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# Bayes minimax estimation under power priors of location parameters for a wide class of spherically symmetric distributions

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**Abstract:** We complement the results of Fourdrinier, Mezoued and Strawderman in [5] who considered Bayesian estimation of the location parameter $\theta$ of a random vector $X$ having a unimodal spherically symmetric density $f(\|x-\theta\|^2)$ for a spherically symmetric prior density $\pi(\|\theta\|^2)$. In [5], expressing the Bayes estimator as $\delta_\pi(X) = X + \nabla M(\|X\|^2)/m(\|X\|^2)$, where $m$ is the marginal associated to $f(\|x-\theta\|^2)$ and $M$ is the marginal with respect to $F(\|x-\theta\|^2) = 1/2 \int_{\|x-\theta\|^2}^{\infty} f(t)\,dt$, it was shown that, under quadratic loss, if the sampling density $f(\|x-\theta\|^2)$ belongs to the Berger class (i.e. there exists a positive constant $c$ such that $F(t)/f(t) \geq c$ for all $t$), conditions, dependent on the monotonicity of the ratio $F(t)/f(t)$, can be found on $\pi$ in order that $\delta_\pi(X)$ is minimax.

The main feature of this paper is that, in the case where $F(t)/f(t)$ is nonincreasing, if $\pi(\|\theta\|^2)$ is a superharmonic power prior of the form $\|\theta\|^{-2k}$ with $k > 0$, the membership of the sampling density to the Berger class can be droped out. Also, our techniques are different from those in [5]. First, writing $\delta_\pi(X) = X + g(X)$ with $g(X) \propto \nabla M(\|X\|^2)/m(\|X\|^2)$, we follow Brandwein and Strawderman [4] proving that, for some $b > 0$, the function $h = b\,\Delta M/m$ is subharmonic and satisfies $\|g\|^2/2 \leq -h \leq -\text{div} g$. Also, we adapt their approach using the fact that $R^{2(k+1)} \int_{B_{\theta,R}} h(x)\,dV_{\theta,R}(x)$ is

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nonincreasing in $R$ for any $\theta \in \mathbb{R}^p$, when $V_{\theta,R}$ is the uniform distribution on the ball $B_{\theta,R}$ of radius $R$ and centered at $\theta$. Examples illustrate the theory.

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## Contents



## 1. Introduction

We recall the framework considered by Fourdrinier, Mezoued and Strawderman in [5]. Let $X$ be a random vector in $\mathbb{R}^p$ with spherically symmetric density

$$f(\|x-\theta\|^2) \tag{1.1}$$

around an unknown location parameter $\theta$ that we wish to estimate. Any estimator $\delta$ is evaluated under the square error loss

$$\|\delta-\theta\|^2, \tag{1.2}$$

through the corresponding quadratic risk $\mathcal{R}(\theta,\delta(X)) = E_\theta[\|\delta(X)-\theta\|^2]$, where $E_\theta$ denotes the expectation with respect to the density in (1.1). As soon as $E_0[\|X\|^2] < \infty$, the standard estimator $X$ is minimax, and has constant risk (actually, equal to $E_0[\|X\|^2]$), which entails that minimaxity of $\delta$ will be obtained by proving that the risk of $\delta$ is less than or equal to the risk of $X$, that is, if $E_\theta[\|\delta(X)-\theta\|^2] \leq E_0[\|X\|^2]$ for any $\theta \in \mathbb{R}^p$ (domination of $\delta$ over $X$ being obtained if, furthermore, this inequality is strict for some $\theta$).

Brandwein and Strawderman [4] gave general conditions on estimators of the form

$$\delta_{a,g}(X) = X + a\,g(X) \tag{1.3}$$

to dominate the standard estimator $X$, when $p \geq 3$. Here $g = (g_1, g_2, \ldots, g_p)$ is a function from $\mathbb{R}^p$ into $\mathbb{R}^p$ and $a$ is a positive constant. Besides the finiteness risk condition

$$E_\theta[\|g(X)\|^2] < \infty \tag{1.4}$$

for $\delta_{a,g}(X)$, these conditions require that

(j) $0 < a \leq 1/[p\, E_0(\|X\|^{-2})]$

and involve the existence of a subharmonic function $h$ such that

(jj) $\|g\|^2/2 \leq -h \leq -\text{div} g$,

where div is the divergence operator defined, for $x = (x_1, \ldots, x_p) \in \mathbb{R}^p$, by $\text{div} g(x) = \sum_{i=1}^p \partial g_i(x)/\partial x_i$, and, denoting by $V_{\theta,R}$ the uniform distribution on the ball $B_{\theta,R} = \{x \in \mathbb{R}^p / \|x - \theta\| \leq R\}$ of radius $R$ and centred at $\theta$, such that the function

(jjj) $R \mapsto R^2\, E_{V_{\theta,R}}[h(X)]$ is nonincreasing in $R$.

Note that (jjj), as (jj), is just a condition on the function $h$, while (j) is a distributional condition on $f$ in (1.1).

In that context, it should be noticed that, although Brandwein and Strawderman [4] gave examples for which (jjj) holds, this condition may be difficult to prove, in particular, in the situation where Bayesian estimators are concerned. In that situation, a more flexible condition is needed and it would be easier, for instance, to deal with a monotonicity condition such that, for some $q \geq 1$, $R^{2q}\, E_{V_{\theta,R}}[h(X)]$ is nonincreasing in $R$.

In this paper, we will show that a modification of the Brandwein and Strawderman approach in [4], taking into account the above new monotonicity condition, can be used to obtain domination over $X$, and hence minimaxity, of generalized Bayes estimators for density prior of the form

$$\|\theta\|^{-2k}, \tag{1.5}$$

where $k$ is a positive constant.

As stated in [8] and [5], for spherical prior densities, the generalized Bayes estimator associated to (1.5) is the posterior mean and can be written under the form

$$\delta_k(X) = X + \frac{\nabla M(\|X\|^2)}{m(\|X\|^2)} \tag{1.6}$$

where $\nabla$ denotes the gradient operator and where

$$m(\|x\|^2) = \int_{\mathbb{R}^p} f(\|x - \theta\|^2)\, \|\theta\|^{-2k}\, d\theta \tag{1.7}$$

and

$$M(\|x\|^2) = \int_{\mathbb{R}^p} F(\|x - \theta\|^2)\, \|\theta\|^{-2k}\, d\theta \tag{1.8}$$

are respectively the marginal densities with respect to the density in (1.1) and the density proportional to $F(\|x - \theta\|^2)$ with

$$F(t) = \frac{1}{2} \int_t^\infty f(u)\, du \tag{1.9}$$

for $t \geq 0$. Note that, in (1.6), (1.7) and (1.8) the marginals depend on the observable $X$ through its squared norm since the prior in (1.5) is spherically symmetric around 0.

It is easily seen that the finiteness risk condition of $X$ is $\mu_2 = E_0[\|X\|^2] < \infty$. It is worth noting that this condition is sufficient to guarantee the finiteness of the risk of Bayesian estimator $\delta_k(X)$ in (1.6) (see [5]).

In the literature, minimaxity of Bayesian estimators is mainly addressed when the sampling in (1.1) is in the Berger class, that is, when there exists a positive constant $c$ such that $F(t)/f(t) \geq c$ for any $t$. Thus, for that class, Fourdrinier, Mezoued and Strawderman [5] provide two wide sets of sampling densities (according to the nondecreasing/nonincreasing monotonicity of the ratio $F(t)/f(t)$) and a wide class of prior densities of the form $\pi(\|\theta\|^2)$ for which the corresponding Bayes estimators $\delta_\pi$ are minimax (it is worth noting that, in [8], the fundamental harmonic prior $\|\theta\|^{2-p}$ is shown to be a member of that class when $F(t)/f(t)$ is nondecreasing in $t$). When one is not restricted to the Berger class, minimaxity of $\delta_\pi$ is much more complicated to obtain (and the techniques used in [8] and [5] fail). That difficulty prompted us to consider a different approach based on a modification of the minimaxity result from Brandwein and Strawderman [4] as mentioned above. Also, dealing only with the case where $F(t)/f(t)$ is nonincreasing in $t$, we only prove minimaxity when the prior densities are power priors of the form (1.5).

As a last remark, note that, if we consider $f(t)$ to be proportional to a density of a positive random variable, then $2\,F(t)/f(t)$ is the reciprocal of the hazard rate. Its monotonicity may be determined in many cases by studying the log-convexity or the log-concavity of $f(t)$ (see e.g. Barlow and Proschan [2]).

In Section 2, we propose a new version of Theorem 2.1 of Brandwein and Strawderman [4]. In Section 3, we give general results on Bayes estimators $\delta_k$ in (1.6) with respect to the spherical priors in (1.5). In Section 4, we focus on the case where the function $F(t)/f(t)$ is nonincreasing and we show that our result in Section 2 can be applied to obtain minimaxity of $\delta_k$. Section 5 is devoted to examples which illustrate the theory while Section 6 is a concluding section. Finally, we provide an appendix which contains technical results needed in the development of the paper.

## 2. A minimaxity theorem

To prove the version of Theorem 2.1 of Brandwein and Strawderman [4] below, we will follow the lines of their proof making use of the radial distribution of $R = \|X - \theta\|^2$ with density related to $f$ by

$$\xi : r \mapsto \frac{2\,\pi^{p/2}}{\Gamma(p/2)}\, r^{p-1}\, f(r^2) \tag{2.1}$$

and whose expectation will be denoted by $E$.

