Brazilian Journal of Probability and Statistics
2014, Vol. 28, No. 3, 424–445
DOI: 10.1214/13-BJPS217

A bivariate CLT under rho-prime mixing

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Abstract. The main result is a bivariate central limit theorem for "rectangular" sums of dependent complex-valued random variables, indexed by $\mathbb{Z}^d$, which are ρ'-mixing. An interesting corollary concerning the limiting behavior of the moments of the sums is then proved.

1 Introduction

The basic background material will be provided in this section.

1.1 A brief history

In the early 19th Century, Legendre published a paper in which he proved the first central limit theorem. For the next 140 years, it was the object of nearly continuous study by mathematicians. The classical result states that a normalized sum of independent, identically distributed random variables converges in distribution to a normal random variable. In 1951, Donsker published the proof of his weak invariance principle, basically giving the final word on CLTs under independence assumptions.

The independence assumption having been exhausted, in 1956 Murray Rosenblatt published a paper in which he allowed dependence in the random sequence. His method involved developing a way to measure the dependence, and then insisting that if the random variables were far removed from each other, then their measured dependence should be small. This, as well as a few additional assumptions, permitted the proof of a CLT. Since that time, a number of new measures of dependence have been developed, each measuring the dependence in a more or less intuitive way.

1.2 The setting

This paper will deal with a very specialized set of hypotheses. First, instead of using sequences (or even two-sided sequences) of random variables, *fields* of random variables shall be the object of study. A *field* of random variables is a collection $X := \{X_k, k \in \mathbb{Z}^d\}$, where each of the X_k's is a complex-valued random variable,

Key words and phrases. Strictly stationary random fields, ρ'-mixing, central limit theorem, spectral density.

Received April 2011; accepted January 2013.

and d is a fixed whole number. Usually, it is best to have $EX_k = 0$ for every k, in which case the field is called *centered*. Moreover, it will usually be necessary to assume that X is *strictly stationary*, which means that for every $S \subset \mathbb{Z}^d$, $S \neq \varnothing$, and any fixed vector $p \in \mathbb{Z}^d$, the collections $\{X_k, k \in S\}$ and $\{X_{k+p}, k \in S\}$ have the same distribution. In other words, the joint distributions are fixed under translations of the indexing sets. So, if X is strictly stationary, the assumption that X is centered requires no loss of generality.

Various methods of studying these fields have been developed, and the following has yielded a number of interesting results:

Consider vectors $\lambda = (\lambda_1, \ldots, \lambda_d) \in [-\pi, \pi]^d$, and write $e^{i\lambda} := (e^{i\lambda_1}, e^{i\lambda_2}, \ldots, e^{i\lambda_d})$, and notice that each coordinate is an element of the unit circle in $\mathbb{C}$. (The notation is abused a bit, but hopefully the context will clarify the meaning below.) Now define the random field $X^{(\lambda)} := \{X_k^{(\lambda)} := e^{-ik\cdot\lambda} X_k : k \in \mathbb{Z}^d\}$, where "$\cdot$" denotes the usual dot product. It is not hard to see that if X is a centered, complex (i.e., the random variables X_k are complex-valued), strictly stationary random field ("*CCSS*" for short), and if $E|X_k|^2 = \sigma^2 < \infty$, then the covariance $EX_k^{(\lambda)}\overline{X_l^{(\lambda)}}$ depends only on the vector $k - l$. This latter criterion is the definition of what it means to be *weakly stationary*, and so the field $X^{(\lambda)}$ is called centered, complex, weakly stationary (*CCWS*).

The interesting thing about these weakly stationary random fields is that they often have spectral densities. To understand what that means, first take the normalized Lebesgue measure on the complex unit circle (denoted by $\mathbb{T}$), $\mu_{\mathbb{T}} := dz/2\pi i z$, and consider the d-dimensional product measure $\mu_{\mathbb{T}}^d = \mu_{\mathbb{T}} \times \mu_{\mathbb{T}} \times \cdots \times \mu_{\mathbb{T}}$. Now you can define the spectral density of the CCWS random field $Y := \{Y_k : k \in \mathbb{Z}^d\}$ to be the nonnegative Borel function $f : \mathbb{T}^d \to \mathbb{R}$ which satisfies

$$EY_k\overline{Y}_l = \int_{\mathbb{T}^d} e^{i(k-l)\cdot\theta} f(\theta)\, d\mu_{\mathbb{T}}^d(e^{i\theta}).$$

The spectral density doesn't always exist, but the spectral measure always does. And as you may have already guessed, the spectral density is just the Radon–Nicodym derivative of the spectral measure. More will be said about the spectral density later.

1.3 Mixing conditions

Intuitively, many phenomena in the real-world are dependent, and independence is often a lot to ask. Take, for instance, the disturbance in a radio signal measured every minute. If there is a large amount of static in the current signal, our expectation for the next signal is that it will also have a lot of static, perhaps due to a storm or other phenomenon. Thus, the time-dependence of this sequence is apparent in small time intervals. On the other hand, if there is a lot of signal disruption in the present signal, what can be said about the measurement taken a day or a month from now? In this case, it seems reasonable to assume that measurements separated by large intervals of time should be more or less independent.

Capturing this idea in a rigorous form means first of all that the dependence must be measured. Consider then the *correlation coefficient* for two sigma-fields $\mathcal{A}$ and $\mathcal{B}$:

$$\rho(\mathcal{A}, \mathcal{B}) = \sup_{f,g} \operatorname{Corr}(f, g),$$

where the supremum is taken over all $f \in L^2(\mathcal{A})$ and $g \in L^2(\mathcal{B})$, and $\operatorname{Corr}(f, g) = (Efg - Ef\,Eg)/\|f\|_2\|g\|_2$.

To apply the correlation coefficient to the context of a random field X, take a nonempty subset $V \subset \mathbb{Z}^d$, and let $\sigma(V)$ denote the sigma field generated by the random variables $X_k \in X$ with indices in $k \in V$. Then, define

$$\rho'(X, n) = \sup_{S,T} \rho\big(\sigma(S), \sigma(T)\big),$$

where now the supremum is taken over all finite nonempty sets S and T which are separated by n units in (at least) one dimension. That is to say, there is a subscript u, $1 \le u \le d$ so that if $S \ni k = (k_1, \dots, k_d)$ and $T \ni l = (l_1, \dots, l_d)$, then $|k_u - l_u| \ge n$. It is important to note that the sets S and T can be "interlaced," meaning there may be $k, j \in S$ and $l \in T$ such that $k_u \le l_u \le j_u$, and vice versa.

It is now possible to describe what is meant by "ρ'-mixing." A random field X is said to be *ρ'-mixing* if $\rho'(X, n) \to 0$ as $n \to \infty$. Again, what is being said here is that the random variables are "asymptotically independent," insofar as ρ' measures dependence.

Another, perhaps better-known, measure of dependence is the ρ^* condition. It is very similar to ρ':

$$\rho^*(X, n) = \sup_{S,T} \rho\big(\sigma(S), \sigma(T)\big).$$

The only difference is that the elements k and l of the finite nonempty sets S and T (respectively) must satisfy $\|k - l\| \ge n$:

$$\min_{k \in S, l \in T} \|k - l\| \ge n.$$

(Here and below, $\|\cdot\|$ is the standard Euclidean norm.) Note that in one dimension, the two mixing conditions are equivalent.

It is easy to see that ρ^*-mixing implies ρ'-mixing, since $\rho^*(n) \ge \rho'(n)$. Therefore, since only ρ'-mixing is assumed below, all of the results proved in this paper apply to ρ^*-mixing fields as well.

1.4 A bit more history

The ρ^* dependence measure has a number of well-known applications. One important example is the class of stationary fields of Gaussian random variables. These fields are the source of a significant body of research, dating back to Ibragimov and Rozanov (1978). In addition, Murray Rosenblatt (1985) implies in that if such a field has a positive continuous spectral density function, then the field is ρ^*-mixing.

