar{\Omega}$ since

$$\bar{X} \circ \bar{\theta}(\omega, x) = \varphi_{\omega}(x) = \bar{\varphi}_{\omega,x}(x) = \bar{\varphi}_{\omega,x}\big(\bar{X}(\omega, x)\big), \qquad \bar{\mathbf{P}}\text{-a.s.} \qquad \square$$

EXAMPLE 1. Consider on $E = \mathbb{R}+$ a recursion of the type

$$X \circ \theta = \varphi(X) := \big[f(X) + \xi\big]^{+},$$

where f is an a.s. continuous mapping from $\mathbb{R}+$ to itself such that $f(x) \le x$ a.s., and ξ is an integrable r.v. such that $\mathbf{E}[\xi] < 0$. As stated by Loynes's theorem, Lindley's equation

$$Y \circ \theta = \psi(Y) := [Y + \xi]^{+}$$

then admits a unique proper solution. So (H1) and (H2) hold true, and Proposition 1 applies.

4. Explicit construction. Under certain conditions, we can construct explicitly an extension solving equation (1). For doing so, we follow an argument related to that of Flipo [12] and Neveu [19], for the workload sequence of the loss queueing system. We start with a nonempty random set G satisfying

$$\varphi(G) \subseteq G \circ \theta \qquad \text{a.s.,} \tag{4}$$

which is checked, for example, by $G \equiv E$, or any set that is a.s. stable by φ.

REMARK 1. Condition (4) can be seen as a randomized version of the *spatial contraction property* introduced by Bhatthacharya and Lee (see, e.g., [6]), and formalized by Borovkov for stochastic recursions valued in a metric space (condition (II*c)–(a), page 82 in [9]). The same idea is used by Vlasiou in [22] to study a nonmonotonic version of Lindley's equation.

Denote a.s. for all $n \in \mathbb{N}^*$,

$$H^n := \Phi^n\big(G \circ \theta^{-n}\big), \tag{5}$$

the set of all possible values of the recursion driven by φ at 0, when letting the value at $-n$ vary over the set $G \circ \theta^{-n}$.

LEMMA 1. *The sequence of random sets $\{H^n\}_{n\in\mathbb{N}}$ is decreasing*,

$$G \supseteq H^1 \supseteq H^2 \supseteq \cdots \supseteq H^n \supseteq \cdots \qquad \textit{a.s.} \tag{6}$$

PROOF. That $H^1 \subseteq G$ a.s., simply follows from (4). Let now $n \in \mathbb{N}^*$. We have a.s. for all $x \in H^{n+1}_\omega$, that for some $y \in G_{\theta^{-(n+1)}\omega}$,

$$x = \Phi^{n+1}_\omega(y) = \Phi^n_\omega\big(\varphi_{\theta^{-(n+1)}\omega}(y)\big).$$

By (4), we have that $\varphi_{\theta^{-(n+1)}\omega}(y) \in G_{\theta^{-n}\omega}$, hence $x \in H^{n+1}_\omega$. $\square$

We can thus define the random set

$$H = \lim_{n\to\infty} H^n = \bigcap_{n\geq 1} H^n \subseteq G. \tag{7}$$

REMARK 2. Lemma 1 can be rephrased in terms of *internal monotonicity*, using the terminology of Stoyan (Chapter 2 of [21]; see as well [18, 20]): the random sequence $\{H^n\}$ is monotone decreasing for the partial ordering "$\subseteq$", provided that $H^1 \subseteq H^0 := G$. It is thus internally monotone, as in Theorem 2.2.8 of [21]. Further, the stochastic recursion $\{W_n\}$ is itself internally monotone on E whenever φ is so, which is the framework of Loynes's theory; see below. Notice that the latter references mainly address distributional (such as strong, convex or concave) orderings. These distributional counterparts of our pathwise monotonicity results are obtained from Strassen's theorem, on a reference probability space on which the sequence $\{\varphi_n\}$ is primarily defined; see as well Section 4.2 of [3].

LEMMA 2. *Almost surely*, $\varphi(H) \subset H \circ \theta$.

PROOF. Fix a sample ω in the event $\{H \text{ is nonempty}\}$. It suffices to notice that for any $x \in H_\omega$, there exists for all $n \geq 1$ an element $y_n \in G_{\theta^{-n}\omega}$ such that $\varphi_\omega(x) = \Phi^{n+1}_{\theta\omega}(y_n)$, so $\varphi_\omega(x)$ clearly belongs to $H_{\theta\omega}$. $\square$

4.1. *Construction of the extension.* We are now in position to construct in some cases an enrichment of the original probability space on which the existence of a solution to (1) is granted.