**Theorem 2.1.** *Let $X$ be a random vector in $\mathbb{R}^p$ with spherically symmetric density as in (1.1) such that, for some fixed $q \geq 1$, $\mu_{-2q} = E_0(\|X\|^{-2q}) < \infty$. Let $\delta_{a,g}(X)$ an estimator as in (1.3). Assume that*

$$0 < a \leq \frac{1}{p} \frac{\mu_{-2(q-1)}}{\mu_{-2q}} . \tag{2.2}$$

*Assume also that there exists a subharmonic function $h$ such that*

$$\|g\|^2 / 2 \leq -h \leq -\mathrm{div} g \tag{2.3}$$

*and such that*

$$R^{2q} \, E_{V_{\theta,R}}[h(X)] \quad \text{is nonincreasing in } R . \tag{2.4}$$

*Then $\delta_{a,g}(X)$ has a risk smaller than or equal to that of $X$.*

*Proof.* In the proof of Theorem 2.1 of [4], it is shown that the risk difference at $\theta$ between $\delta_{a,g}(X)$ and $X$ satisfies

$$\begin{aligned} d_\theta &= \mathcal{R}(\theta, \delta_{a,g}(X)) - \mathcal{R}(\theta, X) \\ &\leq E\left[\left(-\frac{2\,a^2}{R^2} + \frac{2\,a}{p}\right) R^2 \, E_{V_{\theta,R}}[h(X)]\right] . \end{aligned} \tag{2.5}$$

Introducing $R^{2(q-1)}$ in its right hand side Inequality (2.5) can be written as,

$$d_\theta \leq E[\varphi(R)\,\psi(R)] \tag{2.6}$$

where

$$\varphi(R) = \frac{2\,a}{R^{2(q-1)}} \left(\frac{1}{p} - \frac{a}{R^2}\right) \quad \text{and} \quad \psi(R) = R^{2q} \, E_{V_{\theta,R}}[h(X)] . \tag{2.7}$$

As $\varphi(R)$ changes sign once from $-$ to $+$ at $R_0 = a\,p$ and as, by assumption, $\psi(R)$ is nonincreasing, we have

$$d_\theta \leq E[\varphi(R)]\,\psi(R_0) . \tag{2.8}$$

Also, by nonpositivity of $h$, we have $\psi(R_0) \leq 0$, and hence, we will have $d_\theta \leq 0$ when $E[\varphi(R)] \geq 0$, that is, as soon as

$$\frac{1}{p} E\left[\frac{1}{R^{2(q-1)}}\right] \geq a\,E\left[\frac{1}{R^{2q}}\right]$$

or, equivalently, as soon as

$$a \leq \frac{1}{p} \frac{E\left[\frac{1}{R^{2(q-1)}}\right]}{E\left[\frac{1}{R^{2q}}\right]} = \frac{1}{p} \frac{\mu_{-2(q-1)}}{\mu_{-2q}} ,$$

which is the desired result. □

**Remark 2.1.** Note that, for $q > 0$, the ratio $E[R^{-2(q-1)}]/E[R^{-2q}]$ is nonincreasing in $q$ so that, for $q \geq 1$, we have

$$\frac{\mu_{-2(q-1)}}{\mu_{-2q}} = \frac{E\left[\frac{1}{R^{2(q-1)}}\right]}{E\left[\frac{1}{R^{2q}}\right]} \leq \left(E\left[\frac{1}{R^2}\right]\right)^{-1} . \tag{2.9}$$

Thus, comparing with Condition (j) in Section 1, we can see that the range of values of $a$ is smaller with Condition 2.2 than with Condition (j), since it is respectively described by $0 < a < (1/p)\, E[R^{-2(q-1)}]/E[R^{-2q}]$ and $0 < a < (1/p)\, 1/E[R^{-2}]$ .

## 3. Generalized Bayes estimators

As mentioned in Section 1, our aim is to study the minimaxity of the Bayes estimator $\delta_k(X)$ in (1.6) using Theorem 2.1, that is, Conditions (2.2), (2.3) and (2.4) in Section 2. Linking the forms of the estimators in (1.6) and (1.3) leads, for any $x \in \mathbb{R}^p$, to $g(x) = \nabla M(\|x\|^2)/(a\, m(\|x\|^2))$. Hence, insofar as Condition (2.2) is concerned, just fix

$$0 < a \leq \frac{1}{p} \frac{\mu_{-2(q-1)}}{\mu_{-2q}}$$

and, as a candidate for a suitable function $h$, set $h(x) = \Delta M(\|x\|^2)/(a\, m(\|x\|^2))$, so that, clearly, it suffices to prove that Conditions (2.3) and (2.4) are satisfied for that function $h$ to obtain improvement of $\delta_k(X)$ over $X$. Actually, in Section 4, it will turn out that the choice of the upper bound of the value of $a$ will be appropriate for $q = k + 1$, that is,

$$a = \frac{1}{p} \frac{\mu_{-2k}}{\mu_{-2(k+1)}} . \tag{3.1}$$

The first inequality in Condition (2.3) reduces to

$$\frac{1}{2\, a} \left\| \frac{\nabla M(\|x\|^2)}{m(\|x\|^2)} \right\|^2 \leq - \frac{\Delta M(\|x\|^2)}{m(\|x\|^2)} \tag{3.2}$$

and the second one to

$$0 \leq \nabla m(\|x\|^2) \cdot \nabla M(\|x\|^2) \tag{3.3}$$

since

$$-\mathrm{div} \frac{\nabla M(\|x\|^2)}{m(\|x\|^2)} = \frac{\nabla M(\|x\|^2) . \nabla m(\|x\|^2)}{m^2(\|x\|^2)} - \frac{\Delta M(\|x\|^2)}{m(\|x\|^2)} .$$

Note that the superharmonicity condition on $-h$ reduces to the subharmonicity of the ratio $\Delta M(\|x\|^2)/m(\|x\|^2)$, that is, $\Delta\big(\Delta M(\|x\|^2)/m(\|x\|^2)\big) \geq 0$, and it will be convenient to write this Laplacian as

$$\Delta\left(\frac{\Delta M(\|x\|^2)}{m(\|x\|^2)}\right) = A(\|x\|^2) + B(\|x\|^2) \tag{3.4}$$

where

$$A(\|x\|^2) = \frac{\Delta^{(2)} M(\|x\|^2)}{m(\|x\|^2)} \quad - \quad \frac{\Delta M(\|x\|^2) \Delta m(\|x\|^2)}{m^2(\|x\|^2)} \\ + \quad 2 \, \frac{\Delta M(\|x\|^2)}{m(\|x\|^2)} \left\| \frac{\nabla m(\|x\|^2)}{m(\|x\|^2)} \right\|^2 \qquad (3.5)$$

and

$$B(\|x\|^2) = -2 \, \frac{\nabla \left( \Delta M(\|x\|^2) \right) \cdot \nabla m(\|x\|^2)}{m^2(\|x\|^2)} \qquad (3.6)$$

and to prove separately that

$$A(\|x\|^2) \geq 0 \qquad (3.7)$$

and

$$B(\|x\|^2) \geq 0 \,. \qquad (3.8)$$

The above conditions involve the marginals $m(\|x\|^2)$ and $M(\|x\|^2)$ through the ratios

$$\nabla m(\|x\|^2)/m(\|x\|^2), \quad \Delta m(\|x\|^2)/m(\|x\|^2), \quad \nabla M(\|x\|^2)/m(\|x\|^2), \\ \Delta M(\|x\|^2)/m(\|x\|^2), \quad \nabla(\Delta M(\|x\|^2))/m(\|x\|^2),$$

and

$$\Delta^{(2)} M(\|x\|^2)/m(\|x\|^2) \,.$$

As noticed by Fourdrinier, Mezoued and Strawderman in [5], where general spherical priors $\pi(\|\theta\|^2)$ are considered, conditions on $\pi$ and $f$ are needed to express these quantities as expectations with respect the a posteriori distribution given $x$. To this end, they rely on formulas of the type

$$\int_{\mathbb{R}^p} D^\alpha \psi \left( \|x-\theta\|^2 \right) \pi(\|\theta\|^2) \, d\theta = (-1)^\alpha \int_{\mathbb{R}^p} \psi \left( \|x-\theta\|^2 \right) D^\alpha \pi(\|\theta\|^2) \, d\theta \quad (3.9)$$

where $\psi$ is, either the function $f$, or the function $F$ and where, for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_p)$ (a p-uple of nonnegative integers) with lengh $|\alpha| = \alpha_1 + \cdots + \alpha_p$, the operator $D^\alpha = \partial\alpha / \partial x_1^{\alpha_1} \ldots \partial x_p^{\alpha_p}$ is the corresponding partial derivative operator. In order (3.9) to hold the required conditions in [5] are

- $f\,(t) \in S^{\alpha-1, p/2+1+\epsilon} \left( \mathbb{R}_+ \setminus \{0\} \right)$ for a certain $\epsilon > 0$

and

- $\pi(\|\theta\|^2) \in W^{\alpha,1}_{loc}(\mathbb{R}^p) \cap C^\alpha_b \left( \mathbb{R}^p \setminus B_r \right)$ for a certain $r > 0$,

where $S^{\alpha, p+\epsilon} \left( \mathbb{R}^p \setminus B_r \right)$ is the space of functions $\alpha$-times continuously differentiable on $\mathbb{R}^p \setminus B_r$ such that $\sup_{x \in \mathbb{R}^p \setminus B_r; |\beta| \leq \alpha; \gamma \leq p+\epsilon} \|x\|^\gamma \left| D^\beta \psi \left( \|x-\theta\|^2 \right) \right| < \infty$, $W^{\alpha,1}_{loc}(\mathbb{R}^p)$ is the Sobolev space $\left\{ u \in L^1_{loc}(\mathbb{R}^p) \big/ \forall \beta, |\beta| \leq \alpha, \, D^\beta u \in L^1_{loc}(\mathbb{R}^p) \right\}$, and $C_b^\alpha(\mathbb{R}^p \setminus B_r)$ is the space of functions $\alpha$-times continuously differentiable and bounded on $\mathbb{R}^p \setminus B_r$ where $B_r$ is the ball of radius $r$ centered at 0.