Regarding the ρ' coefficient, Bradley (1994) constructed strictly stationary random fields for which $\rho^*(n) = 1$ for all $n \geq 1$, but $\rho'(2) = 0$ (which implies that $\rho'(n) = 0$ for all $n \geq 3$). The result proves that there exist random fields which are ρ'-mixing, but not ρ^*-mixing. Bradley (2010) further shows that if $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are two sequences such that $0 \leq a_n \leq b_n \leq 1$ for every n, then there is a strictly stationary random field such that $\rho^*(n) = b_n$ and $\rho'(n) = a_n$. This implies that there is no relationship between the two conditions, except that $0 \leq \rho'(n) \leq \rho^*(n) \leq 1$. Consequently, the results proved in this paper apply not only to ρ^*-mixing fields, but to the strictly larger class of ρ'-mixing fields.

To understand the specific motivation for this paper, a bit of notation will be necessary. First of all, take an arbitrary vector $v = (v_1, v_2, \dots, v_d)$ in $\mathbb{N}^d$. Now consider the d-dimensional "box"

$$\mathfrak{B}(v) = \{u \in \mathbb{Z}^d : 1 \leq u_j \leq v_j,\ j = 1, 2, \dots, d\}.$$

Next, since CLTs require an analysis of the sums of random variables, take an arbitrary, nonempty subset $\mathfrak{S} \subset \mathbb{Z}^d$ and write

$$S_{\mathfrak{S}}^{(\lambda)} := \sum_{k \in \mathfrak{S}} X_k^{(\lambda)}.$$

NB: From this point forward, assume that $\{v^{(n)} = (v_1^{(n)}, v_2^{(n)}, \dots, v_d^{(n)})\}_{n=1}^{\infty}$ is some fixed sequence of vectors in $\mathbb{N}^d$ which satisfies the following property:

$$\lim_{n \to \infty} \min\{v_1^{(n)}, v_2^{(n)}, \dots, v_d^{(n)}\} = \infty. \tag{1.1}$$

The reason for insisting on condition (1.1) will be explained momentarily.

Since the sequence of vectors is fixed, substitute the notations

$$\mathfrak{B}(n) := \mathfrak{B}(v^{(n)}) \quad \text{and} \quad S_n^{(\lambda)} := S_{\mathfrak{B}(n)}^{(\lambda)}$$

and name the product of the components of $v^{(n)}$:

$$V_n := \prod_{j=1}^{d} v_j^{(n)}.$$

Notice that $V_n = \mathrm{Card}(\mathfrak{B}(v^{(n)}))$.

Finally, the *periodogram* $I_n^{(\lambda)}$ is defined to be

$$I_n^{(\lambda)} := \frac{|S_n^{(\lambda)}|^2}{V_n}.$$

The periodogram is fairly interesting, even though it is not the main object of study in this paper.

So, getting back to the specific motivation of this paper, in 1995 Curtis Miller proved that for the specific sequence of vectors $v^{(n)} := (n, n, \dots, n)$, $n = 1, 2, \dots$

$$\lim_n E\big(I_n^{(\lambda)}\big)^2 = 2\big(f(\lambda)\big)^2 + 4\big|h(\lambda)\big|^2. \tag{1.2}$$

The function f in (1.2) is the (continuous) spectral density, and h is another function which is zero except when $e^{i\lambda} \in \{-1, 1\}^d$, and will be defined more specifically at a later point. To prove (1.2), Miller assumed that the random field X is ρ^*-mixing, and that $E[(X_k)^4] < \infty$ for all k.

In 2006, Frederic Picard proved for $v^{(n)} := (n, n, \dots, n)$ that

$$\lim_n E\big(I_n^{(\lambda)}\big)^3 = 6\big(f(\lambda)\big)^3 + 36\big|h(\lambda)\big|^2 f(\lambda). \tag{1.3}$$

Here, Picard again assumed that X was ρ^*-mixing, and that the X_k had finite sixth moments. The function h in (1.3) is the same as in (1.2).

The results herein are similar to Miller's and Picard's results. The two formulae (1.2) and (1.3) are special cases of the theorem I intend to prove in this paper (though technically speaking, neither Miller's nor Picards results are special cases of the results in this work). Basically, I would like to show that for most $\lambda \in [-\pi, \pi]^d$,

$$\lim_n E\big(I_n^{(\lambda)}\big)^r = E\chi^r,$$

where χ is an exponential $(f(\lambda))$ random variable. What I mean by "most" is that the choice of λ is restricted so that $e^{i\lambda} \notin \{-1, 1\}^d$. The general result requires a few minor details which need not be presented at this time.

The most crucial part of proving this will be proving a Bivariate CLT.

Theorem 1. *Suppose X is a CCSS random field such that $E|X_0|^2 < \infty$. Suppose also that $\rho'(n) \to 0$ as $n \to \infty$. Let $f(\lambda)$ denote the (continuous) spectral density of X. If*

$$\lim_{n\to\infty} \min\{v_1^{(n)}, v_2^{(n)}, \dots, v_d^{(n)}\} = \infty,$$

then whenever $e^{i\lambda} \notin \{-1, 1\}^d$,

$$\frac{1}{\sqrt{V_n}}\big(\Re S_n^{(\lambda)}, \Im S_n^{(\lambda)}\big) \Rightarrow N\big(0, \Sigma^{(\lambda)}\big),$$

where "⇒" denotes weak convergence (convergence in distribution), and

$$\Sigma^{(\lambda)} := \begin{bmatrix} \frac{f(\lambda)}{2} & 0 \\ 0 & \frac{f(\lambda)}{2} \end{bmatrix}.$$

1.5 Important results

The proofs for the theorems in this paper do not require the complex machinery that Miller and Picard used, and they apply to the strictly more general collection of ρ'-mixing random fields. The most important new tool which neither Miller nor Picard had is the following "Rosenthal" inequality (see Bradley (2007), Theorem 29.30).

Theorem 2 (A Rosenthal inequality). *Suppose β is a number in the interval $[2, \infty)$, and that X is a random field of complex-valued random variables (X does not have to be stationary). Suppose further that X is such that for each $k \in \mathbb{Z}^d$, $EX_k = 0$ and $E|X_k|^\beta < \infty$. Finally, assume that $\rho'(n) < 1$ for some $n \in \mathbb{Z}$. Then, for any finite set $S \subset \mathbb{Z}^d$,*

$$E\left|\sum_S X_k\right|^\beta \le C \cdot \left[\sum_S E|X_k|^\beta + \left(\sum_S E|X_k|^2\right)^{\beta/2}\right],$$

where C is a constant that depends on d, n, $\rho'(n)$ and β.

Note the simplification of Rosenthal's inequality when $\beta = 2$.

Rosenthal's inequality will be very useful in controlling the moments of $S_n^{(\lambda)}$, and makes the application of Lyapounov's CLT much easier. It will also be used in proving the final corollary, in addition to the following theorem, which may not be well known (see Billingsley (1995), the corollary to Theorem 25.12).

Theorem 3. *Let r be a positive integer. If $X_n \Rightarrow X$ and $\sup_n E|X_n|^{r+\varepsilon} < \infty$, where $\varepsilon > 0$, then $E|X|^r < \infty$ and $E(X_n)^r \to EX^r$.*

The theorem above is quite handy in showing the convergence of the moments, provided that the supremum is indeed finite. The next result displays an interesting connection between the field $X^{(\lambda)}$ and the spectral density (see Bradley (2007), Theorem 28.21).

Theorem 4. *Suppose that $v^{(n)} \equiv (n, n, \ldots, n)$ for all n. If X is a CCWS random field such that $\rho'(n) \to 0$ as $n \to \infty$, then X has a* continuous *spectral density $f(\lambda)$ on $\mathbb{T}^d$ and*

$$f(\lambda) = \lim_{n\to\infty} EI_n^{(\lambda)},$$

and the convergence is uniform over all $\lambda \in [-\pi, \pi]^d$.

In Section 2, an analogue of Theorem 4 is provided which basically says the same thing, except that the condition (1.1) holds, instead of the condition $v^{(n)} \equiv (n, n, \dots, n)$.