THEOREM 1. *Let G be a random set satisfying* (4), *and suppose that the sets $\{H^n\}_{n\in\mathbb{N}^*}$ are such that*

$$\mathbf{P}\Big[\bigcup_{n\in\mathbb{N}^*} \{H^n \textit{ is finite}\}\Big] > 0. \tag{8}$$

Then, there exists a positive integer c such that the following quadruple $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}}, \tilde{\theta})$ *defines a stationary dynamical system*:

- $\tilde{\Omega} = \{(\omega, x) \in \Omega \times E; x \in H_\omega\}$;
- *$\tilde{\mathcal{F}}$ is the trace of $\mathcal{F} \otimes \mathcal{E}$ on $\tilde{\Omega}$, that is,*

$$\tilde{\mathcal{F}} = \big\{\tilde{A} := \{(\omega, x) \in \Omega \times E; \omega \in A, x \in B \cap H_\omega; A \in \mathcal{F}, B \in \mathcal{E}\}\big\};$$

- *for all $\tilde{\mathcal{F}}$-measurable function F,*

$$\int_{\tilde{\Omega}} F\, d\tilde{\mathbf{P}} = \int_{\Omega}\int_E F\, d\mu_\omega\, d\mathbf{P} \qquad \textit{where } \mu = \frac{1}{c}\sum_{i\in H(\omega)} \delta_i \textit{ a.s.};$$

- *for all $(\omega, x) \in \tilde{\Omega}$, $\tilde{\theta}(\omega, x) = (\theta\omega, \varphi_\omega(x))$.*

Moreover, (1) *admits on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}}, \tilde{\theta})$, a solution $\tilde{X}$.*

PROOF. Let $\mathcal{B} := \bigcup_{n\in\mathbb{N}^*}\{H^n \text{ is finite}\}$. First, as the sets $\{H^n\}_{n\in\mathbb{N}^*}$ are nonempty by construction, Lemma 1 and (8) entail that

$$\mathcal{B} \subset \left(\bigcup_{n\in\mathbb{N}^*}\{H^n = H\}\right) \cap \{H \text{ is finite and nonempty}\}. \tag{9}$$

For almost all samples $\omega \in \mathcal{B}$, there exists $N(\omega)$ such that $H^n(\omega)$ is finite for any $n \geq N(\omega)$. Therefore for any $n \geq N(\omega) + 1$, the set H_ω^{n-1} and hence, the set $H_{\theta\omega}^n = \varphi_\omega(H_\omega^{n-1})$, are finite. In other words, $\theta\omega \in \mathcal{B}$. Consequently, $\mathcal{B}$ is θ-contracting and thus from (8), it is almost sure.

This implies the almost sure surjectivity of φ. To see this, fix $\omega \in \mathcal{B}$ and $y \in H_{\theta\omega}$. For all $n \geq 1$, for some $x_{n+1}(\omega) \in G_{\theta^{-(n+1)}\theta\omega}$, we have that

$$y = \Phi_{\theta\omega}^{n+1}\big(x_{n+1}(\omega)\big) = \varphi_\omega\big(y_n(\omega)\big),$$

where $y_n(\omega) = \Phi_\omega^n(x_{n+1}(\omega)) \in H_\omega^n$. But from (9), for some index $\tau(\omega)$, $H_\omega = H_\omega^n$ for all $n \geq \tau(\omega)$. Thus, setting $y_*(\omega) = y_{\tau(\omega)}(\omega)$, we have that $y_*(\omega) \in H_\omega$ and $y = \varphi_\omega(y_*(\omega))$. So φ is a.s. surjective.

Moreover, it follows from Lemma 2 together with (9) that

$$0 < \operatorname{Card} H \circ \theta \leq \operatorname{Card} H < \infty \qquad \text{a.s.}$$

But $\mathbf{P}[\operatorname{Card} H \circ \theta < \operatorname{Card} H] > 0$ would contradict the ergodic lemma ([3], Lemma 2.2.1). Thus $\operatorname{Card} H = \operatorname{Card} H \circ \theta$ a.s., which implies (i) that φ is a.s. bijective from H to $H \circ \theta$ and (ii) that $\operatorname{Card} H$ is deterministic, say equal to $c > 0$ a.s., from the ergodicity of $\mathbf{P}$ under θ.

In particular, $\tilde{\theta}$ defines an automorphism of $\tilde{\Omega}$. On the other hand, $\tilde{\mathbf{P}}$ defines a probability measure, since it clearly is a σ-finite measure such that

$$\begin{aligned}\tilde{\mathbf{P}}[\tilde{\Omega}] &= \frac{1}{c}\int_\Omega\int_E \mathbf{1}_{H_\omega}(y)\, d\mu_\omega(y)\, d\mathbf{P}(\omega)\\ &= \frac{1}{c}\int_\Omega \operatorname{Card}(E \cap H_\omega)\, d\mathbf{P}(\omega) = 1.\end{aligned}$$

Notice as well that $\tilde{\mathbf{P}}$ has Ω-marginal $\mathbf{P}$ since for all $A \in \mathcal{F}$,

$$\tilde{\mathbf{P}}[A \times E] = \frac{1}{c}\int_\Omega \mathbf{1}_A(\omega)\operatorname{Card}(E \cap H_\omega)\, d\mathbf{P}(\omega) = \mathbf{P}[A]. \tag{10}$$

Fix now $A \in \mathcal{F}$, $B \in \mathcal{E}$ and set

$$\tilde{A} := \{(\omega, x) \in \tilde{\Omega}; \omega \in A, x \in B \cap H_\omega\}.$$

