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Order statistics and exceedances for some models of INID random variables

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Abstract. Order statistics and exceedances for some general models of independent but not necessarily identically distributed (INID) random variables are considered. The distributions of order statistics from INID sample are described in terms of symmetric functions. Some exceedance models based on order statistics from INID random variables are considered, the limit distributions of exceedance statistics are obtained. For the model of INID random variables referred as F^{α}-scheme introduced by Nevzorov (*Zapiski Nauchnykh Seminarov LOMI* **142** (1985) 109–118) the limiting distribution of exceedance statistic has been derived. This distribution is expressed in terms of permutations with inversions, Gaussian Hypergeometric function and incomplete beta functions.

1 Introduction

The theory of order statistics from independent and identically distributed (i.i.d.) random variables is well developed. The first fundamental book describing this theory is David (1981). Arnold et al. (1992) and David and Nagaraja (2003) include new developments on order statistics from i.i.d., dependent and INID random variables. There are not much results on the theory of order statistics from arbitrarily dependent random variables, because of the joint p.d.f.'s, which are not factorized like in i.i.d. case, causing technical difficulties in calculations. The distribution function of a single order statistic from arbitrary dependent random variables is given in David and Nagaraja (2003) and involves distribution functions of maximal order statistics from sample sizes less than the sample size of original sample (see David and Nagaraja (2003), formula (5.3.1), page 99). The joint distribution of two or more order statistics from dependent random variables is studied in Maurer and Margolin (1976). The distribution theory of order statistics from INID random variables first described in Vaughan and Venables (1972) involves the permanent, a concept defined similar to the determinant except that it does not have an alternating sign, that is, taking all terms in the summation of the definition of the determinant to be positive. For a recent review describing the theory of order statistics from INID case and also including interesting results on outliers and robustness, we refer to Balakrishnan (2007). The mean residual life

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functions for INID random variables in the system level is studied in Gurler and Bairamov (2009). Permanent expressions for the distribution function of INID order statistics allows us to obtain some recurrence relations using the expansion of the permanent by some of the rows. However, in some cases when the applications of order statistics from the INID random variables are considered, the usage of the permanent expressions for the distributions of INID order statistics causes some difficulties connected with the complexity of operations. For instance, the mean residual life function of parallel and k-out-of-n coherent systems when the life length of the components are INID random variables can not be easily calculated using permanent expressions. Therefore, the calculations involving the joint distributions of order statistics from INID random variables face technical difficulties, the results are complicated and are not convenient for applications.

In this paper, firstly the INID random variables are considered and distributions of order statistics are described in terms of permanents and the symmetric functions defined as the n-variate functions of products of some permutations of variables. The representations of distributions of order statistics from INID random variables in terms of symmetric functions have an advantage if one uses the derivatives and integration in calculations. Secondly, as the major contribution of the paper the exceedance model from INID random variables is considered. The asymptotic distribution of exceedance statistic has been derived and for a special model of INID random variables the limiting distribution is studied.

1.1 INID random variables

Let $X_1, X_2, \dots, X_n$ be independent but not necessarily identically distributed random variables with cumulative distribution functions (c.d.f.'s) $F_1(x), F_2(x), \dots, F_n(x)$ and $X_{1:n}, X_{2:n}, \dots, X_{n:n}$ be corresponding order statistics. If $F_1, F_2, \dots, F_n$ are absolutely continuous with corresponding probability density functions (p.d.f.'s) $f_1, f_2, \dots, f_n$, then the joint p.d.f. of $X_{1:n}, X_{2:n}, \dots, X_{n:n}$ is

$$f_{1,2,\dots,n}(x_1, x_2, \dots, x_n) = \sum_{\wp_{1,2,\dots,n}} \prod_{i=1}^{n} f_{j_i}(x_i),$$

where the summation $\wp_{1,2,\dots,n}$ extends over all $n!$ permutations $(j_1, j_2, \dots, j_n)$ of $1, 2, \dots, n$. For any borel set $B \in \Re$, where $\Re$ is the Borel σ-algebra of subsets of the set of real numbers $\mathbb{R}$ consider indicators

$$I_{X_i}(B) = \begin{cases} 1, & X_i \in B, \\ 0, & X_i \notin B, \end{cases} \qquad i = 1, 2, \dots, n$$

and let $\nu^*(B) = \sum_{i=1}^{n} I_{X_i}(B)$. Define the empirical distribution of the INID sample $X_1, X_2, \dots, X_n$ as $P_n^*(B) = \frac{\nu^*(B)}{n}$. It is clear that $EI_{X_i}(B) = P\{X_i \in B\} = \int_B dF_i(x) \equiv P_i(B)$ and $\mathrm{Var}(I_{X_i}(B)) = P_i(B)(1 - P_i(B))$ and $EP_n^*(B) =$

$\frac{1}{n}\sum_{i=1}^{n} P_i(B)$ and $\mathrm{var}(P_n^*(B)) = \frac{1}{n^2}\sum_{i=1}^{n} P_i(B)(1 - P_i(B))$. The empirical distribution function of the INID sample then is defined as

$$F_n^*(x) = P_n^*\big((-\infty, x]\big) = \frac{1}{n}\sum_{i=1}^{n} I_{X_i}(x),$$

where $I_{X_i}(x) = 1$ if $X_i \le x$ and $I_{X_i}(x) = 0$, otherwise. According to the Kolmogorov's theorem, the sequence of mutually independent random variables $\xi_1, \xi_2, \dots, \xi_n, \dots$ obeys the strong law of large numbers, if $\sum_{n=1}^{\infty} \frac{\mathrm{Var}(\xi_n)}{n^2} < \infty$ (see Gnedenko (1978), page 215). Since

$$\frac{\mathrm{Var}(I_{X_n}(B))}{n^2} = \frac{P_n(B)(1 - P_n(B))}{n^2} \le \frac{1}{n^2}$$

then the series

$$\sum_{n=1}^{\infty} \frac{\mathrm{Var}(I_{X_n}(B))}{n^2}$$

converges. Then the sequence of mutually independent random variables $I_{X_1}(B), \dots, I_{X_n}(B), \dots$ obeys the strong law of large numbers, that is, as $n \to \infty$, with probability 1

$$\frac{1}{n}\sum_{i=1}^{n} I_{X_i}(B) - \frac{1}{n}\sum_{i=1}^{n} E I_{X_i}(B) \to 0. \tag{1.1}$$

From (1.1), we have

$$P_n^*(B) \overset{\text{a.s.}}{\to} \frac{1}{n}\sum_{i=1}^{n} P_i(B), \qquad B \in \Re$$

and

$$F_n^*(x) \overset{\text{a.s.}}{\to} \frac{1}{n}\sum_{i=1}^{n} F_i(x), \qquad x \in \mathbb{R}.$$

Lemma 1. *For any $B \in \Re$ and $x \in \mathbb{R}$*

$$P\{nP_n^*(B) = k\} = \frac{1}{k!(n-k)!}\sum_{\wp_{1,2,\dots,n}} \prod_{i=1}^{k} P_{j_i}(B) \prod_{i=k+1}^{n} \big(1 - P_{j_i}(B)\big)$$

and

$$P\{nF_n^*(x) = k\} = \frac{1}{k!(n-k)!}\sum_{\wp_{1,2,\dots,n}} \prod_{i=1}^{k} F_{j_i}(x) \prod_{i=k+1}^{n} \big(1 - F_{j_i}(x)\big),$$

where the summation $\wp_{1,2,\dots,n}$ extends over all $n!$ permutations $(j_1, j_2, \dots, j_n)$ of $1, 2, \dots, n$.