The last major result was developed by Miller with the Cramer–Wold device (see Billingsley (1995), Theorem 29.4) in mind: In the following, let a and b be two arbitrary but fixed numbers. Then, define $G_{a,b} : \mathbb{C} \to \mathbb{R}$ by $G_{a,b}(z) = a\Re z + b\Im z$. The following theorem comes from Miller (1995), Lemma 4.1:

Lemma 1. *Suppose X is a CCWS random field such that $\rho^*(n) \to 0$ as $n \to \infty$ and $EX_kX_j = EX_{k-j}X_0$ for all $k, j \in \mathbb{Z}^d$. Let $v^{(n)} = (n, n, \dots, n)$ and let $f(\lambda)$ denote the (continuous) spectral density of X. If $\lambda \in [(-\pi, 0) \cup (0, \pi)]^d$, then for any $a, b \in \mathbb{R}$,*

$$\lim_{n\to\infty} \sup_{\beta\in\mathbb{Z}^d} \left| \frac{(a^2 + b^2) f(\lambda)}{2} - \frac{\sum_{k\in\mathfrak{B}(n)} G_{a,b}(e^{-i(k+\beta)\cdot\lambda} X_{k+\beta})}{n^d} \right| = 0. \tag{1.4}$$

Note that (1.4) implies that the "location" of the box in $\mathbb{Z}^d$ makes no difference to the convergence. Suffice it to say for now that this will be useful in Section 3, where a Bernstein blocking argument is used.

The final result might be an appropriate problem for 400-level analysis. Nevertheless, it will be quite useful.

Lemma 2. *Suppose that $a \in [0, \infty)$ and $\{a_k : k \in \mathbb{Z}^d\}$ is a field of nonnegative numbers such that for every $\varepsilon > 0$ there exists an $M = M(\varepsilon) > 0$ so that whenever the Euclidean norm of any vector $k \in \mathbb{Z}^d$ is greater than M, it holds that $|a_k - a| < \varepsilon$. Then, as $n \to \infty$,*

$$\lim_n \frac{\sum_{k\in\mathfrak{B}(n)} a_k}{V_n} = a.$$

2 Preliminaries

This section consists of some basic calculations which will serve a greater purpose later. Most of these results are minor modifications of others' work, and in those cases the proofs are left to the reader to verify.

In Section 2.2, I will state and prove a theorem which resembles Lemma 1, though the two results are not comparable (their respective hypotheses are a bit different).

2.1 Some adaptations

Lemma 3. *Suppose that X is a CCWS random field which satisfies $\rho'(n) \to 0$. Let $v^{(n)}$ be a sequence which satisfies (1.1). Let $f(\lambda)$ be the continuous spectral* *density of X. Then,*

$$\lim_{n\to\infty} I_n^{(\lambda)} = f(\lambda).$$

Moreover, the convergence is uniform over all $\lambda \in [-\pi, \pi]^d$.

Proof. This proof is essentially the same as the proof of the standard Fejer Theorem, and so it is omitted. (For a proof of the Fejer theorem, see Rudin (1964), p. 176, Theorem 8.15.) □

Lemma 4. *Suppose X is a complex and centered random field which is ρ'-mixing, and suppose further that $E|X_0|^2 < \infty$ and $EX_kX_j = EX_{k-j}X_0$ for all $k, j \in \mathbb{Z}^d$. Suppose $v^{(n)}$ is a sequence of vectors which satisfies (1.1). If there is a subscript s so that $\exp\{i\lambda_s\} \neq \pm 1$, then*

$$\lim_{n\to\infty} \frac{E(S_n^{(\lambda)})^2}{V_n} = 0.$$

Proof. The proof is a minor modification of one of Curtis Miller's results (1995, Lemma 3.6), and so it is also omitted. □

The next result is named after Curtis Miller, who proved an analogous result under ρ^*-mixing (Lemma 1 in this paper). Recall that $G_{a,b}(z) = a\Re z + b\Im z$.

Lemma 5. *Suppose X is a CCWS random field which is ρ'-mixing. Suppose further that $EX_kX_j = EX_{k-j}X_0$ for any $k, j, \in \mathbb{Z}$. Suppose the sequence of vectors $v^{(n)}$ satisfies (1.1). Let $\lambda \in [-\pi, \pi]^d$ be such that $e^{i\lambda} \notin \{-1, 1\}^d$, and let $f(\lambda)$ denote the (continuous) spectral density of X. Then*

$$\lim_{n\to\infty}\left[\sup_{v\in\mathbb{Z}^d}\left|\frac{1}{2}(a^2+b^2)f(\lambda) - \frac{1}{V_n}E\Big(\sum_{k\in\mathfrak{B}(n)} G_{a,b}\big(X_{k+v}^{(\lambda)}\big)\Big)^2\right|\right] = 0.$$

Proof. In his 1995 paper, Miller used his analogues of Lemmas 3 and 4 to prove his version of Lemma 5 (see Miller (1995), Lemma 4.1). The proof of the current lemma is therefore nearly identical to Miller's own proof, and so again it is omitted. □

2.2 Implications

Assume now that X is CCSS. Let $\eta_k^{(\lambda)} := \Re X_k^{(\lambda)}$ and let $\xi_k^{(\lambda)} := \Im X_k^{(\lambda)}$. Lemma 3 implies that

$$\lim_{n\to\infty} E\left[\frac{(\sum_{\mathfrak{B}(n)} \eta_k^{(\lambda)})^2}{V_n} + \frac{(\sum_{\mathfrak{B}(n)} \xi_k^{(\lambda)})^2}{V_n}\right] = f(\lambda).$$

In addition, if $\lambda \notin \{-\pi, 0, \pi\}^d$, then Lemma 4 implies that

$$\lim_{n\to\infty} \operatorname{Cov}\left(\frac{\sum_{\mathfrak{B}(n)} \eta_k^{(\lambda)}}{\sqrt{V_n}}, \frac{\sum_{\mathfrak{B}(n)} \xi_k^{(\lambda)}}{\sqrt{V_n}}\right) = 0.$$

Therefore, if $e^{i\lambda} \notin \{-1, 1\}^d$, then

$$f_\eta(\lambda) := \lim_n E\frac{(\sum_{\mathfrak{B}(n)} \eta_k^{(\lambda)})^2}{V_n} = \frac{1}{2} f(\lambda)$$

and also

$$f_\xi(\lambda) := \lim_n E\frac{(\sum_{\mathfrak{B}(n)} \xi_k^{(\lambda)})^2}{V_n} = \frac{1}{2} f(\lambda).$$

2.3 The missing values

The question at this point concerns the missing values from Lemma 5; namely those $\lambda \in [-\pi, \pi]^d$ where $e^{i\lambda} \in \{-1, 1\}^d$. Obviously, the reason they have been omitted from Lemma 5 is on account of Lemma 4.

However, in his 1995 paper, Curtis Miller showed that when $e^{i\lambda} \in \{-1, 1\}^d$, then the random fields $\eta_k^{(\lambda)}$ and $\xi_k^{(\lambda)}$ defined above are both weakly stationary, and hence have their own spectral densities. Thus, Lemma 3 implies that

$$f_\eta(\lambda) := \lim_n E\frac{(\sum_{\mathfrak{B}(n)} \eta_k^{(\lambda)})^2}{V_n} \quad \text{and} \quad f_\xi(\lambda) := \lim_n E\frac{(\sum_{\mathfrak{B}(n)} \xi_k^{(\lambda)})^2}{V_n}$$

both exist. Moreover, a simple adaptation to Miller's proof of his Lemma 3.7 (from 1995) shows that

$$h(\lambda) := \lim_{n\to\infty} \operatorname{Cov}\left(\frac{\sum_{\mathfrak{B}(n)} \eta_k^{(\lambda)}}{\sqrt{V_n}}, \frac{\sum_{\mathfrak{B}(n)} \xi_k^{(\lambda)}}{\sqrt{V_n}}\right)$$

still exists when $e^{i\lambda} \in \{-1, 1\}^d$. These facts lead to the following conclusion.