Clearly this is the highest order of derivation which matters so that, as the bi-Laplacian is involved, we have to consider $\alpha = 4$. Insofar as the prior in (1.5) is concerned, the membership of $\pi(\|\theta\|^2) = (\|\theta\|^2)^{-k}$ to $C_b^\alpha(\mathbb{R}^p \setminus B_r)$ is guaranteed while it can be checked that its membership to the Sobolev space $W_{loc}^{4,1}(\mathbb{R}^p)$ holds for $k < p/2 - 2$. Hence, in conjunction with

- $\left\| \frac{\nabla\pi(\|\theta\|^2)}{\pi(\|\theta\|^2)} \right\|^2 = \frac{4\,k^2}{\|\theta\|^2}$,
- $\frac{\Delta\pi(\|\theta\|^2)}{\pi(\|\theta\|^2)} = \frac{-2\,k\,(p-2\,(k+1))}{\|\theta\|^2}$

and

- $\frac{\Delta^{(2)}\pi(\|\theta\|^2)}{\pi(\|\theta\|^2)} = \frac{4\,k\,(k+1)\,(p-2\,(k+1))(p-2\,(k+2))}{\|\theta\|^4}$,

this leads to the following lemma where $E^x$ denotes the conditional expectation of $\theta$ given $x$. Note that there is no confusion with the notation $E_\theta$ introduced after Formula (1.2) since, in the former notation, $x$ is a superscript while, in the latter, $\theta$ is a subscript.

**Lemma 3.1.** *Consider a prior as in (1.5) with $k < p/2 - 2$ and $\epsilon > 0$ such that $f(t) \in S^{3,p/2+1+\epsilon}(\mathbb{R}_+^*)$. We have*

$$\frac{\nabla m(\|x\|^2)}{m(\|x\|^2)} = -2\,k\,E^x\left[\frac{\theta}{\|\theta\|^2}\right] \tag{3.10}$$

$$\frac{\Delta m(\|x\|^2)}{m(\|x\|^2)} = -2\,k\,(p-2\,(k+1))\,E^x\left[\frac{1}{\|\theta\|^2}\right] \tag{3.11}$$

$$\frac{\nabla M(\|x\|^2)}{m(\|x\|^2)} = -2\,k\,E^x\left[\frac{F(\|x-\theta\|^2)}{f(\|x-\theta\|^2)}\,\frac{\theta}{\|\theta\|^2}\right] \tag{3.12}$$

$$\frac{\Delta M(\|x\|^2)}{m(\|x\|^2)} = -2\,k\,(p-2\,(k+1))\,E^x\left[\frac{F(\|x-\theta\|^2)}{f(\|x-\theta\|^2)}\,\frac{1}{\|\theta\|^2}\right] \tag{3.13}$$

$$\frac{\nabla\left(\Delta M(\|x\|^2)\right)}{m(\|x\|^2)} = 4\,k\,(k+1)\,(p-2\,(k+1))\,E^x\left[\frac{F(\|x-\theta\|^2)}{f(\|x-\theta\|^2)}\,\frac{\theta}{\|\theta\|^4}\right] \tag{3.14}$$

$$\frac{\Delta^{(2)} M(\|x\|^2)}{m(\|x\|^2)} = 4\,k\,(k+1)(p-2\,(k+1))\,(p-2\,(k+2))\; E^x\left[\frac{F(\|x-\theta\|^2)}{f(\|x-\theta\|^2)}\,\frac{1}{\|\theta\|^4}\right]. \tag{3.15}$$

**Remark 3.1.** Similarly to what was noticed in [6], Lemma 3.1 still holds when requiring only that the assumptions on the generating function $f(t)$ are satisfied except on a finite set $T$ of values of $t$. This will be implicit in the following and will be used in an example of Section 5.

In order to obtain minimaxity of $\delta_k$ in (1.6), expressions in (3.10) and (3.14) will play an important role through their inner product with $x$. This can be seen in the following lemma whose proof is postponed to Appendix A.1.

**Lemma 3.2.** *For any $x \in \mathbb{R}^p$, let*

$$\begin{aligned} \omega(\|x\|^2) &= -\frac{\Delta M(\|x\|^2)}{m(\|x\|^2)} \\ &= 2\,k\,(p-2(k+1))\,E^x\left[\frac{F(\|x-\theta\|^2)}{f(\|x-\theta\|^2)}\,\frac{1}{\|\theta\|^2}\right]. \end{aligned} \tag{3.16}$$

*We have*

$$x \cdot \frac{\nabla m(\|x\|^2)}{m(\|x\|^2)} = \omega(\|x\|^2) - 2k \tag{3.17}$$

*and*

$$x \cdot \frac{\nabla(\Delta M(\|x\|^2))}{m(\|x\|^2)} = (-p+2(k+1))\,\omega(\|x\|^2) + \delta(\|x\|^2) \tag{3.18}$$

*where*

$$\delta(\|x\|^2) = 2\,k\,(p-2(k+1))E^x\left[\|x-\theta\|^2\,\frac{1}{\|\theta\|^2}\right]. \tag{3.19}$$

## 4. Bayes minimax estimators

We can now formulate our main results about the minimaxity of the generalized Bayes estimators $\delta_k(X)$ in (1.6). This minimaxity will be obtained through improvement on the usual estimator $X$.

**Theorem 4.1.** *Assume that $X$ has a spherically symmetric unimodal density $f(\|x-\theta\|^2)$ as in (1.1) such that $\mu_2 = E_0[\|X\|^2] < \infty$, $f(t) \in S^{3,p/2+1+\epsilon}\left(\mathbb{R}_+^*\right)$ for some $\epsilon > 0$ and the function $F(t)/f(t)$ is nonincreasing. Consider a prior as in (1.5) with $0 < k \le p/2 - 3$.*

*Then the Bayes estimator $\delta_k$ in (1.6) dominates $X$ (and hence is minimax), under the quadratic loss (1.2), as soon as $\mu_{-2\,(k+1)} = E_0[\|X\|^{-2\,(k+1)}] < \infty$,*

$$p\,\frac{F(0)}{f(0)}\,\frac{\mu_{-2\,(k+1)}}{\mu_{-2\,k}}\,k - [p-2\,(k+1)] \le 0 \tag{4.1}$$

*and*

$$2\left[\frac{F(0)}{f(0)} + \frac{\mu_2}{p}\right]k^2 + \left[(p-2)\,\frac{F(0)}{f(0)} - (p-6)\,\frac{\mu_2}{p}\right]k - (p-4)\,\frac{\mu_2}{p} \le 0. \tag{4.2}$$

**Remark 4.1.** The assumption $0 < k \leq p/2 - 3$ imposes that $p \geq 7$ and is made to guarantee the superharmonicity of the function $\|\theta\|^{-2\,(k+2)}$ (which implies the superharmonicity of the pior density in (1.5)). It is worth noting that this upper bound for $k$ is implied by Condition (4.2) (see Appendix A.4).

Conditions (4.1) and (4.2) express that $k$ lies in a neighborhood of 0 since, in (4.1), according to Remark 2.1, the ratio $\mu_{-2\,(k+1)}/\mu_{-2\,k}$ is nondecreasing in $k$.

*Proof.* As it was noticed in Section 3, we are reduced to prove Conditions (2.3) and (2.4). Condition (2.3) will follow in proving Inequalities (3.2), (3.7), (3.3) and (3.8) successively.

Assume Conditions (4.1) and (4.2) and consider first Inequality (3.2). According to (3.12) and (3.13) in Lemma 3.1, taking

$$a = \frac{1}{p}\,\frac{\mu_{-2k}}{\mu_{-2(k+1)}}$$

in (3.1), this one can be expressed as

$$\frac{p}{2}\,\frac{\mu_{-2\,(k+1)}}{\mu_{-2\,k}} \times 4\,k^2 \left\| E^x\!\left[\frac{F(\|x-\theta\|^2)}{f(\|x-\theta\|^2)}\,\frac{\theta}{\|\theta\|^2}\right]\right\|^2 - 2\,k\,(p-2(k+1)) \\ E^x\!\left[\frac{F(\|x-\theta\|^2)}{f(\|x-\theta\|^2)}\,\frac{1}{\|\theta\|^2}\right] \leq 0\,.$$

After simplifying, it is clear that, through Jensen's inequality, it is sufficient to prove that

$$p\,\frac{\mu_{-2\,(k+1)}}{\mu_{-2\,k}}\,k E^x\!\left[\left(\frac{F(\|x-\theta\|^2)}{f(\|x-\theta\|^2)}\right)^2 \frac{1}{\|\theta\|^2}\right] - (p-2(k+1)) \\ E^x\!\left[\frac{F(\|x-\theta\|^2)}{f(\|x-\theta\|^2)}\,\frac{1}{\|\theta\|^2}\right] \leq 0\,. \tag{4.3}$$

In the left hand side of Inequality (4.3), as for the first expectation, using the fact that $F(t)/f(t) \leq F(0)/f(0)$ since $F(t)/f(t)$ is nonincreasing, this inequality is satisfied if

$$\left(p\,\frac{F(0)}{f(0)}\,\frac{\mu_{-2\,(k+1)}}{\mu_{-2\,k}}\,k - [p-2\,(k+1)]\right) E^x\!\left[\frac{F(\|x-\theta\|^2)}{f(\|x-\theta\|^2)}\,\frac{1}{\|\theta\|^2}\right] \leq 0. \tag{4.4}$$

Therefore it follows from (4.4) that Inequality (4.3) holds, and hence Inequality (3.2) holds as well, as soon as Inequality (4.1) is satisfied.