Denote now

$$\mathbf{A}(n,k;x)=\binom{n}{k}x^k(1-x)^{n-k},\qquad 0\le x\le 1, k=0,1,2,\ldots,n; n\ge 1$$

and the symmetric function

$$\mathbf{B}(n,k;x_1,x_2,\ldots,x_n)=\frac{1}{k!(n-k)!}\sum_{\wp_{1,2,\ldots,n}}\prod_{i=1}^{k}x_{j_i}\prod_{i=k+1}^{n}(1-x_{j_i}),$$

$$k=0,1,2,\ldots,n; n\ge 1, 0\le x_l\le 1, l=1,2,\ldots,n,$$

where the summation $\wp_{1,2,\ldots,n}$ extends over all $n!$ permutations $(j_1, j_2,\ldots, j_n)$ of $1,2,\ldots,n$, assuming 0^0 and $\prod_{i=j+1}^{j}a_i$ are equal to 1. Note that $\mathbf{B}(n,n;x_1,x_2,\ldots,x_n)=x_1x_2\cdots x_n$ and $\mathbf{B}(n,0;x_1,x_2,\ldots,x_n)=(1-x_1)(1-x_2)\cdots(1-x_n)$. It is clear that $\mathbf{B}(n,k;x_{j_1},x_{j_2},\ldots,x_{j_n})=\mathbf{B}(n,k;x_1,x_2,\ldots,x_n)$ for all $n!$ permutations $(j_1, j_2,\ldots, j_n)$ of $(1,2,\ldots,n)$ and

$$P\{nF_n^*(x)=k\}=\mathbf{B}\big(n,k;F_1(x),F_2(x),\ldots,F_n(x)\big).$$

It is clear that if $F_1=F_2=\cdots=F_n=F$ then

$$P\{nF_n^*(x)=k\}=\mathbf{A}\big(n,k;F(x)\big).$$

The following recurrence relation will be useful.

Lemma 2. *Let $0\le x_l\le 1, l=1,2,\ldots,n$. The following recurrence relation is valid for $k=1,2,\ldots,n-1$ and $n\ge 2$*

$$\mathbf{B}(n,k;x_1,x_2,\ldots,x_n)=\mathbf{B}(n-1,k;x_1,x_2,\ldots,x_{n-1})\bar{x}_n+\mathbf{B}(n-1,k-1;x_1,x_2,\ldots,x_{n-1})x_n,$$

where $\bar{x}_n=1-x_n$.

Proof. See the Appendix. □

1.2 The order statistics from INID random variables and symmetric functions

The c.d.f. of rth order statistic $X_{r:n}$ is

$$\begin{aligned}F_{r:n}(x)&=P\{X_{r:n}\le x\}\\&=\sum_{k=r}^{n}\frac{1}{k!(n-k)!}\sum_{\wp_{1,2,\ldots,n}}\prod_{i=1}^{k}F_{j_i}(x)\prod_{i=k+1}^{n}\big(1-F_{j_i}(x)\big)\end{aligned}\tag{1.2}$$

(see David and Nagaraja (2003)) and in terms of $\mathbf{B}(n,k;x_1,x_2,\ldots,x_n)$ it can be written as

$$F_{r:n}(x)=\sum_{i=r}^{n}\mathbf{B}\big(n,i,F_1(x),F_2(x),\ldots,F_n(x)\big).\tag{1.3}$$

Using Lemma 2, we can write for $1 \le r < n$

$$\begin{aligned}
F_{r:n}(x) &= \sum_{i=r}^{n} \mathbf{B}\big(n, i, F_1(x), F_2(x), \dots, F_n(x)\big) \\
&= \sum_{i=r}^{n-1} \mathbf{B}\big(n, i, F_1(x), F_2(x), \dots, F_n(x)\big) \\
&\quad + \mathbf{B}\big(n, n, F_1(x), F_2(x), \dots, F_n(x)\big) \\
&= \bar{F}_n(x) \sum_{i=r}^{n-1} \mathbf{B}\big(n-1, i, F_1(x), F_2(x), \dots, F_{n-1}(x)\big) \\
&\quad + F_n(x) \sum_{i=r}^{n-1} \mathbf{B}\big(n-1, i-1, F_1(x), F_2(x), \dots, F_{n-1}(x)\big) \\
&\quad + F_n(x)\mathbf{B}\big(n-1, n-1, F_1(x), F_2(x), \dots, F_{n-1}(x)\big) \\
&= \bar{F}_n(x) F_{r:n-1}(x) \\
&\quad + F_n(x) \sum_{j=r-1}^{n-2} \mathbf{B}\big(n-1, j, F_1(x), F_2(x), \dots, F_{n-1}(x)\big) \\
&\quad + F_n(x)\mathbf{B}\big(n-1, n-1, F_1(x), F_2(x), \dots, F_{n-1}(x)\big),
\end{aligned}$$

where $F_{i:n-1}$ denote the c.d.f. of the ith order statistic from INID random variables $X_1, X_2, \dots, X_{n-1}$ with corresponding c.d.f.'s $F_1, F_2, \dots, F_{n-1}$. Therefore, one can write

$$F_{r:n}(x) = \bar{F}_n(x) F_{r:n-1}(x) + F_n(x) F_{r-1:n-1}(x), \qquad 1 \le r < n. \tag{1.4}$$

For $r = n$, we have $F_{n:n}(x) = F_{n-1:n-1}(x) F_n(x)$. Note that (1.4) and related recurrence equalities can be found in David and Nagaraja (2003), page 105. Since,

$$P\{nF_n^*(x) = i\} = \mathbf{B}\big(n, i, F_1(x), F_2(x), \dots, F_n(x)\big), \qquad i = 0, 1, 2, \dots, n,$$

then

$$\sum_{i=0}^{n} \mathbf{B}\big(n, i, F_1(x), F_2(x), \dots, F_n(x)\big) = 1. \tag{1.5}$$

We also have,

$$P\{X_{r:n} \le x\} = \sum_{i=r}^{n} P\{nF_n^*(x) = i\}.$$

The distribution of order statistics obtained from INID random variables can be expressed in terms of the permanent functions. For the recent nice description of the distribution theory of order statistics obtained from INID random variables we refer Balakrishnan (2007).

The joint p.d.f. of $X_{r:n}$ and $X_{s:n}$ is

$$f_{r,s}(x,y) = \frac{1}{(r-1)!(s-r-1)!(n-s)!} \sum_{\wp_{1,2,\ldots,n}} \prod_{l=1}^{r-1} F_{t_l}(x) f_{t_r}(x) \times \prod_{l=r+1}^{s-1} \big(F_{t_l}(y) - F_{t_l}(x)\big) f_{l_s}(y) \prod_{l=s+1}^{n} \big(1 - F_{t_l}(y)\big). \tag{1.6}$$

Now, for $0 \le x_t < y_t \le 1, t = 1, 2, \ldots, n$ denote by

$$\mathbf{C}(n, i, j; x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_n) = \frac{1}{i!(j-i)!(n-j)!} \sum_{\wp_{1,2,\ldots,n}} \prod_{l=1}^{i} x_{t_l} \times \prod_{l=i+1}^{j} (y_{t_l} - x_{t_l}) \prod_{l=j+1}^{n} (1 - y_{t_l}), \tag{1.7}$$

where the summation $\wp_{1,2,\ldots,n}$ extends over all $n!$ permutations $(t_1, t_2, \ldots, t_n)$ of $1, 2, \ldots, n$.

