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The integral of the product of a power and Bessel's K_ν function

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Abstract. Withers and Nadarajah [*Braz. J. Probab. Stat.* **28** (2014) 140–149] gave new expressions for hypergeometric functions when two arguments differ by an integer. Here, we give new expressions for $\int_{x_0}^{x} x^\mu K_\nu(x)\,dx$ when $\mu \pm \nu$ is an integer, where $K_\nu(\cdot)$ denotes the modified Bessel function of order ν. Each new expression is a finite sum of terms involving only the gamma function and the modified Bessel function.

1 Introduction

Let $K_\nu = K_\nu(x)$ denote the modified Bessel function of order ν. This function has applications in many areas of the sciences and engineering. Some examples are distribution theory in statistics, oscillations and stability of the theory of plates, rods and shells, heat wave propagation, wave propagation in electric transmission lines and oscillations of liquid in a vertical cylindrical and prismatic tank. We refer the readers to Watson (1995) and Korenev (2002) for most excellent accounts of the theory and applications of the K_ν function. See also Sections 6.5–6.7 of Gradshteyn and Ryzhik (2007) and Sections 1.12 and 2.11 of Prudnikov et al. (1986).

The aim of this short note is to provide some new expressions for finite integrals of the product of a power and the K_ν function. The few cases, where

$$d_{\mu,\nu} = d_{\mu,\nu}(x_0, x) = \int_{x_0}^{x} x^\mu K_\nu(x)\,dx$$

is available are given by Wheelon (1968, pages 108–109) and Prudnikov et al. (1986, equations (2.16.2.3), (2.16.2.4), (2.16.3.2) and (2.16.3.7)). In terms of $D_{\mu,\nu} = d_{\mu,\nu}(0, x)$ and $\overline{D}_{\mu,\nu} = d_{\mu,\nu}(x, \infty)$, the ones given by Wheelon (1968) are:

$$D_{\nu,\nu-1} = 2^{\nu-1}\Gamma_\nu - x^\nu K_\nu \tag{1.1}$$

for $\mathrm{Re}(\nu) > 0$ and $\Gamma_\nu = \Gamma(\nu)$;

$$D_{-\nu,\nu+1} = -x^{-\nu} K_\nu \tag{1.2}$$

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for $\mathrm{Re}(\nu) > 0$; and,

$$D_{\nu,\nu} = 2^{\nu-1}\pi^{1/2}\Gamma(\nu+1/2)xX_\nu \tag{1.3}$$

for $X_\nu = K_\nu L_{\nu-1} + L_\nu K_{\nu-1}$ and L_ν the modified Struve function defined by

$$\begin{aligned} L_\nu = L_\nu(x) &= \sum_{k=0}^{\infty}(x/2)^{2k+\nu+1}\Gamma(k+3/2)^{-1}\Gamma(k+\nu+3/2)^{-1} \\ &= (x/2)^{\nu+1}\Gamma(3/2)^{-1}\Gamma(\nu+3/2)^{-1}{}_1F_2\big(1;3/2,\nu+3/2:x^2/4\big), \end{aligned}$$

where ${}_1F_2$ is the hypergeometric function. The expressions for $D_{\mu,\nu}$ and $\overline{D}_{\mu,\nu}$ given by Prudnikov et al. (1986) appear complicated and involve linear combinations of hypergeometric functions: equation (2.16.2.3) expresses $D_{\mu,\nu}$ as a linear combination of two different ${}_2F_3$ hypergeometric functions; equation (2.16.2.4) expresses $\overline{D}_{\mu,\nu}$ as a linear combination of four different ${}_2F_3$ hypergeometric functions; equation (2.16.3.2) expresses $D_{\mu,\nu}$ as a linear combination of two different ${}_1F_2$ hypergeometric functions; equation (2.16.3.7) expresses $\overline{D}_{\mu,\nu}$ as a linear combination of three different ${}_1F_2$ hypergeometric functions. Prudnikov et al. (1986) also give some simpler expressions for the particular cases $\mu-\nu=-1,0,1$. Finally, by equation (3.1.45) of Mathai and Saxena (1973, page 78),

$$d_{\mu-1,\nu}(0,\infty) = 2^{\mu-2}\Gamma\big((\mu\pm\nu)/2\big) \tag{1.4}$$

for $\mathrm{Re}(\mu\pm\nu) > 0$. Also given in Mathai and Saxena (1973) are references to tables for $\overline{D}_{0,0}$ and $\overline{D}_{-1,0}$.