We then have that

$$\begin{aligned}
\tilde{\mathbf{P}}[\tilde{\theta}^{-1}\tilde{A}] &= \int \int_{\tilde{\Omega}} \mathbf{1}_{\theta^{-1}A}(\omega)\mathbf{1}_{(\varphi_\omega)^{-1}(B\cap H_{\theta\omega})}(y)\, d\mu_\omega(y)\, d\mathbf{P}(\omega) \\
&= \frac{1}{c}\int_\Omega \mathbf{1}_A(\theta\omega)\biggl(\sum_{i\in H_\omega} \mathbf{1}_{B\cap H_{\theta\omega}}(\varphi_\omega(i))\biggr)\, d\mathbf{P}(\omega) \\
&= \frac{1}{c}\int_\Omega \mathbf{1}_A(\theta\omega)\,\mathrm{Card}\bigl((\varphi_\omega)^{-1}(B\cap H_{\theta\omega})\cap H_\omega\bigr)\, d\mathbf{P}(\omega) \\
&= \frac{1}{c}\int_\Omega \mathbf{1}_A(\theta\omega)\,\mathrm{Card}(B\cap H_{\theta\omega})\, d\mathbf{P}(\omega) \\
&= \frac{1}{c}\int_\Omega \mathbf{1}_A(\omega)\,\mathrm{Card}(B\cap H_\omega)\, d\mathbf{P}(\omega) = \tilde{\mathbf{P}}[\tilde{A}],
\end{aligned}$$

where the fourth equality follows from the bijectivity of φ from H into $H \circ \theta$, and the last one, from the θ-invariance of $\mathbf{P}$. This first shows the measurability of $\tilde{\theta}^{-1}\tilde{A}$ and second, the $\tilde{\theta}$-invariance of $\tilde{\mathbf{P}}$.

Finally, we obtain a solution to (1) by letting $\tilde{X}$ (resp., $\tilde{\varphi}$) be the restriction on $\tilde{\Omega}$ of the r.v. $\bar{X}$ (resp., $\bar{\varphi}$) defined by (3), that is,

$$\tilde{X}(\omega, x) = x \quad \text{and} \quad \tilde{\varphi}_{\omega,x}(y) = \varphi_\omega(y), \qquad y \in E, \tilde{\mathbf{P}}\text{-a.s.} \qquad \square$$

4.2. *Resolution on the original space.* Let us return to the original problem. Fix G as above, and assume that (8) holds. We start by investigating the invariant σ-algebra of $\tilde{\Omega}$. Any invariant $\tilde{I}$ can be written under the form

$$\tilde{I} = \{(\omega, x) \in \tilde{\Omega}; \omega \in A, x \in I_\omega\},$$

where I is a random set belonging a.s. to $\mathcal{E}$ and $A \in \mathcal{F}$. We then have $\theta^{-1}A = A$ and $\varphi(I) = I \circ \theta$ on A. So as θ is ergodic, up to a $\tilde{\mathbf{P}}$-negligible event,

$$\tilde{I} = \{(\omega, x) \in \tilde{\Omega}; x \in I_\omega\} \qquad \text{where } \varphi(I) = I \circ \theta \text{ a.s.} \tag{11}$$

THEOREM 2. *Let*

$$\mathcal{K} := \{\textit{sets } \tilde{I} \textit{ of the form } (11) \textit{ s.t. } \mathrm{Card}(I \cap H) = 1 \textit{ a.s.}\}.$$

There exists a solution to (1) *on* Ω, *taking values in* G *with positive probability, if and only if* $\mathcal{K}$ *is nonempty, which is the unique such solution if and only if* $\mathcal{K}$ *is reduced to a singleton. In particular there exists a unique such solution whenever* $c = 1$.

PROOF. The result follows from the existence of a one-to-one relation between $\mathcal{K}$ and the set $\mathcal{J} := \{E\text{-valued solutions } X \text{ of (1) s.t. } \mathbf{P}[X \in G] > 0\}$. To see this, let be X an element of $\mathcal{J}$, $I_\omega = \{X(\omega)\}$ a.s., and $\tilde{I} = \{(\omega, X(\omega)); \omega \in \Omega\}$. To check that $\tilde{I}$ satisfies (11), it suffices to notice that $X \in H$ a.s. Indeed, $\{X \in G\}$ is θ-contracting since on $\{X \in G\}$,

$$X \circ \theta = \varphi(X) \in \varphi(G) \subseteq G \circ \theta.$$

So the latter event is almost sure and by θ-invariance,

$$\mathbf{P}[X \in H] = \mathbf{P}\left[\bigcap_{n \geq 1} \theta^n \{X \in G\}\right] = 1.$$

Conversely, given i a E-valued r.v. and $\tilde{I} = \{(\omega, i(\omega)) \in \tilde{\Omega}\} \in \mathcal{K}$, i is clearly a E-valued solution to (1). It belongs to $\mathcal{J}$ since $i \in H \subseteq G$ a.s. $\square$

4.3. *Applications*. At first glance, assumptions (4) and (8) might appear abstract and difficult to check in practical examples. We present hereafter several comprehensive cases, in which the latter holds true.

THEOREM 3. *Conditions* (4) *and* (8) *are met*, *and then Theorem* 1 *applies*, *in the following cases*:

(i) *For some collection $\mathcal{G}$ of E-valued r.v.'s*, *there exists an integer p*, *a finite random set B and an event $\mathcal{D}$ of positive probability*, *such that for all $Z \in \mathcal{G}$ and all $n \geq p$*,

$$W_n^{[Z]} \in B \circ \theta^n \qquad \text{on } \theta^{-n}\mathcal{D}.$$

If additionally, $\mathbf{P}[\mathrm{Card}\, B = 1] > 0$, *a solution X to* (1) *exists on the original space*, *to which all sequences $\{W_n^{[Z]}\}_{n \in \mathbb{N}}$, $Z \in \mathcal{G}$, converge with strong backward coupling*.