Lemma 3. *Let $0 \le x_t < y_t \le 1, t = 1, 2, \ldots, n$. The following recurrence relation is valid for $1 \le i < j < n$ and $n \ge 3$*

$$\begin{aligned}\mathbf{C}(n, i, j; x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_n) &= x_n \mathbf{C}(n-1, i-1, j-1; x_1, x_2, \ldots, x_{n-1}; y_1, y_2, \ldots, y_{n-1}) \\ &\quad + (y_n - x_n)\mathbf{C}(n-1, i, j-1; x_1, x_2, \ldots, x_{n-1}; y_1, y_2, \ldots, y_{n-1}) \\ &\quad + (1 - y_n)\mathbf{C}(n-1, i, j; x_1, x_2, \ldots, x_{n-1}; y_1, y_2, \ldots, y_{n-1}).\end{aligned}$$

Proof. The proof of the lemma can be made by using permanent expressions for symmetric functions similar to the proof of Lemma 2. □

The joint distribution function of order statistics $X_{r:n}$ and $X_{s:n}$ can be expressed in terms of symmetric function $\mathbf{C}(n, i, j; x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_n)$ as follows:

$$F_{r,s}(x,y) = \sum_{j=s}^{n} \sum_{i=r}^{j} \mathbf{C}\big(n, i, j; F_1(x), F_2(x), \ldots, F_n(x); F_1(y), F_2(y), \ldots, F_n(y)\big).$$

Using Lemma 3 one can write the following recurrence relation for $1 \le r < s < n, n \ge 3$

$$F_{r,s:n}(x,y) = F_n(x)F_{r-1,s-1:n-1}(x,y) + [F_n(y) - F_n(x)]F_{r,s-1:n-1}(x,y) + (1 - F_n(y))F_{r,s:n-1}(x,y),$$

where $F_{r,s-1:n-1}(x,y)$ is the joint c.d.f. rth and $s-1$th order statistics from INID random variables $X_1, X_2, \dots, X_{n-1}$ with corresponding c.d.f.'s $F_1, F_2, \dots, F_{n-1}$. (See also David and Nagaraja (2003), page 106).

2 Exceedance statistics for INID random variables and their asymptotic distributions

Assume that $X_1, X_2, \dots, X_n$ are independent random variables with continuous distribution functions $F_1(t), F_2(t), \dots, F_n(t)$, respectively and $Y_1, Y_2, \dots, Y_m$ are independent copies of random variable Y with continuous distribution function G. Let $X_{1:n} \le \dots \le X_{n:n}$ be the order statistics constructed from $X_1, X_2, \dots, X_n$. For $1 \le r < s \le n$ define the following random variables

$$\xi_i = \begin{cases} 1 & \text{if } Y_i \in (X_{r:n}, X_{s:n}), \\ 0 & \text{otherwise}, \end{cases}$$
$$i = 1, 2, \dots, n.$$

Let $S_m = \sum_{i=1}^m \xi_i$. It is clear that S_m is the number of observations falling into random threshold $(X_{r:n}, X_{s:n})$ and is the exceedance statistic. The distribution theory of exceedance statistics have been studied in numerous papers which appeared in recent years in statistical literature. See, for example, Bairamov (1997), Wesolowski and Ahsanullah (1998), Bairamov and Eryilmaz (2000), Bairamov and Eryilmaz (2001, 2004), Bairamov and Kotz (2001), Bairamov and Khan (2007), Bairamov (2007).

In general, the derivation of the distribution of exceedance statistic S_m faces technical difficulties connected with the permanent expressions for joint distribution function. Indeed, one has

$$\begin{aligned} P\{S_m = k\} &= \sum_{i_1,i_2,\dots,i_m} P\{\xi_{i_1} = 1, \dots, \xi_{i_k} = 1, \xi_{i_{k+1}} = 0, \dots, \xi_{i_m} = 0\} \\ &= \sum_{i_1,i_2,\dots,i_m} P\{Y_{i_1} \in (X_{r:n}, X_{s:n}), \dots, Y_{i_k} \in (X_{r:n}, X_{s:n}), \\ &\qquad Y_{i_{k+1}} \notin (X_{r:n}, X_{s:n}), \dots, Y_{i_m} \notin (X_{r:n}, X_{s:n})\} \qquad (2.1) \\ &= \sum_{i_1,i_2,\dots,i_m} \int\int P\{Y_{i_1} \in (x,y), \dots, Y_{i_k} \in (x,y), \\ &\qquad Y_{i_{k+1}} \notin (x,y), \dots, Y_{i_m} \notin (x,y)\} f_{r,s}(x,y)\,dx\,dy, \end{aligned}$$

where the summation extends over all $m!$ permutations $(i_1, i_2, \dots, i_m)$ of $1, 2, \dots, m$. Considering formula (1.6) one can observe that even for the special distributions $F_1, F_2, \dots, F_n$ the calculation of $P\{S_m = k\}$ meets with great difficulties and the formula that can be obtained is not convincing for applications. However, the asymptotic distribution of $\frac{S_m}{m}$ can be found by using the functional representations using empirical distribution functions.

In this paper, we focus on asymptotic distributions of exceedance statistics based on INID random variables. We show that $\frac{S_m}{m}$ converges in distribution to the random variable $G(X_{s:n}) - G(X_{r:n})$. Afterwards, we investigate some special distributions for which the distribution of exceedance statistics can be expressed in a good form. More precisely, we consider the F^α scheme introduced by Nevzorov (1985) (see also Nevzorov (1987), Pfeifer (1989, 1991)) and in a special case when $r = 1$ and $s = n$ derive the distribution function of $G(X_{n:n}) - G(X_{1:n})$.

Theorem 1. *Assume that $X_1, X_2, \dots, X_n, \dots$ and $Y_1, Y_2, \dots, Y_n, \dots$ are independent. It is true that*

$$\lim_{m\to\infty} \sup_{0\le x\le 1} \left\{ \left| P\left\{ \frac{S_m}{m} \le x \right\} - P\{W_{rs} \le x\} \right| \right\} = 0,$$

where $W_{rs} = G(X_{s:n}) - G(X_{r:n})$.