To prove (3.7) note that, similarly, by (3.15), (3.13), (3.11) and (3.10) in Lemma 3.1, the term $A(x)$ in (3.5) equals

$$A(x) = 4\,k\,(k+1)(p-2\,(k+1))(p-2\,(k+2))\,E^x\!\left[\frac{F(\|x-\theta\|^2)}{f(\|x-\theta\|^2)}\,\frac{1}{\|\theta\|^4}\right] - 4\,k^2(p-2(k+1))^2\,E^x\!\left[\frac{F(\|x-\theta\|^2)}{f(\|x-\theta\|^2)}\,\frac{1}{\|\theta\|^2}\right]E^x\!\left[\frac{1}{\|\theta\|^2}\right] - 16\,k^3\,(p-2\,(k+1))\,E^x\!\left[\frac{F(\|x-\theta\|^2)}{f(\|x-\theta\|^2)}\,\frac{1}{\|\theta\|^2}\right]\left\|E^x\!\left[\frac{\theta}{\|\theta\|^2}\right]\right\|^2 . \tag{4.5}$$

By applying Jensen's inequality to the last expectation of the right hand side of (4.5) and using the fact that $F(t)/f(t) \le F(0)/f(0)$, we have $A(x) \ge 0$ as soon as

$$(k+1)(p-2\,(k+2))E^x\!\left[\frac{F(\|x-\theta\|^2)}{f(\|x-\theta\|^2)}\,\frac{1}{\|\theta\|^4}\right] - k\,(p-2\,(k+1))\,\frac{F(0)}{f(0)}\left(E^x\!\left[\frac{1}{\|\theta\|^2}\right]\right)^2 - 4\,k^2\,\frac{F(0)}{f(0)}\left(E^x\!\left[\frac{1}{\|\theta\|^2}\right]\right)^2 \ge 0\,. \tag{4.6}$$

The first expectation in the left hand side of (4.6) can be written as

$$E^x\!\left[\frac{F(\|x-\theta\|^2)}{f(\|x-\theta\|^2)}\,\frac{1}{\|\theta\|^4}\right] = \frac{1}{m(\|x\|^2)}\int_{\mathbb{R}^p}\frac{F(\|x-\theta\|^2)}{f(\|x-\theta\|^2)}\,\frac{1}{\|\theta\|^4}\,f(\|x-\theta\|^2)\,\|\theta\|^{-2\,k}\,d\theta = \frac{1}{m(\|x\|^2)}\int_0^\infty\int_{S_{r,x}}\|\theta\|^{-2\,(k+2)}\,dU_{r,x}(\theta)\,\frac{F(r^2)}{f(r^2)}\xi(r)\,dr \tag{4.7}$$

where $U_{r,x}$ is the uniform distribution on the sphere of radius $r$ centered at $x$ and $\xi(r)$ is the radial density in (2.1). Note that, as $0 < k \le p/2 - 3$, the function $\|\theta\|^{-2\,(k+2)}$ is superharmonic, so that the function $\int_{S_{r,x}}\|\theta\|^{-2\,(k+2)}\,dU_{r,x}$ is nonincreasing in $r$. Note also that, by assumption, $F(r^2)/f(r^2)$ is nonincreasing in $r$. Hence, by the covariance inequality, it follows from (4.7) that

$$E^x\!\left[\frac{F(\|x-\theta\|^2)}{f(\|x-\theta\|^2)}\,\frac{1}{\|\theta\|^4}\right] \ge E\!\left[\frac{F(R^2)}{f(R^2)}\right]\,E^x\!\left[\frac{1}{\|\theta\|^4}\right] = \frac{\mu_2}{p}\,E^x\!\left[\frac{1}{\|\theta\|^4}\right], \tag{4.8}$$

since

$$\begin{aligned} E\left[\frac{F(r^2)}{f(r^2)}\right] &= \int_0^\infty F(r^2)\,\frac{2\pi^{p/2}}{\Gamma(p/2)}r^{p-1}\,dr \\ &= \frac{\pi^{p/2}}{\Gamma(p/2)}\int_0^\infty\int_{r^2}^\infty f(u)\,du\,r^{p-1}\,dr \\ &= \frac{\pi^{p/2}}{\Gamma(p/2)}\int_0^\infty\int_0^{\sqrt{u}} r^{p-1}\,dr\,f(u)\,du \\ &= \frac{\pi^{p/2}}{p\,\Gamma(p/2)}\int_0^\infty u^{p/2}\,f(u)\,du \\ &= \frac{\mu_2}{p}, \end{aligned}$$

where the second equality follows from (1.9), the third one is obtained from an application of Fubini's theorem and, for the fourh one, a change of variable is used.

Using (4.8) and applying Jensen's inequality, (4.6) is satisfied if

$$(k+1)\,(p-2\,(k+2))\,\frac{\mu_2}{p}E^x\left[\frac{1}{\|\theta\|^4}\right]-k\,(p-\,2\,(k+1))\,\frac{F(0)}{f(0)}E^x\left[\frac{1}{\|\theta\|^4}\right]$$
$$-4\,k^2\,\frac{F(0)}{f(0)}E^x\left[\frac{1}{\|\theta\|^4}\right]\geq 0\,,$$

that is, if

$$(k+1)\,(p-2\,(k+2))\frac{\mu_2}{p}-k\,(p-2\,(k+1))\,\frac{F(0)}{f(0)}-4\,k^2\,\frac{F(0)}{f(0)}\geq 0\,. \tag{4.9}$$

As (4.9) is equivalent to (4.2), Inequality (3.7) is proved.

We now turn our attention to Inequality (3.3). Note that $F(t)$ is nonincreasing and that, by unimodality of $f(\|x-\theta\|^2)$, the function $f(t)$ is nonincreasing in $t$ as well. As an immediate consequence of (3.10) and (3.12) in Lemma 3.1, the right hand side of (3.3) equals, for any $x\in\mathbb{R}^p$,

$$\nabla m(\|x\|^2)\cdot\nabla M(\|x\|^2)=$$
$$4\,k^2\int_{\mathbb{R}^p}f(\|x-\theta\|^2)\,\|\theta\|^{-2(k+1)}\,\theta\,d\theta\cdot\int_{\mathbb{R}^p}F(\|x-\theta\|^2)\,\|\theta\|^{-2(k+1)}\,\theta\,d\theta\,. \tag{4.10}$$

Now, since $f(\cdot)$ and $F(\cdot)$ are nonincreasing functions and since $\|\theta\|^{-2(k+1)}$ is a nonnegative function, Lemma A.3 in Appendix A.1 guarantees that each integral in (4.10) equals $x$ multiplied by a nonnegative function of $x$. Hence the left hand side of (4.10) is nonnegative and (3.3) is satisfied.

Similarly, by (3.10) and (3.14) in Lemma 3.1, the inner product term in the right hand side of (3.6) equals, for any $x\in\mathbb{R}^p$,

$$\nabla(\Delta M(\|x\|^2))\cdot\nabla m(\|x\|^2)=$$
$$\zeta(k)\int_{\mathbb{R}^p}F(\|x-\theta\|^2)\|\theta\|^{-2(k+2)}\theta d\theta\cdot\int_{\mathbb{R}^p}f(\|x-\theta\|^2)\|\theta\|^{-2(k+1)}\theta d\theta \tag{4.11}$$

where $\zeta(k)=-8k^2(k+1)\,(p-2(k+1))$. As the function $\zeta(k)$ is nonpositive, we can conclude, as in (4.10), that the left hand side of (4.11) is nonpositive. Therefore we obtain that $B(x)$ in (3.6) is nonnegative, that is, Inequality (3.8) which, with Inequality (3.7) proved above, provides the subharmonicity of $\Delta M(\|x\|^2)/m(\|x\|^2)$ according to (3.4). Finally, gathering Conditions (3.2) and (3.3) obtained above, we have completely proved Condition (2.3).

It remains to address Condition (2.4). As it will be more convenient to deal with nonnegative functions, we will be interested in proving the nondecreasing monotonicity in $R$ of

$$R^{2(k+1)} \int_{B_{\theta,R}} \omega(\|x\|^2)\, dV_{\theta,R}(x)\,, \tag{4.12}$$

for $\omega(\|x\|^2)$ defined in (3.16) which was above shown to be superharmonic, so that the function $t \mapsto \omega(t)$ is necessarily nonincreasing. Hence, according to Corollary A in Appendix A.1, this desired result will be obtained if we prove that $t \mapsto r(t) = t^{k+1}\, \omega(t)$ is nondecreasing. Note that the monotonicity of the functions $\omega$ and $r$ can be expressed respectively as

$$\frac{1}{2}\, \frac{1}{\|x\|^2}\, x \cdot \nabla\omega(\|x\|^2) \leq 0 \tag{4.13}$$

and

$$\frac{1}{2}\, \frac{1}{\|x\|^2}\, x \cdot \nabla r(\|x\|^2) \geq 0 \tag{4.14}$$

since the quantities in the left hand side of (4.13) and (4.14) are the derivatives of $\omega(t)$ and $r(t)$ at $t = \|x\|^2$ respectively.