(ii) *Some deterministic finite subset F of E is a.s. stable by φ*.

(iii) (H1) *and* (H3) *hold*.

PROOF. (i) Set

$$G_\omega = \{Z(\omega); Z \in \mathcal{G}\} \qquad \text{a.s.,} \tag{12}$$

and let $Z \in \mathcal{G}$ and $Y = \varphi(Z) \circ \theta^{-1}$. Fix $n \geq p$ and $\omega \in \theta^{-n}\mathcal{D}$. Then

$$\begin{aligned} W_n^{[Y]}(\omega) &= \varphi_{\theta^{n-1}\omega} \circ \cdots \circ \varphi_{\theta\omega} \circ \varphi_\omega(\varphi_{\theta^{-1}\omega}(Z(\theta^{-1}\omega))) \\ &= W_{n+1}^{[Z]}(\theta^{-1}\omega) \in B_{\theta^{n+1}(\theta^{-1}\omega)} = B_{\theta^{-n}\omega}. \end{aligned}$$

This is true on $\theta^{-n}\mathcal{D}$ for all $n \geq p$, so the r.v. Y belongs to $\mathcal{G}$. Hence

$$\mathbf{P}[\theta\{\varphi(G) \subset G \circ \theta\}] = 1,$$

which amounts to (4). It remains to check (8). Let $\omega \in \mathcal{D}$ and $x_\omega \in G_{\theta^{-n}\omega}$, which means that for some r.v. $Z \in \mathcal{G}$, $x_\omega = Z(\theta^{-n}\omega)$. Hence, for all $n \geq p$,

$$\Phi_\omega^n(x_\omega) = W_n^{[Z]}\big(\theta^{-n}\omega\big) \in B_{\theta^n(\theta^{-n}\omega)} = B_\omega.$$

Therefore, on $\mathcal{D}$, we have $H \subset H^n \subset B$ which implies that the H^n's are finite after a certain index. Hence (8) holds true since $\mathcal{D}$ is of positive probability.

Now, on the event $\{\operatorname{Card} B = 1\}$, $H = H^n = B$ for all $n \geq p$, so $c = 1$. Whenever this event is of positive probability, it is almost sure so we can set $H_\omega = \{X(\omega)\}$ a.s. Moreover, for all $n \geq p$, $H^n = H$ a.s. thus for any $Z \in \mathcal{G}$, $X = \Phi^n(Z) = W_n^{[Z]} \circ \theta^{-n}$ a.s. In other words, there is strong backward coupling between $\{W_n^{[Z]}\}_{n\in\mathbb{N}}$ and $\{X \circ \theta^n\}_{n\in\mathbb{N}}$, with coupling time p.

(ii) Take $G = F$ a.s., so (4) and (8) trivially hold.

(iii) Suppose that (H1) holds together with (H3), and let Y be an arbitrary E-valued solution to (2). Set $G = [0, Y] \cap L$ a.s. Then, a.s. for all $y \in \varphi(G)$, $y = \varphi(x)$ for some $x \in E$ such that $x \preceq Y$. But in view of (H1),

$$y \preceq \psi(x) \preceq \psi(Y) = Y \circ \theta, \tag{13}$$

so that $y \in G \circ \theta$. Hence G satisfies (4). Moreover, G is a.s. of finite cardinality in view of the locally-finiteness of L, so (8) holds true. $\square$

On Loynes's theorem. Assume that the structural properties of the recursion are such that the set H given by (7) is necessarily a.s. reduced to a singleton. Then, the construction of Theorem 1 is valid, even when (8) is not satisfied. Indeed, in that case Lemma 2 readily entails that φ is a.s. bijective from H to $H \circ \theta$, and we can construct the extension $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbf{P}}, \tilde{\theta})$ exactly as in Theorem 1, for $c = 1$.

This is typically the case for Loynes's theorem: assume that φ is a.s. nondecreasing and continuous on E. It is routine to check that Loynes's sequence $\{\Phi^n(0_E)\}_{n\in\mathbb{N}}$ is a.s. nondecreasing. Thus, its (possibly infinite) supremum Y clearly is a solution to (1). Set $G = [0_E, Y]$ a.s., and let $n \geq 1$. Then, $\Phi^n(Y \circ \theta^{-n}) = Y$ a.s., so as Φ^n is a.s. nondecreasing,

$$H^n = \big\{\Phi^n(x); x \in \big[0_E; Y \circ \theta^{-n}\big]\big\} = \big[\Phi^n(0_E); Y\big] \qquad \text{a.s.}$$

As Y is the a.s. limit of Loynes's sequence, it readily follows that $H = \{Y\}$ a.s. So the only solution Z to (1) on the original space, such that $\mathbf{P}[Z \leq Y] > 0$, is the r.v. Y itself. This is the exact statement of Loynes's theorem for stochastic recursions [3, 16].