Proof. We have

$$S_m = \sum_{i=1}^{m} \xi_i = \sum_{i=1}^{m} I_{\{(X_{r:n}, X_{s:n})\}}(Y_i), \tag{2.2}$$

where $I_A(x) = 1$ if $x \in A$ and $I_A(x) = 1$ if $x \notin A$. Using the representation (2.2) and conditioning on $X_{r:n}$ and $X_{s:n}$ we can write

$$\begin{aligned}
P\left\{ \frac{S_m}{m} \le x \right\} &= P\left\{ \frac{1}{m} \sum_{i=1}^{m} I_{\{(X_{r:n}, X_{s:n})\}}(Y_i) \le x \right\} \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P\left\{ \frac{1}{m} \sum_{i=1}^{m} I_{\{(X_{r:n}, X_{s:n})\}}(Y_i) \le x \right| \\
&\qquad\qquad \left. X_{r:n} = t, X_{s:n} = z \right\} dF_{X_{r:n}, X_{s:n}}(t, z) \qquad (2.3) \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P\left\{ \frac{1}{m} \sum_{i=1}^{m} I_{\{(t,z)\}}(Y_i) \le x \right\} dF_{X_{r:n}, X_{s:n}}(t, z) \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P\left\{ \int_{-\infty}^{\infty} I_{\{(t,z)\}}(u)\, dG_m^*(u) \le x \right\} dF_{X_{r:n}, X_{s:n}}(t, z),
\end{aligned}$$

where $G_m^*(u) = \frac{1}{m}\sum_{i=1}^m I_{\{Y_i \le u\}}$ is the empirical distribution function of sample $Y_1, Y_2, \dots, Y_m$. Denote the functional of G as

$$\Im(G) = \int_{-\infty}^{\infty} I_{(t,z)}(u)\, dG(u) \tag{2.4}$$

and then

$$\Im(G_m^*) = \int_{-\infty}^{\infty} I_{(t,z)}(u)\, dG_m^*(u).$$

Since the functional $\Im(\cdot)$ is continuous according to uniform metric, and using Glivenko–Cantelli Theorem $P\{w : \sup_u |G_m^*(u) - G(u)| \to 0\} = 1$ we have

$$\Im(G_m^*) \to \Im(G), \qquad \text{a.s. as } m \to \infty,$$

that is,

$$P\Big\{w : \lim_{m\to\infty} \Im(G_m^*) = \Im(G)\Big\} = 1.$$

Then from (2.3), we have for $m \to \infty$

$$\begin{aligned}
P\left\{\frac{S_m}{m} \le x\right\} & \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P\left\{\int_{-\infty}^{\infty} I_{(t,z)}(u)\, dG_m^*(u) \le x\right\} dF_{X_{r:n},X_{s:n}}(t,z) \\
&\to \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P\left\{\int_{-\infty}^{\infty} I_{(t,z)}(u)\, dG(u) \le x\right\} dF_{X_{r:n},X_{s:n}}(t,z) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} P\left\{\int_{-\infty}^{\infty} I_{(t,z)}(u)\, dG(u) \le x \,\middle|\, X_{r:n} = t,\, X_{s:n} = z\right\} dF_{X_{r:n},X_{s:n}}(t,z) \\
&= P\left\{\int_{-\infty}^{\infty} I_{(X_{r:n},X_{s:n})\}}(u)\, dG(u) \le x\right\} \\
&= P\left\{\int_{X_{r:n}}^{X_{s:n}} dG(u) \le x\right\} = P\{G(X_{s:n}) - G(X_{r:n}) \le x\}.
\end{aligned}$$

Thus, the theorem is proved. □

Remark 1. The distribution function of the random variable W_{rs} in case of independent and identically distributed random variables can be found in David (1981). For INID random variables, the distribution of W_{rs} in general has complicated form. However, for some special cases this distribution can be easily calculated. In the following theorem, the p.d.f. of $W_{1n} = G(X_{n:n}) - G(X_{1:n})$ is derived. It is interesting that this p.d.f. can be expressed in terms of permutations with inversions, incomplete beta function and hypergeometric functions. For permutation with inversions see Knuth (1973) and for incomplete beta function and hypergeometric functions see Bateman (1953). Below we provide information about permutations with inversions which can be found for example, Margolius (2001).

2.1 Asymptotic distributions of exceedance statistics based on INID random variables

Definition 1. Let $a_1, a_2, \ldots, a_n$ be a permutation of the set $\{1, 2, \ldots, n\}$. If $i < j$ and $a_i > a_j$, the pair (a_i, a_j) is called an "inversion" of the permutation. For example, the permutation 4312 has five inversions which are $(4, 3)$, $(4, 1)$, $(4, 2)$, $(3, 1)$ and $(3, 2)$. Each inversion is a pair of elements that is *out of order*, and it's clear that the only permutation with no inversions is the unordered permutation. Let $I_n(k)$ denotes the number of permutations of length n with k inversions. In the following, an explicit formula for $I_n(k)$ when $k \le n$ (see Knuth (1973)) is given

$$I_n(k) = \binom{n+k-1}{k} + \sum_{j=1}^{\infty} (-1)^j \binom{n+k-u_j-j-1}{k-u_j-j} + \sum_{j=1}^{\infty} (-1)^j \binom{n+k-u_j-1}{k-u_j}.$$

The binomial coefficients are defined to be zero when the lower index is negative. The u_j are the pentagonal numbers defined as

$$u_j = \frac{j(3j-1)}{2}, \qquad j = 1, 2, \ldots.$$

In Table 1, the exact value of $I_n(k)$ for specific n and k values are given.

Table 1 *The exact value of $I_n(k)$ for specific n and k values*

$I_n(k) = I_n(\binom{n}{2} - k)$												
	k, number of inversions											
$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11
1	**1**											
2	1	**1**										
3	1	2	**2**	1								
4	1	3	5	**6**	5	3	1					
5	1	4	9	15	**20**	22	20	15	9	4	1	
6	1	5	14	29	49	**71**	90	101	101	90	71	49
7	1	6	20	49	98	169	**259**	359	455	531	573	573
8	1	7	27	76	174	343	602	**961**	1415	1940	2493	3017
9	1	8	35	111	285	628	1230	2191	**3606**	5545	8031	11,021
10	1	9	44	155	440	1068	2298	4489	8095	**13,640**	21,670	32,683

Theorem 2. *Let $X_1, X_2, \ldots, X_n, \ldots$ be a sequence of independent random variables with the continuous distribution functions $F_1, F_2, \ldots, F_n, \ldots,$ respectively and $Y_1, Y_2, \ldots, Y_m$ be independent copies of random variable Y with continuous distribution function G where*

$$F_i(t) = G^i(t) \qquad \forall t \in R, i = 1, 2, \ldots, n. \tag{2.5}$$

Consider $W_n \equiv W_{1n} = G(X_{n:n}) - G(X_{1:n})$. Then the p.d.f. of W_{1n} is

$$\begin{aligned} f_{W_n}(w) &= I_n(0)\Big[w^{n-2}\frac{2\binom{n}{2}}{(\binom{n}{2}+1)}(1-w^{\binom{n}{2}+1}) \\ &\qquad + nw^{n-1}(1-w^{\binom{n}{2}})\Big] \\ &+ \sum_{i=1}^{\binom{n}{2}} I_n(i)\Big\{2(-1)^{-(i+1)}\binom{n}{2}w^{\binom{n+1}{2}-1} \\ &\qquad \times \text{Beta}\Big[1-\frac{1}{w}, i+1, \binom{n}{2}-i+1\Big] \\ &\qquad + n(-1)^{-(i+1)}\Big(\binom{n}{2}-i\Big)w^{\binom{n+1}{2}} \\ &\qquad \times \text{Beta}\Big[1-\frac{1}{w}, i+1, \binom{n}{2}-i\Big] \\ &\qquad - n(1-w)^i w^{n-2}{}_2F_1\Big[1, \binom{n}{2}+1, i+1, 1-\frac{1}{w}\Big] \\ &\qquad + i(-1)^{-(i+1)}\Big(\binom{n}{2}-i\Big)w^{\binom{n+1}{2}-1} \\ &\qquad \times \text{Beta}\Big[1-\frac{1}{w}, i, \binom{n}{2}-1\Big]\Big\}, \end{aligned} \tag{2.6}$$

