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On consistency of the least squares estimators in linear errors-in-variables models with infinite variance errors

Yuliya V. Martsynyuk*

Department of Statistics, University of Manitoba
338 Machray Hall, Winnipeg, MB R3T 2N2, Canada
e-mail: Yuliya.Martsynyuk@UManitoba.CA

Abstract: This paper deals simultaneously with linear structural and functional errors-in-variables models (SEIVM and FEIVM), revisiting in this context the ordinary least squares estimators (LSE) for the slope and intercept of the corresponding simple linear regression. It has been known that, subject to some model conditions, these estimators become weakly and strongly consistent in the linear SEIVM and FEIVM with the measurement errors having finite variances when the explanatory variables have an infinite variance in the SEIVM, and a similar infinite spread in the FEIVM, while otherwise, the LSE's require an adjustment for consistency with the so-called reliability ratio. In this paper, weak and strong consistency, with and without the possible rates of convergence being determined, is proved for the LSE's of the slope and intecept, assuming that the measurement errors are in the domain of attraction of the normal law (DAN) and thus are, for the first time, allowed to have infinite variances. Moreover, these results are obtained under the conditions that the explanatory variables are in DAN, have an infinite variance, and dominate the measurement errors in terms of variation in the SEIVM, and under appropriately matching versions of these conditions in the FEIVM. This duality extends a previously known interplay between SEIVM's and FEIVM's.

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Contents

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1. Introduction and main results

1.1. *Linear structural and functional errors-in-variables models (***SEIVM** *and* **FEIVM***)*

We consider the linear errors-in-variables model (EIVM)

$$y_i = \beta\xi_i + \alpha + \delta_i, \quad x_i = \xi_i + \varepsilon_i, \tag{1.1}$$

where $(y_i, x_i) \in \mathbb{R}^2$ are vectors of observations, ξ_i are unknown explanatory/latent variables, the real-valued slope β and intercept α are to be estimated, and δ_i and ε_i are unknown measurement error terms/variables, $1 \leq i \leq n$, $n \in \mathbb{N}$. EIVM (1.1) is also known as a measurement error model, or regression with errors in variables. It is a generalization of the simple linear regression of the form $y_i = \beta\xi_i + \alpha + \delta_i$ in that in (1.1) it is assumed that, in addition to the two variables $\eta := \beta\xi + \alpha$ and ξ being linearly related, now not only η, but also ξ, are observed with respective measurement errors δ_i and ε_i.

This paper deals simultaneously with structural and functional versions of EIVM (1.1) (SEIVM and FEIVM). In an SEIVM the explanatory variables ξ_i are assumed to be independent identically distributed (i.i.d.) random variables (r.v.'s) that are independent of the error terms, while in case of an FEIVM, one treats them as deterministic variables. The vectors of the error terms $\{(\delta,\varepsilon),(\delta_i,\varepsilon_i), i \geq 1\}$ are usually, and also presently, assumed to be i.i.d. mean zero random vectors.

Hereafter, the following notations will be used:

$$\overline{u}_n = \frac{1}{n}\sum_{i=1}^{n} u_i, \quad \overline{u^2}_n = \frac{1}{n}\sum_{i=1}^{n} u_i^2, \quad \overline{uv}_n = \frac{1}{n}\sum_{i=1}^{n} u_i v_i,$$

$$s_{i,uv} = (u_i - c\,\overline{u}_n)(v_i - c\,\overline{v}_n), \quad \text{and} \quad S_{uv} = \frac{1}{n}\sum_{i=1}^{n} s_{i,uv},$$

where $\{u_i,\, 1 \leq i \leq n\}$ and $\{v_i,\, 1 \leq i \leq n\}$ are real-valued variables and constant

$$c = \begin{cases} 0, & \text{if the intercept } \alpha \text{ is known to be zero,} \\ 1, & \text{if the intercept } \alpha \text{ is unknown.} \end{cases} \tag{1.2}$$

1.2. Least squares estimators for the slope and intercept in **SEIVM***'s*

It is well-known that the ordinary least squares estimators (LSE's) of the slope and intercept of the simple linear regression $y_i = \beta x_i + \alpha + \delta_i$, $1 \le i \le n$, that is

$$\hat{\beta}_n = \frac{S_{xy}}{S_{xx}} \quad \text{and} \quad \hat{\alpha}_n = \overline{y}_n - \hat{\beta}_n \overline{x}_n, \tag{1.3}$$

are inconsistent in SEIVM (1.1) when $0 < \operatorname{Var}\xi, \operatorname{Var}\delta, \operatorname{Var}\varepsilon < \infty$. However, if $E(\delta\varepsilon) = 0$, using the so-called reliability ratio k_ξ that is defined via what is known as the signal-to-noise ratio k as

$$k := \frac{E\xi^2 - c(E\xi)^2}{\operatorname{Var}\varepsilon} \quad \text{and} \quad k_\xi := \frac{k}{k+1} = \frac{E\xi^2 - c(E\xi)^2}{E\xi^2 - c(E\xi)^2 + \operatorname{Var}\varepsilon}, \tag{1.4}$$

one can adjust $\hat{\beta}_n$ and $\hat{\alpha}_n$ and obtain consistent estimators for β and α as follows:

$$\tilde{\beta}_n = k_\xi^{-1} \hat{\beta}_n \quad \text{and} \quad \tilde{\alpha}_n = \overline{y}_n - \tilde{\beta}_n \overline{x}_n. \tag{1.5}$$

The reliability ratio k_ξ is a measure of relative spread of the explanatory variables ξ_i to that of the observables x_i, and, clearly, $0 < k_\xi < 1$. Larger values of k lead to larger values of k_ξ and to that ξ_i are more dominant over the measurement errors ε_i, and to that x_i, and hence the statistical inference in SEIVM (1.1), are more precise.

To ensure identifiability and the possibility of consistent estimation of unknown parameters in the model (1.1), such as β and α for example, it is common in the literature to make use of some side conditions in this regard. Assuming prior knowledge of k_ξ of (1.4) in SEIVM (1.1) is one of the several standard conditions of this kind (cf. Cheng and Van Ness (1999) for further details on identifiability in (1.1)). In practice, this assumption is usually unrealistic. Hence, (consistent) estimation of the reliability ratio k_ξ has become a standard practice in SEIVM (1.1). The estimators $\tilde{\beta}_n$ and $\tilde{\alpha}_n$ in (1.5), with known or estimated k_ξ, are also known as the correction-for-attenuation estimators for β and α.

A new type of SEIVM (1.1), with new asymptotic methodologies and results, was introduced in Martsynyuk (2004), and then studied also in Martsynyuk (2005, 2007a, 2007b, 2009), where the explanatory variables ξ_i are, for the first time, assumed to belong to the domain of attraction of the normal law (DAN) with a possibly infinite variance. In particular, this enriched the traditional two-moment space of the explanatory variables that had been used for consistency and central limit theorems studies in SEIVM (1.1) in the literature.

Remark 1.1. For i.i.d. r.v.'s $\{\xi, \xi_i, i \ge 1\}$, $\xi \in$ DAN means that there are constants a_n and b_n, $b_n > 0$, for which $(\sum_{i=1}^n \xi_i - a_n) b_n^{-1} \xrightarrow{\mathcal{D}} N(0,1)$, $n \to \infty$, where a_n can be taken as $nE\xi$ and $b_n = \sqrt{n}\ell_\xi(n)$, where $\ell_\xi(n)$ is a slowly varying function at infinity defined by the distribution of ξ, that is $\ell_\xi(az)/\ell_\xi(z) \to 1$, as $z \to \infty$, for any $a > 0$. If $\xi \in$ DAN, then $E|\xi|^\nu < \infty$ for all $\nu \in (0,2)$, and $\ell_\xi(n) = \sqrt{\operatorname{Var}\xi} > 0$, if $\operatorname{Var}\xi < \infty$, and $\ell_\xi(n) \nearrow \infty$, as $n \to \infty$, if

$\operatorname{Var}\xi = \infty$. Also, $\xi \in \mathrm{DAN}$ with some nonstochastic constants a_n and $b_n > 0$ if and only if $\sum_{i=1}^n (\xi_i - E\xi)^2/b_n^2 \xrightarrow{P} 1$, $n \to \infty$, with some nonstochastic constants $b_n > 0$ (cf. Feller (1971, p. 236, Theorem 2)), and hence, $\xi \in \mathrm{DAN}$ implies that $\sum_{i=1}^n (\xi_i - \overline{\xi}_n)^2/(n\ell_\xi^2(n)) \xrightarrow{P} 1$ and, if also $\operatorname{Var}\xi = \infty$, that $\sum_{i=1}^n \xi_i^2/(n\ell_\xi^2(n)) \xrightarrow{P} 1$, as $n \to \infty$. In addition, $\xi \in \mathrm{DAN}$ if and only if $\max_{1\le i\le n} \xi_i^2 / \sum_{i=1}^n \xi_i^2 \xrightarrow{P} 0$, $n \to \infty$ (cf. Breiman (1965)).

Example 1.1. From Remark 1.1, all the distributions with finite positive variances are in DAN. As to some examples of the distributions in DAN that have infinite variances, a Pareto distribution and its modification that has a somewhat heavier tail, with the respective probability density functions (pdf's)

$$f_1(u) = \begin{cases} \dfrac{2}{u^3}, & \text{if } u > 1, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$f_2(u) = \begin{cases} \dfrac{4\log u}{u^3}, & \text{if } u > 1, \\ 0, & \text{otherwise,} \end{cases}$$

were shown to belong to DAN and to have the following slowly varying functions at infinity in the respective norming constants b_n as in Remark 1.1:

$$\ell_1(n) = \sqrt{\log n} \quad \text{and} \quad \ell_2(n) = \frac{\log n}{\sqrt{2}}$$

(cf. Martsynyuk (2013, Example 1)).

Among other things, it was observed in Martsynyuk (2005, Remark 1.1.6) that the LSE's $\hat{\beta}_n$ and $\hat{\alpha}_n$ of (1.3), as well as the estimators

$$\tilde{\tilde{\beta}}_n = \frac{S_{yy}}{S_{xy}} \quad \text{and} \quad \tilde{\tilde{\alpha}}_n = \overline{y}_n - \tilde{\tilde{\beta}}_n \overline{x}_n, \tag{1.6}$$

are strongly consistent in SEIVM (1.1) with $0 < \operatorname{Var}\delta < \infty$ and $0 < \operatorname{Var}\varepsilon < \infty$ when $\operatorname{Var}\xi = \infty$ (independently of whether $E(\delta\varepsilon) = 0$ or not). Thus, unlike in the traditional model with $0 < \operatorname{Var}\xi < \infty$, the LSE's do not require any adjustments for consistency if $\operatorname{Var}\xi = \infty$, when one can formally put $k_\xi := 1$. This can be interpreted as follows: the impact of the finite variance measurement errors ε_i in the observables x_i is negligible as compared to that of the infinite variance explanatory variables ξ_i, so much so that the model becomes close in spirit to, and behaves as if it were, the simple linear regression $y_i = \beta x_i + \alpha + \delta_i$, $1 \le i \le n$. The LSE's of (1.3) and estimators in (1.6) in SEIVM (1.1) with $\operatorname{Var}\xi = \infty$ add to a handful of examples of consistent estimators in special SEIVM's that do not require any additional information, such as prior knowledge of k_ξ for example (cf. Van Montfort (1988) and texts Kendall and Stuart (1979) and Cheng and Van Ness (1999) for details on these examples). It is interesting to note that the existence of these consistent estimators for β implies that β is identifiable, and the latter fact, when (δ, ε) has a normal distribution, can also be concluded from Reiersøl (1950), as it accordingly holds if and only if ξ is not normally distributed.