Lemma 6. *Let X be a CCSS random field which is ρ'-mixing, and suppose further that $EX_0^2 < \infty$. Let $f(\lambda)$ denote the continuous spectral density of X, and define f_η, f_ξ, and h as above. Suppose finally that the sequence of vectors $v^{(n)}$ satisfies* (1.1). *Then, for any $a, b \in \mathbb{R}$,*

$$\lim_n \left[\sup_{\nu\in\mathbb{Z}^d} \left| a^2 f_\eta(\lambda) + b^2 f_\xi(\lambda) + 2h(\lambda)ab - \frac{1}{V_n} E\left(\sum_{\mathfrak{B}(n)} G_{a,b}(X_{k+\nu}^{(\lambda)})\right)^2 \right|\right] = 0.$$

Proof. There are two cases to consider. The first case is when $e^{i\lambda} \notin \{-1, 1\}^d$. Obviously, this case is covered by Lemma 5. Therefore, assume that $e^{i\lambda} \in \{-1, 1\}^d$.

In this case,

$$E\Big(\sum_{k\in\mathfrak{B}(n)} G_{a,b}(X_{k+\nu}^{(\lambda)})\Big)^2$$

$$= a^2 E\Big(\sum_{\mathfrak{B}(n)} \eta_{k+\nu}^{(\lambda)}\Big)^2 + b^2 E\Big(\sum_{\mathfrak{B}(n)} \xi_{k+\nu}^{(\lambda)}\Big)^2 \qquad (2.1)$$

$$+ 2ab\,\mathrm{Cov}\Big(\sum_{\mathfrak{B}(n)} \eta_k^{(\lambda)}, \sum_{\mathfrak{B}(n)} \xi_k^{(\lambda)}\Big).$$

However, since $e^{i\lambda} \in \{-1, 1\}^d$,

$$\sum_{\mathfrak{B}(n)} \Re[e^{-i\nu\cdot(\lambda)} e^{-ik\cdot\lambda} X_{k+\nu}] = \pm \sum_{\mathfrak{B}(n)} \Re[e^{-ik\cdot\lambda} X_{k+\nu}]. \qquad (2.2)$$

Notice that because of the stationarity of the field X, the right-hand side of (2.2) has the same distribution as $\pm\sum_{\mathfrak{B}(n)} \Re[e^{ik\cdot\lambda} X_k]$. Thus,

$$E\Big(\sum_{\mathfrak{B}(n)} \Re[e^{-i\nu\cdot(\lambda)} e^{-ik\cdot\lambda} X_{k+\nu}]\Big)^2 = E\Big(\sum_{\mathfrak{B}(n)} \Re[e^{-ik\cdot\lambda} X_k]\Big)^2.$$

Similarly, it is easy to see that

$$E\Big(\sum_{\mathfrak{B}(n)} \Im[e^{-i\nu\cdot(\lambda)} e^{-ik\cdot\lambda} X_{k+\nu}]\Big)^2 = E\Big(\sum_{\mathfrak{B}(n)} \Im[e^{-ik\cdot\lambda} X_k]\Big)^2$$

and

$$E\Big[\Big(\sum_{\mathfrak{B}(n)} \eta_{k+\nu}^{(\lambda)}\Big)\Big(\sum_{\mathfrak{B}(n)} \xi_{k+\nu}^{(\lambda)}\Big)\Big] = E\Big[\Big(\sum_{\mathfrak{B}(n)} \eta_k^{(\lambda)}\Big)\Big(\sum_{\mathfrak{B}(n)} \xi_k^{(\lambda)}\Big)\Big].$$

It should now be clear that Equation (2.1) is the same no matter what the value of ν is. Hence, Lemma 6 is proved. □

3 The main result

3.1 Presentation

Here is the main result of this paper.

Theorem 5. *Let X be a CCSS, ρ'-mixing random field such that $E|X_0|^2 < \infty$. Let $f(\lambda)$ denote the continuous spectral density of X, and let $v^{(n)}$ be a sequence which satisfies (1.1). Define the quantities $f_\eta(\lambda)$, $f_\xi(\lambda)$, and $h(\lambda)$ as in Section 2.3. Then, for each $\lambda \in [-\pi, \pi]^d$,*

$$\frac{1}{\sqrt{V_n}}(\Re S_n^{(\lambda)}, \Im S_n^{(\lambda)}) \Rightarrow N(0, \Sigma^{(\lambda)}), \qquad (3.1)$$

where

$$\Sigma^{(\lambda)} := \begin{bmatrix} f_\eta(\lambda) & h(\lambda) \\ h(\lambda) & f_\xi(\lambda) \end{bmatrix}.$$

Remark. Miller (1995, Lemma 4.2) proved an analogue of Theorem 5 under ρ^*-mixing and $E|X_0|^4 < \infty$.

Remark. Christina Tone (2011) proved a central limit theorem for strictly stationary, ρ'-mixing random fields $X := \{X_k, k \in \mathbb{Z}^d\}$ with the random variables taking their values in a separable real Hilbert space. Theorem 5 involves a different context in that the complex-valued random fields $X^{(\lambda)}$, $\lambda \in [-\pi, \pi]^d$ are in view, and these fields are not generally strictly stationary.

Remark. Notice that if λ is such that $e^{i\lambda} \notin \{-1, 1\}^d$, then the limiting distribution is the joint distribution of two independent normal random variables, since the covariance matrix in that case is

$$\Sigma^{(\lambda)} := \frac{1}{2}\begin{bmatrix} f(\lambda) & 0 \\ 0 & f(\lambda) \end{bmatrix},$$

and uncorrelated normals are independent.

3.2 Description of the proof

Generally speaking, the proof consists of two reductions. The first involves truncating the individual random variables (i.e., the X_k's). The second reduction involves the Bernstien blocking argument. In both reductions, it must be borne in mind that L^2-convergence to zero implies weak convergence to zero. This fact, when combined with Slutsky's lemma (see Ibragimov and Linnik (1971), Lemma 18.4.1), will provide the means to show that both reductions are valid.

The proof of Theorem 5 begins below. However, the proof does not conclude until the end of Section 3.8.

Proof of Theorem 5. Fix $\lambda \in [-\pi, \pi]^d$. If $f_\eta(\lambda) = f_\xi(\lambda) = h(\lambda) = 0$, then Miller's lemma shows that

$$\frac{1}{\sqrt{V_n}}(\Re S_n^{(\lambda)}, \Im S_n^{(\lambda)}) \Rightarrow 0.$$

(Since L^2-convergence to zero implies weak convergence to zero.) So, from here on out, assume that at least one of $f_\eta(\lambda)$, $f_\xi(\lambda)$, or $h(\lambda)$ is nonzero. The Cramer–Wold device (see Billingsley (1995), Theorem 29.4) implies that proving (3.1) is equivalent to proving that

$$\frac{G_{a,b}(S_n^{(\lambda)})}{\sqrt{V_n}} \Rightarrow N\big(0, a^2 f_\eta(\lambda) + 2ab \cdot h(\lambda) + b^2 f_\xi(\lambda)\big) \tag{3.2}$$

for an arbitrary point $(a, b) \in \mathbb{R}^2$.

Therefore, let $(a, b) \in \mathbb{R}^2$ be arbitrary but fixed. Since it is fixed, write "G" instead of "$G_{a,b}$." To reiterate, if (3.2) can be proved for these values of a and b, then (3.1) will also hold.

The next sections should be considered as parts of the (lengthy) proof, which concludes in Section 3.8.