if $0 \le w \le 1$ and $f_W(w) = 0$, otherwise. Above, in (2.6) ${}_2F_1[a, b, c, z]$ is the Gaussian hypergeometric function which is defined as

$$\begin{aligned} {}_2F_1[a, b, c, z] &= 1 + \frac{ab}{1!c}z + \frac{a(a+1)b(b+1)}{2!c(c+1)}z^2 + \cdots \\ &= \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!} \end{aligned}$$

and $(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} = x(x+1)\cdots(x+n-1)$ *for* $n \geq 0$, *and* Beta$[z,a,b]$ *is the incomplete beta function defined by*

$$\text{Beta}[z,a,b] \equiv \int_0^z u^{a-1}(1-u)^{b-1}\,du.$$

Remark 2. In special cases when $n=2, n=3$,the p.d.f. given in (2.6) is

$$f_{W_2}(w) = 2-2w, \qquad 0 \leq w \leq 1$$
$$f_{W_3}(w) = 9w - 18w^2 + 14w^3 - 5w^4, \qquad 0 \leq w \leq 1,$$

respectively. Plots of this density functions for $n = 2, 3, \ldots, 9$ are presented in Figure 1 in Section 2.2. It is clear that these densities are polynomial for any n.

Before proving Theorem 2, we need some auxiliary lemmas. Also the following theorem due to Bourget (1871) will be used in the proof.

Theorem A (Bourget (1871), see Comtet (1974), Section 6.4, Theorem B). *The number* $I_n(k)$ *of permutations of* n *with* k *inversions satisfies the following recurrence relations*:

$$I_n(k) = \sum_{\max(0,k-n+1)\leq j\leq k} I_{n-1}(j)$$

for each $n \geq 1$. $I_n(0) = 1$ *for each* $n \geq 1$ *and* $I_0(k) = 1$ *for each* $k \geq 1$.

Lemma 4. *For any* $x, y \in \mathbb{R}$ *and positive integer* n *the following identity is valid*

$$\prod_{i=1}^{n} (y^i - x^i) = (y-x)^n \sum_{i=0}^{\binom{n}{2}} I_n(i) y^{\binom{n}{2}-i} x^i.$$

Proof. See the Appendix. □

Remark 3. Muir (1882) showed that

$$\frac{1}{(1-t)^n} \prod_{i=1}^{n} (1-t^i) = \sum_{i=0}^{\binom{n}{2}} I_n(i) t^i. \tag{2.7}$$

Using (2.7) one has

$$\prod_{i=1}^{n} (y^i - x^i) = y^{n(n+1)/2} \prod_{i=1}^{n} \left(1 - \left(\frac{x}{y}\right)^i\right)$$
$$= y^{n(n+1)/2} \left(1 - \left(\frac{x}{y}\right)\right)^n \sum_{i=0}^{\binom{n}{2}} I_n(i) \left(\frac{x}{y}\right)^i$$

and

$$
\begin{aligned}
H(x, y) &= \prod_{i=1}^{n} y^i - \prod_{i=1}^{n} (y^i - x^i) \\
&= y^{n(n+1)/2} \left\{ 1 - \left(1 - \left(\frac{x}{y}\right)\right)^n \sum_{i=0}^{\binom{n}{2}} I_n(i) \left(\frac{x}{y}\right)^i \right\}.
\end{aligned} \tag{2.8}
$$

Lemma 5. *let the conditions of the Theorem 2 be satisfied. Denote by $H(x, y)$ the joint distribution function of $G(X_{1:n})$ and $G(X_{n:n})$. It is true that*

$$
H(x, y) = y^{n(n+1)/2} - (y - x)^n \sum_{i=0}^{\binom{n}{2}} I_n(i) y^{\binom{n}{2}-i} x^i \tag{2.9}
$$

for $0 \le x \le y$ and $0 \le y \le 1$, where $I_n(k)$ denotes the number of permutations of length n with k inversions.

Proof.

$$
\begin{aligned}
&H(x, y) \\
&\quad = P\{G(X_{1:n}) \le x, G(X_{n:n}) \le y\} \\
&\quad = P\{X_{1:n} \le G^{-1}(x), X_{n:n} \le G^{-1}(y)\} \\
&\quad = P\{X_{n:n} \le G^{-1}(y)\} - P\{G^{-1}(x) < X_{1:n}, X_{n:n} \le G^{-1}(y)\} \\
&\quad = P\{X_1 \le G^{-1}(y), \ldots, X_n \le G^{-1}(y)\} \\
&\qquad - P\{G^{-1}(x) < X_1 \le G^{-1}(y), \ldots, G^{-1}(x) < X_n \le G^{-1}(y)\} \\
&\quad = \prod_{i=1}^{n} F_i(G^{-1}(y)) - \prod_{i=1}^{n} [F_i(G^{-1}(y)) - F_i(G^{-1}(x))] \\
&\quad = \prod_{i=1}^{n} y^i - \prod_{i=1}^{n} [y^i - x^i],
\end{aligned} \tag{2.10}
$$

where $G^{-1}(x) = \min\{t \in \mathbb{R} : G(t) \ge x\}$ is the quintile function of G.

Then using Lemma 4 the last expression in (2.10) can be written in terms of permutation with inversions as

$$
H(x, y) = y^{n(n+1)/2} - (y - x)^n \sum_{i=0}^{\binom{n}{2}} I_n(i) y^{\binom{n}{2}-i} x^i.
$$

For calculations, the formula (2.9) is more convenient than (2.8) because of polynomial form with respect to x and y. □

Corollary 1. *Let the function $h(x, y)$ denotes the joint probability density function of $G(X_{1:n})$ and $G(X_{n:n})$. Then*

$$h(x,y) = (y-x)^n \times \sum_{i=0}^{\binom{n}{2}} I_n(i) y^{\binom{n}{2}-i} x^i \times \left[\frac{2\binom{n}{2}}{(y-x)^2} + \frac{n(\binom{n}{2}-i)}{(y-x)y} - \frac{ni}{(y-x)x} - \frac{(\binom{n}{2}-i)i}{yx}\right] \tag{2.11}$$

for $0 \le x \le y$ and $0 \le y \le 1$, where $I_n(k)$ denotes the number of permutations of length n with k inversions.