1.3. Least squares estimators in **FEIVM***'s*

We now turn our attention to FEIVM (1.1), a companion to SEIVM (1.1), and describe a parallel picture on the LSE's of (1.3) in it.

When $0 < \operatorname{Var}\delta < \infty$ and $0 < \operatorname{Var}\varepsilon < \infty$, just like in SEIVM (1.1) with $0 < \operatorname{Var}\xi < \infty$, the LSE's of (1.3) are inconsistent in FEIVM (1.1) with the deterministic explanatory variables $\{\xi_i\}_{i\geq 1}$ satisfying the assumptions

$$\Big|\lim_{n\to\infty} \overline{\xi}_n\Big| < \infty \quad \text{and} \quad 0 < \lim_{n\to\infty} (\overline{\xi^2}_n - (\overline{\xi}_n)^2) < \infty, \tag{1.7}$$

which have been most common for FEIVM (1.1). The estimators in (1.5), with

$$\frac{\lim_{n\to\infty} S_{\xi\xi}}{\lim_{n\to\infty} S_{\xi\xi} + \operatorname{Var}\varepsilon} \tag{1.8}$$

in place of k_ξ of (1.4), are adjustments of the LSE's for strong consistency when $E(\delta\varepsilon) = 0$, where, similarly to k_ξ, the ratio in (1.8) usually requires estimation and may be viewed as a measure of relative spread of the explanatory variables to that of the error terms.

In Martsynyuk (2005, 2007b, 2009), simultaneously with SEIVM (1.1) with $\xi \in \mathrm{DAN}$, we studied FEIVM (1.1) and established new asymptotics in it under the conditions on the deterministic explanatory variables that match the condition $\xi \in \mathrm{DAN}$, and hence are also new and most general in the context. Accordingly, we assumed that

$$\begin{aligned}
&\Big|\lim_{n\to\infty} \overline{\xi}_n\Big| < \infty,\ 0 < \lim_{n\to\infty} (\overline{\xi^2}_n - (\overline{\xi}_n)^2) \text{ and,} \\
&\text{if } \lim_{n\to\infty} (\overline{\xi^2}_n - (\overline{\xi}_n)^2) = \infty, \text{ also } \lim_{n\to\infty} \frac{\max_{1\leq i\leq n} \xi_i^2}{\sum_{i=1}^n \xi_i^2} = 0.
\end{aligned} \tag{1.9}$$

This also led to the obtained asymptotics being very similar in form for the SEIVM and FEIVM in hand that, in turn, extended a previously known interplay between SEIVM's and FEIVM's (cf. Martsynyuk (2005, pp. 158–159) and Martsynyuk (2007b, Section 2.2)).

It was argued in Martsynyuk (2005, Remark 2.1.10 (e)) that the LSE's $\hat{\beta}_n$ and $\hat{\alpha}_n$ of (1.3), as well as the esimators in (1.6), are weakly consistent estimators of the slope and intercept in FEIVM (1.1), provided that $|\lim_{n\to\infty} \overline{\xi}_n| < \infty$, $\lim_{n\to\infty}(\overline{\xi^2}_n - (\overline{\xi}_n)^2) = \infty$, $0 < \operatorname{Var}\delta < \infty$, and $0 < \operatorname{Var}\varepsilon < \infty$, while these estimators are strongly consistent if, additionally, all the four limits in (1.9) are satisfied and $E|\delta|^{2+\Delta}, E|\varepsilon|^{2+\Delta} < \infty$ for some $\Delta > 0$. The limit $\lim_{n\to\infty}(\overline{\xi^2}_n - (\overline{\xi}_n)^2) = \infty$ parallels the condition $\operatorname{Var}\xi = \infty$ in SEIVM (1.1) discussed above, makes the ratio in (1.8) take its maximal possible value 1, and results in FEIVM (1.1) resembling the corresponding simple linear regression $y_i = \beta x_i + \alpha + \delta_i$, $1 \leq i \leq n$, due to the effect of the measurement errors ε_i being less pronounced.

Liu and Chen (2005) proved that in FEIVM (1.1) with $0 < \operatorname{Var}\delta < \infty$ and $0 < \operatorname{Var}\varepsilon < \infty$, the LSE $\hat{\beta}_n$ of (1.3) is consistent, both strongly and weakly if and only if $\lim_{n\to\infty}(\overline{\xi^2}_n - (\overline{\xi}_n)^2) = \infty$, while the LSE $\hat{\alpha}_n$ is a weakly consistent estimator of α if and only if $\lim_{n\to\infty} n\overline{\xi}_n/\max(n, \sum_{i=1}^n(\xi_i - \overline{\xi}_n)^2) = 0$.

Miao *et al.* (2011), among other things, refined the results of Liu and Chen (2005) by obtaining rates of strong and weak consistency for $\hat{\beta}_n$ and $\hat{\alpha}_n$ in FEIVM (1.1) (with $c = 1$ in (1.2)) as follows:

$$\begin{aligned}&\text{if } 0 < E|\delta|^p < \infty \text{ and } 0 < E|\varepsilon|^p < \infty \text{ for some } p \geq 2, \text{ and}\\ &\lim_{n\to\infty} S_{\xi\xi}/n^{1-2/p} = \infty, \text{ then } n^{-1/p}\sqrt{nS_{\xi\xi}}(\hat{\beta}_n - \beta) \xrightarrow{a.s.} 0,\\ &\text{and, if also } n^{1/2-\theta+1/p}|\overline{\xi}_n|/\sqrt{S_{\xi\xi}} = O(1) \text{ for some } \theta \in (1/2, 1],\\ &\text{then } n^{1-\theta}(\hat{\alpha}_n - \alpha) \xrightarrow{a.s.} 0, \text{ as } n \to \infty;\end{aligned} \tag{1.10}$$

$$\begin{aligned}&\text{if } 0 < \operatorname{Var}\delta < \infty,\ 0 < \operatorname{Var}\varepsilon < \infty,\ \lim_{n\to\infty} S_{\xi\xi} = \infty, \text{ and}\\ &\lim_{n\to\infty} S_{\xi\xi}\tilde{b}_n^2/n = \infty \text{ for some real numbers } \tilde{b}_n \text{ such that } 0 < \tilde{b}_n \to \infty,\\ &\text{then } \tilde{b}_n^{-1}\sqrt{nS_{\xi\xi}}(\hat{\beta}_n - \beta) \xrightarrow{P} 0 \text{ and, if also } \lim_{n\to\infty}(\overline{\xi}_n)^2/(\tilde{b}_n^2 S_{\xi\xi}) = 0 \text{ and}\\ &\lim_{n\to\infty} n^{1/2}|\overline{\xi}_n|/(\tilde{b}_n S_{\xi\xi}) = 0, \text{ then } \tilde{b}_n^{-1}\sqrt{n}(\hat{\alpha}_n - \alpha) \xrightarrow{P} 0, \text{ as } n \to \infty.\end{aligned} \tag{1.11}$$

1.4. Model assumptions and introduction to main results

The results of Martsynyuk (2004, 2005), Liu and Chen (2005) and Miao *et al.* (2011) in connection with consistency of the LSE's of (1.3) in EIVM (1.1), which were discussed in sections 1.2 and 1.3, are all for the model with $0 < \operatorname{Var}\delta < \infty$ and $0 < \operatorname{Var}\varepsilon < \infty$. In contrast, in this paper we deal with SEIVM and FEIVM (1.1) where both measurement errors δ and ε are, for the first time, allowed to have infinite variances via assuming that

(A1) $\{(\delta, \varepsilon), (\delta_i, \varepsilon_i)\}_{i\geq 1}$ are i.i.d. mean zero random vectors with $\delta, \varepsilon \in$DAN and the respective slowly varying functions at infinity $\ell_\delta(n)$ and $\ell_\varepsilon(n)$ that are such that $\sum_{i=1}^n \delta_i/(\sqrt{n}\ell_\delta(n)) \xrightarrow{\mathcal{D}} N(0, 1)$ and $\sum_{i=1}^n \varepsilon_i/(\sqrt{n}\ell_\varepsilon(n)) \xrightarrow{\mathcal{D}} N(0, 1)$, as $n \to \infty$ (cf. Remark 1.1).

Concerning our conditions on the explanatory variables, throughout the paper,

$$\text{(A2)}\quad \begin{cases} \{\xi, \xi_i\}_{i\geq 1} \text{ are i.i.d. r.v.'s with } \xi \in \text{DAN},\ \operatorname{Var}\xi = \infty, & \\ \text{and the slowly varying function at infinity } \ell_\xi(n) \text{ as} & \\ \text{in Remark 1.1, and } \xi \text{ is independent of } (\delta, \varepsilon), & \text{in SEIVM (1.1)},\\ \{\xi_i\}_{i\geq 1} \text{ are deterministic and } \lim\limits_{n\to\infty} S_{\xi\xi} = \infty, & \text{in FEIVM (1.1)}. \end{cases}$$

Sometimes, we will also assume that

$$\text{(A3)} \qquad \limsup_{n\to\infty} |\overline{\xi}_n| < \infty \quad \text{in FEIVM (1.1)}.$$

The main Theorems 1.1 and 1.2 of the present paper prove respectively weak and strong consistency of the LSE's $\hat{\beta}_n$ and $\hat{\alpha}_n$ under (A1)–(A3), and some additional assumptions that ensure that the explanatory variables dominate the measurement errors in terms of variation, a natural requirement for obtaining meaningful inference in the model (1.1). In our main Theorems 1.3 and 1.4, we refine the results of Theorems 1.1 and 1.2 and establish possible rates of weak and strong consistency of $\hat{\beta}_n$ and $\hat{\alpha}_n$.

To the best of our knowledge, EIVM (1.1) with the explanatory variables having an infinite variance or spread (as in (A2)), and with the error terms possibly having infinite variances (as in (A1)), as well as estimation problems in this model, have not been previously studied in the literature. On the other hand, various authors have studied practical and theoretical aspects of linear regression when both errors and regressors may have infinite variances, and established asymptotics for the LSE's for the slope and intercept in it. Initial work in this regard was offered in Blattberg and Sargent (1971) and Smith (1973), under the condition that the errors followed stable laws. Andrews (1987a, 1987b) provides, among other things, a complete list of references for applications of infinite variance regression, particularly in economics. Assuming that the regressors are in a stable domain of attraction (in particular, in DAN), Cline (1989) considers the LSE's for the slope and intercept in linear regression and determines necessary and sufficient conditions for their weak consistency in terms of a relationship between the regressors' and errors' distributions. The latter relationship roughly amounts to a certain asymptotic dominance of the tail probabilities of the regressors over those of the errors. For some further related works on infinite variance linear regression models and asymptotics for the LSE's in them, we refer to a useful survey of the literature in Cline (1989).