3.3 The first reduction: Truncation

For any vector $k \in \mathbb{Z}^d$, define $\langle k \rangle := k_1 \cdot k_2 \cdots k_d$. Then let $0 < q$ be a real number (later q will be chosen in the interval $(0, 1/4)$), and define the random variables

$$B_{k,q}^{(\lambda)} = X_k^{(\lambda)} \mathbb{I}\{|X_k| \le \langle k \rangle^q\} - E X_k^{(\lambda)} \mathbb{I}\{|X_k| \le \langle k \rangle^q\}, \tag{3.3}$$

$$T_{k,q}^{(\lambda)} = X_k^{(\lambda)} \mathbb{I}\{|X_k| > \langle k \rangle^q\} - E X_k^{(\lambda)} \mathbb{I}\{|X_k| > \langle k \rangle^q\}. \tag{3.4}$$

Notice that $E B_{k,q}^{(\lambda)} = E T_{k,q}^{(\lambda)} = 0$ for every k and q. Next, define

$$S_{n,q}^{(\lambda)} := \sum_{k \in \mathfrak{B}(n)} B_{k,q}^{(\lambda)},$$

$$R_{n,q}^{(\lambda)} := \Re[S_{n,q}^{(\lambda)}],$$

$$Q_{n,q}^{(\lambda)} := \Im[S_{n,q}^{(\lambda)}].$$

The following lemma constitutes the substance of the first reduction:

Lemma 7. *In the same context and with the same notations as Theorem 5 and Section 3.3, and for any $q \in (0, 1/4)$,*

$$\frac{(R_{n,q}^{(\lambda)}, Q_{n,q}^{(\lambda)})}{\sqrt{V_n}} \Rightarrow N\big(0, \Sigma^{(\lambda)}\big), \tag{3.5}$$

and

$$\frac{E|\sum_{k \in \mathfrak{B}(n)} T_{k,q}^{(\lambda)}|^2}{V_n} \to 0 \tag{3.6}$$

as $n \to \infty$.

In light of Slutsky's lemma, equation (3.6) implies that equation (3.5) will prove the main result (i.e., (3.1)), since L^2-convergence to zero implies weak convergence to zero).

The proof of Lemma 7 is in two parts. Proving (3.6) is relatively simple and so that will be done first. The proof of (3.5) is more involved (it requires a number of other auxiliary results), and so it will be proved over the course of the next few sections.

Proof of equation (3.6). First of all, notice that because $E|X_0|^2$ is finite, the values $ET_{k,q}^{(\lambda)}$ satisfy the conditions of Lemma 2 (when $\langle k\rangle$ is large, $ET_{k,q}^{(\lambda)} \approx 0$). If the Rosenthal inequality (Theorem 2) is applied to the right-hand side of (3.6):

$$\frac{E|\sum_{k\in\mathfrak{B}(n)} T_{k,q}^{(\lambda)}|^2}{V_n} \leq \frac{C\cdot\sum_{\mathfrak{B}(n)} E|T_{k,q}^{(\lambda)}|^2}{V_n}. \tag{3.7}$$

Application of Lemma 2 to the right-hand side of (3.7) proves (3.6). □

3.4 The blocking argument

What follows is an argument that involves the Bernstein Blocking technique (which actually dates back to at least Markov). Heuristically, the gist of the argument involves slicing $S_{n,q}^{(\lambda)}$ like a loaf of bread, except that the "width" of the slices is not to be uniform. Instead, the first slice should be thick, the second thin, the third thick, the fourth thin, and so on. The thin slices should grow in thickness as $S_{n,q}^{(\lambda)}$ grows in size, and since they come between the thick slices, these latter pieces should be quasi-independent because of the mixing condition. However, when taken all together, the thin slices can't account for too much of $S_{n,q}^{(\lambda)}$, since it will be desirable to be able to neglect the small slices and focus on the big ones. Thus, much care must be taken to ensure that both of these criteria are satisfied.

The most crucial part of the process is in determining the "dimensions" of the various "slices." To that end, define the width of the "thin" slices (a.k.a. the "small blocks"):

$$w(n) := \lfloor \sqrt[3]{v_1^{(n)}} \rfloor, \tag{3.8}$$

where $\lfloor\cdot\rfloor$ denotes the largest integer less than or equal to the argument. Next, the number of large blocks will be

$$l(n) := \min\left\{w(n), \left\lfloor \frac{1}{\sqrt{\rho'(X, w(n))}} \right\rfloor\right\}. \tag{3.9}$$

Finally, the width of the large blocks is pretty much determined; the width of the thick slices will be the unique integer $p(n)$ satisfying

$$\big(p(n) - 1 + w(n)\big)l(n) \leq v_1^{(n)} < \big(p(n) + w(n)\big)l(n). \tag{3.10}$$

Notice that each of $w(n)$, $l(n)$ and $p(n)$ goes to infinity with n.

Now let's define the big blocks. For each $j = 1, 2, \ldots, l(n)$, define the sets

$$\mathfrak{B}(j,n) := \big\{k \in \mathfrak{B}(n) : (j-1)\big(p(n)+w(n)\big) < k_1 \leq jp(n) + (j-1)w(n)\big\}$$

(k_1 is the first coordinate of k). Also define the random variables

$$\Gamma(j,n,\lambda,q) = \Gamma(j,n) := \sum_{k\in\mathfrak{B}(j,n)} G\big(B_{k,q}^{(\lambda)}\big).$$

It will also be convenient to collect the leftovers:

$$\mathfrak{Z}(n) := \mathfrak{B}(n) \Big\backslash \left\{ \bigcup_{j=1}^{l(n)} \mathfrak{B}(j,n) \right\}.$$

With the above notations, it is possible to show that

Lemma 8. *In the current context,*

$$\frac{\sum_{k \in \mathfrak{Z}(n)} G(B_{k,q}^{(\lambda)})}{\sqrt{V_n}} \Rightarrow 0.$$

Proof. Observe that the cardinality of $\mathfrak{Z}(n)$ is $(v_1^{(n)} - l(n)p(n))v_2^{(n)} \cdots v_d^{(n)}$. Next, from (3.8)–(3.10), it is not hard to see that

$$v_1^{(n)} - \big(v_1^{(n)}\big)^{2/3} \le v_1^{(n)} - l(n)w(n) \le l(n)p(n), \tag{3.11}$$

and therefore that

$$\begin{aligned} V_n - l(n)p(n)v_2^{(n)} \cdots v_d^{(n)} &\le V_n - \big(v_1^{(n)} - \big(v_1^{(n)}\big)^{2/3}\big)v_2^{(n)} \cdots v_d^{(n)} \\ &= \big(v_1^{(n)}\big)^{2/3} v_2^{(n)} \cdots v_d^{(n)} \\ &= o(V_n). \end{aligned} \tag{3.12}$$

One final consequence of these definitions is that

$$\lim_n \frac{V_n}{p(n) \cdot l(n) \cdot v_2^{(n)} \cdots v_d^{(n)}} = 1. \tag{3.13}$$

The trick now is to apply the Rosenthal inequality (Theorem 2); the field $B_q^{(\lambda)} := \{B_{k,q}^{(\lambda)} : k \in \mathbb{Z}^d\}$ is ρ'-mixing, so the application is justified.

$$E\left|\frac{\sum_{k \in \mathfrak{Z}(n)} B_{k,q}^{(\lambda)}}{\sqrt{V_n}}\right|^2 \le K \frac{\sum_{k \in \mathfrak{Z}(n)} E|B_{k,q}^{(\lambda)}|^2}{V_n} \le K \frac{\sum_{k \in \mathfrak{Z}(n)} E|X_0|^2}{V_n}. \tag{3.14}$$

However, the extreme right-hand side of equation (3.14) is equal to $K \cdot \mathrm{Card}(\mathfrak{Z}(n)) \times E|X_0|^2/V_n = K\,E|X_0|^2/(v_1^{(n)})^{1/3} \to 0$ as $n \to \infty$. Since L^2-convergence to zero implies weak convergence to zero, Lemma 8 is proved. □

3.5 The second reduction: Independent big blocks

The goal now is to show that it is possible to consider independent copies of the big blocks, instead of the dependent ones. Lemma 9 accomplishes this task by showing that the characteristic function of the independent copies is sufficiently close to the characteristic function of the dependent ones. In Section 3.8, I will explain how this fact is used in the proof of Theorem 5.

Consider

$$\frac{\sum_{j=1}^{l(n)} \Gamma(j,n)}{\sqrt{V_n}}$$

and the corresponding characteristic function

$$\phi^{(n)}(t) := E\left[\exp\left\{it\frac{\sum_{j=1}^{l(n)} \Gamma(j,n)}{\sqrt{V_n}}\right\}\right].$$

Also, for each n, define the functions $\psi^{(n)}(t) := \prod_{j=1}^{l(n)} E[it\Gamma(j,n)/\sqrt{V_n}]$.