The expressions for $D_{\mu,\nu}$ and $\overline{D}_{\mu,\nu}$ given in this short note are specialized for integer values of $\mu\pm\nu$. The given expressions are new and perhaps the simplest known to date. They do not involve complicated hypergeometric functions, as in Prudnikov et al. (1986, Section 2.11). They involve only gamma functions and modified Bessel functions.

The results of this short note are organized as follows. In Section 2, we show how to obtain $d_{\mu,\nu}$ from those for $\mu-\nu=0,1,2,\ldots$ and $\mu+\nu=1,3,5,\ldots$, see Theorems 2.1, 2.2 and 2.3. We also show how to express $d_{\mu,\nu}$ in terms of $d_{-\nu,\nu}$ for $\mu+\nu=2,4,6,\ldots$. An expression for $D_{-\nu,\nu}$ analogous to (1.3) is obtainable.

By Watson (1995, page 133), we have the recurrence formula for $C_\nu = J_\nu$ or Y_ν:

$$\begin{aligned} \int_0^x x^{\mu+1}C_\nu(x)\,dx = -c\big(\mu^2-\nu^2\big)\int_0^x x^{\mu-1}C_\nu(x)\,dx + x^{\mu+1}C_{\nu+1}(x) \\ + c(\mu-\nu)x^\mu C_\nu(x), \end{aligned}$$

where $c=1$. One may derive from this the same relation for $C_\nu = I_\nu$ with $c=-1$. This suggests that it may also hold for $C_\nu = \exp(\nu\pi\mathrm{i})K_\nu$ with $c=-1$ and $\mathrm{i}=\sqrt{-1}$: c.f. pages 361, 376 of Abramowitz and Stegun (1964).

2 Main results

Again, we set $K_\nu = K_\nu(x)$. Its derivative is $\dot{K}_\nu = -K_{\nu-1} - x^{-1}\nu K_\nu$. So, integrating by parts

$$d_{\mu,\nu} = (\mu+1)^{-1}\left\{x^{\mu+1}K_\nu|_{x_0}^{x} + \int_{x_0}^{x} x^{\mu+1}\left(K_{\nu-1} + x^{-1}\nu K_\nu\right)dx\right\}.$$

So,

$$(\mu+1-\nu)d_{\mu,\nu} = x^{\mu+1}K_\nu|_{x_0}^{x} + d_{\mu+1,\nu-1},$$

and

$$d_{\mu,\nu} = (\mu-\nu-1)d_{\mu-1,\nu+1} - x^{\mu}K_{\nu+1}|_{x_0}^{x}. \tag{2.1}$$

Replacing $\nu+1$ by $-\nu-1$ and using $K_{-\nu} = K_\nu$ gives

$$d_{\mu,\nu+2} = (\mu+\nu+1)d_{\mu-1,\nu+1} - x^{\mu}K_{\nu+1}|_{x_0}^{x},$$

that is

$$d_{\mu,\nu} = (\mu+\nu-1)d_{\mu-1,\nu-1} - x^{\mu}K_{\nu-1}|_{x_0}^{x}. \tag{2.2}$$

Note that (2.1) and (2.2) are consistent with (1.2). A special case of (2.1) is

$$d_{\nu+1,\nu} = -x^{-\nu+1}K_{\nu+1}|_{x_0}^{x}.$$

So, by (1.1), $x^{\nu}K_\nu \to 2^{\nu-1}\Gamma_\nu$ as $x \to 0$ for $\mathrm{Re}(\nu) > 0$. Since $K_\nu \to 0$ exponentially as $x \to \infty$, $x^{\mu}K_\nu|_{x}^{\infty} = -x^{\mu}K_\nu$. So, for example,

$$d_{\nu+1,\nu}(x,\infty) = x^{\nu+1}K_{\nu+1}, \qquad \text{that is } d_{\nu,\nu-1}(x,\infty) = x^{\nu}K_\nu. \tag{2.3}$$

Wheelon (1968) does not give the condition $\mathrm{Re}(\nu) > 0$ in (1.2). If it were not necessary, then we could replace ν by $-\nu$ in (1.1) to obtain

$$d_{\nu,\nu+1}(0,x) = -x^{\nu}K_\nu.$$

Adding this to (2.3) gives $0 = d_{\nu,\nu-1}(0,\infty)$, which by (1.4) is false for $\mathrm{Re}(\nu) > 0$.