1.5. Main results with remarks

Theorem 1.1 (weak consistency of the LSE's)**.** *Let* (A1) *and* (A2) *be satisfied. Assume also that, as $n \to \infty$,*

$$\begin{cases} \dfrac{\ell_\varepsilon^2(n)}{\ell_\xi^2(n)} \to 0, & \textit{in } \text{SEIVM (1.1)}, \\[2ex] \dfrac{\ell_\varepsilon^2(n)}{S_{\xi\xi}} \to 0, & \textit{in } \text{FEIVM (1.1)}, \end{cases} \tag{1.12}$$

and, if $\operatorname{Var}\delta = \infty$ *and* $E|\delta\varepsilon| = \infty$*, that*

$$\begin{cases} \dfrac{\ell_\varepsilon(n)\ell_\delta(n)}{\ell_\xi^2(n)} \to 0, & \textit{in } \text{SEIVM (1.1)}, \\[2ex] \dfrac{\ell_\varepsilon(n)\ell_\delta(n)}{S_{\xi\xi}} \to 0, & \textit{in } \text{FEIVM (1.1)}. \end{cases} \tag{1.13}$$

Then,

$$\hat{\beta}_n \xrightarrow{P} \beta, \quad n \to \infty. \tag{1.14}$$

If (A3) *is also valid in* FEIVM (1.1)*, then*

$$\hat{\alpha}_n \xrightarrow{P} \alpha, \quad n \to \infty. \tag{1.15}$$

Hereafter, without loss of generality, we assume for convenience that $S_{\xi\xi} > 0$ for *all* $n \geq 1$ in FEIVM (1.1), in view of having $\lim_{n\to\infty} S_{\xi\xi} = \infty$ in (A2).

Theorem 1.2 (strong consistency of the LSE's)**.** *Let* (A1) *and* (A2) *hold true. In* SEIVM (1.1)*, assume that* $\operatorname{Var}\varepsilon < \infty$ *and, if* $\operatorname{Var}\delta = \infty$*, that* $E|\delta\varepsilon| < \infty$*. In* FEIVM (1.1)*, if* $\operatorname{Var}\varepsilon = \infty$*, let*

$$\sum_{n=1}^{\infty} \frac{1}{n} \frac{E\varepsilon^2 \mathbb{1}_{\{|\varepsilon| \leq n^{1/2+d}\}}}{S_{\xi\xi}} < \infty \quad \textit{for some} \quad d > 0, \tag{1.16}$$

and, if $\operatorname{Var}\delta = \infty$*, suppose additionally that* (A3) *(if* α *is not known to be zero) and one of* (2.2)–(2.5) *are satisfied, and that either* $E|\delta\varepsilon| < \infty$*, or* $E|\delta\varepsilon| = \infty$ *and*

$$\sum_{n=1}^{\infty} \frac{1}{n} \frac{\left(E\varepsilon^2 \mathbb{1}_{\{|\varepsilon| \leq n^{1/2+\nu}\}}\right)^{1/2} \left(E\delta^2 \mathbb{1}_{\{|\delta| \leq n^{1/2+\eta}\}}\right)^{1/2}}{S_{\xi\xi}} < \infty \quad \textit{for some} \quad \nu, \eta > 0. \tag{1.17}$$

Then,

$$\hat{\beta}_n \xrightarrow{a.s.} \beta, \quad n \to \infty. \tag{1.18}$$

If (A3) *is also valid in* FEIVM (1.1)*, then*

$$\hat{\alpha}_n \xrightarrow{a.s.} \alpha, \quad n \to \infty. \tag{1.19}$$

Remark 1.2. When $\operatorname{Var}\xi, \operatorname{Var}\varepsilon < \infty$ in SEIVM (1.1) and $\lim_{n\to\infty} S_{\xi\xi}, \operatorname{Var}\varepsilon < \infty$ in FEIVM (1.1), the respective ratios $\ell_\xi^2(n)/\ell_\varepsilon^2(n)$ and $\lim_{n\to\infty} S_{\xi\xi}/\ell_\varepsilon^2(n)$ that appear in (1.12) coincide with the signal-to-noise ratio k of (1.4) and its prototype $\lim_{n\to\infty} S_{\xi\xi}/\operatorname{Var}\varepsilon$ in FEIVM's (cf. (1.8) and the lines below it). Otherwise, $\ell_\xi^2(n)/\ell_\varepsilon^2(n)$ and $\lim_{n\to\infty} S_{\xi\xi}/\ell_\varepsilon^2(n)$ extend the notion of the signal-to-noise ratio as, in view of Remark 1.1, if $\xi \in$ DAN with $\operatorname{Var}\xi = \infty$ (or if $\lim_{n\to\infty} S_{\xi\xi} = \infty$) and $\varepsilon \in$ DAN with $\operatorname{Var}\varepsilon \leq \infty$, then $\ell_\xi^2(n)$ (or $\lim_{n\to\infty} S_{\xi\xi}$) and $\ell_\varepsilon^2(n)$ continue to be the respective measures of spread of the explanatory variables ξ_i and measurement errors ε_i. Condition (1.12) states that ξ_i must vary substantialy more than ε_i do, so much so that the signal-to-noise ratio converges to infinity, as $n \to \infty$. It agrees with the intuitive notion that in (1.1) the signals ξ_i should dominate the errors ε_i in order to diminish the effect of the latter and thus observe more precise data x_i resulting in more precise estimators of β and α. For example, in SEIVM (1.1), when ε_i follow the Pareto distribution with the pdf $f_1(u)$ as in Example 1.1, while ξ_i have the pdf $f_2(u)$ of that example, with a heavier tail and thus a larger variation, then $\ell_\varepsilon^2(n)/\ell_\xi^2(n) = \log n/(\log^2 n/2) \to 0$, as $n \to \infty$, that is (1.12) is satisfied. It is also natural and desirable to control the effect of the measurement errors δ_i on inference in EIVM (1.1). In Theorem 1.1, if $\operatorname{Var}\delta = \infty$ and $E|\delta\varepsilon| = \infty$, we have condition (1.13) in this regard. For example, in SEIVM (1.1), (1.13) amounts to saying that, further to having $\ell_\varepsilon^2(n)/\ell_\xi^2(n) \to 0$, as in (1.12), we assume that $(\ell_\delta^2(n)/\ell_\xi^2(n))(\ell_\varepsilon^2(n)/\ell_\xi^2(n)) \to 0$, as $n \to \infty$.

Remark 1.3. Conditions (1.16) and (1.17) for strong consistency of $\hat{\beta}_n$ and $\hat{\alpha}_n$ in FEIVM (1.1) in Theorem 1.2 are of the same essence as, and amount to stronger versions of, respective conditions (1.12) and (1.13) required for weak consistency of these estimators in Theorem 1.1. Indeed, in view of (2.26), functions $\ell_\varepsilon^2(n)$ and $\ell_\delta^2(n)$ in (1.12) and (1.13) can be replaced with $E\varepsilon^2 \mathbb{1}_{\{|\varepsilon| \le \sqrt{n}\ell_\varepsilon(n)\}}$ and $E\delta^2 \mathbb{1}_{\{|\delta| \le \sqrt{n}\ell_\delta(n)\}}$ that are also slowly varying functions at infinity (cf. (2.10)). Thus, if $\operatorname{Var}\varepsilon = \infty$, from (1.16) for example, the ratio $E\varepsilon^2 \mathbb{1}_{\{|\varepsilon| \le n^{1/2+d}\}}/S_{\xi\xi}$ approaches zero, as $n \to \infty$, and does so at an appropriate rate, and this and (2.11) clearly imply that $\ell_\varepsilon^2(n)/S_{\xi\xi} = (1 + o(1))E\varepsilon^2 \mathbb{1}_{\{|\varepsilon| \le \sqrt{n}\ell_\varepsilon(n)\}}/S_{\xi\xi} \to 0$, $n \to \infty$, as in (1.12). As to how fast $E\varepsilon^2 \mathbb{1}_{\{|\varepsilon| \le n^{1/2+d}\}}/S_{\xi\xi}$ in (1.16) and $(E\varepsilon^2 \mathbb{1}_{\{|\varepsilon| \le n^{1/2+\nu}\}} E\delta^2 \mathbb{1}_{\{|\delta| \le n^{1/2+\eta}\}})^{1/2}/S_{\xi\xi}$ in (1.17) may converge to zero, we note that the series in (1.16) and (1.17) are both of form $\sum_{n=1}^{\infty} f(n)/n$. By the MacLaurin-Cauchy integral criterion, examples of convergent series of this form are

$$\sum_{n>1} \frac{1}{n(\ln n)^q}, \quad \sum_{n>e} \frac{1}{n \ln n (\ln\ln n)^q}, \quad \sum_{n>e^e} \frac{1}{n \ln n \ln\ln n (\ln\ln\ln n)^q}, \ldots, \text{for } q > 1. \tag{1.20}$$

Thus, if $\operatorname{Var}\varepsilon = \infty$ and the slowly varying function $E\varepsilon^2 \mathbb{1}_{\{|\varepsilon| \le n^{1/2+d}\}}$ converges to infinity, as $n \to \infty$, then in (1.16), $S_{\xi\xi}$ may also be a slowly varying function, but with the rate of convegence to infinity being, for example, at least $(\ln n)^q$ times as fast as that of $E\varepsilon^2 \mathbb{1}_{\{|\varepsilon| \le n^{1/2+d}\}}$. In particular, if ε has the Pareto distribution with an infinite variance and the pdf $f_1(u)$ as in Example 1.1, then $E\varepsilon^2 \mathbb{1}_{\{|\varepsilon| \le n^{1/2+d}\}} = (1 + 2d)\log n$, and the rate of convergence to infinity of $S_{\xi\xi}$ may be as slow as $(\log n)^{q+1}$ for example, where $q > 1$. For discussions on conditions (2.2)–(2.5) that are used in Theorem 1.2, we refer to Remark 2.1 after the proof of Lemma 2.1.

Remark 1.4. As briefly mentioned in the introduction, SEIVM (1.1) and FEIVM (1.1) have exhibited some interplay in the literature in that they share similar asymptotic results provided that the respective conditions on error and explanatory variables in FEIVM (1.1) resemble those in SEIVM (1.1). Theorem 1.1 of the present paper adds further to this interplay: $\hat{\beta}_n$ and $\hat{\alpha}_n$ are weakly consistent both in SEIVM (1.1) and FEIVM (1.1), under the same assumptions on the error terms in (A1), while the conditions on the explanatory variables in (A2) and those on ξ_i versus $(\delta_i, \varepsilon_i)$ in (1.12) and (1.13) in FEIVM (1.1) are simply deterministic versions of the respective conditions in SEIVM (1.1). In view of such duality between the two models in Theorem 1.1, we believe that strong consistency for $\hat{\beta}_n$ and $\hat{\alpha}_n$ in SEIVM (1.1) in Theorem 1.2 should also hold true when $\operatorname{Var}\varepsilon = \infty$ and $E|\delta\varepsilon|$ is not necessarily finite, under similar assumptions to those in FEIVM (1.1) in Theorem 1.2. However, we were unable to prove this, since we could not verify one of the key convergence for the proof of Theorem 1.2 when $\operatorname{Var}\varepsilon = \operatorname{Var}\xi = \infty$, namely that $S_{\varepsilon\varepsilon}/S_{\xi\xi} \xrightarrow{a.s.} 0$, $n \to \infty$.