Now it would be good to show that

Lemma 9. *In the same context as Theorem 5, and with all the current notations,*

$$\lim_n \left|\phi^{(n)}(t) - \psi^{(n)}(t)\right| = 0$$

for every $t \in \mathbb{R}$.

Proof. Fix $t \in \mathbb{R}$. For every m, $1 \le m \le l(n)$, let $A_m := E[\exp\{it\Gamma(m,n)/\sqrt{V_n}\}]$. Then, for every $1 < m < l(n)$, define $B_m := E[\exp\{it\sum_{j=m}^{l(n)} \Gamma(j,n)/\sqrt{V_n}\}]$. It is not hard to see that

$$\begin{aligned} |A_1 \cdots A_{m-1}B_m - A_1 \cdots A_m B_{m+1}| &= |A_1 \cdots A_{m-1}||B_m - A_m B_{m+1}| \\ &\le |B_m - A_m B_{m+1}|. \end{aligned}$$

Because of Theorem 1.1 in Withers (1981), it holds that $|B_m - A_m B_{m+1}| \le \rho'(w(n))\|A_{m+1}\|_2\|B_{m+1}\|_2 \le \rho'(w(n))$. Therefore, by adding and subtracting the appropriate terms, it follows that $|\phi^{(n)}(t) - \psi^{(n)}(t)| \le l(n)\rho'(w(n))$. But, because of the definition of $l(n)$ (see (3.9)), $l(n)\rho'(w(n)) \to 0$, which proves Lemma 9. □

3.6 Preparation for Lyapounov's condition

For every n, take a family $\{\Theta(j,n)\}_{j=1}^{l(n)}$ of *independent* random variables so that $\Theta(j,n)$ has the same distribution as $\Gamma(j,n)$. The last major step in the proof of Theorem 5 is proving that Lyapounov's condition holds for the $\Theta(j,n)$'s, which is Lemma 11 below. The proof of Lemma 11 relies on the following claim.

Lemma 10. *In the current context, and with all the current notations,*

$$\lim_{n\to\infty} \frac{\sum_{j=1}^{l(n)} E|\Theta(j,n)|^2}{(a^2 f_\eta(\lambda) + 2ab\cdot h(\lambda) + b^2 f_\xi(\lambda))V_n} = 1.$$

Proof. Define $U(j,n) := \sum_{k\in\mathfrak{B}(j,n)} G(X_k^{(\lambda)})$ and $Z(j,n) := \sum_{k\in\mathfrak{B}(j,n)} G(T_{k,q}^{(\lambda)})$. Let $K' = a^2 f_\eta(\lambda) + 2ab\cdot h(\lambda) + b^2 f_\xi(\lambda)$. Finally, let $V_n' = l(n)p(n)v_2^{(n)}\cdots v_d^{(n)}$.

First of all, notice that because of (3.11),

$$\lim_n \left|1 - \frac{V_n'}{V_n}\right| \le \lim_n \left|\frac{o(V_n)}{V_n}\right| = 0. \tag{3.15}$$

Therefore,

$$\lim_{n\to\infty} \frac{\sum_j E|\Theta(j,n)|^2}{K'V_n} = \lim_{n\to\infty} \frac{\sum_j E|\Theta(j,n)|^2}{K'l(n)p(n)v_2^{(n)}\cdots v_d^{(n)}}. \tag{3.16}$$

Secondly, observe that Rosenthal's inequality implies

$$\begin{aligned} E\big|Z(j,n)\big|^2 &\le C \sum_{k\in\mathfrak{B}(j,n)} E\big|T_{k,q}^{(\lambda)}\big|^2 = C \sum_{k\in\mathfrak{B}(j,n)} \mathrm{Var}\big(X_k^{(\lambda)}\mathbb{I}\{|X_k| > \langle k\rangle^q\}\big) \\ &\le C \sum_{k\in\mathfrak{B}(j,n)} E\big|X_k^{(\lambda)}\mathbb{I}\{|X_k| > \langle k\rangle^q\}\big|^2. \end{aligned}$$

(It is important to note that the constant C does not depend on j.) Now let β be any element of $\mathfrak{B}(1,n)$, and define the vector $\alpha := (p(n)+w(n),0,0,\dots,0)$. Then, $\beta + j\alpha \in \mathfrak{B}(j,n)$. Furthermore, $\langle\beta\rangle \le \langle\beta + j\alpha\rangle$ for all $\beta\in\mathfrak{B}(1,n)$ and all j. Therefore (since $E|X_0|^2 < \infty$), $E|X_k^{(\lambda)}\mathbb{I}\{|X_k| > \langle\beta + j\alpha\rangle^q\}|^2 < E|X_k^{(\lambda)}\mathbb{I}\{|X_k| > \langle\beta\rangle^q\}|^2$. This implies that

$$\frac{\sum_{k\in\mathfrak{B}(j,n)} E|X_k^{(\lambda)}\mathbb{I}\{|X_k| > \langle k\rangle^q\}|^2}{p(n)v_2^{(n)}\cdots v_d^{(n)}} \le \frac{\sum_{k\in\mathfrak{B}(1,n)} E|X_k^{(\lambda)}\mathbb{I}\{|X_k| > \langle k\rangle^q\}|^2}{p(n)v_2^{(n)}\cdots v_d^{(n)}},$$

and the right-hand side converges to zero by Lemma 2. Therefore, the terms

$$\frac{E|Z(j,n)|^2}{(p(n)v_2^{(n)}\cdots v_d^{(n)})} \to 0 \tag{3.17}$$

uniformly (i.e., uniformly over j).

Finally, notice that Lemma 6 implies that

$$1 = \lim_n \frac{\sum_{j=1}^{l(n)} E|U(j,n)|^2}{K'V_n'}.$$

But, $E|U(j,n)|^2 = E|\Theta(j,n)|^2 + 2E|\Gamma(j,n)||Z(j,n)| + E|Z(j,n)|^2$, so consider the term $\sum_{j=1} l(n)E|Z(j,n)|^2/V_n'$. This obviously converges to zero because it is a Cesaro mean of the terms from (3.17).

To deal with the second term, apply the Cauchy–Schwarz and Minkowski inequalities:

$$
\begin{aligned}
&\frac{E|\Gamma(j,n)||Z(j,n)|}{p(n)v_2^{(n)}\cdots v_d^{(n)}}\\
&\qquad \le \frac{\sqrt{E|\Gamma(j,n)|^2}\sqrt{E|Z(j,n)|^2}}{p(n)v_2^{(n)}\cdots v_d^{(n)}}\\
&\qquad \le \left(\sqrt{\frac{E|U(j,n)|^2}{p(n)v_2^{(n)}\cdots v_d^{(n)}}}+\sqrt{\frac{E|Z(j,n)|^2}{p(n)v_2^{(n)}\cdots v_d^{(n)}}}\right)\\
&\qquad\quad \cdot\sqrt{\frac{E|Z(j,n)|^2}{p(n)v_2^{(n)}\cdots v_d^{(n)}}}\\
&\qquad = \sqrt{\frac{E|U(j,n)|^2}{p(n)v_2^{(n)}\cdots v_d^{(n)}}}\sqrt{\frac{E|Z(j,n)|^2}{p(n)v_2^{(n)}\cdots v_d^{(n)}}}+\frac{E|Z(j,n)|^2}{p(n)v_2^{(n)}\cdots v_d^{(n)}}.
\end{aligned}
\tag{3.18}
$$

Now, uniformly over all j, $\sqrt{E|Z(j,n)|^2/p(n)v_2^{(n)}\cdots v_d^{(n)}}\to 0$. Lemma 6 implies that $\sqrt{E|U(j,n)|^2/p(n)v_2^{(n)}\cdots v_d^{(n)}}\to K'$, uniformly over all j, and so the last line of (3.18) converges to zero as well. Therefore,

$$\frac{1}{K'l(n)}\sum_{j=1}^{l(n)}\frac{2E|\Gamma(j,n)||Z(j,n)|+E|Z(j,n)|^2}{p(n)v_2^{(n)}\cdots v_d^{(n)}}\to 0$$

as well. This implies that the right-hand side (and hence both sides) of (3.16) converge to 1, which proves Lemma 10. □