Renovating events. Condition (i) of Theorem 3 is clearly related to Borovkov and Foss's theory of renovation. Recall that the sequence $\{\theta^{-n}\mathcal{A}\}_{n\in\mathbb{N}}$ (where $\mathcal{A}$ is of positive probability) is termed *sequence of renovating events* of length $m \in \mathbb{N}^*$ for the recursion $\{W_n\}_{n\in\mathbb{N}}$ whenever for some E'-valued r.v. β (where E' is some auxiliary space), some deterministic mapping $\Psi : (E')^m \to E$, for all $n \geq m$,

$$W_n = \Psi\big(\beta \circ \theta^{n-m}, \ldots, \beta \circ \theta^{n-2}, \beta \circ \theta^{n-1}\big) \qquad \text{on } \theta^{-(n-m)}\mathcal{A}.$$

Let $\mathcal{Z}$ a collection of r.v.'s such that all sequences $\{W_n^{[Z]}\}_{n\in\mathbb{N}}$, $Z \in \mathcal{Z}$ admit the same sequence of renovating events $\{\theta^{-n}\mathcal{A}\}_{n\in\mathbb{N}}$, with the same length m and same function Ψ. Take $G := \mathcal{Z}$, $\mathcal{D} := \theta^m\mathcal{A}$ and $p := m$. Then condition (i) is satisfied if we set

$$B = \{\Psi(\beta \circ \theta^{-m}, \dots, \beta \circ \theta^{-2}, \beta \circ \theta^{-1})\} \qquad \text{a.s.}$$

In particular, $c = 1$, so there is a unique solution to (1) on the original probability space, to which all sequences $\{W_n^{[Z]}\}_{n\in\mathbb{N}}$, $Z \in \mathcal{Z}$, converge with strong backward coupling. This is Borovkov and Foss's theorem (Theorem 1, page 260 in [8], Theorem 4 in [10]; see also Corollary 2.5.1 in [3]).

5. The queue with impatient customers. We now consider a queuing model with impatient customers: the nth customer C_n enters the system at time T_n and requests a service duration σ_n. There is a single server operating in first in, first out (FIFO). Moreover, C_n requires to enter service before a given deadline $T_n + D_n$ (D_n thus denotes the *patience time* of C_n). As the discipline is FIFO, the waiting time proposed to C_n before reaching the server is known upon arrival (and equals the *workload* W_n, i.e., the quantity of work present in the system at T_n, in time unit). So C_n actually enters the system if only and if $W_n \le D_n$. If not, C_n does not enter and is lost forever.

We assume that the input in this queue is of the G/G type, and work on the Palm space $(\Omega, \mathcal{F}, \mathbf{P}, \theta)$ of the arrival process $\cdots < T_{-2} < T_{-1} < T_0 = 0 < T_1 < \cdots$; see, for instance, [3]. The stationary sequence of inter-arrival times $\{\xi_n\}_{n\in\mathbb{Z}} := \{T_{n+1} - T_n\}_{n\in\mathbb{Z}}$ is then compatible with θ, that is, $\xi_n = \xi \circ \theta^n$ for all n. The service times $\{\sigma_n\}_{n\in\mathbb{Z}}$ and patience times $\{D_n\}_{n\in\mathbb{Z}}$ of the successive customers form a sequence of marks of the arrival process, and are hence compatible with θ; see, for example, [3] and [11] for the ergodic-theoretical representation of stationary queues. The generic service and patience times are respectively denoted by σ and D, and we assume that $\xi > 0$ almost surely.

The workload sequence $\{W_n\}_{n\in\mathbb{Z}}$ is a stochastic recursion, driven by the mapping

$$x \mapsto \varphi(x) := [x + \sigma \mathbf{1}_{\{x \le D\}} - \xi]^+.$$

We aim to solve (1) for φ as above. Let

$$Y = \left[\max_{i\ge 1}\left((\sigma \wedge D) \circ \theta^{-i} - \sum_{j=1}^{i} \xi \circ \theta^{-j}\right)\right]^+; \tag{14}$$

$$Z = \left[\max_{i\ge 1}\left((\sigma + D) \circ \theta^{-i} - \sum_{j=1}^{i} \xi \circ \theta^{-j}\right)\right]^+; \tag{15}$$

$$A = \left\{ i > 0; (D \circ \theta^{-i}) - \sum_{j=1}^{i} \xi \circ \theta^{-j} > 0 \right\} \quad \text{and} \quad \tau^- = \sup A;$$

$$B = \left\{ i > 0; (\sigma + D) \circ \theta^{-i} - \sum_{j=1}^{i} \xi \circ \theta^{-j} > 0 \right\} \quad \text{and} \quad \rho = \sup B;$$

$$\tau^{+} = \min\left\{ i > 0; \sum_{j=0}^{i-1} \xi \circ \theta^{j} \geq D \right\}.$$

The random set A (resp., B) thus contains all the absolute values of the indexes of the customers possibly in the waiting line (resp., in the total system) at time 0. As immediate consequences of Birkhoff's theorem, the random variables ρ, τ^{-} and τ^{+} are a.s. finite, and the following two minima exist:

$$p = \min\{n > 0; \mathbf{P}[\rho \leq n] > 0\};$$

$$t = \min\{n > 0; \mathbf{P}[\tau^{-} \leq n] > 0\}.$$

THEOREM 4. *A stationary workload exists on Ω whenever*

$$\mathbf{P}[Z = 0] > 0. \tag{16}$$

If not, if both σ and ξ take value in a set of the form

$$L_\alpha := \{n\alpha; n \in \mathbb{N}\}, \qquad \alpha \in \mathbb{R}+, \tag{17}$$

then Theorem 1 applies, and the size c of the extension $\tilde{\Omega}$ is such that

$$\begin{aligned} c \leq \frac{1}{\alpha} \min\Bigg\{ &\max_{i=1,\dots,p} (\sigma \circ \theta^{-i}) + \sum_{j=1}^{t} (\sigma \circ \theta^{-j}); \sum_{j=1}^{p} (\sigma \circ \theta^{-j}); \\ &\max_{i=1,\dots,p} \big((\sigma \circ \theta^{-i}) \vee (D \circ \theta^{-i})\big) \Bigg\} + 1 \qquad \textit{a.s.} \end{aligned} \tag{18}$$