Remark 1.5. We note that in the special case of (A1) when $\operatorname{Var}\delta, \operatorname{Var}\varepsilon < \infty$, Theorems 1.1 and 1.2 hold true simply under (A2) and, in case of consistency of $\hat{\alpha}_n$, also under (A3). Hence, Theorems 1.1 and 1.2 extend weak and strong consistency results for $\hat{\beta}_n$ and $\hat{\alpha}_n$ in SEIVM and FEIVM (1.1) that were previously obtained in Martsynyuk (2004, 2005) and Liu and Chen (2005) (cf. sections 1.2 and 1.3).

The following theorem provides refinements of (1.14) and (1.15) of Theorem 1.1, under some stronger model assumptions than those used in Theorem 1.1.

Theorem 1.3 (rates of weak consistency of the LSE's)**.** *Let* (A1)–(A3) *be satisfied. Suppose that there exist a sequence of positive real numbers* $\{b_n\}_{n\geq 1}$ *such that, as* $n \to \infty$, $b_n \to \infty$ *and*

$$\begin{cases} b_n \dfrac{\ell_\varepsilon^2(n)}{\ell_\xi^2(n)} \to 0, & \textit{in } \text{SEIVM (1.1)}, \\ b_n \dfrac{\ell_\varepsilon^2(n)}{S_{\xi\xi}} + \dfrac{b_n^2}{n} \dfrac{\ell_\varepsilon^2(n) + \ell_\delta^2(n)}{S_{\xi\xi}} \to 0, & \textit{in } \text{FEIVM (1.1)}, \end{cases} \tag{1.21}$$

and, if $\operatorname{Var}\delta = \infty$ *and* $E|\delta\varepsilon| = \infty$, *that*

$$\begin{cases} b_n \dfrac{\ell_\varepsilon(n)\ell_\delta(n)}{\ell_\xi^2(n)} \to 0, & \textit{in } \text{SEIVM (1.1)}, \\ b_n \dfrac{\ell_\varepsilon(n)\ell_\delta(n)}{S_{\xi\xi}} \to 0, & \textit{in } \text{FEIVM (1.1)}. \end{cases} \tag{1.22}$$

Then,

$$b_n(\hat{\beta}_n - \beta) \xrightarrow{P} 0, \quad n \to \infty. \tag{1.23}$$

If also

$$\frac{b_n^2}{n}\left(\ell_\varepsilon^2(n) + \ell_\delta^2(n)\right) \to 0 \quad \textit{in } \text{FEIVM (1.1)}, \quad n \to \infty, \tag{1.24}$$

then

$$b_n(\hat{\alpha}_n - \alpha) \xrightarrow{P} 0, \quad n \to \infty. \tag{1.25}$$

Remark 1.6. In FEIVM (1.1), if the explanatory variables $\{\xi_i\}_{i\geq 1}$ behave as if they were i.i.d. r.v.'s in DAN and had an infinite variance, like in SEIVM (1.1), so that, like in Remark 1.1,

$$\lim_{n\to\infty} S_{\xi\xi}/\nu(n) = 1, \tag{1.26}$$

with some slowly varying function at infinity $\nu(\cdot)$, then the respective condition in (1.21) reduces to having only $b_n \ell_\varepsilon^2(n)/S_{\xi\xi} \to 0$, $n \to \infty$, while that of (1.24) is automatically satisfied. Indeed, (1.26) and convergence to zero of the first summand in (1.21) imply that b_n cannot converge to infinity as fast as, or faster than, a slowly varying function $S_{\xi\xi}$ does. This, via (2.11), implies (1.24) and that $b_n^2(\ell_\varepsilon^2(n)+\ell_\delta^2(n))/(nS_{\xi\xi}) = (b_n/S_{\xi\xi})[b_n(\ell_\varepsilon^2(n)+\ell_\delta^2(n))/n] \to 0$ in (1.21), as $n\to\infty$.

Remark 1.7. In the special case of FEIVM (1.1) with $0<\operatorname{Var}\delta<\infty$ and $0<\operatorname{Var}\varepsilon<\infty$, the results and respective conditions of Theorem 1.3 reduce to those of Miao *et al.* (2011, Theorems 2.3 and 2.4) quoted in (1.11), provided (A3) is assumed when dealing with $\hat{\alpha}_n$. Accordingly, (1.23) with $b_n = \tilde{b}_n^{-1}\sqrt{nS_{\xi\xi}}$ holds true if $S_{\xi\xi}\to\infty$ and $S_{\xi\xi}\tilde{b}_n^2/n\to\infty$, and, if additionally (A3) is satisfied, we have (1.25) with $b_n=\tilde{b}_n^{-1}\sqrt{n}$, where positive real numbers $\tilde{b}_n$ are such that $\tilde{b}_n\to\infty$, as $n\to\infty$.

Theorem 1.4 (rates of strong consistency of the LSE's)**.** *Let* (A1)–(A3) *hold true. In* SEIVM (1.1)*, on assuming that* $\operatorname{Var}\varepsilon<\infty$ *and, if* $\operatorname{Var}\delta=\infty$*, that* $E|\delta\varepsilon|<\infty$*, we have*

$$S_{\xi\xi}^{1-a}(\hat{\beta}_n-\beta)\xrightarrow{a.s.}0 \quad \textit{and} \quad S_{\xi\xi}^{1-a}(\hat{\alpha}_n-\alpha)\xrightarrow{a.s.}0 \quad \textit{for any } a\in(0,1], \quad n\to\infty. \tag{1.27}$$

In FEIVM (1.1)*, if there exist a sequence of positive real numbers* $\{b_n\}_{n\geq1}$ *such that* $b_n\to\infty$*, as* $n\to\infty$*, and*

$$\begin{cases} \limsup\limits_{n\to\infty}\dfrac{b_n}{\sqrt{S_{\xi\xi}}}<\infty, & \textit{if} \quad \operatorname{Var}\delta,\operatorname{Var}\varepsilon<\infty,\\ \displaystyle\sum_{n=1}^{\infty}\frac{1}{n}\frac{b_n^2\left(E\varepsilon^2\mathbb{1}_{\{|\varepsilon|\leq n^{1/2+\nu}\}}+E\delta^2\mathbb{1}_{\{|\delta|\leq n^{1/2+\eta}\}}\right)}{S_{\xi\xi}}<\infty & \textit{for some } \nu,\eta>0,\\ & \textit{otherwise}, \end{cases} \tag{1.28}$$

then

$$b_n(\hat{\beta}_n-\beta)\xrightarrow{a.s.}0, \quad n\to\infty, \tag{1.29}$$

and, provided also that for some $d,\theta>0$*,*

$$\sum_{n=1}^{\infty}\frac{1}{n}\frac{b_n^2\left(E\varepsilon^2\mathbb{1}_{\{|\varepsilon|\leq n^{1/2+d}\}}+E\delta^2\mathbb{1}_{\{|\delta|\leq n^{1/2+\theta}\}}\right)}{n}<\infty \tag{1.30}$$

and

$$\frac{b_n\left(E\varepsilon^2\mathbb{1}_{\{|\varepsilon|\leq n^{1/2+d}\}}+E\delta^2\mathbb{1}_{\{|\delta|\leq n^{1/2+\theta}\}}\right)}{\sqrt{n}}\to0, \quad n\to\infty, \tag{1.31}$$

we have

$$b_n\left(\hat{\alpha}_n-\alpha\right)\xrightarrow{a.s.}0, \quad n\to\infty. \tag{1.32}$$

Remark 1.8. In FEIVM (1.1), (1.29) and (1.32) of Theorem 1.4 are refinements of (1.18) and (1.19) that were obtained under (2.3) if $\operatorname{Var}\delta=\infty$ in Theorem 1.2, and condition (1.28) of Theorem 1.4 implies those in (2.3) (if $\operatorname{Var}\delta=\infty$) and (1.16) (if $\operatorname{Var}\varepsilon=\infty$), and hence the one in (1.17) as well. One can also obtain versions of Theorem 1.4 corresponding to when $\operatorname{Var}\delta=\infty$ and, instead of

(2.3), one of (2.2), (2.4) and (2.5) is assumed in Theorem 1.2. In FEIVM (1.1), similarly to comparing the assumptions of Theorem 1.2 to those of Theorem 1.1 in Remark 1.3, the conditions of Theorem 1.4 are seen to be stronger than the ones of Theorem 1.3: (1.28), (1.30) and (1.31) imply that convergence to zero in (1.21), (1.22) and (1.24) hold true at an appropriate rate.

Remark 1.9. In view of being unable to estimate with a deterministic sequence how fast $S_{\xi\xi}$ could possibly converge to infinity almost surely when $\xi \in$ DAN and $\operatorname{Var}\xi = \infty$, we provided the stochastic rate of $S_{\xi\xi}^{1-a}$ for strong consistency in (1.27), for any $a \in (0, 1]$. In the special case of $a = 1$, (1.27) reduces to (1.18) and (1.19) obtained for SEIVM (1.1) in Theorem 1.2.

Remark 1.10. Further to Remark 1.7, we compare the results of Theorem 1.4 for FEIVM (1.1) with $\delta, \varepsilon \in$ DAN to (1.10) that is due to Miao *et al.* (2011, Theorems 2.1 and 2.2) and proved under $E(|\delta|^p + |\varepsilon|^p) < \infty$ for some $p \geq 2$. Thus, if $p = 2$, then the speed of strong consistency for the LSE $\hat{\beta}_n$ in (1.10) is $\sqrt{S_{\xi\xi}}$, which is the maximum possible rate b_n in (1.29) of Theorem 1.4 when $\operatorname{Var}\varepsilon, \operatorname{Var}\delta < \infty$, in view of having $\limsup_{n\to\infty} b_n/\sqrt{S_{\xi\xi}} < \infty$ in (1.28). Both consistency results hold true in this case simply if $\lim_{n\to\infty} S_{\xi\xi} = \infty$. If at least one of the error variances is assumed to be infinite in Theorem 1.4, then b_n in (1.29) for strong consistency of $\hat{\beta}_n$ is slower than $\sqrt{S_{\xi\xi}}$ (cf. (1.28)). As to the respective rates of strong consistency of the LSE $\hat{\alpha}_n$ in (1.10) and (1.32), we note that while they are both slower than $\sqrt{n}$, the rate b_n in (1.32) can sometimes be a bit faster than the rate of $n^{1-\theta}$ in (1.10), where $\theta \in (1/2, 1]$: for example, when $\operatorname{Var}\varepsilon, \operatorname{Var}\delta < \infty$, we can have $b_n = n^{1/2}/(\ln n)^{q/2}$ in (1.32), with $q > 1$, under (A2), (A3) and (1.28). Moreover, when (A3) is satisfied and, in particular, $\operatorname{Var}\varepsilon, \operatorname{Var}\delta < \infty$ and $b_n = n^{1-a}$ for some $a \in (1/2, 1]$ in Theorem 1.4, then (1.32) holds true under the conditions of $\lim_{n\to\infty} S_{\xi\xi} = \infty$ and $n^{1-a}/\sqrt{S_{\xi\xi}} = O(1)$, and this amounts to (1.10) for $\hat{\alpha}_n$.