3.7 Statement of Lyapounov's condition

Lemma 11. *In the same context as Theorem 5, and with all the current notations,*

$$\lim_{n\to\infty}\frac{\sum_{j=1}^{l(n)}E|\Theta(j,n)|^4}{(\sum_{j=1}^{l(n)}E|\Theta(j,n)|^2)^2}=0. \tag{3.19}$$

Proof. Apply Lemma 10 and Rosenthal's inequality:

$$
\begin{aligned}
&\lim_{n\to\infty}\frac{\sum_{j=1}^{l(n)}E|\Theta(j,n)|^4}{(\sum_{j=1}^{l(n)}E|\Theta(j,n)|^2)^2}\\
&\qquad =\lim\cdot\frac{\sum_j E|\Gamma(j,n)|^4}{(K'l(n)p(n)v_2^{(n)}\cdots v_d^{(n)})^2}
\end{aligned}
\tag{3.20}
$$

$$\leq \lim_n \frac{C_0}{K'} \frac{\sum_j (\sum_{\mathfrak{B}(j,n)} E|G(B_{k,q}^{(\lambda)})|^4 + (\sum_{\mathfrak{B}(j,n)} E|G(B_{k,q}^{(\lambda)})|^2)^2)}{l(n)p(n)v_1^{(n)} \cdots v_d^{(n)}}$$

$$\leq \lim_n \frac{C_1}{K'} \frac{\sum_j (\sum_{\mathfrak{B}(j,n)} E|B_{k,q}^{(\lambda)}|^4 + (\sum_{\mathfrak{B}(j,n)} E|B_{k,q}^{(\lambda)}|^2)^2)}{l(n)p(n)v_1^{(n)} \cdots v_d^{(n)}}.$$

The final inequality follows since for any complex-valued random variable Y, $E|G(Y)|^r \leq C(r)E|Y|^r$. Now, from the definition of $B_{k,q}^{(\lambda)}$, it is clear that $E|B_{k,q}^{(\lambda)}|^4 \leq \langle k\rangle^{4q} \leq (V_n)^{4q}$. However, $\lim_n (V_n)^{4q}/(l(n)p(n)v_1^{(n)} \cdots v_d^{(n)})^{4q} = 1$, hence

$$\lim_n \frac{\sum_j \sum_{\mathfrak{B}(j,n)} E|B_{k,q}^{(\lambda)}|^4}{l(n)p(n)v_1^{(n)} \cdots v_d^{(n)}} \to 0.$$

Similarly, $E|B_{k,q}^{(\lambda)}|^2 \leq E|X_0|^2$, and so

$$\frac{\sum_j (\sum_{\mathfrak{B}(j,n)} E|B_{k,q}^{(\lambda)}|^2)^2}{(l(n)p(n)v_1^{(n)} \cdots v_d^{(n)})^2} \leq \frac{l(n)(p(n)v_2^{(n)} \cdots v_d^{(n)} \cdot E|X_0|^2)^2}{(l(n)p(n)v_1^{(n)} \cdots v_d^{(n)})^2} \to 0.$$

Thus, Lemma 11 holds. □

3.8 Conclusion of the Proof of Theorem 5

Before Theorem 5 can be completed, we need to finish the proof of Lemma 7.

Proof of Lemma 7, equation (3.5). Lemmas 11 and 10 together imply that

$$\frac{\sum_j \Theta(j,n)}{\sqrt{(a^2 f_\eta(\lambda) + 2abh(\lambda) + b^2 f_\xi(\lambda))V_n}} \Rightarrow N(0,1). \tag{3.21}$$

However, Lemma 9 proves that (3.21) is equivalent to

$$\frac{\sum_j \Gamma(j,n)}{\sqrt{(a^2 f_\eta(\lambda) + 2abh(\lambda) + b^2 f_\xi(\lambda))V_n}} \Rightarrow N(0,1). \tag{3.22}$$

Next, an application of Slutsky's lemma together with Lemma 8 implies that

$$\frac{G(S_{n,q}^{(\lambda)})}{\sqrt{(a^2 f_\eta(\lambda) + 2abh(\lambda) + b^2 f_\xi(\lambda))V_n}} \Rightarrow N(0,1), \tag{3.23}$$

and the Cramer–Wold device takes care of the rest. □

It is now time to prove Theorem 5.

Proof of Theorem 5. Lemma 7 implies that the entire sum of the tails $(T_{k,q}^{(\lambda)})$ converges weakly to zero. Therefore, another application of Slutsky's Lemma, together with Lemma 7, and equation (3.6) shows that Theorem 5 holds (since L^2-convergence to zero implies weak convergence to zero). □

4 A corollary about moments

As promised in the Introduction, I need to show why

$$E\big(I_n^{(\lambda)}\big)^r = E\bigg(\frac{|S_n^{(\lambda)}|^2}{V_n}\bigg)^r \Rightarrow E\chi^r, \tag{4.1}$$

where χ is an exponential $(f(\lambda))$ random variable, and $\lambda \notin \{-\pi, 0, \pi\}^d$. Of course, it must be assumed that $E|X_0|^{2r} < \infty$, since otherwise there is no reason to believe that the expectation on the left hand side of the equal sign exists. However, this is the only additional assumption which needs to be added to the hypotheses of Theorem 5.

4.1 The intuition

There is good reason to believe this convergence should hold. Notice first of all that $|S_n^{(\lambda)}|^2 = (\Re S_n^{(\lambda)})^2 + (\Im S_n^{(\lambda)})^2$. Therefore, define the function $H : \mathbb{R}^2 \to \mathbb{R}$ by $H(x, y) = x^2 + y^2$. This function is continuous, hence it satisfies the conditions of the Mapping Theorem (see Billingsley (1995), Theorem 25.7). Therefore, for arbitrary λ, Theorem 5 and the Mapping Theorem imply that

$$I_n^{(\lambda)} \Rightarrow Z_\eta^2 + Z_\xi^2, \tag{4.2}$$

where Z_η is a normal $(0, f_\eta(\lambda))$, Z_ξ is a normal $(0, f_\xi(\lambda))$, and $\mathrm{Cov}(Z_\eta, Z_\xi) = h(\lambda)$.

This may seem to go against (4.1), but remember that when $h(\lambda) = 0$ (which happens whenever $\lambda \notin \{-\pi, 0, \pi\}^d$), then Z_η and Z_ξ are independent (it is a standard fact of probability theory that uncorrelated normals are independent) and identically distributed (since $f_\eta(\lambda) = f_\xi(\lambda) = f(\lambda)/2$). This implies that $Z_\eta^2 + Z_\xi^2$ has a scaled chi-squared distribution with two degrees of freedom, which is an exponential $(f(\lambda))$ random variable.

So, the more general statement of (4.1) is (for arbitrary $\lambda \in (-\pi, \pi]^d$)

$$E\big(I_n^{(\lambda)}\big)^r \to E\big(Z_\eta^2 + Z_\xi^2\big)^r.$$

It must be borne in mind that weak convergence does not imply convergence of moments, and so further work must be done. However, the weak convergence, combined with Miller's and Picard's work (see (1.2) and (1.3)), suggests that it might be possible.

4.2 The corollary

Corollary 1. *Suppose that X is a CCSS ρ'-mixing field and $\lambda \in (-\pi, \pi]^d$. Assuming that $v^{(n)}$ is a sequence that satisfies (1.1), and that $f_\eta(\lambda)$, $f_\xi(\lambda)$, and $h(\lambda)$ are as discussed in Section 2.3. Suppose further that $E|X_0|^{2s} < \infty$ for some $s > 1$. Then*

$$\lim_n E\big(I_n^{(\lambda)}\big)^r = E\big(Z_\eta^2 + Z_\xi^2\big)^r \tag{4.3}$$

for all $r \le s$, where Z_η is a normal $(0, f_\eta(\lambda))$, Z_ξ is a normal $(0, f_\xi(\lambda))$, and $\mathrm{Cov}(Z_\eta, Z_\xi) = h(\lambda)$.

Proof. It must be shown that (4.3) holds for $r = s$; all other values follow from Theorem 3.