Proof.

$$\begin{aligned} h(x,y) &= \frac{\partial^2 H(x,y)}{\partial x\, \partial y} \\ &= -\frac{\partial^2 (y-x)^n \sum_{i=0}^{\binom{n}{2}} [I_n(i) y^{\binom{n}{2}-i} x^i]}{\partial x\, \partial y} \\ &= -\sum_{i=0}^{\binom{n}{2}} I_n(i) \frac{\partial^2 [(y-x)^n y^{\binom{n}{2}-i} x^i]}{\partial x\, \partial y} \\ &= (y-x)^n \sum_{i=0}^{\binom{n}{2}} I_n(i) y^{\binom{n}{2}-i} x^i \\ &\qquad \times \left[\frac{2\binom{n}{2}}{(y-x)^2} + \frac{n(\binom{n}{2}-i)}{(y-x)y} - \frac{ni}{(y-x)x} - \frac{(\binom{n}{2}-i)i}{yx}\right]. \qquad \square \end{aligned}$$

Now using Lemma 4 and Corollary 1 we are ready to prove the Theorem 2.

Proof of Theorem 2. It is clear that the probability density function of $W_n = G(X_{n:n}) - G(X_{1:n})$ can be obtained from the joint probability density function of $G(X_{1:n})$ and $G(X_{n:n}) - G(X_{1:n})$ using the linear transformation. Denote $G(X_{1:n}) = Z_1$ and $G(X_{n:n}) = Z_2$. Using transformation $T_1 = Z_1,\ T_2 = Z_2 - Z_1$,we have $Z_1 = T_1,\ Z_2 = T_1 + T_2$. The Jacobian of this transformation equals to 1 and therefore

$$f_{T_1,T_2}(y_1, y_2) = f_{Z_1,Z_2(y_1,y_1+y_2)|J|^{-1}},$$

$$f_{T_2}(w) = \int_0^{1-w} f_{Z_1,Z_2}(y_1, y_1+y_2)\, dy_1,$$

$$
\begin{aligned}
f_{W_n}(w) &= \int_0^{1-w} f_{G(X_{1:n}),G(X_{n:n})}(x, x+w)\,dx, \\
f_{W_n}(w) &= \int_0^{1-w} h(x, x+w)\,dx \\
&= \sum_{i=0}^{\binom{n}{2}} I_n(i) \int_0^{1-w} (x+w)^{\binom{n}{2}-i} x^i \Bigg[2\binom{n}{2} w^{n-2} + \frac{n(\binom{n}{2}-i)w^{n-1}}{(x+w)} \\
&\qquad - \frac{niw^{n-1}}{x} - \frac{(\binom{n}{2}-i)iw^n}{(x+w)x} \Bigg] dx \\
&= I_n(0)\Bigg[\frac{2\binom{n}{2}}{(\binom{n}{2}+1)} w^{n-2}\big(1-w^{\binom{n}{2}+1}\big) + nw^{n-1}\big(1-w^{\binom{n}{2}}\big) \Bigg] \\
&\quad + \sum_{i=1}^{\binom{n}{2}} I_n(i) \Bigg\{ 2(-i)^{-(i+1)} \binom{n}{2} w^{\binom{n+1}{2}-1} \\
&\qquad \times \mathrm{Beta}\Bigg[1-\frac{1}{w}, i+1, \binom{n}{2}-i+1\Bigg] \\
&\qquad + n(-1)^{-(i+1)} \Bigg(\binom{n}{2}-i\Bigg) w^{\binom{n+1}{2}-1} \\
&\qquad \times \mathrm{Beta}\Bigg[1-\frac{1}{w}, i+1, \binom{n}{2}-i\Bigg] \\
&\qquad - n(1-w)^i w^{(n-2)} {}_2F_1\Bigg[1, 1+\binom{n}{2}, i+1, 1-\frac{1}{w}\Bigg] \\
&\qquad + i(-1)^{-(i+1)} \Bigg(\binom{n}{2}-i\Bigg) w^{\binom{n+1}{2}-1} \\
&\qquad \times \mathrm{Beta}\Bigg[1-\frac{1}{w}, i, \binom{n}{2}-1\Bigg]\Bigg\}
\end{aligned}
$$

for $n \geq 1$, $0 \leq w \leq 1$.

Thus the theorem is proved. □

2.2 Some numerical results

In Theorem 2 for special case $F_i = G^i$, we have obtained the expression of $f_{W_n}(w)$, the p.d.f. of the limiting distribution $P\{W_{1n} \leq x\} = P\{G(X_{n:n}) - G(X_{1:n}) \leq x\}$ which is given in (2.6). This p.d.f. presents an independent interest and below we provide some numerical results and graphs concerning the numerical characteristics, as first, second and third moments, variance and skewness.

Below we provide the graphs of the p.d.f. and c.d.f. of W_n for different values of n. It can be easily seen in Figure 1 that $f_{W_2}(w) = 2 - 2w$, that is, for $n = 2$ the p.d.f. is linear.

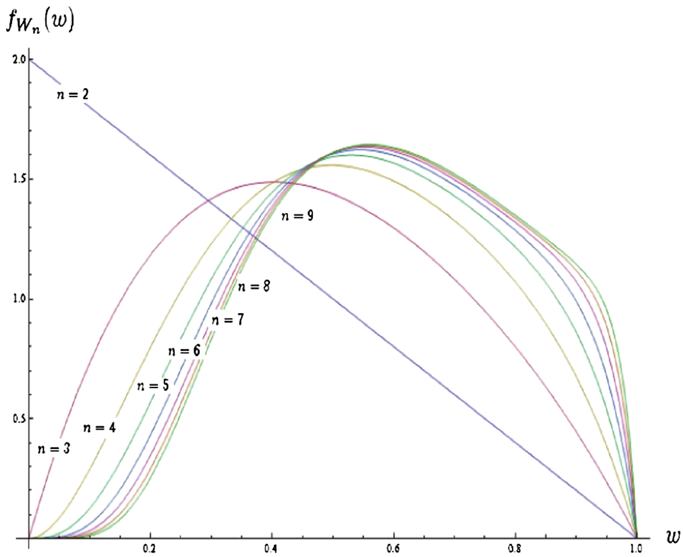

Figure 1 *The graphs of* $f_{W_n}(w)$ *for* $n = 2, \ldots, 9$.

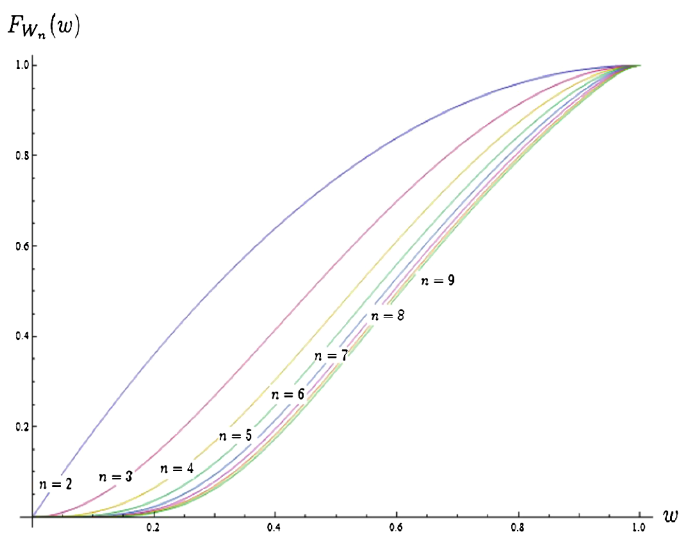

Figure 2 *The graphs of* $F_{W_n}(w)$ *for* $n = 2, \ldots, 9$.