Remark 1.11. By adapting accordingly the conditions of Theorems 1.1–1.4, we can also prove weak and strong consistency, with and without determining the respective possible rates of convergence, for the estimators $\tilde{\tilde{\beta}}_n$ and $\tilde{\tilde{\alpha}}_n$ of (1.6). The statements and proofs of these results are omitted here.

2. Auxiliary results and proofs

2.1. Auxiliary results

Lemma 2.1. *In* FEIVM (1.1)*, let* $\{\delta, \delta_i\}_{i\geq 1}$ *be i.i.d. mean zero r.v.'s and* $\delta \in$ DAN *with* $\operatorname{Var}\delta = \infty$*, and let*

$$S_{\xi\xi} \geq const > 0 \quad \text{for all } n \geq 1. \tag{2.1}$$

Assume that one of the following conditions (2.2)–(2.5) *is satisfied:*

$$\sum_{n=1}^{\infty} \frac{1}{n} \frac{|\xi_n|}{\overline{\xi^2}_n} < \infty; \tag{2.2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \frac{E\delta^2 \mathbb{1}_{\{|\delta| \le n^{1/2+b}\}}}{\overline{\xi^2}_n} < \infty \quad \textit{for some } b > 0; \tag{2.3}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \frac{E\delta^2 \mathbb{1}_{\{|\delta| \le n^{1/2+b}\}}}{\overline{\xi^2}_n} \frac{\xi_n^2}{n\overline{\xi^2}_n} < \infty \quad \textit{for some } b > 0; \tag{2.4}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+a}} \frac{E\delta^2 \mathbb{1}_{\{|\delta| \le (n+1)^{(1+a)(1/2+b)}\}}}{\overline{\xi^2}_{n^{1+a}}} \frac{\sum_{i=n^{1+a}+1}^{(n+1)^{1+a}-1} \xi_i^2}{\sum_{i=1}^{n^{1+a}+1} \xi_i^2} < \infty \quad \textit{for some } a, b > 0. \tag{2.5}$$

If the intercept α of (1.1) *is not known to be zero, suppose also that* (A3) *holds true. Then,*

$$\frac{S_{\xi\delta}}{S_{\xi\xi}} \xrightarrow{a.s.} 0, \quad n \to \infty. \tag{2.6}$$

Proof of Lemma 2.1. The proof of (2.6) reduces to showing that

$$\frac{\sum_{i=1}^n \xi_i \delta_i}{\sum_{i=1}^n \xi_i^2} \xrightarrow{a.s.} 0, \quad n \to \infty, \tag{2.7}$$

on account of having

$$\frac{S_{\xi\delta}}{S_{\xi\xi}} = \left(\frac{\overline{\xi\delta}_n}{\overline{\xi^2}_n} - \frac{c\,\overline{\xi}_n \overline{\delta}_n}{\overline{\xi^2}_n} \right) \left(1 + \frac{c(\overline{\xi}_n)^2}{S_{\xi\xi}} \right)$$

and, if $\alpha \ne 0$ (and c of (1.2) is 1), also applying the strong law of large numbers (SLLN) for $\overline{\delta}_n$ and conditions (2.1) and (A3).

If (2.2) holds true, then (2.7) follows directly from Lemma A.2 of Appendix (with $r = 1$), using only that $E|\delta| < \infty$, while the rest of the proof is dedicated to establishing (2.7) when one of the conditions (2.3)–(2.5) is satisfied.

First, we note that the following three statements are equivalent:

$$\delta \in \text{DAN}; \tag{2.8}$$

$$\frac{z^2 P(|\delta| > z)}{E\delta^2 \mathbb{1}_{\{|\delta| \le z\}}} \to 0, \quad z \to \infty; \tag{2.9}$$

$$\ell(z) := E\delta^2 \mathbb{1}_{\{|\delta| \le z\}} \text{ is a slowly varying function at } \infty. \tag{2.10}$$

That (2.8)⇔(2.9) is due to Lévy (1937), while it follows from Feller (1971, p. 313, Theorem 1a) that (2.8)⇔(2.10).

It is not hard to see next that for any nondecreasing slowly varying function at infinity $\ell(\cdot)$, including $\ell(\cdot)$ of (2.10),

$$\forall a > 0, \quad \frac{\ell(z)}{z^a} \le \textit{const} \quad \text{for all } z \ge \text{some } z_0 > 0. \tag{2.11}$$

Indeed, since $\lim_{n\to\infty} \ell(2^n)/\ell(2^{n+1}) = 1$ (cf. Remark 1.1), we have that for all $n \ge$ some n_0,

$$\frac{\ell(2^n)}{(2^n)^a} = \frac{\ell(2^{n+1})}{(2^{n+1})^a} \cdot \frac{2^a \ell(2^n)}{\ell(2^{n+1})} > \frac{\ell(2^{n+1})}{(2^{n+1})^a} \tag{2.12}$$

and, if $2^n < z < 2^{n+1}$,

$$\frac{\ell(z)}{z^a} < \frac{\ell(2^n)}{(2^n)^a} \cdot \frac{\ell(z)}{\ell(2^n)} \le \frac{\ell(2^n)}{(2^n)^a} \cdot \frac{\ell(2^{n+1})}{\ell(2^n)} < 2^a \frac{\ell(2^n)}{(2^n)^a}. \tag{2.13}$$

Combining (2.12) and (2.13) results in

$$\frac{\ell(z)}{z^a} \le \max\left\{ \frac{\ell(2^{n_0})}{(2^{n_0})^a}, 2^a \frac{\ell(2^{n_0+1})}{(2^{n_0+1})^a} \right\} \quad \text{for all } z \ge 2^{n_0+1}.$$

We next observe that, due to (2.8)–(2.11), for any $b > 0$,

$$\sum_{n=1}^{\infty} P(|\delta_n| > n^{\frac{1}{2}+b}) \le const \sum_{n=1}^{\infty} \frac{\ell(n^{\frac{1}{2}+b})}{n^{1+2b}} \le const \sum_{n=1}^{\infty} \frac{1}{n^{1+b}} < \infty. \tag{2.14}$$

Consequently, sequences $\{\xi_n \delta_n\}_{n\ge 1}$ and $\{\xi_n \delta_n \mathbb{1}_{\{|\delta_n| \le n^{\frac{1}{2}+b}\}}\}_{n\ge 1}$ are Khinchin equivalent (that is $\sum_{n=1}^{\infty} P(\xi_n\delta_n \ne \xi_n\delta_n \mathbb{1}_{\{|\delta_n|\le n^{\frac{1}{2}+b}\}}) < \infty$) and, in view of having $\sum_{i=1}^n \xi_i^2 \to \infty$, $n \to \infty$, via Shorack (2000, p. 206, Proposition 2.1) for example, as $n \to \infty$,

$$\frac{\sum_{i=1}^n \xi_i \delta_i}{\sum_{i=1}^n \xi_i^2} \xrightarrow{a.s.} 0 \quad \text{if and only if} \quad \frac{\sum_{i=1}^n \xi_i \delta_i \mathbb{1}_{\{|\delta_i| \le i^{\frac{1}{2}+b}\}}}{\sum_{i=1}^n \xi_i^2} \xrightarrow{a.s.} 0. \tag{2.15}$$

Thus, the proof of (2.7) is now reduced to showing the second convergence in (2.15) that, in turn, amounts to establishing

$$\frac{\sum_{i=1}^n \xi_i \left(\delta_i \mathbb{1}_{\{|\delta_i| \le i^{\frac{1}{2}+b}\}} - E\delta \mathbb{1}_{\{|\delta| \le i^{\frac{1}{2}+b}\}} \right)}{\sum_{i=1}^n \xi_i^2} \xrightarrow{a.s.} 0, \quad n \to \infty, \tag{2.16}$$

with some $b > 0$, provided that

$$\frac{\sum_{i=1}^n \xi_i E\delta \mathbb{1}_{\{|\delta| \le i^{\frac{1}{2}+b}\}}}{\sum_{i=1}^n \xi_i^2} \to 0, \quad n \to \infty. \tag{2.17}$$

From Griffin and Kuelbs (1989, Lemma 6.2 with $\theta \downarrow 0$), (2.9) implies

$$\frac{zE|\delta| \mathbb{1}_{\{|\delta| > z\}}}{\ell(z)} \to 0, \quad z \to \infty, \tag{2.18}$$

with $\ell(z)$ of (2.10). On using (2.18), (2.11) and (2.1),

$$\begin{aligned} \frac{\left|\sum_{i=1}^n \xi_i E\delta \mathbb{1}_{\{|\delta| \le i^{\frac{1}{2}+b}\}}\right|}{\sum_{i=1}^n \xi_i^2} &= \frac{\left|\sum_{i=1}^n \xi_i E\delta \mathbb{1}_{\{|\delta| > i^{\frac{1}{2}+b}\}}\right|}{\sum_{i=1}^n \xi_i^2} \\ &\le \frac{\sum_{i=1}^n |\xi_i| E|\delta| \mathbb{1}_{\{|\delta| > i^{\frac{1}{2}+b}\}}}{\sum_{i=1}^n \xi_i^2} \le const \frac{\left(\sum_{i=1}^n \xi_i^2\right)^{1/2} \left(\sum_{i=1}^n \ell^2(i^{\frac{1}{2}+b})/i^{1+2b}\right)^{1/2}}{\sum_{i=1}^n \xi_i^2} \end{aligned}$$

$$\leq const\, \frac{\ell(n^{\frac{1}{2}+b})}{\sqrt{n}} \left(\sum_{i=1}^{n} \frac{1}{i^{1+2b}} \right)^{1/2} \leq \frac{const}{n^{1/4}} \left(1 + \int_{1}^{n} \frac{dx}{x^{1+2b}} \right)^{1/2}$$
$$= \frac{const}{n^{1/4}} \left(1 + \frac{1}{2b} - \frac{1}{2bn^{2b}} \right)^{1/2} \to 0, \quad n \to \infty, \tag{2.19}$$

that is (2.17) has been verified.

In the rest of the proof, we will establish (2.16), and thus (2.7) as well, assuming that one of the conditions (2.3)–(2.5) is satisfied.

If (2.3) holds true with some $b > 0$, then (2.16) is on account of the Borel-Cantelli lemma and having

$$P\left(\frac{\left| \sum_{i=1}^{n} \xi_i \left(\delta_i \mathbb{1}_{\{|\delta_i| \leq i^{\frac{1}{2}+b}\}} - E\delta \mathbb{1}_{\{|\delta| \leq i^{\frac{1}{2}+b}\}} \right) \right|}{\sum_{i=1}^{n} \xi_i^2} \geq d \right)$$
$$\leq \frac{\sum_{i=1}^{n} \xi^2 E\delta^2 \mathbb{1}_{\{|\delta| \leq i^{\frac{1}{2}+b}\}}}{d^2 (\sum_{i=1}^{n} \xi_i^2)^2} \leq \frac{E\delta^2 \mathbb{1}_{\{|\delta| \leq n^{\frac{1}{2}+b}\}}}{d^2 \sum_{i=1}^{n} \xi_i^2},$$

for any $d > 0$.

The Hájek-Rényi inequality (cf. Lemma A.1 of Appendix) and steps similar to those in Kounias and Weng (1969) that were used for concluding Lemma A.2 of Appendix, give (2.16) when (2.4) is valid.