Let $\varepsilon > 0$ be arbitrary but fixed. Next, consider the truncated random variables from the proof of Lemma 7, where this time q is chosen so that

$$0 < q < \min\left\{\frac{1}{4}, \frac{s+\varepsilon-1}{2s+2\varepsilon}\right\}.$$

Lemma 7 together with the Mapping Theorem implies that $|S_{n,q}^{(\lambda)}|^2/V_n \Rightarrow Z_\eta^2 + Z_\xi^2$.

Next, apply Rosenthal's inequality (Theorem 2):

$$\frac{E|S_{n,q}^{(\lambda)}|^{2s+2\varepsilon}}{(V_n)^{s+\varepsilon}} \le C\frac{\sum_{k\in\mathfrak{B}(n)} E|B_{k,q}^{(\lambda)}|^{2s+2\varepsilon}}{(V_n)^{s+\varepsilon}} + C\frac{\big(\sum_{k\in\mathfrak{B}(n)} E|B_{k,q}^{(\lambda)}|^2\big)^{s+\varepsilon}}{(V_n)^{s+\varepsilon}}. \tag{4.4}$$

Because of the truncation level and the definition of q,

$$\frac{\sum_{k\in\mathfrak{B}(n)} E|B_{k,q}^{(\lambda)}|^{2s+2\varepsilon}}{(V_n)^{s+\varepsilon}} \le \frac{V_n((2V_n)^q)^{2s+2\varepsilon}}{(V_n)^{s_\varepsilon}} \to 0. \tag{4.5}$$

Also notice that

$$\left(\frac{\sum_{k\in\mathfrak{B}(n)} E|B_{k,q}^{(\lambda)}|^2}{V_n}\right)^{s+\varepsilon} \to \big(E|X_0|^2\big)^{s+\varepsilon}, \tag{4.6}$$

since Lemma 2 applies to the fraction on the left-hand side. Equation (4.6) implies that

$$\sup_n\left(\frac{\sum_{k\in\mathfrak{B}(n)} E|B_{k,q}^{(\lambda)}|^2}{V_n}\right)^{s+\varepsilon} < \infty. \tag{4.7}$$

Therefore, if equations (4.4)–(4.7) are combined, it is easy to see that

$$\sup_n E\left(\frac{|S_{n,q}^{(\lambda)}|^2}{V_n}\right)^{s+\varepsilon} < \infty,$$

hence by Theorem 3,

$$\lim_n E\left(\frac{|S_{n,q}^{(\lambda)}|^2}{V_n}\right)^{s} = E\big(Z_\eta^2 + Z_\xi^2\big)^s. \tag{4.8}$$

The burden now is to prove that

$$\lim_n \frac{E|T_{k,q}^{(\lambda)}|^{2s}}{(V_n)^s} = 0. \tag{4.9}$$

The Rosenthal inequality again does the job:

$$\frac{E|T_{k,q}^{(\lambda)}|^{2s}}{(V_n)^s} \le C\frac{1}{(V_n)^{s-1}} \cdot \frac{\sum_{\mathfrak{B}(n)} E|T_{k,q}^{(\lambda)}|^{2s}}{V_n} + C\left(\frac{\sum_{\mathfrak{B}(n)} E|T_{k,q}^{(\lambda)}|^{2}}{V_n}\right)^s. \tag{4.10}$$

Equation (3.6) of Lemma 7 implies that the second summand on the right-hand side of (4.10) converges to zero. As for the other summand, notice that for every $k \in \mathfrak{B}(n)$, $E|T_{k,q}^{(\lambda)}|^{2s} \le E|X_0|^{2s}$, hence so is the average:

$$\frac{\sum_{\mathfrak{B}(n)} E|T_{k,q}^{(\lambda)}|^{2s}}{V_n} \le E|X_0|^{2s} < \infty.$$

Since $s > 1$, $1/(V_n)^{1-s} \to 0$, which proves that (4.9) holds.

Finally, apply Minkowski's inequality to $\|S_n^{(\lambda)}\|_{2s}$:

$$\|S_{n,q}^{(\lambda)}\|_{2s} - \left\|\sum T_{k,q}^{(\lambda)}\right\|_{2s} \le \|S_n^{(\lambda)}\|_{2s} \le \|S_{n,q}^{(\lambda)}\|_{2s} + \left\|\sum T_{k,q}^{(\lambda)}\right\|_{2s}. \tag{4.11}$$

Divide by $\sqrt{V_n}$ in the appropriate places in (4.11) to see that

$$\|I_n^{(\lambda)}\|_s \to \left(E\left(Z_\eta^2 + Z_\xi^2\right)^s\right)^{1/2s},$$

which proves Corollary 1. □

4.3 Coming full circle

In Section 1.4, I mentioned how Miller and Picard proved similar results under the ρ^*-mixing assumption. To reiterate, Miller (1995) showed that when $s = 2$, in (4.3), the formula becomes

$$\lim_n E\big(I_n(\lambda)\big)^2 = 2\big(f(\lambda)\big)^2 + 4\big|h(\lambda)\big|^2,$$

and, Picard (2006) showed that when $s = 3$, it is

$$\lim_n E\big(I_n(\lambda)\big)^3 = 6\big(f(\lambda)\big)^3 + 36\big|h(\lambda)\big|^2 f(\lambda).$$

To see why these two formulae agree with Corollary 1, note that

$$Z_\xi = \frac{h(\lambda)}{f_\eta(\lambda)} Z_\eta + \sqrt{f_\xi(\lambda) - \frac{(h(\lambda))^2}{f_\eta(\lambda)}} \cdot Z,$$

where Z is a $N(0, 1)$ random variable which is independent of Z_η. By symmetry

$$Z_\eta = \frac{h(\lambda)}{f_\eta(\lambda)} Z_\xi + \sqrt{f_\eta(\lambda) - \frac{(h(\lambda))^2}{f_\xi(\lambda)}} \cdot \widetilde{Z},$$

where $\widetilde{Z}$ is a $N(0, 1)$ random variable which is independent of Z_ξ. The remaining work consists of standard calculations, which are left to the reader.

Acknowledgments

The author wishes to thank Richard Bradley for posing the question, and also Frederic Picard for his generous encouragement. He would also like to thank the reviewers for their helpful comments.

References

Billingsley, P. (1995). *Probability and Measure*, 3rd ed. New York: Wiley. MR1324786

Bradley, R. C. (1994). On regularity conditions for random fields. *Proc. Amer. Math. Soc.* **121**, 593–598. MR1219721

Bradley, R. C. (2007). *Introduction to Strong Mixing Conditions, Vol. 3*. Heber City, UT: Kendrick Press. MR2325296

Bradley, R. C. (2010). On the dependence coefficients associated with three mixing conditions for random fields. In *Dependence in Analysis, Probability and Number Theory* (I. Berkes, R. C. Bradley, H. Dehling, M. Peligrad and R. Tichy, eds.) 89–121. Heber City, UT: Kendrick Press. MR2731040

Ibragimov, I. A. and Linnik, Y. V. (1971). *Independent and Stationary Sequences of Random Variables*. Groningen: Wolters-Noordhoff Publishing. MR0322926

Ibragimov, I. A. and Rozanov, Y. A. (1978). *Gaussian Random Processes*. New York: Springer-Verlag. MR0543837

Miller, C. (1995). A CLT for the periodograms of a ρ^*-mixing random field. *Stochastic Process. Appl.* **60**, 313–330. MR1376806

Picard, F. (2006). Some problems concerning multilinear forms. Ph.D. thesis, Indiana Univ. MR2708906

Rosenblatt, M. (1985). *Stationary Sequences and Random Fields*. Boston: Birkhäuser. MR0885090

Rudin, W. (1964). *Principles of Mathematical Analysis*, 2nd ed. New York: McGraw-Hill. MR0166310

Tone, C. (2011). Central limit theorems for Hilbert-space valued random field satisfying a strong mixing condition. *ALEA Lat. Am. J. Prob. Math. Stat.* **8**, 77–94. MR2754401

Withers, C. S. (1981). Central limit theorems for dependent random variables I. *Z. Wahrsch. Verw. Gebiete* **57**, 509–534. MR0631374

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