It is easily seen, through the expression of $\omega(\|x\|^2)$, that the inner product in (4.13) equals

$$\begin{aligned} x \cdot \nabla\omega(\|x\|^2) &= -\omega(\|x\|^2)\, x \cdot \frac{\nabla m(\|x\|^2)}{m(\|x\|^2)} - x \cdot \frac{\nabla(\Delta M(\|x\|^2))}{m(\|x\|^2)} \\ &= -\omega(\|x\|^2)\, \{\omega(\|x\|^2) - 2\,k\} + (p - 2(k+1))\, \omega(\|x\|^2) - \delta(\|x\|^2) \\ &= -\omega^2(\|x\|^2) + \{p-2\}\, \omega(\|x\|^2) - \delta(\|x\|^2)\,, \end{aligned} \tag{4.15}$$

according to (3.17) and (3.18) in Lemma 3.2. Therefore the fact that Inequality (4.13) is satisfied (as mentioned above) can be expressed as

$$\omega^2(t) - \{p-2\}\, \omega(t) + \delta(t) \geq 0\,. \tag{4.16}$$

Also, since

$$r'(t) = (t^{k+1}\, \omega(t))' = ((k+1)\, \omega(t) + t\, \omega'(t))\, t^k\,,$$

it follows from the left hand sides of (4.13) and (4.14) that

$$\frac{1}{\|x\|^2}\, x \cdot \nabla r(\|x\|^2) = [2\,(k+1)\, \omega(\|x\|^2) + x \cdot \nabla\omega(\|x\|^2)]\, (\|x\|^2)^k$$

and hence Inequality (4.14) will be satisfied if and only if

$$\omega^2(t) - \{p - 2 + 2\,(k+1)\}\, \omega(t) + \delta(t) \leq 0\,, \tag{4.17}$$

according to (4.15).

Finally, we have, according to (A.17) in Appendix A.3,

$$\delta(t) \leq p\, \omega(t)\,. \tag{4.18}$$

Therefore it follows from (4.18) that

$$\omega^2(t) - \{p + 2\,k\}\,\omega(t) + \delta(t) \le \omega^2(t) - \{2\,k\}\,\omega(t)$$

so that a sufficient condition for (4.17) to hold is

$$0 \le \omega(t) \le 2\,k\,.$$

As $\omega(t)$ is nonincreasing in $t$ and $\omega(0) = 2\,k$ (see Lemma A.4 in Appendix A.3), this is clearly satisfied. □

**Remark 4.2.** In [5], the minimaxity conditions are weaker but the sampling densities are restricted to the Berger class, that is, there exists a positive constant $c$ such that $F(t)/f(t) > c$ for any $t \ge 0$. Here our approach allows to include the case where $\lim_{t\to\infty} F(t)/f(t) = 0$.

## 5. Examples of sampling densities

As a consequence of Remark 4.2, all the examples in [5] work. So, we focus on sampling densities for which

$$\lim_{t\to\infty} F(t)/f(t) = 0. \tag{5.1}$$

Note that, in the examples below, it is guaranteed that $f \in S^{3,p/2+1+\epsilon}\left(\mathbb{R}_+^*\right)$ for some $\epsilon > 0$ since the densities are elementary functions of the exponential function (with the exception of Example 1 for which this assumption is not fulfilled at one point; see Remark 3.1).

**Example 1.** Let

$$f(t) = \frac{\Gamma(p/2 + A)}{\pi^{p/2}\,\Gamma(A)\,\mathbb{R}^p}\left(1 - \frac{t}{R^2}\right)^{A-1} \mathbb{1}_{[0,R^2]}(t)$$

where $A \ge 1$ and $R > 0$. Clearly $f(t)$ and

$$\frac{F(t)}{f(t)} = \frac{R^2}{2\,A}\left(1 - \frac{t}{R^2}\right)\mathbb{1}_{[0,R^2]}(t)$$

are nonincreasing in $t$. Also

$$\frac{F(0)}{f(0)} = \frac{R^2}{2\,A},$$

and the condition (5.1) is fulfilled, since

$$\lim_{t\to R^2} \frac{F(t)}{f(t)} = 0\,.$$

As, for any $i > -p$, it can be easily checked that

$$\mu_i = R^i\,\frac{\Gamma((p+i)/2)\,\Gamma(p/2+A)}{\Gamma((p+i)/2+A)\,\Gamma(p/2)},$$

then we have

$$\frac{\mu_2}{p} = \frac{R^2}{2A + p}$$

and, for $k < p/2 - 1$,

$$\frac{\mu_{-2\,(k+1)}}{\mu_{-2\,k}} = \frac{p - 2(k+1) + 2\,A}{R^2\,[p - 2(k+1)]}\,.$$

Consequently, one can see, after straightforward calculations, that Condition (4.1) reduces to

$$2\,(p + 4\,A)\,k^2 - [8\,(p-2)\,A + p\,(p - 2 + 2\,A)]\;k + 2\,(p-2)^2\,A \geq 0,$$

and hence (see Remark 4.1) to

$$0 < k \leq \psi(p, A) =: \frac{8\,(p-2)\,A + p\,(2\,A + p - 2) - \sqrt{p^2\,(2\,A + p - 2)^2 + 32\,p\,(p-2)\,A^2}}{4\;[4\,A + p]}\,. \tag{5.2}$$

On the other hand, Condition (4.2) expresses as

$$\left(\frac{2}{p + 2\,A} + \frac{1}{A}\right) k^2 + \left(\frac{p-2}{2\,A} - \frac{p-6}{p+2\,A}\right) k - \frac{p-4}{2\,A + p} \leq 0\,,$$

and is satisfied if

$$0 < k \leq \xi(p, A) =: \frac{\frac{p-6}{p+2\,A} - \frac{p-2}{2\,A} + \sqrt{\left(\frac{p-2}{2\,A} - \frac{p-6}{p+2\,A}\right)^2 + 4\left(\frac{2}{p+2\,A} + \frac{1}{A}\right)\frac{p-4}{2\,A+p}}}{2\left(\frac{2}{p+2\,A} + \frac{1}{A}\right)}\,. \tag{5.3}$$

Finally, according to (5.2) and (5.3), Conditions (4.1) and (4.2) are satisfied as soon as

$$0 < k \leq \min\{\psi(p, A), \xi(p, A)\} =: k_{max}(p, A)\,. \tag{5.4}$$

It is quite involved to investigate formally this upper bound of the values of $k$. However, for different values of $p$ and $A$, Table 1 provides the values $k_{max}(p, A)$ in (5.4).

We can see that the upper bound[1] $k_{max}(p, A)$ is increasing in $A$, for any fixed $p$.

**Example 2.** Consider

$$f(t) = \frac{\Gamma(p/2)\,\beta\,\gamma^{p/2\beta}}{\pi^{p/2}\Gamma(p/2\beta)}\;\exp(-\gamma\,t^{\beta})$$

[1] Note that, for the set of values of $p$ and $A$ in Table 1, $k_{max}(p, A)$ always equals $\xi(p, A)$.

TABLE 1
*Values of $k_{max}(p, A)$ for different values of $p$ and $A$*

| $p \backslash A$ | 1 | 2 | 3 | 5 | 10 | 20 | 30 | 50 | 75 | 100 |
|---|---|---|---|---|---|---|---|---|---|---|
| 7 | 0.130 | 0.209 | 0.261 | 0.324 | 0.394 | 0.441 | 0.459 | 0.474 | 0.483 | 0.487 |
| 10 | 0.130 | 0.203 | 0.306 | 0.414 | 0.556 | 0.666 | 0.712 | 0.753 | 0.775 | 0.786 |
| 15 | 0.106 | 0.199 | 0.281 | 0.416 | 0.637 | 0.851 | 0.953 | 1.050 | 1.106 | 1.135 |
| 20 | 0.085 | 0.166 | 0.240 | 0.372 | 0.621 | 0.911 | 1.067 | 1.229 | 1.326 | 1.379 |

with $\gamma > 0$ and $\beta > 1$. Clearly $f(t)$ is nonincreasing. Furthermore we have

$$\begin{aligned} 2\,\frac{F(t)}{f(t)} &= \int_t^\infty \exp(-\gamma\,[u^\beta - t^\beta])\,du \\ &= \int_0^\infty \exp(-\gamma\,[(v+t)^\beta - t^\beta])\,dv \\ &\leq \int_0^\infty \exp(-\gamma\,[(v+t)\,t^{\beta-1} - t^\beta])\,dv \\ &= \int_0^\infty \exp(-\gamma\,v\,t^{\beta-1})\,dv \end{aligned}$$

which shows that $F(t)/f(t)$ is nonincreasing as well and, by the Lebesgue dominated convergence theorem, implies that

$$\lim_{t\to\infty} \frac{F(t)}{f(t)} = 0\,,$$

since $\beta > 1$.