If moreover, σ is a.s. upper-bonded by $\bar{\sigma}$,

$$c \leq \left\lceil \frac{\bar{\sigma}(\mathbf{E}[D] + \mathbf{E}[\xi])}{\alpha \mathbf{E}[\xi]} \right\rceil. \tag{19}$$

PROOF. First notice that a.s. for all x,

$$\begin{aligned} \chi(x) &:= [x \vee (\sigma \wedge D) - \xi]^{+} \leq \varphi(x) \leq \psi(x) \\ &:= [x \vee (\sigma + D) - \xi]^{+}; \end{aligned} \tag{20}$$

see equations (9) and (12) in [17]. The r.v.'s Y and Z defined by (14), (15) are the only proper solutions of the recursions respectively driven by the monotone maps χ and ψ. Then the existence of a solution to (1) on Ω under condition (16) follows from (20) using renovating events; see Theorem 2 of [17].

Assume now that (16) does not hold. In this case, Theorem 3 of [17] shows the existence of a solution on $\Omega \times \mathbb{R}+$ provided that σ and ξ both take value in L_α.

Theorem 1 allows us to construct that extension explicitly. Clearly, L_α is a.s. stable by φ. Hence, when initiated in L_α, the recursion remains in this set forever. We are thus in the case (iii) of Theorem 3, taking $L = L_\alpha$. More precisely, we have a.s. in view of (20) that for any $Y \leq x \leq Z$,

$$Y \circ \theta = \chi(Y) \leq \chi(x) \leq \varphi(x) \leq \psi(x) \leq \psi(Z) = Z \circ \theta.$$

Hence, the random set $G = L_\alpha \cap [Y, Z]$ meets condition (4). Moreover, Z is a.s. finite in view of Birkhoff's theorem; thus G is a.s. of finite cardinality. So does H: (8) holds true and Theorem 1 applies.

We now turn to the proof of (18) and (19). A quick computation first shows that

$$\begin{aligned} c \leq \operatorname{Card} G &= \left\lceil \frac{Z - Y}{\alpha} \right\rceil \\ &\leq \left\lceil \frac{1}{\alpha} \max_{i=1,\dots,p} \{(\sigma \vee D) \circ \theta^{-i}\} \right\rceil \qquad \text{on } \{\rho \leq p\}. \end{aligned} \tag{21}$$

On the other hand, the largest possible workload at time 0 is less than the sum of the service time of the customer in service and the service times requested by the customers in the waiting line at 0. Thus

$$H \subseteq L_\alpha \cap \left[0; \sigma \circ \theta^{-\rho} + \sum_{j=1}^{+\infty} (\sigma \circ \theta^{-j}) \mathbf{1}_A\{j\}\right], \tag{22}$$

which implies in turn that

$$c \leq \frac{1}{a} \left(\sum_{j=1}^{p} \sigma \circ \theta^{-j} \right) + 1 \qquad \text{on } \{\rho \leq p\}; \tag{23}$$

$$c \leq \frac{1}{\alpha} \left(\max_{i=1,\dots,p} \left(\sigma \circ \theta^{-i} + \sum_{j=1}^{t} \sigma \circ \theta^{-j} \right) \right) + 1 \qquad \text{on } \{\tau^- \leq t\}. \tag{24}$$

Inequalities (21), (23) and (24) hold true with positive probability, so they are true almost surely since c is deterministic, hence (18).

Finally, remarking that $j \in A \Leftrightarrow \tau^+ \circ \theta^{-j} > j$ for all $j > 0$ we obtain that

$$\mathbf{E}[D] > \mathbf{E}\left[\sum_{j=1}^{\infty} \xi \circ \theta^{j-1} \mathbf{1}_{\tau^+ > j} \right] = \mathbf{E}\big[(\xi \circ \theta^{-1}) \operatorname{Card} A\big]. \tag{25}$$

So if the service times are bounded by $\bar{\sigma}$, (22) and (25) entail (19). $\square$

The following two simple examples illustrate how the extension technique can be used to solve explicitly the stability problem of the queue:

EXAMPLE 2. We work on the following elementary ergodic quadruple:

$$\begin{cases} \Omega = \{\omega_1, \omega_2\}; \\ \mathcal{F} = \mathcal{P}(\Omega); \\ \mathbf{P} = \text{ uniform on } \Omega; \\ \theta\omega_1 = \omega_2 \text{ and } \theta\omega_2 = \omega_1. \end{cases}$$

We first consider (a particular case of) Example 1, page 303 in [17]. Define

$$\xi(\omega_1) = \xi(\omega_2) = 1; \qquad \sigma(\omega_1) = 0.5, \qquad \sigma(\omega_2) = 1.5;$$
$$D(\omega_1) = 1.51, \qquad D(\omega_2) = 2.01.$$