Below, we give also the graphs of $E(W_n)$, $E(W_n^2)$, $E(W_n^3)$ and skewness with respect to n.

As you can see in Figures 1–4 and in Table 2, the moments $E(W_n)$, $E(W_n^2)$, $E(W_n^3)$ increase as n increases and the graph of $f_{W_n}(w)$ is right skewed for $n = 2, 3$ and left skewed for $n > 3$.

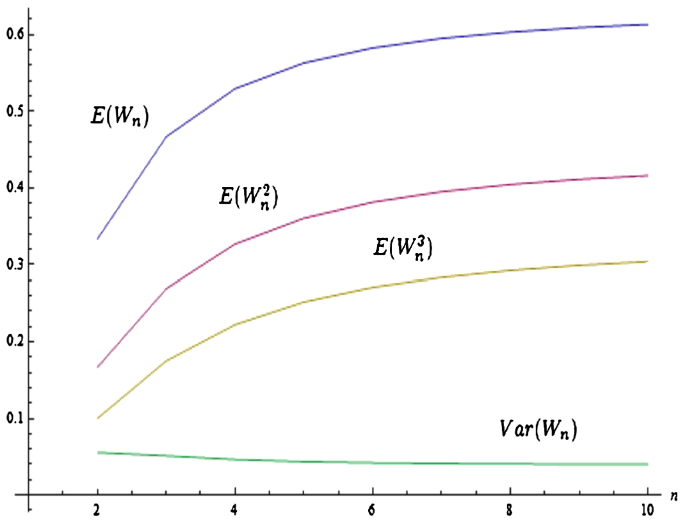

Figure 3 *The graphs of moments and variance with respect to n.*

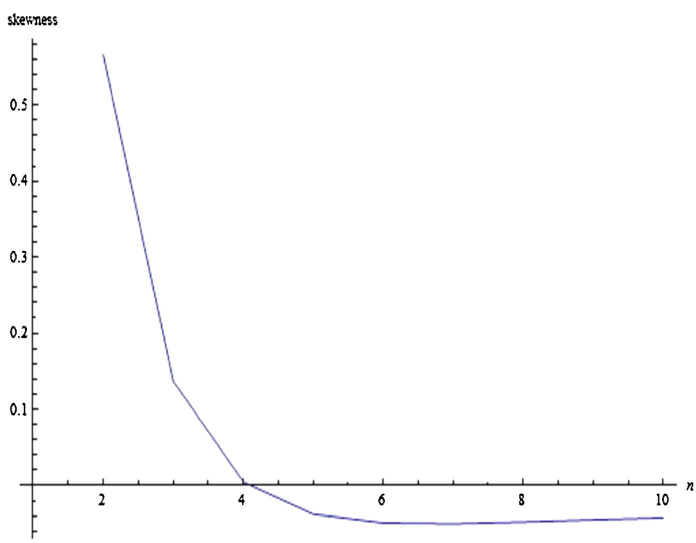

Figure 4 *The graph of skewness with respect to n.*

Table 2 *Moments, variance and skewness of* $f_{W_n}(w)$ *for* $n = 2, \dots, 10$

n	$E(W_n)$	$E(W_n^2)$	$E(W_n^3)$	$\mathrm{Var}(W_n)$	Skewness
2	0.333333	0.166667	0.100000	0.0555556	0.565685
3	0.466667	0.269048	0.175000	0.0512698	0.137187
4	0.529293	0.326647	0.222161	0.0464964	0.00484184
5	0.562734	0.360383	0.251653	0.0437131	−0.0377184
6	0.582385	0.381309	0.270722	0.0421366	−0.0492966
7	0.594799	0.395001	0.283555	0.0412154	−0.0502751
8	0.603094	0.404379	0.292525	0.0406568	−0.0480019
9	0.60889	0.411053	0.299006	0.0403066	−0.0451022
10	0.613089	0.415959	0.303826	0.0400806	−0.0424507

Appendix

Proof of Lemma 2. Let $\mathbf{A} = \{a_{ij}\}$ be a square matrix of order n. Then the permanent of the matrix $\mathbf{A}$ is defined to be

$$\mathrm{Per}(\mathbf{A}) = \sum_{\wp_{1,2,\dots,n}} \prod_{i=1}^{n} a_{ij_i},$$

where $\sum_{\wp_{1,2,\dots,n}}$ denotes the sum over all $n!$ permutations $(j_1, j_2, \dots, j_n)$ of $(1, 2, \dots, n)$. The permamnent is similar to the determinant except that it does not have the alternating sign depending on whether the permutation is of odd or even order. Note that, similar to determinant the permanent of a matrix can also be expanded by any row or column. For simplicity of expressions we denote

$$\mathrm{Per}(\mathbf{A}) = \left\| \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \right\|.$$

It is clear that

$$\begin{aligned} B(n, k; x_1, x_2, \dots, x_n) &= \frac{1}{k!(n-k)!} \sum_{\wp_{1,2,\dots,n}} x_{j_1} x_{j_2} \cdots x_{j_k} \bar{x}_{j_{k+1}} \cdots \bar{x}_{j_n} \\ &= \frac{1}{k!(n-k)!} \left\| \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ \cdots & \cdots & \cdots & \cdots \\ x_1 & x_2 & \cdots & x_n \\ \bar{x}_1 & \bar{x}_2 & \cdots & \bar{x}_n \\ \cdots & \cdots & \cdots & \cdots \\ \bar{x}_1 & \bar{x}_2 & \cdots & \bar{x}_n \end{bmatrix} \right\| \begin{matrix} \left.\vphantom{\begin{matrix} a \\ a \\ a \end{matrix}}\right\} k \text{ times} \\ \left.\vphantom{\begin{matrix} a \\ a \\ a \end{matrix}}\right\} n-k \text{ times}, \end{matrix} \end{aligned} \tag{A.1}$$

where $\bar{x}_i = 1 - x_i$, $i = 1, 2, \dots, n$.

It is easy to see that expanding the permanent along the last column we have

$$
\left\| \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ \cdots & \cdots & \cdots & \cdots \\ x_1 & x_2 & \cdots & x_n \\ & & & \\ \bar{x}_1 & \bar{x}_2 & \cdots & \bar{x}_n \\ \cdots & \cdots & \cdots & \cdots \\ \bar{x}_1 & \bar{x}_2 & \cdots & \bar{x}_n \end{bmatrix} \right\| \begin{matrix} \left.\vphantom{\begin{matrix} a \\ a \\ a \end{matrix}}\right\} k \text{ times} \\ \\ \left.\vphantom{\begin{matrix} a \\ a \\ a \end{matrix}}\right\} n-k \text{ times} \end{matrix}
$$

$$
= k x_n \left\| \begin{bmatrix} x_1 & x_2 & \cdots & x_{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ x_1 & x_2 & \cdots & x_{n-1} \\ & & & \\ \bar{x}_1 & \bar{x}_2 & \cdots & \bar{x}_{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ \bar{x}_1 & \bar{x}_2 & & \bar{x}_{n-1} \end{bmatrix} \right\| \begin{matrix} \left.\vphantom{\begin{matrix} a \\ a \\ a \end{matrix}}\right\} k-1 \text{ times} \\ \\ \left.\vphantom{\begin{matrix} a \\ a \\ a \end{matrix}}\right\} n-k \text{ times} \end{matrix} \tag{A.2}
$$

$$
+ (n-k)\bar{x}_n \left\| \begin{bmatrix} x_1 & x_2 & \cdots & x_{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ x_1 & x_2 & \cdots & x_{n-1} \\ & & & \\ \bar{x}_1 & \bar{x}_2 & \cdots & \bar{x}_{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ \bar{x}_1 & \bar{x}_2 & \cdots & \bar{x}_{n-1} \end{bmatrix} \right\| \begin{matrix} \left.\vphantom{\begin{matrix} a \\ a \\ a \end{matrix}}\right\} k \text{ times} \\ \\ \left.\vphantom{\begin{matrix} a \\ a \\ a \end{matrix}}\right\} n-k-1 \text{ times.} \end{matrix}
$$