Finally, suppose that (2.5) is satisfied with some $a, b > 0$. Due to (2.1), for any $d > 0$,

$$P\left(\frac{\left| \sum_{i=1}^{m^{1+a}} \xi_i \left(\delta_i \mathbb{1}_{\{|\delta_i| \leq i^{\frac{1}{2}+b}\}} - E\delta \mathbb{1}_{\{|\delta| \leq i^{\frac{1}{2}+b}\}} \right) \right|}{\sum_{i=1}^{m^{1+a}} \xi_i^2} \geq d \right)$$
$$\leq \frac{\sum_{i=1}^{m^{1+a}} \xi_i^2 E\delta^2 \mathbb{1}_{\{|\delta| \leq i^{\frac{1}{2}+b}\}}}{d^2 (\sum_{i=1}^{m^{1+a}} \xi_i^2)^2} \leq \frac{\ell\left(m^{(1+a)(\frac{1}{2}+b)} \right)}{d^2 \sum_{i=1}^{m^{1+a}} \xi_i^2} \leq const \frac{\ell\left(m^{(1+a)(\frac{1}{2}+b)} \right)}{m^{1+a}}, \tag{2.20}$$

with $\ell(\cdot)$ of (2.10), and via Kolmogorov's inequality, we similarly have

$$P\left(\max_{m^{1+a} < n < (m+1)^{1+a}} \frac{\left| \sum_{i=m^{1+a}+1}^{n} \xi_i \left(\delta_i \mathbb{1}_{\{|\delta_i| \leq i^{\frac{1}{2}+b}\}} - E\delta \mathbb{1}_{\{|\delta| \leq i^{\frac{1}{2}+b}\}} \right) \right|}{\sum_{i=1}^{n} \xi^2} \geq d \right)$$
$$\leq P\left(\max_{m^{1+a} < n < (m+1)^{1+a}} \left| \sum_{i=m^{1+a}+1}^{n} \xi_i \left(\delta_i \mathbb{1}_{\{|\delta_i| \leq i^{\frac{1}{2}+b}\}} - E\delta \mathbb{1}_{\{|\delta| \leq i^{\frac{1}{2}+b}\}} \right) \right| \geq d \sum_{i=1}^{m^{1+a}+1} \xi_i^2 \right)$$
$$\leq \frac{\ell\left((m+1)^{(1+a)(\frac{1}{2}+b)} \right) \sum_{i=m^{1+a}+1}^{(m+1)^{1+a}-1} \xi_i^2}{d^2 \left(\sum_{i=1}^{m^{1+a}+1} \xi_i^2 \right)^2}. \tag{2.21}$$

We note that for any $n \in \mathbb{N}$, there exist $m \in \mathbb{N}$, such that

$$\frac{\left|\sum_{i=1}^{n} \xi_i \left(\delta_i \mathbb{1}_{\{|\delta_i| \le i^{\frac{1}{2}+b}\}} - E\delta \mathbb{1}_{\{|\delta| \le i^{\frac{1}{2}+b}\}}\right)\right|}{\sum_{i=1}^{n} \xi_i^2}$$
$$\le \frac{\left|\sum_{i=1}^{m^{1+a}} \xi_i \left(\delta_i \mathbb{1}_{\{|\delta_i| \le i^{\frac{1}{2}+b}\}} - E\delta \mathbb{1}_{\{|\delta| \le i^{\frac{1}{2}+b}\}}\right)\right|}{\sum_{i=1}^{m^{1+a}} \xi_i^2}$$
$$+ \max_{m^{1+a} < n < (m+1)^{1+a}} \frac{\left|\sum_{i=m^{1+a}+1}^{n} \xi_i \left(\delta_i \mathbb{1}_{\{|\delta_i| \le i^{\frac{1}{2}+b}\}} - E\delta \mathbb{1}_{\{|\delta| \le i^{\frac{1}{2}+b}\}}\right)\right|}{\sum_{i=1}^{n} \xi_i^2}. \tag{2.22}$$

Now, on combining (2.20)–(2.22) and using (2.11), (2.5) and the Borel-Cantelli lemma, we obtain (2.16) under (2.5) and thus conclude the proof of Lemma 2.1. □

Remark 2.1. According to (2.3), we must have $\overline{\xi^2}_n \to \infty$, $n \to \infty$, and, furthermore, the relative variability of δ to that of $\xi_1, \ldots, \xi_n$, namely $E\delta^2 \mathbb{1}_{\{|\delta| \le n^{1/2+b}\}} / \overline{\xi^2}_n$, must converge to zero at an appropriate rate. This ratio also plays a role in (2.4), but less prominently so when $\xi_n^2 / \sum_{i=1}^n \xi_i^2 \to 0$, $n \to \infty$. If the latter convergence has a fast enough speed in (2.4), then there is no need for having $\overline{\xi^2}_n \to \infty$, $n \to \infty$, even when $\operatorname{Var}\delta = \infty$. Concerning condition (2.5), it is satisfied for any $a, b > 0$ when $\lim_{n\to\infty} \overline{\xi^2}_n$ exists and is finite, regardless of whether $\operatorname{Var}\delta < \infty$ or $\operatorname{Var}\delta = \infty$ (cf. (2.11)). If $\lim_{n\to\infty} \overline{\xi^2}_n = \infty$ in such a way that $\lim_{n\to\infty} \overline{\xi^2}_n / \nu(n) = 1$, with some slowly varying function at infinity $\nu(\cdot)$, as if $\{\xi_n\}_{n\ge1}$ were i.i.d. r.v.'s in DAN (cf. Remark 1.1), then

$$\begin{aligned}\frac{\sum_{i=n^{1+a}+1}^{(n+1)^{1+a}-1} \xi_i^2}{\sum_{i=1}^{n^{1+a}+1} \xi_i^2} &= \frac{((n+1)^{1+a}-1)\nu((n+1)^{1+a}-1) - n^{1+a}\nu(n^{1+a})}{(n^{1+a}+1)\nu(n^{1+a}+1)} + o(1) \\ &= o(1), \quad n \to \infty.\end{aligned}$$

Consenquently, (2.5) holds true in this case as well, both when $\operatorname{Var}\delta < \infty$ and $\operatorname{Var}\delta = \infty$. We also note that the series in (2.2)–(2.4) are all of form $\sum_{n=1}^{\infty} f(n)/n$. For examples of convergent series of this form we refer to (1.20) in Remark 1.3.

2.2. Proofs of the main results

Proof of Theorem 1.1. We have

$$\hat{\beta}_n - \beta = \frac{S_{xy} - \beta S_{xx}}{S_{xx}} = \frac{S_{\xi\delta} - \beta S_{\xi\varepsilon} + S_{\delta\varepsilon} - \beta S_{\varepsilon\varepsilon}}{S_{\xi\xi} + 2S_{\xi\varepsilon} + S_{\varepsilon\varepsilon}}, \tag{2.23}$$

where, in view of the weak law of large numbers (WLLN), (A1), (A2), Remark 1.1, and (1.12),

$$\frac{S_{\varepsilon\varepsilon}}{S_{\xi\xi}} = \frac{\overline{\varepsilon^2}_n}{S_{\xi\xi}} - c\frac{(\overline{\varepsilon}_n)^2}{S_{\xi\xi}} = \frac{\overline{\varepsilon^2}_n}{S_{\xi\xi}} + o_P(1)$$

$$= \left\{ \begin{array}{ll} (1+o_P(1))\dfrac{\ell_\varepsilon^2(n)}{\ell_\xi^2(n)} + o_P(1), & \text{in SEIVM (1.1)}, \\ (1+o_P(1))\dfrac{\ell_\varepsilon^2(n)}{S_{\xi\xi}} + o_P(1), & \text{in FEIVM (1.1)}, \end{array} \right. = o_P(1), \tag{2.24}$$

with c of (1.2), and, since $|S_{\xi\varepsilon}|/S_{\xi\xi} \le (S_{\varepsilon\varepsilon}/S_{\xi\xi})^{1/2}$,

$$\frac{S_{\xi\varepsilon}}{S_{\xi\xi}} = o_P(1), \tag{2.25}$$

as $n \to \infty$.

For $\delta \in \text{DAN}$,

$$\frac{E\delta^2 \mathbb{1}_{\{|\delta| \le \sqrt{n}\ell_\delta(n)\}}}{\ell_\delta^2(n)} \underset{n \to \infty}{\longrightarrow} \left\{ \begin{array}{ll} \dfrac{E\delta^2}{\operatorname{Var}\delta}, & \text{if } \operatorname{Var}\delta < \infty, \\ 1, & \text{if } \operatorname{Var}\delta = \infty, \end{array} \right. \tag{2.26}$$

(cf., e.g., Giné *et al.* (1997, proof of Lemma 3.2)). On combining (2.26) with (2.9),

$$nP\left(|\delta| > \sqrt{n}\ell_\delta(n)\right) \to 0, \quad n \to \infty. \tag{2.27}$$

In FEIVM (1.1), we have

$$\begin{aligned} \frac{S_{\xi\delta}}{S_{\xi\xi}} &= \frac{\sum_{i=1}^n (\xi_i - c\overline{\xi}_n)\delta_i}{\sum_{i=1}^n (\xi_i - c\overline{\xi}_n)^2} \\ &= \frac{\sum_{i=1}^n (\xi_i - c\overline{\xi}_n)\left(\delta_i \mathbb{1}_{\{|\delta_i| \le \sqrt{n}\ell_\delta(n)\}} - E\delta \mathbb{1}_{\{|\delta| \le \sqrt{n}\ell_\delta(n)\}}\right)}{\sum_{i=1}^n (\xi_i - c\overline{\xi}_n)^2} \\ &\quad + \frac{E\delta \mathbb{1}_{\{|\delta| \le \sqrt{n}\ell_\delta(n)\}} \sum_{i=1}^n (\xi_i - c\overline{\xi}_n)}{\sum_{i=1}^n (\xi_i - c\overline{\xi}_n)^2} + \frac{\sum_{i=1}^n (\xi_i - c\overline{\xi}_n)\delta_i \mathbb{1}_{\{|\delta_i| > \sqrt{n}\ell_\delta(n)\}}}{\sum_{i=1}^n (\xi_i - c\overline{\xi}_n)^2}, \end{aligned} \tag{2.28}$$

where, in view of (2.11), (2.1), and (2.26), for any $d > 0$,

$$\begin{aligned} &P\left(\frac{\left|\sum_{i=1}^n (\xi_i - c\overline{\xi}_n)\left(\delta_i \mathbb{1}_{\{|\delta_i| \le \sqrt{n}\ell_\delta(n)\}} - E\delta \mathbb{1}_{\{|\delta| \le \sqrt{n}\ell_\delta(n)\}}\right)\right|}{\sum_{i=1}^n (\xi_i - c\overline{\xi}_n)^2} \ge d\right) \\ &\le \frac{E\delta^2 \mathbb{1}_{\{|\delta| \le \sqrt{n}\ell_\delta(n)\}}}{d^2 \sum_{i=1}^n (\xi_i - c\overline{\xi}_n)^2} \le const \frac{\ell\left(\sqrt{n}\ell_\delta(n)\right)}{n} \to 0, \end{aligned} \tag{2.29}$$

while, similarly to (2.19),

$$\frac{\left|E\delta \mathbb{1}_{\{|\delta| \le \sqrt{n}\ell_\delta(n)\}} \sum_{i=1}^n (\xi_i - c\overline{\xi}_n)\right|}{\sum_{i=1}^n (\xi_i - c\overline{\xi}_n)^2} \le \frac{E|\delta| \mathbb{1}_{\{|\delta| > \sqrt{n}\ell_\delta(n)\}}}{\sqrt{S_{\xi\xi}}} \to 0, \tag{2.30}$$

and, due to (2.27), for any $d > 0$,

$$P\left(\frac{|\sum_{i=1}^{n}(\xi_i - c\overline{\xi}_n)\delta_i \mathbb{1}_{\{|\delta_i|>\sqrt{n}\ell_\delta(n)\}}|}{\sum_{i=1}^{n}(\xi_i - c\overline{\xi}_n)^2} \geq d\right) \leq P\left(\bigcup_{i=1}^{n}\{|\delta_i| > \sqrt{n}\ell_\delta(n)\}\right) \leq nP(|\delta| > \sqrt{n}\ell_\delta(n)) \to 0, \quad (2.31)$$

as $n \to \infty$. Putting together (2.28)–(2.31) gives

$$\frac{S_{\xi\delta}}{S_{\xi\xi}} = o_P(1), \quad n \to \infty, \quad (2.32)$$

which is also true in SEIVM (1.1), since $S_{\xi\delta} = \overline{\xi\delta}_n - c\,\overline{\xi}_n\overline{\delta}_n = o_P(1)$ and $S_{\xi\xi}^{-1} = o_P(1)$ in this model, as $n \to \infty$, simply by the WLLN, (A1) and (A2).