Then, setting $G = [Y, Z] \cap 0.5\mathbb{N}$, we obtain that $c = 3$. Moreover the extension is not ergodic, and there are three solutions on Ω, corresponding respectively to the three invariant events of $\mathcal{K}$,

$$\mathcal{I} = \{(\omega_1, 0.5); (\omega_2, 0)\}; \qquad \mathcal{I}' = \{(\omega_1, 1); (\omega_2, 0.5)\};$$
$$\mathcal{I}'' = \{(\omega_1, 1.5); (\omega_2, 1)\}.$$

Define now on the same probability space,

$$\xi(\omega_1) = \xi(\omega_2) = 1; \qquad \sigma(\omega_1) = 3, \qquad \sigma(\omega_2) = 2;$$
$$D(\omega_1) = 3.01, \qquad D(\omega_2) = 1.99.$$

Set $G = [Y, Z] \cap \mathbb{N}$ a.s. Then the invariant σ-algebra of $\tilde{\mathcal{F}}$ is $\{\varnothing, \tilde{\Omega}\}$, so the extension is ergodic, but there is no stationary workload on Ω.

Independent case. Consider now a $GI/GI/1/1 + GI$ queue: the service, patience and inter-arrivals times form three independent i.i.d. sequences. Under such conditions, [4, 5] show the existence of a unique stationary workload if

$$\mathbf{P}[\sigma < \xi] > 0. \tag{26}$$

We can easily retrieve this result with our method. From (26), for some $y, \varepsilon > 0$ we have $\mathbf{P}[\sigma \le y - \varepsilon] > 0$ and $\mathbf{P}[\xi \ge y] > 0$. Moreover, Z defined by (15) is finite, so there exists $n \in \mathbb{N}$ such that $\mathbf{P}[Z \circ \theta^{-n} < n\varepsilon] > 0$. Let

$$\mathcal{E}_i^x = \bigcap_{j=1}^{i} \{\xi \circ \theta^{-j} \ge x\} \quad \text{and} \quad \mathcal{F}_i^x = \bigcap_{j=1}^{i} \{\sigma \circ \theta^{-j} \le x\}, \qquad i \in \mathbb{N}^*, x \in \mathbb{R}+;$$
$$\mathcal{A}_n = \{Z \circ \theta^{-n} < n\varepsilon\} \cap \mathcal{E}_n^y \cap \mathcal{F}_n^{y-\varepsilon}, \qquad n \in \mathbb{N}.$$

For any sample ω in $\mathcal{A}_n$, if C_{-n} finds upon arrival a workload equal to $u \in G_{\theta^{-n}\omega}$, we have the following alternative:

(i) either the server never idles before the end of service of C_{-1}, so

$$\Phi_\omega^n(u) \le \left[u + \sum_{j=1}^{n} \sigma \circ \theta^{-j}(\omega) - \sum_{j=1}^{n} \xi \circ \theta^{-j}(\omega) \right]^+ = 0,$$

(ii) or for some $q \in [\![1, n-1]\!]$, the system is empty at the arrival of C_{-q}. Then, each subsequent customer is immediately attended upon arrival and leaves the system before the next arrival. In particular $\Phi^n_\omega(u) = 0$.

Therefore, $H = \{0\}$ on $\mathcal{A}_n$. As $\mathbf{P}[\mathcal{A}_n] > 0$ by independence, we have $c = 1$. So the extension is ergodic, and there exists a unique solution on Ω.

6. The loss queue. We now address the classical, but challenging stability problem of the *loss queue* $G/G/1/1$. There is one server and no waiting room, so that each customer is either immediately accepted for service (if the system is empty), or rejected upon arrival (if the server is busy). As easily checked, the workload sequence is a $\mathbb{R}+$-valued recursion driven by

$$x \mapsto \varphi(x) = [x + \sigma \mathbf{1}_{\{x=0\}} - \xi]^+,$$

so this can be seen as a particular case of queue with impatient customers for which $D = 0$ a.s. Here again, it is quite simple to exhibit examples for which uniqueness, and even existence of a solution to (1) do not hold; see [3], page 121. A solution on $\Omega \times \mathbb{N}$ is constructed explicitly in [12, 19], whereas the existence on $\Omega \times (\mathbb{R}+)$ is proven in [1, 2] using an approach based on tightness, closely related to our Proposition 1. Hereafter, we use Theorem 1 to construct explicitly this solution, and relate it to that of [12, 19]. Denote

$$\gamma = \sup\left\{ i > 0; \sigma \circ \theta^{-i} - \sum_{j=1}^{i} \xi \circ \theta^{-j} > 0 \right\},$$

the absolute value of the index of the oldest possible customer in service at time 0. As above, the following minimum exists:

$$g = \min\{n > 0; \mathbf{P}[\gamma \leq n] > 0\}.$$

On the event $\{\gamma \leq g\}$, the workload at time 0 belongs to the random set

$$B := \left\{ \sigma \circ \theta^{-i} - \sum_{j=1}^{i} \xi \circ \theta^{-j}; i = 1, \dots, g \right\},$$

so for any E-valued r.v. Z and for all $n \geq g$, $\Phi^n(Z \circ \theta^{-n}) \in B$ on $\{\gamma \leq g\}$, that is to say that $W_n^{[Z]} \in C \circ \theta^n$ on $\theta^{-n}\{\gamma \leq g\}$. We are thus in the case (i) of Theorem 3 taking $G := \mathbb{R}+$ a.s., $p := g$ and $\mathcal{D} = \{\gamma \leq g\}$, so there exists an extension $\tilde{\Omega}$ on which (1) admits a solution.