Adding (1.1) to (2.3) gives $d_{\nu,\nu-1}(0,\infty) = 2^{\nu-1}\Gamma_\nu$ for $\mathrm{Re}(\nu) > 0$, consistent with (1.4). By (2.1), for $\mathrm{Re}(\mu-\nu) \geq 1$

$$D_{\mu,\nu} = (\mu-\nu-1)D_{\mu-1,\nu+1} - x^{\mu}K_{\nu+1} + 2^{\nu-1}\Gamma_\nu\delta_{\mu,\nu+1},$$

where $\delta_{\mu,\nu} = 1$ if $\mu = \nu$ and 0 if $\mu \neq \nu$. Putting $\mu = \nu+2, \nu+3, \ldots$ gives

$$\begin{aligned}
D_{\nu+2,\nu} &= D_{\nu+1,\nu+1} - x^{\nu+2}K_{\nu+1},\\
D_{\nu+3,\nu} &= 2D_{\nu+2,\nu+1} - x^{\nu+3}K_{\nu+1} = 2^{\nu+2}\Gamma_{\nu+2} - 2x^{\nu+2}K_{\nu+2} - x^{\nu+3}K_{\nu+1},\\
D_{\nu+4,\nu} &= 3D_{\nu+3,\nu+1} - x^{\nu+4}K_{\nu+1} = 3D_{\nu+2,\nu+2} - 3x^{\nu+3}K_{\nu+2} - x^{\nu+4}K_{\nu+1}.
\end{aligned}$$

In general, we have the following.

Theorem 2.1. *We have $D_{\mu,\nu} = d_{\mu,\nu}(0, x)$ for $\mu - \nu = 0, 1, \ldots$ given by*

$$D_{\nu+2i+1,\nu} = [2i]_i^2 2^{\nu+i}\Gamma_{\nu+i+1} - \sum_{j=0}^{i}[2i]_j^2 x^{\nu+2i+1-j}K_{\nu+j+1},$$

$$D_{\nu+2i,\nu} = [2i-1]_{i-1}^2 D_{\nu+i,\nu+i} - \sum_{j=0}^{i}[2i-1]_j^2 x^{\nu+2i-j}K_{\nu+j+1}$$

for $i = 0, 1, \ldots$, where $[a]_j^2 = a(a-2)(a-4)\cdots(a-2j+2)$ if $j > 0$, $[a]_0^2 = 1$, and $D_{\nu+i,\nu+i}$ is given by (1.3).

By (1.4), $d_{\nu,\nu}(0,\infty) = 2^{\nu-1}\pi^{1/2}\Gamma_{\nu+1/2}$ for $\mathrm{Re}(\nu) > -1/2$. So, by (1.3),

$$\overline{D}_{\nu,\nu} = 2^{\nu-1}\pi^{1/2}\Gamma_{\nu+1/2}(1 - xX_\nu) \tag{2.4}$$

for $\mathrm{Re}(\nu) > -1/2$. By (2.1),

$$\overline{D}_{\mu,\nu} = (\mu - \nu - 1)\overline{D}_{\mu-1,\nu+1} + x^\mu K_{\nu+1}.$$

Putting $\mu = \nu + 1, \nu + 2, \ldots$ gives

$$\begin{aligned}
\overline{D}_{\nu+1,\nu} &= x^{\nu+1}K_{\nu+1},\\
\overline{D}_{\nu+2,\nu} &= \overline{D}_{\nu+1,\nu+1} + x^{\nu+2}K_{\nu+1},\\
\overline{D}_{\nu+3,\nu} &= 2\overline{D}_{\nu+2,\nu+1} + x^{\nu+3}K_{\nu+1} = 2x^{\nu+2}K_{\nu+2} + x^{\nu+3}K_{\nu+1}.
\end{aligned}$$

In general, we have the following.

Theorem 2.2. *We have $\overline{D}_{\mu,\nu} = d_{\mu,\nu}(x,\infty)$ for $\mu - \nu = 1, 2, \ldots$ given by*

$$\overline{D}_{\nu+2i-1,\nu} = \sum_{j=0}^{i-1}[2i-2]_j^2 x^{\nu+2i-1-j}K_{\nu+j+1},$$

$$\overline{D}_{\nu+2i,\nu} = [2i-1]_{i-1}^2\overline{D}_{\nu+1,\nu+1} + \sum_{j=0}^{i-1}[2i-1]_j^2 x^{\nu+2i-j}K_{\nu+j+1}$$

for $i = 1, 2, \ldots$, where $\overline{D}_{\nu+i,\nu+i}$ is given by (2.4).