Now, through straightforward calculations, we have

$$\frac{F(0)}{f(0)} = \frac{1}{2}\,\frac{1}{\beta}\,\frac{\Gamma(1/\beta)}{\gamma^{1/\beta}}\,, \tag{5.5}$$

and, for $i > -p$,

$$\mu_i = \frac{1}{\gamma^{i/2\beta}}\,\frac{\Gamma([p+i]/2\beta)}{\Gamma(p/2\beta)}\,, \tag{5.6}$$

so that, for $k < p/2 - 1$,

$$\frac{\mu_{-2\,(k+1)}}{\mu_{-2\,k}} = \gamma^{1/\beta}\,\frac{\Gamma([p-2\,(k+1)]/2\beta)}{\Gamma([p-2\,k]/2\beta)}\,. \tag{5.7}$$

Then Condition (4.1) reduces to

$$G_{\beta,p}(k) =: p\,\frac{\Gamma(1/\beta)}{2\,\beta}\,\frac{\Gamma([p-2\,(k+1)]/2\beta)}{\Gamma([p-2\,k]/2\beta)}\,k - [p-2(k+1)] \;\leq 0 \tag{5.8}$$

Also, according to (5.6), we have

$$\frac{\mu_2}{p} = \frac{1}{p}\,\frac{1}{\gamma^{1/\beta}}\frac{\Gamma([p+2]/2\beta)}{\Gamma(p/2\beta)}$$

TABLE 2

*Values of $K_{max}(\beta, p)$ for different values of $p$ and $\beta$*

| $p \backslash \beta$ | 1 | 1.1 | 1.3 | 1.5 | 2 | 3 | 5 |
|---|---|---|---|---|---|---|---|
| 7 | 0.500 | 0.435 | 0.349 | 0.296 | 0.227 | 0.177 | 0.150 |
| 10 | 0.822 | 0.677 | 0.497 | 0.390 | 0.275 | 0.198 | 0.158 |
| 15 | 1.232 | 0.934 | 0.605 | 0.444 | 0.277 | 0.182 | 0.136 |
| 20 | 1.561 | 1.101 | 0.645 | 0.446 | 0.259 | 0.160 | 0.115 |

so that, after simplifying by $1/\gamma^{1/\beta}$, Condition (4.2) is expressed as

$$H_{\beta,p}(k) =: \; 2\left[\frac{\Gamma(1/\beta)}{2\,\beta} + \frac{1}{p}\,\frac{\Gamma([p+2]/2\beta)}{\Gamma(p/2\beta)}\right] k^2 + \left[(p-2)\,\frac{\Gamma(1/\beta)}{2\,\beta} - \frac{p-6}{p}\,\frac{\Gamma([p+2]/2\beta)}{\Gamma(p/2\beta)}\right] k - \frac{p-4}{p}\,\frac{\Gamma([p+2]/2\beta)}{\Gamma(p/2\beta)} \le 0\,, \tag{5.9}$$

which is satisfied if

$$0 < k \le \frac{-\left[(p-2)\,\frac{\Gamma(1/\beta)}{2\,\beta} - \frac{p-6}{p}\,\frac{\Gamma([p+2]/2\beta)}{\Gamma(p/2\beta)}\right] + \sqrt{\Delta}}{4\left[\frac{\Gamma(1/\beta)}{2\,\beta} + \frac{1}{p}\,\frac{\Gamma([p+2]/2\beta)}{\Gamma(p/2\beta)}\right]}\,, \tag{5.10}$$

with

$$\Delta = \left[(p-2)\,\frac{\Gamma(1/\beta)}{2\,\beta} - \frac{p-6}{p}\,\frac{\Gamma([p+2]/2\beta)}{\Gamma(p/2\beta)}\right]^2 + 8\left[\frac{\Gamma(1/\beta)}{2\,\beta} + \frac{1}{p}\,\frac{\Gamma([p+2]/2\beta)}{\Gamma(p/2\beta)}\right] \times \frac{p-4}{p}\,\frac{\Gamma([p+2]/2\beta)}{\Gamma(p/2\beta)}\,.$$

Even if we know, according to (5.7) and to Remark 4.1, that the ratio $\Gamma([p - 2\,(k+1)]/2\beta)/\Gamma([p-2\,k]/2\beta)$ is nondecreasing in $k$, it is difficult to compare the upper bound for $k$ implicit in (5.8) and the upper bound in (5.10). However the upper bound for $k$ combining Conditions (5.8) and (5.10), that is,

$$K_{max}(\beta,p) = \min\{\max\{k>0/G_{\beta,p}(k)=0\}, \max\{k>0/H_{\beta,p}(k)=0\}\} \tag{5.11}$$

can be evaluated specifying values of $p$ and $\beta$. For fixed values of $p$ and $\beta$, Table 2 provides the corresponding values of $K_{max}(\beta, p)$.

Although, in this example, we assume $\beta > 1$ (since we deal with the case where $\lim_{t\to\infty} F(t)/f(t) = 0$), in Table 2, we provide the value $\beta = 1$ corresponding to the normal case (which is covered by Theorem 4.1 which does put any restriction on the limit $\lim_{t\to\infty} F(t)/f(t)$). The upper bound of the value of $k$ for which the Bayes estimator $\delta_k$ is minimax, given in (5.11), is decreasing in $\beta$, for any fixed $p$.

**Example 3.** We now give a fairly general example for which we do not express explicitly Conditions (4.1) and (4.2). Let

$$f(t) \propto \exp(-\,\alpha\,t\,g(t))$$

where $g(t)$ is a nondecreasing and convex function such that $\lim_{t\to\infty} g(t) = \infty$ and $\alpha > 0$.

It is clear that $f(\|x-\theta\|^2)$ is unimodal. Also, expressing $F(t)/f(t)$ as follows

$$\begin{aligned}\frac{F(t)}{f(t)} &\propto \frac{\int_t^\infty f(s)\,ds}{f(t)} \\ &= \int_t^\infty \exp(-\alpha\,[s\,g(s) - t\,g(t)])\,ds \\ &= \int_0^\infty \exp(-\alpha\,[(v+t)\,g(v+t) - t\,g(t)])\,dv\,, \qquad (5.12)\end{aligned}$$

we can see, first, that the function $F(t)/f(t)$ is nonincreasing. Indeed, the function

$$\psi(t) = (v+t)\,g(v+t) - t\,g(t)$$

in (5.12) is nondecreasing, since

$$\begin{aligned}\psi'(t) &= g(v+t) - g(t) + t\,[g'(v+t) - g'(t)] + v\,g'(v+t) \\ &\geq 0\,,\end{aligned}$$

by nondecreasing montonicity and convexity of the function $g$.

Secondly, for such class of generated functions $f(t)$, it has been shown in [3] that the condition $\inf_{t\geq 0} \int_t^\infty f(s)\,ds/f(t) > 0$ is violated and therefore $\lim_{t\to\infty}(F(t)/f(t)) = 0$ since, in our case, $F(t)/f(t)$ is nonincreasing.

Note that a simple example of function $g$ is $g(t) = \exp(t)$ so that $f(t) \propto \exp(-\alpha\,t\,e^t)$.

It is worth noting that $f(t) \propto \exp(-\alpha\,e^t)$ is not included in the previous class but gives rise to another example satisfying the conditions of Theorem 4.1. Indeed $f(t)$ is clearly nonincreasing and $f'(t)/f(t) = -\alpha e^t$ is nonincreasing so that the function $F(t)/f(t)$ is nonincreasing as well. We also have

$$\begin{aligned}\lim_{t\to\infty}\frac{F(t)}{f(t)} &\propto \lim_{t\to\infty}\int_t^\infty \exp(-\alpha\,[e^s - e^t])ds \\ &= \lim_{t\to\infty}\int_0^\infty \exp(-\alpha\,e^t[e^v - 1])dv \\ &= 0\,,\end{aligned}$$

from an application of the Lebesgue dominated convergence theorem.

## 6. Concluding remarks

We have seen that, for a sampling density $f(\|x-\theta\|^2)$ which is unimodal and such that the ratio $F(t)/f(t)$ is nonincreasing (with $F(t)$ in (1.9)), minimaxity of generalized Bayes estimators $\delta_k(X)$ can be obtained for spherical prior densities $\|\theta\|^{-2k}$ under conditions involving constants depending on $f(\cdot)$ and $k$.

We complement the results of Fourdrinier, Mezoued and Strawderman [5] in so far as these authors reduce their framework to the Berger class. However we only deal with the case where $F(t)/f(t)$ is nonincreasing. Indeed, in the case where $F(t)/f(t)$ is nondecreasing, our techniques are unsuitable for this monotonicity. Also, we adopted here a completely different approach since we relied on a modification of the Brandwein and Strawderman approach [4], which may be of an independent interest. Various examples of sampling densities illustrate our findings while the basic example of prior densities is formed of the class of $\|\theta\|^{-2k}$ with $k > 0$. A natural scope of a future work is to extend that class of priors.

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## Appendix

### *A.1. Properties of some expectations and integrals*

In the following lemma and its corollary, we mention in our notations the dimension of the spaces in which spheres and balls lie. Thus $U^{p+k}_{R,\theta}$ denotes the uniform distribution on the sphere $S^{p+k}_{R,\theta} = \{(x,u) \in \mathbb{R}^{p+k} / \|(x,u) - (\theta,0)\| = R\}$, in $\mathbb{R}^{p+k}$, of radius $R$ and centered at $(\theta,0) \in \mathbb{R}^{p+k}$, while $V^p_{R,\theta}$ holds for the uniform distribution on the ball $B^p_{R,\theta} = \{x \in \mathbb{R}^p / \|x - \theta\| \leq R\}$, in $\mathbb{R}^p$, of radius $R$ and centered at $\theta \in \mathbb{R}^p$. In that context, we extend a result given by Fourdrinier and Strawderman [9].