Convergence

$$\frac{S_{\delta\varepsilon}}{S_{\xi\xi}} = o_P(1), \quad n \to \infty, \quad (2.33)$$

follows directly from the WLLN, (A1) and (A2), if $E|\delta\varepsilon| < \infty$, or from (1.13), Remark 1.1 and having

$$\frac{|S_{\delta\varepsilon}|}{S_{\xi\xi}} \leq \frac{(S_{\delta\delta}S_{\varepsilon\varepsilon})^{1/2}}{S_{\xi\xi}} = \begin{cases} (1+o_P(1))\dfrac{\ell_\delta(n)\ell_\varepsilon(n)}{\ell_\xi^2(n)}, & \text{in SEIVM (1.1)}, \\ (1+o_P(1))\dfrac{\ell_\delta(n)\ell_\varepsilon(n)}{S_{\xi\xi}}, & \text{in FEIVM (1.1)}. \end{cases}$$

Finally, weak consistency of the LSE $\hat{\beta}_n$ is concluded from (2.23)–(2.25), (2.32) and (2.33), and then, that of $\hat{\alpha}_n$ is easily seen via

$$\hat{\alpha}_n - \alpha = \overline{y}_n - \alpha - \hat{\beta}_n\overline{x}_n = \overline{\delta}_n - \beta\overline{\varepsilon}_n - (\hat{\beta}_n - \beta)(\overline{\xi}_n + \overline{\varepsilon}_n). \quad (2.34)$$

□

Proof of Theorem 1.2. Similarly to the proof of Theorem 1.1, we obtain strong consistency of $\hat{\beta}_n$ on account of (2.23) and arguing that

$$\frac{S_{\varepsilon\varepsilon}}{S_{\xi\xi}} \xrightarrow{a.s.} 0, \quad (2.35)$$

$$\frac{S_{\xi\varepsilon}}{S_{\xi\xi}} \xrightarrow{a.s.} 0, \quad (2.36)$$

$$\frac{S_{\xi\delta}}{S_{\xi\xi}} \xrightarrow{a.s.} 0, \quad (2.37)$$

and

$$\frac{S_{\delta\varepsilon}}{S_{\xi\xi}} \xrightarrow{a.s.} 0, \quad (2.38)$$

as $n \to \infty$. The proof of strong consistency of $\hat{\alpha}_n$ goes via (2.34) and is omitted.

In SEIVM (1.1) with $\operatorname{Var}\varepsilon < \infty$ and $E|\delta\varepsilon| < \infty$, (2.35)–(2.38) are immediate.

Consider FEIVM (1.1) now. Convergence in (2.35), seen from the SLLN if $\operatorname{Var}\varepsilon < \infty$, reduces to proving that for some $d > 0$,

$$\frac{\sum_{i=1}^{n} \varepsilon_i^2 \mathbb{1}_{\{|\varepsilon_i| \le i^{1/2+d}\}}}{n S_{\xi\xi}} \xrightarrow{a.s.} 0, \quad n \to \infty, \tag{2.39}$$

if $\operatorname{Var}\varepsilon = \infty$, similarly to (2.24) and (2.15). Due to (1.16), (2.39) is on account of Lemma A.2 of Appendix with $r = 1$. Clearly, (2.35) $\Rightarrow$ (2.36). As to (2.37) and (2.38), if $\operatorname{Var}\delta < \infty$, then $S_{\delta\delta}/S_{\xi\xi} \xrightarrow{a.s.} 0$ and hence, $S_{\xi\delta}/S_{\xi\xi} \xrightarrow{a.s.} 0$ and $|S_{\delta\varepsilon}|/S_{\xi\xi} \le (S_{\delta\delta}/S_{\xi\xi})^{1/2}(S_{\varepsilon\varepsilon}/S_{\xi\xi})^{1/2} \xrightarrow{a.s.} 0, n \to \infty$. Suppose now that $\operatorname{Var}\delta = \infty$. Using (A3) (if $\alpha \ne 0$), one of the conditions (2.2)–(2.5) and Lemma 2.1, we obtain (2.37), while convergence in (2.38) follows from the SLLN, if $E|\delta\varepsilon| < \infty$, and from the condition (1.17) and Lemma A.2 of Appendix (with $r = 1$), if $E|\delta\varepsilon| = \infty$, similarly to concluding (2.35). □

Proof of Theorem 1.3. Consider first SEIVM (1.1). By the WLLN, (A1), (A2), Remark 1.1, and (1.21), as $n \to \infty$,

$$b_n \frac{S_{\xi\delta}}{S_{\xi\xi}} = \frac{b_n}{\ell_\xi^2(n)} o_P(1) = o_P(1) \quad \text{and} \quad b_n \frac{S_{\xi\varepsilon}}{S_{\xi\xi}} = o_P(1), \tag{2.40}$$

while

$$b_n \frac{S_{\varepsilon\varepsilon}}{S_{\xi\xi}} = b_n \frac{\ell_\varepsilon^2(n)}{\ell_\xi^2(n)}(1 + o_P(1)) = o_P(1) \tag{2.41}$$

and, using also (1.22) when $\operatorname{Var}\delta = \infty$ and $E|\delta\varepsilon| = \infty$,

$$b_n \frac{|S_{\delta\varepsilon}|}{S_{\xi\xi}} \le \begin{cases} b_n \dfrac{O_P(1)}{\ell_\xi^2(n)} = o_P(1), & \text{if } E|\delta\varepsilon| < \infty, \\ b_n \dfrac{(S_{\delta\delta}S_{\varepsilon\varepsilon})^{1/2}}{S_{\xi\xi}} = b_n \dfrac{\ell_\varepsilon(n)\ell_\delta(n)}{\ell_\xi^2(n)}(1 + o_P(1)) = o_P(1), & \text{if } E|\delta\varepsilon| = \infty. \end{cases} \tag{2.42}$$

Combining (2.40)–(2.42) and (2.23), we obtain (1.23) for the LSE $\hat{\beta}_n$.

As to (1.25) for $\hat{\alpha}_n$, arguing similarly and applying also (2.34), (A1), (1.21), (2.11), and (1.23), we get

$$\begin{aligned} b_n(\hat{\alpha}_n - \alpha) &= b_n(\overline{\delta}_n - \beta\overline{\varepsilon}_n) - b_n(\hat{\beta}_n - \beta)(\overline{\xi}_n + \overline{\varepsilon}_n) \\ &= b_n\left(\frac{\ell_\delta(n)}{\sqrt{n}} O_P(1) + \frac{\ell_\varepsilon(n)}{\sqrt{n}} O_P(1)\right) + o_P(1)O_P(1) = o_P(1), \quad n \to \infty. \end{aligned} \tag{2.43}$$

We will now prove (1.23) and (1.25) in FEIVM (1.1). Similarly to (2.41) and (2.42), we have

$$b_n \frac{S_{\varepsilon\varepsilon}}{S_{\xi\xi}} = b_n \frac{\ell_\varepsilon^2(n)}{S_{\xi\xi}}(1 + o_P(1)) = o_P(1) \quad \text{and} \quad b_n \frac{S_{\delta\varepsilon}}{S_{\xi\xi}} = o_P(1), \tag{2.44}$$

as $n \to \infty$. The respective version of (2.40) is obtained similarly to the lines in (2.28)–(2.31), which amount to the proof of (2.32). Thus, in order to prove that $b_n S_{\xi\delta}/S_{\xi\xi} = o_P(1)$, it suffices to adapt only (2.29) and (2.30). Accordingly, on account of (2.26), (A3) and having $b_n/S_{\xi\xi} \to 0$ and $b_n^2 \ell_\delta^2(n)/(nS_{\xi\xi}) \to 0$ from (1.21), as $n \to \infty$,

$$P\left(b_n \frac{\left|\sum_{i=1}^n (\xi_i - c\overline{\xi}_n)\left(\delta_i \mathbb{1}_{\{|\delta_i| \le \sqrt{n}\ell_\delta(n)\}} - E\delta \mathbb{1}_{\{|\delta| \le \sqrt{n}\ell_\delta(n)\}}\right)\right|}{\sum_{i=1}^n (\xi_i - c\overline{\xi}_n)^2} \ge d\right)$$
$$\le b_n^2 \frac{E\delta^2 \mathbb{1}_{\{|\delta| \le \sqrt{n}\ell_\delta(n)\}}}{d^2 \sum_{i=1}^n (\xi_i - c\overline{\xi}_n)^2} \le const \frac{b_n^2 \ell_\delta^2(n)}{nS_{\xi\xi}} \to 0 \tag{2.45}$$

and

$$b_n \frac{|E\delta \mathbb{1}_{\{|\delta| \le \sqrt{n}\ell_\delta(n)\}} \sum_{i=1}^n (\xi_i - c\overline{\xi}_n)|}{\sum_{i=1}^n (\xi_i - c\overline{\xi}_n)^2} \le const \frac{b_n}{S_{\xi\xi}} E|\delta| \mathbb{1}_{\{|\delta| > \sqrt{n}\ell_\delta(n)\}} \to 0. \tag{2.46}$$

Likewise, we conclude that $b_n S_{\xi\varepsilon}/S_{\xi\xi} = o_P(1)$, $n \to \infty$, and then, via (2.23) and (2.44), that (1.23) holds true. The proof of (1.25) is as in (2.43), with the difference that one has to use (A3) and that convergence $b_n \ell_\delta(n)/\sqrt{n} \to 0$ and $b_n \ell_\varepsilon(n)/\sqrt{n} \to 0$ used in (2.43) has to be guaranteed by condition (1.24) now, since in (1.21), $S_{\xi\xi}$ and hence b_n may not necessarily converge to infinity respectively as, and at most as fast as, slowly varying functions, like they do in SEIVM (1.1), as $n \to \infty$. □