This construction can be compared to that of [12]: define a.s. for all $i \in \mathbb{N}$,

$$\ell_\omega(i) = \begin{cases} i+1, & \text{if } C_i \text{ is still in service at } T_0-, \\ & \text{provided he/she found an empty system upon arrival;} \\ 0, & \text{else;} \end{cases}$$

$$L^n_\omega(i) = \ell_{\theta^{-1}\omega} \circ \ell_{\theta^{-2}\omega} \circ \cdots \circ \ell_{\theta^{-n}\omega}(i), \qquad n \geq 1.$$

In words, $L^n(i)$ represents the index of the customers present in the system at T_0- when assuming that customer C_{-n-i} found an empty system upon arrival. Denoting then $\hat{H}^n_\omega = L^n_\omega(\mathbb{N})$ and $\hat{H}_\omega = \bigcap_{n\geq 1} \hat{H}^n_\omega$, similarly to Theorem 1 one can construct on $\Omega \times \mathbb{N}$ an extension $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbf{P}}, \hat{\theta})$, replacing φ by ℓ and H by $\hat{H}$. The recursion is then solved by setting

$$\hat{X}(\omega, i) = \left[\sigma\left(\theta^{-i}\omega\right) - \sum_{j=1}^{i} \xi\left(\theta^{-j}\omega\right)\right]^+ \quad \text{and} \quad \hat{\varphi}_{\omega,i} = \varphi_\omega, \qquad \hat{\mathbf{P}}\text{-a.s.}$$

As shown below, the extension $\hat{\Omega}$ can be projected onto $\tilde{\Omega}$:

PROPOSITION 2. *Set* $\Phi^0_\omega(0) = 0$. *The following mapping is a.s. surjective*:

$$F_\omega : \begin{cases} \hat{H}_\omega \longrightarrow H_\omega, \\ i \longmapsto \Phi^i_\omega(0). \end{cases}$$

PROOF. Fix a sample ω. We first check that F_ω maps $\hat{H}_\omega$ onto H_ω. Let $j \in \hat{H}_\omega$. For all $n \geq 1$, there exists $i_n \in \mathbb{N}$ such that $j = L^n_\omega(i_n)$. In other words, for the sample ω, C_{-j} is in service just before time T_0 whenever C_{n+i_n} entered an empty system, hence $\Phi^{n+i_n}_\omega(0) = \Phi^j_\omega(0) = F_\omega(j)$. Therefore, $F_\omega(j) \in \Phi^{n+i_n}_\omega(\mathbb{R}+)$, so there exists $n' = n + i_n \geq n$ such that $F_\omega(j) \in H^{n'}_\omega$. This is true for all $n \geq 1$, hence $F_\omega(j) \in H_\omega$.

To check that F_ω is surjective, take $x \in H_\omega$ and let for all $n \geq 1$, $x_n \in [0, Y(\theta^{-n}\omega)]$ be such that $x = \Phi^n_\omega(x_n)$. First, as shown above, there exists $j \in [\![0, \gamma]\!]$ such that $x = \Phi^j_\omega(0) = F_\omega(j)$. Fix now $n \geq 1$. Then, assuming that for all $\tilde{n} \geq n$,

$$x_{\tilde{n}}(\omega) - \sum_{j=n+1}^{\tilde{n}} \xi\left(\theta^{-j}\omega\right) \geq 0$$

would contradict Birkhoff's theorem since $\mathbf{E}[\xi] > 0$. Thus, there exists $\tilde{n} \geq n$ such that $x_{\tilde{n}}(\omega) - \sum_{j=n+1}^{\tilde{n}} \xi(\theta^{-j}\omega) < 0$, which means that either (i) $x_{\tilde{n}} = 0$ and the system was empty upon the arrival of $C_{-\tilde{n}}$ or (ii) $C_{-\tilde{n}}$ found a busy server upon arrival, having a residual workload equal to $x_{\tilde{n}}$, and the customer in service at that instant has left the system before the arrival of C_{-n}. In both cases, whenever $C_{-\tilde{n}}$ found a workload equal to $x_{\tilde{n}}$, there exists an index $\hat{n} \in [\![n, \tilde{n}]\!]$ such that the system is empty at the arrival of $C_{-\hat{n}}$. In other words, $\Phi^{\tilde{n}}_\omega(x_{\tilde{n}}) = \Phi^{\hat{n}}_\omega(0)$. As a consequence, there exists a nonnegative integer $i_n := \hat{n} - n$ such that

$$\Phi^j_\omega(0) = x = \Phi^{\tilde{n}}_\omega(x_{\tilde{n}}) = \Phi^{n+i_n}_\omega(0),$$

that is to say $j = L^n_\omega(i_n)$. This is true for all $n \geq 1$, hence $j \in \hat{H}_\omega$. $\square$

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LABORATOIRE DE MATHÉMATIQUES APPLIQUÉES DE COMPIÈGNE
UNIVERSITÉ DE TECHNOLOGIE DE COMPIÈGNE
DÉPARTEMENT GÉNIE INFORMATIQUE
CENTRE DE RECHERCHES DE ROYALLIEU
BP 20 529
60 205 COMPIEGNE CEDEX
FRANCE
E-MAIL: pascal.moyal@utc.fr