**Lemma A.1.** *Let $r(t)$ be a nonnegative and nondecreasing function on $[0,\infty]$ such that $r(t)/t^q$ is nonincreasing, for some $q \geq 1$. Then, for any fixed $\theta \in \mathbb{R}^p$, the function*

$$f_\theta : R \longmapsto R^{2q} \int_{S^{p+k}_{R,\theta}} \frac{r(\|x\|^2)}{\|x\|^{2q}} \, dU^{p+k}_{R,\theta}(x,u) \tag{A.1}$$

*is nondecreasing for $p \geq 1$ and $k \geq 2$.*

*Proof.* Under $U^{p+k}_{R,\theta}$, it is well known that the marginal distribution of $(x,u) \mapsto x$ is absolutely continuous with unimodal density $\psi(\|x-\theta\|^2/R^2)/R^p$ for all $k \geq 2$ where $\psi(t) = (1-t)^{k/2-1}\mathbf{1}_{[0,1]}(t)$. Then $f_\theta$ can be written as

$$f_\theta(R) = \int_{B^p_{1,0}} \frac{r(R^2 \, \|z + \frac{\theta}{R}\|^2)}{\|z + \frac{\theta}{R}\|^{2q}} \, \psi(\|z\|^2) \, dz \,.$$

For any $R_1 \leq R_2$, we have, by nondecreasing monotonicity of $r(t)$,

$$f_\theta(R_1) \leq \int_{B^p_{1,0}} \frac{r(R_2^2 \|z + \frac{\theta}{R_1}\|^2)}{\|z + \frac{\theta}{R_1}\|^{2q}} \, \psi(\|z\|^2) \, dz \,.$$

Furthermore nonincreasing monotonicity of $r(t)/t^q$ implies that the function $r(R_2^2\,\|z+\theta/R_1\|^2)/\|z+\theta/R_1\|^{2q}$ is symmetric and unimodal in $z$ about $-\theta/R_1$. Hence, by Anderson's theorem (see [1]),

$$\begin{aligned}\int_{B^p_{1,0}} \frac{r(R_2^2\|z+\frac{\theta}{R_1}\|^2)}{\|z+\frac{\theta}{R_1}\|^{2q}}\,\psi(\|z\|^2)\,dz &\le \int_{B^p_{1,0}} \frac{r(R_2^2\|z+\frac{\theta}{R_2}\|^2)}{\|z+\frac{\theta}{R_2}\|^{2q}}\,\psi(\|z\|^2)\,dz \\ &= f_\theta(R_2)\,.\end{aligned}$$

□

As, if $(X,U)\sim U^{p+2}_{R,\theta}$ then $X\sim V^p_{R,\theta}$, the following corollary of Lemma A.1 is immediate.

**Corollary A.** *Under the conditions of Lemma A.1, the function*

$$R\longmapsto R^{2q}\int_{B^p_{R,\theta}} \frac{r(\|x\|^2)}{\|x\|^{2q}}\,dV^p_{R,\theta}(x) \tag{A.2}$$

*is nondecreasing for $p\ge 1$.*

When integrating with respect to $F(\|\theta-x\|^2)$, the following result is useful.

**Lemma A.2.** *For any function $\gamma$, we have*

$$\int_{R^p} \gamma(\theta)\,F(\|\theta-x\|^2)\,d\theta = \lambda(B)\int_0^\infty \int_{B_{r,x}} \gamma(\theta)\,dV_{r,x}(\theta)\,r^{p+1}\,f(r^2)\,dr\,. \tag{A.3}$$

*Proof.* We have the successive equalities

$$\begin{aligned}\int_{R^p} \gamma(\theta)\,F(\|\theta-x\|^2)\,d\theta &= \int_{R^p} \gamma(\theta)\,\frac{1}{2}\int_{\|\theta-x\|^2}^\infty f(u)\,du\,d\theta \\ &= \int_{R^p} \gamma(\theta)\int_{\|\theta-x\|}^\infty r\,f(r^2)\,dr\,d\theta \\ &= \int_0^\infty \int_{B_{r,x}} \gamma(\theta)\,d\theta\,r\,f(r^2)\,dr \\ &= \lambda(B)\int_0^\infty \int_{B_{r,x}} \gamma(\theta)\,dV_{r,x}(\theta)\,r^{p+1}\,f(r^2)\,dr\end{aligned}$$

the second inequality following from the change of variable $u=r^2$, the third one from the Fubini theorem; in the fourth one, $\lambda(B)$ is the volume of the unit ball. □

The following lemma can be found in Fourdrinier and Righi [7] where the density in (1.2) is considered with fixed $x$ as a function of $\theta$, say $\theta\mapsto f(\|\theta-x\|^2)$, so that $E_x$ denotes the expectation with respect to that density.

**Lemma A.3.** *Let $x\in\mathbb{R}^p$ fixed and let $\Theta$ a random vector in $\mathbb{R}^p$ with spherically symmetric density $f(\|\theta-x\|^2)$. Let $g$ be a function from $\mathbb{R}_+$ into $\mathbb{R}$.*

*Then there exists a function* $\Gamma$ *from* $\mathbb{R}^p$ *into* $\mathbb{R}$ *such that*

$$E_x[g(\|\Theta\|^2)\,\Theta] = \Gamma(x)\cdot x\,, \tag{A.4}$$

*provided this expectation exists. Moreover, if the function* $f$ *is nonincreasing and if the function* $g$ *is nonnegative, then the function* $\Gamma$ *is nonnegative.*

### *A.2. Proof of Lemma 3.2*

For any $x \in \mathbb{R}^p$, we have

$$\begin{aligned} x\cdot\nabla m(\|x\|^2) &= x\cdot\int_{\mathbb{R}^p} -2\,k\,\theta\,\|\theta\|^{-2(k+1)}\,f(\|x-\theta\|^2)\,d\theta \\ &= -2\,k\int_{\mathbb{R}^p}(x-\theta)\cdot\theta\,\|\theta\|^{-2(k+1)}\,f(\|x-\theta\|^2)\,d\theta \\ &\qquad -2\,k\int_{\mathbb{R}^p}\|\theta\|^{-2\,k}\,f(\|x-\theta\|^2)\,d\theta \end{aligned} \tag{A.5}$$

substracting and adding $\theta$. Now, denoting by $\sigma_{r,x}$ the uniform measure on the sphere $S_{r,x}$ of radius $r$ and of center $x$, the first integral in the right hand side of (A.5) can be written as

$$\begin{aligned} &\int_{\mathbb{R}^p}(x-\theta)\cdot\theta\,\|\theta\|^{-2(k+1)}\,f(\|x-\theta\|^2)\,d\theta \\ &\quad= \int_0^\infty\!\!\int_{S_{r,x}}(x-\theta)\cdot\theta\,\|\theta\|^{-2(k+1)}\,d\sigma_{r,x}(\theta)\,f(r^2)\,dr \\ &\quad= -\int_0^\infty\!\!\int_{S_{r,x}}\frac{\theta-x}{\|\theta-x\|}\cdot\theta\,\|\theta\|^{-2(k+1)}\,d\sigma_{r,x}(\theta)\,r\,f(r^2)\,dr \\ &\quad= -\int_0^\infty\!\!\int_{B_{r,x}}\operatorname{div}(\theta\,\|\theta\|^{-2(k+1)})\,r\,f(r^2)\,dr \\ &\quad= -\,(p-2(k+1))\int_0^\infty\!\!\int_{B_{r,x}}\|\theta\|^{-2(k+1)}d\theta\,r\,f(r^2)\,dr\,, \end{aligned} \tag{A.6}$$

by the Stokes theorem. Hence, according to (A.3), (A.6) gives

$$\begin{aligned} \int_{\mathbb{R}^p}(x-\theta)\cdot\theta\,\|\theta\|^{-2(k+1)}\,f(\|x-\theta\|^2)\,d\theta &= -(p-2\,(k+1)) \\ &\qquad \int_{\mathbb{R}^p}\|\theta\|^{-2(k+1)}F(\|\theta-x\|^2)\,d\theta\,. \end{aligned} \tag{A.7}$$

Finally, substituting (A.7) in (A.5) yields

$$\begin{aligned} x\cdot\nabla m(\|x\|^2) = 2\,k\,(p-2(k+1))\;&\int_{\mathbb{R}^p}\|\theta\|^{-2(k+1)}\,F(\|\theta-x\|^2)\,d\theta \\ &-2k\int_{R^p}\|\theta\|^{-2k}f(\|\theta-x\|^2)\,d\theta \end{aligned} \tag{A.8}$$

and hence, according to the definition of the expectation $E^x$, we obtain

$$x \cdot \frac{\nabla m(\|x\|^2)}{m(\|x\|^2)} = 2\,k\,(p-2(k+1))\,E^x\Big[\frac{F(\|x-\theta\|^2)}{f(\|x-\theta\|^2)}\,\frac{1}{\|\theta\|^2}\Big] - \,2\,k,$$

which gives (3.17) in Lemma 3.2.