From (A.1) and (A.2) we obtain the assertion of the lemma. □

Proof of Lemma 4. We use mathematical induction. For $n = 1$, we have

$$
\begin{aligned}
\prod_{i=1}^{1} (y^i - x^i) = (y - x) = (y - x) \sum_{i=0}^{\binom{1}{2}} I_1(i) y^{\binom{1}{2} - i} x^i \\
= (y - x)[I_1(0) y^0 x^0] = (y - x),
\end{aligned}
$$

and for $n = 2$ we have

$$
\begin{aligned}
\prod_{i=1}^{2} (y^i - x^i) = (y - x)(y^2 - x^2) &= (y - x)^2 \sum_{i=0}^{\binom{2}{2}} I_2(i) y^{\binom{2}{2} - i} x^i \\
&= (y - x)^2 [I_2(0) y^1 x^0 + I_2(1) y^0 x^1] = (y - x)^2 (x + y) \\
&= (y - x)(y^2 - x^2).
\end{aligned}
$$

Therefore, the assertion of the lemma is clearly true for $n = 1$ and $n = 2$.

Now, using induction we will show that if for each $n \geq 1$ it is true that

$$\prod_{i=1}^{n} (y^i - x^i) = (y-x)^n \sum_{i=0}^{\binom{n}{2}} I_n(i) y^{\binom{n}{2}-i} x^i,$$

then

$$\prod_{i=1}^{n+1} (y^i - x^i) = (y-x)^{n+1} \sum_{i=0}^{\binom{n+1}{2}} I_{n+1}(i) y^{\binom{n+1}{2}-i} x^i.$$

Indeed, one has

$$\begin{aligned}
\prod_{i=1}^{n+1} (y^i - x^i) &= (y^{n+1} - x^{n+1}) \prod_{i=1}^{n} (y^i - x^i) \\
&= (y-x)(y^n + y^{n-1}x + \cdots + yx^{n-1} + x^n) \prod_{i=1}^{n} (y^i - x^i) \\
&= (y-x)^{n+1}(y^n + y^{n-1}x + \cdots + yx^{n-1} + x^n) \sum_{i=0}^{\binom{n}{2}} I_n(i) y^{\binom{n}{2}-i} x^i \\
&= (y-x)^{n+1} \sum_{k=0}^{n} \sum_{i=0}^{\binom{n}{2}} I_n(i) y^{\binom{n}{2}-i+n-k} x^{i+k} \\
&= (y-x)^{n+1} \\
&\quad \times \sum_{k=0}^{n} \Big[I_n(0) y^{\binom{n}{2}+n-k} x^k + I_n(1) y^{\binom{n}{2}+n-k-1} x^{k+1} + \cdots \\
&\qquad + I_n\left(\binom{n}{2} - 1\right) y^{n+k+1} x^{\binom{n}{2}+k-1} + I_n\left(\binom{n}{2}\right) x^{\binom{n}{2}+k} \Big].
\end{aligned}$$

Therefore,

$$\begin{aligned}
\prod_{i=1}^{n+1} (y^i - x^i) &= (y^{n+1} - x^{n+1}) \\
&\quad \times \Big[I_n(0) y^{\binom{n}{2}+n} + I_n(1) y^{\binom{n}{2}+n-1} x + \cdots + I_n\left(\binom{n}{2}\right) y^n x^{\binom{n}{2}} \\
&\qquad + I_n(0) y^{\binom{n}{2}+n-1} x + I_n(1) y^{\binom{n}{2}+n-2} x^2 + \cdots \\
&\qquad + I_n\left(\binom{n}{2}\right) y^{n-1} x^{\binom{n}{2}+1} + \cdots \\
&\qquad + I_n(0) y^{\binom{n}{2}+1} x^{n-1} + I_n(1) y^{\binom{n}{2}} x^n + \cdots
\end{aligned}$$

$$+ I_n\left(\binom{n}{2}\right) y x^{\binom{n}{2}+n-1}$$

$$+ I_n(0) y^{\binom{n}{2}} x^n + I_n(1) y^{\binom{n}{2}-1} x^{n+1} + \cdots + I_n\left(\binom{n}{2}\right) x^{\binom{n}{2}+n}\Bigg]$$

and

$$\prod_{i=1}^{n+1}(y^i - x^i) = (y^{n+1} - x^{n+1})$$

$$\times \Big[I_n(0) y^{\binom{n}{2}+n}$$

$$+ \big(I_n(0) + I_n(1)\big) y^{\binom{n}{2}+n-1} x$$

$$+ \big(I_n(0) + I_n(1) + I_n(2)\big) y^{\binom{n}{2}+n-2} x^2 + \cdots$$

$$+ \big(I_n(0) + I_n(1) + \cdots + I_n(n)\big) y^{\binom{n}{2}} x^n$$

$$+ \big(I_n(1) + I_n(2) + \cdots + I_n(n+1)\big) y^{\binom{n}{2}-1} x^{n+1}$$

$$+ \big(I_n(2) + I_n(3) + \cdots + I_n(n+2)\big) y^{\binom{n}{2}-2} x^{n+2} + \cdots$$

$$+ I_n\left(\binom{n}{2} + n\right) x^{\binom{n}{2}+n}\Big].$$

Finally, using Theorem A one can write

$$\prod_{i=1}^{n+1}(y^i - x^i) = (y^{n+1} - x^{n+1})$$

$$\times \Bigg[\sum_{\max(0,0-(n+1)+1)\le j\le 0} I_n(j) \quad y^{\binom{n}{2}+n}$$

$$+ \sum_{\max(0,1-(n+1)+1)\le j\le 1} I_n(j) \quad y^{\binom{n}{2}+n-1} x + \cdots$$

$$+ \sum_{\max(0,\binom{n}{2}+n-(n+1)+1)\le j\le \binom{n}{2}+n} I_n(j) \quad x^{\binom{n}{2}+n}\Bigg]$$

$$= (y^{n+1} - x^{n+1})\Big[I_{n+1}(0) y^{\binom{n+1}{2}} + I_{n+1}(1) y^{\binom{n+1}{2}-1} x + \cdots$$

$$+ I_{n+1}\left(\binom{n+1}{2}\right) x^{\binom{n+1}{2}}\Big]$$

$$= (y - x)^{n+1} \sum_{i=0}^{\binom{n+1}{2}} I_{n+1}(i) y^{\binom{n+1}{2}-i} x^i,$$

which proves the lemma. □

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