To summarize, Theorems 2.1 and 2.2 give $\int_{x_0}^{x} x^\mu K_\nu(x)\,dx$ for $\mu - \nu = 0, 1, 2, \ldots$. We now turn to the case $\mu + \nu = 1, 3, 5, \ldots$. By (2.2),

$$D_{\mu,\nu} = (\mu + \nu - 1)D_{\mu-1,\nu-1} - x^\mu K_{\nu-1} + \delta_{\mu,\nu-1}2^{\nu-2}\Gamma_{\nu-1}$$

and

$$\overline{D}_{\mu,\nu} = (\mu + \nu - 1)\overline{D}_{\mu-1,\nu-1} + x^\mu K_{\nu-1} \tag{2.5}$$

for $\mathrm{Re}(\mu-\nu)\geq 1$ and $\mathrm{Re}(\nu)>1$. Putting $\mu=1-\nu$ gives $\overline{D}_{1-\nu,\nu}=x^{1-\nu}K_{\nu-1}$, so $\overline{D}_{-\nu,\nu+1}=x^{-\nu}K_\nu$. Adding this to (1.2) gives

$$d_{-\nu,\nu+1}(0,\infty)=0 \tag{2.6}$$

for $\mathrm{Re}(\nu)>0$. This does not contradict (1.4) because its parameter values are out of range. Putting $\mu=2-\nu,3-\nu$ in (2.5) gives

$$\begin{aligned}\overline{D}_{2-\nu,\nu}&=\overline{D}_{1-\nu,\nu-1}+x^{2-\nu}K_{\nu-1},\\ \overline{D}_{3-\nu,\nu}&=2\overline{D}_{2-\nu,\nu-1}+x^{3-\nu}K_{\nu-1}.\end{aligned} \tag{2.7}$$

Similarly, $\mu=4-\nu$ in (2.5) gives

$$\begin{aligned}\overline{D}_{4-\nu,\nu}&=3\overline{D}_{3-\nu,\nu-1}+x^{4-\nu}K_{\nu-1}\\ &=3d_{3-\nu,\nu-1}(0,\infty)-3D_{3-\nu,\nu-1}+x^{4-\nu}K_{\nu-1}.\end{aligned}$$

The first term on the right-hand side is given by (1.4) for $\mathrm{Re}(\nu)<5/2$. The second term is given by (2.7) for $2<\mathrm{Re}(\nu)<5/2$ in terms of $D_{-\nu,\nu}$. Also $D_{1-\nu,\nu}=-x^{1-\nu}K_{\nu-1}$ for $\mathrm{Re}(\nu)>1$ and $D_{1-\nu,\nu}=d_{1-\nu,\nu}(0,\infty)-\overline{D}_{1-\nu,\nu}=-\overline{D}_{1-\nu,\nu}$ by (2.6). So, $\overline{D}_{1-\nu,\nu}=x^{1-\nu}K_{\nu-1}$ for $\mathrm{Re}(\nu)>1$, $\overline{D}_{2-\nu,\nu-1}=x^{2-\nu}K_{\nu-2}$ for $\mathrm{Re}(\nu)>2$, and $\overline{D}_{3-\nu,\nu}=2x^{2-\nu}K_{\nu-2}+x^{3-\nu}K_{\nu-1}$ for $\mathrm{Re}(\nu)>2$. Putting $\mu=5-\nu,7-\nu,\ldots$ in (2.5) gives

$$\overline{D}_{5-\nu,\nu}=4\overline{D}_{4-\nu,\nu-1}+x^{5-\nu}K_{\nu-1}=4\cdot 2x^{3-\nu}K_{\nu-3}+4x^{4-\nu}K_{\nu-2}+x^{5-\nu}K_{\nu-1}$$

for $\mathrm{Re}(\nu)>3$. In general, we have the following.

Theorem 2.3. *We have $\overline{D}_{\mu,\nu}=d_{\mu,\nu}(x,\infty)$ for $\mu+\nu=1,3,\ldots$ given by*

$$\overline{D}_{2i+1-\nu,\nu}=\sum_{j=0}^{i}[2i]_j^2x^{-\nu+2i+1-j}K_{\nu-1-j}$$

for $i=0,1,\ldots$ and $\mathrm{Re}(\nu)>i+1$.

Similarly, $D_{2i-\nu,\nu}$ can be reduced to $D_{-\nu,\nu}$. An expression for $D_{-\nu,\nu}$ is obtainable analogous to (1.3).

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