Proof of Theorem 1.4. Clearly, if $\operatorname{Var} \varepsilon < \infty$ and $E|\delta\varepsilon| < \infty$, then (2.35)–(2.38) hold true also when $S_{\xi\xi}$ is replaced with $S_{\xi\xi}^a$, for any $a \in (0, 1]$, and, via (2.23), this results in the first convergence in (1.27). Consequently, for the second convergence in (1.27), we have

$$\begin{aligned} S_{\xi\xi}^{1-a}(\hat{\alpha}_n - \alpha) &= S_{\xi\xi}^{1-a}(\overline{\delta}_n - \beta\overline{\varepsilon}_n) - S_{\xi\xi}^{1-a}(\hat{\beta}_n - \beta)(\overline{\xi}_n + \overline{\varepsilon}_n) \\ &\overset{a.s.}{=} S_{\xi\xi}^{1-a}(\overline{\delta}_n - \beta\overline{\varepsilon}_n) + o(1) \overset{a.s.}{=} o(1), \quad n \to \infty, \end{aligned}$$

where, by using the Hartman-Wintner law of the iterated logarithm for $\sum_{i=1}^n \varepsilon_i$ and, if $\operatorname{Var} \delta < \infty$, for $\sum_{i=1}^n \delta_i$, and applying the Marcinkiewicz-Zygmund SLLN for $S_{\xi\xi}$ and, if $\operatorname{Var} \delta = \infty$, for $\sum_{i=1}^n \delta_i$,

$$S_{\xi\xi}^{1-a}(\overline{\delta}_n - \beta\overline{\varepsilon}_n) = \left(\frac{S_{\xi\xi}}{n^{1/4}}\right)^{1-a} \frac{n^{(1-a)/4}}{n^{1/4}} \left(\frac{\sum_{i=1}^n \delta_i}{n^{3/4}} - \beta \frac{\sum_{i=1}^n \varepsilon_i}{n^{3/4}}\right) \overset{a.s.}{=} o(1), \quad n \to \infty. \tag{2.47}$$

Consider FEIVM (1.1) now. First, let $\operatorname{Var} \delta, \operatorname{Var} \varepsilon < \infty$. Then, on account of Lemma A.3 of Appendix, (1.28), the SLLN, and (A2), as $n \to \infty$,

$$b_n \frac{S_{\xi\delta}}{S_{\xi\xi}} = \frac{b_n}{\sqrt{S_{\xi\xi}}} \frac{\sum_{i=1}^n (\xi_i - \overline{\xi}_n)\delta_i}{\sqrt{n \sum_{i=1}^n (\xi_i - \overline{\xi}_n)^2}} \overset{a.s.}{=} o(1), \tag{2.48}$$

$$b_n \frac{S_{\xi\varepsilon}}{S_{\xi\xi}} \overset{a.s.}{=} o(1), \quad b_n \frac{S_{\varepsilon\varepsilon}}{S_{\xi\xi}} = \frac{b_n}{\sqrt{S_{\xi\xi}}} \frac{S_{\varepsilon\varepsilon}}{\sqrt{S_{\xi\xi}}} \overset{a.s.}{=} o(1), \tag{2.49}$$

and

$$b_n \frac{S_{\delta\varepsilon}}{S_{\xi\xi}} \overset{a.s.}{=} o(1). \tag{2.50}$$

Suppose now that at least one of the error variances is infinite. Condition (1.28) implies that

$$\sum_{n=1}^{\infty} b_n^2 E\delta^2 \mathbb{1}_{\{|\delta| \le n^{1/2+\eta}\}} / (n \overline{\xi^2}_n) < \infty \quad \text{and hence that} \quad b_n^2 / \overline{\xi^2}_n \to 0, \quad n \to \infty. \tag{2.51}$$

Using (2.51) and adapting the steps of the proof of (2.6) under (2.3) and (A3) ($\alpha \neq 0$), we note first that convergence in (2.48) reduces to

$$b_n \frac{\sum_{i=1}^n \xi_i \left(\delta_i \mathbb{1}_{\{|\delta_i| \le i^{1/2+\eta}\}} - E\delta \mathbb{1}_{\{|\delta| \le i^{1/2+\eta}\}} \right)}{\sum_{i=1}^n \xi_i^2} \overset{a.s.}{\longrightarrow} 0, \quad n \to \infty, \tag{2.52}$$

since

$$b_n \frac{S_{\xi\delta}}{S_{\xi\xi}} = \left(b_n \frac{\overline{\xi\delta}_n}{\overline{\xi^2}_n} - c \frac{b_n}{\overline{\xi^2}_n} \overline{\xi}_n \overline{\delta}_n \right) \left(1 + \frac{c(\overline{\xi}_n)^2}{S_{\xi\xi}} \right) \overset{a.s.}{=} \left(b_n \frac{\overline{\xi\delta}_n}{\overline{\xi^2}_n} + o(1) \right) (1 + o(1)),$$

due to (2.51), (A3) and the SLLN, and since we have (2.15), with $b_n / \sum_{i=1}^n \xi_i^2$ replacing $1/\sum_{i=1}^n \xi_i^2$, as well as

$$b_n \frac{|\sum_{i=1}^n \xi_i E\delta \mathbb{1}_{\{|\delta| \le i^{1/2+\eta}\}}|}{\sum_{i=1}^n \xi_i^2} \le const \frac{b_n}{\sqrt{\overline{\xi^2}_n}} \frac{E\delta^2 \mathbb{1}_{\{|\delta| \le n^{1/2+\eta}\}}}{\sqrt{n}} \to 0, \quad n \to \infty,$$

where the latter convergence is argued similarly to (2.19), by using (2.51) and (2.11). The Borel-Cantelli lemma and convergence of the series in (2.51) give (2.52). Condition (1.28) also implies that $\sum_{n=1}^{\infty} b_n E\delta^2 \mathbb{1}_{\{|\delta| \le n^{1/2+\eta}\}} / (n S_{\xi\xi}) < \infty$, leading to

$$b_n \frac{S_{\delta\delta}}{S_{\xi\xi}} \overset{a.s.}{=} o(1), \quad n \to \infty, \tag{2.53}$$

(cf. the proof of (2.35) via (2.39)). Similarly to the proofs of (2.48) and (2.53), we argue (2.49) via (1.28). Finally, as to (2.50),

$$b_n \frac{|S_{\delta\varepsilon}|}{S_{\xi\xi}} \le \left(b_n \frac{S_{\delta\delta}}{S_{\xi\xi}} \right)^{1/2} \left(b_n \frac{S_{\varepsilon\varepsilon}}{S_{\xi\xi}} \right)^{1/2} \overset{a.s.}{=} o(1), \quad n \to \infty.$$

Thus, we obtain (1.29) using (2.23) and (2.48)–(2.50).

Concerning (1.32) for the LSE $\hat{\alpha}_n$ in FEIVM (1.1), using the expansion in (2.43), we only need to show that

$$b_n (\overline{\delta}_n - \beta \overline{\varepsilon}_n) \overset{a.s.}{=} o(1), \quad n \to \infty. \tag{2.54}$$

Similarly to (2.15)–(2.17), with $\theta > 0$ as in (1.30) and (1.31), as $n \to \infty$,

$$b_n \overline{\delta}_n \xrightarrow{a.s.} 0 \quad \text{if and only if} \quad \frac{b_n \sum_{i=1}^n \delta_i \mathbb{1}_{\{|\delta_i| \le i^{1/2+\theta}\}}}{n} \xrightarrow{a.s.} 0$$
$$\text{if and only if} \quad \frac{b_n \sum_{i=1}^n \left(\delta_i \mathbb{1}_{\{|\delta_i| \le i^{1/2+\theta}\}} - E\delta \mathbb{1}_{\{|\delta| \le i^{1/2+\theta}\}}\right)}{n} \xrightarrow{a.s.} 0, \tag{2.55}$$

since, on taking $\xi_i = 1$ in (2.19) and using (1.31),

$$\begin{aligned} \frac{b_n \left| \sum_{i=1}^n E\delta \mathbb{1}_{\{|\delta| \le i^{1/2+\theta}\}} \right|}{n} &\le const \frac{b_n}{\sqrt{n}} \left(\sum_{i=1}^n \frac{\left(E\delta^2 \mathbb{1}_{\{|\delta| \le i^{1/2+\theta}\}}\right)^2}{i^{1+2\theta}} \right)^{1/2} \\ &\le const \frac{b_n E\delta^2 \mathbb{1}_{\{|\delta| \le n^{1/2+\theta}\}}}{\sqrt{n}} \to 0. \end{aligned}$$

Convergence in (2.55) is concluded by applying Lemma A.1 of Appendix and condition (1.30). Convergence $b_n \overline{\varepsilon}_n \xrightarrow{a.s.} 0$, $n \to \infty$, is proved in the same way. Thus, the proof of (2.54) and hence that of (1.32) is now complete. □

Appendix

This section contains auxiliary results from the literature that are used for the proofs in Section 2.

The well-known Hájek-Rényi inequality can be found in, for example, Petrov (1987).

Lemma A.1 (the Hájek-Rényi inequality)**.** *Let $X_1, \ldots, X_n$ be independent r.v.'s such that $EX_i = 0$ and $EX_i^2 < \infty$ for all $i = 1, \ldots, n$, and let $0 < c_n \le c_{n-1} \le \cdots \le c_1$. Then, for any $x > 0$ and $m < n$,*

$$P\left(\max_{m \le k \le n} c_k \left| \sum_{i=1}^k X_i \right| \ge x \right) \le \frac{1}{x^2} \left(c_m^2 \sum_{k=1}^m EX_k^2 + \sum_{k=m+1}^n c_k^2 EX_k^2 \right).$$

Kounias and Weng (1969) generalized the Hájek-Rényi inequality and, as a consequence, proved the following almost sure convergence.

Lemma A.2 (Kounias and Weng (1969))**.** *Let $\{X_i\}_{i \ge 1}$ be a sequence of r.v.'s such that $E|X_i|^r < \infty$ for some $r > 0$ and all $i \ge 1$, and let $\{b_i\}_{i \ge 1}$ be a nondecreasing sequence of positive constants. Suppose that*

$$\sum_{n=1}^\infty \frac{E|X_n|^r}{b_n^r} < \infty \quad \text{for } 0 < r \le 1, \quad \text{or} \quad \sum_{n=1}^\infty \frac{E^{1/r}|X_n|^r}{b_n} < \infty \quad \text{for } 1 \le r,$$

then

$$\frac{\sum_{i=1}^n X_i}{b_n} \xrightarrow{a.s.} 0, \quad n \to \infty.$$

The following almost sure convergence for weighted partial sums comes handy for us in the proof of Theorem 1.4.

Lemma A.3 (Chow (1966))**.** *If $\{X_i\}_{i\geq 1}$ are i.i.d. r.v.'s with zero mean and finite variance and $\{a_{n,i}, 1 \leq i \leq n, n \geq 1\}$ is a sequence of real numbers satisfying $\sum_{i=1}^n a_{n,i}^2 = 1$ for $n \geq 1$, then*

$$\frac{\sum_{i=1}^n a_{n,i} X_i}{n^{1/2}} \xrightarrow{a.s.} 0, \quad n \to \infty.$$

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