As for (3.18), we have

$$\begin{aligned} x\cdot\nabla(\Delta M(\|x\|^2)) &= -2\,k\,(p-2(k+1))x\cdot\nabla\int_{R^p}\|\theta\|^{-2(k+1)}\,F(\|\theta-x\|^2)\,d\theta \\ &= -2\,k\,(p-2(k+1))\int_{R^p} x\cdot(\theta-x)\,\|\theta\|^{-2(k+1)}\,f(\|\theta-x\|^2)\,d\theta \\ &= B_1(\|x\|^2)+B_2(\|x\|^2) \end{aligned} \tag{A.9}$$

where

$$B_1(\|x\|^2) = 2\,k\,(p-2(k+1))\int_{R^p}\|\theta-x\|^2\,\|\theta\|^{-2(k+1)}\,f(\|\theta-x\|^2)\,d\theta \tag{A.10}$$

and

$$B_2(\|x\|^2) = -2\,k\,(p-2(k+1))\int_{R^p}\theta\cdot(\theta-x)\,\|\theta\|^{-2(k+1)}\,f(\|\theta-x\|^2)\,d\theta \tag{A.11}$$

since

$$x\cdot(\theta-x) = -(\theta-x-\theta)\cdot(\theta-x) = -\|\theta-x\|^2+\theta\cdot(\theta-x)\,.$$

As above, a decomposition through the spheres $S_{r,x}$ gives rise to

$$\begin{aligned} B_2(\|x\|^2) &= -2\,k\,(p-2(k+1))\int_0^\infty\int_{S_{r,x}}\frac{\theta-x}{\|\theta-x\|}\cdot\theta\,\|\theta\|^{-2(k+1)}\,d\sigma_{r,x}(\theta)\,r\,f(r^2)\,dr \\ &= -2\,k\,(p-2(k+1))\int_0^\infty\int_{B_{r,x}}\mathrm{div}(\theta\,\|\theta\|^{-2(k+1)})\,d\theta\,r\,f(r^2)\,dr \\ &= -2\,k\,(p-2(k+1))\int_{\mathbb{R}^p}\mathrm{div}(\theta\,\|\theta\|^{-2(k+1)})\,F(\|\theta-x\|^2)\,d\theta \\ &= -2\,k\,(p-2(k+1))^2\int_{\mathbb{R}^p}\|\theta\|^{-2(k+1)}\,F(\|\theta-x\|^2)\,d\theta\,, \end{aligned} \tag{A.12}$$

according to the Stokes theorem and to (A.3), and expanding the divergence term. Dividing $B_1(\|x\|^2)$ and $B_2(\|x\|^2)$ in (A.10) and (A.12) by $m(\|x\|^2)$ we obtain

$$\frac{B_1(\|x\|^2)}{m(\|x\|^2)} = 2\,k\,(p-2(k+1))\,E^x\left[\|\theta-x\|^2\;\frac{1}{\|\theta\|^2}\right], \tag{A.13}$$

and

$$\begin{aligned} \frac{B_2(\|x\|^2)}{m(\|x\|^2)} &= -2k(p-2(k+1))^2\,E^x\left[\frac{F(\|\theta-x\|^2)}{f(\|\theta-x\|^2)}\frac{1}{\|\theta\|^2}\right] \\ &= (-p+2(k+1))\,\omega(\|x\|^2)\,. \end{aligned} \tag{A.14}$$

Finally, according to (A.9)and by using (A.13) and (A.14), we have obtained (3.18) in Lemma 3.2. □

### *A.3. Expressing the functions in Lemma 3.2*

We now give expressions, in terms of sphere mean and ball mean, for the functions $\omega(\|x\|^2)$, $\gamma(\|x\|^2)$ and $\delta(\|x\|^2)$ defined in Lemma 3.2.

**Lemma A.4.** *For a prior as in (1.5) and for any $x \in R^p$, setting $t = \|x\|^2$, we have*

$$\omega(t) = \frac{2\,k\,(p-2-2\,k)}{p}\;\frac{\int_0^\infty \int_{B_{r,x}} \|\theta\|^{-2(k+1)}\, dV_{r,x}(\theta)\, r^{p+1}\, f(r^2)\, dr}{\int_0^\infty \int_{S_{r,x}} \|\theta\|^{-2\,k}\, dU_{r,x}(\theta)\, r^{p-1}\, f(r^2)\, dr} \tag{A.15}$$

*and*

$$\delta(t) = 2\,k\,(p-2-2\,k)\;\frac{\int_0^\infty \int_{S_{r,x}} \|\theta\|^{-2(k+1)}\, dU_{r,x}(\theta)\, r^{p+1}\, f(r^2)\, dr}{\int_0^\infty \int_{S_{r,x}} \|\theta\|^{-2\,k}\, dU_{r,x}(\theta)\, r^{p-1}\, f(r^2)\, dr}\,. \tag{A.16}$$

*Also, for $k \le p/2 - 2$,*

$$\omega(t) \ge \frac{1}{p}\,\delta(t) \tag{A.17}$$

*and*

$$\omega(0) = 2\,k\,, \qquad \text{and} \qquad \delta(0) = 2\,k\,(p-2\,-2\,k)\,. \tag{A.18}$$

*Proof.* According to the definition of the conditional expectation $E^x$ the function $\omega(t)$ in Lemma 3.2 can be written as

$$\begin{aligned}
\omega(t) &= 2\,k\,(p-2(k+1))\,\frac{\int_{R^p} \|\theta\|^{-2(k+1)}\, F(\|\theta-x\|^2)\, d\theta}{\int_{R^p} \|\theta\|^{-2k}\, f(\|\theta-x\|^2)\, d\theta} \\
&= 2\,k\,(p-2(k+1))\,\frac{\lambda(B)\, \int_0^\infty \int_{B_{r,x}} \|\theta\|^{-2(k+1)}\, dV_{r,x}(\theta)\, r^{p+1}\, f(r^2)\, dr}{\sigma(S)\, \int_0^\infty \int_{S_{r,x}} \|\theta\|^{-2k}\, dU_{r,x}(\theta)\, r^{p-1}\, f(r^2)\, dr}\,,
\end{aligned}$$

applying (A.3), where $\sigma(S)$ denote the surface of the unit sphere. Then (A.15) follows from the fact that $\lambda(B)/\sigma(S) = 1/p$.

Similarily, for (A.16), the result follows directly from the representation of $\delta(t)$ in (3.19) as

$$\delta(t) = 2\,k\,(p-2(k+1))\,\frac{\int_{R^p} \|\theta\ - x\|^2\, \|\theta\|^{-2(k+1)}\, f(\|\theta-x\|^2)\, d\theta}{\int_{R^p} \|\theta\|^{-2k}\, f(\|\theta-x\|^2)\, d\theta}\,.$$

Now, note that, by superharmonicity of $\|\theta\|^{-2(k+1)}$ for $k \le p/2-2$, we have

$$\int_{B_{r,x}} \|\theta\|^{-2(k+1)}\, dV_{r,x}(\theta) \ge \int_{S_{r,x}} \|\theta\|^{-2(k+1)}\, dU_{r,x}(\theta)$$

so that comparing (A.15) and (A.16) gives (A.17).

Finally the values at 0 of the functions $\omega$ and $\delta$ will be derived expressing the integrals on the balls $B_r$ and the spheres centered at 0. Considering the right-hand side of (A.15), the innermost integral in the numerator of the second ratio satisfies

$$\begin{aligned} r^{p+1} \int_{B_r} \|\theta\|^{-2(k+1)} \, dV_r(\theta) &= \frac{r}{\lambda(B)} \int_0^r \tau^{-2(k+1)} \, \sigma(S) \, \tau^{p-1} \, d\tau \\ &= p \, r \int_0^r \tau^{p-3-2k} \, d\tau \\ &= \frac{p}{p-2-2k} \, r^{p-1-2k} \, . \end{aligned}$$

As for the innermost integral in the denominator, we have

$$r^{p-1} \int_{S_r} \|\theta\|^{-2k} \, dU_r(\theta) = r^{p-1-2k}$$

so that the same power of $r$ appears in the integrals in $r$ and, after simplifying, we obtain that $\omega(0) = 2\,k$.

Finally, obtaining the value of $\delta(0)$ follows the same way and gives

$$\delta(0) = 2\,k\,(p - 2(k+1)) \, .$$

□

### A.4. Proof of the statement in Remark 4.1

As $k > 0$, Inequality (4.2) holds if and only if

$$k \leq \frac{-B + \sqrt{B^2 - 4AC}}{2A}$$

where

$$A = 2\left(\frac{F(0)}{f(0)} + \frac{\mu_2}{p}\right), \quad B = \frac{F(0)}{f(0)}\,(p-2) - \frac{\mu_2}{p}\,(p-6) \quad \text{and} \quad C = -\frac{\mu_2}{p}\,(p-4) \, .$$

This upper bound for $k$ is less than or equal to $p/2 - 3$ if and only if

$$B^2 - 4\,A\,C \leq [(p-6)\,A + B]^2$$

which is equivalent to

$$0 \leq (p-6)^2\,A + 2\,(p-6)\,B + 4\,C \, .$$

Then, according to the values of $A$, $B$ and $C$, this equality results in

$$0 \leq 2\,(p-6)^2\,\frac{F(0)}{f(0)} + 2\,(p-6)^2\,\frac{\mu_2}{p} + 2\,(p-6)\,\frac{F(0)}{f(0)}\,(p-2) - 2\,(p-6)^2\,\frac{\mu_2}{p} - 4\,(p-4)\,\frac{\mu_2}{p}$$

which, after simplication, gives rise to

$$0 \leq (p-6)\,\frac{F(0)}{f(0)} - \frac{\mu_2}{p}\,.$$

Now, this last equality holds since $p \geq 7$, the ratio $F(t)/f(t)$ is nonincreasing in $t$ and $E[F(R^2)/f(R^2)] = \mu_2/p$. Therefore we have proved that Condition (4.2) implies that $k \leq p/2 - 3$.

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